Ground states and concentration phenomena for the fractional Schrödinger equation

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Abstract. We consider here solutions of the nonlinear fractional Schrödinger equation
\[ \varepsilon^{2s}(-\Delta)^s u + V(x)u = u^p. \]
We show that concentration points must be critical points for \( V \). We also prove that, if the potential \( V \) is coercive and has a unique global minimum, then ground states concentrate suitably at such minimal point as \( \varepsilon \) tends to zero. In addition, if the potential \( V \) is radial, then the minimizer is unique provided \( \varepsilon \) is small.

1. INTRODUCTION

In this paper we will study standing waves for a nonlinear differential equation driven by the fractional Laplacian. We will focus on the so-called fractional Schrödinger equation
\[ i\hbar \frac{\partial \psi}{\partial t} = \hbar^{2s}(-\Delta)^s \psi + V(x)\psi - |\psi|^{p-1}\psi \] (1.1)
where \( \hbar \) is the Planck constant, \((x,t) \in \mathbb{R}^N \times (0, +\infty), 0 < s < 1, \) and \( V: \mathbb{R}^N \to \mathbb{R} \) is an external potential function. The operator \((-\Delta)^s\) is the fractional Laplacian of order \( s \), which, for a function \( \varphi \in C_0^\infty \) (here and in the sequel when omitting the space of definition we are meaning \( \mathbb{R}^N \)) may be defined via Fourier transform:
\[ \mathcal{F}(-\Delta)^s \varphi(\xi) = |\xi|^{2s} \hat{\varphi}(\xi) \quad \text{for } \xi \in \mathbb{R}^N, \]
where we used the standard notation
\[ \hat{\varphi}(\xi) := \mathcal{F}(\varphi)(\xi) := \frac{1}{(2\pi)^{\frac{N}{2}}} \int_{\mathbb{R}^N} e^{-i\xi \cdot x} \varphi(x) dx \]
for the Fourier transform of a function \( \varphi \in L^2 \). As customary, we will focus on the standing wave situation of equation (1.1), namely on the case in which \( \psi(x,t) = u(x)e^{it} \), with \( u \geq 0 \) under this further assumption (and replacing \( V + 1 \) with \( V \) and \( \hbar \) with the small parameter \( \varepsilon > 0 \)), equation (1.1) reduces to
\[ \varepsilon^{2s}(-\Delta)^s u + V(x)u - u^p = 0. \] (1.2)
This is the main equation studied in this paper and it will be set in the whole of \( \mathbb{R}^N \), with \( N > 2s \) and \( p \) subcritical\(^1\), namely
\[ 1 < p < \frac{N + 2s}{N - 2s}. \] (1.3)
As for the potential \( V \) in (1.2), we suppose that is smooth, positive, and bounded from zero, namely we assume that
\[ \|V\|_{C^2} < \infty, \quad \bar{V} = \inf_{\mathbb{R}^N} V > 0. \] (1.4)
The weak formulation of the fractional Laplacian naturally leads to the study of the fractional Sobolev spaces
\[ \mathcal{H}^s := \left\{ u \in L^2 : \int_{\mathbb{R}^N} |\xi|^{2s} |\hat{u}|^2 d\xi < \infty \right\}, \] (1.5)
endowed with the norm
\[ \|u\|_{\mathcal{H}^s}^2 := \|u\|_{L^2}^2 + \|u\|_{L^2}^{2s}, \]
where
\[ \|u\|_{L^2}^2 := \int_{\mathbb{R}^N} |\xi|^{2s} |\hat{u}|^2 d\xi. \]
Notice that all the functional spaces \( L^2, \mathcal{H}^s \) etc. are set in the whole of \( \mathbb{R}^n \) unless explicitly mentioned. In this functional setting, a weak solution of equation (1.2) is a function \( u_\varepsilon \in \mathcal{H}^s \) such that
\[ \varepsilon^{2s} \int_{\mathbb{R}^N} |\xi|^{2s} \hat{u}_\varepsilon(\xi) \hat{\varphi}(\xi) d\xi = \int_{\mathbb{R}^N} \left( V(x)u_\varepsilon(x) - u_\varepsilon^p(x) \right) \varphi(x) dx \]
for any \( \varphi \in \mathcal{H}^s \). For the existence of weak solutions for special cases of (1.2), see e.g. [4–6,10,12,14,15,23]; in this circumstance, the solutions found are indeed positive, bounded and \( C^{2,\alpha} \) (see Theorem 3.4 in [14] and Lemma 4.4
\[^1\]When \( N \leq 2s \), one can say that \( p \) is subcritical when \( p \in (1, +\infty) \).
in [3]). In this case, equation (1.2) holds pointwise and the fractional Laplace of \( u \) at the point \( x \in \mathbb{R}^N \) has the integral representation
\[
(-\Delta)^s u(x) = c(n, s) \int_{\mathbb{R}^N} \frac{2u(x) - u(x + y) - u(x - y)}{|y|^{n+2s}} \, dy
\] 
for a suitable \( c(n, s) > 0 \), see e.g. Proposition 3.3 in [9].

The first result that we provide characterizes the points at which solutions of (1.2) concentrate for small \( \varepsilon \), stating that these points are critical for the potential. This is somehow an extension to the nonlocal setting of Wang’s result, see [25]. To state this first result, given a sequence of positive solutions \( u_\varepsilon \) for equation (1.2) in the whole of \( \mathbb{R}^N \), we say that \( x_0 \in \mathbb{R}^N \) is a strong concentration point for this sequence (or that the sequence \( u_\varepsilon \) strongly concentrates at \( x_0 \)) if
\[
\text{for any } \delta > 0 \text{ there exist } \varepsilon_0 \text{ and } R > 0 \text{ such that, for any } \varepsilon \in (0, \varepsilon_0),
\]
\[
u_\varepsilon(x) < \delta \text{ for all } x \in \mathbb{R}^N \setminus B(x_0, \varepsilon R).
\] 

With this setting, the following result holds:

**Theorem 1.1.** Assume (1.4) and let \( u_\varepsilon \in H^s \) be a sequence of positive solutions of (1.2) in the whole of \( \mathbb{R}^N \) that strongly concentrate at \( x_0 \). Then \( \nabla V(x_0) = 0 \).

