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Hamiltonian framework for short optical pulses

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Abstract

Physics of short optical pulses is an important and active research area in nonlinear optics. In what follows we theoretically consider the most extreme representatives of short pulses that contain only several oscillations of electromagnetic field. Description of such pulses is traditionally based on envelope equations and slowly varying envelope approximation, despite the fact that the envelope is not "slow" and, moreover, there is no clear definition of such a "fast" envelope. This happens due to another paradoxical feature: the standard (envelope) generalized nonlinear Schrödinger equation yields very good correspondence to numerical solutions of full Maxwell equations even for few-cycle pulses, a thing that should not be.

In what follows we address ultrashort optical pulses using Hamiltonian framework for nonlinear waves. As it appears, the standard optical envelope equation is just a reformulation of general Hamiltonian equations. In a sense, no approximations are required, this is why the generalized nonlinear Schrödinger equation is so effective. Moreover, the Hamiltonian framework greatly contributes to our understanding of "fast" envelope, ultrashort solitons, stability and radiation of optical pulses. Even the inclusion of dissipative terms is possible making the Hamiltonian approach an universal theoretical tool also in extreme nonlinear optics.

1 Introduction

1.1 Ultrashort pulses

Remarkable recent progress in pulse generation with femtosecond [4, 17] and sub-femtosecond [33, 57, 20, 65] durations has resulted in rapidly growing interest to ultrashort or so-called few-cycle optical pulses. These pulses are yielded by modern mode-locking techniques, e.g., a readily accessible pulse duration of 6 fs at a near-infrared wavelength of 900 nm corresponds to two optical cycles. On the other hand, spatially localized field "bursts" with extreme amplitudes and short durations can self-organize in a variety of nonlinear systems at unexpectedly high rate [56]. They are referred to as rogue waves. Similar bursts have been observed in nonlinear fibers and interpreted as the optical rogue waves [63, 37]. Here optical setting provides researches with a non-destructive tool to measure statistics of such ultrashort extreme events. Two examples of numerically calculated ultrashort pulses yielded by different pulse propagation models are shown in Fig. 1.

Turning to the applications one should stress that few-cycle pulses yield the possibility to excite and follow fast relaxation processes with a spatial resolution of order of one micron (a single wavelength) and to study light-matter interactions at extreme intensity levels. For instance, currently available temporal and spatial confinement results in peak intensities higher

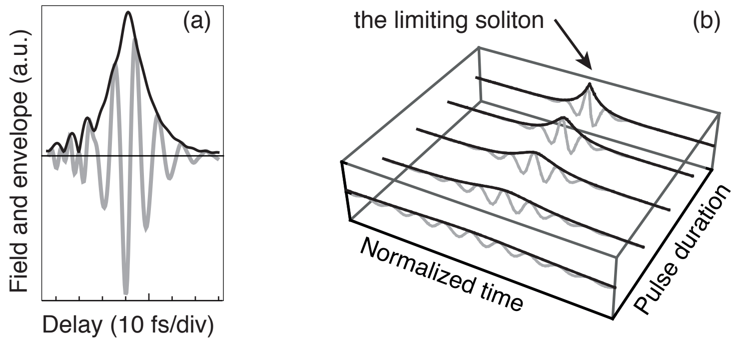

Figure 1: Examples of ultrashort pulses. (a) an exemplary single-cycle pulse resulting from pulse compression in a ZBLAN fiber (numerical solution of the general pulse propagation equation after [18]). (b) A family of solitons in a cubic media with the Drude dispersion law (exact solutions of the simplified propagation equation after [62, 6]). The shortest limiting soliton contains approximately one and half oscillations at half maximum.

than 10^{15} W$/$cm^2 for pulse energies of the order of one microjoule [17]. The corresponding field strength is comparable to that inside atoms. In particular, intense, few-cycle optical pulses are used to trigger and trace chemical reactions, to test high-speed semiconductor devices, and for precision processing of materials. More sophisticated applications include modeling of event horizons of white and black holes [58], recent measurements of Hawking radiation [9, 24], and recent experimental observations of negative-frequency radiation [59, 11].

Theory of the ultrashort optical pulses has been developed in several directions. For small space scales, e.g., propagation lengths of several tenths of a wavelength, one can address the numerical solution of the fundamental Maxwell equations equipped by a suitable medium response model, e.g., Bloch equations [32, 36, 55, 53, 61]. On the other hand, if some approximate but simple medium dispersion law applies, it becomes possible to derive a simplified propagation equation. A typical example is the so-called short pulse equation [38, 43, 60, 10]. Other settings yield the modified Korteweg-de Vries and sine-Gordon equations [8, 48, 49, 46], and more sophisticated models [51, 52, 47]. However, simple models are not available for the real-world dispersion laws and especially in the presence of dissipative effects. An envelope equation is the method of choice for realistic situations, either the fundamental nonlinear Schrödinger equation or its generalizations [2, 15, 3]. These envelope equations have an unexpected behavior: (i) they seem to describe few- and even sub-cycle pulses that should have no envelope [16, 30, 40, 31] and (ii) they show good correspondence to the solutions of more general unidirectional field equations, which are independent of the envelope concept [34, 42, 41, 39].

In what follows we consider pulse propagation using the Hamiltonian point of view for systems with infinitely many degrees of freedom [74, 71]. The promoted approach applies when pulses, otherwise arbitrary, propagate in the transparency window of optical materials, such that dissipation provides small contribution to pulse dynamics. As it appears, the generalized envelope equation is just equivalent to the underlying Maxwell equation [5], and the complex envelope is just a combination of the corresponding canonical coordinate and momentum. Apart from explanation of the paradoxical durability of the envelope equations, the Hamiltonian approach provides a convenient framework for investigation of integrals of motion, solitons, and numerics.

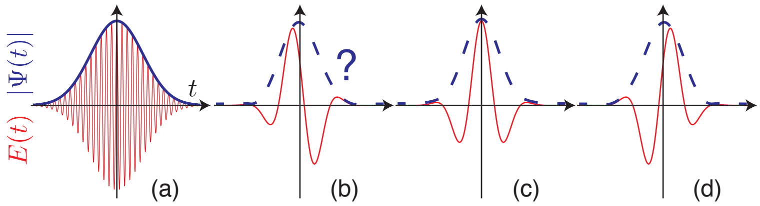

Figure 2: Pulse field (red) and its envelope (blue) are shown for a gaussian pulse shape. (a) For a multi-cycle pulse both $|\Psi|$ is well-defined and the actual value of $\arg\Psi$ in Eq. (1) is unimportant. (b–d) On the contrary, to derive the envelope from the field of a single-cycle pulse, one should revisit the general envelope definition. Moreover, $\arg\Psi$ noticeably affects the peak electric field calculated from Eq. (1).

1.2 Envelope definition

From the mathematical side a correct description of the ultrafast phenomena is a challenge because the involved time-scales may differ in many orders of magnitude making direct numerical solution of the fundamental Maxwell and material equations impractical. A common approach to such multi-scale optical systems is based on the slowly varying envelope approximation (SVEA). For instance, let us consider a scalar electric field $E(t)$ at some given point in space. SVEA presupposes the representation

$$E(t) = \frac{1}{2}\Psi(t)e^{-i\omega_0 t} + \text{c.c.} = |\Psi|\cos(\omega_0 t - \arg\Psi) \tag{1}$$

where ω_0 is referred to as the carrier frequency and the complex-valued function $\Psi(t)$ is the envelope. SVEA assumes that both $|\Psi|$ and $\arg\Psi$ are slow, i.e., $\Psi(t)$ does not change on the time scale $1/\omega_0$.

It is usually sufficient to think about optical pulses in terms of the observed quantities such as the instant power. The latter is proportional to $|\Psi|^2$ and independent on $\arg\Psi$ for a "normal" multi-cycle pulse like one in Fig. 2a. Also the local frequency

$$\omega = \omega_0 - \frac{d}{dt}\arg\Psi \tag{2}$$

takes no notice of a global shift in phase.

The situation is different for a few-cycle pulse: $\arg\Psi$ significantly affects the peak electric field that is actually experienced by an atom (Fig. 2b–d). It is clear that an adequate propagation model for such an ultrashort pulse should the treat field phase with a great care.

Another difficulty appears if we consider a standard derivation [54, 12, 1] of the SVEA propagation equation that includes the following typical step

$$\frac{d^2E}{dt^2} + \omega_0^2 E = \frac{1}{2}\left(\frac{d^2\Psi}{dt^2} - 2i\omega_0\frac{d\Psi}{dt}\right)e^{-i\omega_0 t} + \text{c.c.} \approx -i\omega_0\frac{d\Psi}{dt}e^{-i\omega_0 t} + \text{c.c.}, \tag{3}$$

in which one ignores the second derivative of $\Psi(t)$ because the latter is "slow". This is why all envelope equations are first-order equations, which are simple and suitable for numerical treatment. For an ultrashort pulse, however, both the field and the envelope coexist and evolve on the same scale (Fig. 2b–d). Strictly speaking, the envelope may remain stationary for a single stable soliton, but it is subject to quick changes for, e.g., colliding pulses or higher-order solitons. Therefore approximation (3) becomes invalid and the derivation of the first-order propagation equation should be reconsidered.

Finally, the very definition (1) is ambivalent for short pulses. Namely, if the SVEA applies one can invert (1) and express the complex envelope in terms of the field

$$2E(t)e^{i\omega_0 t} = \Psi(t) + \Psi^*(t)e^{2i\omega_0 t} \quad \Rightarrow \quad \Psi(t) = 2\left\langle E(t)e^{i\omega_0 t}\right\rangle, \tag{4}$$

where $\langle\rangle$ denotes a sliding average over several oscillations of the carrier field. Here, the SVEA indicates that $\langle\Psi(t)\rangle$ remains unaffected and that $\langle\Psi^*(t)e^{2i\omega_0 t}\rangle$ vanishes. Clearly Eq. (4) cannot be applied to a short pulse with the fast envelope and definition of the complex envelope should be reconsidered.

One possible redefinition of the complex envelope explores the fact that the operator in Eq. (3) can be factorized

$$\frac{d^2E}{dt^2} + \omega_0^2 E = \left(\omega_0 - i\frac{d}{dt}\right)\left(\omega_0 + i\frac{d}{dt}\right)E.$$

Using this factorization we define a generalized complex envelope $\overset{\leftrightsquigarrow}{\Psi}(t)$ directly from the equation

$$E + i\omega_0^{-1}\frac{dE}{dt} = \overset{\leftrightsquigarrow}{\Psi} e^{-i\omega_0 t}, \tag{5}$$

such that the standard relations of the theory of linear oscillations

$$E = |\overset{\leftrightsquigarrow}{\Psi}|\cos(\omega_0 t - \arg\overset{\leftrightsquigarrow}{\Psi}), \qquad \frac{dE}{dt} = -\omega_0|\overset{\leftrightsquigarrow}{\Psi}|\sin(\omega_0 t - \arg\overset{\leftrightsquigarrow}{\Psi}),$$

are just forced by a suitable definition of $\overset{\leftrightsquigarrow}{\Psi}$ (see [13]).

Equation (1) still holds and Eq. (3) is replaced with

$$\frac{d^2E}{dt^2} + \omega_0^2 E = \omega_0\left(\omega_0 - i\frac{d}{dt}\right)\overset{\leftrightsquigarrow}{\Psi} e^{-i\omega_0 t} = -i\omega_0\frac{d\overset{\leftrightsquigarrow}{\Psi}}{dt}e^{-i\omega_0 t}$$

The latter relation is exact, one does not have to neglect the second derivative as opposed by Eq. (3). Moreover, the definition (5) is very convenient if combined with the standard sliding average over the fast time. For instance, considering an oscillator with a small driving "force" $f(E, dE/dt)$

$$\frac{d^2E}{dt^2} + \omega_0^2 E = f\left(E, \frac{dE}{dt}\right) \tag{6}$$

one immediately obtains an exact equation

$$i\omega_0\frac{d\overset{\leftrightsquigarrow}{\Psi}}{dt} + e^{i\omega_0 t} f\left(\frac{\overset{\leftrightsquigarrow}{\Psi} e^{-i\omega_0 t} + \text{c.c.}}{2}, \frac{-i\omega_0\overset{\leftrightsquigarrow}{\Psi} e^{-i\omega_0 t} + \text{c.c.}}{2}\right) = 0, \tag{7}$$

where the further averaging is trivial for any polynomial or Taylor expanded $f(E, dE/dt)$.

On the other hand, if the succeeded averaging of Eq. (7) is inappropriate, the new-defined envelope always contains unphysical quickly oscillating terms. Indeed, combining (1) and (5) we obtain

$$\overset{\leftrightsquigarrow}{\Psi} = \Psi + \frac{i}{2\omega_0}\frac{d\Psi}{dt} + \frac{i}{2\omega_0}\frac{d\Psi^*}{dt}e^{2i\omega_0 t}$$

where the second-harmonic term on the right-hand-side appears not because of physical reasons, like quadratic nonlinearities, but simply because of unlucky definition (5).

Another alternative for envelope definition was suggested by Gabor. A real-valued $E(t)$ is replaced by a complex-valued $\mathcal{E}(t)$ following the instruction [28]:

> “Suppress the amplitudes belonging to negative frequencies, and multiply the amplitudes of positive frequencies by two”.

The complex field $\mathcal{E}(t)$ will be referred to as the complex or analytic signal. In what follows, we consider $e^{-i\omega t}$ as a harmonic oscillation with the (angular) frequency ω. A monochromatic wave with the wave vector $\mathbf{k}$ and frequency ω is defined by $e^{i(\mathbf{kr}-\omega t)}$. Consequently, we write the continuous Fourier transform of $E(t)$ as

$$E(\omega) = \int_{-\infty}^{\infty} E(t)e^{i\omega t}dt \qquad \text{and} \qquad E(t) = \int_{-\infty}^{\infty} E(\omega)e^{-i\omega t}\frac{d\omega}{2\pi}, \tag{8}$$

two latter equations become completely symmetric if one switches to the physical frequency $f = \omega/(2\pi)$. In accord to Gabor's rule, the analytic signal is given by the relation

$$\mathcal{E}(t) = \int_{0}^{\infty} E(\omega)e^{-i\omega t}\frac{d\omega}{\pi}, \tag{9}$$

where

$$E(t) = \frac{\mathcal{E}(t) + \mathcal{E}^*(t)}{2},$$

and $\mathcal{E}^*(t)$ accumulates contributions of all negative frequencies in $E(t)$.

