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A nonlocal free boundary problem

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Abstract. Given $s, \sigma \in (0,1)$ and a bounded domain $\Omega \subset \mathbb{R}^n$, we consider the following minimization problem of s-Dirichlet plus σ-perimeter type

$$[u]_{H^s(\mathbb{R}^{2n}\setminus(\Omega^c)^2)} + \operatorname{Per}_\sigma(\{u>0\}, \Omega),$$

where $[\cdot]_{H^s}$ is the fractional Gagliardo seminorm and $\operatorname{Per}_\sigma$ is the fractional perimeter.

Among other results, we prove a monotonicity formula for the minimizers, glueing lemmata, uniform energy bounds, convergence results, a regularity theory for the planar cones and a trivialization result for the flat case.

Several classical free boundary problems are limit cases of the one that we consider in this paper, as $s \nearrow 1$, $\sigma \nearrow 1$ or $\sigma \searrow 0$.

1. Introduction

In this paper we deal with a free boundary problem driven by some nonlocal features. The nonlocal structures that we consider appear both in the term that is sometimes related to "elastic" atomic interactions and in the so-called "surface tension" potential.

These two features are allowed to have different nonlocal behaviors, namely we parameterize them with two different fractional parameters $s, \sigma \in (0,1)$. Several classical free boundary problems appear in the limit of our framework by taking limits either in s (as $s \nearrow 1$) or in σ (as $\sigma \nearrow 1$ or $\sigma \searrow 0$), or both.

More precisely, we will consider here the minimization of an energy functional that involves a fractional gradient and a nonlocal perimeter. Given $s, \sigma \in (0,1)$ and a bounded and Lipschitz domain $\Omega \subset \mathbb{R}^n$, we consider

$$\mathcal{F}(u,E) := \iint_{\mathbb{R}^{2n}\setminus(\Omega^c)^2} \frac{|u(x)-u(y)|^2}{|x-y|^{n+2s}}\,dx\,dy + \operatorname{Per}_\sigma(E,\Omega), \tag{1.1}$$

where E is the positivity set for u (more precisely, $u \geqslant 0$ a.e. in $E \cap \Omega$ and $u \leqslant 0$ a.e. in $E^c \cap \Omega$). As customary, the superscript c used here above denotes the complementary set operation, i.e. $\Omega^c := \mathbb{R}^n \setminus \Omega$. The σ-fractional perimeter $\operatorname{Per}_\sigma(E,\Omega)$ of a set E in Ω was introduced in [6] and it is defined as

$$\begin{aligned}\operatorname{Per}_\sigma(E,\Omega) :=& \mathcal{L}(E\cap\Omega, E^c\cap\Omega) \\ &+ \mathcal{L}(E\cap\Omega, E^c\cap\Omega^c) + \mathcal{L}(E\cap\Omega^c, E^c\cap\Omega),\end{aligned} \tag{1.2}$$

where the interaction $\mathcal{L}$ is the following

$$\mathcal{L}(A,B) := \iint_{A\times B} \frac{dx\,dy}{|x-y|^{n+\sigma}} \tag{1.3}$$

for any disjoint, measurable sets A and B.

The nonlocal perimeter converges to the classical perimeter as $\sigma \nearrow 1$ and to the Lebesgue measure of E as $\sigma \searrow 0$ (up to multiplicative constants), see [10, 4, 17, 14] for precise statements.

In [8] the authors consider a minimization problem that corresponds to (1.1) in the case $s = 1$, namely

$$\int_\Omega |\nabla u(x)|^2\, dx + \operatorname{Per}_\sigma(\{u > 0\}, \Omega)\,. \tag{1.4}$$

They use blow-up analysis to obtain regularity results for minimizers and for the free boundaries. When $\sigma \searrow 0$, the functional in (1.4) reduces to a classical free boundary problem related to fluid dynamics and that has been extensively studied in the literature after the pioneer work in [2, 3]. On the other hand, when $\sigma \nearrow 1$, the energy in (1.4) reduces to the problem studied in [5], where the energy functional is a competition between the classical Dirichlet form and the perimeter of the interface.

The energy functional in (1.1) that we study here is thus a variation of these type of problems, in which both the quadratic form and the interface energy appearing in the functional are of nonlocal type.

For other recent results on fractional free boundary problems see, for instance, [7, 11, 12, 1].

The variational notion of minimizers that we consider in this paper is the following. Fixed $E_0 \subseteq \mathbb{R}^n$ with locally finite σ-perimeter and $\varphi \in H^s_{\mathrm{loc}}(\mathbb{R}^n)$ with $\varphi \geqslant 0$ a.e. in E_0 and $\varphi \leqslant 0$ a.e. in E_0^c, we say that (u, E) is a minimizing pair (in the domain Ω with external datum φ) if $\mathcal{F}(u, E)$ attains the minimal possible value among all the functions v such that

$$v - \varphi \in H^s(\mathbb{R}^n) \text{ with } v = \varphi \text{ a.e. in } \Omega^c \tag{1.5}$$

and all the measurable sets $F \subseteq \mathbb{R}^n$ with $F \setminus \Omega = E \setminus \Omega$ and such that

$$v \geqslant 0 \text{ a.e. in } F \cap \Omega \text{ and } v \leqslant 0 \text{ a.e. in } F^c \cap \Omega. \tag{1.6}$$

In spite of its technical flavor, the definition above can be intuitively understood by saying, roughly speaking, that the function u minimizes the energy functional among all the competitors v that coincide with u outside the domain Ω (the technicality is to formally state that F is the positivity set of v for which we need to compute the σ-perimeter).

The existence of minimizing pairs will be guaranteed by the forthcoming Lemma 3.1 and it follows from the direct method joined with a suitable fractional compact embedding.

We will show that the energy of a minimizing pair can be bounded uniformly: more precisely, if (u, E) is a minimizing pair in a given ball, then the energy in a smaller ball is bounded, according to the next result:

Theorem 1.1 (Uniform energy estimates)**.** *Let (u, E) be a mininimizing pair in B_2. Then*

$$\iint_{\mathbb{R}^{2n}\setminus(B_1^c)^2} \frac{\big|u(x) - u(y)\big|^2}{|x - y|^{n+2s}}\, dx\, dy + \operatorname{Per}_\sigma(E, B_1) \leqslant C \left(1 + \int_{\mathbb{R}^n} \frac{|u(y)|^2}{1 + |y|^{n+2s}}\, dy\right),$$

for some $C > 0$ only depending on n and s.

The proof of Theorem 1.1 relies on appropriate gluing results that are interesting in themselves (roughly speaking, they allow us to change an admissible pair outside a given domain, by controlling the energy produced by the interpolation).

For this, it is useful to consider an associated extension problem. That is, we set $\mathbb{R}^{n+1}_+ := \{(x,z) \in \mathbb{R}^n \times \mathbb{R} \text{ s.t. } z > 0\}$, and, given a function $u : \mathbb{R}^n \to \mathbb{R}$, we associate a function $\overline{u}$ defined in $\mathbb{R}^{n+1}_+$ as

$$\overline{u}(\cdot, z) = u * P_s(\cdot, z), \text{ where } P_s(x,z) := c_{n,s} \frac{z^{2s}}{(|x|^2 + z^2)^{(n+2s)/2}}. \tag{1.7}$$

Here $c_{n,s}$ is a normalizing constant depending on n and s.

Moreover, given a measurable set $E \subset \mathbb{R}^n$ we associate a function U defined in $\mathbb{R}^{n+1}_+$ as

$$U(\cdot, z) = (\chi_E - \chi_{E^c}) * P_\sigma(\cdot, z), \text{ where } P_\sigma(x,z) := c_{n,\sigma} \frac{z^{\sigma}}{(|x|^2 + z^2)^{(n+\sigma)/2}}, \tag{1.8}$$

and $c_{n,\sigma}$ is a normalizing constant depending on n and σ (these constants are only needed to normalize the integral of P_s and P_σ).

We will denote the extended variable as $X := (x,z) \in \mathbb{R}^{n+1}_+$, where $x \in \mathbb{R}^n$ and $z > 0$. Moreover, $B_r := \{|x| < r\}$ is the ball of radius r in $\mathbb{R}^n$ and $\mathcal{B}^+_r := \{|X| < r\}$ is the ball of radius r in $\mathbb{R}^{n+1}_+$.

We will study in detail the extended problem in Section 4, where we will also find equivalent minimizing conditions between the original functional in (u, E) and an extended functional in $(\overline{u}, U)$ (see in particular Proposition 4.1). Here we just mention that the notion of minimization in the extended variables in a domain $\Omega \subset \mathbb{R}^{n+1}$ requires not only that the competing functions agree near $\partial\Omega$, but also a consistency condition on the trace $\{z = 0\}$, where the functions reduce to characteristic functions of sets. Namely, we say that $(\overline{u}, U)$ is a minimizing pair for the extended problem in $\Omega \subset \mathbb{R}^{n+1}$ if

$$\begin{aligned} &\int_{\Omega_+} z^{1-2s} |\nabla \overline{u}|^2 \, dX + c_{n,s,\sigma} \int_{\Omega_+} z^{1-\sigma} |\nabla U|^2 \, dX \\ &\qquad \leqslant \int_{\Omega_+} z^{1-2s} |\nabla \overline{v}|^2 \, dX + c_{n,s,\sigma} \int_{\Omega_+} z^{1-\sigma} |\nabla V|^2 \, dX \end{aligned}$$

for every functions $\overline{v}$ and V that satisfy the following conditions:

i) $V = U$ in a neighborhood of $\partial\Omega$,
ii) the trace of V on $\{z = 0\}$ is $\chi_F - \chi_{F^c}$ for some set $F \subset \mathbb{R}^n$,
iii) $\overline{v} = \overline{u}$ in a neighborhood of $\partial\Omega$, and $\overline{v}\big|_{\{z=0\}} \geqslant 0$ a.e. in F, $\overline{v}\big|_{\{z=0\}} \leqslant 0$ a.e. in F^c.

In this setting, we can use glueing techniques to prove convergence of minimizing pairs of the extended problem, as stated in the following result:

Theorem 1.2 (Convergence of minimizers)**.** *Let $(\overline{u}_m, U_m)$ be a sequence of minimizing pairs for the extended problem in $\mathcal{B}^+_2$. Suppose that $\overline{u}_m$ is the extension of u_m as in* (1.7)*, and*

$$u_m \to u \ \text{ in } L^\infty(B_2), \quad \overline{u}_m \to \overline{u} \ \text{ in } L^\infty(\mathcal{B}^+_2) \ \text{ and } \ U_m \to U \ \text{ in } L^2_{\sigma/2}(\mathcal{B}^+_2), \tag{1.9}$$

as $m \to +\infty$, for some couple $(\overline{u}, U)$, with $\overline{u}$ continuous in $\overline{\mathbb{R}^{n+1}_+}$.

Then $(\overline{u}, U)$ is a minimizing pair in $\mathcal{B}^+_{1/2}$.

Moreover

$$\lim_{m\to+\infty}\int_{\mathcal{B}_1^+} z^{1-2s}|\nabla\overline{u}_m|^2\,dX = \int_{\mathcal{B}_1^+} z^{1-2s}|\nabla\overline{u}|^2\,dX \tag{1.10}$$
$$\text{and}\qquad \lim_{m\to+\infty}\int_{\mathcal{B}_1^+} z^{1-\sigma}|\nabla U_m|^2\,dX = \int_{\mathcal{B}_1^+} z^{1-\sigma}|\nabla U|^2\,dX.$$

A particularly important case of convergence is given by the blow-up limit. This is also related to the study of the minimizing pairs that possess suitable homogeneity properties, and in particular the ones induced by the natural scaling of the functional. For this, we say that a minimizing pair (u, E) is a minimizing cone if u is homogeneous of degree $s - \frac{\sigma}{2}$ and E is a cone (i.e., for any $t > 0$, $tx \in E$ if and only if $x \in E$).

In this framework, we exploit Theorems 1.1 and 1.2, combined with some arguments in [6], and we obtain the following relation between blow-up limits and minimizing cones:

Theorem 1.3 (Blow-up cones)**.** *Let $s > \sigma/2$ and (u, E) be a minimizing pair in B_1, with $0 \in \partial E$. For any $r > 0$ let*

$$u_r(x) := r^{\frac{\sigma}{2}-s}u(rx) \quad \text{and} \quad E_r := \frac{1}{r}E. \tag{1.11}$$

Assume that $u \in C^{s-\frac{\sigma}{2}}(\mathbb{R}^n)$. Then there exist a minimizing cone (u_0, E_0) and a sequence $r_k \to 0$ such that $u_{r_k} \to u_0$ in $L^\infty_{\mathrm{loc}}(\mathbb{R}^n)$ and $E_{r_k} \to E_0$ in $L^1_{\mathrm{loc}}(\mathbb{R}^n)$.

We remark that the rescaling in (1.11) is the one induced by the energy, since if (u, E) is a minimizing pair for $\mathcal{F}$ in Ω, then (u_r, E_r) is a minimizing pair for $\mathcal{F}$ in $\frac{1}{r}\Omega$. Moreover, the exponent $\frac{\sigma}{2} - s$ in (1.11) corresponds to the one obtained in [8] in the case $s = 1$.

A complete classification of the minimal cones in dimension 2 holds true, according to the following result:

Theorem 1.4 (Classification of minimizing cones in the plane)**.** *Let $n = 2$ and let (u, E) be a minimizing pair in any domain.*

Assume that u is homogeneous of degree $s - \frac{\sigma}{2}$ and that E is the union of finitely many closed conical sectors, with both E and E^c nonempty.

Then E is a halfplane.

The proof of Theorem 1.4 uses a second order domain variation, in the spirit of the technique introduced in [18, 19] (since the main ideas of the proof are the same, but some technical differences arise here due to the presence of minimizing pairs rather than functions, we give the full details of the proof in Appendix A).

As a final remark, we point out that the natural scaling of the problem does not exhaust the complexity of the minimizers. This fact is typical for fractional free boundary problems (for instance, in [5], the natural scaling would produce a power $1/2$ and it is related to $C^{1/2}$-regularity, but Lipschitz regularity holds true in the end: compare, e.g., Theorem 3.1 and 4.1 in [5]).

For example, in our framework, a special scaling feature occurs when $s = \sigma/2$: in this case the Gagliardo seminorm and the fractional perimeter have exactly the same dimensional properties and one may think that, under this circumstance, a minimizing pair reduces to the characteristic function of a set, consistently with the fact that the blow-up limits are homogeneous of degree zero. But it turns out that this is not the case, as next observation points out:

Remark 1.5. *Let $s \in (0,1/2)$ and $\sigma = 2s$. Fix a set $E_0 \subseteq \mathbb{R}^n$ with locally finite σ-perimeter, and let $u_0 := \chi_{E_0} - \chi_{E_0^c}$.*

Let (u,E) be a minimizing pair in B_1 with respect to the datum (u_0,E_0) outside B_1.

Then, it is not true that $u = \chi_E - \chi_{E^c}$ (unless either $E = \mathbb{R}^n$ or $E = \varnothing$).

We also observe that the problem we consider may develop plateaus, i.e. fattening of the zero level set of minimizers. For instance, we point out that, in dimension 1 and for $s = 1/2$, it is not possible that $\{u = 0\}$ is just (locally) a single point, unless u is $(1/2)$-harmonic across the free boundary, as shown by the following simple example:

Remark 1.6. *Let $n = 1$, $s = 1/2$ and (u,E) be a minimizing pair in $(-1,1)$, with $u \in C([-1,1]) \cap H^{1/2}(\mathbb{R})$.*

Then either $(-\Delta)^{1/2}u = 0$ in $(-1,1)$ or the set $\{u = 0\} \cap (-1,1)$ contains infinitely many points.

We recall that the fattening of the zero level set of the minimizers also occur in other free boundary problems, see in particular Theorem 9.1 in [1].

In the subsequent section, we present some additional results that are auxiliary to the ones presented till now, but that we believe may have independent interest. A detailed plan about the organization of the paper will then be presented at the end of Section 2.

2. Additional results

Here we collect some further results that complete the picture described in Section 1 and that possess some independent interest. First of all, we obtain a Weiss-type monotonicity formula for minimizing pairs (u,E) (see [22] for the original monotonicity formula in the setting of classical free boundaries):

Theorem 2.1 (Monotonicity formula)**.** *Let (u,E) be a minimizing pair in B_ρ, and let $\overline{u}$ and U be as in (1.7) and (1.8). Then*

$$\begin{aligned}\Phi_u(r) := r^{\sigma-n}\left(\int_{\mathcal{B}_r^+} z^{1-2s}|\nabla\overline{u}|^2\,dX + c_{n,s,\sigma}\int_{\mathcal{B}_r^+} z^{1-\sigma}|\nabla U|^2\,dX\right)\\ -\left(s-\frac{\sigma}{2}\right) r^{\sigma-n-1}\int_{\partial\mathcal{B}_r^+} z^{1-2s}\,\overline{u}^2\,d\mathcal{H}^n\end{aligned} \tag{2.1}$$

is increasing in $r \in (0,\rho)$.

Moreover, Φ_u is constant if and only if $\overline{u}$ is homogeneous of degree $s - \frac{\sigma}{2}$ and U is homogeneous of degree 0.

We also show that the minimizing pairs enjoy a dimensional reduction property. Namely, if a minimizing pair is trivial in a given direction, then it can be sliced to a minimizing pair in one dimension less. Conversely, given a minimizing pair in $\mathbb{R}^n$, one obtains a minimizing pair in $\mathbb{R}^{n+1}$ by adding the trivial action of one dimension more. The formal statement of this property sounds as follows:

Theorem 2.2 (Dimensional reduction)**.** *The pair (u, E) is minimizing in any domain of $\mathbb{R}^n$ if and only if the pair $(u^\star, E^\star)$ is minimizing in any domain of $\mathbb{R}^{n+1}$, where $u^\star(x, x_{n+1}) := u(x)$ and $E^\star := E \times \mathbb{R}$.*

In the study of the local free boundary problems and minimal surfaces, homogeneous solutions and minimizing cones are often explicit and they constitute the easiest possible nontrivial example. In our case, the existence of nontrivial minimizing cones is not obvious, since the example of the halfspace trivializes, according to the following result:

Theorem 2.3 (Trivialization of halfspaces)**.** *Let (u, E) be a minimizing cone, with $u \in C(\mathbb{R}^n)$ and $[u]_{C^\gamma(\mathbb{R}^n)} < +\infty$, for some $\gamma \in (0, 1]$.*

If E is contained in a halfspace then $u \leqslant 0$.

Similarly, if E^c is contained in a halfspace then $u \geqslant 0$.

In particular, if E is a halfspace then u vanishes identically.

The proof of Theorem 2.3 relies on a suitable nonlocal maximum principle in unbounded domains that we explicitly state as follows:

Theorem 2.4 (Nonlocal maximum principle in a halfspace)**.** *Let D be an open set of $\mathbb{R}^n$, contained in the halfspace $\{x_n > 0\}$. Let $v \in L^\infty(D) \cap C^2(D)$ be continuous on $\overline{D}$ and such that*

$$\begin{cases} (-\Delta)^s v \leqslant 0 & \text{in } D, \\ v \leqslant 0 & \text{in } D^c. \end{cases} \tag{2.2}$$

Then $v \leqslant 0$ in D.

The rest of the paper will present all the material necessary to the proofs of the results presented here above and in Section 1. More precisely, in Section 3 we show some preliminary properties of the minimizing pairs.

In Section 4 we deal with an equivalent minimization problem on the extended variables and we use it to prove Theorem 2.1. The proof of the dimensional reduction of Theorem 2.2 is contained in Section 5.

Section 6 contains some glueing results that are interesting in themselves and that are used to prove the uniform energy estimates of Theorem 1.1, which are contained in Section 7, and the convergence result of Theorem 1.2, which is contained in Section 8. The convergence to blow-up cones, as detailed in Theorem 1.3, is proved in Section 9. Then, in Section 10 we prove Theorems 2.3 and 2.4. Finally, the proofs of Remarks 1.5 and 1.6 are contained in Sections 11 and 12, respectively.

3. Preliminaries

Here we discuss some basic properties of the minimizing pairs, such as existence and s-harmonicity.

Lemma 3.1. *The minimizing pair exists.*

Proof. Let (u_j, E_j) be a minimizing sequence. By compactness (see e.g. Theorem 7.1 in [13]) we infer that, up to subsequences, u_j converges to some u and χ_{E_j} converges to some χ_E in $L^2(\Omega)$ and a.e. in Ω. In fact, since u_j and χ_{E_j} are fixed outside Ω, the convergence holds a.e. in $\mathbb{R}^n$ and so, by Fatou Lemma, $\mathcal{F}(u, E)$ attains the desired minimum of the energy. It remains to show that this pair is admissible, i.e. $u \geqslant 0$ a.e. in $E \cap \Omega$ and $u \leqslant 0$ a.e. in $E^c \cap \Omega$. Indeed, let $x \in E \cap \Omega$.

Up to a set of null measure we have that $\chi_{E_j}(x) \to \chi_E(x) = 1$. Since the image of the characteristic function is a discrete set, it follows that $\chi_{E_j}(x) = 1$ for large j, hence $u_j(x) \geqslant 0$ and therefore $u(x) \geqslant 0$. Similarly, one can prove that $u \leqslant 0$ a.e. in $E^c \cap \Omega$. □

Lemma 3.2. *Let (u, E) be a minimizing pair. If Ω is an open subset of either $\{u > 0\}$ or $\{u < 0\}$, then $(-\Delta)^s u(x) = 0$ for any $x \in \Omega$. In particular, if $u \in C(\mathbb{R}^n)$, then $(-\Delta)^s u(x) = 0$ for any $x \in \{u > 0\} \cup \{u < 0\}$.*

Proof. Fix $x_o \in \Omega \subset \{u > 0\}$ (the case $\Omega \subset \{u < 0\}$ is similar). Then there exists $r > 0$ such that $B_r(x_o) \Subset \Omega$ and therefore

$$\mu := \min_{B_r(x_o)} u > 0.$$

Let $\eta \in C_0^\infty(B_r(x_o))$ and $\varepsilon \in \mathbb{R}$ with $|\varepsilon| < \mu \|\eta\|^{-1}_{L^\infty(\mathbb{R}^n)}$. We define $u_\varepsilon := u + \varepsilon\eta$. Notice that $u_\varepsilon = u$ outside $B_r(x_o)$ and $u_\varepsilon \geqslant \mu - |\varepsilon| \, \|\eta\|_{L^\infty(\mathbb{R}^n)} > 0$ in $B_r(x_o)$.