We remark that, if we perform a translation and a spacial dilation of factor \( 1/\varepsilon \), equation (1.2) becomes
\[
(-\Delta)^s u + V(\varepsilon x + x_0)u - u^p = 0.
\] 

Thus, to study the concentration phenomena of this equation, it is convenient to define
\[
V_\varepsilon(x) := V(\varepsilon x + x_0),
\]
\[
\|u\|_{V_\varepsilon}^2 := \int_{\mathbb{R}^N} V_\varepsilon(x) u^2(x) \, dx,
\]
\[
\|u\|_{D,s,2}^2 := \|u\|_{D,s,2}^2 + \|u\|_{V_\varepsilon}^2,
\]
and \( \nu(V_\varepsilon) := \inf_{u \neq 0} \frac{\|u\|_{V_\varepsilon}^2}{\|u\|_{L,p+1}^2} \).

Notice that these definitions also make sense when \( \varepsilon = 0 \), namely, one has
\[
\|u\|_{D,s,2}^0 := V(x_0) \int_{\mathbb{R}^N} u^2(x) \, dx
\]
and so on. Moreover, we remark that if \( u \) is a minimizer for \( \nu(V_\varepsilon) \) then \( u_\varepsilon(x) := u((x - x_0)/\varepsilon) \) is a minimizer for
\[
\nu_\varepsilon(V) := \varepsilon^{N(1-p)/(1+p)} \inf_{u \neq 0} \varepsilon^{2s} \|u\|_{D,s,2}^2 + \int_{\mathbb{R}^N} V(x) u^2(x) \, dx
\]
\[
\|u\|_{L,p+1}^2
\].

In this setting, we can better determine the variational properties of the concentration point \( x_0 \). Namely, while we know from Theorem 1.1 that \( x_0 \) is a stationary point for the potential, now we give conditions under which it is a minimum. For this scope, given a sequence of positive solutions \( u_\varepsilon \) for equation (1.2) in the whole of \( \mathbb{R}^N \), we say that \( x_0 \in \mathbb{R}^N \) is a weak concentration point for this sequence (or that the sequence \( u_\varepsilon \) weakly concentrates at \( x_0 \)) if there exists a sequence of points \( x_\varepsilon \rightarrow x_0 \) such that
\[
\text{for any } \delta > 0 \text{ there exist } \varepsilon_0 \text{ and } R > 0 \text{ such that, for any } \varepsilon \in (0, \varepsilon_0),
\]
\[
u_\varepsilon(x) \leq \delta \text{ for all } x \in \mathbb{R}^N \setminus B(x_\varepsilon, \varepsilon R).
\] 

By comparing (1.7) and (1.9), we notice that strong concentration implies weak concentration (by choosing \( x_\varepsilon := x_0 \) for every \( \varepsilon \)). Then, the following result holds:

**Theorem 1.2.** Suppose that \( V \) has a unique global minimum point and that \( u_\varepsilon \) is a minimizer for \( \nu_\varepsilon(V) \). Assume in addition that \( V \) at infinity stays above such minimal value, i.e.
\[
\liminf_{|x| \rightarrow +\infty} V(x) > \min_{\mathbb{R}^N} V.
\] 

Then \( u_\varepsilon \) weakly concentrates at the global minimum point \( x_0 \) of \( V \). More precisely, the point \( x_\varepsilon \) in (1.9) is the unique global maximum point of \( u_\varepsilon \).
We emphasize that, in the above theorem, an additional complication is that the nonlocal operator $(-\Delta)^s$ does not “see” local maximum points. Namely if $y_\varepsilon$ is a local maximum point for $u_\varepsilon$, it is not necessarily true that $(-\Delta)^s u_\varepsilon(y_\varepsilon) \geq 0$ (and, as a matter of fact, the “local” behavior of “nonlocal” equations can be very wild: for instance all functions are locally $s$-harmonic up to an arbitrarily small error, see [11]). This feature makes the proof of the uniqueness of the global maximum point of $u_\varepsilon$ more delicate than in the classical case. About characterization of concentration sets for minimizers of singular perturbation problems we refer the reader to [1, 7, 19–22, 25] and some references therein.

Next result establishes a uniqueness property for the minimizers:

**Theorem 1.3.** Assume that $V \in C^1(\mathbb{R}^N)$, with $\inf_{\mathbb{R}^N} V > 0$ and it is radial. Let $v_\varepsilon$ be a minimizer for $\nu_\varepsilon(V)$. Then $v_\varepsilon$ is unique, provided that $\varepsilon$ small enough.

The rest of the paper is organized as follows. In Section 2 we study the concentration phenomena at given points of the space and we prove Theorem 1.1. The proof of Theorem 1.2 requires some preliminary work, that is carried out in Section 3. In particular, we obtain there an expansion of the minimizers of $\nu(V_\varepsilon)$ as perturbation of a suitable translation of the ground state (for this, no condition on the concentration point is required). The proof of Theorem 1.2 is then completed in Section 4. Then, Section 5 contains the preliminaries needed for the proof of Theorem 1.3, which, in turn, will be completed in Section 7.