Of course, the analytic signal can be defined without any reference to frequencies

$$\mathcal{E}(t) = E(t) + \frac{i}{\pi}\fint_{-\infty}^{\infty}\frac{E(\tau)d\tau}{\tau - t}, \tag{10}$$

where integration in the last term is referred to as Hilbert transform and is performed using the principal value. More details on analytic signal and Hilbert transform can be found in the Appendix. Here we would like to stress the following key points.

1 The analytic signal behaves as expected from the envelope in all simple cases. Taking for instance a carrier cosine oscillation modulated with the frequency ν

$$E(t) = \cos(\nu t)\cos(\omega_0 t) = \frac{1}{4}(e^{i\nu t} + e^{-i\nu t})(e^{i\omega_0 t} + e^{-i\omega_0 t}),$$

with $\omega_0 > \nu > 0$ we derive

$$\mathcal{E}(t) = \frac{1}{2}\left[e^{-i(\omega_0+\nu)t} + e^{-i(\omega_0-\nu)t}\right] = \cos\nu t e^{-i\omega_0 t}.$$

In particular, $|\mathcal{E}(t)|$ is a natural envelope for $E(t)$.

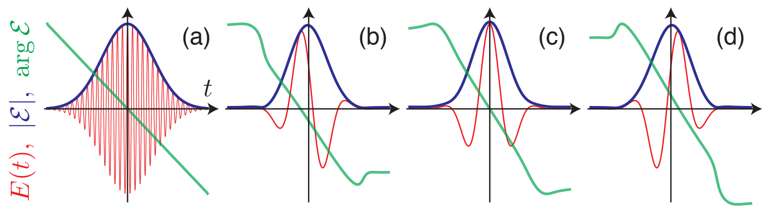

Figure 3: Use of the analytic signal: $|\mathcal{E}|$ (blue lines) and $\arg\mathcal{E}$ (green lines) are shown for pulses from Fig. 2. For a multi-cycle pulse (a) $\arg\mathcal{E}$ is perfectly approximated by $-\omega_0 t$ in favor of Eq. (11). Even for the few-cycle pulses (b–d) the regions with "fast" $\Psi(t)$ are localized outside the pulses.

2 Moreover, there is an intrinsic relation between the definitions (1) and (9). To show this let us assume that spectrum of the envelope $\Psi(t)$ in Eq. (1) completely belongs to the interval $[-\omega_0, \omega_0]$. The assumption is much less restrictive than the standard SVEA with its narrow spectral lines. This relaxed assumption still ensures that $\Psi(t)e^{-i\omega_0 t}$ contains only positive (and $\Psi(t)e^{i\omega_0 t}$ only negative) frequency components. We immediately derive that

$$\mathcal{E}(t) = \Psi(t)e^{-i\omega_0 t} \tag{11}$$

such that the complex envelope $\Psi(t)$ is uniquely defined by the analytic signal $\mathcal{E}(t)$ provided that one has a reasonable definition of the carrier frequency ω_0.

3 The precise definition of ω_0 in Eq. (11) may differ and is not critical. A reasonable choice is to avoid fast oscillations of $\Psi(t)$ as good as possible. $\arg\mathcal{E}$ is then approximated by a straight line, $\arg\mathcal{E} \approx -\omega_0 t$. The approximation is perfect for a many-cycle pulse like one in Fig. 3a, but not for the few-cycle pulses in Fig. 3b–d. However, deviations of $\arg\mathcal{E}$ from $-\omega_0 t$ are localized "outside" the pulses. The splitting of $\mathcal{E}(t)$ into $\Psi(t)$ and $e^{-i\omega_0 t}$ is then still reasonable [16].

4 The values of $|\mathcal{E}|^2$ can be used as weights when approximating $\arg\mathcal{E}$ by $-\omega_0 t$. The resulting expression [19]

$$\omega_0 = \frac{\int_0^\infty \omega|\mathcal{E}(\omega)|^2 d\omega}{\int_0^\infty |\mathcal{E}(\omega)|^2 d\omega}$$

will be assumed in what follows.

5 Analytic signal can be formally considered for a complex argument $t + it'$

$$\mathcal{E}(t+it') = \int_0^\infty E(\omega)e^{-i\omega t}e^{\omega t'}\frac{d\omega}{\pi},$$

where the resulting function is holomorphic for $t' < 0$ and quickly vanishes for $t' \to -\infty$. In other words, a real $E(t)$ is equipped by an imaginary part such that the resulting complex $\mathcal{E}(t)$ is holomorphic in a half-plane of "complex times". The analytic signal can be investigated using all powerful tools provided by complex analysis. For instance, it is subject to Kramers–Kronig relations in a full similarity to the standard response functions [45, 35].

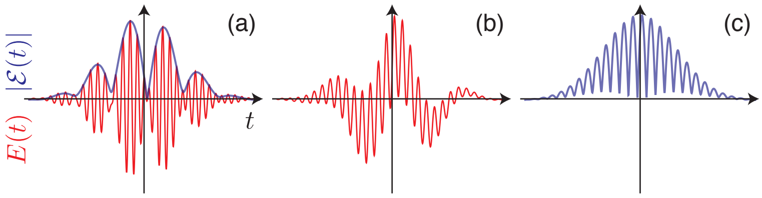

Figure 4: The field $E(t)$ (red) and its analytic signal $|\mathcal{E}(t)|$ (blue) for superposition of two gaussian pulses. (a) the pulses have similar frequencies, the analytic signal is perfectly shaped to the corresponding beat oscillations. (b-c) the pulses have considerably different frequencies. The field (b) and the envelope (c) look very different, the latter is neither smooth nor slow.

The above properties of the analytic signal are so attractive that $\mathcal{E}(t)$ is usually considered as the "correct" envelope [14, 67]. Unfortunately, Gabor's definition has its own difficulties. First, the analytic signal is neither smooth nor slow when the field in question contains considerably different frequency components, like in Fig. 4b–c. This always happens in, e.g., the so-called optical supercontinuum [23]. Of course, correct description of optical fields with wide spectrum is of crucial importance for ultrashort pulses as well. Another difficulty appears due to nonlinearities. Even if $\mathcal{E}(t)$ does contain exclusively positive frequencies, any simple nonlinear expression does not, e.g., the standard cubic term $|\mathcal{E}|^2\mathcal{E}$ always contains a small negative frequency tail. One has either to come around with such an addition to the complex signal or to cut it off. Both possibilities are not quite appropriate. (a) the negative-frequency tail may quickly grow due to nonlinear resonant interactions making definition (9) questionable. What is more astonishing, it can lead to observable physical effects like the negative-frequency radiation [59, 11]. (b) cutting off the negative frequencies (i.e., taking only the positive-frequency-part of $|\mathcal{E}|^2\mathcal{E}$), makes propagation equations unnecessarily complicated for analytical treatment and explanation of the just mentioned effects.

Our approach to the description of ultrashort pulses explores the fact that the pulse field in optics is not just an observable, the field results from the well-known fundamental equations. Moreover, in the region of frequencies that is of interest for, e.g., pulse transmission in optical fibers, these equations are to a good approximation dissipation-free. Ignoring dissipation in a first step, one can (i) find the Poisson bracket for the fundamental equations and (ii) introduce the canonical coordinate $Q(\mathbf{r},t)$ and momentum $P(\mathbf{r},t)$. Both quantities are continuous fields with possibly more than one component. They are governed by the canonical equations in which the standard derivatives are replaced with the functional derivatives

$$\partial_t Q = \frac{\delta\mathscr{H}}{\delta P} \qquad \text{and} \qquad \partial_t P = -\frac{\delta\mathscr{H}}{\delta Q}, \tag{12}$$

the Poisson bracket $\{P,Q\}$ is proportional to a generalized function. Being in possession of Eq. (12) one can treat pulse propagation benefiting from techniques that have been developed for Hamiltonian systems with infinitely many degrees of freedom [74, 73, 71, 69]. The dissipative terms are included to the final equations as small perturbations.

In particular, a natural complex-valued field variable for the envelope-type description is given by a suitable combination of $Q(\mathbf{r},t)$ and $P(\mathbf{r},t)$. The combination is taken in such a way that the Hamiltonian $\mathscr{H}[Q,P]$ takes some simple form, the latter is dictated by the fact that in the frequency domain the optical field is described by a set of coupled weakly nonlinear oscillators. The above definition (5) is an example of such a combination of position and momentum, more generally, the variables in question are classical analogies of the creation and annihilation operators in the second quantization formalism.

Actually there are many competing complex variables that transform the quadratic part of the Hamiltonian to a standard form, a fundamental feature that dictates the next step: to make a sequence of canonical transforms to remove quick oscillations from the complex field. This occurs in a full analogy with the classical Hamiltonian perturbation theory that step by step kills non-resonant nonlinearities [44, 50].

2 Poisson brackets

In the this section we briefly outline some key facts from the Hamiltonian mechanics of discrete [44, 7] and continuous [71, 26, 25] systems. The Poisson bracket is regarded as the cornerstone of the theory.

2.1 Discrete systems

We consider a phase-space manifold with a finite set of local coordinates $\xi = (\xi^1,\, \xi^2,\, \dots\, \xi^N)$. The key mathematical structure on the manifold is given by the so-called Poisson bracket. Namely, having two observables $f(\xi)$ and $g(\xi)$ one can calculate the third one, $\{f, g\}$, which is bilinear with respect to its arguments. Moreover, the following three rules should be respected

$$\{f, g\} + \{g, f\} = 0, \tag{13}$$

$$\{f, gh\} = \{f, g\}h + g\{f, h\}, \tag{14}$$

$$\{f, \{g, h\}\} + \{h, \{f, g\}\} + \{g, \{h, f\}\} = 0, \tag{15}$$

where, the latter two equations are referred to as the Leibniz and Jacobi identities respectively. In addition, the Poisson bracket should vanish if one of its arguments is a constant observable.

To specify such a Poison structure one may first set the brackets between the coordinates

$$\Lambda^{\alpha\beta}(\xi) = \{\xi^\alpha, \xi^\beta\}, \qquad 1 \le \alpha, \beta \le N, \tag{16}$$

and then define the bracket between two arbitrary observables by setting

$$\{f, g\} = \sum_{\alpha,\beta} \Lambda^{\alpha\beta} \partial_\alpha f \partial_\beta g, \tag{17}$$

where ∂_α stays for $\partial/\partial\xi^\alpha$. Equation (13) simply indicates that $\Lambda^{\alpha\beta}$ is antisymmetric. The Leibniz rule (14) is satisfied automatically because $\{f,\ \}$ is a differential operator, its special case

$$\{\xi^\alpha,\ \} = \sum_{\beta} \Lambda^{\alpha\beta} \partial_\beta \tag{18}$$

will play an important role in what follows. The Jacobi identity is first tested for the coordinates

$$\{\xi^\alpha, \{\xi^\beta, \xi^\gamma\}\} + \{\xi^\beta, \{\xi^\gamma, \xi^\alpha\}\} + \{\xi^\gamma, \{\xi^\alpha, \xi^\beta\}\} = 0, \tag{19}$$

where all Greek indices change from 1 to N.

Equation (19) in accord with (16) and (18) is equivalent to the following nonlinear matrix relation

$$\Lambda^{\alpha\mu}\partial_\mu\Lambda^{\beta\gamma} + \Lambda^{\beta\mu}\partial_\mu\Lambda^{\gamma\alpha} + \Lambda^{\gamma\mu}\partial_\mu\Lambda^{\alpha\beta} = 0, \tag{20}$$

the latter identity should be checked directly. The task is simplified for a non-degenerate Λ. One can change to the inverse Λ^{-1} and obtain for its components, $\Lambda^{-1}_{\alpha\beta}(\xi)$, that Eq. (20) is equivalent to a linear relation

$$\partial_\alpha\Lambda^{-1}_{\beta\gamma} + \partial_\beta\Lambda^{-1}_{\gamma\alpha} + \partial_\gamma\Lambda^{-1}_{\alpha\beta} = 0 \quad \Leftrightarrow \quad d\left(\sum_{\alpha,\beta}\Lambda^{-1}_{\alpha\beta}d\xi^\alpha \wedge d\xi^\beta\right) = 0, \tag{21}$$

such that a non-singular Poisson structure yields a symplectic structure and vice versa [7]. We prefer the formulation in which the symplectic structure is the derived one. The Poisson language is promoted because $\Lambda^{\alpha\beta}(\xi)$ may be degenerate, destroying the symplectic form (21) and making a direct test of Eq. (20) more complicated. In any case, after Eq. (20) is validated, the general Eq. (15) is validated as well because all newly appearing terms that does not enter into (20) contain second derivatives and cancel each other due to the antisymmetry condition (13).

In order to define dynamical equations we specify a special observable, the Hamiltonian $\mathscr{H}(\xi)$, and consider the following system of equations

$$\frac{d\xi^\alpha}{dt} = \{\mathscr{H}, \xi^\alpha\} \qquad \text{or} \qquad \frac{d\xi^\alpha}{dt} = -\sum_\beta \Lambda^{\alpha\beta}\partial_\beta\mathscr{H}. \tag{22}$$

In particular, one can check that time evolution of any observable $f(\xi)$ along the solutions of the system (22) is yielded by the equation

$$\frac{df}{dt} = \{\mathscr{H}, f\},$$

where for the sake of brevity our observables have no explicit time dependence. The special relation $\{\mathscr{H}, f\} = 0$ implies that f is an integral of motion for the Hamiltonian system (22).