Therefore $u_\varepsilon \geqslant 0$ in E and $u_\varepsilon \leqslant 0$ in E^c, since the same holds for u. This says that (u_ε, E) is an admissible competitor, therefore

$$0 \leqslant \mathcal{F}(u_\varepsilon, E) - \mathcal{F}(u, E) = 2\varepsilon \iint_{\mathbb{R}^{2n}} \frac{\big(u(x) - u(y)\big)\big(\eta(x) - \eta(y)\big)}{|x - y|^{n+2s}} \, dx \, dy + o(\varepsilon).$$

Dividing by ε and taking the limit we conclude that $(-\Delta)^s u(x_o) = 0$ in the weak sense, and thus in the classical sense (see e.g. [20]). □

We prove also the following comparison principle.

Lemma 3.3. *Let (u, E) be a minimizing pair and let $A \in \mathbb{R}$. If $\varphi \geqslant A$ (respectively $\varphi \leqslant A$), then $u \geqslant A$ (respectively $u \leqslant A$).*

Proof. We prove the case $\varphi \geqslant A$, the case $\varphi \leqslant A$ is analogous.

Notice that if (v, E) is an admissible competitor against (u, E), then we have

$$0 \leqslant \mathcal{F}(v, E) - \mathcal{F}(u, E) = \iint_{\mathbb{R}^{2n}} \frac{|v(x) - v(y)|^2 - |u(x) - u(y)|^2}{|x - y|^{n+2s}} \, dx \, dy. \tag{3.1}$$

Suppose first that $A = 0$. We denote by $\tilde{u} := \max\{u, 0\}$ and we notice that $\tilde{u} = u \geqslant 0$ in E and $\tilde{u} = 0$ in E^c. Therefore $(\tilde{u}, E)$ is an admissible competitor, and so (3.1) holds with $v := \tilde{u}$, that is

$$0 \leqslant \iint_{\mathbb{R}^{2n}} \frac{|\tilde{u}(x) - \tilde{u}(y)|^2 - |u(x) - u(y)|^2}{|x - y|^{n+2s}} \, dx \, dy. \tag{3.2}$$

On the other hand, we have that $|\tilde{u}(x) - \tilde{u}(y)|^2 \leqslant |u(x) - u(y)|^2$. This, together with (3.2), implies that

$$\iint_{\mathbb{R}^{2n}} \frac{|\tilde{u}(x) - \tilde{u}(y)|^2 - |u(x) - u(y)|^2}{|x - y|^{n+2s}} \, dx \, dy = 0,$$

which gives that

$$\begin{aligned} &\text{there exists a set } \mathcal{Z} \subset \mathbb{R}^{2n} \text{ of measure zero} \\ &\text{such that } |\tilde{u}(x) - \tilde{u}(y)|^2 = |u(x) - u(y)|^2 \text{ for every } (x, y) \in \mathbb{R}^{2n} \setminus \mathcal{Z}. \end{aligned} \tag{3.3}$$

Now, we claim that

$$\text{there exist } \bar{y} \in \mathbb{R}^n \text{ and } \mathcal{V} \subset \mathbb{R}^n \text{ such that} \tag{3.4}$$
$$|\mathcal{V}| = 0, \text{ and}$$
$$(x, \bar{y}) \in \mathbb{R}^{2n} \setminus \mathcal{Z} \text{ for any } x \in \mathbb{R}^n \setminus \mathcal{V}.$$

Indeed, for any $y \in \mathbb{R}^n$, we define

$$b(y) := \int_{\mathbb{R}^n} \chi_{\mathcal{Z}}(x, y)\, dx.$$

Then, by Fubini's theorem, b is a nonnegative and measurable function, and

$$\int_{\mathbb{R}^n} b(y)\, dy = \iint_{\mathbb{R}^{2n}} \chi_{\mathcal{Z}}(x, y)\, dx\, dy = |\mathcal{Z}| = 0.$$

Therefore, $b(y) = 0$ for a.e. $y \in \mathbb{R}^n$. In particular, we can fix $\bar{y} \in \mathbb{R}^n$ such that $b(\bar{y}) = 0$, that is

$$\int_{\mathbb{R}^n} \chi_{\mathcal{Z}}(x, \bar{y})\, dx = 0.$$

This implies that $\chi_{\mathcal{Z}}(x, \bar{y}) = 0$ for a.e. $x \in \mathbb{R}^n$ (say, for every $x \in \mathbb{R}^n \setminus \mathcal{V}$, for a suitable $\mathcal{V} \subset \mathbb{R}^n$ of zero measure). This concludes the proof of (3.4).

Having established (3.4), we use it together with (3.3) to deduce that $|\tilde{u}(x) - \tilde{u}(\bar{y})|^2 = |u(x) - u(\bar{y})|^2$ for every $x \in \mathbb{R}^n \setminus \mathcal{V}$, which means that $\tilde{u}(x) - \tilde{u}(\bar{y}) = \pm(u(x) - u(\bar{y}))$ for a.e. $x \in \mathbb{R}^n$. Setting $c_\pm := \tilde{u}(\bar{y}) \mp u(\bar{y})$, we obtain that $\tilde{u}(x) = \pm u(x) + c_\pm$ for a.e. $x \in \mathbb{R}^n$. Since $\tilde{u} = u = \varphi$ outside Ω, we get that $u = \tilde{u}$ a.e. in $\mathbb{R}^n$, which implies that $u \geqslant 0$. This concludes the proof in the case $A = 0$.

Now suppose that $A < 0$. In this case we define $\widehat{u} := \max\{u, A\}$. It is not difficult to see that

$$|\widehat{u}(x) - \widehat{u}(y)|^2 \leqslant |u(x) - u(y)|^2. \tag{3.5}$$

Moreover $\widehat{u} = u \geqslant 0$ in E and $\widehat{u} \leqslant 0$ in E^c, which says that the couple $(\widehat{u}, E)$ is an admissible competitor against (u, E). Therefore, from (3.1) with $v := \widehat{u}$ and (3.5) we obtain that

$$\iint_{\mathbb{R}^{2n}} \frac{|\widehat{u}(x) - \widehat{u}(y)|^2 - |u(x) - u(y)|^2}{|x - y|^{n+2s}}\, dx\, dy = 0.$$

Now, we proceed as in the case $A = 0$ and we deduce that $u = \widehat{u}$ a.e. in $\mathbb{R}^n$, which implies that $u \geqslant A$ and concludes the proof in the case $A < 0$.

Finally, we deal with the case $A > 0$. For this, given a function $v : \mathbb{R}^n \to \mathbb{R}$ we use the notation

$$\boldsymbol{\mathcal{E}}(v) := \sqrt{\iint_{\mathbb{R}^{2n} \setminus (\Omega^c)^2} \frac{|v(x) - v(y)|^2}{|x - y|^{n+2s}}\, dx\, dy}.$$

We denote by $u^\star$ the unique minimizer of the Dirichlet energy with datum φ, that is

$$\boldsymbol{\mathcal{E}}^2(u^\star) = \min_{v \in \mathcal{H}} \boldsymbol{\mathcal{E}}^2(v),$$

where $\mathcal{H} := \{v \in H^s(\mathbb{R}^n) \text{ s.t. } v = \varphi \text{ a.e. in } \Omega^c\}$. We observe that the fact that $\varphi \geqslant A$ implies that

$$u^\star \geqslant A > 0 \tag{3.6}$$

(see Lemma 2.4 in [15]). This means that the positivity set of $u^\star$ is the whole $\mathbb{R}^n$. Therefore, we have that

$$\boldsymbol{\mathcal{E}}^2(u^\star) \leqslant \boldsymbol{\mathcal{E}}^2(u) \text{ and } \operatorname{Per}_\sigma(\mathbb{R}^n, \Omega) = 0 \leqslant \operatorname{Per}_\sigma(E, \Omega). \tag{3.7}$$

Now, we claim that

$$\operatorname{Per}_\sigma(E, \Omega) = 0. \tag{3.8}$$

Indeed, suppose by contradiction that $\operatorname{Per}_\sigma(E, \Omega) > 0$. Then,

$$\operatorname{Per}_\sigma(E, \Omega) > \operatorname{Per}_\sigma(\mathbb{R}^n, \Omega),$$

and so, using this and (3.7), we have

$$\mathcal{F}(u^\star, \mathbb{R}^n) = \boldsymbol{\mathcal{E}}^2(u^\star) + \operatorname{Per}_\sigma(\mathbb{R}^n, \Omega) < \boldsymbol{\mathcal{E}}^2(u) + \operatorname{Per}_\sigma(E, \Omega) = \mathcal{F}(u, E),$$

which contradicts the minimality of (u, E). This shows (3.8).

From (3.7), (3.8) and the minimality of (u, E), we obtain

$$\mathcal{F}(u^\star, \mathbb{R}^n) = \boldsymbol{\mathcal{E}}^2(u^\star) \leqslant \boldsymbol{\mathcal{E}}^2(u) = \mathcal{F}(u, E) \leqslant \mathcal{F}(u^\star, \mathbb{R}^n),$$

which implies that $\boldsymbol{\mathcal{E}}^2(u^\star) = \boldsymbol{\mathcal{E}}^2(u)$. Since $u^\star$ is the unique minimizer of the Dirichlet energy with datum φ, this, in turn, gives that $u = u^\star$ a.e. in $\mathbb{R}^n$. Recalling (3.6) we conclude the proof in the case $A > 0$. □

4. An equivalent extended problem, a monotonicity formula and proof of Theorem 2.1

In this section, we discuss a problem on the extended variables that is equivalent to our original minimization problem (this can be seen as a generalization of the extension problem of [9]).

For this, for any bounded Lipschitz domain $\Omega \subset \mathbb{R}^{n+1}$ we set $\Omega_0 := \Omega \cap \{z = 0\}$ and $\Omega_+ := \Omega \cap \{z > 0\}$. Hence, recalling (1.7) and (1.8), we have the following characterization of minimizing pairs (u, E).

Proposition 4.1. *The pair (u, E) is minimizing in B_r if and only if*

$$\begin{aligned} &\int_{\Omega_+} z^{1-2s}|\nabla \overline{u}|^2\, dX + c_{n,s,\sigma} \int_{\Omega_+} z^{1-\sigma}|\nabla U|^2\, dX \\ &\qquad \leqslant \int_{\Omega_+} z^{1-2s}|\nabla \overline{v}|^2\, dX + c_{n,s,\sigma} \int_{\Omega_+} z^{1-\sigma}|\nabla V|^2\, dX \end{aligned} \tag{4.1}$$

for every bounded, Lipschitz domain $\Omega \subset \mathbb{R}^{n+1}$ with $\Omega_0 \subset B_r$, and every functions $\overline{v}$ and V that satisfy the following conditions:

i) *$V = U$ in a neighborhood of $\partial\Omega$,*

ii) *the trace of V on $\{z = 0\}$ is $\chi_F - \chi_{F^c}$ for some set $F \subset \mathbb{R}^n$,*

iii) *$\overline{v} = \overline{u}$ in a neighborhood of $\partial\Omega$, and $\overline{v}\big|_{\{z=0\}} \geqslant 0$ a.e. in F, $\overline{v}\big|_{\{z=0\}} \leqslant 0$ a.e. in F^c.*

Proof. From Lemma 7.2 of [6], we know that, for any $E \subseteq \mathbb{R}^n$ with U as in (1.8), and for any $F \subset \mathbb{R}^n$ that coincides with E outside a compact subset of B_r, we have that

$$\operatorname{Per}_\sigma(F, B_r) - \operatorname{Per}_\sigma(E, B_r) = c_{n,\sigma} \inf_{(\Omega,V)\in I_\sigma} \int_{\Omega_+} z^{1-\sigma}\Big(|\nabla V|^2 - |\nabla U|^2\Big)\, dX. \tag{4.2}$$

The set I_σ above consists of the couples of every bounded Lipschitz set $\Omega \subset \mathbb{R}^{n+1}$ such that $\Omega_0 \subset B_r$ and every function V that coincides with U near $\partial\Omega$ and such that $V(x,0)=(\chi_F-\chi_{F^c})(x)$. Without loss of generality, we can prescribe that $V=U$ outside Ω, since this does not change the above integrals.

Similarly, for any function u, with $\overline{u}$ defined in (1.7), and any v that coincides with u outside a compact subset of B_r, we have that

$$\begin{aligned}&\iint_{\mathbb{R}^{2n}\setminus(B_r^c)^2}\frac{|v(x)-v(y)|^2-|u(x)-u(y)|^2}{|x-y|^{n+2s}}\,dx\,dy\\ &\qquad= c_{n,s}\inf_{(\Omega,\overline{v})\in I_s}\int_{\Omega_+} z^{1-2s}\Big(|\nabla\overline{v}|^2-|\nabla\overline{u}|^2\Big)\,dX,\end{aligned}\tag{4.3}$$

where I_s above consists of the couples of every bounded Lipschitz set Ω such that $\Omega_0\subset B_r$ and every function $\overline{v}$ that coincides with $\overline{u}$ near $\partial\Omega$ and such that $\overline{v}(x,0)=v(x)$. Once again, without loss of generality, we can prescribe that $\overline{v}=\overline{u}$ outside Ω.

Now we define

$$\begin{aligned}&\mathcal{G}_\sigma(\Omega,V):=c_{n,\sigma}\int_{\Omega_+} z^{1-\sigma}\Big(|\nabla V|^2-|\nabla U|^2\Big)\,dX\\ \text{and}\quad&\mathcal{G}_s(\Omega,\overline{v}):=c_{n,s}\int_{\Omega_+} z^{1-2s}\Big(|\nabla\overline{v}|^2-|\nabla\overline{u}|^2\Big)\,dX\end{aligned}\tag{4.4}$$

and we show that

$$\inf_{(\Omega,V)\in I_\sigma}\mathcal{G}_\sigma(\Omega,V)+\inf_{(\Omega,\overline{v})\in I_s}\mathcal{G}_s(\Omega,\overline{v})=\inf_{(\Omega,\overline{v},V)\in\mathcal{I}_{s,\sigma}}\big(\mathcal{G}_\sigma(\Omega,V)+\mathcal{G}_s(\Omega,\overline{v})\big)\tag{4.5}$$

where $\mathcal{I}_{s,\sigma}$ consists of the triplets of every bounded Lipschitz set Ω such that $\Omega_0\subset B_r$, every function $\overline{v}$ that coincides with $\overline{u}$ outside a compact subset of Ω and such that $\overline{v}(x,0)=v(x)$, and every function V that coincides with U outside a compact subset of Ω and such that $V(x,0)=(\chi_F-\chi_{F^c})(x)$. To show (4.5), first take a triplet $(\Omega,\overline{v},V)\in\mathcal{I}_{s,\sigma}$. Then, by construction, $(\Omega,V)\in I_\sigma$ and $(\Omega,\overline{v})\in I_s$, therefore

$$\inf_{(\Omega,V)\in I_\sigma}\mathcal{G}_\sigma(\Omega,V)+\inf_{(\Omega,\overline{v})\in I_s}\mathcal{G}_s(\Omega,\overline{v})\leqslant\mathcal{G}_\sigma(\Omega,V)+\mathcal{G}_s(\Omega,\overline{v})$$

and so

$$\inf_{(\Omega,V)\in I_\sigma}\mathcal{G}_\sigma(\Omega,V)+\inf_{(\Omega,\overline{v})\in I_s}\mathcal{G}_s(\Omega,\overline{v})\leqslant\inf_{(\Omega,\overline{v},V)\in\mathcal{I}_{s,\sigma}}\big(\mathcal{G}_\sigma(\Omega,V)+\mathcal{G}_s(\Omega,\overline{v})\big).$$

This shows one inequality in (4.5) and we now focus on the reverse inequality. For this, we fix $\eta>0$ and we take $(\Omega^{1,\eta},V^\eta)\in I_\sigma$ and $(\Omega^{2,\eta},\overline{v}^\eta)\in I_s$ such that

$$\eta+\inf_{(\Omega,V)\in I_\sigma}\mathcal{G}_\sigma(\Omega,V)+\inf_{(\Omega,\overline{v})\in I_s}\mathcal{G}_s(\Omega,\overline{v})\geqslant\mathcal{G}_\sigma(\Omega^{1,\eta},V^\eta)+\mathcal{G}_s(\Omega^{2,\eta},\overline{v}^\eta).\tag{4.6}$$

Let $\Omega^\eta:=\Omega^{1,\eta}\cup\Omega^{2,\eta}$. Since Ω^η contains both $\Omega^{1,\eta}$ and $\Omega^{2,\eta}$, we have that $\overline{v}^\eta$ coincides with $\overline{u}$ outside a compact subset of Ω^η and V coincides with U outside a compact subset of Ω^η. Accordingly, $(\Omega^\eta,\overline{v}^\eta,V^\eta)\in\mathcal{I}_{s,\sigma}$ and so

$$\begin{aligned}\mathcal{G}_\sigma(\Omega^{1,\eta},V^\eta)+\mathcal{G}_s(\Omega^{2,\eta},\overline{v}^\eta)&=\mathcal{G}_\sigma(\Omega^\eta,V^\eta)+\mathcal{G}_s(\Omega^\eta,\overline{v}^\eta)\\ &\geqslant\inf_{(\Omega,\overline{v},V)\in\mathcal{I}_{s,\sigma}}\big(\mathcal{G}_\sigma(\Omega,V)+\mathcal{G}_s(\Omega,\overline{v})\big).\end{aligned}$$

By plugging this into (4.6), we obtain

$$\eta+\inf_{(\Omega,V)\in I_\sigma}\mathcal{G}_\sigma(\Omega,V)+\inf_{(\Omega,\overline{v})\in I_s}\mathcal{G}_s(\Omega,\overline{v})\geqslant\inf_{(\Omega,\overline{v},V)\in\mathcal{I}_{s,\sigma}}\big(\mathcal{G}_\sigma(\Omega,V)+\mathcal{G}_s(\Omega,\overline{v})\big).$$

So we take η as small as we wish and we complete the proof of the reverse inequality in (4.5).

Having established (4.5), we can sum up (4.2) and (4.3) (taking E as the positivity set of u and recalling (4.4)) and obtain

$$\mathcal{F}(v,F)-\mathcal{F}(u,E)=\inf_{(\Omega,\overline{v},V)\in\mathcal{I}_{s,\sigma}}\left(\mathcal{G}_\sigma(\Omega,V)+\mathcal{G}_s(\Omega,\overline{v})\right). \tag{4.7}$$

From this, we obtain the desired result by arguing as follows. First, suppose that (u,E) is a minimizing pair and take Ω, $\overline{v}$, and V as in the statement of Proposition 4.1. We define $v(x):=\overline{v}(x,0)$. Then, the triplet $(\Omega,\overline{v},V)$ belongs to $\mathcal{I}_{s,\sigma}$ and therefore, by (4.7) we have that

$$\mathcal{F}(v,F)-\mathcal{F}(u,E)\leqslant\mathcal{G}_\sigma(\Omega,V)+\mathcal{G}_s(\Omega,\overline{v}). \tag{4.8}$$

On the other hand, by item iii) in the statement of Proposition 4.1, we have that $v(x)=\overline{v}(x,0)=\overline{u}(x,0)=u(x)$ outside a compact subset of B_r. Similarly, by item i) and ii), we have that $(\chi_F-\chi_{F^c})(x)=V(x,0)=U(x,0)=(\chi_E-\chi_{E^c})(x)$ outside a compact subset of B_r. Moreover, $v(x)=\overline{v}(x,0)\geqslant 0$ for a.e. $x\in F$ and $v(x)=\overline{v}(x,0)\leqslant 0$ for a.e. $x\in F^c$, thanks to item iii). As a consequence, v and F are admissible competitors with respect to (1.5) and (1.6), hence the minimality of (u,E) gives that $\mathcal{F}(u,E)\leqslant\mathcal{F}(v,F)$. By inserting this into (4.8) we obtain

$$0\leqslant\mathcal{G}_\sigma(\Omega,V)+\mathcal{G}_s(\Omega,\overline{v}),$$

which, recalling (4.4), gives (4.1).

Now, viceversa, suppose that

(4.9)

(4.1) holds under conditions i), ii) and iii) of the statement of Proposition 4.1,

and let (v,F) be a competing pair according to (1.5) and (1.6). We show that, in this case,

(4.10) any triplet $(\Omega,\overline{v},V)\in\mathcal{I}_{s,\sigma}$, satisfies conditions i), ii) and iii) of the statement of Proposition 4.1.

Indeed, since (v,F) satisfies (1.5) and (1.6) we have that $\overline{v}(x,0)=v(x)\geqslant 0$ a.e. in F and $\overline{v}(x,0)=v(x)\leqslant 0$ a.e. in F^c. This and the definition of $\mathcal{I}_{s,\sigma}$ give (4.10).

By (4.9) and (4.10) we obtain that (4.1) holds true for any triplet $(\Omega,\overline{v},V)\in\mathcal{I}_{s,\sigma}$. This means, recalling (4.4), that

$$\mathcal{G}_\sigma(\Omega,V)+\mathcal{G}_s(\Omega,\overline{v})\geqslant 0$$

for any triplet $(\Omega,\overline{v},V)\in\mathcal{I}_{s,\sigma}$. Consequently, by (4.7), we obtain that

$$\mathcal{F}(v,F)-\mathcal{F}(u,E)\geqslant 0,$$

which shows that (u,E) is minimizing and thus it completes the proof of Proposition 4.1. □

Now we address the proof of Theorem 2.1, with the aid of some simple but useful lemmata.

Lemma 4.2. *Let $c\in\mathbb{R}$ and $u:B_r\setminus\{0\}\to\mathbb{R}$ be a function satisfying*

$$\nabla u(x)\cdot x=c\,u(x),\qquad \text{for any } x\in B_r\setminus\{0\}.$$

Then u is homogeneous of degree c (more precisely, u can be extended to a function defined in the whole of $\mathbb{R}^n\setminus\{0\}$ that is homogeneous of degree c).

Proof. The function $\varphi(t) := u(tx) - t^c\, u(x)$ satisfies the ODE $\varphi'(t) = \frac{c}{t}\varphi(t)$ for any $t \in (0,1) \cup (1,+\infty)$, with $\varphi(1) = 0$. By uniqueness we get that $\varphi = 0$, as desired. □

In the following lemma we show that Φ_u, defined in (2.1), possesses a natural scaling.