2. Concentrations occurring at critical points of $V$ and proof of Theorem 1.1

In this section, we prove Theorem 1.1. We define

$$v_\varepsilon(x) := u_\varepsilon(\varepsilon x + x_0).$$

By construction, $v_\varepsilon$ is a positive solution of

$$(-\Delta)^s v_\varepsilon + V(\varepsilon x + x_0)v_\varepsilon - v_\varepsilon^p = 0 \quad \text{in } \mathbb{R}^N. \tag{2.1}$$

Roughly speaking, the idea is to take the derivative of (2.1), test it against $v_\varepsilon$, integrate by parts and hence send $\varepsilon \to 0$, in order to see that $\nabla V(x_0) = 0$: but to do these steps, some uniform regularity and decay estimates in $\varepsilon$ are in order. To obtain these estimates, we define

$$m_\varepsilon := \max_{\mathbb{R}^N} v_\varepsilon = \|u_\varepsilon\|_{L^\infty}. \tag{2.2}$$

We claim that

$$m := \sup_{\varepsilon \in (0,1)} m_\varepsilon < +\infty. \tag{2.3}$$

The proof is based on a classical contradiction and scaling arguments. Namely, suppose that

$$m_\varepsilon \to +\infty, \tag{2.3}$$

up to a subsequence. Now we recall (1.4) and we use (1.7) with

$$\delta := \min \left\{ 1, \left( \frac{\bar{V}}{2} \right)^{\frac{1}{p-1}} \right\}. \tag{2.3}$$

Accordingly, we obtain that there exists $R_1 > 0$ for which

$$u_\varepsilon(x) \leq \min \left\{ 1, \left( \frac{\bar{V}}{2} \right)^{\frac{1}{p-1}} \right\} \quad \text{for any } y \in \mathbb{R}^N \text{ such that } |y - x_0| \geq \varepsilon R_1, \tag{2.4}$$

as long as $\varepsilon$ is small enough. Now we notice that if $x \in \mathbb{R}^N \setminus B(0, R_1)$ then $|(\varepsilon x + x_0) - x_0| \geq \varepsilon R_1$: hence (2.4) implies that

$$v_\varepsilon(x) \leq \min \left\{ 1, \left( \frac{\bar{V}}{2} \right)^{\frac{1}{p-1}} \right\} \quad \text{for any } x \in \mathbb{R}^N \setminus B(0, R_1). \tag{2.5}$$

From (2.3) and (2.5), we conclude that, for small $\varepsilon$,

$$1 < m_\varepsilon = \max_{B(0,R_1)} v_\varepsilon, \tag{2.6}$$

and there exists

$$x_\varepsilon \in B(0,R_1) \tag{2.6}$$
maximizing $v_\varepsilon$, that is
\[ m_\varepsilon = \max_{\mathbb{R}^N} v_\varepsilon = v_\varepsilon(x_\varepsilon). \]
So, we set $\mu_\varepsilon := m_{\varepsilon}^{1-p}/2s$ and $w_\varepsilon(x) := m_{\varepsilon}^{-1}v_\varepsilon(x_\varepsilon + \varepsilon\mu_\varepsilon x)$. Then $\|w_\varepsilon\|_{L^{\infty}} = 1 = w_\varepsilon(0)$ and
\[ (-\Delta)^s w_\varepsilon(x) = -\mu_\varepsilon^{2s} V(\varepsilon(x_\varepsilon + \varepsilon\mu_\varepsilon)) w_\varepsilon(x) + w_\varepsilon^p(x). \quad (2.7) \]
Notice that $\mu_\varepsilon \to 0$ as $\varepsilon \to 0$, thanks to (2.3). Therefore, by (2.7), we have that $\|(-\Delta)^s w_\varepsilon\|_{L^{\infty}}$ is bounded uniformly in $\varepsilon$. As a consequence of this and of the regularity results (see e.g. Lemma 4.4 in [3], see also [24]), we deduce that $\|w_\varepsilon\|_{C^{2,\alpha}}$ is bounded uniformly in $\varepsilon$, for some $\alpha \in (0, 1)$. Hence, we can suppose that $w_\varepsilon$ converges to some function $w_0$ in $C^{2,\alpha}_{\text{loc}}$, with
\[ \|w_0\|_{L^{\infty}} = 1 = w_0(0). \quad (2.8) \]
By passing to the limit in (2.7), we obtain that
\[ (-\Delta)^s w_0 = w_0^p \quad \text{in} \quad \mathbb{R}^N. \quad (2.9) \]
Since the only non-negative and bounded solution of (2.9) with $p$ subcritical (according to (1.3)) is the one constantly equal to zero (see Remark 1.2 in [17] or Theorem 1.3 in [13]), we conclude that $w_0$ vanishes identically, in contradiction with (2.8).