Now we consider a situation where $\Lambda^{\alpha\beta}$ is degenerate. This, e.g., happens for any odd N. For instance, if all components $\Lambda^{\alpha\beta}$ are constants and we have found a kernel vector

$$n_\beta \quad \text{such that} \quad \sum_\beta \Lambda^{\alpha\beta}n_\beta \equiv 0,$$

then we have found an integral of motion $\mathscr{C} = \sum_\alpha n_\alpha\xi^\alpha$ simply because

$$\frac{d\mathscr{C}}{dt} = \frac{d}{dt}\left(\sum_\alpha n_\alpha\xi^\alpha\right) = -\sum_{\alpha,\beta} n_\alpha\Lambda^{\alpha\beta}\partial_\beta\mathscr{H} = \sum_{\alpha,\beta}\Lambda^{\alpha\beta}n_\beta\partial_\alpha\mathscr{H} = 0. \tag{23}$$

The latter integral conserves for any Hamiltonian $\mathscr{H}$ due to the degeneracy of Λ. The conservation law is of geometric nature and independent on the choice of a specific system: phase trajectories just cannot leave the hyperplane $\sum_\alpha n_\alpha \xi_\alpha = \text{const}$ on which they started.

This so-called Casimir integral $\mathscr{C}$ is recognized by the relation $\{\mathscr{H}, \mathscr{C}\} = 0$ that should apply to any $\mathscr{H}$. In particular, we will see that pulse area is a geometric integral of motion. It is worth noting that the degeneracy of the Poisson bracket may be eliminated. For instance, one can introduce a reduced Hamiltonian system directly on a fixed hyperplane $\sum_\alpha n_\alpha \xi^\alpha = \text{const}$ to get rid of the degenerate degree of freedom. The trick will be used in what follows.

Dealing with Eq. (22), it might be useful to take N independent observables

$$\Xi^\alpha = \Xi^\alpha(\xi^1, \xi^2, \dots \xi^N), \qquad 1 \le \alpha \le N$$

and announce them as the new coordinates. To this end one has to calculate the Poisson bracket between the new coordinates

$$\{\Xi^\alpha, \Xi^\beta\} = \sum_{\mu\nu} \Lambda^{\mu\nu} \frac{\partial \Xi^\alpha}{\partial \xi^\mu} \frac{\partial \Xi^\beta}{\partial \xi^\nu}, \qquad 1 \le \mu, \nu \le N,$$

and to set

$$\{f, g\} = \sum_{\alpha,\beta} \{\Xi^\alpha, \Xi^\beta\} \frac{\partial f}{\partial \Xi^\beta} \frac{\partial g}{\partial \Xi^\beta}.$$

Indeed, the latter equation is compatible with the definition (17) because

$$\sum_{\alpha,\beta} \{\Xi^\alpha, \Xi^\beta\} \frac{\partial f}{\partial \Xi^\alpha} \frac{\partial g}{\partial \Xi^\beta} = \sum_{\alpha,\beta,\mu,\nu} \Lambda^{\mu\nu} \frac{\partial \Xi^\alpha}{\partial \xi^\mu} \frac{\partial \Xi^\beta}{\partial \xi^\nu} \frac{\partial f}{\partial \Xi^\alpha} \frac{\partial g}{\partial \Xi^\beta} = \sum_{\mu,\nu} \{\xi^\mu, \xi^\nu\} \frac{\partial f}{\partial \xi^\mu} \frac{\partial g}{\partial \xi^\nu},$$

or, in other words, $\{\xi^\mu, \xi^\nu\}$ is a second-order tensor and the Poisson bracket (17) is an invariant convolution [21]. Therefore after switching to the new variables we still deal with the same set of Hamiltonian equations

$$\frac{d\Xi^\alpha}{dt} = \{\mathscr{H}, \Xi^\alpha\} \qquad \text{or} \qquad \frac{d\Xi^\alpha}{dt} = -\sum_\beta \{\Xi^\alpha, \Xi^\beta\} \frac{\partial \mathscr{H}}{\partial \Xi^\beta},$$

where both $\mathscr{H}$ and $\{\Xi^\alpha, \Xi^\beta\}$ are now expressed using the new coordinates.

One natural application of the latter equations is to simplify the Poisson bracket by choosing the most suitable coordinates. Another possibility is to keep the bracket by setting $\{\Xi^\alpha, \Xi^\beta\} = \{\xi^\alpha, \xi^\beta\}$ and to simplify $\mathscr{H}$ instead. One can even first simplify the bracket and then keep the bracket and simplify the Hamiltonian, this is the canonical perturbation theory [50].

2.2 Complex variables

The most simple nontrivial example of a Hamiltonian system is given by $N = 2$ with $\xi^1 = q$ and $\xi^2 = p$, where by construction

$$\Lambda = \begin{pmatrix} \{q, q\} & \{q, p\} \\ \{p, q\} & \{p, p\} \end{pmatrix} = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \quad \Rightarrow \quad \{f, g\} = \frac{\partial f}{\partial p} \frac{\partial g}{\partial q} - \frac{\partial f}{\partial q} \frac{\partial g}{\partial p}.$$

Equation (22) indicates that each Hamiltonian $\mathscr{H}(q,p)$ generates a system

$$\frac{dq}{dt} = \frac{\partial \mathscr{H}}{\partial p} \qquad \text{and} \qquad \frac{dp}{dt} = -\frac{\partial \mathscr{H}}{\partial q}, \tag{24}$$

which is a standard set of two Hamiltonian equations. They can be transformed into a single complex equation using the following variable [64], c.f. Eq. (5)

$$\mathfrak{z} = \frac{q+ip}{\sqrt{2}}, \tag{25}$$

where

$$\{\mathfrak{z},\mathfrak{z}\} = \{\mathfrak{z}^*,\mathfrak{z}^*\} = 0 \quad \text{and} \quad \{\mathfrak{z},\mathfrak{z}^*\} = i. \tag{26}$$

The derivatives with respect to $\mathfrak{z}$ (Wirtinger derivatives [68]) are defined from a natural requirement

$$\frac{\partial \mathscr{H}}{\partial q}dq + \frac{\partial \mathscr{H}}{\partial p}dp \equiv \frac{\partial \mathscr{H}}{\partial \mathfrak{z}}d\mathfrak{z} + \frac{\partial \mathscr{H}}{\partial \mathfrak{z}^*}d\mathfrak{z}^*,$$

such that

$$\frac{\partial \mathscr{H}}{\partial \mathfrak{z}} = \frac{1}{\sqrt{2}}\left(\frac{\partial \mathscr{H}}{\partial q} - i\frac{\partial \mathscr{H}}{\partial p}\right) \quad \text{and} \quad \frac{\partial \mathscr{H}}{\partial \mathfrak{z}^*} = \frac{1}{\sqrt{2}}\left(\frac{\partial \mathscr{H}}{\partial q} + i\frac{\partial \mathscr{H}}{\partial p}\right).$$

One can derive a new expression for the Poison bracket

$$\{f,g\} = \{\mathfrak{z},\mathfrak{z}^*\}\frac{\partial f}{\partial \mathfrak{z}}\frac{\partial g}{\partial \mathfrak{z}^*} + \{\mathfrak{z}^*,\mathfrak{z}\}\frac{\partial f}{\partial \mathfrak{z}^*}\frac{\partial g}{\partial \mathfrak{z}} = i\left(\frac{\partial f}{\partial \mathfrak{z}}\frac{\partial g}{\partial \mathfrak{z}^*} - \frac{\partial f}{\partial \mathfrak{z}^*}\frac{\partial g}{\partial \mathfrak{z}}\right), \tag{27}$$

and the following complex equivalent of the Hamiltonian equations (24)

$$i\frac{d\mathfrak{z}}{dt} = \frac{\partial \mathscr{H}}{\partial \mathfrak{z}^*}. \tag{28}$$

A continuous analogue of (27) and (28) will play an important role for optical systems in what follows.

We are in a good position to stress that the same equations (27) and (28) can be derived for many different definitions of the complex variable. The only thing that matters is the bracket. For instance, one can rescale Eq. (25) and set

$$a = \frac{1}{\sqrt{2}}\left(Cq + \frac{ip}{C^*}\right) \quad \text{with} \quad C \in \mathbb{C}, \tag{29}$$

check that

$$\{a,a\} = \{a^*,a^*\} = 0, \qquad \{a,a^*\} = i,$$

and immediately conclude that

$$\{f,g\} = i\left(\frac{\partial f}{\partial a}\frac{\partial g}{\partial a^*} - \frac{\partial f}{\partial a^*}\frac{\partial g}{\partial a}\right), \qquad i\frac{da}{dt} = \frac{\partial \mathscr{H}}{\partial a^*},$$

where

$$q = \frac{1}{\sqrt{2}}\left(\frac{a}{C} + \frac{a^*}{C^*}\right), \qquad \frac{\partial \mathscr{H}}{\partial a^*} = \frac{1}{\sqrt{2}}\left(\frac{1}{C^*}\frac{\partial \mathscr{H}}{\partial q} + iC\frac{\partial \mathscr{H}}{\partial p}\right).$$
$$p = \frac{i}{\sqrt{2}}(Ca^* - C^*a),$$

The freedom in the definition of the complex variable can be used to simplify $\mathscr{H}(a, a^*)$. In our case, the most important Hamiltonian corresponds to a nonlinear oscillator

$$\mathscr{H} = \frac{p^2}{2m} + \frac{kq^2}{2} + \mathscr{H}_{\text{nonl}}(q, p),$$

where one can set $C = \sqrt[4]{km}$ with

$$a = \frac{1}{\sqrt{2}}\left(q\sqrt[4]{km} + \frac{ip}{\sqrt[4]{km}}\right),$$

and obtain that

$$\mathscr{H} = \omega a a^* + \mathscr{H}_{\text{nonl}}(a, a^*), \qquad i\frac{da}{dt} = \omega a + \frac{\partial}{\partial a^*}\mathscr{H}_{\text{nonl}}(a, a^*),$$

where $\omega = \sqrt{k/m}$ is the linear frequency.

These results are readily extended to any even N. A canonical Poisson bracket requires special coordinates consisting of $N/2$ generalized positions q_i and $N/2$ generalized momentums p_i, where by construction

$$\begin{aligned}\{q_i, q_j\} &= 0,\\ \{p_i, p_j\} &= 0,\end{aligned} \qquad \{p_i, q_j\} = \begin{cases} 1 & \text{for } i = j, \\ 0 & \text{for } i \neq j, \end{cases} \qquad 1 \leq i, j \leq N/2. \tag{30}$$

In other words,

$$\Lambda = \begin{pmatrix} 0 & -I \\ I & 0 \end{pmatrix}, \qquad \{f, g\} = \sum_{i=1}^{N/2}\left(\frac{\partial f}{\partial p_i}\frac{\partial g}{\partial q_i} - \frac{\partial f}{\partial q_i}\frac{\partial g}{\partial p_j}\right), \tag{31}$$

where I is a unit $N/2 \times N/2$ matrix. Any non-degenerate $\Lambda^{\alpha\beta}$ can be locally reduced to the canonical form (31) such that one obtains a textbook pair of the Hamiltonian equations [7]

$$\frac{dq_i}{dt} = \frac{\partial \mathscr{H}}{\partial p_i}, \qquad \frac{dp_i}{dt} = -\frac{\partial \mathscr{H}}{\partial q_i}, \qquad 1 \leq i \leq N/2. \tag{32}$$

For a set of $N/2$ coupled nonlinear oscillators

$$\mathscr{H} = \sum_{s=1}^{N/2}\left(\frac{p_s^2}{2m_s} + \frac{k_s q_s^2}{2}\right) + \mathscr{H}_{\text{nonl}},$$

one can introduce

$$a_s = \frac{1}{\sqrt{2}}\left(q_s\sqrt[4]{k_s m_s} + \frac{ip_s}{\sqrt[4]{k_s m_s}}\right), \qquad \omega_s = \sqrt{\frac{k_s}{m_s}},$$

where the new representation of the canonical Poisson structure is defined by the only non-trivial bracket between the complex coordinates

$$\{a_s, a^*_{s'}\} = \begin{cases} i & \text{for } s = s', \\ 0 & \text{for } s \neq s', \end{cases} \quad \Rightarrow \quad \{f, g\} = i \sum_{s=1}^{N/2} \left(\frac{\partial f}{\partial a_s} \frac{\partial g}{\partial a^*_s} - \frac{\partial f}{\partial a^*_s} \frac{\partial g}{\partial a_s} \right).$$

A complex analogue of Eq. (32) reads

$$\mathscr{H} = \sum_{s=1}^{N/2} \omega_s a_s a^*_s + \mathscr{H}_{\text{nonl}}, \qquad i\frac{da_s}{dt} = \omega_s a_s + \frac{\partial \mathscr{H}_{\text{nonl}}}{\partial a^*_s}. \tag{33}$$

The latter equations give a starting point of the perturbation theory that step by step removes the non-resonant terms from $\mathscr{H}_{\text{nonl}}$ by consequent change to new canonical variables.

2.3 Single continuous field

Now we turn to systems with infinitely many degrees of freedom [71, 26, 25]. First, we consider one scalar filed u with an infinite number of "coordinates" $u(x)$ for all $x \in \mathbb{R}$. We assume that $u(x) \to 0$ for $x \to \pm\infty$. The observables are now given by functionals, such as pulse area

$$A[u] = \int_{-\infty}^{\infty} u(x) dx. \tag{34}$$

Derivatives are replaced by functional derivatives. In analogy with Eq. (17), which we write in the form

$$\{f, g\} = \sum_{\alpha,\beta} \{\xi^\alpha, \xi^\beta\} \partial_\alpha f \partial_\beta g,$$

the Poisson bracket of two observables $F[u]$ and $G[u]$ is defined as

$$\{F, G\} = \iint_{-\infty}^{\infty} \frac{\delta F}{\delta u(x)} \frac{\delta G}{\delta u(y)} \{u(x), u(y)\} dx dy, \tag{35}$$

where integration replaces summation over all α and β. Moreover, in the next sections we will deal with multi-component fields, then

$$\{F, G\} = \iint_{-\infty}^{\infty} \sum_{i,j} \frac{\delta F}{\delta u_i(x)} \frac{\delta G}{\delta u_j(y)} \{u_i(x), u_j(y)\} dx dy, \tag{36}$$

where summation over all components is assumed.

