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Ground states and concentration phenomena for the fractional Schrödinger equation

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Abstract. We consider here solutions of the nonlinear fractional Schrödinger equation

$$\varepsilon^{2s}(-\Delta)^s u + V(x)u = u^p.$$

We show that concentration points must be critical points for V. We also prove that, if the potential V is coercive and has a unique global minimum, then ground states concentrate suitably at such minimal point as ε tends to zero. In addition, if the potential V is radial, then the minimizer is unique provided ε is small.

1. Introduction

In this paper we will study standing waves for a nonlinear differential equation driven by the fractional Laplacian. We will focus on the so-called *fractional Schrödinger equation*

$$\mathrm{i}\hbar\,\frac{\partial\psi}{\partial t} = \hbar^{2s}(-\Delta)^s\psi + V(x)\psi - |\psi|^{p-1}\,\psi \tag{1.1}$$

where $\hbar$ is the Planck constant, $(x,t) \in \mathbb{R}^N \times (0,+\infty)$, $0 < s < 1$, and $V\colon \mathbb{R}^N \to \mathbb{R}$ is an external potential function. The operator $(-\Delta)^s$ is the fractional Laplacian of order s, which, for a function $\varphi \in C_c^\infty$ (here and in the sequel when omitting the space of definition we are meaning $\mathbb{R}^N$) may be defined via Fourier transform:

$$\mathcal{F}(-\Delta)^s\varphi(\xi) = |\xi|^{2s}\widehat{\varphi}(\xi) \qquad \text{for } \xi \in \mathbb{R}^N,$$

where we used the standard notation

$$\widehat{\varphi}(\xi) := \mathcal{F}(\varphi)(\xi) := \frac{1}{(2\pi)^{\frac{N}{2}}}\int_{\mathbb{R}^N} e^{-\imath\xi\cdot x}\varphi(x)dx$$

for the Fourier transform of a function $\varphi \in L^2$. As customary, we will focus on the standing wave situation of equation (1.1), namely on the case in which $\psi(x,t) = u(x)e^{\frac{\mathrm{i}t}{\hbar}}$, with $u \geq 0$: under this further assumption (and replacing $V+1$ with V and $\hbar$ with the small parameter $\varepsilon > 0$), equation (1.1) reduces to

$$\varepsilon^{2s}(-\Delta)^s u + V(x)u - u^p = 0. \tag{1.2}$$

This is the main equation studied in this paper and it will be set in the whole of $\mathbb{R}^N$, with $N > 2s$ and p subcritical[1], namely

$$1 < p < \frac{N+2s}{N-2s}. \tag{1.3}$$

As for the potential V in (1.2), we suppose that is smooth, positive, and bounded from zero, namely we assume that

$$\|V\|_{C^2} < \infty, \qquad \bar{V} = \inf_{\mathbb{R}^N} V > 0. \tag{1.4}$$

The weak formulation of the fractional Laplacian naturally leads to the study of the fractional Sobolev spaces

$$H^s := \left\{u \in L^2 \,:\, \int_{\mathbb{R}^N} |\xi|^{2s}|\widehat{u}|^2\,d\xi < \infty\right\}, \tag{1.5}$$

endowed with the norm

$$\|u\|^2_{H^s} := \|u\|^2_{L^2} + \|u\|^2_{\mathcal{D}^{s,2}},$$

$$\text{where} \qquad \|u\|^2_{\mathcal{D}^{s,2}} := \int_{\mathbb{R}^N} |\xi|^{2s}|\widehat{u}|^2\,d\xi.$$

Notice that all the functional spaces L^2, H^s etc. are set in the whole of $\mathbb{R}^n$ unless explicitly mentioned. In this functional setting, a weak solution of equation (1.2) is a function $u_\varepsilon \in H^s$ such that

$$\varepsilon^{2s}\int_{\mathbb{R}^N} |\xi|^{2s}\widehat{u}_\varepsilon(\xi)\overline{\widehat{\varphi}(\xi)}\,d\xi = \int_{\mathbb{R}^N}\Big(V(x)u_\varepsilon(x) - u_\varepsilon^p(x)\Big)\,\varphi(x)\,dx$$

for any $\varphi \in H^s$. For the existence of weak solutions for special cases of (1.2), see e.g. [4–6, 10, 12, 14, 15, 23]: in this circumstance, the solutions found are indeed positive, bounded and $C^{2,\alpha}$ (see Theorem 3.4 in [14] and Lemma 4.4

[1] When $N \leq 2s$, one can say that p is subcritical when $p \in (1,+\infty)$.

in [3]). In this case, equation (1.2) holds pointwise and the fractional Laplace of u at the point $x \in \mathbb{R}^N$ has the integral representation

$$(-\Delta)^s u(x) = c(n,s) \int_{\mathbb{R}^N} \frac{2u(x) - u(x+y) - u(x-y)}{|y|^{n+2s}}\, dy \tag{1.6}$$

for a suitable $c(n,s) > 0$, see e.g. Proposition 3.3 in [9].

The first result that we provide characterizes the points at which solutions of (1.2) concentrate for small ε, stating that these points are critical for the potential. This is somehow an extension to the nonlocal setting of Wang's result, see [25]. To state this first result, given a sequence of positive solutions u_ε for equation (1.2) in the whole of $\mathbb{R}^N$, we say that $x_0 \in \mathbb{R}^N$ is a strong concentration point for this sequence (or that the sequence u_ε strongly concentrates at x_0) if

$$\begin{gathered}\text{for any } \delta > 0 \text{ there exist } \varepsilon_0 \text{ and } R > 0 \text{ such that, for any } \varepsilon \in (0, \varepsilon_0),\\ u_\varepsilon(x) \le \delta \text{ for all } x \in \mathbb{R}^N \setminus B(x_0, \varepsilon R).\end{gathered} \tag{1.7}$$

With this setting, the following result holds:

Theorem 1.1. *Assume* (1.4) *and let $u_\varepsilon \in H^s$ be a sequence of positive solutions of* (1.2) *in the whole of $\mathbb{R}^N$ that strongly concentrate at x_0. Then $\nabla V(x_0) = 0$.*

We remark that, if we perform a translation and a spacial dilation of factor $1/\varepsilon$, equation (1.2) becomes

$$(-\Delta)^s u + V(\varepsilon x + x_0) u - u^p = 0. \tag{1.8}$$

Thus, to study the concentration phenomena of this equation, it is convenient to define

$$\begin{aligned} & V_\varepsilon(x) := V(\varepsilon x + x_0),\\ & \|u\|^2_{\varepsilon,V} := \int_{\mathbb{R}^N} V_\varepsilon(x)\, u^2(x)\, dx,\\ & \|u\|^2_\varepsilon := \|u\|^2_{\mathcal{D}^{s,2}} + \|u\|^2_{\varepsilon,V},\\ \text{and}\quad & \nu(V_\varepsilon) := \inf_{u \neq 0} \frac{\|u\|^2_\varepsilon}{\|u\|^2_{L^{p+1}}}. \end{aligned}$$

Notice that these definitions also make sense when $\varepsilon = 0$, namely, one has

$$\|u\|^2_{0,V} := V(x_0) \int_{\mathbb{R}^N} u^2(x)\, dx$$

and so on. Moreover, we remark that if u is a minimizer for $\nu(V_\varepsilon)$ then $u_\varepsilon(x) := u((x - x_0)/\varepsilon)$ is a minimizer for

$$\nu_\varepsilon(V) := \varepsilon^{\frac{N\,(1-p)}{1+p}} \inf_{u \neq 0} \frac{\varepsilon^{2s}\|u\|^2_{\mathcal{D}^{s,2}} + \displaystyle\int_{\mathbb{R}^N} V(x)\, u^2(x)\, dx}{\|u\|^2_{L^{p+1}}}.$$

In this setting, we can better determine the variational properties of the concentration point x_0. Namely, while we know from Theorem 1.1 that x_0 is a stationary point for the potential, now we give conditions under which it is a minimum. For this scope, given a sequence of positive solutions u_ε for equation (1.2) in the whole of $\mathbb{R}^N$, we say that $x_0 \in \mathbb{R}^N$ is a weak concentration point for this sequence (or that the sequence u_ε weakly concentrates at x_0) if there exists a sequence of points $x_\varepsilon \to x_0$ such that

$$\begin{gathered}\text{for any } \delta > 0 \text{ there exist } \varepsilon_0 \text{ and } R > 0 \text{ such that, for any } \varepsilon \in (0, \varepsilon_0),\\ u_\varepsilon(x) \le \delta \text{ for all } x \in \mathbb{R}^N \setminus B(x_\varepsilon, \varepsilon R).\end{gathered} \tag{1.9}$$

By comparing (1.7) and (1.9), we notice that strong concentration implies weak concentration (by choosing $x_\varepsilon := x_0$) for every ε. Then, the following result holds:

Theorem 1.2. *Suppose that V has a unique global minimum point and that u_ε is a minimizer for $\nu_\varepsilon(V)$. Assume in addition that V at infinity stays above such minimal value, i.e.*

$$\liminf_{|x| \to +\infty} V(x) > \min_{\mathbb{R}^n} V. \tag{1.10}$$

Then u_ε weakly concentrates at the global minimum point x_0 of V. More precisely, the point x_ε in (1.9) *is the unique global maximum point of u_ε.*

We emphasize that, in the above theorem, an additional complication is that the nonlocal operator $(-\Delta)^s$ does not "see" local maximum points. Namely if y_ε is a local maximum point for u_ε, it is not necessarily true that $(-\Delta)^s u_\varepsilon(y_\varepsilon) \geq 0$ (and, as a matter of fact, the "local" behavior of "nonlocal" equations can be very wild: for instance all functions are locally s-harmonic up to an arbitrarily small error, see [11]). This feature makes the proof of the uniqueness of the global maximum point of u_ε more delicate than in the classical case. About characterization of concentration sets for minimizers of singular perturbation problems we refer the reader to [1, 7, 19–22, 25] and some references therein.

Next result establishes a uniqueness property for the minimizers:

Theorem 1.3. *Assume that $V \in C^1(\mathbb{R}^N)$, with $\inf_{\mathbb{R}^N} V > 0$ and it is radial. Let v_ε be a minimizer for $\nu_\varepsilon(V)$. Then v_ε is unique, provided that ε small enough.*

The rest of the paper is organized as follows. In Section 2 we study the concentration phenomena at given points of the space and we prove Theorem 1.1. The proof of Theorem 1.2 requires some preliminary work, that is carried out in Section 3. In particular, we obtain there an expansion of the minimizers of $\nu(V_\varepsilon)$ as perturbation of a suitable translation of the ground state (for this, no condition on the concentration point is required).

The proof of Theorem 1.2 is then completed in Section 4. Then, Section 5 contains the preliminaries needed for the proof of Theorem 1.3, which, in turn, will be completed in Section 7.