This completes the proof of (2.2). As a consequence of (2.1), (2.2) and of the regularity results (see e.g. Lemma 4.4 in [3]), we conclude that
\[ \|v_\varepsilon\|_{C^{2,\alpha}} \quad \text{is bounded uniformly in} \quad \varepsilon, \quad (2.10) \]
hence we may suppose that
\[ v_\varepsilon \quad \text{converges to some function} \quad v_0 \quad \text{in} \quad C^{2,\alpha}_{\text{loc}}. \quad (2.11) \]
Now, since $x_\varepsilon$ maximizes $v_\varepsilon$, we have that
\[ 2v_\varepsilon(x_\varepsilon) - v_\varepsilon(x_\varepsilon + y) - v_\varepsilon(x_\varepsilon - y) \geq 0 \quad \text{for any} \quad y \in \mathbb{R}^N, \]
and so, using (1.6) and (2.1),
\[ 0 \leq (-\Delta)^s v_\varepsilon(x_\varepsilon) = v_\varepsilon^p(x_\varepsilon) - V(\varepsilon x_\varepsilon + x_0) v_\varepsilon(x_\varepsilon). \]
Accordingly,
\[ V(\varepsilon x_\varepsilon + x_0) \leq v_\varepsilon^{p-1}(x_\varepsilon). \quad (2.12) \]
Since $|x_\varepsilon|$ is bounded uniformly in $\varepsilon$, in light of (2.6), we suppose, up to a subsequence, that $x_\varepsilon \to \bar{x}$, for some $\bar{x} \in B(0, R_1)$, as $\varepsilon \to 0$. Thus, by taking the limit as $\varepsilon \to 0$ in (2.12), we obtain that
\[ 0 < \bar{V} \leq V(x_0) \leq v_0^{p-1}(\bar{x}). \]
In particular,
\[ v_0 \quad \text{is not identically zero.} \quad (2.13) \]
Next we claim that there exists $\varepsilon_0 > 0$ such that
\[ v_\varepsilon(x) \leq \frac{\text{Const}}{1 + |x|^{N+2s}} \quad \forall \varepsilon \in (0, \varepsilon_0). \quad (2.14) \]
To prove this, we use Lemma 4.2 of [14], according to which there exists a function $\bar{w}$ such that
\[ 0 \leq \bar{w}(x) \leq \frac{\text{Const}}{1 + |x|^{N+2s}} \quad (2.15) \]
and
\[ (-\Delta)^s \bar{w} + \frac{\bar{V}}{2} w \geq 0 \quad \text{in} \quad \mathbb{R}^N \setminus B(0, \bar{R}), \quad (2.16) \]
for a suitable $\bar{R} > 0$. Now, we take
\[ R_2 := \min\{R_1, \bar{R}\}, \quad (2.17) \]
where $R_1$ is the one in (2.5). Thanks to (1.7), we have that $v_\varepsilon$ converges to zero as $|x| \to \infty$ uniformly with respect to $\varepsilon$. From (2.5), we obtain
\[ (-\Delta)^s v_\varepsilon + \frac{\bar{V}}{2} v_\varepsilon = (-\Delta)^s v_\varepsilon + V v_\varepsilon - \left( V - \frac{\bar{V}}{2} \right) v_\varepsilon \]
\[ = v_\varepsilon^p - \left( V - \frac{\bar{V}}{2} \right) v_\varepsilon \leq v_\varepsilon^p - \frac{\bar{V}}{2} v_\varepsilon = v_\varepsilon \left( v_\varepsilon^{p-1} - \frac{\bar{V}}{2} \right) \leq 0. \quad (2.18) \]
Now we set
\[ b := \inf_{B(0,R_2)} \bar{w} > 0 \]  
(2.19)
and
\[ z_\varepsilon := (m+1)\bar{w} - bv_\varepsilon, \]  
(2.20)
where \( m \) is given in (2.2). Our goal is to show that
\[ z_\varepsilon \geq 0 \quad \text{in} \quad \mathbb{R}^N. \]  
(2.21)
For this we argue by contradiction and suppose that
\[ 0 > \inf_{\mathbb{R}^N} z_\varepsilon = \lim_{j \to +\infty} z_\varepsilon(x_{j,\varepsilon}), \]  
(2.22)
for a suitable sequence \( x_{j,\varepsilon} \). Notice that
\[ \lim_{|x| \to +\infty} |x|^{n+2s} u_\varepsilon(x) = 0, \]  
due to (2.15), and
\[ \lim_{|x| \to +\infty} v_\varepsilon(x) = 0, \]  
due to our integrability and continuity assumptions on \( u_\varepsilon \), and therefore
\[ \lim_{|x| \to +\infty} z_\varepsilon(x) = 0. \]  
Consequently, the sequence \( x_{j,\varepsilon} \) is bounded and therefore, up to subsequence, we suppose that \( x_{j,\varepsilon} \to x_{*,\varepsilon} \) as \( j \to +\infty \), for some \( x_{*,\varepsilon} \in \mathbb{R}^N \). So (2.22) becomes
\[ 0 > \min_{\mathbb{R}^N} z_\varepsilon = z_\varepsilon(x_{*,\varepsilon}). \]  
(2.23)
The minimality property of \( x_{*,\varepsilon} \) and (1.6) give that
\[ (-\Delta)^s z_\varepsilon(x_{*,\varepsilon}) = c(n,s) \int_{\mathbb{R}^N} \frac{2z_\varepsilon(x_{*,\varepsilon}) - z_\varepsilon(x_{*,\varepsilon}+y) - z_\varepsilon(x_{*,\varepsilon}-y)}{|y|^{n+2s}} \, dy \leq 0. \]  
(2.24)
Now notice that, by (2.2) and (2.19),
\[ z_\varepsilon \geq mb + \bar{w} - bm > 0 \quad \text{in} \quad B(0,R_2). \]  
Comparing this with (2.23), we see that
\[ x_{*,\varepsilon} \in \mathbb{R}^N \setminus B(0,R_2). \]  
(2.25)
Moreover, from (2.16), (2.17) and (2.18), we obtain that
\[ (-\Delta)^s z_\varepsilon + \frac{V}{2} z_\varepsilon \geq 0 \quad \text{in} \quad \mathbb{R}^N \setminus B(0,R_2). \]  
(2.26)
Thanks to (2.25), we can evaluate (2.26) at the point \( x_{*,\varepsilon} \): in this way, and recalling (2.23) and (2.24), we obtain that
\[ 0 \leq (-\Delta)^s z_\varepsilon(x_{*,\varepsilon}) + \frac{V}{2} z_\varepsilon(x_{*,\varepsilon}) < 0. \]  
This is a contradiction, so (2.21) is established.
From (2.21), we deduce that \( v_\varepsilon \leq (m+1)b^{-1}\bar{w} \), which, together with (2.15), completes the proof of (2.14).
Using (2.11) and (2.14) and the dominated convergence theorem, we see that
\[ \lim_{\varepsilon \to 0} \int_{\mathbb{R}^N} \partial_i V(\varepsilon x + x_0) v_\varepsilon^2 = \partial_i V(x_0) \int_{\mathbb{R}^N} v_0^2, \]  
(2.27)
for any \( i \in \{1, \ldots, N\} \).
Now we show that
\[ |\nabla v_\varepsilon| \in L^2. \]  
(2.28)
For this, we write (2.1) as
\[ (-\Delta)^sv_\varepsilon = -V(\varepsilon x + x_0) v_\varepsilon + v_\varepsilon^p =: \psi_\varepsilon. \]  
(2.29)
We know from (2.14) that
\[ \psi_\varepsilon(x) \leq \frac{\text{Const}}{1 + |x|^{N+2s}}. \]
Thus we take the fundamental solution \( \Gamma \) of the operator in (2.29) and we obtain that
\[ v_\varepsilon(x) = \text{Const} \int_{\mathbb{R}^N} \frac{\psi_\varepsilon(y)}{|x - y|^{N-2s}} \, dy. \]
Therefore
\[ |\nabla v_\varepsilon(x)| \leq \text{Const} \int_{\mathbb{R}^N} \frac{1}{|x - y|^{N-2s+1}} \, dy \leq \frac{\text{Const}}{(1 + |y|^{N+2s})} \int_{\mathbb{R}^N} \frac{1}{|x - y|^{N-2s+1}} \, dy. \]
So, fixing \( x \in \mathbb{R}^N \setminus B(0, 2) \), we observe that
\[ \int_{B(x, 1)} \frac{1}{|x - y|^{N-2s+1}} \, dy \leq \frac{\text{Const}}{|x|^{N+2s}} \int_{B(0, 1)} \frac{1}{|\xi|^{N-2s+1}} \, d\xi \leq \frac{\text{Const}}{|x|^{N+2s}}, \]
therefore we obtain that
\[ |\nabla v_\varepsilon(x)| \leq \text{Const} \left[ \int_{\mathbb{R}^N \setminus B(x, 1)} \frac{1}{(1 + |y|^{N+2s})} \, dy + \frac{1}{|x|^{N+2s}} \right]. \]
Accordingly, by the properties of the convolution of decaying kernels (see e.g. Lemma 5.1 in [8]), we obtain that
\[ |\nabla v_\varepsilon(x)| \leq \frac{\text{Const}}{|x|^{\kappa}}, \quad (2.30) \]
with \( \kappa := \min\{N + 2s, N - 2s + 1\} \). Notice that
\[ 2\kappa = \min\{2N + 4s, N + N - 4s + 2\} > \min\{2N, 2s + N - 4s + 2\} > N, \]
hence (2.30) implies (2.28), as desired.