Here and in what follows we deliberately avoid use of $\delta F/\delta u$ or $\delta F/\delta u(x,t)$ and prefer to use $\delta F/\delta u(x)$ even if the field in question evolves with time. This happens because $u(x)$ is considered as a direct generalization of $\xi = (\xi^1, \xi^1, \cdots \xi^N)$. The components of such a "super-point" in the phase space with the infinite number of dimensions are numerated by x. Therefore the notation $\delta/\delta u(x)$ is a direct analog of $\partial/\partial \xi^i$.

As a specific example of $\{u(x), u(y)\}$ we take the so-called Gardner-Zakharov-Faddeev bracket (GZF bracket, see [71, 25] and references cited therein)

$$\{u(x), u(y)\} = \delta'(x-y). \tag{37}$$

Replacing $\delta'(x-y)$ with $-\frac{\partial}{\partial y}\delta(x-y)$, integrating (35) by parts, and finally integrating over dx, one calculates that

$$\{F, G\} = \int_{-\infty}^{\infty} \frac{\delta F}{\delta u(y)} \frac{\partial}{\partial y} \frac{\delta G}{\delta u(y)} dy. \tag{38}$$

For instance, considering the pulse area (34) we obtain

$$\frac{\delta A}{\delta u(x)} = 1 \quad \Rightarrow \quad \{F, A\} = 0,$$

for any observable $F[u]$. The pulse area is then a geometric integral of motion for any Hamiltonian system generated by the bracket (37).

To write down an equation of motion analogous to Eq. (22), one should be able to calculate the bracket $\{F, u(x)\}$. To this end we consider $u(x)$ as a functional that takes any observable $u(y)$ and returns its value at some fixed point x

$$u(x) = \int_{-\infty}^{\infty} u(y)\delta(x-y)dy \quad \Rightarrow \quad \frac{\delta u(x)}{\delta u(y)} = \delta(x-y).$$

Therefore Eq. (38) indicates that

$$\{F, u(x)\} = \int_{-\infty}^{\infty} \left[\frac{\delta F}{\delta u(y)} \frac{\partial}{\partial y}\delta(x-y)\right] dy = -\frac{\partial}{\partial x}\frac{\delta F}{\delta u(x)}.$$

System evolution in the continuous setting means that all "coordinates" $u(x)$ go time-dependent. Therefore one looks for $u = u(x,t)$. The Hamiltonian equation of motion for a given $\mathscr{H}[u]$ reads

$$\partial_t u = \{\mathscr{H}, u(x)\} \quad \Rightarrow \quad \partial_t u = -\frac{\partial}{\partial x}\frac{\delta \mathscr{H}}{\delta u(x)}. \tag{39}$$

Equation (39) looks provoking, because we have a Hamiltonian system with one variable. The corresponding canonic pair [70] is determined on an invariant subspace of pulses $u(x)$ with the area $A[u] = 0$. For each $k > 0$ we define

$$q(k) = \frac{1}{\sqrt{\pi}k}\int_{-\infty}^{\infty} u(x)\cos(kx)dx, \quad p(k) = -\frac{1}{\sqrt{\pi}}\int_{-\infty}^{\infty} u(x)\sin(kx)dx, \tag{40}$$

and calculate that

$$\{p(k), q(k')\} = \int_{-\infty}^{\infty} \frac{\delta p(k)}{\delta u(y)} \frac{\partial}{\partial y} \frac{\delta q(k')}{\delta u(y)} dy = \frac{1}{\pi}\int_{-\infty}^{\infty} \sin(ky)\sin(k'y)dy.$$

Now we take advantage of the identity

$$\int_{-\infty}^{\infty} \cos(\kappa y)dy = 2\pi\delta(\kappa),$$

and derive that

$$\{p(k), q(k')\} = \delta(k - k') - \delta(k + k') = \delta(k - k'),$$

because $k + k' \neq 0$ by construction. In the same way one can obtain the remaining brackets

$$\{q(k), q(k')\} = \{p(k), p(k')\} = 0, \qquad \{q(k), p(k')\} = -\delta(k - k'),$$

in a full analogue with the discrete Eq. (30).

Following Eq. (29) the complex field can be introduced as

$$a(k) = \frac{1}{\sqrt{2}} \left[q(k)\sqrt{k} + \frac{ip(k)}{\sqrt{k}} \right] = \frac{1}{\sqrt{2\pi k}} \int_{-\infty}^{\infty} u(x) e^{-ikx} dx = \frac{u(k)}{\sqrt{2\pi k}}, \tag{41}$$

where we recall that k is positive. The already known brackets between the canonical variables $q(k)$ and $p(k)$ indicate that

$$\{a(k), a(k')\} = 0 \quad \text{and} \quad \{a(k), a^*(k')\} = i\delta(k - k'), \tag{42}$$

in a full similarity with Eq. (26). The general Poisson bracket between two real-valued observables in the complex formulation

$$\{F, G\} = \iint_{-\infty}^{\infty} \left[\frac{\delta F}{\delta a(k)} \frac{\delta G}{\delta a^*(k')} \{a(k), a^*(k')\} + \text{c.c.} \right] dk dk',$$

takes the form

$$\{F, G\} = i \int_{-\infty}^{\infty} \left[\frac{\delta F}{\delta a(k)} \frac{\delta G}{\delta a^*(k)} - \frac{\delta F}{\delta a^*(k)} \frac{\delta G}{\delta a(k)} \right] dk, \tag{43}$$

which is a continuous analogue of Eq. (27).

The complex representation will be introduced for several continuous systems in what follows. Noteworthy, the fundamental equations (42) and (43) apply to all of them. In all such systems the Hamiltonian equations

$$\partial_t a = \{\mathscr{H}, a(k)\} = -i \frac{\delta \mathscr{H}}{\delta a^*(k)}, \qquad \partial_t a^* = \{\mathscr{H}, a^*(k)\} = i \frac{\delta \mathscr{H}}{\delta a(k)},$$

are conjugated to each other such that one can deal with a single complex equation for $a(k, t)$

$$i\partial_t a = \frac{\delta \mathscr{H}}{\delta a^*(k)}, \tag{44}$$

in a full analogy with Eq. (28).

For example, consider the following Hamiltonian

$$\mathscr{H}[u] = \int_{-\infty}^{\infty} \left[\frac{(\partial_x u)^2}{2} + u^3 \right] dx = \mathscr{H}_2[u] + \mathscr{H}_3[u].$$

For the corresponding dynamic system we apply Eq. (39) and obtain a nonlinear wave equation

$$\partial_t u = \partial_x^3 u - 6u\partial_x u,$$

which is equivalent to the famous Korteweg-de Vries equation [29]. On the other hand, according to the Parseval theorem and definition (41)

$$\mathscr{H}_2 = \int_{-\infty}^{\infty} \frac{k^2|u(k)|^2}{2}\frac{dk}{2\pi} = \int_0^{\infty} k^3|a(k)|^2 dk,$$

such that for $a(k,t)$ we have

$$i\partial_t a = \omega(k)a + \frac{\delta \mathscr{H}_3}{\delta a^*(k)}, \qquad \omega(k) = k^3, \qquad k > 0,$$

in a full analogy with Eq. (33). Here all wave-vectors are positive because of unidirectionality.

2.4 Canonical bracket for two fields

In this section we deal with a more traditional Hamiltonian that depends on two scalar fields, $\mathscr{H}[u,v]$. A multicomponent Eq. (36) should be used instead of the single-component Eq. (35), and one should specify all possible brackets between the fields. A direct generalization of the canonical discrete Poisson bracket (30) with $i \to x$, $q_i \to u(x)$, and $p_i \to v(x)$ is given by

$$\{u(x), u(y)\} = \{v(x), v(y)\} = 0, \qquad \{u(x), v(y)\} = -\delta(x-y).$$

The derived Poisson bracket between two observables $F[u,v]$ and $G[u,v]$ is

$$\{F,G\} = \iint_{-\infty}^{\infty} \left[\frac{\delta F}{\delta u(x)}\frac{\delta G}{\delta v(y)}\{u(x), v(y)\} + \frac{\delta F}{\delta v(x)}\frac{\delta G}{\delta u(y)}\{v(x), u(y)\}\right] dxdy,$$

and is reduced to the form

$$\{F,G\} = \int_{-\infty}^{\infty} \left[\frac{\delta F}{\delta v(y)}\frac{\delta G}{\delta u(y)} - \frac{\delta F}{\delta u(y)}\frac{\delta G}{\delta v(y)}\right] dy, \tag{45}$$

which is a continuous analogue of Eq. (31). The corresponding dynamic equations read

$$\partial_t u = \{\mathscr{H}, u(x)\} = \int_{-\infty}^{\infty} \frac{\delta \mathscr{H}}{\delta v(y)}\frac{\delta u(x)}{\delta u(y)} dy = \frac{\delta \mathscr{H}}{\delta v(x)},$$

and

$$\partial_t v = \{\mathscr{H}, v(x)\} = -\int_{-\infty}^{\infty} \frac{\delta \mathscr{H}}{\delta u(y)}\frac{\delta v(x)}{\delta v(y)} dy = -\frac{\delta \mathscr{H}}{\delta u(x)},$$

in a full analogy with Eq. (12).

Exactly like in the discrete case (25), one can replace u and v with a single complex field $\psi = (u+iv)/\sqrt{2}$. The coefficients in the definition of ψ can be chosen differently, the only important thing is the bracket that should correspond to Eq. (42)

$$\{\psi(x), \psi(y)\} = 0, \qquad \{\psi(x), \psi^*(y)\} = i\delta(x-y).$$

The Poisson bracket between two real-valued observables that depend on $\psi(x)$ and $\psi^*(x)$ is derived like Eq. (43)

$$\{F, G\} = i \int_{-\infty}^{\infty} \left[\frac{\delta F}{\delta \psi(y)} \frac{\delta G}{\delta \psi^*(y)} - \frac{\delta F}{\delta \psi^*(y)} \frac{\delta G}{\delta \psi(y)} \right] dy,$$

and yields the Hamiltonian equation

$$i \partial_t \psi = i\{\mathscr{H}, \psi(x)\} = \frac{\delta \mathscr{H}}{\delta \psi^*(x)},$$

in full analogy with Eq. (44).

Both the Poisson bracket in the complex representation and the resulting complex Hamiltonian equation are structurally identical to those from the previous section. The only difference is that they now apply to the observables in the physical (x, t) space, as opposed by the observables in Fourier $(k > 0, t)$ space from the previous section. A key example in the physical space is given by

$$\mathscr{H} = \frac{1}{2} \int_{-\infty}^{\infty} \left(|\partial_x \psi|^2 - |\psi|^4 \right) dx$$

which leads to the (focusing) nonlinear Schrödinger equation

$$i \partial_t \psi + \frac{1}{2} \partial_x^2 \psi + |\psi|^2 \psi = 0. \tag{46}$$

Equation (46) is commonly abbreviated as NLSE and is of fundamental importance for nonlinear optics as well as the generalized one (GNLSE) which in the simplest case is generated by the Hamiltonian

$$\mathscr{H} = -\frac{1}{2} \int_{-\infty}^{\infty} \left[\psi^* (\hat{L} \psi) + \psi (\hat{L} \psi)^* + \gamma |\psi|^4 \right] dx.$$

Here $\gamma =$ const and operator $\hat{L}$ is a polynomial with respect to $i\partial_x$. The resulting GNLSE reads

$$i \partial_t \psi + \hat{L} \psi + \gamma |\psi|^2 \psi = 0. \tag{47}$$

In more complicates situations γ is an operator as well, and moreover, Eq. (47) is equipped with non-Hamiltonian terms describing linear and nonlinear damping. Note, that both Eq. (46) and Eq. (47) are complex envelope equations. Poisson brackets that are relevant for the non-envelope equations will be discussed in the next section.

2.5 GZF bracket for two fields

A more sophisticated example of the Poisson bracket with two fields, the one that is relevant for the field-level description of optical pulses, is given by the Gardner-Zakharov-Faddeev construction. The non-vanishing brackets are set to be

$$\{u(x), v(y)\} = \{v(x), u(y)\} = \delta'(x - y), \tag{48}$$

where the bracket (48) is antisymmetric because $\delta'(x)$ is an odd function. The resulting bracket between two observables $F[u,v]$ and $G[u,v]$ is calculated from Eq. (36) by partial integration exactly as in Eq. (38)

$$\{F,G\} = \int_{-\infty}^{\infty} \left[\frac{\delta F}{\delta u(y)} \frac{\partial}{\partial y} \frac{\delta G}{\delta v(y)} + \frac{\delta F}{\delta v(y)} \frac{\partial}{\partial y} \frac{\delta G}{\delta u(y)} \right] dy. \tag{49}$$

To derive the corresponding equations of motion we calculate

$$\{F, u(x)\} = \int_{-\infty}^{\infty} \left[\frac{\delta F}{\delta v(y)} \frac{\partial}{\partial y} \delta(x-y) \right] dy = -\frac{\partial}{\partial x} \frac{\delta F}{\delta v(x)},$$

and

$$\{F, v(x)\} = \int_{-\infty}^{\infty} \left[\frac{\delta F}{\delta u(y)} \frac{\partial}{\partial y} \delta(x-y) \right] dy = -\frac{\partial}{\partial x} \frac{\delta F}{\delta u(x)}.$$

Therefore each Hamiltonian $\mathscr{H}[u,v]$ generates the following two equations

$$\partial_t u = -\frac{\partial}{\partial x} \frac{\delta \mathscr{H}}{\delta v(x)} \qquad \text{and} \qquad \partial_t v = -\frac{\partial}{\partial x} \frac{\delta \mathscr{H}}{\delta u(x)}. \tag{50}$$

For instance, choosing

$$\mathscr{H} = \int_{-\infty}^{\infty} \left(\frac{u^2}{2} + \frac{v^2}{2} \right) dx, \tag{51}$$

we derive a common linear wave equation

$$\begin{aligned} \partial_t u &= -\partial_x v, \\ \partial_t v &= -\partial_x u, \end{aligned} \quad \Rightarrow \quad \partial_t^2 u - \partial_x^2 u = 0,$$

with the dispersion law

$$\omega(k) = |k|. \tag{52}$$

More complicated Hamiltonians of the type (51) can account for dispersion and nonlinearity in the wave equation. This situation is important for pulse propagation in fibers, so it's worth taking the time for complex formulation of the system (50). A possible choice of the canonical variables reads

$$\begin{aligned} q(k) &= \frac{1}{\sqrt{2\pi|k|}} \int_{-\infty}^{\infty} \left[\Lambda(k) u(x) + \frac{\sigma(k)}{\Lambda(k)} v(x) \right] \cos(kx) dx, \\ p(k) &= -\frac{1}{\sqrt{2\pi|k|}} \int_{-\infty}^{\infty} \left[\Lambda(k) u(x) + \frac{\sigma(k)}{\Lambda(k)} v(x) \right] \sin(kx) dx, \end{aligned} \tag{53}$$

where we use the sign function

$$\sigma(k) = \frac{k}{|k|},$$

and the scaling factor $\Lambda(k)$. The latter is a real-valued and even but otherwise arbitrary function

$$\Lambda(k) = \Lambda(-k) \in \mathbb{R}. \tag{54}$$

Note, that the wave vector k in Eq. (53) takes all real values as opposed by only positive ones in Eq. (40).