2. Concentrations occurring at critical points of V and proof of Theorem 1.1

In this section, we prove Theorem 1.1. We define

$$v_\varepsilon(x) := u_\varepsilon(\varepsilon x + x_0).$$

By construction, v_ε is a positive solution of

$$(-\Delta)^s v_\varepsilon + V(\varepsilon x + x_0) v_\varepsilon - v_\varepsilon^p = 0 \qquad \text{in } \mathbb{R}^N. \tag{2.1}$$

Roughly speaking, the idea is to take the derivative of (2.1), test it against v_ε, integrate by parts and hence send $\varepsilon \to 0$, in order to see that $\nabla V(x_0) = 0$: but to do these steps, some uniform regularity and decay estimates in ε are in order. To obtain these estimates, we define

$$m_\varepsilon := \max_{\mathbb{R}^N} v_\varepsilon = \|u_\varepsilon\|_{L^\infty}.$$

We claim that

$$m := \sup_{\varepsilon \in (0,1)} m_\varepsilon < +\infty. \tag{2.2}$$

The proof is based on a classical contradiction and scaling arguments. Namely, suppose that

$$m_\varepsilon \to +\infty, \tag{2.3}$$

up to a subsequence. Now we recall (1.4) and we use (1.7) with

$$\delta := \min\left\{1, \left(\frac{\bar{V}}{2}\right)^{\frac{1}{p-1}}\right\}.$$

Accordingly, we obtain that there exists $R_1 > 0$ for which

$$u_\varepsilon(x) \leq \min\left\{1, \left(\frac{\bar{V}}{2}\right)^{\frac{1}{p-1}}\right\} \quad \text{for any } y \in \mathbb{R}^N \text{ such that } |y - x_0| \geq \varepsilon R_1, \tag{2.4}$$

as long as ε is small enough. Now we notice that if $x \in \mathbb{R}^N \setminus B(0, R_1)$ then $|(\varepsilon x + x_0) - x_0| \geq \varepsilon R_1$: hence (2.4) implies that

$$v_\varepsilon(x) \leq \min\left\{1, \left(\frac{\bar{V}}{2}\right)^{\frac{1}{p-1}}\right\} \quad \text{for any } x \in \mathbb{R}^N \setminus B(0, R_1). \tag{2.5}$$

From (2.3) and (2.5), we conclude that, for small ε,

$$1 < m_\varepsilon = \max_{B(0,R_1)} v_\varepsilon,$$

and there exists

$$x_\varepsilon \in \overline{B(0, R_1)} \tag{2.6}$$

maximizing v_ε, that is

$$m_\varepsilon = \max_{\mathbb{R}^N} v_\varepsilon = v_\varepsilon(x_\varepsilon).$$

So, we set $\mu_\varepsilon := m_\varepsilon^{\frac{1-p}{2s}}$ and $w_\varepsilon(x) := m_\varepsilon^{-1} v_\varepsilon(x_\varepsilon + \varepsilon\mu_\varepsilon x)$. Then $\|w_\varepsilon\|_{L^\infty} = 1 = w_\varepsilon(0)$ and

$$(-\Delta)^s w_\varepsilon(x) \;= -\,\mu_\varepsilon^{2s} V(\varepsilon(x_\varepsilon + \varepsilon\mu_\varepsilon))\, w_\varepsilon(x) + w_\varepsilon^p(x). \tag{2.7}$$

Notice that $\mu_\varepsilon \to 0$ as $\varepsilon \to 0$, thanks to (2.3). Therefore, by (2.7), we have that $\|(-\Delta)^s w_\varepsilon\|_{L^\infty}$ is bounded uniformly in ε. As a consequence of this and of the regularity results (see e.g. Lemma 4.4 in [3], see also [24]), we deduce that $\|w_\varepsilon\|_{C^{2,\alpha}}$ is bounded uniformly in ε, for some $\alpha \in (0,1)$. Hence, we can suppose that w_ε converges to some function w_0 in $C^{2,\alpha}_{loc}$, with

$$\|w_0\|_{L^\infty} = 1 = w_0(0). \tag{2.8}$$

By passing to the limit in (2.7), we obtain that

$$(-\Delta)^s w_0 = w_0^p \;\text{ in } \mathbb{R}^N. \tag{2.9}$$

Since the only non-negative and bounded solution of (2.9) with p subcritical (according to (1.3)) is the one constantly equal to zero (see Remark 1.2 in [17] or Theorem 1.3 in [13]), we conclude that w_0 vanishes identically, in contradiction with (2.8).

This completes the proof of (2.2). As a consequence of (2.1), (2.2) and of the regularity results (see e.g. Lemma 4.4 in [3]), we conclude that

$$\|v_\varepsilon\|_{C^{2,\alpha}} \text{ is bounded uniformly in } \varepsilon, \tag{2.10}$$

hence we may suppose that

$$v_\varepsilon \text{ converges to some function } v_0 \text{ in } C^{2,\alpha}_{loc}. \tag{2.11}$$

Now, since x_ε maximizes v_ε, we have that

$$2v_\varepsilon(x_\varepsilon) - v_\varepsilon(x_\varepsilon + y) - v_\varepsilon(x_\varepsilon - y) \geq 0 \qquad \text{for any } y \in \mathbb{R}^N,$$

and so, using (1.6) and (2.1),

$$0 \leq (-\Delta)^s v_\varepsilon(x_\varepsilon) = v_\varepsilon^p(x_\varepsilon) - V(\varepsilon x_\varepsilon + x_0) v_\varepsilon(x_\varepsilon).$$

Accordingly,

$$V(\varepsilon x_\varepsilon + x_0) \leq v_\varepsilon^{p-1}(x_\varepsilon). \tag{2.12}$$

Since $|x_\varepsilon|$ is bounded uniformly in ε, in light of (2.6), we suppose, up to a subsequence, that $x_\varepsilon \to \bar{x}$, for some $\bar{x} \in \overline{B(0, R_1)}$, as $\varepsilon \to 0$. Thus, by taking the limit as $\varepsilon \to 0$ in (2.12), we obtain that

$$0 < \bar{V} \leq V(x_0) \leq v_0^{p-1}(\bar{x}).$$

In particular,

$$v_0 \text{ is not identically zero.} \tag{2.13}$$

Next we claim that there exists $\varepsilon_0 > 0$ such that

$$v_\varepsilon(x) \leq \frac{Const}{1 + |x|^{N+2s}} \qquad \forall \varepsilon \in (0, \varepsilon_0). \tag{2.14}$$

To prove this, we use Lemma 4.2 of [14], according to which there exists a function $\bar{w}$ such that

$$0 \leq \bar{w}(x) \leq \frac{Const}{1 + |x|^{N+2s}} \tag{2.15}$$

and

$$(-\Delta)^s \bar{w} + \frac{\bar{V}}{2} w \geq 0 \;\text{ in } \mathbb{R}^N \setminus B(0, \tilde{R}), \tag{2.16}$$

for a suitable $\tilde{R} > 0$. Now, we take

$$R_2 := \min\{R_1, \tilde{R}\}, \tag{2.17}$$

where R_1 is the one in (2.5). Thanks to (1.7), we have that v_ε converges to zero as $|x| \to \infty$ uniformly with respect to ε. From (2.5), we obtain

$$\begin{aligned}(-\Delta)^s v_\varepsilon + \frac{\bar{V}}{2} v_\varepsilon &= (-\Delta)^s v_\varepsilon + V v_\varepsilon - \left(V - \frac{\bar{V}}{2}\right) v_\varepsilon \\ &= v_\varepsilon^p - \left(V - \frac{\bar{V}}{2}\right) v_\varepsilon \leq v_\varepsilon^p - \frac{\bar{V}}{2} v_\varepsilon = v_\varepsilon \left(v_\varepsilon^{p-1} - \frac{\bar{V}}{2}\right) \leq 0.\end{aligned} \tag{2.18}$$

Now we set

$$b := \inf_{B(0,R_2)} \bar{w} > 0 \tag{2.19}$$

and

$$z_\varepsilon := (m+1)\bar{w} - bv_\varepsilon, \tag{2.20}$$

where m is given in (2.2). Our goal is to show that

$$z_\varepsilon \geq 0 \quad \text{in } \mathbb{R}^N. \tag{2.21}$$

For this we argue by contradiction and suppose that

$$0 > \inf_{\mathbb{R}^N} z_\varepsilon = \lim_{j\to+\infty} z_\varepsilon(x_{j,\varepsilon}), \tag{2.22}$$

for a suitable sequence $x_{j,\varepsilon}$. Notice that

$$\lim_{|x|\to+\infty} \bar{w}(x) = 0,$$

due to (2.15), and

$$\lim_{|x|\to+\infty} u_\varepsilon(x) = 0,$$

due to our integrability and continuity assumptions on u_ε, and therefore

$$\lim_{|x|\to+\infty} v_\varepsilon(x) = 0,$$

and so

$$\lim_{|x|\to+\infty} z_\varepsilon(x) = 0.$$

Consequently, the sequence $x_{j,\varepsilon}$ is bounded and therefore, up to subsequence, we suppose that $x_{j,\varepsilon} \to x_{\star,\varepsilon}$ as $j \to +\infty$, for some $x_{\star,\varepsilon} \in \mathbb{R}^N$. So (2.22) becomes

$$0 > \min_{\mathbb{R}^N} z_\varepsilon = z_\varepsilon(x_{\star,\varepsilon}). \tag{2.23}$$

The minimality property of $x_{\star,\varepsilon}$ and (1.6) give that

$$(-\Delta)^s z_\varepsilon(x_{\star,\varepsilon}) = c(n,s) \int_{\mathbb{R}^N} \frac{2z_\varepsilon(x_{\star,\varepsilon}) - z_\varepsilon(x_{\star,\varepsilon}+y) - z_\varepsilon(x_{\star,\varepsilon}-y)}{|y|^{n+2s}}\, dy \leq 0. \tag{2.24}$$

Now notice that, by (2.2) and (2.19),

$$z_\varepsilon \geq mb + \bar{w} - bm > 0 \quad \text{in } B(0,R_2).$$

Comparing this with (2.23), we see that

$$x_{\star,\varepsilon} \in \mathbb{R}^N \setminus B(0,R_2). \tag{2.25}$$

Moreover, from (2.16), (2.17) and (2.18), we obtain that

$$(-\Delta)^s z_\varepsilon + \frac{\bar{V}}{2} z_\varepsilon \geq 0 \quad \text{in } \mathbb{R}^N \setminus B(0,R_2). \tag{2.26}$$

Thanks to (2.25), we can evaluate (2.26) at the point $x_{\star,\varepsilon}$: in this way, and recalling (2.23) and (2.24), we obtain that

$$0 \leq (-\Delta)^s z_\varepsilon(x_{\star,\varepsilon}) + \frac{\bar{V}}{2} z_\varepsilon(x_{\star,\varepsilon}) < 0.$$

This is a contradiction, so (2.21) is established.

From (2.21), we deduce that $v_\varepsilon \leq (m+1)b^{-1}\bar{w}$, which, together with (2.15), completes the proof of (2.14).

Using (2.11) and (2.14) and the dominated convergence theorem, we see that

$$\lim_{\varepsilon\to0} \int_{\mathbb{R}^N} \partial_i V(\varepsilon x + x_0) v_\varepsilon^2 = \partial_i V(x_0) \int_{\mathbb{R}^N} v_0^2, \tag{2.27}$$

for any $i \in \{1,\dots,N\}$.

Now we show that

$$|\nabla v_\varepsilon| \in L^2. \tag{2.28}$$

For this, we write (2.1) as

$$(-\Delta)^s v_\varepsilon = -V(\varepsilon x + x_0)\, v_\varepsilon + v_\varepsilon^p =: \psi_\varepsilon. \tag{2.29}$$

We know from (2.14) that

$$\psi_\varepsilon(x) \le \frac{Const}{1+|x|^{N+2s}}.$$

Thus we take the fundamental solution Γ of the operator in (2.29) and we obtain that

$$v_\varepsilon(x) = Const \int_{\mathbb{R}^N} \frac{\psi_\varepsilon(y)}{|x-y|^{N-2s}}\,dy.$$

Therefore

$$|\nabla v_\varepsilon(x)| \le Const \int_{\mathbb{R}^N} \frac{|\psi_\varepsilon(y)|}{|x-y|^{N-2s+1}}\,dy \le \int_{\mathbb{R}^N} \frac{Const}{(1+|y|^{N+2s})\,|x-y|^{N-2s+1}}\,dy.$$

So, fixing $x \in \mathbb{R}^N \setminus B(0,2)$, we observe that

$$\int_{B(x,1)} \frac{|\psi_\varepsilon(y)|}{|x-y|^{N-2s+1}}\,dy \le \frac{Const}{|x|^{N+2s}} \int_{B(0,1)} \frac{1}{|\xi|^{N-2s+1}}\,d\xi \le \frac{Const}{|x|^{N+2s}},$$

therefore we obtain that

$$\begin{aligned} |\nabla v_\varepsilon(x)| &\le Const \left[\int_{\mathbb{R}^N\setminus B(x,1)} \frac{1}{(1+|y|^{N+2s})\,|x-y|^{N-2s+1}}\,dy + \frac{1}{|x|^{N+2s}}\right] \\ &\le Const \left[\int_{\mathbb{R}^N\setminus B(x,1)} \frac{1}{(1+|y|^{N+2s})\,(1+|x-y|^{N-2s+1})}\,dy + \frac{1}{|x|^{N+2s}}\right]. \end{aligned}$$

Accordingly, by the properties of the convolution of decaying kernels (see e.g. Lemma 5.1 in [8]), we obtain that

$$|\nabla v_\varepsilon(x)| \le \frac{Const}{|x|^{\kappa}}, \tag{2.30}$$

with $\kappa := \min\{N+2s,\ N-2s+1\}$. Notice that

$$2\kappa = \min\{2N+4s,\ N+N-4s+2\} > \min\{2N,\ 2s+N-4s+2\} > N,$$

hence (2.30) implies (2.28), as desired.