Lemma 4.3. *Let (u,E) be a minimizing pair in B_ρ and let Φ_u be as in* (2.1). *Let also*

$$G_u(r) := r^{\sigma-n}\left(\int_{\mathcal{B}_r^+} z^{1-2s}|\nabla\overline{u}|^2\,dX + c_{n,s,\sigma}\int_{\mathcal{B}_r^+} z^{1-\sigma}|\nabla U|^2\,dX\right), \tag{4.11}$$

and let (u_r, E_r) be the rescaled pair defined in (1.11). *Then, for any $t > 0$,*

$$G_u(rt) = G_{u_r}(t) \quad \textit{and} \quad \Phi_u(rt) = \Phi_{u_r}(t). \tag{4.12}$$

Proof. The claim follows by observing that $\overline{u}_r(X) = r^{\frac{\sigma}{2}-s}\overline{u}(rX)$ and $U_r(X) = U(rX)$. □

With this, we are in the position of proving Theorem 2.1.

Proof of Theorem 2.1. We will prove that

$$\frac{d}{dr}\Phi_u(r) \geqslant 0 \quad \text{for a.e. } r.$$

We write

$$\Phi_u(r) = G_u(r) - H_u(r),$$

where G_u is as in (4.11) and

$$H_u(r) := \left(s - \frac{\sigma}{2}\right) r^{\sigma-n-1}\int_{\partial\mathcal{B}_r^+} z^{1-2s}\,\overline{u}^2\,d\mathcal{H}^n.$$

Thanks to the scaling properties in Lemma 4.3, it is sufficient to prove the theorem when $r = 1$.

Given a small $\varepsilon > 0$, we consider a competitor $(\overline{u}^\varepsilon, U^\varepsilon)$ for $(\overline{u}, U)$ defined as follows

$$\overline{u}^\varepsilon(X) := \begin{cases} (1-\varepsilon)^{s-\frac{\sigma}{2}}\,\overline{u}\left(\frac{X}{1-\varepsilon}\right) & \text{if } X \in \mathcal{B}_{1-\varepsilon}^+, \\ |X|^{s-\frac{\sigma}{2}}\,\overline{u}\left(\frac{X}{|X|}\right) & \text{if } X \in \mathcal{B}_1^+ \setminus \mathcal{B}_{1-\varepsilon}^+, \\ \overline{u}(X) & \text{if } X \in (\mathcal{B}_1^+)^c, \end{cases}$$

and

$$U^\varepsilon(X) := \begin{cases} U\left(\frac{X}{1-\varepsilon}\right) & \text{if } X \in \mathcal{B}_{1-\varepsilon}^+, \\ U\left(\frac{X}{|X|}\right) & \text{if } X \in \mathcal{B}_1^+ \setminus \mathcal{B}_{1-\varepsilon}^+, \\ U(X) & \text{if } X \in (\mathcal{B}_1^+)^c. \end{cases}$$

Since the pair (u,E) is a minimizer and $\overline{u}^\varepsilon$ and U^ε satisfy conditions i), ii) and iii) in the statement of Proposition 4.1, from (4.1) we have that

$$G_u(1) \leqslant G_{u^\varepsilon}(1), \tag{4.13}$$

where $u^\varepsilon(x) := \overline{u}^\varepsilon(x,0)$. Now, we compute $G_u(1)$ and $G_{u^\varepsilon}(1)$ by splitting the integrals in $\mathcal{B}_1^+$ into integrals in $\mathcal{B}_{1-\varepsilon}^+$ and $\mathcal{B}_1^+\setminus\mathcal{B}_{1-\varepsilon}^+$. Therefore, we have

(4.14)

$$\begin{aligned} G_u(1) =& \int_{\mathcal{B}_{1-\varepsilon}^+} z^{1-2s}|\nabla\overline{u}|^2\,dX + \varepsilon\int_{\partial\mathcal{B}_1^+} z^{1-2s}|\nabla\overline{u}|^2\,d\mathcal{H}^n \\ &+ c_{n,s,\sigma}\left(\int_{\mathcal{B}_{1-\varepsilon}^+} z^{1-\sigma}|\nabla U|^2\,dX + \varepsilon\int_{\partial\mathcal{B}_1^+} z^{1-\sigma}|\nabla U|^2\,d\mathcal{H}^n\right) + o(\varepsilon) \\ =& (1-\varepsilon)^{n-\sigma}G_u(1-\varepsilon) + \varepsilon\int_{\partial\mathcal{B}_1^+} z^{1-2s}\left(|\overline{u}_\tau|^2+|\overline{u}_\nu|^2\right)d\mathcal{H}^n \\ &+ \varepsilon\, c_{n,s,\sigma}\int_{\partial\mathcal{B}_1^+} z^{1-\sigma}\left(|U_\tau|^2+|U_\nu|^2\right)d\mathcal{H}^n + o(\varepsilon), \end{aligned}$$

where, as usual, u_τ and u_ν stand for the tangential and the normal gradient of u on $\partial\mathcal{B}_1^+$.

To compute $G_{u^\varepsilon}(1)$ we notice that $\overline{u}^\varepsilon$ and U^ε coincide with the rescaling $\overline{u}_{1/(1-\varepsilon)}$ and $U_{1/(1-\varepsilon)}$, respectively, in $\mathcal{B}_{1-\varepsilon}^+$, as given in (1.11), hence

$$\begin{aligned} G_{u^\varepsilon}(1) =& (1-\varepsilon)^{n-\sigma}G_{u_{1/(1-\varepsilon)}}(1-\varepsilon) + \varepsilon\, c_{n,s,\sigma}\int_{\partial\mathcal{B}_1^+} z^{1-\sigma}|U_\tau|^2\,d\mathcal{H}^n \\ &+ \varepsilon\int_{\partial\mathcal{B}_1^+} z^{1-2s}\left(|\overline{u}_\tau|^2 + \left(s-\frac{\sigma}{2}\right)^2\overline{u}^2\right)d\mathcal{H}^n + o(\varepsilon). \end{aligned} \tag{4.15}$$

Also, from Lemma 4.3 (used here with $t := 1-\varepsilon$ and $r := 1/(1-\varepsilon)$), we see that

$$G_{u_{1/(1-\varepsilon)}}(1-\varepsilon) = G_u(1).$$

Therefore, (4.15) becomes

$$\begin{aligned} G_{u^\varepsilon}(1) =& (1-\varepsilon)^{n-\sigma}G_u(1) + \varepsilon\, c_{n,s,\sigma}\int_{\partial\mathcal{B}_1^+} z^{1-\sigma}|U_\tau|^2\,d\mathcal{H}^n \\ &+ \varepsilon\int_{\partial\mathcal{B}_1^+} z^{1-2s}\left(|\overline{u}_\tau|^2 + \left(s-\frac{\sigma}{2}\right)^2\overline{u}^2\right)d\mathcal{H}^n + o(\varepsilon). \end{aligned} \tag{4.16}$$

Plugging (4.14) and (4.16) into (4.13) we obtain

$$\begin{aligned} (1-\varepsilon)^{n-\sigma}G_u(1) \geqslant (1-\varepsilon)^{n-\sigma}G_u(1-\varepsilon) \quad &+ \quad \varepsilon\int_{\partial\mathcal{B}_1^+} z^{1-2s}\left[|\overline{u}_\nu|^2 - \left(s-\frac{\sigma}{2}\right)^2\overline{u}^2\right]d\mathcal{H}^n \\ &+ \quad \varepsilon\, c_{n,s,\sigma}\int_{\partial\mathcal{B}_1^+} z^{1-\sigma}|U_\nu|^2\,d\mathcal{H}^n + o(\varepsilon), \end{aligned}$$

which implies

(4.17)

$$G_u'(1) \geqslant \int_{\partial\mathcal{B}_1^+} z^{1-2s}\left[|\overline{u}_\nu|^2 - \left(s-\frac{\sigma}{2}\right)^2\overline{u}^2\right]d\mathcal{H}^n + c_{n,s,\sigma}\int_{\partial\mathcal{B}_1^+} z^{1-\sigma}|U_\nu|^2\,d\mathcal{H}^n.$$

Now, we claim that

$$H_u'(1) = \left(s-\frac{\sigma}{2}\right)\int_{\partial\mathcal{B}_1^+} z^{1-2s}\left(2\,\overline{u}\,\overline{u}_\nu + (\sigma-2s)\overline{u}^2\right)d\mathcal{H}^n. \tag{4.18}$$

For this, we notice that, by using the change of variable $X = rY$, with $z = rw$, we can rewrite $H_u(r)$ as

$$H_u(r) = \left(s - \frac{\sigma}{2}\right) r^{\sigma-2s} \int_{\partial\mathcal{B}_1^+} w^{1-2s}\, \overline{u}^2(rY)\, d\mathcal{H}^n.$$

Taking the derivative with respect to r and then setting $r = 1$ we obtain (4.18).

From (4.17) and (4.18) we deduce that

$$\begin{aligned} \Phi_u'(1) \quad &\geqslant \quad \int_{\partial\mathcal{B}_1^+} z^{1-2s} \left[|\overline{u}_\nu|^2 - \left(s - \frac{\sigma}{2}\right)^2 \overline{u}^2 \right] d\mathcal{H}^n + c_{n,s,\sigma} \int_{\partial\mathcal{B}_1^+} z^{1-\sigma} |U_\nu|^2\, d\mathcal{H}^n \\ &\qquad - \left(s - \frac{\sigma}{2}\right) \int_{\partial\mathcal{B}_1^+} z^{1-2s} \left(2\,\overline{u}\,\overline{u}_\nu + (\sigma - 2s)\overline{u}^2\right) d\mathcal{H}^n. \end{aligned}$$

Notice that

$$|\overline{u}_\nu|^2 - \left(s - \frac{\sigma}{2}\right)^2 \overline{u}^2 - 2\left(s - \frac{\sigma}{2}\right) \overline{u}\,\overline{u}_\nu - \left(s - \frac{\sigma}{2}\right)(\sigma - 2s)\overline{u}^2 = \left(\overline{u}_\nu - \left(s - \frac{\sigma}{2}\right)\overline{u}\right)^2,$$

and so

$$\Phi_u'(1) \geqslant \int_{\partial\mathcal{B}_1^+} z^{1-2s} \left(\overline{u}_\nu - \left(s - \frac{\sigma}{2}\right)\overline{u}\right)^2 d\mathcal{H}^n + c_{n,s,\sigma} \int_{\partial\mathcal{B}_1^+} z^{1-\sigma} |U_\nu|^2\, d\mathcal{H}^n. \tag{4.19}$$

This implies that Φ_u is increasing in $(0, \rho)$.

Moreover, if Φ_u is constant, then (4.19) and Lemma 4.2 give that $\overline{u}$ is homogeneous of degree $s - \frac{\sigma}{2}$ and U is homogeneous of degree 0. Conversely, suppose that $\overline{u}$ and U are homogeneous of degree $s - \frac{\sigma}{2}$ and 0 respectively. Then, $u = u_r$ for any $r > 0$, and therefore from Lemma 4.3 we have that $\Phi_u(rt) = \Phi_u(t)$, which implies that Φ_u is constant. This concludes the proof of Theorem 2.1. □

5. Dimensional reduction and proof of Theorem 2.2

In order to establish the dimensional reduction property, as stated in Theorem 2.2, we recall a useful gluing result from Lemma 10.2 of [6] (this is indeed just the translation of such result by some a in the $(n+1)$th component):

Lemma 5.1. *Fix R, $a > 0$. Let $\alpha \in (-1, 1)$. Let W be a bounded function in $\mathcal{B}_R^+ \subset \mathbb{R}^{n+1}$. Suppose that*

$$W = 0 \text{ in a neighborhood of } \partial\mathcal{B}_R \text{ and } \int_{\mathcal{B}_R^+} z^\alpha |\nabla W|^2 dX < \infty,$$

where $X = (x, z) \in \mathbb{R}^{n+1}$. Then there exists a function $\mathcal{W} = \mathcal{W}(x, x_{n+1}, z)$ defined on $\mathcal{B}_R^+ \times [a-1, a+1]$ with the following properties:

$$\mathcal{W} = 0 \text{ if } x_{n+1} < a - \frac{1}{2}, \tag{5.1}$$

$$\mathcal{W} = W \text{ if } x_{n+1} > a + \frac{1}{2}, \tag{5.2}$$

$$\mathcal{W} = 0 \text{ in a neighborhood of } \partial\mathcal{B}_R^+ \times [a-1, a+1], \tag{5.3}$$

$$\mathcal{W}(x, x_{n+1}, 0) = \begin{cases} 0 & \text{if } x_{n+1} \leqslant a, \\ W(x, 0) & \text{if } x_{n+1} > a. \end{cases} \tag{5.4}$$

with

$$C(\mathcal{W}) := \int_{\mathcal{B}_R^+ \times [a-1, a+1]} z^\alpha |\nabla \mathcal{W}|^2 d\mathcal{X} \text{ finite and independent of } a,$$

where $\mathcal{X} = (x, x_{n+1}, z) \in \mathbb{R}^{n+2}$.

From the geometric point of view, Lemma 5.1 states that one can interpolate 0 with a given function W by performing a sharp switch at $\{x_{n+1} = a\} \cup \{z = 0\}$, maintaining the energy finite. As a consequence, we obtain:

Corollary 5.2. *Fix* R*,* $a > 0$*. Let* $\alpha \in (-1, 1)$*. Let* $\mathcal{U}$ *and* $\mathcal{V}$ *be bounded functions in* $\mathcal{B}_R^+ \subset \mathbb{R}^{n+1}$ *with* $\mathcal{U} = \mathcal{V}$ *in a neighborhood of* $\partial\mathcal{B}_R$ *and*

$$\int_{\mathcal{B}_R^+} z^\alpha \Big(|\nabla \mathcal{U}|^2 + |\nabla \mathcal{V}|^2\Big) dX < \infty,$$

where $X = (x, z) \in \mathbb{R}^{n+1}$*. Then there exists a function* $\mathcal{Z} = \mathcal{Z}(x, x_{n+1}, z)$ *defined on* $\mathcal{B}_R^+ \times [-(a+1), a+1]$ *with the following properties:*

$$\mathcal{Z} \text{ is even in } x_{n+1}, \tag{5.5}$$

$$\mathcal{Z} = \mathcal{V} \text{ if } |x_{n+1}| < a - \frac{1}{2}, \tag{5.6}$$

$$\mathcal{Z} = \mathcal{U} \text{ in a neighborhood of } \partial\mathcal{B}_R^+ \times [-(a+1), a+1], \tag{5.7}$$

$$\mathcal{Z}(x, x_{n+1}, 0) = \begin{cases} \mathcal{V}(x, 0) & \text{if } |x_{n+1}| \in [0, a], \\ \mathcal{U}(x, 0) & \text{if } |x_{n+1}| \in (a, a+1], \end{cases} \tag{5.8}$$

with

$$C(\mathcal{Z}) := \int_{\mathcal{B}_R^+ \times [a-1, a+1]} z^\alpha |\nabla \mathcal{Z}|^2 d\mathcal{X} \text{ finite and independent of } a, \tag{5.9}$$

where $\mathcal{X} = (x, x_{n+1}, z) \in \mathbb{R}^{n+2}$.

Proof. Let $\mathcal{W}$ be the function obtained by applying Lemma 5.1 to the function $W := \mathcal{U} - \mathcal{V}$, and let $\tilde{\mathcal{W}}(x, x_{n+1}, z) := \mathcal{W}(x, |x_{n+1}|, z)$. Then let

$$\mathcal{Z}(x, x_{n+1}, z) := \begin{cases} \tilde{\mathcal{W}}(x, x_{n+1}, z) + \mathcal{V}(x, z) & \text{if } |x_{n+1}| \in (a-1, a+1], \\ \mathcal{V}(x, z) & \text{if } |x_{n+1}| \in [0, a-1]. \end{cases}$$

We remark that (5.5) holds true by construction, while (5.6) and (5.7) follow from (5.1) and (5.3) respectively. Also, (5.8) is a consequence of (5.4). □

With this, we are ready for the proof of Theorem 2.2:

Proof of Theorem 2.2. If (u, E) is a minimizer in $\mathbb{R}^n$, then $(u^\star, E^\star)$ is a minimizer in $\mathbb{R}^{n+1}$: this follows easily from Proposition 4.1 by slicing, using that for a function $\overline{v}(x, x_{n+1}, z)$ one has that $|\nabla_{\mathcal{X}} \overline{v}|^2 \geqslant |\nabla_X \overline{v}|^2$ for any fixed x_{n+1}, where $\mathcal{X} := (x, x_{n+1}, z)$ and $X := (x, z)$.

Now we suppose that $(u^\star, E^\star)$ is minimizing in $\mathbb{R}^{n+1}$ and we show that (u, E) is minimizing in any domain of $\mathbb{R}^n$. To this extent, we use again Proposition 4.1. For this, we fix a competitor triplet V, $\overline{v}$ and $\Omega := \mathcal{B}_R \subset \mathbb{R}^{n+1}$ as prescribed by Proposition 4.1 (in particular, we also have a set F given in item ii) there), and our goal is to show that (4.1) holds true in this case. The idea is to construct a competitor in one dimension more with respect to $(u^\star, E^\star)$ and thus to use the minimality of $(u^\star, E^\star)$ for this competitor. The details of the computation go as follows. Fix $a > 0$, to be taken arbitrarily large at the end of the argument. We take

$\mathcal{Z}_s$ to be the function constructed in Corollary 5.2, applied here with $\alpha := 1-2s$, $\mathcal{U} := \overline{u}$ and $\mathcal{V} := \overline{v}$. By (5.6),

$$\begin{aligned}\int_{\mathcal{B}_R^+\times[-(a-1),a-1]} z^{1-2s}|\nabla_{\mathcal{X}}\mathcal{Z}_s|^2 d\mathcal{X} &= \int_{\mathcal{B}_R^+\times[-(a-1),a-1]} z^{1-2s}|\nabla_X\overline{v}|^2 d\mathcal{X} \\ &= 2(a-1)\int_{\mathcal{B}_R^+} z^{1-2s}|\nabla_X\overline{v}|^2 dX,\end{aligned}$$

since $\overline{v}$ does not depend on x_{n+1}. Therefore, by (5.5),

$$\begin{aligned}&\int_{\mathcal{B}_R^+\times[-(a+1),a+1]} z^{1-2s}|\nabla\mathcal{Z}_s|^2 d\mathcal{X} \\ &\quad = 2\int_{\mathcal{B}_R^+\times[a-1,a+1]} z^{1-2s}|\nabla\mathcal{Z}_s|^2 d\mathcal{X} + 2(a-1)\int_{\mathcal{B}_R^+} z^{1-2s}|\nabla\overline{v}|^2 dX.\end{aligned} \tag{5.10}$$

Similarly, one can define $\mathcal{Z}_\sigma$ to be the function constructed in Corollary 5.2, applied here with $\alpha := 1-\sigma$, $\mathcal{U} := U$ and $\mathcal{V} := V$. In analogy with (5.10), we obtain

$$\begin{aligned}&\int_{\mathcal{B}_R^+\times[-(a+1),a+1]} z^{1-\sigma}|\nabla\mathcal{Z}_\sigma|^2 d\mathcal{X} \\ &\quad = 2\int_{\mathcal{B}_R^+\times[a-1,a+1]} z^{1-\sigma}|\nabla\mathcal{Z}_\sigma|^2 d\mathcal{X} + 2(a-1)\int_{\mathcal{B}_R^+} z^{1-\sigma}|\nabla V|^2 dX.\end{aligned} \tag{5.11}$$

Now we point out that

(5.12) $\mathcal{Z}_\sigma$, $\mathcal{Z}_s$ and $\mathcal{B}_R \times [-(a+1), a+1] \subset \mathbb{R}^{n+2}$ are an admissible triplet (with respect to $(u^\star, E^\star)$, as prescribed by Proposition 4.1).

For this, we observe that $\mathcal{Z}_\sigma = U$ and $\mathcal{Z}_s = \overline{u}$ on $\partial\big(\mathcal{B}_R \times [-(a+1), a+1]\big)$, thanks to (5.7) (the first of these observations takes care of item i) in the statement of Proposition 4.1, while the second is involved in item iii)).

Furthermore, by (5.8), we see that $\mathcal{Z}_\sigma\big|_{\{z=0\}} = \chi_{\tilde{F}} - \chi_{\tilde{F}^c}$, where

$$\tilde{F} := \Big(F \cap \{x_{n+1} \leqslant a\}\Big) \cup \Big(E \cap \{x_{n+1} > a\}\Big),$$

and

$$\mathcal{Z}_s\big|_{\{z=0\}} = \begin{cases} \overline{v}\big|_{\{z=0\}} & \text{if } x_{n+1} \leqslant a, \\ \overline{u}\big|_{\{z=0\}} & \text{if } x_{n+1} > a. \end{cases}$$

Accordingly, $\mathcal{Z}_s\big|_{\{z=0\}} \geqslant 0$ a.e. in $\tilde{F}$ and $\mathcal{Z}_s\big|_{\{z=0\}} \leqslant 0$ a.e. in $\tilde{F}^c$. This proves (5.12).