Let us check the Poisson bracket between the new defined $q(k)$ and $p(k)$. Using Eq. (49) we calculate that

$$\{p(k), q(k')\} = \int_{-\infty}^{\infty} \left[\frac{\delta p(k)}{\delta u(y)} \frac{\partial}{\partial y} \frac{\delta q(k')}{\delta v(y)} + \frac{\delta p(k)}{\delta v(y)} \frac{\partial}{\partial y} \frac{\delta q(k')}{\delta u(y)} \right] dy =$$
$$\frac{\sqrt{|k'/k|}}{2\pi} \left[\frac{\Lambda(k)}{\Lambda(k')} + \sigma(k)\sigma(k') \frac{\Lambda(k')}{\Lambda(k)} \right] \int_{-\infty}^{\infty} \sin(ky)\sin(k'y) dy =$$
$$\frac{1+\sigma(k)\sigma(k')}{2} [\delta(k-k') - \delta(k+k')] = \delta(k-k').$$

Furthermore, it is easy to obtain that both $\{q(k), q(k')\}$ and $\{p(k), p(k')\}$ vanish, such that $q(k)$ and $p(k)$ are proper canonical variables.

Following Eq. (29) the complex field can be introduced as

$$a(k) = \frac{q(k) + ip(k)}{\sqrt{2}} = \frac{1}{2\sqrt{\pi |k|}} \int_{-\infty}^{\infty} \left[\Lambda(k) u(x) + \frac{\sigma(k)}{\Lambda(k)} v(x) \right] e^{-ikx} dx,$$

where the fundamental relations (42) and (43) are satisfied automatically. The original real-valued fields can be reconstructed from the complex-valued one using the relations

$$\Lambda(k) \frac{u(k)}{\sqrt{\pi |k|}} = a(k) + a^*(-k), \qquad \frac{\sigma(k)}{\Lambda(k)} \frac{v(k)}{\sqrt{\pi |k|}} = a(k) - a^*(-k),$$

where $u(k)$ and $v(k)$ are the spectral components

$$u(k) = \int_{-\infty}^{\infty} u(x) e^{-ikx} dx, \qquad v(k) = \int_{-\infty}^{\infty} v(x) e^{-ikx} dx.$$

For instance, using Parseval's theorem for the Hamiltonian (51) and setting $\Lambda = 1$, one directly obtains that

$$\mathscr{H} = \int_{-\infty}^{\infty} \left[\frac{|u(k)|^2}{2} + \frac{|v(k)|^2}{2} \right] \frac{dk}{2\pi} = \int_{-\infty}^{\infty} \omega(k) |a(k)|^2 dk,$$

in accord with the dispersion law (52). In more complex situations both $|u(k)|^2$ and $|v(k)|^2$ may come with some weights, the latter will be equal to each other after a proper choice of $\Lambda(k)$.

3 Pulses in optical fibers

The electric field $\mathbf{E}(\mathbf{r}, t)$ and magnetic induction $\mathbf{B}(\mathbf{r}, t)$ created by any optical pulse are, of course, governed by the fundamental microscopic Maxwell equations [14]

$$\begin{aligned} \epsilon_0 \nabla \mathbf{E} &= \rho, & \nabla \times \mathbf{E} &= -\partial_t \mathbf{B}, \\ \nabla \mathbf{B} &= 0, & \mu_0^{-1} \nabla \times \mathbf{B} &= \mathbf{j} + \epsilon_0 \partial_t \mathbf{E}. \end{aligned} \tag{55}$$

Here the so-called vacuum permittivity ϵ_0 and vacuum permeability μ_0 are physical constants, whereas $\rho(\mathbf{r},t)$ and $\mathbf{j}(\mathbf{r},t)$ are exact microscopic charge and current densities. On a macroscopic level, however, one can avoid a tremendous task of solving of additional equations for each elementary charge composing $\rho(\mathbf{r},t)$ and $\mathbf{j}(\mathbf{r},t)$. Instead, one deals with an appropriate simplified material model. For a set of non-destructive electromagnetic waves propagating in a dielectric it is sufficient to introduce a single macroscopic polarization vector $\mathbf{P}(\mathbf{r},t)$, such that the inhomogeneous pair of Maxwell equations (55) changes to the form

$$\nabla(\epsilon_0\mathbf{E}+\mathbf{P})=0, \qquad \frac{1}{\mu_0}\nabla\times\mathbf{B}=\partial_t(\epsilon_0\mathbf{E}+\mathbf{P}),$$

where a macroscopic-level relation between $\mathbf{P}(\mathbf{r},t)$ and $\mathbf{E}(\mathbf{r},t)$ is assumed. This so-called material relation $\mathbf{P}(\mathbf{E})$ may be complicated, even given in terms of additional equations, the one that is suitable for the optical fibers will be specified in the next section. Here, we note that the use of polarization is, of course, automatically compatible to the charge conservation equation

$$\partial_t\rho+\nabla\mathbf{j}=0, \tag{56}$$

that is why four quantities $\rho(\mathbf{r},t)$ and $\mathbf{j}(\mathbf{r},t)$ are safely replaced by three components of $\mathbf{P}(\mathbf{r},t)$.

3.1 Problem posing

In what follows we will consider a linearly polarized wave in which all involved fields have only one nontrivial component

$$\mathbf{E}=(E,0,0), \qquad \mathbf{B}=(0,B,0), \qquad \mathbf{P}=(P,0,0). \tag{57}$$

Moreover, all fields depend on time and a single spatial variable, the propagation coordinate z. In the first place such approximation applies to bulk propagation of plane electromagnetic pulses in dispersive media. On the other hand, the approximation applies to a so-called single-mode polarization-preserving fiber [1, 66]. Namely, a single-mode fiber means that the radial structure of the pulse is approximately the same for all frequencies in question, such that the radial degrees of freedom can be integrated out leaving us with quantities depending on (z,t). In addition, the fiber can be purposely made slightly asymmetric introducing a small difference between two possible polarizations of a plane electromagnetic wave. Such a fiber preserves polarization of the input pulse, that is presupposed by Eq. (57).

For a plane wave described by Eq. (57), the full system (55) is reduced to only two scalar equations

$$\partial_z E=-\partial_t B \qquad \text{and} \qquad -\frac{1}{\mu_0}\partial_z B=\partial_t(\epsilon_0 E+P). \tag{58}$$

To proceed we need a specific expression for $P(E)$. The common general expression is given by the sequence [45, 15]

$$P=\epsilon_0\left(\hat{\chi}^{(1)}E+\hat{\chi}^{(2)}[E,E]+\hat{\chi}^{(3)}[E,E,E]+\cdots\right), \tag{59}$$

where $\hat{\chi}^{(1)}$ is a linear operator, $\hat{\chi}^{(2)}$ is a bilinear one, $\hat{\chi}^{(3)}$ is trilinear and so on. This sequence of linear and nonlinear susceptibilities encodes solution of an additional equation for $P(z,t)$ induced by $E(z,t)$.

In what follows we consider the following reduction of (59)

$$\frac{1}{\epsilon_0}P = \hat{\chi}^{(1)}E + \chi^{(3)}E^3. \tag{60}$$

Equation (60) assumes that $P(-E) = -P(E)$ such that the second-order susceptibility vanishes due to symmetry arguments, which is a typical situation in optical fibers. The nonlinear susceptibility of the third order $\hat{\chi}^{(3)}$ is approximated by a constant, $\chi^{(3)}$. The linear part of the medium response is determined by the standard delay integral [45]

$$\hat{\chi}^{(1)}E(t) = \int_0^\infty \chi^{(1)}(\tau)E(t-\tau)d\tau, \tag{61}$$

with the memory function $\chi^{(1)}(\tau)$. It is convenient to introduce the linear dispersion operator $\hat{\epsilon} = 1 + \hat{\chi}^{(1)}$ and to combine the system (58) into a single propagation equation

$$\partial_z^2 E - \frac{1}{c^2}\partial_t^2\left(\hat{\epsilon}E + \chi^{(3)}E^3\right) = 0 \tag{62}$$

which is a self-consistent nonlinear wave equation with $c = 1/\sqrt{\mu_0\epsilon_0}$.

Considering operator $\hat{\epsilon}$ in the frequency domain one derives from Eq. (61) that

$$\hat{\epsilon}e^{-i\omega t} = \epsilon(\omega)e^{-i\omega t} \quad \text{with} \quad \epsilon(\omega) = 1 + \int_0^\infty \chi^{(1)}(\tau)e^{i\omega\tau}d\tau.$$

The just defined relative permeability $\epsilon(\omega)$ is a complex-valued function with the property

$$\epsilon(\omega) = [\epsilon(-\omega)]^*, \tag{63}$$

and it is subject to the Kramers–Kronig relations [45]. Equation (63) together with the relation

$$\hat{\epsilon}E(t) = \hat{\epsilon}\int_{-\infty}^{\infty} E(\omega)e^{-i\omega t}\frac{d\omega}{2\pi} = \int_{-\infty}^{\infty}\epsilon(\omega)E(\omega)e^{-i\omega t}\frac{d\omega}{2\pi}, \tag{64}$$

guaranties that operator $\hat{\epsilon}$ transforms a real-valued field into a real-valued one. Moreover, Eq. (64) indicates that

$$\hat{\epsilon} = \epsilon(i\partial_t), \tag{65}$$

if $\epsilon(\omega)$ is approximated by a polynomial in the frequency range where $E(\omega) \neq 0$.

In a more complex situation $\epsilon(\omega)$ is approximated by a polynomial in some interval of only positive frequencies, e.g., by a Taylor expansion of the "true" $\epsilon(\omega)$ near the pulse carrier frequency. Such an expansion typically violates the fundamental Eq. (63), and in any case it has a finite convergence radius. To avoid the difficulty one can split positive and negative frequencies in the electric field in accord with Gabor's rule (9) and then apply $\epsilon(i\partial_t)$ only to the positive-frequency part

$$E(z,t) = \frac{1}{2}\mathcal{E}(z,t) + \text{c.c.} \quad \Rightarrow \quad \hat{\epsilon}E(z,t) = \frac{1}{2}\epsilon(i\partial_t)\mathcal{E}(z,t) + \text{c.c.}, \tag{66}$$

such a trick is an important step in derivation of the complex envelope equation.

To conclude this section we stress that Eq. (62) is on one hand a nonlinear wave equation and on the other hand a delay differential equation. Not only the initial conditions, $E(t = 0, z)$ and $\partial_t E(t = 0, z)$, are required to find $E(t > 0, z)$ but also the prehistory $E(t < 0, z)$. Moreover, despite of the simple $1 + 1$ geometry, the full numerical solution of Eq. (62) may be immense, e.g., for a wavelength of $1\,\mu$m and fiber length of 1 km. A much more simple formulation can be gained if one deals with a single wave packet or a dense sequence of wave-packets that propagate in one direction. One can then neglect the backward waves and follow the pulses only in the moving frame. This approach leads to the envelope NLSE and and to the "less-envelope" GNLSE, as described below.

3.2 Forward and backward waves

Neglecting for a while the nonlinear term in Eq. (62), one can apply a standard substitution $E \sim e^{i(kz-\omega t)}$ and derive the following dispersion relation

$$k^2c^2 = \omega^2\epsilon(\omega) \tag{67}$$

for the linear waves. A given (positive) frequency yields two wave vectors, $k = \pm\beta(\omega)$, for the forward and the backward wave respectively. In fiber optics $\beta(\omega)$ is referred to as the propagation constant. The real-valued field of a monochromatic forward wave is written as

$$E(z,t) = \frac{1}{2}Ae^{i\beta(\omega)z-i\omega t} + \text{c.c.}, \tag{68}$$

where A is the complex amplitude. It is convenient to define the propagation constant for negative frequencies in such a way that

$$\beta(-\omega) = -\beta^*(\omega), \tag{69}$$

the latter definition is compatible with the dispersion relation (67) and with the fundamental Eq. (63). Complex conjugation of $e^{i\beta(\omega)z-i\omega t}$ is then equivalent to the replacement $\omega \to -\omega$, which is a convenient property of the Fourier coefficients.