Now we perform some calculations on integrals that involve v_ε. For this, we let e_i be the ith vector of the standard Euclidean base, we fix $R>1$ and we use the divergence theorem to see that, for any $\in \{1,\dots,N\}$,

$$\int_{B(0,R)} \partial_i v_\varepsilon^{p+1} = \int_{B(0,R)} \operatorname{div}(v_\varepsilon^{p+1} e_i) = \int_{\partial B(0,R)} v_\varepsilon^{p+1} \frac{x_i}{R}.$$

Thus, from (2.14), we have

$$\int_{B(0,R)} \partial_i v_\varepsilon^{p+1} = O(R^{N-1-(p+1)(N+2s)}). \tag{2.31}$$

Similarly,

$$\begin{aligned} \int_{B(0,R)} V(\varepsilon x+x_0)\partial_i v_\varepsilon^2 &= \int_{B(0,R)} \Big[\operatorname{div}\Big(V(\varepsilon x+x_0)v_\varepsilon^2 e_i\Big) - \varepsilon \partial_i V(\varepsilon x+x_0) v_\varepsilon^2\Big] \\ &= \int_{\partial B(0,R)} V(\varepsilon x+x_0) v_\varepsilon^2 \frac{x_i}{R} - \varepsilon \int_{B(0,R)} \partial_i V(\varepsilon x+x_0) v_\varepsilon^2 \end{aligned}$$

which, together with (2.14), gives that

$$\int_{B(0,R)} V(\varepsilon x+x_0)\partial_i v_\varepsilon^2 = -\varepsilon \int_{B(0,R)} \partial_i V(\varepsilon x+x_0) v_\varepsilon^2 + O(R^{N-1-2(N+2s)}). \tag{2.32}$$

We summarize the estimates in (2.31) and (2.32) by writing

$$\begin{aligned} &\int_{B(0,R)} \Big(\varepsilon \partial_i V(\varepsilon x+x_0) v_\varepsilon^2 + \frac{1}{2} V(\varepsilon x+x_0)\partial_i v_\varepsilon^2 - \frac{p}{p+1}\partial_i v_\varepsilon^{p+1}\Big) \\ &\quad = \frac{\varepsilon}{2}\int_{B(0,R)} \partial_i V(\varepsilon x+x_0) v_\varepsilon^2 + O(R^{N-1-2(N+2s)}). \end{aligned} \tag{2.33}$$

Now, we point out that $(-\Delta)^s v_\varepsilon$ is C^2 and bounded, due to (1.4), (2.1), (2.2) and (2.10) (recall also (1.4), therefore we can speak about $\partial_i(-\Delta)^s v_\varepsilon$ in the classical sense. Accordingly, we can take a derivative, say in the ith coordinate direction, of (2.1): we get

$$\partial_i(-\Delta)^s v_\varepsilon + \varepsilon \partial_i V(\varepsilon x + x_0) v_\varepsilon + V(\varepsilon x + x_0)\partial_i v_\varepsilon - p v_\varepsilon^{p-1}\partial_i v_\varepsilon = 0. \tag{2.34}$$

So, recalling (2.10), (2.14) and (2.28), we see that

$$\partial_i(-\Delta)^s v_\varepsilon \in L^2. \tag{2.35}$$

Consequently, by Plancherel theorem, we obtain

$$\begin{aligned}\int_{\mathbb{R}^N} v_\varepsilon \partial_i(-\Delta)^s v_\varepsilon &= \int_{\mathbb{R}^N} \overline{\widehat{v_\varepsilon}}\, \mathcal{F}(\partial_i(-\Delta)^s v_\varepsilon)\\ &= -\int_{\mathbb{R}^N} \xi_i \overline{\widehat{v_\varepsilon}}\, \mathcal{F}((-\Delta)^s v_\varepsilon) = -\int_{\mathbb{R}^N} \xi_i |\xi|^{2s} |\widehat{v_\varepsilon}(\xi)|^2\\ &= -\int_{\mathbb{R}^N} \xi_i \big|\mathcal{F}\big((-\Delta)^{s/2} v_\varepsilon\big)\big|^2.\end{aligned} \tag{2.36}$$

We remark that

$$(-\Delta)^{s/2} v_\varepsilon \in L^2. \tag{2.37}$$

Indeed, since $u_\varepsilon \in H^s$ (hence $v_\varepsilon \in H^s$), we have that

$$\int_{\mathbb{R}^N} |\xi|^{2s} |\widehat{v_\varepsilon}|^2 \le \int_{\mathbb{R}^N} (1+|\xi|^{2s}) |\widehat{v_\varepsilon}|^2 < +\infty,$$

therefore $\mathcal{F}((-\Delta)^{s/2} v_\varepsilon) = |\xi|^s \widehat{v_\varepsilon} \in L^2$ and so (2.37) follows from the Plancherel theorem.

Also, for any $g \in L^2$, we have that

$$\text{the map } \xi \mapsto |\widehat{g}(\xi)|^2 \text{ is even.} \tag{2.38}$$

To check this, notice that

$$\overline{\widehat{g}(\xi)} = \overline{\int_{\mathbb{R}^N} g(x) e^{-\mathrm{i}x\cdot\xi}\, dx} = \int_{\mathbb{R}^N} g(x) e^{\mathrm{i}x\cdot\xi}\, dx = \widehat{g}(-\xi)$$

and so

$$\overline{\widehat{g}(-\xi)} = \widehat{g}(\xi).$$

As a consequence

$$|\widehat{g}(-\xi)|^2 = \overline{\widehat{g}(-\xi)}\widehat{g}(-\xi) = \widehat{g}(\xi)\overline{\widehat{g}(\xi)} = |\widehat{g}(\xi)|^2,$$

that proves (2.38).

So, from (2.37) and (2.38), we see that the map $\xi \mapsto \xi_i\, |\mathcal{F}((-\Delta)^{s/2} v_\varepsilon)|^2$ is odd, and therefore

$$\int_{B(0,R)} \xi_i \,\big|\mathcal{F}\big((-\Delta)^{s/2} v_\varepsilon\big)\big|^2 = 0$$

for any $R > 0$. By plugging this into (2.36) and recalling (2.35) we obtain

$$\int_{\mathbb{R}^N} v_\varepsilon \partial_i(-\Delta)^s v_\varepsilon = \lim_{R\to+\infty} \int_{B(0,R)} v_\varepsilon \partial_i(-\Delta)^s v_\varepsilon = 0. \tag{2.39}$$

Now we go back to (2.34) and we multiply this equation by v_ε: in this way we obtain that

$$\begin{aligned}0 &= v_\varepsilon \partial_i(-\Delta)^s v_\varepsilon + \varepsilon \partial_i V(\varepsilon x + x_0) v_\varepsilon^2 + V(\varepsilon x + x_0) v_\varepsilon \partial_i v_\varepsilon - p v_\varepsilon^p \partial_i v_\varepsilon\\ &= v_\varepsilon \partial_i(-\Delta)^s v_\varepsilon + \varepsilon \partial_i V(\varepsilon x + x_0) v_\varepsilon^2 + \frac{1}{2} V(\varepsilon x + x_0)\partial_i v_\varepsilon^2 - \frac{p}{p+1}\partial_i v_\varepsilon^{p+1}.\end{aligned}$$

We fix $R > 1$ and we integrate the above equation on $B(0,R)$: thus, exploiting (2.33) we obtain

$$\int_{B(0,R)} v_\varepsilon \partial_i(-\Delta)^s v_\varepsilon + \frac{\varepsilon}{2}\int_{B(0,R)} \partial_i V(\varepsilon x + x_0) v_\varepsilon^2 = O(R^{N-1-2(N+2s)}).$$

So we send $R \to +\infty$, recalling also (2.39) and we divide by ε, we get

$$\int_{\mathbb{R}^N} \partial_i V(\varepsilon x + x_0) v_\varepsilon^2 = 0.$$

Now we send $\varepsilon \to 0$: recalling (2.27) we conclude that

$$\partial_i V(x_0) \int_{\mathbb{R}^N} v_0^2 = 0.$$

Therefore, by (2.13), we obtain that $\partial_i V(x_0) = 0$ for any $i \in \{1, \dots, N\}$, and this completes the proof of Theorem 1.1. $\square$

3. Concentration points of ground-states: preliminary work for the proof of Theorem 1.2

In this section we discuss some basic concentration properties of the minimizers. For this, we recall that, for any $\lambda > 0$ there exists a unique function U_λ that attains the following minimization problem

$$\nu(\lambda) := \inf_{\|u\|_{L^{p+1}}=1} \|u\|^2_{\mathcal{D}^{s,2}} + \lambda \|u\|^2_{L^2}.$$

In addition, such minimizer is unique radially symmetric and belongs to $C^\infty \cap H^{2s+1}(\mathbb{R}^N)$ (we refer to [16] for further details on this, see in particular Theorem 4 there). Thus, we will denote by $\widetilde{U}$ the radially symmetric function that attains

$$\inf_{\|u\|_{L^{p+1}}=1} \|u\|^2_{\mathcal{D}^{s,2}} + \widetilde{V} \, \|u\|^2_{L^2},$$

where

$$\widetilde{V} := \inf_{\mathbb{R}^N} V. \tag{3.1}$$

With this notation, we provide an asymptotic expansion for the minimizers of $\nu(V_\varepsilon)$. It is worth pointing out that this expansion is valid without assuming any structural condition on the potential V (in particular the point $x_0 \in \mathbb{R}^n$ can be fixed, without assuming that is minimal or critical):

Lemma 3.1. *Let v_ε be a positive minimizer for $\nu(V_\varepsilon)$, with $\|v_\varepsilon\|_{L^{p+1}} = 1$. Then there exists a sequence of points $a_\varepsilon, c \in \mathbb{R}^N$ such that, up to a subsequence,*

$$\begin{aligned} & v_\varepsilon(x + a_\varepsilon) = \widetilde{U}(x - c) + \omega_\varepsilon(x), \\ \textit{with} \qquad & \|\omega_\varepsilon\|_{H^s} \to 0 \quad \textit{as} \quad \varepsilon \to 0. \end{aligned}$$

Also

$$\lim_{\varepsilon \to 0} \nu(V_\varepsilon) = \nu(\widetilde{V})$$

and, for any $x \in \mathbb{R}^n$,

$$\lim_{\varepsilon \to 0} V(\varepsilon x + \varepsilon a_\varepsilon + x_0) = \widetilde{V}. \tag{3.2}$$

Proof. We observe that

$$\|u\|^2_{\varepsilon,V} \in \left[\bar{V} \, \|u\|^2_{L^2}, \ \|V\|_{L^\infty} \|u\|^2_{L^2} \right]$$

thanks to (1.4), and therefore $\nu(V_\varepsilon)$ is bounded (and bounded from zero) uniformly in ε. Hence, up to a subsequence, we suppose that

$$\nu(V_\varepsilon) \to \widetilde{\nu} \tag{3.3}$$

as $\varepsilon \to 0$, for some $\widetilde{\nu} > 0$.

Also, v_ε is bounded in H^s and, using Lemma 2.2 in [14], we have that there exists $a_\varepsilon \in \mathbb{R}^N$ and positive real numbers R and γ such that

$$\liminf_{\varepsilon \to 0} \int_{B_R(a_\varepsilon)} v_\varepsilon(x) \, dx \geq \gamma. \tag{3.4}$$

Thus, setting $w_\varepsilon(x) = v_\varepsilon(x + a_\varepsilon)$, we have that w_ε is bounded in H^s so it converges, up to a subsequence, to a function $w \in H^s$ weakly in H^s, strongly in L^{p+1}_{loc} and a.e.; furthermore, by (3.4), we have that $w \neq 0$.

We also notice that, by (1.4) and the theorem of Ascoli, there exists $\lambda : \mathbb{R}^N \to \mathbb{R}$ such that, up to a subsequence,

$$\lim_{\varepsilon \to 0} V(\varepsilon x + \varepsilon a_\varepsilon + x_0) = \lambda(x). \tag{3.5}$$

We set $\lambda := \lambda(0)$ and we claim that

$$\lambda(x) = \lambda \tag{3.6}$$

for any $x \in \mathbb{R}^N$. Indeed, for any $x \in \mathbb{R}^N$,

$$\begin{aligned}|\lambda(x) - \lambda| &= \lim_{\varepsilon\to 0} \left|V(\varepsilon x + \varepsilon a_\varepsilon + x_0) - V(\varepsilon a_\varepsilon + x_0)\right| \\ &\le Const \lim_{\varepsilon\to 0} |\varepsilon x| = 0,\end{aligned}$$

thanks to (1.4), and this proves (3.6).