Now we perform some calculations on integrals that involve \( v_\varepsilon \). For this, we let \( e_i \) be the \( i \)th vector of the standard Euclidean base, we fix \( R > 1 \) and we use the divergence theorem to see that, for any \( i \in \{1, \ldots, N\} \),
\[ \int_{B(0,R)} \partial_i v_\varepsilon^{p+1} = \int_{B(0,R)} \text{div}(v_\varepsilon^{p+1} e_i) = \int_{\partial B(0,R)} v_\varepsilon^{p+1} \frac{x_i}{R}. \]
Thus, from (2.14), we have
\[ \int_{B(0,R)} \partial_i v_\varepsilon^{p+1} = O(R^{N-1-(p+1)(N+2s)}). \quad (2.31) \]
Similarly,
\[ \int_{B(0,R)} V(\varepsilon x + x_0) \partial_i v_\varepsilon^2 = \int_{B(0,R)} \left[ \text{div} \left( V(\varepsilon x + x_0) v_\varepsilon^2 e_i \right) - \varepsilon \partial_i V(\varepsilon x + x_0) v_\varepsilon^2 \right] \]
\[ = \int_{\partial B(0,R)} V(\varepsilon x + x_0) v_\varepsilon^2 x_i \frac{1}{R} - \varepsilon \int_{B(0,R)} \partial_i V(\varepsilon x + x_0) v_\varepsilon^2 \]
which, together with (2.14), gives that
\[ \int_{B(0,R)} V(\varepsilon x + x_0) \partial_i v_\varepsilon^2 = -\varepsilon \int_{B(0,R)} \partial_i V(\varepsilon x + x_0) v_\varepsilon^2 + O(R^{N-1-2(N+2s)}). \quad (2.32) \]
We summarize the estimates in (2.31) and (2.32) by writing
\[ \int_{B(0,R)} \left( \varepsilon \partial_i V(\varepsilon x + x_0) v_\varepsilon^2 + \frac{1}{2} V(\varepsilon x + x_0) \partial_i v_\varepsilon^2 - \frac{p}{p+1} \partial_i v_\varepsilon^{p+1} \right) \]
\[ = \frac{\varepsilon}{2} \int_{B(0,R)} \partial_i V(\varepsilon x + x_0) v_\varepsilon^2 + O(R^{N-1-2(N+2s)}). \quad (2.33) \]
Now, we point out that \((-\Delta)^s v_\varepsilon\) is \(C^2\) and bounded, due to (1.4), (2.1), (2.2) and (2.10) (recall also (1.4), therefore we can speak about \(\partial_i (-\Delta)^s v_\varepsilon\) in the classical sense. Accordingly, we can take a derivative, say in the \(i\)th coordinate direction, of (2.1): we get
\[
\partial_i (-\Delta)^s v_\varepsilon + \varepsilon \partial_i V(\varepsilon x + x_0) v_\varepsilon + V(\varepsilon x + x_0) \partial_i v_\varepsilon - p v_\varepsilon^{p-1} \partial_i v_\varepsilon = 0. \tag{2.34}
\]
So, recalling (2.10), (2.14) and (2.28), we see that
\[
\partial_i (-\Delta)^s v_\varepsilon \in L^2. \tag{2.35}
\]
Consequently, by Plancherel theorem, we obtain
\[
\int_{\mathbb{R}^N} v_\varepsilon \partial_i (-\Delta)^s v_\varepsilon = \int_{\mathbb{R}^N} \hat{v}_\varepsilon \mathcal{F}(\partial_i (-\Delta)^s v_\varepsilon)
= - \int_{\mathbb{R}^N} \xi_i \hat{v}_\varepsilon \mathcal{F}((-\Delta)^s v_\varepsilon) - \int_{\mathbb{R}^N} \xi_i |\xi|^2 \hat{v}_\varepsilon(\xi)|^2
= - \int_{\mathbb{R}^N} \xi_i |\mathcal{F}((-\Delta)^{s/2} v_\varepsilon)|^2.
\tag{2.36}
\]
We remark that
\[
(-\Delta)^{s/2} v_\varepsilon \in L^2. \tag{2.37}
\]
Indeed, since \(u_\varepsilon \in H^s\) (hence \(v_\varepsilon \in H^s\)), we have that
\[
\int_{\mathbb{R}^N} |\xi|^2 |\hat{v}_\varepsilon|^2 \leq \int_{\mathbb{R}^N} (1 + |\xi|^{2s}) |\hat{v}_\varepsilon|^2 < +\infty,
\]
therefore \(\mathcal{F}((-\Delta)^{s/2} v_\varepsilon) = |\xi|^s \hat{v}_\varepsilon \in L^2\) and so (2.37) follows from the Plancherel theorem.