Using (5.12) and the minimality of $(u^\star, E^\star)$, we deduce from Proposition 4.1 that

$$\begin{aligned}&\int_{\mathcal{B}_R^+\times[-(a+1),a+1]} z^{1-2s}|\nabla\overline{u}^\star|^2 d\mathcal{X} + c_{n,s,\sigma}\int_{\mathcal{B}_R^+\times[-(a+1),a+1]} z^{1-\sigma}|\nabla U^\star|^2 d\mathcal{X} \\ &\quad \leqslant \int_{\mathcal{B}_R^+\times[-(a+1),a+1]} z^{1-2s}|\nabla\mathcal{Z}_s|^2 d\mathcal{X} + c_{n,s,\sigma}\int_{\mathcal{B}_R^+\times[-(a+1),a+1]} z^{1-\sigma}|\nabla\mathcal{Z}_\sigma|^2 d\mathcal{X},\end{aligned}$$

where $\overline{u}^\star(x, x_{n+1}, z) := \overline{u}^\star(x, z)$ and $U(x, x_{n+1}, z) := U(x, z)$. Thus, we can compute the integrals on the left hand side in the $(n+1)$th variable and use (5.10) and

(5.11): we obtain

$$
\begin{aligned}
&2(a+1)\int_{\mathcal{B}_R^+} z^{1-2s}|\nabla\overline{u}^\star|^2 dX + 2(a+1)c_{n,s,\sigma}\int_{\mathcal{B}_R^+} z^{1-\sigma}|\nabla U^\star|^2 dX \\
&\quad\leqslant 2(a-1)\int_{\mathcal{B}_R^+} z^{1-2s}|\nabla\overline{v}|^2 dX + 2(a-1)c_{n,s,\sigma}\int_{\mathcal{B}_R^+} z^{1-\sigma}|\nabla V|^2 dX + O(1),
\end{aligned}
$$

where $O(1)$ is a quantity independent on a (recall (5.9)). Hence, we divide by $2a$ and we take a as large as we wish: we conclude that

$$
\begin{aligned}
&\int_{\mathcal{B}_R^+} z^{1-2s}|\nabla\overline{u}^\star|^2 dX + c_{n,s,\sigma}\int_{\mathcal{B}_R^+} z^{1-\sigma}|\nabla U^\star|^2 dX \\
&\quad\leqslant \int_{\mathcal{B}_R^+} z^{1-2s}|\nabla\overline{v}|^2 dX + c_{n,s,\sigma}\int_{\mathcal{B}_R^+} z^{1-\sigma}|\nabla V|^2 dX.
\end{aligned}
$$

By Proposition 4.1, this says that (u, E) is a minimal pair in $\mathbb{R}^n$, as desired. □

6. Some glueing lemmata

Here we present some results that glue two admissible pairs together by estimating the excess of energy produced by this surgery.

Lemma 6.1. *Let $\beta \in (0,1)$ and $\varepsilon \in (0, 1/4)$. Then there exists a function*

$$\phi := \phi_\varepsilon : \overline{\mathbb{R}^{n+1}_+} \to [0,1]$$

such that

$$\phi(x,z) = 0 \quad \textit{for any } (x,z) \in \mathcal{B}^+_{1-\varepsilon}, \tag{6.1}$$

$$\phi(x,z) = 1 \quad \textit{for any } (x,z) \in \mathbb{R}^{n+1}_+ \setminus \mathcal{B}^+_{1+\varepsilon}, \tag{6.2}$$

$$\phi(x,0) = \chi_{\mathbb{R}^n\setminus B_1}(x) \quad \textit{a.e. } x \in \mathbb{R}^n, \tag{6.3}$$

$$\textit{and} \qquad \int_{\mathbb{R}^{n+1}_+} z^\beta |\nabla\phi(x,z)|^2\,dx\,dz \leqslant \frac{C}{\varepsilon}, \tag{6.4}$$

for some $C > 0$.

Proof. Let $\varepsilon' := \varepsilon/2$ and, for any $X = (x,z) \in \mathbb{R}^{n+1}_+$, we define

$$\phi_1(X) := \begin{cases} 0 & \text{if } |X| < 1-\varepsilon', \\ \big(|X| - 1 + \varepsilon'\big)/(2\varepsilon') & \text{if } |X| \in [1-\varepsilon',\, 1+\varepsilon'), \\ 1 & \text{if } |X| \geqslant 1+\varepsilon'. \end{cases}$$

Let also

$$\phi_2(X) := \begin{cases} 0 & \text{if } |x| < 1-z, \\ \big(|x| - 1 + z\big)/(2z) & \text{if } |x| \in [1-z,\, 1+z), \\ 1 & \text{if } |x| \geqslant 1+z \end{cases}$$

and

$$\eta(X) := \begin{cases} z/\varepsilon' & \text{if } z \in (0, \varepsilon'), \\ 1 & \text{if } z \geqslant \varepsilon'. \end{cases}$$

We also set

$$\phi := \eta\phi_1 + (1-\eta)\phi_2.$$

We remark that $\eta\big|_{\{z=0\}} = 0$, thus

$$\phi\big|_{\{z=0\}} = \phi_2\big|_{\{z=0\}} = \chi_{\mathbb{R}^n\setminus B_1},$$

which proves (6.3).

Now we prove (6.1). For this, we fix $X \in \mathbb{R}^{n+1}_+$, with $|X| < 1 - \varepsilon = 1 - 2\varepsilon'$. Then $\phi_1(X) = 0$, hence

$$\phi(X) = (1 - \eta(X))\phi_2(X). \tag{6.5}$$

Now, if $|x| < 1 - z$, we have that $\phi_2(X) = 0$, and therefore $\phi(X) = 0$, that proves (6.1) in this case. Accordingly, we may suppose that $|x| \geqslant 1 - z$. So we have that

$$z \geqslant \frac{1 - (1 - z)^2 - z^2}{2} \geqslant \frac{1 - |x|^2 - z^2}{2} = \frac{1 - |X|^2}{2} > \frac{1 - (1 - 2\varepsilon')^2}{2} > \varepsilon',$$

and so $\eta(X) = 1$. As a consequence of this and (6.5), we obtain that $\phi(X) = 0$, and this establishes (6.1).

Now we prove (6.2). To this goal, we fix $X \in \mathbb{R}^{n+1}_+$ with $|X| \geqslant 1 + \varepsilon = 1 + 2\varepsilon'$. In this case, we have that

$$\phi_1(X) = 1. \tag{6.6}$$

Now, if $z \geqslant \varepsilon'$, we have that $\eta(X) = 1$ and so

$$\phi(X) = \phi_1(X) = 1.$$

Thus, we can assume that $z < \varepsilon'$. In this case, we have that $|x|^2 = |X|^2 - z^2 > (1 + z)^2$, which implies that $\phi_2(X) = 1$. Combining this and (6.6) we conclude that $\phi(X) = \eta(X) + (1 - \eta(X)) = 1$, which proves (6.2).

Now we prove (6.4). For this, we first observe that

$$\nabla\phi_2(X) = \left(\frac{x}{2z\,|x|}, \frac{1 - |x|}{2z^2}\right) \chi_{(1-z,\,1+z)}(|x|)$$

and therefore

$$|\nabla\phi_2(X)| \leqslant \frac{C}{z}\, \chi_{(1-z,\,1+z)}(|x|), \tag{6.7}$$

for some $C > 0$. Moreover

$$|\nabla\phi_1(X)| \leqslant \frac{C}{\varepsilon}\, \chi_{(1-\varepsilon',\,1+\varepsilon')}(|X|).$$

As a consequence

$$\int_{\mathbb{R}^{n+1}_+} z^\beta |\nabla\phi_1|^2\, dX \leqslant \frac{C\,\Big|\big\{|X| \in (1 - \varepsilon',\, 1 + \varepsilon')\big\}\Big|}{\varepsilon^2} = \frac{C}{\varepsilon}, \tag{6.8}$$

up to renaming $C > 0$. Also, $\phi = \phi_1$ if $z > \varepsilon'$, therefore we deduce from (6.8) that

$$\int_{\{z>\varepsilon'\}} z^\beta |\nabla\phi|^2\, dX = \int_{\{z>\varepsilon'\}} z^\beta |\nabla\phi_1|^2\, dX \leqslant \frac{C}{\varepsilon}. \tag{6.9}$$

Furthermore, if $z \leqslant \varepsilon'$ and $|X| > 2$, we have that $\phi_1(X) = 1$ and $|x|^2 = |X|^2 - z^2 > (1 + z)^2$, that gives $\phi_2(X) = 1$.

As a consequence, $\phi_1 - \phi_2 = 0$ if $z \leqslant \varepsilon'$ and $|X| > 2$, therefore

$$\begin{aligned} \int_{\{z\leqslant\varepsilon'\}} z^\beta |\nabla\eta|^2\, |\phi_1 - \phi_2|^2\, dX &\leqslant \frac{C}{\varepsilon^2} \int_{\{z\leqslant\varepsilon'\}\cap\{|X|\leqslant 2\}} z^\beta\, dX \\ &\leqslant \frac{C}{\varepsilon^2} \int_{\{z\leqslant\varepsilon'\}} z^\beta\, dz = C\varepsilon^{\beta-1}. \end{aligned} \tag{6.10}$$

In addition, using (6.7), we obtain that

$$\begin{aligned}(6.11)\qquad \int_{\{z\leqslant\varepsilon'\}} z^{\beta}|\nabla\phi_2|^2\,dX &\leqslant C\int_{\{z\leqslant\varepsilon'\}\cap\{|x|\in(1-z,\,1+z)\}} z^{\beta-2}\,dX\\ &\leqslant C\int_{\{z\leqslant\varepsilon'\}} z^{\beta-1}\,dz = C\varepsilon^{\beta}.\end{aligned}$$

Notice also that

$$\nabla\phi = \nabla\eta(\phi_1-\phi_2)+\eta\nabla\phi_1+(1-\eta)\nabla\phi_2.$$

Consequently, by gathering the estimates in (6.8), (6.10) and (6.11) and using Young inequality, we deduce that

$$\begin{aligned}&\int_{\{z\leqslant\varepsilon'\}} z^{\beta}|\nabla\phi|^2\,dX\\ \leqslant\;& C\int_{\{z\leqslant\varepsilon'\}} z^{\beta}\Big(|\nabla\eta|^2\,|\phi_1-\phi_2|^2+\eta^2|\nabla\phi_1|^2+(1-\eta)^2|\nabla\phi_2|^2\Big)\,dX\\ \leqslant\;& \frac{C}{\varepsilon},\end{aligned}$$

up to renaming C. This and (6.9) imply (6.4). □

Next we give a glueing result: namely, given any admissible pair in B_1, we glue it to another admissible pair outside B_1, keeping the energy contribution under control.

Lemma 6.2. *Let $\varepsilon>0$. Let (u_i,E_i), $i\in\{1,2\}$, be admissible pairs in B_2, and let $\overline{u}_i$ and U_i be their extensions according to* (1.7) *and* (1.8). *Let also ϕ be the function introduced in Lemma 6.1.*

Then there exist $F\subseteq\mathbb{R}^n$, $\overline{v}:\mathbb{R}^{n+1}_+\to\mathbb{R}$ and $V:\mathbb{R}^{n+1}_+\to\mathbb{R}$ such that

$$(6.12)\qquad F\cap B_1=E_1\cap B_1\quad\text{and}\quad F\setminus B_1=E_2\setminus B_1,$$

and $\overline{v}$ and V satisfy the properties listed in Proposition 4.1, namely

- i) *$V=U_2$ in a neighborhood of $\partial\mathcal{B}^+_{3/2}$,*
- ii) *the trace of V on $\{z=0\}$ is $\chi_F-\chi_{F^c}$,*
- iii) *$\overline{v}=\overline{u}_2$ in a neighborhood of $\partial\mathcal{B}^+_{3/2}$, and $\overline{v}|_{\{z=0\}}\geqslant 0$ a.e. in F and $\overline{v}|_{\{z=0\}}\leqslant 0$ a.e. in F^c.*

Also,

(6.13)

$$\begin{aligned}\int_{\mathcal{B}^+_{3/2}} z^{1-2s}\left(|\nabla\overline{v}|^2-|\nabla\overline{u}_2|^2\right)dX &\leqslant \int_{\mathcal{B}^+_1} z^{1-2s}\left(|\nabla\overline{u}_1|^2-|\nabla\overline{u}_2|^2\right)dX\\ &+C\varepsilon^{-2}\int_{\mathcal{B}^+_{1+\varepsilon}\setminus\mathcal{B}^+_{1-\varepsilon}} z^{1-2s}|\overline{u}_1-\overline{u}_2|^2\,dX + C\int_{\mathcal{B}^+_{1+\varepsilon}\setminus\mathcal{B}^+_{1-\varepsilon}} z^{1-2s}\left(|\nabla\overline{u}_1|^2+|\nabla\overline{u}_2|^2\right)dX\end{aligned}$$

and

(6.14)

$$\begin{aligned}\int_{\mathcal{B}^+_{3/2}} z^{1-\sigma}\left(|\nabla V|^2-|\nabla U_2|^2\right)dX &\leqslant \int_{\mathcal{B}^+_1} z^{1-\sigma}\left(|\nabla U_1|^2-|\nabla U_2|^2\right)dX\\ &+C\int_{\mathcal{B}^+_{1+\varepsilon}\setminus\mathcal{B}^+_{1-\varepsilon}} z^{1-\sigma}|\nabla\phi|^2\,|U_1-U_2|^2\,dX + C\int_{\mathcal{B}^+_{1+\varepsilon}\setminus\mathcal{B}^+_{1-\varepsilon}} z^{1-\sigma}\left(|\nabla U_1|^2+|\nabla U_2|^2\right)dX\end{aligned}$$

for some $C > 0$.

Proof. We set

$$F := (E_1 \cap B_1) \cup (E_2 \setminus B_1). \tag{6.15}$$

With this we have established (6.12).

Now, we define

$$\overline{w}_\pm := \min\{\overline{u}_1^\pm, \overline{u}_2^\pm\}.$$

Let also $\eta_1, \eta_2 \in C^\infty(\mathbb{R}^{n+1}, [0,1])$, with

$$\begin{aligned} &\eta_1(X) = 1 \text{ if } |X| \leqslant 1-\varepsilon,\\ &\eta_1(X) = 0 \text{ if } |X| \geqslant 1-\varepsilon/2,\\ &\eta_2(X) = 1 \text{ if } |X| \leqslant 1+\varepsilon/2,\\ &\eta_2(X) = 0 \text{ if } |X| \geqslant 1+\varepsilon\\ \text{and}\qquad &|\nabla\eta_1| + |\nabla\eta_2| \leqslant \frac{8}{\varepsilon}. \end{aligned}$$

We define[1]

$$\overline{v}_\pm := \eta_1\,\eta_2\,\overline{u}_1^\pm + \eta_2(1-\eta_1)\,\overline{w}_\pm + (1-\eta_2)\,\overline{u}_2^\pm$$

and $\overline{v} := \overline{v}_+ - \overline{v}_-$.

By construction, $\overline{v} = \overline{u}_2$ near $\partial\mathcal{B}^+_{3/2}$.

Also, if $x \in F$, then $x \in E_1 \cup E_2$, and so either $\overline{u}_1(x,0) \geqslant 0$ or $\overline{u}_2(x,0) \geqslant 0$ (up to sets of zero measure), and then either $\overline{u}_1^-(x,0) = 0$ or $\overline{u}_2^-(x,0) = 0$, so $\overline{w}_-(x,0) = 0$. This gives that for a.e. $x \in F$

$$\overline{v}_-(x,0) = \eta_1(x,0)\,\eta_2(x,0)\,\overline{u}_1^-(x,0) + (1-\eta_2(x,0))\,\overline{u}_2^-(x,0),$$

and so

$$\begin{aligned} &\overline{v}(x,0)\\ &= \eta_1(x,0)\,\eta_2(x,0)\,\overline{u}_1(x,0) + (1-\eta_2(x,0))\,\overline{u}_2(x,0) + \eta_2(x,0)(1-\eta_1(x,0))\,\overline{w}_+(x,0)\\ &\geqslant \eta_1(x,0)\,\eta_2(x,0)\,\overline{u}_1(x,0) + (1-\eta_2(x,0))\,\overline{u}_2(x,0). \end{aligned} \tag{6.16}$$

Now, for a.e. $x \in F \cap B_1 = E_1 \cap B_1$ we have that $\eta_2(x,0) = 1$, thus (6.16) implies that

$$\overline{v}(x,0) \geqslant \eta_1(x,0)\,\overline{u}_1(x,0) \geqslant 0.$$

Similarly, for a.e $x \in F \setminus B_1 = E_2 \setminus B_1$ we have that $\eta_1(x,0) = 0$, thus (6.16) gives that

$$\overline{v}(x,0) \geqslant (1-\eta_2(x,0))\,\overline{u}_2(x,0) \geqslant 0.$$

This shows that, for a.e. $x \in F$, $\overline{v}(x,0) \geqslant 0$.

Conversely, if $x \in F^c$, then $x \in E_1^c \cup E_2^c$ and so either $\overline{u}_1(x,0) \leqslant 0$ or $\overline{u}_2(x,0) \leqslant 0$ (up to sets of zero measure), that is either $\overline{u}_1^+(x,0) = 0$ or $\overline{u}_2^+(x,0) = 0$, so $\overline{w}_+(x,0) = 0$. As a consequence, for a.e. $x \in F^c$,

$$\overline{v}_+(x,0) = \eta_1(x,0)\,\eta_2(x,0)\,\overline{u}_1^+(x,0) + (1-\eta_2(x,0))\,\overline{u}_2^+(x,0),$$

---

[1] We put $\pm$ as a subscript (rather than a superscript) in $\overline{v}_\pm$ and $\overline{w}_\pm$ not to confuse in principle the notation with the positive/negative part of a function.

and so

$$
\begin{aligned}
& \overline{v}(x,0) \\
= \;& \eta_1(x,0)\,\eta_2(x,0)\,\overline{u}_1(x,0) + (1-\eta_2(x,0))\,\overline{u}_2(x,0) - \eta_2(x,0)(1-\eta_1(x,0))\,\overline{w}_-(x,0) \\
\leqslant \;& \eta_1(x,0)\,\eta_2(x,0)\,\overline{u}_1(x,0) + (1-\eta_2(x,0))\,\overline{u}_2(x,0).
\end{aligned}
$$

In particular, for a.e. $x \in F^c \cap B_1 = E_1^c \cap B_1$, we have that $\eta_2(x,0) = 1$, so $\overline{v}(x,0) \leqslant \eta_1(x,0)\,\overline{u}_1(x,0) \leqslant 0$, and for a.e. $x \in F \setminus B_1 = E_2^c \setminus B_1$, we have that $\eta_1(x,0) = 0$, so $\overline{v}(x,0) \leqslant (1-\eta_2(x,0))\overline{u}_2(x,0) \leqslant 0$. This shows that $\overline{v}(x,0) \leqslant 0$ for a.e. $x \in F^c$, thus completing the proof of iii).

Now we prove (6.13). For this, we notice that

$$
\begin{aligned}
\nabla \overline{v}_\pm \;=\;& \eta_1\,\eta_2\,\nabla \overline{u}_1^\pm + \eta_2(1-\eta_1)\nabla \overline{w}_\pm + (1-\eta_2)\nabla \overline{u}_2^\pm \\
& + \nabla\left(\eta_1\,\eta_2\right)\overline{u}_1^\pm + \nabla\left(\eta_2(1-\eta_1)\right)\overline{w}_\pm + \nabla(1-\eta_2)\overline{u}_2^\pm,
\end{aligned}
$$

so

$$
\begin{aligned}
\nabla \overline{v} \;=\;& \eta_1\,\eta_2\,\nabla \overline{u}_1 + \eta_2(1-\eta_1)\left(\nabla \overline{w}_+ - \nabla \overline{w}_-\right) + (1-\eta_2)\nabla \overline{u}_2 \\
& + \nabla\left(\eta_1\,\eta_2\right)\overline{u}_1 + \nabla\left(\eta_2(1-\eta_1)\right)\left(\overline{w}_+ - \overline{w}_-\right) + \nabla(1-\eta_2)\overline{u}_2.
\end{aligned}
$$

Now we notice that

$$
\begin{aligned}
& \nabla\left(\eta_1\,\eta_2\right)\overline{u}_1^+ + \nabla\left(\eta_2(1-\eta_1)\right)\overline{w}_+ + \nabla(1-\eta_2)\overline{u}_2^+ \\
=\;& \left(\eta_1\,\nabla\eta_2 + \nabla\eta_1\,\eta_2\right)\overline{u}_1^+ + \left((1-\eta_1)\nabla\eta_2 - \nabla\eta_1\,\eta_2\right)\overline{w}_+ - \nabla\eta_2\,\overline{u}_2^+ \\
=\;& \left(\eta_1\,\overline{u}_1^+ + (1-\eta_1)\overline{w}_+ - \overline{u}_2^+\right)\nabla\eta_2 + \left(\overline{u}_1^+ - \overline{w}_+\right)\eta_2\,\nabla\eta_1 \\
=\;& O\left(\varepsilon^{-1}|\overline{u}_1^+ - \overline{u}_2^+|\right)\chi_{\mathcal{B}_{1+\varepsilon}^+\setminus\mathcal{B}_{1-\varepsilon}^+} \\
=\;& O\left(\varepsilon^{-1}|\overline{u}_1 - \overline{u}_2|\right)\chi_{\mathcal{B}_{1+\varepsilon}^+\setminus\mathcal{B}_{1-\varepsilon}^+},
\end{aligned}
$$

and similarly for the negative parts. Hence,

$$
\begin{aligned}
\nabla \overline{v} \;=\;& \chi_{\mathcal{B}_1^+}\nabla \overline{u}_1 + \Big(\eta_1\,\eta_2 - \chi_{\mathcal{B}_1^+}\Big)\nabla \overline{u}_1 + (1-\eta_2)\nabla \overline{u}_2 \\
& + \eta_2(1-\eta_1)O\left(|\nabla \overline{u}_1| + |\nabla \overline{u}_2|\right) + O\left(\varepsilon^{-1}|\overline{u}_1 - \overline{u}_2|\right)\chi_{\mathcal{B}_{1+\varepsilon}^+\setminus\mathcal{B}_{1-\varepsilon}^+}.
\end{aligned}
$$

That is

$$
\begin{aligned}
|\nabla \overline{v}|^2 \;\leqslant\;& \chi_{\mathcal{B}_1^+}|\nabla \overline{u}_1|^2 + (1-\eta_2)^2|\nabla \overline{u}_2|^2 \\
& + C\chi_{\mathcal{B}_{1+\varepsilon}^+\setminus\mathcal{B}_{1-\varepsilon}^+}\left(|\nabla \overline{u}_1|^2 + |\nabla \overline{u}_2|^2\right) + C\chi_{\mathcal{B}_{1+\varepsilon}^+\setminus\mathcal{B}_{1-\varepsilon}^+}\varepsilon^{-2}|\overline{u}_1 - \overline{u}_2|^2,
\end{aligned}
$$

for some constant $C > 0$. Since $\overline{v} = \overline{u}_2$ outside $\mathcal{B}_{1+\varepsilon}^+$, we conclude that

$$
\begin{aligned}
& \int_{\mathcal{B}_{3/2}^+} z^{1-2s}\left(|\nabla \overline{v}|^2 - |\nabla \overline{u}_2|^2\right)dX \\
&= \int_{\mathcal{B}_{1+\varepsilon}^+} z^{1-2s}\left(|\nabla \overline{v}|^2 - |\nabla \overline{u}_2|^2\right)dX \\
&\leqslant \int_{\mathcal{B}_1^+} z^{1-2s}\left(|\nabla \overline{u}_1|^2 - |\nabla \overline{u}_2|^2\right)dX + C\int_{\mathcal{B}_{1+\varepsilon}^+\setminus\mathcal{B}_{1-\varepsilon}^+} z^{1-2s}\left(|\nabla \overline{u}_1|^2 + |\nabla \overline{u}_2|^2\right)dX \\
&\qquad + C\varepsilon^{-2}\int_{\mathcal{B}_{1+\varepsilon}^+\setminus\mathcal{B}_{1-\varepsilon}^+} z^{1-2s}|\overline{u}_1 - \overline{u}_2|^2\,dX,
\end{aligned}
$$

that concludes the proof of (6.13).