By (3.5) and (3.6), we can write

$$\lim_{\varepsilon\to 0} V(\varepsilon x + \varepsilon a_\varepsilon + x_0) = \lambda. \tag{3.7}$$

Since, by (2.1),

$$(-\Delta)^s w_\varepsilon + V(\varepsilon x + \varepsilon a_\varepsilon + x_0) w_\varepsilon = \nu(V_\varepsilon) w_\varepsilon^p,$$

we can pass to the limit and obtain

$$(-\Delta)^s w + \lambda w = \widetilde{\nu} w^p. \tag{3.8}$$

By testing (3.8) against w we obtain that

$$\widetilde{\nu} = \frac{\|w\|^2_{\mathcal{D}^{s,2}} + \lambda \|w\|^2_{L^2}}{\|w\|^2_{L^{p+1}}} \ge \inf_{u\neq 0} \frac{\|u\|^2_{\mathcal{D}^{s,2}} + \lambda \|u\|^2_{L^2}}{\|u\|^2_{L^{p+1}}} = \nu(\lambda). \tag{3.9}$$

On the other hand, by the dominated convergence theorem, we see that, for any $u \in C_c^\infty$,

$$\lim_{\varepsilon\to 0} \int_{\mathbb{R}^N} \left|V(\varepsilon x + \varepsilon a_\varepsilon + x_0) - \lambda\right| u^2(x)\, dx = 0.$$

As a consequence, for any $u \in C_c^\infty$, $u \neq 0$, we set $\tilde{u}_\varepsilon(x) := u(x - a_\varepsilon)$ and we observe that

$$\begin{aligned}\widetilde{\nu} &= \lim_{\varepsilon\to 0} \nu(V_\varepsilon) \\ &\le \frac{\|u\|^2_{\mathcal{D}^{s,2}} + \|u\|^2_{\varepsilon,V}}{\|u\|^2_{L^{p+1}}} \\ &= \lim_{\varepsilon\to 0} \frac{\|\tilde{u}_\varepsilon\|^2_{\mathcal{D}^{s,2}} + \int_{\mathbb{R}^N} V(\varepsilon x + x_0)\, \tilde{u}_\varepsilon^2(x)\, dx}{\|\tilde{u}_\varepsilon\|^2_{L^{p+1}}} \\ &= \lim_{\varepsilon\to 0} \frac{\|u\|^2_{\mathcal{D}^{s,2}} + \int_{\mathbb{R}^N} V(\varepsilon x + x_0)\, u^2(x - a_\varepsilon)\, dx}{\|u\|^2_{L^{p+1}}} \\ &= \lim_{\varepsilon\to 0} \frac{\|u\|^2_{\mathcal{D}^{s,2}} + \int_{\mathbb{R}^N} V(\varepsilon x + \varepsilon a_\varepsilon + x_0)\, u^2(x)\, dx}{\|u\|^2_{L^{p+1}}} \\ &= \frac{\|u\|^2_{\mathcal{D}^{s,2}} + \lambda \|u\|^2_{L^2}}{\|u\|^2_{L^{p+1}}}.\end{aligned}$$

By density, this is valid for any $u \in H^s$, and so, taking the infimum over $u \neq 0$, we obtain that $\widetilde{\nu} \le \nu(\lambda)$. This and (3.9) give that

$$\widetilde{\nu} = \nu(\lambda). \tag{3.10}$$

This, (3.8) and the uniqueness of the ground state (see Theorem 4 in [16]) give that w is a translation of U_λ, namely $w(x) = U_\lambda(x - c)$, for some $c \in \mathbb{R}^N$.

Now we claim that

$$\lambda = \widetilde{V} \tag{3.11}$$

To prove this, let us fix $p \in \mathbb{R}^n$. Then, for any $u \in C_c^\infty$, we set $\underline{u}_\varepsilon(x) := u(x + \varepsilon^{-1}(x_0 - p))$ and we use the change of variable $y := x + \varepsilon^{-1}(p - x_0)$ to obtain that

$$\begin{aligned}
& \frac{\|u\|^2_{\mathcal{D}^{s,2}} + \displaystyle\int_{\mathbb{R}^N} V(\varepsilon x + p)\, u^2(x)\, dx}{\|u\|^2_{L^{p+1}}} \\
= & \frac{\|u\|^2_{\mathcal{D}^{s,2}} + \displaystyle\int_{\mathbb{R}^N} V(\varepsilon y + x_0)\, u^2(y + \varepsilon^{-1}(x_0 - p))\, dy}{\|u\|^2_{L^{p+1}}} \\
= & \frac{\|\underline{u}_\varepsilon\|^2_{\mathcal{D}^{s,2}} + \displaystyle\int_{\mathbb{R}^N} V(\varepsilon y + x_0)\, \underline{u}_\varepsilon^2(y)\, dy}{\|\underline{u}_\varepsilon\|^2_{L^{p+1}}} \\
= & \frac{\|\underline{u}_\varepsilon\|^2_{\varepsilon,V}}{\|\underline{u}_\varepsilon\|^2_{L^{p+1}}} \\
\geq & \ \nu(V_\varepsilon).
\end{aligned}$$

So, by (3.3), (3.10) and the dominated convergence theorem, we obtain

$$\begin{aligned}
\frac{\|u\|^2_{\mathcal{D}^{s,2}} + V(p)\|u\|^2_{L^2}}{\|u\|^2_{L^{p+1}}} &= \lim_{\varepsilon \to 0} \frac{\|u\|^2_{\mathcal{D}^{s,2}} + \displaystyle\int_{\mathbb{R}^N} V(\varepsilon x + p)\, u^2(x)\, dx}{\|u\|^2_{L^{p+1}}} \\
&\geq \lim_{\varepsilon \to 0} \nu(V_\varepsilon) = \widetilde{\nu} = \nu(\lambda) = \frac{\|U_\lambda\|^2_{\mathcal{D}^{s,2}} + \lambda\|U_\lambda\|^2_{L^2}}{\|U_\lambda\|^2_{L^{p+1}}}.
\end{aligned}$$

This is valid for any $u \in C_c^\infty$ and so, by density, also for U_λ. Thus we conclude that

$$\frac{\|U_\lambda\|^2_{\mathcal{D}^{s,2}} + V(p)\|U_\lambda\|^2_{L^2}}{\|U_\lambda\|^2_{L^{p+1}}} \geq \frac{\|U_\lambda\|^2_{\mathcal{D}^{s,2}} + \lambda\|U_\lambda\|^2_{L^2}}{\|U_\lambda\|^2_{L^{p+1}}}$$

and therefore

$$V(p) \geq \lambda.$$

Now, this is valid for any $p \in \mathbb{R}^N$, thus, recalling (3.1), we obtain that

$$\widetilde{V} = \inf_{p \in \mathbb{R}^N} V(p) \geq \lambda.$$

The other inequality follows from (3.7), and so the proof of (3.11) is complete.

Then, (3.11) and the definition of $\widetilde{U}$ give that $U_\lambda = \widetilde{U}$. Accordingly,

$$v_\varepsilon(x + a_\varepsilon) = w_\varepsilon(x) \to w(x) = U_\lambda(x - c) = \widetilde{U}(x - c)$$

weakly in H^s, strongly in L^{p+1}_{loc} and a.e., so to complete the proof of Lemma 3.1 it only remains to show that the convergence occurs strongly in H^s. To see this, we use the fact that w is a minimizer for the quotient $\nu(\widetilde{V}) = \nu(\lambda) = \widetilde{\nu}$, hence

$$\widetilde{\nu} = \frac{\|w\|^2_{\mathcal{D}^{s,2}} + \lambda\|w\|^2_{L^2}}{\|w\|^2_{L^{p+1}}}. \tag{3.12}$$

On the other hand, by testing (3.8) against w, we obtain that

$$\|w\|^2_{\mathcal{D}^{s,2}} + \lambda\|w\|^2_{L^2} = \widetilde{\nu}\, \|w\|^{p+1}_{L^{p+1}}.$$

By comparing this with (3.12), we conclude that $\|w\|_{L^{p+1}} = 1$. Therefore

$$\begin{aligned}
\|w_\varepsilon\|^2_{\mathcal{D}^{s,2}} + \lambda\|w_\varepsilon\|^2_{L^2} &= \|v_\varepsilon\|^2_{\mathcal{D}^{s,2}} + \lambda\|v_\varepsilon\|^2_{L^2} \\
&= \|v_\varepsilon\|^2_{\mathcal{D}^{s,2}} + \lambda\|v_\varepsilon\|^2_{\varepsilon,V} + \int_{\mathbb{R}^N} \big(\lambda - V(\varepsilon x + x_0)\big)\, v_\varepsilon(x)\, dx \\
&= \nu(V_\varepsilon) + \int_{\mathbb{R}^N} \big(\lambda - V(\varepsilon x + x_0)\big)\, v_\varepsilon(x)\, dx.
\end{aligned}$$

Moreover, by (3.1) and (3.11),

$$\lambda = \inf_{\mathbb{R}^N} V \leq V(\varepsilon x + x_0),$$

thus we obtain that

$$\|w_\varepsilon\|^2_{\mathcal{D}^{s,2}} + \lambda \|w_\varepsilon\|^2_{L^2} \le \nu(V_\varepsilon).$$

So, from the weak convergence and Fatou lemma, passing to the limit we obtain that

$$\begin{aligned}\widetilde{\nu} = \|w\|^2_{\mathcal{D}^{s,2}} + \lambda\|w\|^2_{L^2} &\le \liminf_{\varepsilon\to 0} \|w_\varepsilon\|^2_{\mathcal{D}^{s,2}} + \lambda\|w_\varepsilon\|^2_{L^2} \\ &\le \limsup_{\varepsilon\to 0} \|w_\varepsilon\|^2_{\mathcal{D}^{s,2}} + \lambda\|w_\varepsilon\|^2_{L^2} \le \limsup_{\varepsilon\to 0} \nu(V_\varepsilon) = \widetilde{\nu}.\end{aligned}$$

This gives that

$$\lim_{\varepsilon\to 0} \|w_\varepsilon\|^2_{\mathcal{D}^{s,2}} + \lambda\|w_\varepsilon\|^2_{L^2} = \|w\|^2_{\mathcal{D}^{s,2}} + \lambda\|w\|^2_{L^2}.$$

By making use of this and of the weak convergence of w_ε, we infer that $w_\varepsilon \to w$ in the Hilbert norm $\sqrt{\|\cdot\|^2_{\mathcal{D}^{s,2}} + \lambda\|\cdot\|^2_{L^2}}$.

Since this norm is equivalent to the one in H^s, we have proved that $w_\varepsilon \to w$ in H^s. □

4. Completion of the proof of Theorem 1.2

Now we finish the proof of Theorem 1.2. For this, we suppose that V has a unique global minimum point at x_0 Let u_ε be a minimizer for $\nu_\varepsilon(V)$. Then $v_\varepsilon(x) := u_\varepsilon(x_0 + \varepsilon x)$ is a minimizer for $\nu(V_\varepsilon)$. By Lemma 3.1, there are points $a_\varepsilon, c \in \mathbb{R}^N$ such that, up to a subsequence,

$$w_\varepsilon(x) := v_\varepsilon(x + a_\varepsilon) = \widetilde{U}(x - c) + \omega_\varepsilon(x),$$

where $\|\omega_\varepsilon\|_{H^s} \to 0$, the function $\widetilde{U}$ is a minimizer for $\nu(\widetilde{V})$, and, comparing (3.1) and (3.2), we have that

$$\lim_{\varepsilon\to 0} V(\varepsilon a_\varepsilon + x_0) = \widetilde{V} = \min_{x\in\mathbb{R}^N} V(x) = V(x_0). \tag{4.1}$$

Now we prove that

$$\lim_{\varepsilon\to 0} \varepsilon a_\varepsilon = 0. \tag{4.2}$$

Suppose not, say $|\varepsilon a_\varepsilon| \ge a_0$ for some $a_0 > 0$ and an infinitesimal sequence of ε's. Then $|\varepsilon a_\varepsilon|$ remains bounded, otherwise, by (1.10), the limit in (4.1) would be strictly larger than $V(x_0)$.

Accordingly, there exists an infinitesimal sequence of ε's for which $\varepsilon a_\varepsilon \to \alpha$, for some $\alpha \in \mathbb{R}^N$ with $|\alpha| \ge a_0 > 0$. From this and (4.1), we obtain that

$$V(x_0) = \lim_{\varepsilon\to 0} V(\varepsilon a_\varepsilon + x_0) = V(\alpha + x_0).$$

This contradicts the uniqueness of the minimal point for V, and so it proves (4.2).