Also, for any \(g \in L^2\), we have that
\[
\text{the map } \xi \mapsto |\hat{g}(\xi)|^2 \text{ is even.} \tag{2.38}
\]
To check this, notice that
\[
\hat{g}(\xi) = \int_{\mathbb{R}^N} g(x)e^{-ix\xi} \, dx = \int_{\mathbb{R}^N} g(x)e^{ix\xi} \, dx = \hat{g}(-\xi)
\]
and so
\[
\hat{g}(-\xi) = \hat{g}(\xi).
\]
As a consequence
\[
|\hat{g}(-\xi)|^2 = \hat{g}(-\xi)\hat{g}(-\xi) = \hat{g}(\xi)\hat{g}(\xi) = |\hat{g}(\xi)|^2,
\]
that proves (2.38).

So, from (2.37) and (2.38), we see that the map \(\xi \mapsto \xi_i |\mathcal{F}((-\Delta)^{s/2} v_\varepsilon)|^2\) is odd, and therefore
\[
\int_{B(0,R)} \xi_i |\mathcal{F}((-\Delta)^{s/2} v_\varepsilon)|^2 = 0
\]
for any \(R > 0\). By plugging this into (2.36) and recalling (2.35) we obtain
\[
\int_{\mathbb{R}^N} v_\varepsilon \partial_i (-\Delta)^s v_\varepsilon = \lim_{R \to +\infty} \int_{B(0,R)} v_\varepsilon \partial_i (-\Delta)^s v_\varepsilon = 0. \tag{2.39}
\]
Now we go back to (2.34) and we multiply this equation by \(v_\varepsilon\): in this way we obtain that
\[
0 = v_\varepsilon \partial_i (-\Delta)^s v_\varepsilon + \varepsilon \partial_i V(\varepsilon x + x_0) v_\varepsilon^2 + V(\varepsilon x + x_0) \partial_i v_\varepsilon - p v_\varepsilon^{p-1} \partial_i v_\varepsilon
= v_\varepsilon \partial_i (-\Delta)^s v_\varepsilon + \varepsilon \partial_i V(\varepsilon x + x_0) v_\varepsilon^2 + \frac{1}{2} V(\varepsilon x + x_0) \partial_i v_\varepsilon^2 - \frac{p}{p+1} \partial_i v_\varepsilon^{p+1}.
\]
We fix \(R > 1\) and we integrate the above equation on \(B(0,R)\): thus, exploiting (2.33) we obtain
\[
\int_{B(0,R)} v_\varepsilon \partial_i (-\Delta)^s v_\varepsilon + \frac{\varepsilon}{2} \int_{B(0,R)} \partial_i V(\varepsilon x + x_0) v_\varepsilon^2 = O(R^{N-1-2(N+2s)}).
\]
So we send \(R \to +\infty\), recalling also (2.39) and we divide by \(\varepsilon\), we get
\[
\int_{\mathbb{R}^N} \partial_i V(\varepsilon x + x_0) v_\varepsilon^2 = 0.
\]
Now we send $\varepsilon \to 0$: recalling (2.27) we conclude that
\[ \partial_i V(x_0) \int_{\mathbb{R}^N} v^2 = 0. \]
Therefore, by (2.13), we obtain that $\partial_i V(x_0) = 0$ for any $i \in \{1, \ldots, N\}$, and this completes the proof of Theorem 1.1.

3. Concentration points of ground-states: preliminary work for the proof of Theorem 1.2

In this section we discuss some basic concentration properties of the minimizers. For this, we recall that, for any $\lambda > 0$ there exists a unique function $U_\lambda$ that attains the following minimization problem
\[ \nu(\lambda) := \inf_{\|u\|_{L^{p+1}} = 1} \|u\|_{L^{p+2}}^2 + \lambda \|u\|_{L^2}^2. \]
In addition, such minimizer is unique radially symmetric and belongs to $C^\infty \cap H^{2s+1}(\mathbb{R}^N)$ (we refer to [16] for further details on this, see in particular Theorem 4 there). Thus, we will denote by $\overline{U}$ the radially symmetric function that attains
\[ \inf_{\|u\|_{L^{p+1}} = 1} \|u\|_{L^{p+2}} + \overline{V} \|u\|_{L^2}, \]
where
\[ \overline{V} := \inf_{\mathbb{R}^N} V. \]

With this notation, we provide an asymptotic expansion for the minimizers of $\nu(V_\varepsilon)$. It is worth pointing out that this expansion is valid without assuming any structural condition on the potential $V$ (in particular the point $x_0 \in \mathbb{R}^n$ can be fixed, without assuming that is minimal or critical):

**Lemma 3.1.** Let $v_\varepsilon$ be a positive minimizer for $\nu(V_\varepsilon)$, with $\|v_\varepsilon\|_{L^{p+1}} = 1$. Then there exists a sequence of points $a_\varepsilon, c \in \mathbb{R}^N$ such that, up to a subsequence,
\[ v_\varepsilon(x + a_\varepsilon) = \overline{U}(x - c) + \omega_\varepsilon(x), \]
with $\|\omega_\varepsilon\|_{H^s} \to 0$ as $\varepsilon \to 0$.

Also
\[ \lim_{\varepsilon \to 0} \nu(V_\varepsilon) = \nu(\overline{V}) \]
and, for any $x \in \mathbb{R}^n$,
\[ \lim_{\varepsilon \to 0} V(\varepsilon x + \varepsilon a_\varepsilon + x_0) = \overline{V}. \]

**Proof.** We observe that
\[ \|u\|^2_{L^2} \in \left[ \overline{V} \|u\|^2_{L^2}, \|V\|_{L^\infty} \|u\|^2_{L^2} \right] \]
thanks to (1.4), and therefore $\nu(V_\varepsilon)$ is bounded (and bounded from zero) uniformly in $\varepsilon$. Hence, up to a subsequence, we suppose that
\[ \nu(V_\varepsilon) \to \overline{\nu} \]
as $\varepsilon \to 0$, for some $\overline{\nu} > 0$.