Now, let ϕ be as in Lemma 6.1, and set $\tilde\chi_E := \chi_E - \chi_{E^c}$. We define $V := (1-\phi)U_1 + \phi U_2$. We observe that, for a.e. $x \in \mathbb{R}^n$, $\phi(x,0) = \chi_{\mathbb{R}^n\setminus B_1}$ and therefore

$$V\big|_{\{z=0\}} = \chi_{B_1}\tilde\chi_{E_1} + \chi_{\mathbb{R}^n\setminus B_1}\tilde\chi_{E_2} = \tilde\chi_F, \tag{6.17}$$

where F is defined in (6.15). This establishes ii).

Also, $\phi = 1$ outside $\mathcal{B}^+_{1+\varepsilon}$ hence

$$V = U_2 \text{ outside } \mathcal{B}^+_{1+\varepsilon}, \tag{6.18}$$

thus proving i).

Now we show (6.14). We observe that

$$\nabla V = \nabla U_1 + (U_2 - U_1)\nabla\phi + \phi\nabla(U_2 - U_1).$$

Therefore, by Young inequality we have

$$|\nabla V|^2 \leqslant C\left(|\nabla\phi|^2|U_2-U_1|^2 + |\nabla U_1|^2 + |\nabla U_2|^2\right),$$

for suitable $C > 0$. Hence, integrating over $\mathcal{B}^+_{1+\varepsilon}\setminus\mathcal{B}^+_{1-\varepsilon}$ we get

$$\begin{aligned} &\int_{\mathcal{B}^+_{1+\varepsilon}\setminus\mathcal{B}^+_{1-\varepsilon}} z^{1-\sigma}|\nabla V|^2\,dX \\ &\qquad\leqslant C\int_{\mathcal{B}^+_{1+\varepsilon}\setminus\mathcal{B}^+_{1-\varepsilon}} z^{1-\sigma}\left(|\nabla\phi|^2|U_2-U_1|^2 + |\nabla U_1|^2 + |\nabla U_2|^2\right)dX, \end{aligned} \tag{6.19}$$

for some $C > 0$. Furthermore, $V = U_1$ in $\mathcal{B}^+_{1-\varepsilon}$. Thus, using (6.18) and (6.19) we obtain that

$$\begin{aligned} &\int_{\mathcal{B}^+_{3/2}} z^{1-\sigma}|\nabla V|^2\,dX \\ =\;& \int_{\mathcal{B}^+_{1-\varepsilon}} z^{1-\sigma}|\nabla U_1|^2\,dX + \int_{\mathcal{B}^+_{3/2}\setminus\mathcal{B}^+_{1+\varepsilon}} z^{1-\sigma}|\nabla U_2|^2\,dX + \int_{\mathcal{B}^+_{1+\varepsilon}\setminus\mathcal{B}^+_{1-\varepsilon}} z^{1-\sigma}|\nabla V|^2\,dX \\ \leqslant\;& \int_{\mathcal{B}^+_{1}} z^{1-\sigma}|\nabla U_1|^2\,dX + \int_{\mathcal{B}^+_{3/2}} z^{1-\sigma}|\nabla U_2|^2\,dX - \int_{\mathcal{B}^+_{1+\varepsilon}} z^{1-\sigma}|\nabla U_2|^2\,dX \\ &\quad + C\int_{\mathcal{B}^+_{1+\varepsilon}\setminus\mathcal{B}^+_{1-\varepsilon}} z^{1-\sigma}\left(|\nabla\phi|^2|U_2-U_1|^2 + |\nabla U_1|^2 + |\nabla U_2|^2\right)dX. \end{aligned}$$

This implies (6.14) and concludes the proof of Lemma 6.2. ☐

Lemma 6.3. *Let E_1, $E_2 \subseteq \mathbb{R}^n$ and $F := (E_1\cap B_1)\cup(E_2\setminus B_1)$. Then*

$$F\cap B_1 = E_1\cap B_1 \quad\textit{and}\quad F\setminus B_1 = E_2\setminus B_1, \tag{6.20}$$

and

$$\begin{aligned} &\mathrm{Per}_\sigma(F, B_{3/2}) - \mathrm{Per}_\sigma(E_2, B_{3/2}) \\ &\qquad\leqslant \mathrm{Per}_\sigma(E_1, B_1) - \mathrm{Per}_\sigma(E_2, B_1) + \mathcal{L}(B_1, (E_1\Delta E_2)\setminus B_1). \end{aligned} \tag{6.21}$$

Proof. It is clear that F satisfies (6.20). Now we prove (6.21). For this, we use (6.20) to see that

$$\mathrm{Per}_\sigma(F, B_{3/2}) - \mathrm{Per}_\sigma(E_2, B_{3/2}) = \mathrm{Per}_\sigma(F, B_1) - \mathrm{Per}_\sigma(E_2, B_1). \tag{6.22}$$

Furthermore, (6.20) also gives that

$$
\begin{aligned}
&\mathrm{Per}_\sigma(F,B_1)-\mathrm{Per}_\sigma(E_1,B_1)\\
&\quad=\mathcal{L}(F\cap B_1,F^c\cap B_1)+\mathcal{L}(F\cap B_1,F^c\cap B_1^c)+\mathcal{L}(F^c\cap B_1,F\cap B_1^c)\\
&\qquad-\mathcal{L}(E_1\cap B_1,E_1^c\cap B_1)-\mathcal{L}(E_1\cap B_1,E_1^c\cap B_1^c)-\mathcal{L}(E_1^c\cap B_1,E_1\cap B_1^c)\\
&\quad=\mathcal{L}(E_1\cap B_1,E_1^c\cap B_1)+\mathcal{L}(E_1\cap B_1,E_2^c\cap B_1^c)+\mathcal{L}(E_1^c\cap B_1,E_2\cap B_1^c)\\
&\qquad-\mathcal{L}(E_1\cap B_1,E_1^c\cap B_1)-\mathcal{L}(E_1\cap B_1,E_1^c\cap B_1^c)-\mathcal{L}(E_1^c\cap B_1,E_1\cap B_1^c)\\
&\quad=\mathcal{L}(E_1\cap B_1,E_2^c\cap B_1^c)-\mathcal{L}(E_1\cap B_1,E_1^c\cap B_1^c)\\
&\qquad+\mathcal{L}(E_1^c\cap B_1,E_2\cap B_1^c)-\mathcal{L}(E_1^c\cap B_1,E_1\cap B_1^c)\\
&\quad\leqslant\mathcal{L}(E_1\cap B_1,(E_1\setminus E_2)\cap B_1^c)+\mathcal{L}(E_1^c\cap B_1,(E_2\setminus E_1)\cap B_1^c)\\
&\quad\leqslant\mathcal{L}(B_1,(E_1\setminus E_2)\cap B_1^c)+\mathcal{L}(B_1,(E_2\setminus E_1)\cap B_1^c)\\
&\quad=\mathcal{L}(B_1,(E_1\Delta E_2)\cap B_1^c).
\end{aligned}
$$

By combining this and (6.22), we conclude that

$$
\begin{aligned}
&\mathrm{Per}_\sigma(F,B_{3/2})-\mathrm{Per}_\sigma(E_2,B_{3/2})\\
&\quad=\mathrm{Per}_\sigma(F,B_1)-\mathrm{Per}_\sigma(E_2,B_1)+\mathrm{Per}_\sigma(E_1,B_1)-\mathrm{Per}_\sigma(E_1,B_1)\\
&\quad\leqslant\mathrm{Per}_\sigma(E_1,B_1)-\mathrm{Per}_\sigma(E_2,B_1)+\mathcal{L}(B_1,(E_1\Delta E_2)\cap B_1^c),
\end{aligned}
$$

which establishes (6.21). □

7. Uniform energy bounds for minimizing pairs and proof of Theorem 1.1

Here we prove that if (u,E) is a minimizing pair in some ball then its energy in a smaller ball is bounded uniformly, only in dependence of a weighted L^2 norm of u. For this, we start with some technical observations:

Lemma 7.1. *Let $\eta\in C_0^\infty(B_1)$ and $u:\mathbb{R}^n\to\mathbb{R}$ be a measurable function. Then*

$$
\iint_{\mathbb{R}^{2n}\setminus(B_1^c)^2}|u(y)|^2\,\frac{|\eta(x)-\eta(y)|^2}{|x-y|^{n+2s}}\,dx\,dy\leqslant C\int_{\mathbb{R}^n}\frac{|u(y)|^2}{1+|y|^{n+2s}}\,dy. \tag{7.1}
$$

Here $C>0$ only depends on $\|\eta\|_{C^1(\mathbb{R}^n)}$, n and s.

Proof. We suppose that the right-hand side of (7.1) is finite, otherwise we are done. Then we observe that, for any $y\in\mathbb{R}^n$,

$$
\begin{aligned}
\int_{\mathbb{R}^n}\frac{|\eta(x)-\eta(y)|^2}{|x-y|^{n+2s}}\,dx&\leqslant\int_{\mathbb{R}^n}\frac{\min\{4\|\eta\|^2_{L^\infty(\mathbb{R}^n)},\ \|\nabla\eta\|^2_{L^\infty(\mathbb{R}^n)}|x-y|^2\}}{|x-y|^{n+2s}}\,dx\\
&\leqslant C\int_{\mathbb{R}^n}\frac{\min\{1,|z|^2\}}{|z|^{n+2s}}\,dz\leqslant C,
\end{aligned} \tag{7.2}
$$

for some $C>0$ (that may be different from step to step). Similarly, we have that

$$
\begin{aligned}
\sup_{y\in B_2\setminus B_1}\int_{B_1}\frac{|\eta(x)-\eta(y)|^2}{|x-y|^{n+2s}}\,dx&\leqslant\sup_{y\in B_2\setminus B_1}\int_{B_1}\frac{\|\nabla\eta\|^2_{L^\infty(\mathbb{R}^n)}|x-y|^2}{|x-y|^{n+2s}}\,dx\\
&\leqslant C\int_{B_3}\frac{|z|^2}{|z|^{n+2s}}\,dz\leqslant C.
\end{aligned} \tag{7.3}
$$

Furthermore, if $y \in \mathbb{R}^n \setminus B_2$ and $x \in B_1$, we have that $|x-y| \geqslant |y|-|x| \geqslant |y|/2$, therefore

$$\begin{aligned}\int_{B_1} \frac{|\eta(x)-\eta(y)|^2}{|x-y|^{n+2s}}\,dx &\leqslant 4\|\eta\|^2_{L^\infty(\mathbb{R}^n)} \cdot 2^{n+2s} \int_{B_1} \frac{dx}{|y|^{n+2s}} \\ &\leqslant \frac{C}{|y|^{n+2s}} \qquad \text{for any } y \in \mathbb{R}^n \setminus B_2.\end{aligned} \tag{7.4}$$

Accordingly, using (7.2), (7.3) and (7.4), we see that

$$\begin{aligned}&\iint_{\mathbb{R}^{2n}\setminus(B_1^c)^2} |u(y)|^2 \frac{|\eta(x)-\eta(y)|^2}{|x-y|^{n+2s}}\,dx\,dy \\ =\;& \iint_{\mathbb{R}^n\times B_1} |u(y)|^2 \frac{|\eta(x)-\eta(y)|^2}{|x-y|^{n+2s}}\,dx\,dy + \iint_{B_1\times(B_2\setminus B_1)} |u(y)|^2 \frac{|\eta(x)-\eta(y)|^2}{|x-y|^{n+2s}}\,dx\,dy \\ &\quad + \iint_{B_1\times(\mathbb{R}^n\setminus B_2)} |u(y)|^2 \frac{|\eta(x)-\eta(y)|^2}{|x-y|^{n+2s}}\,dx\,dy \\ \leqslant\;& C\left(\int_{B_1} |u(y)|^2\,dy + \int_{B_2\setminus B_1} |u(y)|^2\,dy + \int_{\mathbb{R}^n\setminus B_2} \frac{|u(y)|^2}{|y|^{n+2s}}\,dy\right),\end{aligned}$$

that gives (7.1). □

Corollary 7.2. *Let (u,E) be a minimizing pair in B_2. Then*

$$\iint_{\mathbb{R}^{2n}\setminus(B_1^c)^2} \frac{|u(x)-u(y)|^2}{|x-y|^{n+2s}}\,dx\,dy \leqslant C \int_{\mathbb{R}^n} \frac{|u(y)|^2}{1+|y|^{n+2s}}\,dy,$$

for some $C>0$ only depending on n and s.

Proof. Let $\eta \in C_0^\infty(B_2)$ with $\eta = 1$ in B_1. Let $\varepsilon \in \mathbb{R}$ and $u_\varepsilon := (1+\varepsilon\eta^2)u$. We observe that the sign of u_ε is the same as the one of u, as long as ε is sufficiently small, and so (u_ε, E) is an admissible competitor. Therefore $\mathcal{F}(u_\varepsilon,E)-\mathcal{F}(u,E) \geqslant 0$. Dividing by ε and taking the limit as $\varepsilon \to 0$, we obtain that

$$\iint_{\mathbb{R}^{2n}\setminus(B_2^c)^2} \frac{\big(u(x)-u(y)\big)\big(\eta^2(x)u(x)-\eta^2(y)u(y)\big)}{|x-y|^{n+2s}}\,dx\,dy = 0. \tag{7.5}$$

Moreover

(7.6)

$$\begin{aligned}&\big(u(x)-u(y)\big)\big(\eta^2(x)u(x)-\eta^2(y)u(y)\big) \\ =&\big(u(x)-u(y)\big)\big(\eta^2(x)u(x)-\eta^2(x)u(y)+\eta^2(x)u(y)-\eta^2(y)u(y)\big) \\ =&\,\eta^2(x)\big|u(x)-u(y)\big|^2 + u(y)\big(u(x)-u(y)\big)\big(\eta(x)-\eta(y)\big)\big(\eta(x)+\eta(y)\big) \\ \geqslant&\,\eta^2(x)\big|u(x)-u(y)\big|^2 - \frac{1}{8}\big|\eta(x)+\eta(y)\big|^2\big|u(x)-u(y)\big|^2 - 8u^2(y)\big|\eta(x)-\eta(y)\big|^2.\end{aligned}$$

Also, if we use the symmetry of the kernel, we see that

$$\begin{aligned}
&\iint_{\mathbb{R}^{2n}\setminus(B_2^c)^2} \frac{\big|\eta(x)+\eta(y)\big|^2\big|u(x)-u(y)\big|^2}{|x-y|^{n+2s}}\,dx\,dy \\
\leqslant\;& 2\iint_{\mathbb{R}^{2n}\setminus(B_2^c)^2} \frac{\big(\eta^2(x)+\eta^2(y)\big)\,\big|u(x)-u(y)\big|^2}{|x-y|^{n+2s}}\,dx\,dy \\
=\;& 4\iint_{\mathbb{R}^{2n}\setminus(B_2^c)^2} \frac{\eta^2(x)\big|u(x)-u(y)\big|^2}{|x-y|^{n+2s}}\,dx\,dy.
\end{aligned}$$

Consequently, if we integrate (7.6) and we use the latter estimate, we conclude that

$$\begin{aligned}
&\iint_{\mathbb{R}^{2n}\setminus(B_2^c)^2} \frac{\big(u(x)-u(y)\big)\,\big(\eta^2(x)u(x)-\eta^2(y)u(y)\big)}{|x-y|^{n+2s}}\,dx\,dy \\
\geqslant\;& \iint_{\mathbb{R}^{2n}\setminus(B_2^c)^2} \frac{\eta^2(x)\big|u(x)-u(y)\big|^2-\frac18\big|\eta(x)+\eta(y)\big|^2\big|u(x)-u(y)\big|^2-8u^2(y)\big|\eta(x)-\eta(y)\big|^2}{|x-y|^{n+2s}}\,dx\,dy \\
\geqslant\;& \iint_{\mathbb{R}^{2n}\setminus(B_2^c)^2} \frac{\frac12\eta^2(x)\big|u(x)-u(y)\big|^2-8u^2(y)\big|\eta(x)-\eta(y)\big|^2}{|x-y|^{n+2s}}\,dx\,dy.
\end{aligned}$$

By inserting this into (7.5) and using that $\eta=1$ in B_1 we obtain

$$\iint_{B_1\times\mathbb{R}^n} \frac{\big|u(x)-u(y)\big|^2}{|x-y|^{n+2s}}\,dx\,dy \leqslant 16\iint_{\mathbb{R}^{2n}\setminus(B_2^c)^2} \frac{u^2(y)\big|\eta(x)-\eta(y)\big|^2}{|x-y|^{n+2s}}\,dx\,dy.$$

By interchanging the variable we obtain a similar estimates with $\mathbb{R}^n\times B_1$ as domain in the left-hand side, and therefore, by summing up

$$\iint_{\mathbb{R}^{2n}\setminus(B_1^c)^2} \frac{\big|u(x)-u(y)\big|^2}{|x-y|^{n+2s}}\,dx\,dy \leqslant 32\iint_{\mathbb{R}^{2n}\setminus(B_2^c)^2} \frac{u^2(y)\big|\eta(x)-\eta(y)\big|^2}{|x-y|^{n+2s}}\,dx\,dy.$$

This and Lemma 7.1 imply the desired result. □

Now we are ready for the completion of the proof of Theorem 1.1:

Proof of Theorem 1.1. We use Lemma 6.3 with $E_1:=\mathbb{R}^n$ and $E_2:=E$, and we obtain that there exists F such that $F\setminus B_1=E\setminus B_1$ and

$$\operatorname{Per}_\sigma(F,B_{3/2})-\operatorname{Per}_\sigma(E,B_{3/2})\leqslant -\operatorname{Per}_\sigma(E,B_1)+\mathcal{L}(B_1,B_1^c). \tag{7.7}$$

In addition, we take $\eta\in C_0^\infty(B_{3/2},[0,1])$ with $\eta=1$ in B_1, and we define $v:=(1-\eta)u$. We observe that $v=u$ outside $B_{3/2}$. Also, the positive set of u and v are the same and $v=0$ in B_1. This implies that $v\geqslant 0$ in F and $v\leqslant 0$ in F^c, thus (v,F) is an admissible competitor in $B_{3/2}$, which gives that

$$\begin{aligned}
&\iint_{\mathbb{R}^{2n}\setminus(B_{3/2}^c)^2} \frac{\big|v(x)-v(y)\big|^2}{|x-y|^{n+2s}}\,dx\,dy+\operatorname{Per}_\sigma(F,B_{3/2}) \\
&\quad-\iint_{\mathbb{R}^{2n}\setminus(B_{3/2}^c)^2} \frac{\big|u(x)-u(y)\big|^2}{|x-y|^{n+2s}}\,dx\,dy-\operatorname{Per}_\sigma(E,B_{3/2})\geqslant 0.
\end{aligned} \tag{7.8}$$

Now we observe that

$$\begin{aligned}&\big|v(x)-v(y)\big|^2=\big|(1-\eta(x))u(x)-(1-\eta(x))u(y)+(1-\eta(x))u(y)-(1-\eta(y))u(y)\big|^2\\&\quad\leqslant 2\left(\big(1-\eta(x)\big)^2\big|u(x)-u(y)\big|^2+u^2(y)\big|\eta(x)-\eta(y)\big|^2\right).\end{aligned}$$

Integrating this inequality and using Lemma 7.1 and Corollary 7.2 we obtain that

$$\iint_{\mathbb{R}^{2n}\setminus(B^c_{3/2})^2}\frac{\big|v(x)-v(y)\big|^2}{|x-y|^{n+2s}}\,dx\,dy\leqslant C\int_{\mathbb{R}^n}\frac{|u(y)|^2}{1+|y|^{n+2s}}\,dy,$$

for some $C>0$. This, (7.7) and (7.8) imply that

$$\begin{aligned}0&\leqslant\iint_{\mathbb{R}^{2n}\setminus(B^c_{3/2})^2}\frac{\big|v(x)-v(y)\big|^2-\big|u(x)-u(y)\big|^2}{|x-y|^{n+2s}}\,dx\,dy-\operatorname{Per}_\sigma(E,B_1)+C\\&\leqslant C\int_{\mathbb{R}^n}\frac{|u(y)|^2}{1+|y|^{n+2s}}\,dy-\iint_{\mathbb{R}^{2n}\setminus(B^c_{3/2})^2}\frac{\big|u(x)-u(y)\big|^2}{|x-y|^{n+2s}}\,dx\,dy-\operatorname{Per}_\sigma(E,B_1)+C,\end{aligned}$$

up to renaming C, and this implies the thesis of Theorem 1.1. □

8. Convergence results and proof of Theorem 1.2

In the sequel, given $\alpha\in(0,1)$ and $r>0$, we denote by $L^2_\alpha(\mathcal{B}^+_r)$ the weighted Lebesgue space with respect to the weight $z^{1-2\alpha}$, i.e. the Lebesgue space with norm

$$\|v\|_{L^2_\alpha(\mathcal{B}^+_r)}:=\sqrt{\int_{\mathcal{B}^+_r}z^{1-2\alpha}|v(X)|^2\,dX}.$$

Now we study the convergence of the energy for a sequence of minimizing pairs.

For this, we first obtain a useful "integration by parts" formula.

Lemma 8.1. *Let $R>0$. Let $u:\mathbb{R}^n\to\mathbb{R}$ be such that $|u(x)|\leqslant C\,|x|^\alpha$, with $\alpha<2s$ and $C>0$. Suppose that*

$$(-\Delta)^s u=0 \text{ in } B_R\cap\{u\neq 0\},\tag{8.1}$$

and let $\overline{u}$ be as in (1.7). Assume also that $\overline{u}$ is continuous in $\overline{\mathcal{B}^+_R}$ and

$$|\nabla\overline{u}|\in L^2_s(\mathcal{B}^+_R).\tag{8.2}$$

Then

$$\int_{\mathbb{R}^{n+1}_+}z^{1-2s}\nabla\overline{u}\cdot\nabla(\overline{u}\phi)\,dX\;=\;0\tag{8.3}$$

for any $\phi\in C^\infty_0(\mathcal{B}^+_R)$.