Now we claim that

$$\sup_\varepsilon \int_{|x|\ge R} w_\varepsilon^r \, dx \to 0 \quad \text{ as } R \to \infty, \tag{4.3}$$

with[2] $r := \frac{2N}{N-2s}$. To see this, we can assume by contradiction that there exists δ positive and a sequence of $R_n \to \infty$ such that

$$\sup_\varepsilon \int_{|x|\ge R_n} w_\varepsilon^r \, dx \ge \delta \quad \text{ as } n \to \infty,$$

This implies that for a sequence of $\varepsilon_n \to 0$, we have

$$\int_{|x|\ge R_n} w_{\varepsilon_n}^r \, dx \ge \delta \quad \text{ as } n \to \infty.$$

Because w_{ε_n} converges strongly in L^r, we have (see e.g. [2, Theorem 4.9]) that there exists $h \in L^r$ and a subsequence, still denoted by ε_n such that $w_{\varepsilon_n} \le h$ a.e. in $\mathbb{R}^N$. But then

$$0 < \delta \le \int_{|x|\ge R_n} w_{\varepsilon_n}^r \, dx \le \int_{|x|\ge R_n} h^r \, dx \to 0 \text{ as } n \to \infty.$$

This leads to a contradiction. We thus have proved (4.3).

Next we observe that $(-\Delta)^s w_\varepsilon - \nu(\widetilde{V}_\varepsilon) w_\varepsilon^{p-1} w_\varepsilon \le 0$ in $\mathbb{R}^N$. Since $w_\varepsilon^{p-1} \in L^q_{loc}$ for some $q > \frac{N}{2s}$, we deduce from [17, Proposition 2.6] that for any compact set K, we have

$$\max_K w_\varepsilon \le C \int_K w_\varepsilon^r \, dx,$$

[2] If $N \le 2s$, the above definition of r can be replaced by just fixing $r \in (1, +\infty)$.

where r is as above. We therefore conclude from (4.3) that

$$\sup_{\varepsilon} w_\varepsilon(x) \to 0 \quad \text{as } |x| \to \infty.$$

This together with Lemma C.2 in [16] also imply that

$$w_\varepsilon(x) \le \frac{Const}{1+|x|^{N+2s}}. \tag{4.4}$$

By scaling back, we obtain

$$u_\varepsilon(x) = v_\varepsilon\left(\frac{x-x_0}{\varepsilon}\right) = w_\varepsilon\left(\frac{x-x_0-\varepsilon a_\varepsilon}{\varepsilon}\right) \le \frac{Const\ \varepsilon^{N+2s}}{\varepsilon^{N+2s}+|x-x_0-\varepsilon a_\varepsilon|^{N+2s}}. \tag{4.5}$$

It is then clear that u_ε concentrates at x_0 in the sense of (1.9).

Now to prove the last statement of the theorem (u_ε has a unique global maximum point), we observe that $u_\varepsilon \in C^{2,\alpha}_{loc}$ and by (4.5), we have $\lim_{|x|\to\infty} u_\varepsilon(x) = 0$ for every fixed and positive ε. We can therefore let $u(x_\varepsilon) = \max_{\mathbb{R}^N} u_\varepsilon$. Then $(-\Delta)^s u_\varepsilon(x_\varepsilon) \ge 0$ and thus from (1.2) (recalling (1.4)), we deduce that

$$u(x_\varepsilon) \ge \left(\frac{\bar V}{\nu(V)}\right)^{\frac{1}{p-1}} =: C_0.$$

Hence by (4.5), we get

$$C_0 \le \frac{Const\ \varepsilon^{N+2s}}{\varepsilon^{N+2s}+|x_\varepsilon-x_0-\varepsilon a_\varepsilon|^{N+2s}}$$

so that

$$|x_\varepsilon - x_0 - \varepsilon a_\varepsilon| \le C_1\varepsilon. \tag{4.6}$$

From this we conclude, provided $|x-x_\varepsilon| \ge \varepsilon R \ge 2\varepsilon C_1$, that

$$u_\varepsilon(x) \le \frac{Const}{1+R-C_1} \le \frac{Const}{1+R/2}$$

and this completes the proof of concentration of u_ε at x_0.
We now prove the uniqueness of x_ε. Indeed, we observe that

$$\begin{aligned}(-\Delta)^s\omega_\varepsilon = (-\Delta)^s w_\varepsilon - (-\Delta)^s\widetilde U(\cdot - c) &= V(\varepsilon x+\varepsilon a_\varepsilon + x_0)[\widetilde U(\cdot - c) - w_\varepsilon] + [V(x_0) - V(\varepsilon x + \varepsilon a_\varepsilon + x_0)]\widetilde U(\cdot - c)\\ &\quad + [\nu(V_\varepsilon) - \nu(V(x_0))]w_\varepsilon^p + \nu(V(x_0))[w_\varepsilon^p - \widetilde U^p(\cdot - c)].\end{aligned}$$

We rewrite this as

$$(-\Delta)^s\omega_\varepsilon + \beta_\varepsilon(x)\omega_\varepsilon = [V(x_0) - V(\varepsilon x+\varepsilon a_\varepsilon + x_0)]\widetilde U(\cdot - c) + [\nu(V_\varepsilon) - \nu(V(x_0))]w_\varepsilon^p,$$

where we have set

$$\beta_\varepsilon(x) = V(\varepsilon x + \varepsilon a_\varepsilon + x_0) - \frac{\nu(V(x_0))[w_\varepsilon^p - \widetilde U^p(\cdot - c)]}{w_\varepsilon - \widetilde U(\cdot - c)}.$$

By (4.4), we have $|w_\varepsilon^p - \widetilde U^p(\cdot - c)| \le C|w_\varepsilon - \widetilde U(\cdot - c)|$ and thus $|\beta_\varepsilon(x)| \le Const.$. Applying [17, Proposition 2.6], we deduce that $\omega_\varepsilon \to 0$ in $C^{0,\alpha}_{loc}(\mathbb{R}^N)$ for some $\alpha \in (0,1)$. Now by a bootstrap argument and using Proposition 2.1.8 in [24], we conclude that $\omega_\varepsilon = w_\varepsilon - \widetilde U(\cdot - c) \to 0$ in $C^{2,\alpha}_{loc}(\mathbb{R}^N)$ for some $\alpha \in (0,1)$.
We now set $\bar w_\varepsilon(x) = w_\varepsilon(x + \bar x_\varepsilon)$ with $\bar x_\varepsilon = \frac{x_\varepsilon - x_0 - \varepsilon a_\varepsilon}{\varepsilon}$. We notice that 0 is the global maximum point of $\bar w_\varepsilon$ and so we have

$$0 = \nabla \bar w_\varepsilon(0) = \nabla \widetilde U(\bar x_\varepsilon - c) + \nabla\omega_\varepsilon(\bar x_\varepsilon).$$

Recalling that $\widetilde U$ is symmetric decreasing with respect to the origin, that has a unique critical point and also $\omega_\varepsilon \to 0 \in C^{2,\alpha}_{loc}(\mathbb{R}^N)$. Therefore from (4.6) we deduce that

$$|\bar x_\varepsilon - c| \to 0.$$

It is clear that any other global maximum point of $\bar w_\varepsilon$ must stay in a neighborhood of c. We then observe that

$$\bar w_\varepsilon(x) = \widetilde U(x) + \bar\omega_\varepsilon(x),$$

where $\bar\omega_\varepsilon(x) = [\widetilde U(x + \bar x_\varepsilon - c) - \widetilde U(x)] + \omega_\varepsilon(x)$. Since $\widetilde U \in C^\infty(\mathbb{R}^N)$, we obtain $\bar\omega_\varepsilon \to 0 \in C^{2,\alpha}_{loc}$. Now using Lemma 4.2 in [20], we conclude that the only critical point for $\bar w_\varepsilon$ is the origin. □

Remark 4.1. We remark that from the above proof, the minimizers u_ε for $\nu_\varepsilon(V)$ has the following precise form:

$$u_\varepsilon(\varepsilon x + x_\varepsilon) = \widetilde{U}(x) + \bar{\omega}_\varepsilon(x),$$

where $\bar{\omega}_\varepsilon \to 0$ in $H^s(\mathbb{R}^N) \cap C^{2,\alpha}_{loc}(\mathbb{R}^N) \cap L^\infty(\mathbb{R}^N)$ with x_ε the unique global maximizer for u_ε and x_ε converges to x_0 which is the global minimum point for V. Also $\widetilde{U}$ is the unique minimizer for $\nu(V(x_0))$.

5. Non-degeneracy and uniqueness: preliminaries for the proof of Theorem 1.3

Now we will deal with the functional

$$J_\varepsilon(u, \nu(V_\varepsilon)) := \frac{1}{2}\|u\|_\varepsilon^2 - \frac{\nu(V_\varepsilon)}{p+1}\int_{\mathbb{R}^N} |u|^{p+1}\,dx \tag{5.1}$$

and we will consider the scalar products that induce the norms of the fractional spaces used in this paper, namely we set

$$\begin{aligned}
\langle u, v\rangle_{\mathcal{D}^{s,2}} &:= \int_{\mathbb{R}^N} |\xi|^{2s}\widehat{u}\,\widehat{v}\,d\xi,\\
\langle u, v\rangle_{\varepsilon,V} &:= \int_{\mathbb{R}^N} V(\varepsilon x + x_0)\,u(x)\,v(x)\,dx,\\
\text{and}\qquad \langle u, v\rangle_\varepsilon &:= \langle u, v\rangle_{\mathcal{D}^{s,2}} + \langle u, v\rangle_{\varepsilon,V}.
\end{aligned}$$

The Hilbert space associated with $\langle\cdot,\cdot\rangle_\varepsilon$ will be denoted by H^s_ε and, as usual, we say that $u \perp_\varepsilon v$ whenever $\langle u, v\rangle_\varepsilon = 0$. One simple, but important feature, is that the radially symmetric minimizer U for $\nu(V_0)$ is perpendicular in H^s_0 (that is H^s_ε with $\varepsilon = 0$) to its derivatives, and the derivatives themselves are perpendicular to each other, according to the following result:

Lemma 5.1. *For any $i \in \{1, \dots, N\}$, we have that*

$$\begin{aligned}
&\langle U, \partial_i U\rangle_0 = 0\\
\textit{and}\quad &\int_{\mathbb{R}^N} U^p \partial_i U = 0.
\end{aligned} \tag{5.2}$$

Moreover, for any $i, j \in \{1, \dots, N\}$, with $i \neq j$, we have that

$$\int_{\mathbb{R}^N} U^{p-1}\,\partial_i U\,\partial_j U = 0 \tag{5.3}$$

and

$$\langle \partial_i U, \partial_j U\rangle_0 = 0. \tag{5.4}$$

Proof. By construction

$$(-\Delta)^s U + V(0)U = U^p \tag{5.5}$$

and so, taking derivatives,

$$(-\Delta)^s(\partial_i U) + V(0)\partial_i U = pU^{p-1}\partial_i U. \tag{5.6}$$

We multiply (5.5) by $\partial_i U$ and (5.6) by U and integrate: we obtain, respectively,

$$\langle U, \partial_i U\rangle_0 = \int_{\mathbb{R}^N} U^p\,\partial_i U \quad\text{and}\quad \langle U, \partial_i U\rangle_0 = p\int_{\mathbb{R}^N} U^p\,\partial_i U.$$

By comparing these two equations we obtain that

$$p\int_{\mathbb{R}^N} U^p\,\partial_i U = \int_{\mathbb{R}^N} U^p\,\partial_i U,$$

and so

$$\langle U, \partial_i U\rangle_0 = \int_{\mathbb{R}^N} U^p\,\partial_i U = 0,$$

that proves (5.2).

Now we use the rotational invariance of U to write $U(x) = \bar{U}(|x|)$, for some $\bar{U}: \mathbb{R} \to \mathbb{R}$. Then we have that $\partial_i U(x) = \bar{U}'(|x|)\,|x|^{-1}\,x_i$ and so, by symmetry

$$\int_{\mathbb{R}^N} U^{p-1}(x)\,\partial_i U(x)\,\partial_j U(x)\,dx = \int_{\mathbb{R}^N} \bar{U}^{p-1}(|x|)\,\left|\bar{U}'(|x|)\right|^2 |x|^{-2}\,x_i x_j\,dx = 0.$$

This establishes (5.3). Then, formula (5.4) follows multiplying (5.6) by $\partial_j U$ and integrating over $\mathbb{R}^N$. ∎

Our next result is of coercivity type. It is stronger than what we will need in the following of the paper we expose it here because we believe that it might be of interest.

We also mention that in the rest of the paper, the regularity assumption can be relaxed to $V \in C^1(\mathbb{R}^N)$.