In what follows we will use the index of refraction $n(\omega) = \sqrt{\epsilon(\omega)}$, the corresponding operator $\hat{n}$ is defined as in Eq. (64). It is also convenient to relate a suitable operator $\hat{\beta}$ with the propagation constant. The problem is that for a real-valued $E(t)$, the expression $\int_{-\infty}^{\infty}\beta(\omega)E(\omega)e^{-i\omega t}d\omega$ is complex-valued. So instead, we define

$$\hat{\beta} = \frac{1}{c}\hat{n}\partial_t \quad \Rightarrow \quad \hat{\beta}e^{-i\omega t} = -i\beta(\omega)e^{-i\omega t}.$$

The key observation is that with the help of $\hat{\beta}$ and $\hat{n}$ one can split Eq. (62) in two first-order equations

$$(\hat{\beta} + \partial_z)\underset{\rightarrow}{E} + \frac{\chi^{(3)}}{2c}\hat{n}^{-1}\partial_t\left(\underset{\rightarrow}{E} + \underset{\leftarrow}{E}\right)^3 = 0, \tag{70}$$

$$(\hat{\beta} - \partial_z)\underset{\leftarrow}{E} + \frac{\chi^{(3)}}{2c}\hat{n}^{-1}\partial_t\left(\underset{\rightarrow}{E} + \underset{\leftarrow}{E}\right)^3 = 0, \tag{71}$$

where by construction

$$E(z,t) = \underset{\rightarrow}{E}(z,t) + \underset{\leftarrow}{E}(z,t).$$

Indeed, after we apply $\hat{\beta} - \partial_z$ to (70) and $\hat{\beta} + \partial_z$ to (71), we add them together and return to Eq. (62). Moreover, it is easy to see that the linearized equations (70) and (71) describe the pure forward and the pure backward wave respectively. These waves are coupled by nonlinearity.

Now let us consider a sequence of forward pulses. Assuming that the nonlinear excitation of the backward wave is non-resonant, one can neglect the backward field and replace Eq. (62) by the so-called forward Maxwell equation

$$\partial_z E + \hat{\beta} E + \frac{\chi^{(3)}}{2c} \hat{n}^{-1} \partial_t (E)^3 = 0, \tag{72}$$

where from now on we do not distinguish between the forward field and E. In optical context Eq. (72) was first applied in [34], the splitting technique was discussed in [42, 41, 27, 39]. Equation (72), being of interest on its own, is a good starting point for derivation of the GNLSE, because (72) is already of first order with respect to the propagation coordinate.

3.3 Envelope equations

To derive an envelope equation from Eq. (72) we consider a typical situation in which the forward pulse in question has narrow spectrum localized at the carrier frequency ω_0. For $\omega \approx \omega_0$ we approximate the wave vector $k = \beta(\omega)$ by its Taylor expansion

$$\beta(\omega) = \sum_{m=0}^{M} \frac{\beta_m}{m!} (\omega - \omega_0)^m, \qquad \beta_m = \beta^{(m)}(\omega_0),$$

of order $M \geq 2$ at least. The carrier wave reads $e^{i(\beta_0 z - \omega_0 t)}$, the carrier phase velocity is ω_0/β_0. The group velocity $V_{\text{gr}} = d\omega/dk$, therefore

$$\frac{1}{V_{\text{gr}}(\omega)} = \beta_1 + \beta_2(\omega - \omega_0) + \cdots ,$$

where β_2 is referred to as the group velocity dispersion (GVD) [1]. The GVD parameter describes the frequency dependent correction to the group velocity $1/\beta_1$ of the carrier wave.

We now apply the substitution

$$E(z,t) = \frac{1}{2} \Psi(z,t) e^{i(\beta_0 z - \omega_0 t)} + \text{c.c.}, \tag{73}$$

to remove fast oscillations of the pulse electric field in (72). First, we ignore the third harmonic in the nonlinear term

$$E^3 \approx \frac{3}{8} |\Psi|^2 \Psi e^{i(\beta_0 z - \omega_0 t)} + \text{c.c.}, \tag{74}$$

that is, we assume that medium excitation induced by $\Psi^3 e^{3i(\beta_0 z - \omega_0 t)}$ is non-resonant, $\beta(3\omega_0) \neq 3\beta(\omega_0)$.

Second, we note that Eq. (65) applies also to the expansion of $\beta(\omega)$

$$\hat{\beta} = -i \sum_{m=0}^{M} \frac{\beta_m}{m!} (i\partial_t - \omega_0)^m,$$

and using (73) one can derive that

$$\hat{\beta} E = -\frac{i}{2} \left[\sum_{m=0}^{M} \frac{\beta_m}{m!} (i\partial_t)^m \Psi \right] e^{i(\beta_0 z - \omega_0 t)} + \text{c.c.},$$

in accord with the general principle (66).

Third, the operator $\hat{n}^{-1}\partial_t$ in the nonlinear term in (72) corresponds to $-i\omega/n(\omega)$ and can be treated exactly like $\hat{\beta}$, namely if

$$\frac{\omega}{n(\omega)} = \sum_{m=0}^{M} \frac{\gamma_m}{m!} (\omega - \omega_0)^m \qquad \text{then} \qquad \hat{n}^{-1}\partial_t = -i \sum_{m=0}^{M} \frac{\gamma_m}{m!} (i\partial_t - \omega_0)^m,$$

and in accord with (74)

$$\hat{n}^{-1}\partial_t (E^3) = -\frac{3i}{8} \left[\sum_{m=0}^{M} \frac{\gamma_m}{m!} (i\partial_t)^m (|\Psi|^2 \Psi) \right] e^{i(\beta_0 z - \omega_0 t)} + \text{c.c.}$$

Combining all three steps we see that Eq. (72) reduces to the following propagation equation for $\Psi(z, t)$

$$i\partial_z \Psi + \left[\sum_{m=1}^{M} \frac{\beta_m}{m!} (i\partial_t)^m \right] \Psi + \frac{3\chi^{(3)}}{8c} \left[\sum_{m=0}^{M} \frac{\gamma_m}{m!} (i\partial_t)^m \right] |\Psi|^2 \Psi = 0, \tag{75}$$

in which the β_0-term is cancelled out. The latter equation is simplified in two further steps. First, we recall that β_1 is the inverse carrier group velocity. It is then convenient to define the so-called retarded time

$$\tau = t - \beta_1 z, \tag{76}$$

and to introduce $\psi(z, \tau) = \Psi(z, t)$, i.e., it is convenient to change to a moving frame that follows the pulse. Thereafter the β_1-term is cancelled out from Eq. (75).

Second, the coefficients γ_m are usually calculated by replacing $n(\omega)$ with $n(\omega_0)$, then

$$\gamma_0 = \frac{\omega_0}{n(\omega_0)}, \qquad \gamma_1 = \frac{1}{n(\omega_0)}, \qquad \gamma_{m \geq 2} = 0.$$

Altogether, Eq. (75) reduces to the GNLSE

$$i\partial_z \psi + \hat{\mathcal{D}} \psi + \frac{\omega_0}{c} n_2 (1 + i\omega_0^{-1} \partial_\tau) |\psi|^2 \psi = 0, \tag{77}$$

where following [1] we use the notation $n_2 = (3/8)\chi^{(3)}/n(\omega_0)$ and introduce the so-called dispersion operator

$$\hat{\mathcal{D}} = \sum_{m=2}^{M} \frac{\beta_m}{m!} (i\partial_\tau)^m.$$

After $\psi(z,\tau)$ is calculated, the field is yielded by

$$E(z,t) = \frac{1}{2}\psi(z, t-\beta_1 z)e^{i(\beta_0 z - \omega_0 t)} + \text{c.c.} \tag{78}$$

The standard, classical NLSE, the one that is completely integrable [75], appears if one approximates both operators in (77) by the leading terms

$$i\partial_z\psi - \frac{\beta_2}{2}\partial_\tau^2\psi + \frac{\omega_0}{c}n_2|\psi|^2\psi = 0, \tag{79}$$

where in a focusing ($n_2 > 0$) medium and in the anomalous ($\beta_2 < 0$) dispersion domain one can rescale the variables and obtain the normalized Eq. (46).

An important observation is that the derived propagation equations should be solved with respect to z. In fiber optics both the GNLSE (77) and the NLSE (79) require $E(z = 0, t)$ and yield $E(z > 0, t)$. Physical consequences of this feature will be discussed later on. Another important observation is that in typical settings Eq. (77) provides an exceptionally good agreement with the solutions of the full nonlinear wave Eq. (62) as reported in [16, 31]. To explain this behavior we now consider the Hamiltonian framework for optical pulses.

4 Hamiltonian description of pulses

We now turn to the Hamiltonian structure of equations (58). It is of interest that to some extent such structure can be recognized even without exact knowledge of $P(E)$. To this end we introduce the standard displacement field $D(z,t)$ and the magnetic intensity $H(z,t)$

$$D = \epsilon_0 E + P, \qquad H = \frac{B}{\mu_0},$$

such that equations (58) take the form

$$\partial_z E = -\partial_t B \qquad \text{and} \qquad -\partial_z H = \partial_t D. \tag{80}$$

The relation between H and B is trivial because most optical materials are not magnetic [14]. The Hamiltonian is given by a yet unknown functional $\mathscr{H}[E,H]$ for which the following variation is required

$$\delta\mathscr{H}[E,H] = \int_{-\infty}^{\infty}(D\delta E + B\delta H)dt. \tag{81}$$

Indeed, such a $\mathscr{H}[E,H]$ implies that the system (80) reads

$$\partial_z E = -\partial_t\frac{\delta\mathscr{H}}{\delta H}, \qquad \partial_z H = -\partial_t\frac{\delta\mathscr{H}}{\delta E}, \tag{82}$$

and has the same mathematical structure as the Hamiltonian equations (50), but one should exchange the time and space variables. For instance, the Poisson bracket (48) now reads

$$\{E(t_1), H(t_2)\} = \{H(t_1), E(t_2)\} = \delta'(t_1 - t_2). \tag{83}$$

This simple exchange of variables results in important physical differences and changes the physical meaning of $\mathscr{H}[E,H]$. We will first discuss these differences and then return to Eq. (82) and apply the results that have already been derived for the GZF bracket (83).

4.1 z-propagation

The system (82) is solved with respect to z in the context of the so-called z-propagation picture similar to the GNLSE (77). This problem formulation naturally applies to many optical settings, where some source-device creates the input field $E(z=0,t)$ which is “known” for all times at the beginning of the fiber and some detector-device measures the output field $E(z=L,t)$, or some derived quantity like power, at the end. Here L is the propagation distance. The reflected backward wave is neglected, exactly like in the already derived envelope equation. The pulses are either periodic or localized in time, such that we have a kind of boundary condition for the time axis.

One consequence of the z-propagation is that $\mathscr{H}$ is related not to the energy and system invariance with respect to time shifts [44]. Instead, $\mathscr{H}$ refers to momentum conservation and to invariance of basic equations with respect to space shifts. Another consequence is that now all conserved quantities are obtained by integration of the corresponding flux densities over time. For instance, consider the conservation law (56) for one spatial dimension

$$\partial_t \rho(z,t) + \partial_z j_z(z,t) = 0. \tag{84}$$

The later equation is usually integrated over dz assuming that $\rho(z,t)$ and $j(z,t)$ are localized in space. The result

$$\frac{d}{dt}\int_{-\infty}^{\infty} \rho(z,t)dz = 0,$$

is interpreted as charge conservation. In the z-propagation picture, however, we integrate Eq. (84) over dt assuming that for a given z both the charge density $\rho(z,t)$ and the current density $j(z,t)$ are induced by an isolated pulse and disappear for $t\to\pm\infty$. The resulting conservation law

$$\frac{d}{dz}\int_{-\infty}^{\infty} j(z,t)dt = 0,$$

means that the time-averaged current density is the same for all observation points inside the fiber. Consequently $\mathscr{H}[E,H]$ must be a time-averaged momentum flux.

Before proceeding with the explicit expression for $\mathscr{H}[E,H]$ let us revisit the definition of the complex Hamiltonian variable. In the case of t-propagation the canonical variables are parametrized by the wave vector. The standard complex variable was introduced by

$$a(k) = \frac{q(k)+ip(k)}{\sqrt{2}} \quad \Rightarrow \quad i\partial_t a = \frac{\delta\mathscr{H}}{\delta a^*(k)},$$

such that

$$\mathscr{H} = \int_{-\infty}^{\infty} \omega(k)|a(k)|^2 dk \quad \text{yields} \quad a(k,t)\sim e^{-i\omega(k)t}.$$

The latter expression agrees with the familiar $e^{i(kz-\omega t)}$ representation of monochromatic waves.

In the case of z-propagation the canonical variables are parametrized by frequency and it is convenient to take

$$\mathcal{A}(\omega) = \frac{q(\omega)-ip(\omega)}{\sqrt{2}} \quad \Rightarrow \quad i\partial_t \mathcal{A} = -\frac{\delta\mathscr{H}}{\delta\mathcal{A}^*(k)}, \tag{85}$$

such that, e.g.,

$$\mathscr{H} = \int_{-\infty}^{\infty} \beta(\omega)|\mathcal{A}(\omega)|^2 d\omega \quad \text{yields} \quad \mathcal{A}(\omega, z) \sim e^{i\beta(\omega)t}.$$

The actual z-Hamiltonian will be a bit more complicated, as it should describe both the forward and the backward waves.

4.2 z-Hamiltonian

Momentum conservation for the electromagnetic field in vacuum in the one-dimensional setting (58) is given by a well-known relation

$$\partial_t \left(\frac{j_P}{c^2} \right) + \partial_z \left(\frac{\epsilon_0 E^2}{2} + \frac{B^2}{2\mu_0} \right) = 0,$$

where $j_P = EB/\mu_0$ is the Poynting vector and the second bracket contains the vacuum momentum flux density. Motivated by this example, we consider the following functional [72]

$$\mathscr{H}[E, H] = \int_{-\infty}^{\infty} \left[\epsilon_0 \left(\frac{E\hat{\epsilon}E}{2} + \frac{\chi^{(3)}E^4}{4} \right) + \frac{\mu_0 H^2}{2} \right] dt. \tag{86}$$

In what follows we will also use the frequency components of the involved fields

$$E(\omega) = \int_{-\infty}^{\infty} E(t) e^{i\omega t} dt, \qquad H(\omega) = \int_{-\infty}^{\infty} H(t) e^{i\omega t} dt,$$

and representation of the definition (86) in the frequency domain.