Proof. By Sard's Lemma, we can take a sequence of $\varepsilon\searrow 0$ such that $S_1:=\{\overline{u}=\pm\varepsilon\}$ is a smooth set in $\mathbb{R}^{n+1}_+$. So we write $\mathcal{B}^+_R\cap(\partial\{|\overline{u}|>\varepsilon\})=S_1\cup S_2$, with $S_2\subseteq\mathbb{R}^n\times\{0\}$ and $|\overline{u}(X)|\geqslant\varepsilon$ for any $X\in S_2$. Accordingly, from (8.1), the quantity $z^{1-2s}\partial_z\overline{u}$ vanishes along S_2 and therefore, by the Divergence Theorem,

$$\begin{aligned}&\int_{\{|\overline{u}|>\varepsilon\}}\operatorname{div}(z^{1-2s}\overline{u}\phi\nabla\overline{u})\,dX=-\int_{S_1}z^{1-2s}\overline{u}\,\phi\,\partial_z\overline{u}\,d\mathcal{H}^n=\mp\varepsilon\int_{S_1}z^{1-2s}\phi\partial_z\overline{u}\,d\mathcal{H}^n\\&\quad=\mp\varepsilon\int_{\{|\overline{u}|>\varepsilon\}}\operatorname{div}(z^{1-2s}\phi\nabla\overline{u})\,dX=\mp\varepsilon\int_{\{|\overline{u}|>\varepsilon\}}z^{1-2s}\nabla\phi\cdot\nabla\overline{u}\,dX.\end{aligned}$$

From this and (8.2) we obtain that

$$\lim_{\varepsilon\searrow 0}\int_{\{|\overline{u}|>\varepsilon\}} z^{1-2s}\nabla(\overline{u}\phi)\cdot\nabla\overline{u}\,dX \;=\; \lim_{\varepsilon\searrow 0}\int_{\{|\overline{u}|>\varepsilon\}} \operatorname{div}\left(z^{1-2s}\overline{u}\phi\nabla\overline{u}\right)dX \;=\; 0. \qquad \square$$

The importance of the "integration by parts" formula in (8.3) is exploited in the next observation:

Lemma 8.2. *Let u and $\overline{u}$ be as in Lemma 8.1. Then*

$$\int_{\mathcal{B}_R^+} \overline{u}^2 \operatorname{div}\left(z^{1-2s}\nabla\phi\right)dX = 2\int_{\mathcal{B}_R^+} z^{1-2s}\,\phi\,|\nabla\overline{u}|^2\,dX. \tag{8.4}$$

for any $\phi\in C_0^\infty(\mathcal{B}_R)$ that is even in z.

Proof. Since ϕ is even in z, we have that $\partial_z\phi(x,0)=0$ and so for any $z>0$ we have that

$$\partial_z\phi(x,z)=\partial_z^2\phi(x,\tilde{z})\,z,$$

for some $\tilde{z}\in[0,z]$ and so

$$\lim_{z\to 0} z^{1-2s}\partial_z\phi(x,z)=\lim_{z\to 0} z^{2-2s}\partial_z^2\phi(x,\tilde{z})=0.$$

From this and the Divergence Theorem, we obtain that

$$\int_{\mathcal{B}_R^+}\operatorname{div}\left(z^{1-2s}\overline{u}^2(X)\,\nabla\phi(X)\right)dX \;=\; 0. \tag{8.5}$$

Furthermore a direct computation shows that

$$\overline{u}^2\operatorname{div}\left(z^{1-2s}\nabla\phi\right)-2z^{1-2s}\,\phi\,|\nabla\overline{u}|^2=\operatorname{div}\left(z^{1-2s}\overline{u}^2\,\nabla\phi\right)-2z^{1-2s}\nabla\overline{u}\cdot\nabla(\phi\overline{u}).$$

Consequently, if we integrate this identity and make use of (8.3) and (8.5), we obtain (8.4). $\square$

Lemma 8.3. *Let (u_m,E_m) be a minimizing pair in B_2, and let $\overline{u}_m$ be the extension of u_m as in* (1.7)*. Suppose that u_m converges to some u in $L^\infty(B_2)$ and $\overline{u}_m$ converges to some $\overline{u}$ in $L^\infty(\mathcal{B}_2^+)$, with $\overline{u}$ continuous in $\overline{\mathbb{R}^{n+1}_+}$, being $\overline{u}$ the extension of u as in* (1.7)*. Then*

$$\lim_{m\to+\infty}\int_{\mathcal{B}_1^+} z^{1-2s}|\nabla\overline{u}_m(X)|^2\,dX \;=\; \int_{\mathcal{B}_1^+} z^{1-2s}|\nabla\overline{u}(X)|^2\,dX.$$

Proof. First we observe that, for any $\phi\in C_0^\infty(\mathcal{B}_2^+)$, we have that

$$\int_{\mathbb{R}^{n+1}_+} z^{1-2s}\nabla\overline{u}_m(X)\cdot\nabla(\phi\overline{u}_m)(X)\,dX=0. \tag{8.6}$$

To prove this, we denote by U_m the extension of E_m according to (1.8), and we set $\tilde{u}_m:=(1+\varepsilon\phi)\overline{u}_m$, with $|\varepsilon|<1$ to be taken sufficiently small. We have that the positive set of $\tilde{u}_m$ coincide with the one of $\overline{u}_m$, and $(\tilde{u}_m,U_m)$ is a competing pair with $(\overline{u}_m,U_m)$ in Proposition 4.1.

As a consequence, the minimality property of $(\overline{u}_m,U_m)$ gives that

$$\begin{aligned} 0 \;&\leqslant\; \int_{\mathcal{B}_2^+} z^{1-2s}|\nabla\tilde{u}_m(X)|^2\,dX-\int_{\mathcal{B}_2^+} z^{1-2s}|\nabla\overline{u}_m(X)|^2\,dX \\ &=\; 2\varepsilon\int_{\mathcal{B}_2^+} z^{1-2s}\nabla\overline{u}_m(X)\cdot\nabla(\phi\overline{u}_m)(X)\,dX+o(\varepsilon), \end{aligned}$$

which implies (8.6).

Now we check that $\overline{u}$ satisfies (8.1) and (8.2) (this will allow us to exploit Lemma 8.2 in the sequel). For this, we take $p \in B_R$, with $u(p) \neq 0$. So there exists $r > 0$ such that $u \neq 0$ in $B_r(p)$. By the uniform convergence, for m sufficiently large we have that $u_m \neq 0$ in $B_r(p)$. Then, by minimality and Lemma 3.2, we know that $(-\Delta)^s u_m = 0$ in $B_r(p)$. So, by uniform convergence, we obtain that $(-\Delta)^s u = 0$ in the weak (and so in the strong) sense in $B_r(p)$. This shows that u satisfies (8.1).

Moreover, given any $\psi \in C_0^\infty(\mathcal{B}_2^+)$, if we apply (8.6) with $\phi := \psi^2$ we obtain that

$$0 = \int_{\mathbb{R}^{n+1}_+} 2z^{1-2s}\psi\, \overline{u}_m\, \nabla \overline{u}_m(X) \cdot \nabla \psi \, dX + \int_{\mathbb{R}^{n+1}_+} z^{1-2s}\psi^2\, |\nabla \overline{u}_m|^2 \, dX.$$

Thus, using Young inequality, we see that

$$\int_{\mathbb{R}^{n+1}_+} z^{1-2s}\psi^2\, |\nabla \overline{u}_m|^2 \, dX \leqslant C \int_{\mathbb{R}^{n+1}_+} z^{1-2s}\overline{u}_m^2\, |\nabla \psi|^2 \, dX,$$

for some $C > 0$. In particular, fixing ψ with $\psi = 1$ in $\mathcal{B}^+_{2-(1/10)}$, we obtain that

$$\int_{\mathcal{B}^+_{2-(1/10)}} z^{1-2s}|\nabla \overline{u}_m|^2 \, dX \leqslant C \int_{\mathcal{B}_2^+} z^{1-2s}\overline{u}_m^2 \, dX \leqslant 1 + C \int_{\mathcal{B}_2^+} z^{1-2s}\overline{u}^2 \, dX, \tag{8.7}$$

for large m, up to renaming C. As a consequence, we may suppose that

$$z^{(1-2s)/2}\nabla \overline{u}_m \text{ converges to some } \Phi \text{ weakly in } L^2(\mathcal{B}^+_{2-(1/10)}). \tag{8.8}$$

We claim that

$$\Phi = z^{(1-2s)/2}\nabla \overline{u} \tag{8.9}$$

in the weak sense. Indeed, fixed any ball $B \subset \mathcal{B}^+_{2-(1/10)}$ such that $\overline{B} \subset \mathbb{R}^{n+1}_+$, for any $\Psi \in C_0^\infty(B, \mathbb{R}^n)$ we have that

$$\int_B \operatorname{div}\left(\overline{u}_m \Psi\right) dX = 0,$$

due to the Divergence Theorem, therefore

$$\int_B \nabla \overline{u}_m \cdot \Psi \, dX = \int_B \operatorname{div}\left(\overline{u}_m \Psi\right) dX - \int_B \overline{u}_m \operatorname{div}\Psi \, dX = -\int_B \overline{u}_m \operatorname{div}\Psi \, dX.$$

Also, by (8.8), we have that

$$\lim_{m\to+\infty} \int_B \nabla \overline{u}_m \cdot \Psi \, dX = \lim_{m\to+\infty} \int_B z^{(1-2s)/2}\nabla \overline{u}_m \cdot (z^{(2s-1)/2}\Psi) \, dX = \int_B \Phi \cdot (z^{(2s-1)/2}\Psi) \, dX.$$

On the other hand, by the uniform convergence of $\overline{u}_m$, we have that

$$\lim_{m\to+\infty} \int_B \overline{u}_m \operatorname{div}\Psi \, dX = \int_B \overline{u} \operatorname{div}\Psi \, dX.$$

These observations imply that

$$\int_B \Phi \cdot (z^{(2s-1)/2}\Psi) \, dX = -\int_B \overline{u} \operatorname{div}\Psi \, dX,$$

that is $\nabla \overline{u} = z^{(2s-1)/2}\Phi$ in B, in the weak sense, which concludes the proof of (8.9).

From (8.8) and (8.9) we conclude that $z^{(1-2s)/2}\nabla\overline{u}_m$ converges to $z^{(1-2s)/2}\nabla\overline{u}$ weakly in $L^2(\mathcal{B}^+_{2-(1/10)})$. As a consequence, recalling (8.7), we obtain that

$$\begin{aligned}\int_{\mathcal{B}^+_{2-(1/10)}} z^{1-2s}|\nabla\overline{u}|^2\,dX &= \lim_{m\to+\infty}\int_{\mathcal{B}^+_{2-(1/10)}} z^{1-2s}\Big(|\nabla\overline{u}_m|^2 - |\nabla\overline{u}_m - \nabla\overline{u}|^2\Big)\,dX \\ &\leqslant \lim_{m\to+\infty}\int_{\mathcal{B}^+_{2-(1/10)}} z^{1-2s}|\nabla\overline{u}_m|^2\,dX \leqslant 1 + C\int_{\mathcal{B}^+_2} z^{1-2s}\overline{u}^2\,dX.\end{aligned}$$

This proves that $\overline{u}$ satisfies (8.2) (up to renaming the radius of the ball).

Therefore we are in the position to apply Lemma 8.2, which gives that

$$\int_{\mathcal{B}^+_2} \overline{u}^2\,\mathrm{div}\,\Big(z^{1-2s}\nabla\phi\Big)\,dX = 2\int_{\mathcal{B}^+_2} z^{1-2s}\,\phi\,|\nabla\overline{u}|^2\,dX,$$

for any $\phi\in C^\infty_0(\mathcal{B}_2)$ that is even in z. On the other hand, (8.6) implies that

$$\int_{\mathcal{B}^+_2} \overline{u}_m^2\,\mathrm{div}\,\Big(z^{1-2s}\nabla\phi\Big)\,dX = 2\int_{\mathcal{B}^+_2} z^{1-2s}\,\phi\,|\nabla\overline{u}_m|^2\,dX,$$

for any $\phi\in C^\infty_0(\mathcal{B}_2)$ that is even in z. As a consequence, if we take $\varepsilon>0$ and ϕ with image in $[0,1]$, such that $\phi=1$ in $\mathcal{B}_1$ and $\phi=0$ outside $\mathcal{B}_{1+\varepsilon}$, we obtain that

$$\begin{aligned}\lim_{m\to+\infty} 2\int_{\mathcal{B}^+_1} z^{1-2s}\,|\nabla\overline{u}_m|^2\,dX &\leqslant \lim_{m\to+\infty} 2\int_{\mathcal{B}^+_2} z^{1-2s}\,\phi\,|\nabla\overline{u}_m|^2\,dX \\ &= \lim_{m\to+\infty}\int_{\mathcal{B}^+_2} \overline{u}_m^2\,\mathrm{div}\,\Big(z^{1-2s}\nabla\phi\Big)\,dX = \int_{\mathcal{B}^+_2} \overline{u}^2\,\mathrm{div}\,\Big(z^{1-2s}\nabla\phi\Big)\,dX \\ &= 2\int_{\mathcal{B}^+_{1+\varepsilon}} z^{1-2s}\,\phi\,|\nabla\overline{u}|^2\,dX.\end{aligned}$$

Since ε can be taken as small as we like, we obtain

$$\lim_{m\to+\infty} 2\int_{\mathcal{B}^+_1} z^{1-2s}\,|\nabla\overline{u}_m|^2\,dX \leqslant 2\int_{\mathcal{B}^+_1} z^{1-2s}\,|\nabla\overline{u}|^2\,dX. \tag{8.10}$$

On the other hand, if we take $\varepsilon>0$ and ϕ with image in $[0,1]$, such that $\phi=1$ in $\mathcal{B}_{1-\varepsilon}$ and $\phi=0$ outside $\mathcal{B}_1$, the argument above gives

$$\begin{aligned}\lim_{m\to+\infty} 2\int_{\mathcal{B}^+_1} z^{1-2s}\,|\nabla\overline{u}_m|^2\,dX &\geqslant \lim_{m\to+\infty} 2\int_{\mathcal{B}^+_2} z^{1-2s}\,\phi\,|\nabla\overline{u}_m|^2\,dX \\ &= \lim_{m\to+\infty}\int_{\mathcal{B}^+_2} \overline{u}_m^2\,\mathrm{div}\,\Big(z^{1-2s}\nabla\phi\Big)\,dX = \int_{\mathcal{B}^+_2} \overline{u}^2\,\mathrm{div}\,\Big(z^{1-2s}\nabla\phi\Big)\,dX \\ &= 2\int_{\mathcal{B}^+_2} z^{1-2s}\,\phi\,|\nabla\overline{u}|^2\,dX \geqslant 2\int_{\mathcal{B}^+_{1-\varepsilon}} z^{1-2s}\,|\nabla\overline{u}|^2\,dX,\end{aligned}$$

and so, taking ε as small as we like,

$$\lim_{m\to+\infty} 2\int_{\mathcal{B}^+_1} z^{1-2s}\,|\nabla\overline{u}_m|^2\,dX \geqslant 2\int_{\mathcal{B}^+_1} z^{1-2s}\,|\nabla\overline{u}|^2\,dX.$$

This and (8.10) complete the proof of Lemma 8.3. □

Now we can complete the proof of Theorem 1.2:

Proof of Theorem 1.2. The first relation in (1.10) is a direct consequence of Lemma 8.3. As for the second, it follows as in Proposition 9.1 of [6] (using Lemma 6.3 to control the fractional perimeter in Theorem 3.3 of [6]). This completes the proof of (1.10).

Now, in order to show that $(\overline{u}, U)$ is a minimizing pair in $\mathcal{B}^+_{1/2}$, we take

$$\text{a competitor } (\overline{v}, V) \text{ for } (\overline{u}, U) \text{ in } \mathcal{B}^+_{1/2}, \tag{8.11}$$

according to Proposition 4.1, and we claim that

$$\begin{aligned} &\int_{\mathcal{B}^+_{1/2}} z^{1-2s}|\nabla\overline{u}|^2\,dX + c_{n,s,\sigma}\int_{\mathcal{B}^+_{1/2}} z^{1-\sigma}|\nabla U|^2\,dX \\ &\qquad \leqslant \int_{\mathcal{B}^+_{1/2}} z^{1-2s}|\nabla\overline{v}|^2\,dX + c_{n,s,\sigma}\int_{\mathcal{B}^+_{1/2}} z^{1-\sigma}|\nabla V|^2\,dX. \end{aligned} \tag{8.12}$$

For this, we use Lemma 6.2 (with $(\overline{u}_1, U_1) := (\overline{v}, V)$ and $(\overline{u}_2, U_2) := (\overline{u}_m, U_m)$) to find a pair $(\overline{v}_m, V_m)$ such that $\overline{v}_m = \overline{u}_m$ and $V_m = U_m$ in a neighborhood of $\partial\mathcal{B}^+_{3/2}$. Hence, $(\overline{v}_m, V_m)$ is a competitor for $(\overline{u}_m, U_m)$ in $\mathcal{B}^+_{3/2}$ according to Proposition 4.1, and so

$$\begin{aligned} &\int_{\mathcal{B}^+_{3/2}} z^{1-2s}|\nabla\overline{u}_m|^2\,dX + c_{n,s,\sigma}\int_{\mathcal{B}^+_{3/2}} z^{1-\sigma}|\nabla U_m|^2\,dX \\ &\qquad \leqslant \int_{\mathcal{B}^+_{3/2}} z^{1-2s}|\nabla\overline{v}_m|^2\,dX + c_{n,s,\sigma}\int_{\mathcal{B}^+_{3/2}} z^{1-\sigma}|\nabla V_m|^2\,dX, \end{aligned} \tag{8.13}$$

since $(\overline{u}_m, U_m)$ is a minimizing pair.

Moreover, thanks to (6.13) and (6.14), we have that

$$\begin{aligned} &\int_{\mathcal{B}^+_{3/2}} z^{1-2s}|\nabla\overline{v}_m|^2\,dX + c_{n,s,\sigma}\int_{\mathcal{B}^+_{3/2}} z^{1-\sigma}|\nabla V_m|^2\,dX \\ &\quad \leqslant \int_{\mathcal{B}^+_{1}} z^{1-2s}|\nabla\overline{v}|^2\,dX + c_{n,s,\sigma}\int_{\mathcal{B}^+_{1}} z^{1-\sigma}|\nabla V|^2\,dX \\ &\qquad + \int_{\mathcal{B}^+_{3/2}} z^{1-2s}|\nabla\overline{u}_m|^2\,dX + c_{n,s,\sigma}\int_{\mathcal{B}^+_{3/2}} z^{1-\sigma}|\nabla U_m|^2\,dX \\ &\qquad - \int_{\mathcal{B}^+_{1}} z^{1-2s}|\nabla\overline{u}_m|^2\,dX - c_{n,s,\sigma}\int_{\mathcal{B}^+_{1}} z^{1-\sigma}|\nabla U_m|^2\,dX + c_m(\varepsilon), \end{aligned} \tag{8.14}$$

where

$$\begin{aligned} c_m(\varepsilon) \;:=\; & C\,\varepsilon^{-2}\int_{\mathcal{B}^+_{1+\varepsilon}\setminus\mathcal{B}^+_{1-\varepsilon}} z^{1-2s}|\overline{v}-\overline{u}_m|^2\,dX + C\int_{\mathcal{B}^+_{1+\varepsilon}\setminus\mathcal{B}^+_{1-\varepsilon}} z^{1-2s}\left(|\nabla\overline{v}|^2 + |\nabla\overline{u}_m|^2\right)dX \\ & + C\int_{\mathcal{B}^+_{1+\varepsilon}\setminus\mathcal{B}^+_{1-\varepsilon}} z^{1-\sigma}|\nabla\phi|^2|V-U_m|^2\,dX + C\int_{\mathcal{B}^+_{1+\varepsilon}\setminus\mathcal{B}^+_{1-\varepsilon}} z^{1-\sigma}\left(|\nabla V|^2 + |\nabla U_m|^2\right)dX, \end{aligned}$$

with $\phi := \phi_\varepsilon$ as in Lemma 6.1. Putting together (8.13) and (8.14), we obtain that

$$\begin{aligned} &\int_{\mathcal{B}^+_{1}} z^{1-2s}|\nabla\overline{u}_m|^2\,dX + c_{n,s,\sigma}\int_{\mathcal{B}^+_{1}} z^{1-\sigma}|\nabla U_m|^2\,dX \\ &\qquad \leqslant \int_{\mathcal{B}^+_{1}} z^{1-2s}|\nabla\overline{v}|^2\,dX + c_{n,s,\sigma}\int_{\mathcal{B}^+_{1}} z^{1-\sigma}|\nabla V|^2\,dX + c_m(\varepsilon). \end{aligned} \tag{8.15}$$

Now we take the limit as $m \to +\infty$ in (8.15). Thanks to (1.10) (which has been already proved), we have that

(8.16)
$$\text{the left-hand side converges to } \int_{\mathcal{B}_1^+} z^{1-2s} |\nabla \overline{u}|^2 \, dX + c_{n,s,\sigma} \int_{\mathcal{B}_1^+} z^{1-\sigma} |\nabla U|^2 \, dX.$$

Now we compute the limit of $c_m(\varepsilon)$ as $m \to +\infty$, for a fixed $\varepsilon > 0$ (and then send $\varepsilon \to 0$ at the end). For this, we first observe that $\overline{v} = \overline{u}$ and $V = U$ outside $\mathcal{B}_{1/2}^+$, thanks to (8.11), and so $c_m(\varepsilon)$ can be written as

(8.17)
$$\begin{aligned} c_m(\varepsilon) = {} & C\,\varepsilon^{-2} \int_{\mathcal{B}_{1+\varepsilon}^+ \setminus \mathcal{B}_{1-\varepsilon}^+} z^{1-2s} |\overline{u} - \overline{u}_m|^2 \, dX + C \int_{\mathcal{B}_{1+\varepsilon}^+ \setminus \mathcal{B}_{1-\varepsilon}^+} z^{1-2s} \left(|\nabla \overline{u}|^2 + |\nabla \overline{u}_m|^2 \right) dX \\ & + C \int_{\mathcal{B}_{1+\varepsilon}^+ \setminus \mathcal{B}_{1-\varepsilon}^+} z^{1-\sigma} |\nabla \phi|^2 |U - U_m|^2 \, dX + C \int_{\mathcal{B}_{1+\varepsilon}^+ \setminus \mathcal{B}_{1-\varepsilon}^+} z^{1-\sigma} \left(|\nabla U|^2 + |\nabla U_m|^2 \right) dX. \end{aligned}$$

Now we claim that

(8.18)
$$\int_{\mathcal{B}_{1+\varepsilon}^+ \setminus \mathcal{B}_{1-\varepsilon}^+} z^{1-2s} |\overline{u} - \overline{u}_m|^2 \, dX \to 0 \quad \text{and} \quad \int_{\mathcal{B}_{1+\varepsilon}^+ \setminus \mathcal{B}_{1-\varepsilon}^+} z^{1-\sigma} |\nabla \phi|^2 |U - U_m|^2 \, dX \to 0,$$

as $m \to +\infty$ (for a fixed $\varepsilon > 0$). Indeed, the first limit follows from (1.9). As for the second limit, we observe that $|U_m| \leqslant 1$, since U_m is obtained by convolution between a characteristic function and the Poisson kernel which has integral 1. Hence also $|U| \leqslant 1$ in $\mathcal{B}_2^+$. This means that, for a fixed $\varepsilon > 0$,

$$z^{1-\sigma} |\nabla \phi|^2 |U - U_m|^2 \leqslant 4 z^{1-\sigma} |\nabla \phi|^2,$$

and this function lies in $L^1(\mathbb{R}_+^{n+1})$, thanks to (6.4), applied here with $\beta := 1 - \sigma \in (0,1)$. Moreover, for a fixed $\varepsilon > 0$, we have that $z^{1-\sigma} |\nabla \phi|^2 |U - U_m|^2 \to 0$ as $m \to +\infty$. Then the second limit in (8.18) follows from the Dominated Convergence Theorem. This completes the proof of (8.18).