Given the radially symmetric minimizer U for $\nu(V_0)$ and $a \in \mathbb{R}^N$, we define $U_a(x) := U(x-a)$ and

$$W_\varepsilon := \Big\{ v \in H^s \text{ s.t. } v \perp_\varepsilon U_a \text{ and } v \perp_\varepsilon \partial_j U_a \text{ for any } j = 1, \dots, N \Big\}. \tag{5.7}$$

With this, we can bound the second derivative of $J_\varepsilon(U_a, \nu_\varepsilon)$ from below as follows:

Lemma 5.2. *Let J_ε be as in* (5.1)*. There exists $\varepsilon_0 > 0$ such that for any $\varepsilon \in (0, \varepsilon_0)$, for any $v \in W_\varepsilon$ and for any $a \in \mathbb{R}^N$*

$$J''_\varepsilon(U_a, \nu(V_\varepsilon))[v, v] \geq Const \, \|v\|^2_\varepsilon.$$

The $Const$ above does not depend on a.

Proof. Up to translations, we can suppose that $x_0 = 0$. We consider $\chi \in C^\infty_c(\mathbb{R}^N, (0,2))$ such that $\chi = 1$ in B_1 and $\chi = 0$ in $\mathbb{R}^N \setminus B_2$. Also we take $R > 1$, to be chosen suitably large in the sequel. We define

$$\begin{aligned} & \chi_R(x) = \chi\left(a + \frac{x}{R}\right), \\ & \bar\chi_R := 1 - \chi_R, \\ & v_1 := \chi_R v, \\ & v_2 := \bar\chi_R v, \\ \text{and} \qquad & I_1 := \int_{\mathbb{R}^{2N}} \frac{\chi_R(x)\, \bar\chi_R(x)\, (v(x) - v(y))^2}{|x-y|^{N+2s}} \, dx\, dy. \end{aligned}$$

First we prove that

$$\left| \langle v_1, v_2 \rangle_{\mathcal{D}^{s,2}} \right| \leq \eta_R(v), \tag{5.8}$$

with $\eta_R(v)$ not depending on ε and such that

$$\lim_{R \to +\infty} \eta_R(v) = 0, \tag{5.9}$$

for v fixed. To this goal, we compute

$$\begin{aligned} & (v_1(x) - v_1(y))\,(v_2(x) - v_2(y)) \\ = \;& \Big(v(x)\chi_R(x) - v(y)\chi_R(y)\Big)\Big(v(x)\bar\chi_R(x) - v(y)\bar\chi_R(y)\Big) \\ = \;& \Big(v(x)\Big(\chi_R(x) - \chi_R(y)\Big) + \chi_R(y)\Big(v(x) - v(y)\Big)\Big) \\ & \quad \cdot \Big(\bar\chi_R(x)\Big(v(x) - v(y)\Big) + v(y)\Big(\bar\chi_R(x) - \bar\chi_R(y)\Big)\Big) \\ = \;& \Big(v(x)\Big(\chi_R(x) - \chi_R(y)\Big) + \chi_R(y)\Big(v(x) - v(y)\Big)\Big) \\ & \quad \cdot \Big(\bar\chi_R(x)\Big(v(x) - v(y)\Big) - v(y)\Big(\chi_R(x) - \chi_R(y)\Big)\Big) \\ = \;& -v(x)\, v(y)\Big(\chi_R(x) - \chi_R(y)\Big)^2 + \chi_R(x)\,\bar\chi_R(x)\Big(v(x) - v(y)\Big)^2 \\ & \quad + v(x)\,\bar\chi_R(x)\Big(v(x) - v(y)\Big)\Big(\chi_R(x) - \chi_R(y)\Big) \\ & \quad - v(y)\,\chi_R(y)\Big(v(x) - v(y)\Big)\Big(\chi_R(x) - \chi_R(y)\Big). \end{aligned}$$

Therefore

$$\left| \langle v_1, v_2 \rangle_{\mathcal{D}^{s,2}} \right| \;\leq\; I_1 \,+\, Const\,(J_1 + J_2) \tag{5.10}$$

with

$$\begin{aligned} J_1 \;&:= \int_{\mathbb{R}^{2N}} |v(x)|\,|v(y)|\,\Big(\chi_R(x) - \chi_R(y)\Big)^2 \, d\mu(x,y), \\ J_2 \;&:= \int_{\mathbb{R}^{2N}} |v(x)|\;|v(x) - v(y)|\;|\chi_R(x) - \chi_R(y)|\, d\mu(x,y), \\ \text{and } \; d\mu(x,y) \;&:= |x-y|^{-N-2s}\, dx\, dy. \end{aligned} \tag{5.11}$$

Now we observe that $\|\nabla\chi_R\|_{L^\infty} \le Const\, R^{-1}$, and so

$$|\chi_R(x)-\chi_R(y)| \le Const\ \min\big\{1,\ R^{-1}|x-y|\big\}. \tag{5.12}$$

Therefore, for any $x\in\mathbb{R}^N$,

$$\begin{aligned}
&\int_{\mathbb{R}^n} \frac{|\chi_R(x)-\chi_R(y)|^2}{|x-y|^{N+2s}}\,dy\\
&\le Const\left[\int_{B(0,R)} \frac{R^{-2}\,|x-y|^2}{|x-y|^{N+2s}}\,dy + \int_{\mathbb{R}^N\setminus B(0,R)} \frac{1}{|x-y|^{N+2s}}\,dy\right]\\
&= Const\, R^{-2s}.
\end{aligned}$$

Using this and the Hölder inequality we obtain

$$\begin{aligned}
J_2 &\le \sqrt{\int_{\mathbb{R}^{2N}} |v(x)|^2\,|\chi_R(x)-\chi_R(y)|^2\,d\mu(x,y)}\cdot\sqrt{\int_{\mathbb{R}^{2N}} |v(x)-v(y)|^2\,d\mu(x,y)}\\
&\le \sqrt{Const\, R^{-2s}\int_{\mathbb{R}^N} |v(x)|^2\,dx}\cdot \|v\|_{\mathcal{D}^{s,2}}\\
&\le Const\, R^{-s}\,\|v\|_{L^2}\cdot\|v\|_{\mathcal{D}^{s,2}}\\
&\le Const\, R^{-s}\,\|v\|_\varepsilon^2.
\end{aligned} \tag{5.13}$$

Now we define

$$\begin{aligned}
&\mathbb{R}^{2N}_R := \big\{(x,y)\in\mathbb{R}^{2N} \text{ s.t. } |x-y|<R\big\}\\
\text{and}\qquad &\mathcal{V} := \big\{(x,y)\in\mathbb{R}^{2N} \text{ s.t. } |v(x)|\ge|v(y)|\big\}.
\end{aligned}$$

By symmetry

$$\begin{aligned}
&\int_{\mathbb{R}^{2N}_R} |v(x)|\,|v(y)|\,|x-y|^2\,d\mu(x,y) \le 2\int_{\mathbb{R}^{2N}_R\cap\mathcal{V}} |v(x)|^2\,|x-y|^2\,d\mu(x,y)\\
&\quad\le 2\int_{\mathbb{R}^N} |v(x)|^2\left[\int_{B(x,R)} |x-y|^{2-N-2s}\,dy\right]dx \le Const\, R^{2-2s}\int_{\mathbb{R}^N}|v(x)|^2\,dx\\
&\quad= Const\, R^{2-2s}\,\|v\|_{L^2}^2.
\end{aligned} \tag{5.14}$$

Similarly,

$$\begin{aligned}
&\int_{(\mathbb{R}^{2N}\setminus\mathbb{R}^{2N}_R)} |v(x)|\,|v(y)|\,d\mu(x,y) \le 2\int_{(\mathbb{R}^{2N}\setminus\mathbb{R}^{2N}_R)\cap\mathcal{V}} |v(x)|^2\,d\mu(x,y)\\
&\quad\le 2\int_{\mathbb{R}^N} |v(x)|^2\left[\int_{\mathbb{R}^N\setminus B(0,R)} |x-y|^{-N-2s}\,dy\right]dx \le Const\, R^{-2s}\int_{\mathbb{R}^N}|v(x)|^2\,dx\\
&\quad= Const\, R^{-2s}\,\|v\|_{L^2}^2.
\end{aligned}$$

We use the latter inequality together with (5.12) and (5.14) to conclude that

$$\begin{aligned}
J_1 &\le Const\left[R^{-2}\int_{\mathbb{R}^{2N}_R} |v(x)|\,|v(y)|\,|x-y|^2\,d\mu(x,y) + \int_{\mathbb{R}^{2N}\setminus\mathbb{R}^{2N}_R} |v(x)|\,|v(y)|\,d\mu(x,y)\right]\\
&\le Const\, R^{-2s}\,\|v\|_{L^2}^2.
\end{aligned}$$

Hence, by (5.10) and (5.13),

$$\big|\langle v_1,v_2\rangle_{\mathcal{D}^{s,2}}\big| \le I_1 + Const\,(R^{-2s}+R^{-s})\|v\|_\varepsilon^2. \tag{5.15}$$

Now we estimate I_1. For this we observe that the function $\chi_R\bar\chi_R$ is supported in $B(a,2R)\setminus B(a,R)$, hence

$$I_1 \le \int_{(B(a,2R)\setminus B(a,R))\times\mathbb{R}^N} \frac{(v(x)-v(y))^2}{|x-y|^{N+2s}}\,dx\,dy.$$

Since v is a fixed function of H^s, we have that

$$\lim_{R\to+\infty}\int_{(B(a,2R)\setminus B(a,R))\times\mathbb{R}^N}\frac{(v(x)-v(y))^2}{|x-y|^{N+2s}}\,dx\,dy=0.$$

These considerations and (5.15) imply (5.8), as desired.

From (5.8), we obtain that

$$\begin{aligned}\|v\|^2_{\mathcal{D}^{s,2}}=\|v_1+v_2\|^2_{\mathcal{D}^{s,2}}&=\|v_1\|^2_{\mathcal{D}^{s,2}}+\|v_2\|^2_{\mathcal{D}^{s,2}}+2\langle v_1,v_2\rangle_{\mathcal{D}^{s,2}}\\&\le\|v_1\|_{\mathcal{D}^{s,2}}+\|v_2\|_{\mathcal{D}^{s,2}}+2\eta_R(v).\end{aligned}\tag{5.16}$$

Moreover

$$\begin{aligned}\|v\|^2_{\varepsilon,V}&=\|v_1\|^2_{\varepsilon,V}+\|v_2\|^2_{\varepsilon,V}+\int_{\mathbb{R}^N}V(\varepsilon x)\,v_1(x)\,v_2(x)\,dx\\&\le Const\,\big(\|v_1\|^2_{\varepsilon,V}+\|v_2\|^2_{\varepsilon,V}\big).\end{aligned}$$

This and (5.16) yield that

$$\|v\|^2_\varepsilon\le Const\,\big(\|v_1\|^2_\varepsilon+\|v_2\|^2_\varepsilon+\eta_R(v)\big).\tag{5.17}$$

On the other hand, $v_1v_2=\chi_R\bar\chi_R v^2$, therefore $v_1v_2\ge0$ and it is supported in $B(a,2R)\setminus B(a,R)$. In this domain U_a is of the order $R^{-(N+2s)}$, therefore

$$\int_{\mathbb{R}^N}U_a^{p-1}v_1\,v_2\le Const\,R^{-(p-1)(N+2s)}\int_{B(a,2R)\setminus B(a,R)}|v|^2\le Const\,R^{-(p-1)(N+2s)}\,\|v\|^2_{L^2}.$$

From this and (5.8) we infer that

$$\begin{aligned}J''_\varepsilon(U_a,\nu(V_\varepsilon))[v_1,v_2]&=\langle v_1,v_2\rangle_{\mathcal{D}^{s,2}}+\int_{\mathbb{R}^N}V(\varepsilon x)v_1v_2-p\nu(V_\varepsilon)\int_{\mathbb{R}^N}U_a^{p-1}v_1v_2\\&\ge-Const\,R^{-s}\,\|v\|^2_\varepsilon+0-Const\,R^{-(p-1)(N+2s)}\,\|v\|^2_{L^2}\\&\ge-Const\,R^{-\gamma}\,\|v\|^2_\varepsilon,\end{aligned}\tag{5.18}$$

up to renaming constants, where $\gamma:=\min\{s,\,(p-1)(N+2s)\}>0$ (here we have also used Lemma 3.1 to bound $\nu(V_\varepsilon)$ uniformly in ε). Similarly, v_2 is supported outside $B(0,R)$, hence

$$\int_{\mathbb{R}^N}U_a^{p-1}v_2^2\le Const\,R^{-(p-1)(N+2s)}\int_{\mathbb{R}^N}v^2,$$

and therefore

$$J''_\varepsilon(U_a,\nu(V_\varepsilon))[v_2,v_2]=\|v_2\|^2_\varepsilon-p\nu(V_\varepsilon)\int_{\mathbb{R}^N}U_a^{p-1}v_2^2\ge\|v_2\|^2_\varepsilon-Const\,R^{-(p-1)(N+2s)}\,\|v\|^2_{L^2}.\tag{5.19}$$