Also, $v_\varepsilon$ is bounded in $H^s$ and, using Lemma 2.2 in [14], we have that there exists $a_\varepsilon \in \mathbb{R}^N$ and positive real numbers $R$ and $\gamma$ such that
\[ \liminf_{\varepsilon \to 0} \int_{B_R(a_\varepsilon)} v_\varepsilon(x) \, dx \geq \gamma. \]
Thus, setting $w_\varepsilon(x) = v_\varepsilon(x + a_\varepsilon)$, we have that $w_\varepsilon$ is bounded in $H^s$ so it converges, up to a subsequence, to a function $w \in H^s$ weakly in $H^s$, strongly in $L^{p+1}_{\text{loc}}$ and a.e.; furthermore, by (3.4), we have that $w \neq 0$.

We also notice that, by (1.4) and the theorem of Ascoli, there exists $\lambda : \mathbb{R}^N \to \mathbb{R}$ such that, up to a subsequence,
\[ \lim_{\varepsilon \to 0} V(\varepsilon x + \varepsilon a_\varepsilon + x_0) = \lambda(x). \]
We set $\lambda := \lambda(0)$ and we claim that
\[ \lambda(x) = \lambda \]
(3.6)
for any $x \in \mathbb{R}^N$. Indeed, for any $x \in \mathbb{R}^N$,
\[
|\lambda(x) - \lambda| = \lim_{\varepsilon \to 0} |V(\varepsilon x + \varepsilon a + x_0) - V(\varepsilon a + x_0)|
\leq \text{Const} \lim_{\varepsilon \to 0} |\varepsilon x| = 0,
\]
thanks to (1.4), and this proves (3.6).

By (3.5) and (3.6), we can write
\[
\lim_{\varepsilon \to 0} V(\varepsilon x + \varepsilon a + x_0) = \lambda.
\] (3.7)

Since, by (2.1),
\[
(-\Delta)^sw \pm V(\varepsilon x + \varepsilon a + x_0) = \nu(V_\varepsilon)w_\varepsilon,
\]
we can pass to the limit and obtain
\[
(-\Delta)^sw + \lambda w = \tilde{\nu}w.
\] (3.8)

By testing (3.8) against $w$ we obtain that
\[
\tilde{\nu} = \frac{\|w\|_{D^{s,2}}^2 + \lambda \|w\|_{L^2}^2}{\|w\|_{L^{p+1}}^2} \geq \inf_{u \neq 0} \frac{\|u\|_{D^{s,2}}^2 + \lambda \|u\|_{L^2}^2}{\|u\|_{L^{p+1}}^2} = \nu(\lambda).
\] (3.9)

On the other hand, by the dominated convergence theorem, we see that, for any $u \in C^\infty_c$,
\[
\lim_{\varepsilon \to 0} \int_{\mathbb{R}^N} |V(\varepsilon x + \varepsilon a + x_0) - \lambda| u^2(x) \, dx = 0.
\]
As a consequence, for any $u \in C^\infty_c$, $u \neq 0$, we set $\tilde{u}_\varepsilon(x) := u(x - a_\varepsilon)$ and we observe that
\[
\tilde{\nu} = \lim_{\varepsilon \to 0} \nu(V_\varepsilon)
\leq \frac{\|u\|_{D^{s,2}}^2 + \|u\|_{L^2}^2}{\|u\|_{L^{p+1}}^2}
\leq \frac{\|\tilde{u}_\varepsilon\|_{D^{s,2}}^2 + \int_{\mathbb{R}^N} V(\varepsilon x + x_0) \tilde{u}_\varepsilon^2(x) \, dx}{\|\tilde{u}_\varepsilon\|_{L^{p+1}}^2}
\leq \frac{\|u\|_{D^{s,2}}^2 + \int_{\mathbb{R}^N} V(\varepsilon x + x_0) u^2(x - a_\varepsilon) \, dx}{\|u\|_{L^{p+1}}^2}
\leq \frac{\|u\|_{D^{s,2}}^2 + \lambda \|u\|_{L^2}^2}{\|u\|_{L^{p+1}}^2}.
\]
By density, this is valid for any $u \in H^s$, and so, taking the infimum over $u \neq 0$, we obtain that $\tilde{\nu} \leq \nu(\lambda)$. This and (3.9) give that
\[
\tilde{\nu} = \nu(\lambda).
\] (3.10)

This, (3.8) and the uniqueness of the ground state (see Theorem 4 in [16]) give that $w$ is a translation of $U_\lambda$, namely $w(x) = U_\lambda(x - c)$, for some $c \in \mathbb{R}^N$.

Now we claim that
\[
\lambda = \tilde{V}
\] (3.11)
To prove this, let us fix $p \in \mathbb{R}^n$. Then, for any $u \in C_c^\infty$, we set $u_\varepsilon(x) := u(x + \varepsilon^{-1}(x_0 - p))$ and we use the change of variable $y := x + \varepsilon^{-1}(p - x_0)$ to obtain that

$$
\frac{\|u\|_{L^{p+2},2}^2 + \int_{\mathbb{R}^N} V(\varepsilon x + p) u^2(x) \, dx}{\|u\|_{L^{p+1},2}^2} = \frac{\|u\|_{L^{p+2},2}^2 + \int_{\mathbb{R}^N} V(\varepsilon y + x_0) u^2(y + \varepsilon^{-1}(x_0 - p)) \, dy}{\|u\|_{L^{p+1},2}^2}
$$

$$
= \frac{\|u_\varepsilon\|_{L^{p+2},2}^2 + \int_{\mathbb{R}^N} V(\varepsilon y + x_0) u_\varepsilon^2(y) \, dy}{\|u_\varepsilon\|_{L^{p+1},2}^2}
$$

$$
= \frac{\|u_\varepsilon\|_{L^{p+2},2}^2}{\|u_\varepsilon\|_{L^{p+1},2}^2} \geq \nu(V_\varepsilon).
$$