Now, we claim that

(8.19)
$$\lim_{m \to +\infty} \int_{\mathcal{B}_{1+\varepsilon}^+ \setminus \mathcal{B}_{1-\varepsilon}^+} z^{1-2s} \left(|\nabla \overline{u}|^2 + |\nabla \overline{u}_m|^2 \right) dX \leqslant C \int_{\mathcal{B}_{1+2\varepsilon}^+ \setminus \mathcal{B}_{1-2\varepsilon}^+} z^{1-2s} |\nabla \overline{u}|^2 \, dX,$$

for a suitable $C > 0$. For this, we observe that $\mathcal{B}_{1+\varepsilon}^+ \setminus \mathcal{B}_{1-\varepsilon}^+$ can be covered by a finite overlapping family of N_ε balls of radius ε, say $\mathcal{B}_\varepsilon(X_j)$ with $j = 1, \dots N_\varepsilon$, and so

$$\begin{aligned} \int_{\mathcal{B}_{1+\varepsilon}^+ \setminus \mathcal{B}_{1-\varepsilon}^+} z^{1-2s} |\nabla \overline{u}_m|^2 \, dX & \leqslant \int_{\cup_{j=1}^{N_\varepsilon} \mathcal{B}_\varepsilon(X_j)} z^{1-2s} |\nabla \overline{u}_m|^2 \, dX \\ & \leqslant C \sum_{j=1}^{N_\varepsilon} \int_{\mathcal{B}_\varepsilon(X_j)} z^{1-2s} |\nabla \overline{u}_m|^2 \, dX. \end{aligned}$$

By using (1.10) once again, this implies that

$$\lim_{m\to+\infty}\int_{\mathcal{B}^+_{1+\varepsilon}\setminus\mathcal{B}^+_{1-\varepsilon}} z^{1-2s}|\nabla\overline{u}_m|^2\,dX \leqslant C\sum_{j=1}^{N_\varepsilon}\int_{\mathcal{B}_\varepsilon(X_j)} z^{1-2s}|\nabla\overline{u}|^2\,dX \leqslant C\int_{\mathcal{B}^+_{1+2\varepsilon}\setminus\mathcal{B}^+_{1-2\varepsilon}} z^{1-2s}|\nabla\overline{u}|^2\,dX,$$

which shows (8.19) up to renaming constants.

Analogously, one can prove that

$$\lim_{m\to+\infty}\int_{\mathcal{B}^+_{1+\varepsilon}\setminus\mathcal{B}^+_{1-\varepsilon}} z^{1-\sigma}\left(|\nabla U|^2+|\nabla U_m|^2\right)\,dX \leqslant C\int_{\mathcal{B}^+_{1+2\varepsilon}\setminus\mathcal{B}^+_{1-2\varepsilon}} z^{1-\sigma}|\nabla U|^2\,dX, \tag{8.20}$$

for some $C>0$. Using (8.18), (8.19) and (8.20) into (8.17), we get

$$\lim_{m\to+\infty} c_m(\varepsilon) \leqslant C\int_{\mathcal{B}^+_{1+2\varepsilon}\setminus\mathcal{B}^+_{1-2\varepsilon}} z^{1-2s}|\nabla\overline{u}|^2\,dX + C\int_{\mathcal{B}^+_{1+2\varepsilon}\setminus\mathcal{B}^+_{1-2\varepsilon}} z^{1-\sigma}|\nabla U|^2\,dX,$$

up to relabelling constants. Hence, sending $\varepsilon\to 0$, we obtain that

$$\lim_{\varepsilon\to 0}\lim_{m\to+\infty} c_m(\varepsilon) = 0.$$

Using this and (8.16) into (8.15), we obtain that

$$\int_{\mathcal{B}^+_1} z^{1-2s}|\nabla\overline{u}|^2\,dX + c_{n,s,\sigma}\int_{\mathcal{B}^+_1} z^{1-\sigma}|\nabla U|^2\,dX \leqslant \int_{\mathcal{B}^+_1} z^{1-2s}|\nabla\overline{v}|^2\,dX + c_{n,s,\sigma}\int_{\mathcal{B}^+_1} z^{1-\sigma}|\nabla V|^2\,dX,$$

which implies that $(\overline{u},U)$ is a minimizing pair in $\mathcal{B}^+_{1/2}$, according to Proposition 4.1. This shows (8.12) and concludes the proof of Theorem 1.2. □

9. Limit of the blow-up sequences and proof of Theorem 1.3

Here we show that the blow-up limit of a minimizing pair is a minimizing cone and prove Theorem 1.3.

Proof of Theorem 1.3. First of all, we notice that, for any x, $\tilde{x}\in\mathbb{R}^n$,

$$|u_r(x)-u_r(\tilde{x})| = r^{\frac{\sigma}{2}-s}|u(rx)-u(r\tilde{x})| \leqslant \|u\|_{C^{s-\frac{\sigma}{2}}(\mathbb{R}^n)}|x-\tilde{x}|^{s-\frac{\sigma}{2}}. \tag{9.1}$$

This shows that $u_r\in C^{s-\frac{\sigma}{2}}(\mathbb{R}^n)$, with norm bounded uniformly in r. So, up to a subsequence, we may assume that

$$u_r \text{ converges locally uniformly to some } u_0\in C^{s-\frac{\sigma}{2}}(\mathbb{R}^n). \tag{9.2}$$

We observe that $u\geqslant 0$ in E and $u\leqslant 0$ in E^c: thus, since $0\in\partial E$, we have that $u(0)=0$. As a consequence $u_r(0)=0$ and therefore, by (9.1),

$$|u_r(x)| \leqslant \|u\|_{C^{s-\frac{\sigma}{2}}(\mathbb{R}^n)}|x|^{s-\frac{\sigma}{2}}, \tag{9.3}$$

and so

$$|u_0(x)| \leqslant \|u\|_{C^{s-\frac{\sigma}{2}}(\mathbb{R}^n)}|x|^{s-\frac{\sigma}{2}}. \tag{9.4}$$

Since (u_r, E_r) is a minimizing pair in $B_{1/r}$, we can fix any $R > 0$, take $r \in (0, 1/(4R))$ and use Theorem 1.1: we obtain that

$$\begin{aligned} \iint_{\mathbb{R}^{2n}\setminus(B_R^c)^2} & \frac{|u_r(x) - u_r(y)|^2}{|x-y|^{n+2s}}\, dx\, dy + \operatorname{Per}_\sigma(E_r, B_R) \\ & \leqslant C \left(1 + \int_{\mathbb{R}^n} \frac{|u_r(y)|^2}{1+|y|^{n+2s}}\, dy\right) \leqslant C, \end{aligned}$$

for some $C > 0$, possibly different from step to step, where (9.3) was used in the last passage. In particular, we have that $\operatorname{Per}_\sigma(E_r, B_R)$ is bounded uniformly in r. By compactness, this shows that, up to a subsequence, E_r converges in $L^1_{\text{loc}}(\mathbb{R}^n)$ to some E_0.

Now, let $\overline{u}_r$ and U_r be the extension functions of u_r and E_r, as in (1.7) and (1.8). Similarly, let $\overline{u}_0$ and U_0 be the extension functions of u_0 and E_0.

By (9.3) and (9.4), if we fix $\rho > 0$ and we take $x \in B_\rho$ and $z \in (0, \rho)$, we have that

$$P_s(y,z)\ |u_r(x-y) - u_0(x-y)| \leqslant \frac{2c_{n,s}\|u\|_{C^{s-\frac{\sigma}{2}}(\mathbb{R}^n)}\rho^{2s}(\rho+|y|)^{s-\frac{\sigma}{2}}}{|y|^{n+2s}}.$$

This implies that

$$\int_{\mathbb{R}^n\setminus B_1} P_s(y,z)\ |u_r(x-y) - u_0(x-y)|\, dy < +\infty$$

and therefore for any fixed $\varepsilon > 0$ there exists $R := R_{\rho,\varepsilon} > 0$ such that

$$\int_{\mathbb{R}^n\setminus B_R} P_s(y,z)\ |u_r(x-y) - u_0(x-y)|\, dy \leqslant \varepsilon.$$

Consequently, for any $\rho > 0$ and any $x \in B_\rho$ and $z \in (0, \rho)$, we have that

$$\begin{aligned} |\overline{u}_r(x,z) - \overline{u}_0(x,z)| & \leqslant \int_{B_R} P_s(y,z)\ \|u_r - u_0\|_{L^\infty(B_{R+\rho})}\, dy + \varepsilon \\ & \leqslant \|u_r - u_0\|_{L^\infty(B_{R+\rho})} + \varepsilon. \end{aligned}$$

That is

$$\|\overline{u}_r - \overline{u}_0\|_{L^\infty(B_\rho\times(0,\rho))} \leqslant \|u_r - u_0\|_{L^\infty(B_{R+\rho})} + \varepsilon$$

and therefore, by (9.2),

$$\lim_{r\to 0} \|\overline{u}_r - \overline{u}_0\|_{L^\infty(B_\rho\times(0,\rho))} \leqslant \varepsilon.$$

Since $\varepsilon > 0$ may be taken arbitrarily small, we infer that

$$\lim_{r\to 0} \|\overline{u}_r - \overline{u}_0\|_{L^\infty(B_\rho\times(0,\rho))} = 0,$$

hence $\overline{u}_r$ converges locally uniformly to $\overline{u}_0$.

Moreover, as in Proposition 9.1 in [6], we have that U_r converges, up to subsequence, to some U_0 locally in $L^2_{\sigma/2}$. These observations give that (1.9) is satisfied in this case. Now we claim that $\overline{u}_0$ is continuous on $\overline{\mathbb{R}^{n+1}_+}$. For this, we take a sequence $(x_k, z_k) \in \mathbb{R}^{n+1}_+$, with $(x_k, z_k) \to (x, z) \in \overline{\mathbb{R}^{n+1}_+}$ as $k \to +\infty$. We have

$$\begin{aligned}\overline{u}_0(x_k, z_k) &= \int_{\mathbb{R}^n} P_s(y, z_k)\, u_0(x_k - y)\, dy \\ &= \int_{\mathbb{R}^n} z_k^{-n} P_s(z_k^{-1} y, 1)\, u_0(x_k - y)\, dy = \int_{\mathbb{R}^n} P_s(\tilde{y}, 1)\, u_0(x_k - z_k \tilde{y})\, d\tilde{y}.\end{aligned}$$

Now we observe that

$$\lim_{k \to +\infty} u_0(x_k - z_k \tilde{y}) = u_0(x - z\tilde{y}),$$

due to (9.2). Also, by (9.4),

$$P_s(\tilde{y}, 1)\, |u_0(x_k - z_k \tilde{y})| \leqslant \frac{c_{n,s}\, \|u\|_{C^{s-\frac{\sigma}{2}}(\mathbb{R}^n)} (1 + |x| + z|\tilde{y}|)^{s-\frac{\sigma}{2}}.}{(1 + |\tilde{y}|^2)^{\frac{n+2s}{2}}},$$

for k large, which is integrable in $\tilde{y} \in \mathbb{R}^n$. Accordingly, by the Dominated Convergence Theorem,

$$\lim_{k \to +\infty} \overline{u}_0(x_k, z_k) = \int_{\mathbb{R}^n} P_s(\tilde{y}, 1)\, u_0(x - z\tilde{y})\, d\tilde{y} = \overline{u}_0(x, z),$$

that proves the continuity of $\overline{u}_0$ in $\overline{\mathbb{R}^{n+1}_+}$.

Therefore, we can use Theorem 1.2 and obtain that $(\overline{u}_0, U_0)$ is a minimizing pair. Thus, by Proposition 4.1, we have that (u_0, E_0) is a minimizing pair.

It remains to show that (u_0, E_0) is homogeneous (hence it is a minimizing cone). For this, we recall (2.1) and we use (1.10) to see that

$$\lim_{r \to 0} \Phi_{u_r}(t) = \Phi_{u_0}(t).$$

This and (4.12) give that

$$\lim_{r \to 0} \Phi_u(rt) = \Phi_{u_0}(t).$$

That is

$$\Phi_{u_0}(t) = \lim_{\tau \to 0} \Phi_u(\tau),$$

and this limit exists since Φ_u is monotone (recall Theorem 2.1). In particular, Φ_{u_0} is constant and so, by Theorem 2.1, we have that (u_0, E_0) is homogeneous. This completes the proof of Theorem 1.3. □

10. A maximum principle in unbounded domains for the fractional Laplacian and proof of Theorems 2.3 and 2.4

The purpose of this section is to prove Theorems 2.4 and 2.3.

Proof of Theorem 2.4. First we observe that

$$(-\Delta)^s v^+ \leqslant 0 \text{ in the whole of } \mathbb{R}^n \tag{10.1}$$

in the viscosity sense. To check this, let ϕ be a competing function touching v^+ from above at p.

If $v^+(p) > 0$, then $p \in D$, since $v^+ = 0$ outside D. Notice also that $\phi \geqslant v^+ \geqslant v$ and $\phi(p) = v^+(p) = v(p)$, thus ϕ touches v from above at p, and therefore, by (2.2), we obtain that $(-\Delta)^s \phi(p) \leqslant 0$.

On the other hand, if $v^+(p) = 0$, then we have that $\phi \geqslant v^+ \geqslant 0$ and $\phi(p) = v^+(p) = 0$, which gives directly that

$$\int_{\mathbb{R}^n} \frac{\phi(p+y) + \phi(p-y) - 2\phi(p)}{|y|^{n+2s}}\, dy \geqslant 0.$$

This proves (10.1).

Now we show that

$$v^+ \text{ vanishes identically.} \tag{10.2}$$

Suppose not, then we can define

$$A := \sup_{\mathbb{R}^n} v^+ \in (0, +\infty).$$

So we fix any $q = (q', q_n) \in \mathbb{R}^n$ such that $v^+(q) > 0$. Notice that $q_n > 0$ since $v^+ = 0$ in $\{x_n \leqslant 0\}$. So we can set $r := 2q_n > 0$ and $\tilde{q} := (q', -r/4)$, and we remark that

$$B_{r/4}(\tilde{q}) \subseteq B_r(q) \cap \{x_n \leqslant 0\}.$$

Accordingly,

$$\big|B_r(q) \cap \{v^+ \leqslant 0\}\big| \geqslant \big|B_r(q) \cap \{x_n \leqslant 0\}\big| \geqslant \big|B_{r/4}(\tilde{q})\big| \geqslant \delta r^n$$

for some universal $\delta > 0$. So we are in the position of applying a Harnack-type inequality (see e.g. Corollary 4.5 in [21]) and we conclude that $v^+ \leqslant (1-\gamma)A$ in $B_{r/2}(q)$, and so, in particular $v^+(q) \leqslant (1-\gamma)A$, for some $\gamma \in (0,1)$. As a consequence, since q is arbitrary,

$$A = \sup_{q \in \{v^+ > 0\}} v^+(q) \leqslant (1-\gamma)A,$$

which is a contradiction. This proves (10.2) which in turn implies Theorem 2.4. □

As a consequence of Theorem 2.4 we have the following classification result:

Corollary 10.1. *Let $A > 0$. Let E be a cone in $\mathbb{R}^n$ such that*

$$E \subseteq \{x_n > 0\}. \tag{10.3}$$

Let $u \in C^2(E)$ and continuous on $\overline{E}$, with $[u]_{C^\gamma(E)} < +\infty$ for some $\gamma \in (0, s]$. Assume that

$$\begin{cases} (-\Delta)^s u \leqslant 0 & \text{in } E, \\ u \geqslant 0 & \text{in } E, \\ u \leqslant 0 & \text{in } E^c. \end{cases}$$

Then

$$u(x) \leqslant C_A (x_n)_+^s \tag{10.4}$$

for any $x \in E \cap \{x_n \leqslant A\}$ with

$$C_A := A^{\gamma - s}\, [u]_{C^\gamma(E)}. \tag{10.5}$$

Also, if u is homogeneous of degree $\alpha < s$ then u vanishes identically in E.

Proof. First we focus on the proof of (10.4). For this, we first observe that

$$\text{for every } x \in E,\ u(x) \leqslant [u]_{C^\gamma(E)} x_n^\gamma. \tag{10.6}$$

Indeed, by (10.3), for any $x = (x', x_n) \in E$ there exists $\tau \in [0, x_n)$ such that $y := (x', \tau) \in \partial E$. Therefore $u(y) = 0$ and so

$$u(x) = u(x) - u(y) \leqslant [u]_{C^\gamma(E)} |x - y|^\gamma = [u]_{C^\gamma(E)} |x_n - \tau|^\gamma \leqslant [u]_{C^\gamma(E)} x_n^\gamma,$$

that establishes (10.6). In particular, we have that

$$\text{for every } x \in E \cap \{x_n \leqslant A\},\ u(x) \leqslant A^\gamma [u]_{C^\gamma(E)}. \tag{10.7}$$

So we define $v(x) := u(x) - C_A (x_n)_+^s$, with C_A as in (10.5). By [16], we know that $(-\Delta)^s (x_n)_+^s = 0$ in $\{x_n > 0\}$, therefore $(-\Delta)^s v \leqslant 0$ in $D := E \cap \{x_n \leqslant A\}$.

Moreover, if $x \in E^c$ we have that $v(x) \leqslant -C_A (x_n)_+^s \leqslant 0$, and if $x \in E \cap \{x_n > A\}$ we have that

$$\begin{aligned} v(x) &\leqslant [u]_{C^\gamma(E)} x_n^\gamma - C_A x_n^s = x_n^\gamma \left([u]_{C^\gamma(E)} - C_A x_n^{s-\gamma} \right) \\ &\leqslant x_n^\gamma \left([u]_{C^\gamma(E)} - C_A A^{s-\gamma} \right) \leqslant 0, \end{aligned}$$

thanks to (10.6) and (10.5) (recall also that $\gamma \leqslant s$). As a consequence $v(x) \leqslant 0$ for any $x \in E^c \cup (E \cap \{x_n > A\}) = \big(E \cap (E \cap \{x_n > A\})^c \big)^c = (E \cap \{x_n \leqslant A\})^c = D^c$. Also $v \in L^\infty(D)$ thanks to (10.7). So we can apply Theorem 2.4 and obtain that $v \leqslant 0$ in D, which is (10.4).

Now we establish the second claim in the statement of Corollary 10.1. For this we suppose in addition that u is homogeneous of degree $\alpha < s$: then, fix any $x \in E$ and any $A > x_n$. By (10.4) we have

$$u(x) = t^{-\alpha} u(tx) \leqslant C_A t^{s-\alpha} (x_n)_+^s$$

for any $t \in (0,1)$, hence, by taking $t \to 0$ the second claim of Corollary 10.1 follows. □

Proof of Theorem 2.3. We make some preliminary observations. First, we notice that if u vanishes identically then the thesis trivially follows. Therefore, we can suppose that $u \neq 0$, and so

$$\text{there exists } \omega \in S^{n-1} \text{ such that } u(\omega) \neq 0. \tag{10.8}$$

Now, we claim that $s - \frac{\sigma}{2} \geqslant 0$. For this, we observe that $u \in C^\gamma(\mathbb{R}^n)$, in particular it belongs to $C^\gamma(\overline{B}_2)$. Therefore, from Weierstraß's theorem, we have that u is bounded in B_2. On the other hand, u is homogeneous of degree $s - \frac{\sigma}{2}$, and so

$$u(rx) = r^{s - \frac{\sigma}{2}} u(x) \tag{10.9}$$

for any $x \in B_2$ and $r \in (0,1]$. Since $x, rx \in B_2$, we have that both $u(x)$ and $u(rx)$ are bounded. Therefore, sending $r \searrow 0$ in (10.9), we obtain that $s - \frac{\sigma}{2} \geqslant 0$.

Now, if $s - \frac{\sigma}{2} = 0$, then $u = c$ for some constant $c \in \mathbb{R}$. Then, the claim of the theorem easily follows: indeed, for instance, if the positivity set E is contained in a halfspace then $u = c \leqslant 0$.

Hence, from now on we assume that

$$s - \frac{\sigma}{2} > 0. \tag{10.10}$$

We prove that

$$u(0) = 0. \tag{10.11}$$

Indeed, since u is homogeneous of degree $s - \frac{\sigma}{2}$, we have that

$$u(0) = r^{s - \frac{\sigma}{2}} u(0)$$

for any $r > 0$, which implies (10.11).