Note, that operator $\hat{\epsilon}$ in Eq. (86) is not self-adjoint. Indeed, defining the scalar product of two real fields at some fixed point in space by

$$\int_{-\infty}^{\infty} E_1(t) E_2(t) dt = \int_{-\infty}^{\infty} E_1^*(\omega) E_2(\omega) \frac{d\omega}{2\pi}, \tag{87}$$

we obtain

$$\int_{-\infty}^{\infty} [E_1(t)\hat{\epsilon}E_2(t) - E_2(t)\hat{\epsilon}E_1(t)] dt = \int_{-\infty}^{\infty} [\epsilon(\omega) - \epsilon^*(\omega)] E_1^*(\omega) E_2(\omega) \frac{d\omega}{2\pi},$$

in accord with (63) and (64). The imaginary part of $\epsilon(\omega)$, while always present, can be extremely small for frequencies of interest, as it happens in the transparency region of all fiber-relevant materials. If $\epsilon(\omega)$ can be taken real, the system (58) becomes dissipation-free and $\hat{\epsilon}$ becomes self-adjoint. The variation of $\mathscr{H}[E, H]$ from (86) is then given by (81), and we have found the required Hamiltonian. From the physical side, $\mathscr{H}$ gives the total amount of momentum that is transferred across any given point inside the fiber per unit area of the transverse cross-section. $\mathscr{H}$ is an integral of motion in the sense that $\partial_z \mathscr{H}[E(z, t), H(z, t)] = 0$ if $E(z, t)$

and $H(z,t)$ solve (58). Equation (87) indicates that in the frequency domain the quadratic part of the Hamiltonian (86) is given by the relation

$$\mathscr{H}_2[E,H] = \int_{-\infty}^{\infty} \left[\frac{\epsilon_0 \epsilon(\omega)}{2} E(\omega) E^*(\omega) + \frac{\mu_0}{2} H(\omega) H^*(\omega) \right] \frac{d\omega}{2\pi}.$$

Now following (53) we introduce the canonical variables

$$q(\omega) = \frac{1}{\sqrt{2\pi|\omega|}} \int_{-\infty}^{\infty} \left[\Lambda(\omega) E(t) + \frac{\sigma(\omega)}{\Lambda(\omega)} H(t) \right] \cos(\omega t) dx,$$
$$p(\omega) = -\frac{1}{\sqrt{2\pi|\omega|}} \int_{-\infty}^{\infty} \left[\Lambda(\omega) E(t) + \frac{\sigma(\omega)}{\Lambda(\omega)} H(t) \right] \sin(\omega t) dx,$$

where $\sigma(\omega) = \omega/|\omega|$ and $\Lambda(\omega) = \Lambda(-\omega) \in \mathbb{R}$ similar to Eq. (54). The complex variable is introduced in accord with Eq. (85)

$$\mathcal{A}(\omega) = \frac{q(\omega) - ip(\omega)}{\sqrt{2}} = \frac{1}{2\sqrt{\pi|\omega|}} \int_{-\infty}^{\infty} \left[\Lambda(\omega) E(t) + \frac{\sigma(\omega)}{\Lambda(\omega)} H(t) \right] e^{i\omega t} dt.$$

By construction it is subject to the z-propagation equation

$$i\partial_z \mathcal{A}(\omega, z) + \frac{\delta \mathscr{H}}{\delta \mathcal{A}^*(\omega)} = 0, \tag{88}$$

which is analogous to the t-equation (44). The above definition of $\mathcal{A}(\omega)$ suggests to switch to the frequency domain where we have

$$\mathcal{A}(\omega) = \frac{1}{2\sqrt{\pi|\omega|}} \left[\Lambda(\omega) E(\omega) + \frac{\sigma(\omega)}{\Lambda(\omega)} H(\omega) \right], \tag{89}$$

and

$$\Lambda(\omega) \frac{E(\omega)}{\sqrt{\pi|\omega|}} = \mathcal{A}(\omega) + \mathcal{A}^*(-\omega), \qquad \frac{\sigma(\omega)}{\Lambda(\omega)} \frac{H(\omega)}{\sqrt{\pi|\omega|}} = \mathcal{A}(\omega) - \mathcal{A}^*(-\omega). \tag{90}$$

The scaling factor $\Lambda(\omega)$ is chosen in such a way that

$$\frac{\epsilon_0 \epsilon(\omega)}{\Lambda^2(\omega)} = \mu_0 \Lambda^2(\omega) \quad \Rightarrow \quad \Lambda(\omega) = \sqrt[4]{\frac{\epsilon_0 \epsilon(\omega)}{\mu_0}},$$

then the quadratic part of the Hamiltonian (86) is given by

$$\mathscr{H}_2 = \int_{-\infty}^{\infty} |\beta(\omega)| \mathcal{A}(\omega) \mathcal{A}^*(\omega) d\omega.$$

This is a z-replacement of the standard t-expression $\int_{-\infty}^{\infty} \omega(k)|a(k)|^2 d\omega$. Equation (88) takes the form

$$i\partial_z \mathcal{A}(z,\omega) + |\beta(\omega)| \mathcal{A}(z,\omega) + \frac{\delta \mathscr{H}_{\text{int}}}{\delta \mathcal{A}^*(\omega)} = 0, \tag{91}$$

which is the pulse propagation equation in the Hamiltonian framework. The linear part of Eq. (91) looks unusual just because (91) combines both forward and backward waves. Indeed, ignoring for a moment $\mathscr{H}_{\text{int}}$ we obtain

$$\mathcal{A}(z,\omega) = \mathcal{A}(0,\omega)e^{i|\beta(\omega)|z}, \tag{92}$$

such that

$$E(z,\omega)e^{-i\omega t} = \sqrt{\frac{\pi|\omega|}{\epsilon_0 c n(\omega)}} \left[\mathcal{A}(0,\omega)e^{i|\beta(\omega)|z - i\omega t} + \mathcal{A}^*(0,-\omega)e^{-i|\beta(\omega)|z - i\omega t}\right].$$

Taking some $\omega > 0$ we immediately see that $\mathcal{A}(0,\omega)$ and $\mathcal{A}^*(0,-\omega)$ describe forward and backward waves with the physical frequency ω.

Now we return to the full Eq. (91). Before applying it one has to calculate $\delta\mathscr{H}_{\text{int}}/\delta\mathcal{A}^*(\omega)$ with $\mathscr{H}_{\text{int}} = \frac{1}{4}\epsilon_0\chi^{(3)}\int_{-\infty}^{\infty} E^4 dt$ or

$$\mathscr{H}_{\text{int}} = \frac{\epsilon_0\chi^{(3)}}{32\pi^3}\int_{-\infty}^{\infty} E_1E_2E_3E_4\delta_{1234}d\mho, \tag{93}$$

where

$$d\mho = d\omega_1 d\omega_2 d\omega_3 d\omega_4,$$

E_i abbreviates $E(\omega_i, z)$, and the delta function $\delta(\omega_1+\omega_2+\omega_3+\omega_4)$ is abbreviated by δ_{1234}. Other combinations of frequencies will be denoted by over-bars, e.g.,

$$\delta_{12\bar{3}\bar{4}} = \delta(\omega_1+\omega_2-\omega_3-\omega_4).$$

Equation (93) can be expressed as

$$\mathscr{H}_{\text{int}} = \frac{1}{32\pi}\int_{-\infty}^{\infty} T_{1234}(\mathcal{A}_1+\mathcal{A}_{-1}^*)(\mathcal{A}_2+\mathcal{A}_{-2}^*)(\mathcal{A}_3+\mathcal{A}_{-3}^*)(\mathcal{A}_4+\mathcal{A}_{-4}^*)\delta_{1234}d\mho,$$

where T_{1234} denotes

$$T(\omega_1,\omega_2,\omega_3,\omega_4) = \frac{\mu_0\chi^{(3)}\sqrt{|\omega_1\omega_2\omega_3\omega_4|}}{\sqrt{n(\omega_1)n(\omega_2)n(\omega_3)n(\omega_4)}}, \tag{94}$$

and is completely symmetric with respect to both permutation of indices and replacement of the sign of any frequency, $\omega \to -\omega$. In many cases $n(\omega)$ is close to a constant for all relevant frequencies and one can use a very simple expression

$$T(\omega_1,\omega_2,\omega_3,\omega_4) = \frac{\mu_0\chi^{(3)}}{n^2(\omega_0)}\sqrt{|\omega_1\omega_2\omega_3\omega_4|}, \tag{95}$$

where ω_0 is the carrier frequency. On the other hand, if description of the nonlinearity in terms of a single $\chi^{(3)} = \text{const}$ is inappropriate, $\chi^{(3)}$ is replaced by a full third-order susceptibility $\chi^{(3)}(\omega_1,\omega_2,\omega_3,\omega_4)$ that enters into Eq. (94) for T_{1234}. Also in this case we assume symmetry with respect to permutation of frequencies, the so-called overall permutation symmetry. The latter applies to the fiber transparence window, where all involved frequencies considerably differ from the transition frequencies of the medium [15].

The above derived representation of $\mathscr{H}_{\text{int}}(\mathcal{A}_1, \mathcal{A}_2, \mathcal{A}_3, \mathcal{A}_4)$ is immediately split in three terms

$$\mathscr{H}_{\text{int}} = \mathscr{H}_{40} + \mathscr{H}_{31} + \mathscr{H}_{22},$$

with simple physical interpretations: each term is responsible for a separate 4-wave process, namely

$$\mathscr{H}_{40} = \frac{1}{32\pi} \int_{-\infty}^{\infty} T_{1234} \mathcal{A}_1 \mathcal{A}_2 \mathcal{A}_3 \mathcal{A}_4 \delta_{1234} d\mho + \text{c.c.},$$
$$\mathscr{H}_{31} = \frac{1}{8\pi} \int_{-\infty}^{\infty} T_{1234} \mathcal{A}_1 \mathcal{A}_2 \mathcal{A}_3 \mathcal{A}_4^* \delta_{123\bar{4}} d\mho + \text{c.c.},$$
$$\mathscr{H}_{22} = \frac{3}{16\pi} \int_{-\infty}^{\infty} T_{1234} \mathcal{A}_1 \mathcal{A}_2 \mathcal{A}_3^* \mathcal{A}_4^* \delta_{12\bar{3}\bar{4}} d\mho,$$

where, e.g., the last expression is a classical analogue of the quantum-mechanical process in which two photons disappear and two new photons are born.

Although all three constituents of $\mathscr{H}_{\text{int}}$ have the same order in amplitude, they are of different importance for Eq. (91). For instance, using Eq. (92) one can estimate that $\mathscr{H}_{40}$ is a weighted average of the quickly oscillating $e^{i(|\beta(\omega_1)|+|\beta(\omega_2)|+|\beta(\omega_3)|+|\beta(\omega_4)|)z}$ factor over the hyperplane $\omega_1 + \omega_2 + \omega_3 + \omega_4 = 0$ in the frequency space. Therefore $\mathscr{H}_{40}$ can be just neglected, or more precisely eliminated using the corresponding canonical transform as described in [74, 69]. In a similar way, $\mathscr{H}_{31}$ can essentially contribute to the dynamic if the following conditions are satisfied

$$|\beta(\omega_1)| + |\beta(\omega_2)| + |\beta(\omega_3)| = |\beta(\omega_4)|, \qquad \omega_1 + \omega_2 + \omega_3 = \omega_4. \tag{96}$$

Recall that $\mathcal{A}(z, \omega)$ is related to the forward wave for $\omega > 0$ and to the backward wave for $\omega < 0$. Considering, e.g., only forward waves one can replace $|\beta(\omega)|$ with $\beta(\omega)$ and Eq. (96) reduces to the standard resonance conditions [74, 69]. In what follows we neglect both $\mathscr{H}_{40}$ and $\mathscr{H}_{31}$ such that the propagation Eq. (91) finally reads

$$\Big[i\partial_z + |\beta(\omega)|\Big] \mathcal{A}(z, \omega) + \frac{3}{8\pi} \int_{-\infty}^{\infty} T_{123\omega} \mathcal{A}_1 \mathcal{A}_2 \mathcal{A}_3^* \delta_{12\bar{3}\bar{\omega}} d\omega_1 d\omega_2 d\omega_3 = 0. \tag{97}$$

Equation (97) has the same mathematical structure and can be solved using the same numerical methods as the GNLSE (77). Several examples are given in [5]. On the other hand, Eq. (97) is just a reformulation of the original Maxwell equations (recall that one don't have to neglect the non-resonant terms in the Hamiltonian, they can be removed by a suitable canonical transformation). One don't have to sacrifice the generality. Beside numerics, the presented Hamiltonian framework has important applications with respect to integrals of motion and an intrinsic relation to the GNLSE. These topics are described in the reminder of the manuscript.

4.3 Energy transport

In this section we address the problem of energy transport. The value of the instant power is an important characteristic of optical pulses, in practical applications of GNLSE (77) it is convenient to renormalize the envelope $\psi(z, \tau)$ such that $|\psi(z, \tau)|^2$ gives the instant power [22].

The renormalization, however, depends on the chosen carrier frequency and may be inappropriate for optical supercontinuum in which very different frequencies can provide comparable contributions to the total power.