Next we estimate $J''_\varepsilon(U_a,\nu(V_\varepsilon))[v_1,v_1]$. To this goal, we project v_1 along the space spanned by U_a and its derivatives, i.e. we set

$$\psi:=\frac{1}{\|U_a\|_0^2}\langle v_1,U_a\rangle_0U_a+\frac{1}{\|\partial_iU_a\|_0^2}\langle v_1,\partial_iU_a\rangle_0\partial_iU_a,$$

where the repeated indices convention is used, and $w:=v_1-\psi$. Therefore

$$J''_\varepsilon(U_a,\nu(V_\varepsilon))[v_1,v_1]=J''_\varepsilon(U_a,\nu(V_\varepsilon))[w,w]+J''_\varepsilon(U_a,\nu(V_\varepsilon))[\psi,\psi]+2J''_\varepsilon(U_a,\nu(V_\varepsilon))[w,\psi].\tag{5.20}$$

We observe that the norms $\|\cdot\|_0$ and $\|\cdot\|_\varepsilon$ are comparable, thanks to (1.4). Therefore

$$|\psi|\le\frac{\|v_1\|_0}{\|U\|_0}\,U_a+\frac{\|v_1\|_0}{\|\partial_iU_a\|_0}\,|\partial_iU_a|\le Const\,\|v_1\|_\varepsilon\,\Big(U_a+|\partial_iU_a|\Big),\tag{5.21}$$

hence

$$\int_{\mathbb{R}^N}(1+|x|)\psi^2\le Const\,\|v_1\|^2_\varepsilon.$$

Using this, the fact that $|V(\varepsilon x) - V(0)| \le Const\,\varepsilon\,|x|$, and that v_1 is supported in $B(a,R)$, we conclude that

$$\begin{aligned}
&\int_{\mathbb{R}^N} |V(\varepsilon x) - V(0)| w^2 \\
&= \int_{B(a,R)} |V(\varepsilon x) - V(0)|\, v_1^2 + \int_{\mathbb{R}^N} |V(\varepsilon x) - V(0)|\, \psi^2 - 2\int_{\mathbb{R}^N} |V(\varepsilon x) - V(0)|\, v_1 \psi \\
&\le Const \left[\int_{B(a,R)} |V(\varepsilon x) - V(0)|\, v_1^2 + \int_{\mathbb{R}^N} |V(\varepsilon x) - V(0)|\, \psi^2 \right] \\
&\le Const\,\varepsilon\,(R + |a|)\, \|v_1\|_\varepsilon^2.
\end{aligned} \tag{5.22}$$

Now we remark that w is orthogonal in H_0^s (i.e. in H_ε^s with $\varepsilon = 0$) to any element of the basis $\{U_a,\, \partial_1 U_a, \dots,\, \partial_N U_a\}$, thanks to Lemma 5.1. Hence, from [16], we have that

$$J_0''(U_a, \nu_0)[w,w] \ge Const\, \|w\|_0^2 \ge Const\, \|w\|_\varepsilon^2.$$

As a consequence,

$$\begin{aligned}
J_\varepsilon''(U_a, \nu(V_\varepsilon))[w,w] &= J_0''(U_a, \nu(V_0))[w,w] + \int_{\mathbb{R}^N} [V(\varepsilon x) - V(0)] w^2 - p(\nu(V_\varepsilon) - \nu(V_0)) \int_{\mathbb{R}^N} U_a^{p-1} w^2 \\
&\ge Const\, \|w\|_\varepsilon^2 + \int_{\mathbb{R}^N} [V(\varepsilon x) - V(0)] w^2 - p(\nu(V_\varepsilon) - \nu(V_0)) \int_{\mathbb{R}^N} U^{p-1} w^2 \\
&\ge Const\, \|w\|_\varepsilon^2 - Const\,\varepsilon\,(R + |a|)\, \|v_1\|_\varepsilon^2 - Const\, |\nu(V_\varepsilon) - \nu(V_0)|\, \|v_1\|_\varepsilon,
\end{aligned} \tag{5.23}$$

where both (5.22) and (5.21) were used in the last inequality.

Furthermore, since $v \perp_\varepsilon U_a$, we have

$$\begin{aligned}
\langle v_1, U_a\rangle_0 &= \langle v_1, U_a\rangle_\varepsilon + \int_{\mathbb{R}^N} [V(0) - V(\varepsilon x)] v_1 U_a \\
&= \langle v, U_a\rangle_\varepsilon - \langle v_2, U_a\rangle_\varepsilon + \int_{\mathbb{R}^N} [V(0) - V(\varepsilon x)] v_1 U_a \\
&= -\langle v_2, U_a\rangle_\varepsilon + \int_{\mathbb{R}^N} [V(0) - V(\varepsilon x)] v_1 U_a \\
&= -\langle v_2, U_a\rangle_{\mathcal{D}^{s,2}} - \int_{\mathbb{R}^N} V(\varepsilon x) v_2 U_a + \int_{\mathbb{R}^N} [V(0) - V(\varepsilon x)] v_1 U_a \\
&= -\int_{\mathbb{R}^N} v_2 (-\Delta)^s U_a - \int_{\mathbb{R}^N} V(\varepsilon x) v_2 U_a + \int_{\mathbb{R}^N} [V(0) - V(\varepsilon x)] v_1 U_a \\
&= -\int_{\mathbb{R}^N} v_2 [-V(0) U_a + \nu_0 U_a^p] - \int_{\mathbb{R}^N} V(\varepsilon x) v_2 U_a + \int_{\mathbb{R}^N} [V(0) - V(\varepsilon x)] v_1 U_a \\
&= \int_{\mathbb{R}^N} [V(0) - V(\varepsilon x)] v U_a - \nu_0 \int_{\mathbb{R}^N} U_a^p v_2
\end{aligned}$$

thus, since v_2 is supported outside $B(a,R)$,

$$\left| \langle v_1, U_a\rangle_0 \right| \le Const \left(\varepsilon\,(R + |a|)\, \|v\|_{L^2} + R^{-(p-1)(N+2s)} \|v\|_{L^2} \right). \tag{5.24}$$

In a similar way, since also $v \perp_\varepsilon \partial_i U_a$, we have that

$$\left| \langle v_1, \partial_i U_a\rangle_0 \right| \le Const \left(\varepsilon\,(R + |a|)\, \|v\|_{L^2} + R^{-(p-1)(N+2s)} \|v\|_{L^2} \right). \tag{5.25}$$

We deduce from (5.24) and (5.25) that

$$\|\psi\|_0 \le Const \left(\left| \langle v_1, U_a\rangle_0 \right| + \left| \langle v_1, \partial_i U_a\rangle_0 \right| \right) \le Const \left(\varepsilon (R + |a|)\, \|v\|_{L^2} + R^{-(p-1)(N+2s)} \|v\|_{L^2} \right)$$

and so, since the two norms are comparable,

$$\|\psi\|_\varepsilon \le Const \left(\varepsilon (R + |a|)\, \|v\|_{L^2} + R^{-(p-1)(N+2s)} \|v\|_{L^2} \right).$$

So, we use the fact that

$$2|\langle v_1, \psi\rangle_\varepsilon| = 2|\langle v_1/2,\, 2\psi\rangle_\varepsilon| \le \frac{\|v_1\|_\varepsilon}{4} + 4\|\psi\|_\varepsilon$$

to conclude that

$$\begin{aligned}\|w\|_\varepsilon^2 &= \|v_1\|_\varepsilon^2 + \|\psi\|_\varepsilon^2 - 2\langle v_1, \psi\rangle_\varepsilon \\ &\geq \frac{3}{4}\|v_1\|_\varepsilon^2 - Const\,(\varepsilon(R+|a|) + R^{-(p-1)(N+2s)})^2\,\|v\|_{L^2}^2.\end{aligned}$$

Exploiting this and (5.23) we obtain

$$\begin{aligned}J_\varepsilon''(U_a, \nu(V_\varepsilon))[w,w] \geq Const\,\|v_1\|_\varepsilon^2 - Const\,(\varepsilon(R+|a|) + R^{-(p-1)(N+2s)})^2\,\|v\|_{L^2}^2 \\ - Const\,\varepsilon\,(R+|a|)\,\|v_1\|_\varepsilon^2 - Const\,|\nu(V_\varepsilon) - \nu(V_0)|\,\|v_1\|_\varepsilon.\end{aligned} \tag{5.26}$$

Notice now that

$$J_\varepsilon''(U_a, \nu(V_\varepsilon))[v,v] = J_\varepsilon''(U_a, \nu(V_\varepsilon))[v_1,v_1] + J_\varepsilon''(U_a, \nu(V_\varepsilon))[v_2,v_2] + 2J_\varepsilon''(U_a, \nu(V_\varepsilon))[v_1,v_2].$$

Thus, by collecting (5.18), (5.19) and (5.26), we obtain

$$\begin{aligned}J_\varepsilon''(U_a, \nu(V_\varepsilon))[v,v] \geq\ & Const\,(\|v_1\|_\varepsilon^2 + \|v_2\|_\varepsilon^2) \\ & - Const\,(\varepsilon(R+|a|) + R^{-\gamma})\,\|v\|_\varepsilon^2, - Const\,R^{-(p-1)(N+2s)}\,\|v\|_{L^2}^2 \\ & - Const\,\varepsilon\,(R+|a|)\,\|v_1\|_\varepsilon^2 - Const\,|\nu(V_\varepsilon) - \nu(V_0)|\,\|v_1\|_\varepsilon\end{aligned}$$

Now, recalling (5.17) and (5.9), and sending first $\varepsilon \to 0$ and then $R \to +\infty$, we get the desired result. ☐

6. Uniqueness of radial solutions

In this section we assume that V is radial and we consider the functional in (5.1). We denote by H_r^s the subspace of H^s of radially symmetric function. We will make use of the minimizer U for $\nu(V_0)$, normalized with $\|U\|_{L^{p+1}} = 1$, which is a solution of

$$\langle U, v\rangle_{\mathcal{D}^{s,2}} + V(0)\,\langle U, v\rangle_{L^2} = \nu(V_0) \int_{\mathbb{R}^N} U^p(x)\,v(x)\,dx, \tag{6.1}$$

for every $v \in H^s$.

We also define I_ε as the restriction of $u \mapsto J_\varepsilon(u, \nu(V_\varepsilon))$ on H_r^s. Next, we define the operator $\Phi_\varepsilon : H_r^s \to H_r^s$ by

$$\Phi_\varepsilon(\omega) := I_\varepsilon'\,(U+\omega)\,. \tag{6.2}$$

By (6.2), we mean: for all $w \in H_r^s$

$$\langle \Phi_\varepsilon(\omega), w\rangle = I_\varepsilon'\,(U+\omega)\,[w]. \tag{6.3}$$

Lemma 6.1. *There exists $\delta > 0$ sufficiently small such that: if $\Phi_\varepsilon(w_1) = \Phi_\varepsilon(w_2)$ for some w_1, $w_2 \in H_r^s$ with $\|w_1\|_\varepsilon + \|w_2\|_\varepsilon \leq \delta$, then $w_1 = w_2$.*

Proof. The proof is a consequence of Lemma 5.2. The details go as follows. First we fix the following notation: given $f \in H_r^s$, we define

$$c_f := \frac{\langle f, U\rangle_\varepsilon}{\|U\|_\varepsilon^2} \quad \text{and} \quad \widetilde{f} := f - c_f U.$$

Notice that $\widetilde{f}$ is radial, since so are f and U, and that $\langle \widetilde{f}, U\rangle_\varepsilon = 0$. As a matter of fact, since both $\widetilde{f}$ and U are radial, a direct computation based on odd symmetry shows that also $\langle \widetilde{f}, \partial_i U\rangle_\varepsilon = 0$, that is

$$\widetilde{f} \in W_\varepsilon, \tag{6.4}$$

according to the definition in (5.7).

Notice that $f = \widetilde{f} + c_f U$. We also consider the reflection of f with respect to U, namely

$$f^\star := \widetilde{f} - c_f U. \tag{6.5}$$

Now we observe that

$$J_0''(U, \nu(V_0))[U, v] = (1-p)\,\nu(V_0) \int_{\mathbb{R}^N} U^p(x)\,v(x)\,dx \tag{6.6}$$

for any $v \in H_r^s$. Indeed, for any $v \in H^s$,

$$\begin{aligned}J_0''(U, \nu(V_0))[U, v] &= \langle U, v\rangle_{\mathcal{D}^{s,2}} + V(0)\,\langle U, v\rangle_{L^2} - p\nu(V_0) \int_{\mathbb{R}^N} U^{p-1}(x)\,U(x)\,v(x)\,dx \\ &= (1-p)\,\nu(V_0) \int_{\mathbb{R}^N} U^p(x)\,v(x)\,dx,\end{aligned}$$

thanks to (6.1), and this establishes (6.6).