Now, we recall that $u \in C^\gamma(\mathbb{R}^n)$ and we prove that

$$\gamma = s - \frac{\sigma}{2}. \tag{10.12}$$

For this, we take ω as in (10.8) and we obtain that, for any $r > 0$,

$$|u(r\omega) - u(0)| \leqslant [u]_{C^\gamma(\mathbb{R}^n)} r^\gamma.$$

On the other hand,

$$|u(r\omega) - u(0)| = |u(r\omega)| = r^{s-\frac{\sigma}{2}} |u(\omega)|,$$

thanks to (10.11). Therefore,

$$r^{s-\frac{\sigma}{2}} |u(\omega)| \leqslant [u]_{C^\gamma(\mathbb{R}^n)} r^\gamma. \tag{10.13}$$

Since $|u(\omega)| \neq 0$ (recall (10.8)), this implies that

$$r^{s-\frac{\sigma}{2}-\gamma} \leqslant \frac{[u]_{C^\gamma(\mathbb{R}^n)}}{|u(\omega)|} =: C_1, \tag{10.14}$$

for a suitable positive constant C_1. Moreover, u is not identically a constant, thanks to (10.10), and so $[u]_{C^\gamma(\mathbb{R}^n)} \neq 0$. Hence, (10.13) implies that

$$r^{\gamma-s+\frac{\sigma}{2}} \geqslant \frac{|u(\omega)|}{[u]_{C^\gamma(\mathbb{R}^n)}} =: C_2, \tag{10.15}$$

for some constant $C_2 > 0$. Now, if $\gamma < s - \frac{\sigma}{2}$ then, we send r to $+\infty$ in (10.14) and we obtain a contradiction. If $\gamma > s - \frac{\sigma}{2}$, we send $r \searrow 0$ in (10.15) and we reach again a contradiction. This proves (10.12).

Now, we prove the first claim in Theorem 2.3 (the proof of the second claim is similar, and then the last claim clearly follows). For this, we suppose, up to a rigid motion, that $E \subseteq \{x_n > 0\}$ and we show that $u \leqslant 0$. So we assume, by contradiction, that

$$E^+ := \{u > 0\} \neq \varnothing.$$

By construction, $u \leqslant 0$ in $E^c \supseteq \{x_n \leqslant 0\}$, therefore $E^+ \subseteq \{x_n > 0\}$. Also, by Lemma 3.2, $(-\Delta)^s u = 0$ in E^+, and $u \leqslant 0$ outside E^+. Moreover, $[u]_{C^{s-\frac{\sigma}{2}}(\mathbb{R}^n)} < +\infty$, thanks to (10.12). Therefore, by the second claim in Corollary 10.1, we obtain that u vanishes identically in E^+, hence u^+ is identically zero, and so $u \leqslant 0$. □

11. Functions, sets and proof of Remark 1.5

We observe that the scaling properties in (1.11) suggest that when $s = \sigma/2$, homogeneous functions of degree zero play a crucial role for the problem.

This may lead to the conjecture that, at least in this case, a minimizing pair (u, E) reduces to the set E itself, i.e. $u = \chi_E - \chi_{E^c}$, provided that the boundary data allow such configuration (notice that when $s = \sigma/2$ then $s \in (0, 1/2)$ and so the Gagliardo seminorm of the characteristic function of a smooth set is finite, thus the energy is also well defined).

The content of Remark 1.5 is that this is not true.

Proof of Remark 1.5. Suppose by contradiction that $u = \chi_E - \chi_{E^c}$, with $E \neq \varnothing$ and $E^c \neq \varnothing$, that is, in the measure theoretic sense,

$$|E| > 0 \text{ and } |E^c| > 0. \tag{11.1}$$

Notice that either $|E \cap B_1| > 0$ or $|E^c \cap B_1| > 0$. So, for concreteness, we may suppose that $|E \cap B_1| > 0$. As a consequence, there exists $r \in (0, 1)$ such that

$$|E \cap B_r| > 0. \tag{11.2}$$

Let now $R \in (r, 1)$ and $\tau \in C_0^\infty(B_R, [0,1])$, with

$$\tau = 1 \text{ in } B_r. \tag{11.3}$$

For any $t \in [0, 1)$, let $u_t(x) := (1 - t\tau(x))u(x)$.

We observe that $u_0 = u$. In addition, $u_t = u$ outside B_1. Also $1 - t\tau(x) \geqslant 1 - t > 0$, hence the sign of u is the same as the one of u_t. As a consequence, the pair (u_t, E) is admissible, hence $\mathcal{F}(u, E) \leqslant \mathcal{F}(u_t, E)$ by minimality. Accordingly

$$\begin{aligned} 0 &\leqslant \mathcal{F}(u_t, E) - \mathcal{F}(u, E) \\ &= \iint_{\mathbb{R}^{2n} \setminus (B_1^c)^2} \frac{|u_t(x) - u_t(y)|^2 - |u(x) - u(y)|^2}{|x - y|^{n+2s}} \, dx \, dy \\ &= \iint_{\mathbb{R}^{2n}} \frac{|u_t(x) - u_t(y)|^2 - |u(x) - u(y)|^2}{|x - y|^{n+2s}} \, dx \, dy. \end{aligned} \tag{11.4}$$

Notice that

$$\begin{aligned} & |u_t(x) - u_t(y)|^2 \\ = \; & (1 - t\tau(x))^2 u^2(x) + (1 - t\tau(y))^2 u^2(y) - 2(1 - t\tau(x))\,(1 - t\tau(y))\, u(x)\, u(y) \\ = \; & |u(x) - u(y)|^2 \\ & + 2t\Big[- \tau(x)u^2(x) - \tau(y)u^2(y) + (\tau(x) + \tau(y))\, u(x)\, u(y)\Big] \\ & + t^2 \Big| \tau(x)\, u(x) - \tau(x)\, u(x) \Big|^2 . \end{aligned}$$

By inserting this into (11.4) and dividing by $2t$ we thus obtain

$$0 \leqslant \iint_{\mathbb{R}^{2n}} \frac{-\tau(x)u^2(x) - \tau(y)u^2(y) + (\tau(x) + \tau(y))\, u(x)\, u(y)}{|x - y|^{n+2s}} \, dx \, dy + \Xi t,$$

for some $\Xi \in \mathbb{R}$ depending on u and τ but independent of t. Hence we may send $t \searrow 0$ and we conclude that

$$0 \leqslant \iint_{\mathbb{R}^{2n}} \frac{-\tau(x)u^2(x) - \tau(y)u^2(y) + (\tau(x) + \tau(y))\, u(x)\, u(y)}{|x - y|^{n+2s}} \, dx \, dy. \tag{11.5}$$

Now, if either $(x, y) \in E \times E$ or $(x, y) \in E^c \times E^c$ we have that $u^2(x) = u^2(y) = u(x)u(y) = 1$ hence the integrand in (11.5) vanishes. Hence, since the role of x and y is symmetric, we obtain from (11.5) that

$$\begin{aligned} 0 \; & \leqslant \; 2 \iint_{E \times E^c} \frac{-\tau(x)u^2(x) - \tau(y)u^2(y) + (\tau(x) + \tau(y))\, u(x)\, u(y)}{|x - y|^{n+2s}} \, dx \, dy \\ & = \; 2 \iint_{E \times E^c} \frac{-\tau(x) - \tau(y) - (\tau(x) + \tau(y))}{|x - y|^{n+2s}} \, dx \, dy \\ & = \; -4 \iint_{E \times E^c} \frac{\tau(x) + \tau(y)}{|x - y|^{n+2s}} \, dx \, dy. \end{aligned}$$

Since the integrand above is nonnegative, recalling (11.1) we infer that $\tau(x) + \tau(y) = 0$ for a.e. $(x, y) \in E \times E^c$.

As a consequence, $\tau(x) = 0$ for a.e. $x \in E$, and so, in particular, for a.e. $x \in E \cap B_r$. This set has indeed positive measure, thanks to (11.2), hence we get that there exists $p \in E \cap B_r$ such that $\tau(p) = 0$. But this is in contradiction with (11.3) and thus it proves Remark 1.5. $\square$

12. Removable singularities and proof of Remark 1.6

In this section we give the simple proof of Remark 1.6. As a matter of fact, we stress that Remark 1.6 only aims at pointing out the possible development of plateau in a simple, concrete example, using as little technology as possible (more general results may be obtained by capacity considerations, and with the use of the fundamental solution of the fractional Laplacian when $n \geqslant 2s$).

Proof of Remark 1.6. Assume that

$$\{u=0\} \cap (-1,1) \subseteq \{p_1, \dots, p_N\}. \tag{12.1}$$

We show that $(-\Delta)^{1/2} u = 0$ in $(-1,1)$.

For this, we take $\bar{u}$ to be the harmonic extension of u in $\mathbb{R}^2_+$. Namely $\bar{u}$ has finite $H^1(\mathbb{R}^2_+)$-seminorm and satisfies

$$\begin{cases} \Delta \bar{u}(x,y) = 0 & \text{for any } x \in \mathbb{R} \text{ and any } y > 0, \\ \bar{u}(x,0) = u(x) & \text{for any } x \in \mathbb{R}. \end{cases} \tag{12.2}$$

By Lemma 3.2 and (12.1) we have that $(-\Delta)^{1/2} u(x) = 0$ for any $x \in \mathbb{R} \setminus \{p_1, \dots, p_N\}$, hence

$$\partial_y \bar{u}(x,0) = 0 \text{ for any } x \in \mathbb{R} \setminus \{p_1, \dots, p_N\}. \tag{12.3}$$

Now we take the even symmetric extension of $\bar{u}$, that is, we define

$$u^*(x,y) := \begin{cases} \bar{u}(x,y) = 0 & \text{for any } x \in \mathbb{R} \text{ and any } y \geqslant 0, \\ \bar{u}(x,-y) = 0 & \text{for any } x \in \mathbb{R} \text{ and any } y < 0. \end{cases}$$

We observe that

$$\Delta u^* = 0 \text{ for any } (x,y) \in \mathbb{R}^2 \setminus \{(p_1,0), \dots, (p_N,0)\}.$$

Therefore, by the removal of singularities result for harmonic functions, we conclude that $\Delta u^* = 0$ in the whole of $\mathbb{R}^2$ and therefore $\partial_y u^*$ is continuous also in the vicinity of $(p_1,0), \dots, (p_N,0)$. This implies that $\partial_y \bar{u}(x,0) = 0$ for any $x \in (-1,1)$, which means $(-\Delta)^{1/2} u = 0$ in $(-1,1)$. $\square$

Appendix A. Regularity of cones in the plane and proof of Theorem 1.4

This section is devoted to the regularity of the two-dimensional cones. Namely, in order to prove Theorem 1.4, we follow the methods introduced in [18, 19] to prove the regularity of σ-minimal surfaces and used in [8] to obtain the regularity of the minimizers of the functional (1.4).

We first introduce some notations. We define, for any $r > 0$,

$$\mathcal{E}_r(\overline{v}, V) := \int_{\mathcal{B}_r^+} z^{1-2s} \, |\nabla \overline{v}|^2 \, dX + c_{n,s,\sigma} \int_{\mathcal{B}_r^+} z^{1-\sigma} \, |\nabla V|^2 \, dX. \tag{A.1}$$

We consider a cutoff function $\varphi \in C^\infty(\mathbb{R})$ such that

$$\varphi = 1 \text{ in } [-1/2, 1/2] \text{ and } \varphi = 0 \text{ outside } (-3/4, 3/4),$$

and, for any $R > 0$, we introduce the following diffeomorphism in $\mathbb{R}^{n+1}_+$, defined for every $X \in \mathbb{R}^{n+1}_+$ as

$$X \mapsto Y := X + \varphi\left(\frac{|X|}{R}\right) e_1. \tag{A.2}$$

Then, we define

$$\overline{u}_R^+(Y) := \overline{u}(X) \quad \text{and} \quad U_R^+(Y) := U(X).$$

We may also define $\overline{u}_R^-$ and U_R^- by simply changing e_1 into $-e_1$ in (A.2).

The argument that we perform is similar to the one of Proposition 6.2 in [8]. The main difference here is that the two terms involved in the functional (A.1) are defined in the extension, and therefore we have to consider domain variations in $\mathbb{R}^{n+1}_+$ both for $\overline{u}$ and for U.

First we prove an estimate for the second variation of the energy $\mathcal{E}_R$.

Lemma A.1. *Let $(\overline{u}, U)$ be a minimizer of $\mathcal{E}_R$. Suppose that $\overline{u}$ and U are homogeneous of degree $s - \frac{\sigma}{2}$ and 0, respectively. Then, there exists a constant $C > 0$ independent of R such that*

$$\mathcal{E}_R(\overline{u}_R^+, U_R^+) + \mathcal{E}_R(\overline{u}_R^-, U_R^-) - 2\mathcal{E}_R(\overline{u}, U) \leqslant C\, R^{n-2-\sigma}.$$

Proof. By direct computations (see formula (11) in [18]), one can prove that

$$\begin{aligned} z^{1-2s}\left(|\nabla \overline{u}_R^+|^2 + |\nabla \overline{u}_R^-|^2\right) dY &= 2\, z^{1-2s}\left(1 + O(1/R^2)\,\chi_{\mathcal{B}_R^+ \setminus \mathcal{B}_{R/2}^+}\right) |\nabla \overline{u}|^2\, dX, \\ z^{1-\sigma}\left(|\nabla U_R^+|^2 + |\nabla U_R^-|^2\right) dY &= 2\, z^{1-\sigma}\left(1 + O(1/R^2)\,\chi_{\mathcal{B}_R^+ \setminus \mathcal{B}_{R/2}^+}\right) |\nabla U|^2\, dX. \end{aligned}$$

Therefore

$$\begin{aligned} &\int_{\mathcal{B}_R^+} z^{1-2s}\left(|\nabla \overline{u}_R^+|^2 + |\nabla \overline{u}_R^-|^2\right) dY - 2\int_{\mathcal{B}_R^+} z^{1-2s}\, |\nabla \overline{u}|^2\, dX \\ &\qquad \leqslant C\, R^{-2} \int_{\mathcal{B}_R^+ \setminus \mathcal{B}_{R/2}^+} z^{1-2s}\, |\nabla \overline{u}|^2\, dX. \end{aligned}$$

Now, since $\overline{u}$ is homogeneous of degree $s - \frac{\sigma}{2}$, we have that $z^{1-2s}\, |\nabla \overline{u}|^2$ is homogeneous of degree $-1-\sigma$, and so

$$\int_{\mathcal{B}_R^+} z^{1-2s}\left(|\nabla \overline{u}_R^+|^2 + |\nabla \overline{u}_R^-|^2\right) dY - 2\int_{\mathcal{B}_R^+} z^{1-2s}\, |\nabla \overline{u}|^2\, dX \leqslant C\, R^{-2} \cdot R^{n-\sigma}. \tag{A.3}$$

Similarly, U is homogeneous of degree 0, and therefore $z^{1-\sigma}\, |\nabla U|^2$ is homogeneous of degree $-1-\sigma$. Hence

$$\begin{aligned} &c_{n,s,\sigma} \int_{\mathcal{B}_R^+} z^{1-\sigma}\left(|\nabla U_R^+|^2 + |\nabla U_R^-|^2\right) dY - 2c_{n,s,\sigma} \int_{\mathcal{B}_R^+} z^{1-\sigma}\, |\nabla U|^2\, dX \\ &\qquad \leqslant C\, R^{-2} \int_{\mathcal{B}_R^+ \setminus \mathcal{B}_{R/2}^+} z^{1-\sigma}\, |\nabla U|^2\, dX \leqslant C R^{-2} \cdot R^{n-\sigma}. \end{aligned}$$

By summing up this and (A.3), we obtain the thesis (recall (A.1)). □

Corollary A.2. *Let $(\overline{u}, U)$ be a minimizer of $\mathcal{E}_R$. Suppose that $\overline{u}$ and U are homogeneous of degree $s - \frac{\sigma}{2}$ and 0, respectively. Then, there exists a constant $C > 0$ independent of R such that*

$$\mathcal{E}_R(\overline{u}_R^+, U_R^+) \leqslant \mathcal{E}_R(\overline{u}, U) + C\, R^{n-2-\sigma}.$$

In particular, if $n = 2$, we have

$$\mathcal{E}_R(\overline{u}_R^+, U_R^+) \leqslant \mathcal{E}_R(\overline{u}, U) + \frac{C}{R^\sigma}. \tag{A.4}$$

Proof. Thanks to Proposition 4.1, the minimality of $(\overline{u}, U)$ gives

$$\mathcal{E}_R(\overline{u}, U) \leqslant \mathcal{E}_R(\overline{u}_R^-, U_R^-).$$

From this and Lemma A.1 we get the desired claim. □

Proof of Theorem 1.4. We follow the line of the proof of Proposition 6.2 in [8]. For the sake of completeness we repeat the proof here.

We suppose that $n = 2$ and we argue by contradiction, assuming that E is not a halfplane. Hence, we can find a point $p \in B_M$, for some $M > 0$, say on the e_2-axis, such that p lies in the interior of E but $p + e_1$ and $p - e_1$ lie in E^c. Therefore, recalling the notation introduced at the beginning of this section, we have that, for $R > 4M$,

$$\begin{aligned} &\overline{u}_R^+(X) = \overline{u}(X - e_1) \quad \text{for any } X \in \mathcal{B}_{2M}^+, \\ &U_R^+(X) = U(X - e_1) \quad \text{for any } X \in \mathcal{B}_{2M}^+, \\ &\overline{u}_R^+(X) = \overline{u}(X) \quad \text{for any } X \in \mathbb{R}_+^3 \setminus \mathcal{B}_R^+, \\ &U_R^+(X) = U(X) \quad \text{for any } X \in \mathbb{R}_+^3 \setminus \mathcal{B}_R^+. \end{aligned} \tag{A.5}$$

Now, we define

$$\begin{aligned} &\overline{v}_R(X) := \min\{\overline{u}(X), \overline{u}_R^+(X)\}, \quad \overline{w}_R(X) := \max\{\overline{u}(X), \overline{u}_R^+(X)\}, \\ &V_R(X) := \min\{U(X), U_R^+(X)\} \quad \text{and} \quad W_R(X) := \max\{U(X), U_R^+(X)\}. \end{aligned}$$

Moreover, we set $P := (p, 0) \in \mathbb{R}^3$. We claim that

$$U_R^+ < W_R = U \text{ in a neighborhood of } P \tag{A.6}$$

$$\text{and} \qquad U < W_R = U_R^+ \text{ in a neighborhood of } P + e_1. \tag{A.7}$$

Indeed, by (A.5)

$$\begin{aligned} &U_R^+(P) = U(P - e_1) = (\chi_E - \chi_{E^c})\,(p - e_1) = -1, \\ &U(P) = (\chi_E - \chi_{E^c})\,(p) = 1, \\ &U_R^+(P + e_1) = U(P) = 1 \\ \text{and} \quad &U(P + e_1) = (\chi_E - \chi_{E^c})\,(p + e_1) = -1. \end{aligned}$$

Then, the claim follows from the continuity of the functions U and U_R^+ at P and $P + e_1$.

By Proposition 4.1, the minimality of $(\overline{u}, U)$ gives

$$\mathcal{E}_R(\overline{u}, U) \leqslant \mathcal{E}_R(\overline{v}_R, V_R).$$

Moreover, we have that

$$\mathcal{E}_R(\overline{v}_R, V_R) + \mathcal{E}_R(\overline{w}_R, W_R) = \mathcal{E}_R(\overline{u}, U) + \mathcal{E}_R(\overline{u}_R^+, U_R^+).$$

Therefore

$$\mathcal{E}_R(\overline{w}_R, W_R) \leqslant \mathcal{E}_R(\overline{u}_R^+, U_R^+). \tag{A.8}$$

Now, we prove that $(\overline{w}_R, W_R)$ is not a minimizer for $\mathcal{E}_{2M}$ with respect to compact perturbations in $\mathcal{B}_{2M}^+ \times \mathcal{B}_{2M}^+$. Indeed, if $(\overline{w}_R, W_R)$ was a minimizer, then W_R would be a minimizer for the σ-perimeter, thanks to Proposition 4.1 (one can fix $\overline{w}_R$ and perturb only W_R, and notice that compact perturbations inside $\mathcal{B}_{2M}^+ \times \mathcal{B}_{2M}^+$ do not touch the trace). On the other hand, from the definition of W_R we have that $U \leqslant W_R$. Moreover, U and W_R satisfy the same equation in $\mathcal{B}_{2M}^+ \times \mathcal{B}_{2M}^+$. Hence, (A.6) and the strong maximum principle imply that $U = W_R$ in $\mathcal{B}_{2M}^+$, which is a contradiction to (A.7). Therefore, there exists $\delta > 0$ and a competitor $(\overline{u}_*, U_*)$ that coincides with $(\overline{w}_R, W_R)$ outside $\mathcal{B}^+_{2M} \times \mathcal{B}^+_{2M}$ (actually we take $\overline{u}_* = \overline{w}_R$) and such that

$$\mathcal{E}_{2M}(\overline{u}_*, U_*) + \delta \leqslant \mathcal{E}_{2M}(\overline{w}_R, W_R).$$

Notice that δ does not depend on R, since $(\overline{w}_R, W_R)$ does not depend on R in $\mathcal{B}^+_{2M} \times \mathcal{B}^+_{2M}$, thanks to (A.5). Since $(\overline{u}_*, U_*)$ agrees with $(\overline{w}_R, W_R)$ outside $\mathcal{B}^+_{2M} \times \mathcal{B}^+_{2M}$, we conclude that

$$\mathcal{E}_R(\overline{u}_*, U_*) + \delta \leqslant \mathcal{E}_R(\overline{w}_R, W_R).$$

From this, (A.4) and (A.8) we obtain

$$\mathcal{E}_R(\overline{u}_*, U_*) + \delta \leqslant \mathcal{E}_R(\overline{w}_R, W_R) \leqslant \mathcal{E}_R(\overline{u}^+_R, U^+_R) \leqslant \mathcal{E}_R(\overline{u}, U) + C\, R^{-\sigma}.$$

Therefore, if R is large enough, we have that

$$\mathcal{E}_R(\overline{u}_*, U_*) < \mathcal{E}_R(\overline{u}, U),$$

and this is a contradiction to the minimality of $(\overline{u}, U)$. □

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