Hamiltonian language gives a simple representation for the instant power, the representation correctly accounts for all frequencies. Energy conservation for the electromagnetic field in vacuum in the one-dimensional setting (58) is given by a well-known relation

$$\partial_t \left(\frac{\epsilon_0 E^2}{2} + \frac{B^2}{2\mu_0} \right) + \partial_z j_P = 0,$$

where like in the previous section $j_P = EB/\mu_0$ is the Poynting vector. For the z-propagation picture the quantity $\int_{-\infty}^{\infty} j_P(z,t)dt$ should be constant, the latter yields the total amount of energy transferred by a pulse per unit area that is transversal to the direction of propagation. If we now account for both the fiber dispersion and nonlinearity, the expression for the energy density becomes complicated, however the expression for j_P should be identical to that in the free space [45]. Therefore we consider the following quantity

$$\mathscr{E}[E,H] = \int_{-\infty}^{\infty} E(z,t)H(z,t)dt = \int_{-\infty}^{\infty} E(z,\omega)H^*(z,\omega)\frac{d\omega}{2\pi}.$$

z-conservation of $\mathscr{E}$ can be established directly from Eq. (58) because

$$\frac{d}{dz}\int_{-\infty}^{\infty} EHdt = \int_{-\infty}^{\infty} (-H\partial_t B - E\partial_t D)dt = \int_{-\infty}^{\infty} D(E)\partial_t E dt$$

and therefore any possible z-dependence of $\mathcal{E}$ is related only with the non-instantaneous part of $D(E)$

$$\frac{d\mathcal{E}}{dz} = \epsilon_0 \int_{-\infty}^{\infty} (\hat{\epsilon}E)(\partial_t E)dt = i\epsilon_0 \int_{-\infty}^{\infty} \omega\epsilon(\omega)E(\omega)E^*(\omega)\frac{d\omega}{2\pi}.$$

Equation (63) indicates that real and imaginary parts of $\epsilon(\omega)$ are even and odd functions of frequency. Therefore the last integral is determined only by the imaginary part of $\epsilon(\omega)$. If the imaginary part can be neglected, the last integral vanishes and $\mathscr{E}[E,H]$ becomes an integral of motion.

Now we express the energy flux in terms of the normal variables $\mathcal{A}(z,t)$ and $\mathcal{A}^*(z,t)$ using (90). After simple transformations we obtain

$$\mathscr{E}[E,H] = \int_{-\infty}^{\infty} \omega\mathcal{A}(z,\omega)\mathcal{A}^*(z,\omega)d\omega.$$

The latter expression is, of course, of universal nature and gives a desired expression of total energy transferred by the pulse per unit area of the fiber cross-section.

4.4 Photon number

In this section we consider the following functional

$$\mathcal{N}[\mathcal{A},\mathcal{A}^*] = 2\pi k_0 \int_{-\infty}^{\infty} \mathcal{A}(z,t)\mathcal{A}^*(z,t)dt = k_0 \int_{-\infty}^{\infty} \mathcal{A}(z,\omega)\mathcal{A}^*(z,\omega)d\omega,$$

where k_0 is an arbitrary constant wave vector. $\mathscr{N}[\mathcal{A},\mathcal{A}^*]$ is a rightful Hamiltonian, the corresponding equation of motion yields

$$i\partial_z\mathcal{A}(z,\omega)+\frac{\delta\mathscr{N}}{\delta\mathcal{A}^*(\omega)}=0 \quad\Rightarrow\quad \mathcal{A}(z,\omega)=\mathcal{A}(0,\omega)e^{ik_0z}. \tag{98}$$

Now, we apply a well known theorem from the classical Hamiltonian mechanics [7]. Namely, if some Hamiltonian $\mathscr{H}[\mathcal{A},\mathcal{A}^*]$ is invariant with respect to the one-parametric family of the phase shifts (98), the quantity $\mathscr{N}[\mathcal{A},\mathcal{A}^*]$ is an integral of motion for the dynamic system generated by $\mathscr{H}[\mathcal{A},\mathcal{A}^*]$. This, e.g., applies to the Hamiltonians $\mathscr{H}_2$ and $\mathscr{H}_{22}$, and therefore to the propagation equation generated by

$$\mathscr{H}=\int_{-\infty}^{\infty}|\beta(\omega)|\mathcal{A}(\omega)\mathcal{A}^*(\omega)d\omega+\frac{3}{16\pi}\int_{-\infty}^{\infty}T_{1234}\mathcal{A}_1\mathcal{A}_2\mathcal{A}_3^*\mathcal{A}_4^*\delta_{12\bar{3}\bar{4}}d\mho.$$

The quantity $\mathscr{N}$ can be interpreted as the classical expression for the photon number. An arbitrary factor k_0 remains undetermined, because the photon number cannot be defined completely self-consistently in the framework of classical fields.

4.5 Analytic signal

In this section we establish a relation between the Hamiltonian equation (91) and the GNSE (77). In particular, we will demonstrate that the analytic signal naturally appears also in the Hamiltonian framework. To this end we split the definition (89). First a complex field $\mathcal{E}(z,t)$ is derived from $E(z,t)$ and $H(z,t)$ such that

$$\mathcal{E}(z,\omega)=E(z,\omega)+\frac{\sigma(\omega)}{\epsilon_0cn(\omega)}H(z,\omega)=E(z,\omega)-\frac{i\partial_zE(z,\omega)}{|\beta(\omega)|}, \tag{99}$$

and then the normal variable $\mathcal{A}(z,t)$ is obtained by a simple rescaling in the frequency domain

$$\mathcal{A}(z,\omega)=\frac{1}{2}\sqrt{\frac{\epsilon_0cn(\omega)}{\pi|\omega|}}\mathcal{E}(z,\omega). \tag{100}$$

These new definitions (99) and (100) are compatible with the older one (89). Let us show that GNLSE just stops at the $\mathcal{E}(z,t)$ level, without explicit use of the canonical variable $\mathcal{A}(z,t)$. Indeed Eq. (99) indicates that

$$E(\omega)=\frac{\mathcal{E}(\omega)+\mathcal{E}^*(-\omega)}{2}$$

such that in the physical space $E(z,t)=\frac{1}{2}\mathcal{E}(z,t)+$ c.c., as is should be for the complex amplitude. Moreover, applying Eq. (99) to a forward wave with $E(z,\omega)\sim e^{i\beta(\omega)z}$ we see that $\mathcal{E}(z,\omega)$ vanishes for $\omega<0$. In a similar way, for a backward wave with $E(z,\omega)\sim e^{-i\beta(\omega)z}$ we see that $\mathcal{E}(z,\omega)$ vanishes for $\omega>0$. In other words, if solely forward waves are present, $\mathcal{E}(z,t)$ is just an analytic signal and envelope for the electric field. Moreover, the propagation equation for $\mathcal{E}(z,t)$ appears to be identical to GNLSE [5].

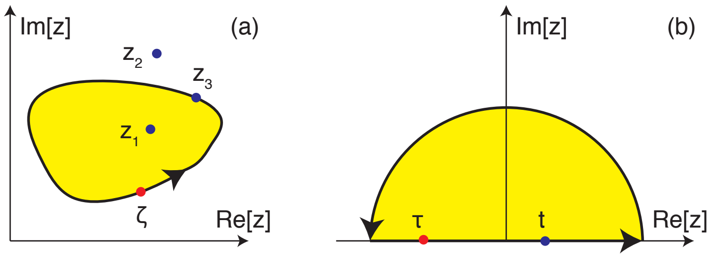

Figure 5: (a) Cauchy integral (101), where $I(z_1) = 2\pi i w(z_1)$ and $I(z_2) = 0$, whereas calculation of $I(z_3) = \pi i w(z_3)$ requires the principal value. (b) A special contour integral (102) that leads to the Hilbert transform (103).

4.6 Conclusions

In conclusions let us summarize the most important results discussed above. We deal with the pulses propagating in optical fibers and use the approach originally developed by V. Zakharov and his coworkers for the continuous Hamiltonian systems. The most important peculiarity is that optical equations are solved with respect to the propagation coordinate, not time. Note, that waves in a fiber can propagate in both directions, as opposed by standard dynamical systems, which evolve forward in time. This difference leads to some changes in the general formalism, e.g., the quadratic part of Hamiltonian looses its standard form $\int_{-\infty}^{\infty} \omega(k) a(k) a^*(k) dk$ and appears as $\int_{-\infty}^{\infty} |\beta(\omega)| \mathcal{A}(\omega) \mathcal{A}^*(\omega) d\omega$. One has an unusual representation of the resonance conditions and, moreover, all integrals of motion get an unusual meaning. The new integrals are determined by the time-averaged fluxes of the relevant physical quantities. However, all core features of the Hamiltonian approach are identical for both systems. This makes possible to develop a new framework for pulse propagation. In particular, one can recognize that the generalized envelope equation, a model of choice in optics, is just a reformulation of the general Hamiltonian equation. In other words, envelope equations can be derived without use of the slowly varying envelope equation and the envelope as such. This explains why GNLSE works better than expected.

5 Appendix: Hilbert transform

In what follows we briefly outline necessary mathematical concepts. The presentation is neither full nor rigorous and by no means is a replacement of standard textbooks. We just fix notations and motivate definitions.

Consider a complex-valued function of a complex variable $w(z) = u(z) + iv(z)$ that is holomorphic in the upper half-plane. Let $w(t) = u(t) + iv(t)$ denote the values of $w(z)$ on the real axis with $t \in \mathbb{R}$. Now consider some domain A in the upper half-plane with a “good” boundary ∂A and Cauchy contour integral (Fig. 5a)

$$I(z) = \oint_{\partial A} \frac{w(\zeta) d\zeta}{\zeta - z}. \tag{101}$$

It is well known that $I(z) = 2\pi i w(z)$ if z is inside A and $I(z) = 0$ if z is outside $A + \partial A$.

If $z \in \partial A$ the integral (101) is singular. One can however define $I(z)$ in the sense of Cauchy principal value and derive that $I(z) = \pi i w(z)$. In a next step we:

1 Let A be a “large” half-disk in the upper half-plane (Fig. 5b);

2 Consider $I(z)$ for $z = t \in \mathbb{R}$ in the sense of Cauchy principal value;

3 Assume that $|w(z)|$ quickly vanishes for $|z| \to \infty$ such that the integral over the half-circle disappears as the latter increases.

We then obtain the following relation

$$u(t) + iv(t) = \frac{1}{\pi i} \fint_{-\infty}^{\infty} \frac{u(\tau) + iv(\tau)}{\tau - t} d\tau, \tag{102}$$

where $\fint$ denotes the principal value. Motivated by this expression we introduce the so-called Hilbert transform $\hat{\mathsf{H}}$ for a suitable complex-valued function $w(t)$ of a real argument

$$w(t) \ \mapsto \ \hat{\mathsf{H}}[w](t) = \frac{1}{\pi} \fint_{-\infty}^{\infty} \frac{w(\tau) d\tau}{\tau - t} \tag{103}$$

Equation (102) simply indicates that

$$\hat{\mathsf{H}}[u + iv] = i(u + iv) \qquad \text{or} \qquad \hat{\mathsf{H}}[u] = -v \qquad \text{and} \qquad \hat{\mathsf{H}}[v] = u. \tag{104}$$

The last two equations are equivalent to Kramers–Kronig relations. It is important to recall that $w(t) = u(t) + iv(t)$ was generated by a function that is holomorphic in the upper half-plane. If one starts with a function holomorphic in the lower half-plane and performs the same steps, one obtains that $\hat{\mathsf{H}}[u + iv] = -i(u + iv)$. In any case $\hat{\mathsf{H}}^2 = -1$. Moreover, a suitable real-valued function $u(t)$ can be decomposed into a sum of two complex-valued functions

$$u = \frac{u + i\hat{\mathsf{H}}[u]}{2} + \frac{u - i\hat{\mathsf{H}}[u]}{2} \tag{105}$$

such that the first term can be extended into the lower half-plane and the second term can be extended into the upper half-plane.

Motivated by Eq. (105) we start with a real pulse field $E(t)$ we define a complex-valued analytic signal $\mathcal{E}(t)$ such that

$$\mathcal{E}(t) = E(t) + i\hat{\mathsf{H}}[E(t)] = E(t) + \frac{i}{\pi} \fint_{-\infty}^{\infty} \frac{E(\tau) d\tau}{\tau - t}. \tag{106}$$

To get a better understanding of Eq. (106) consider Hilbert transform of a single Fourier harmonic $e^{-i\omega t}$. The latter function is holomorphic. It vanishes for $|t| \to \infty$ in the upper half-plane if $\omega < 0$, such that Eq. (104) applies and the action of $\hat{\mathsf{H}}$ is equivalent to multiplication by i. If $\omega > 0$, the action of $\hat{\mathsf{H}}$ is equivalent to multiplication by $-i$, i.e.,

$$\hat{\mathsf{H}}\left[e^{-i\omega t}\right] = \begin{cases} -ie^{-i\omega t}, & \omega > 0, \\ +ie^{-i\omega t}, & \omega < 0. \end{cases} \tag{107}$$

We see that in the frequency domain action of $\hat{\mathsf{H}}$ is equivalent to multiplication by $\mathsf{H}(\omega) = -i\omega/|\omega|$. In particular, $\mathsf{H}(\omega) = \mathsf{H}^*(-\omega)$, i.e., $\hat{\mathsf{H}}$ transforms a real function into a real one. Now it is easy to see that the analytic signal $\mathcal{E}(t)$ of the real field $E(t)$ contains only positive-frequency components

$$E(t) = \int_0^\infty E(\omega)e^{-i\omega t}\frac{d\omega}{2\pi} + \int_{-\infty}^0 E(\omega)e^{-i\omega t}\frac{d\omega}{2\pi} = E^{(+)}(t) + E^{(-)}(t),$$
$$i\hat{\mathsf{H}}[E(t)] = \int_0^\infty E(\omega)e^{-i\omega t}\frac{d\omega}{2\pi} - \int_{-\infty}^0 E(\omega)e^{-i\omega t}\frac{d\omega}{2\pi} = E^{(+)}(t) - E^{(-)}(t),$$
$$\mathcal{E}(t) = E(t) + i\hat{\mathsf{H}}[E(t)] = 2E^{(+)}(t) = \int_0^\infty E(\omega)e^{-i\omega t}\frac{d\omega}{\pi},$$

in accord with the Gabor's rule [28].

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