Furthermore

$$J_\varepsilon''(U,\nu(V_\varepsilon))[U,v]-J_0''(U,\nu(V_0))[U,v]=\int_{\mathbb{R}^N}(V(\varepsilon x)-V(0))U(x)v(x)\,dx-p(\nu(V_\varepsilon)-\nu(V_0))\int_{\mathbb{R}^N}U^p(x)\,v(x)\,dx.$$

This, combined with (6.6) gives that

$$J_\varepsilon''(U,\nu(V_\varepsilon))[U,v]=\int_{\mathbb{R}^N}(V(\varepsilon x)-V(0))U(x)v(x)\,dx-c_\varepsilon\int_{\mathbb{R}^N}U^p(x)\,v(x)\,dx,$$

where

$$c_\varepsilon:=(1-p)\,\nu(V_0)-p(\nu(V_\varepsilon)-\nu(V_0)).$$

Notice that $c_\varepsilon\to(1-p)\,\nu(V_0)>0$ as $\varepsilon\to 0$, due to Lemma 3.1. In particular

$$J_\varepsilon''(U,\nu(V_\varepsilon))[U,U]=\eta_\varepsilon-c_\varepsilon,$$

with

$$\eta_\varepsilon:=\int_{\mathbb{R}^N}(V(\varepsilon x)-V(0))U^2(x)\,dx\to 0,$$

as $\varepsilon\to 0$, by dominated convergence theorem. We conclude that

$$J_\varepsilon''(U,\nu(V_\varepsilon))[U,U]\le-\frac{c_\varepsilon}{2} \tag{6.7}$$

for small ε. Now, for any v, $w\in H^s_r$, we set

$$\begin{aligned}\mathcal{N}_\varepsilon(v)[w] &:= \Phi_\varepsilon(v)[w]-\Phi_\varepsilon(0)[w]-\langle\Phi_\varepsilon'(0)[v],w\rangle\\ &= I_\varepsilon'(U+v)[w]-I_\varepsilon'(U)[w]-I_\varepsilon''(U)[v,w]\\ &= \nu(V_\varepsilon)\left(-\int_{\mathbb{R}^N}|U+v|^p w\,dx+\int_{\mathbb{R}^N}U^p w\,dx+p\int_{\mathbb{R}^N}U^{p-1}w\,dx\right).\end{aligned}$$

Referring to page 128 in [1], we obtain

$$\|\mathcal{N}_\varepsilon(v_1)-\mathcal{N}_\varepsilon(v_2)\|\le\ Const\,(\|v_1\|_\varepsilon+\|v_1\|_\varepsilon^{p-1}+\|v_2\|_\varepsilon+\|v_2\|_\varepsilon^{p-1})\|v_1-v_2\|_\varepsilon. \tag{6.8}$$

Now we take $w:=w_1-w_2$ and we use the notation in (6.5) and the assumption that $\Phi_\varepsilon(w_1)=\Phi_\varepsilon(w_2)$ to compute:

$$\begin{aligned}0 &= \Phi_\varepsilon(w_1)[w^\star]-\Phi_\varepsilon(w_2)[w^\star]\\ &= \mathcal{N}_\varepsilon(w_1)[w^\star]+\Phi_\varepsilon(0)[w^\star]+\langle\Phi_\varepsilon'(0)[w_1],w^\star\rangle-\mathcal{N}_\varepsilon(w_2)[w^\star]-\Phi_\varepsilon(0)[w^\star]-\langle\Phi_\varepsilon'(0)[w_2],w^\star\rangle\\ &= \langle\Phi_\varepsilon'(0)[w_1],w^\star\rangle-\langle\Phi_\varepsilon'(0)[w_2],w^\star\rangle+\mathcal{N}_\varepsilon(w_1)[w^\star]-\mathcal{N}_\varepsilon(w_2)[w^\star]\\ &= J_\varepsilon''(U,\nu(V_\varepsilon))[w,w^\star]+\mathcal{N}_\varepsilon(w_1)[w^\star]-\mathcal{N}_\varepsilon(w_2)[w^\star].\end{aligned}$$

Thus, we write $w=\widetilde{w}+c_wU$ and $w^\star=\widetilde{w}-c_wU$, and we exploit (6.8) and (6.7), to see that

$$\begin{aligned}0 &\ge J_\varepsilon''(U,\nu(V_\varepsilon))[\widetilde{w},\widetilde{w}]-c_w^2J_\varepsilon''(U,\nu(V_\varepsilon))[U,U]-\ Const\,\delta^{\min\{1,p-1\}}\|w\|_\varepsilon\|w^\star\|_\varepsilon\\ &\ge J_\varepsilon''(U,\nu(V_\varepsilon))[\widetilde{w},\widetilde{w}]+\frac{c_\varepsilon c_w^2}{2}-\ Const\,\delta^{\min\{1,p-1\}}\|w\|_\varepsilon\|w^\star\|_\varepsilon.\end{aligned}$$

Now, thanks to (6.4), we can make use of Lemma 5.2 and write that $J_\varepsilon''(U,\nu(V_\varepsilon))[\widetilde{w},\widetilde{w}]\ge\ Const\,\|\widetilde{w}\|_\varepsilon^2$. So we obtain that

$$0\ge\ Const\left(\|\widetilde{w}\|_\varepsilon^2+c_w^2\right)-\ Const\,\delta^{\min\{1,p-1\}}\|w\|_\varepsilon\|w^\star\|_\varepsilon. \tag{6.9}$$

Also, by (6.4),

$$\|w\|_\varepsilon^2=\|\widetilde{w}\|_\varepsilon^2+c_w^2\|U\|_\varepsilon^2+2\langle\widetilde{w},U\rangle_\varepsilon=\|\widetilde{w}\|_\varepsilon^2+c_w^2\|U\|_\varepsilon^2$$

and, similarly,

$$\|\widetilde{w}\|_\varepsilon^2=\|\widetilde{w}\|_\varepsilon^2+c_w^2\|U\|_\varepsilon^2.$$

In particular, $\|w\|_\varepsilon^2\le\ Const\left(\|\widetilde{w}\|_\varepsilon^2+c_w^2\right)$ and (6.9) becomes

$$\begin{aligned}0 &\ge\ Const\left(\|\widetilde{w}\|_\varepsilon^2+c_w^2\right)-\ Const\,\delta^{\min\{1,p-1\}}\|w\|_\varepsilon^2\\ &\ge \left(Const\ -\ Const\,\delta^{\min\{1,p-1\}}\right)\|w\|_\varepsilon^2,\end{aligned}$$

which implies that $\|w\|_\varepsilon=0$ if δ is small enough. $\square$

7. Completeness of the proof of Theorem 1.3

Now we complete the proof of Theorem 1.3. For this, let v_ε^i be a minimizer for $\nu(V_\varepsilon)$, with $i = 1, 2$. Since V is radial then using symmetric decreasing arguments of the moving plane argument, we have that v_ε^i is radial (or using the minimization in the space H_r^s and the compactness in [18]). Then by Lemma 3.1, provided ε is sufficiently small, we have

$$v_\varepsilon^i(x) = U + w_\varepsilon^i \qquad \text{with} \quad \|w_i^\varepsilon\|_\varepsilon \to 0 \quad \text{as} \quad \varepsilon \to 0.$$

It turns out that $\Phi_\varepsilon(w_\varepsilon^i) = I'_\varepsilon(v_\varepsilon^i) = 0$, so we conclude that $w_\varepsilon^1 = w_\varepsilon^2$, due to Lemma 6.1. □

References

[1] A. Ambrosetti and A. Malchiodi, *Perturbation Methods and Semilinear Elliptic Problems on $\mathbb{R}^n$*. Progress in Mathematics, Birkhäuser Verlag, Basel-Boston-Berlin (2005).

[2] H. Brezis, *Functional analysis, Sobolev spaces and partial differential equations*. Universitext. Springer, New York, 2011.

[3] X. Cabré and Y. Sire, *Nonlinear equations for fractional Laplacians, I: Regularity, maximum principles, and Hamiltonian estimates*. Ann. Inst. H. Poincaré Anal. Non Linéaire 31 (2014), no. 1, 23-53.

[4] G. Chen and Y. Zheng, *Concentration phenomenon for fractional nonlinear Schrödinger equations*. Comm. Pure Appl. Anal. 13 (2014), no. 6, 2359-2376.

[5] M. Cheng, *Bound state for the fractional Schrödinger equation with unbounded potential.* J. Math. Phys. 53 (2012), 043507.

[6] Y. Cho and T. Ozawa, *A note on the existence of a ground state solution to a fractional Schrödinger equation*. Kyushu J. Math. 67 (2013), 227-236.

[7] J. Davila, M. del Pino and J. Wei,*Concentrating standing waves for the fractional nonlinear Schrodinger equation.* J. Diff. Eqns. 256(2014), no.2, 858-892.

[8] J. Dávila, M. Del Pino, S. Dipierro and E. Valdinoci, *Concentration phenomena for the nonlocal Schrödinger equation with Dirichlet datum.* Preprint
`http://arxiv.org/pdf/1403.4435.pdf`

[9] E. Di Nezza, G. Palatucci and E. Valdinoci, *Hitchhikers guide to the fractional Sobolev spaces*. Bull. Sci. Math. 136 (2012), no. 5, 521-573.

[10] S. Dipierro, G. Palatucci and E. Valdinoci, *Existence and symmetry results for a Schrödinger type problem involving the fractional Laplacian.* Le Matematiche 68 (2013), no. 1, 201-216.

[11] S. Dipierro, O. Savin and E. Valdinoci, *All functions are locally s-harmonic up to a small error*. Preprint
`http://www.wias-berlin.de/preprint/1944/wias_preprints_1944.pdf`

[12] G. Evéquoz and M. M. Fall, *Positive solutions to some nonlinear fractional Schrödinger equation via a Min-Max procedure*.
`http://arxiv.org/abs/1312.7068`

[13] M. M. Fall and T. Weth, *Monotonicity and nonexistence results for some fractional elliptic problems in the half space*. To appear in Commun. Contemp. Math.
`http://arxiv.org/abs/1309.7230`

[14] P. Felmer, A. Quaas and J. Tan, *Positive solutions of Nonlinear Schrödinger equation with the fractional Laplacian*. Proc. R. Soc. Edinb., Sect. A, Math. 142 (2012), no. 6, 1237-1262.

[15] B. Feng, *Ground states for the fractional Schrödinger equation*, Electron. J. Differential Equations 127 (2013), 1-11.

[16] R. L. Frank, E. Lenzmann and L. Silvestre, *Uniqueness of radial solutions to fractional laplacian.* Preprint
`http://arxiv.org/abs/1302.2652v1`

[17] T. Jin, Y. Y. Li and J. Xiong, *On a fractional Nirenberg problem, part I: blow up analysis and compactness of solutions.* J. Eur. Math. Soc. (JEMS) 16 (2014), no. 6, 1111-1171.

[18] P.-L. Lions, *Symétrie et compacité dans les espaces de Sobolev.* J. Funct. Anal. 49 (1982), no. 3, 315-334.

[19] W-M. Ni; I. Takagi, *Locating the peaks of least-energy solutions to a semilinear Neumann problem.* Duke Math. J. 70 (1993), no. 2, 247-281.

[20] W.-M. Ni and I. Takagi, *On the shape of least-energy solutions to a semilinear Neumann problem.* Comm. Pure Appl. Math. 44 (1991), no. 7, 819-851.

[21] W-M. Ni and J. Wei, *On the location and profile of spike-layer solutions to singularly perturbed semilinear Dirichlet problems*, Comm. Pure Appl. Math. 48, 1995, 731-768.

[22] S. Santra, J. Wei, *Profile of the least energy solution of a singular perturbed Neumann problem with mixed powers.* Ann. Mat. Pura Appl. (4) 193 (2014), no. 1, 39-70.

[23] S. Simone, *Ground state solutions for nonlinear fractional Schrödinger equations in $\mathbb{R}^N$*. J. Math. Phys. 54 (2013), no. 3, 031501, 17 p.

[24] L. Silvestre, *Regularity of the obstacle problem for a fractional power of the Laplace operator*. Comm. Pure Appl. Math. 60 (2007), no. 1, 67-112.

[25] X. Wang, *On concentration of positive bound states of nonlinear Schrödinger equations*, Commun. Math. Phys. 153 (1993), 229-244.