So, by (3.3), (3.10) and the dominated convergence theorem, we obtain

$$
\frac{\|u\|_{L^{p+2},2}^2 + V(p)\|u\|_{L^2}^2}{\|u\|_{L^{p+1},2}^2} = \lim_{\varepsilon \to 0} \frac{\|u\|_{L^{p+2},2}^2 + \int_{\mathbb{R}^N} V(\varepsilon x + p) u^2(x) \, dx}{\|u\|_{L^{p+1},2}^2}
$$

$$
\geq \lim_{\varepsilon \to 0} \nu(V_\varepsilon) = \tilde{\nu} = \nu(\lambda) = \frac{\|U_\lambda\|_{L^{p+2},2}^2 + \lambda\|U_\lambda\|_{L^2}^2}{\|U_\lambda\|_{L^{p+1},2}^2}.
$$

This is valid for any $u \in C_c^\infty$ and so, by density, also for $U_\lambda$. Thus we conclude that

$$
\frac{\|U_\lambda\|_{L^{p+2},2}^2 + V(p)\|U_\lambda\|_{L^2}^2}{\|U_\lambda\|_{L^{p+1},2}^2} \geq \frac{\|U_\lambda\|_{L^{p+2},2}^2 + \lambda\|U_\lambda\|_{L^2}^2}{\|U_\lambda\|_{L^{p+1},2}^2}
$$

and therefore

$$
V(p) \geq \lambda.
$$

Now, this is valid for any $p \in \mathbb{R}^N$, thus, recalling (3.1), we obtain that

$$
\tilde{V} = \inf_{p \in \mathbb{R}^N} V(p) \geq \lambda.
$$

The other inequality follows from (3.7), and so the proof of (3.11) is complete.

Then, (3.11) and the definition of $\tilde{U}$ give that $U_\lambda = \tilde{U}$. Accordingly,

$$
v_\varepsilon(x + a_\varepsilon) = w_\varepsilon(x) \to w(x) = U_\lambda(x - c) = \tilde{U}(x - c)
$$

weakly in $H^s$, strongly in $L^{p+1}_{loc}$ and a.e., so to complete the proof of Lemma 3.1 it only remains to show that the convergence occurs strongly in $H^s$. To see this, we use the fact that $w$ is a minimizer for the quotient $\nu(\tilde{V}) = \nu(\lambda) = \tilde{\nu}$, hence

$$
\tilde{\nu} = \frac{\|w\|_{L^{p+2},2}^2 + \lambda\|w\|_{L^2}^2}{\|w\|_{L^{p+1},2}^2}.
$$

On the other hand, by testing (3.8) against $w$, we obtain that

$$
\|w\|_{L^{p+2},2}^2 + \lambda\|w\|_{L^2}^2 = \tilde{\nu}\|w\|_{L^{p+1},2}^2.
$$

By comparing this with (3.12), we conclude that $\|w\|_{L^{p+1},2} = 1$. Therefore

$$
\|v_\varepsilon\|_{L^{p+2},2}^2 + \lambda\|v_\varepsilon\|_{L^2}^2 = \|v_\varepsilon\|_{L^{p+2},2}^2 + \lambda\|v_\varepsilon\|_{L^2}^2
$$

$$
= \|v_\varepsilon\|_{L^{p+2},2}^2 + \lambda\|v_\varepsilon\|_{L^2}^2 + \int_{\mathbb{R}^N} (\lambda - V(\varepsilon x + x_0)) v_\varepsilon(x) \, dx
$$

$$
= \nu(V_\varepsilon) + \int_{\mathbb{R}^N} (\lambda - V(\varepsilon x + x_0)) v_\varepsilon(x) \, dx.
$$

Moreover, by (3.1) and (3.11),

$$
\lambda = \inf_{\mathbb{R}^N} V \leq V(\varepsilon x + x_0),
$$
thus we obtain that
\[ \|w_\varepsilon\|_{D^s,2}^2 + \lambda \|w_\varepsilon\|_{L^2}^2 \leq \nu(V_\varepsilon). \]
So, from the weak convergence and Fatou lemma, passing to the limit we obtain that
\[ \bar{\nu} = \|w\|_{D^s,2}^2 + \lambda \|w\|_{L^2}^2 \leq \liminf_{\varepsilon \to 0} \|w_\varepsilon\|_{D^s,2}^2 + \lambda \|w_\varepsilon\|_{L^2}^2 \]
\[ \leq \limsup_{\varepsilon \to 0} \|w_\varepsilon\|_{D^s,2}^2 + \lambda \|w_\varepsilon\|_{L^2}^2 \leq \limsup_{\varepsilon \to 0} \nu(V_\varepsilon) = \bar{\nu}. \]
This gives that
\[ \lim_{\varepsilon \to 0} \|w_\varepsilon\|_{D^s,2}^2 + \lambda \|w_\varepsilon\|_{L^2}^2 = \|w\|_{D^s,2}^2 + \lambda \|w\|_{L^2}^2. \]
By making use of this and of the weak convergence of \( w_\varepsilon \), we infer that \( w_\varepsilon \to w \) in the Hilbert norm \( \sqrt{\|\cdot\|_{D^s,2}^2 + \lambda \|\cdot\|_{L^2}^2} \).
Since this norm is equivalent to the one in \( H^s \), we have proved that \( w_\varepsilon \to w \) in \( H^s \).

4. COMPLETION OF THE PROOF OF THEOREM 1.2
Now we finish the proof of Theorem 1.2. For this, we suppose that \( V \) has a unique global minimum point at \( x_0 \).
Let \( u_\varepsilon \) be a minimizer for \( \nu_\varepsilon(V) \). Then \( \nu_\varepsilon(x) := u_\varepsilon(x_0 + \varepsilon x) \) is a minimizer for \( \nu(V_\varepsilon) \). By Lemma 3.1, there are points \( a_\varepsilon, c \in \mathbb{R}^N \) such that, up to a subsequence,
\[ w_\varepsilon(x) := u_\varepsilon(x + a_\varepsilon) = \tilde{U}(x - c) + \omega_\varepsilon(x), \]
where \( \|\omega_\varepsilon\|_{H^s} \to 0 \), the function \( \tilde{U} \) is a minimizer for \( \nu(\tilde{V}) \), and, comparing (3.1) and (3.2), we have that
\[ \lim_{\varepsilon \to 0} V(\varepsilon a_\varepsilon + x_0) = \tilde{V} = \min_{x \in \mathbb{R}^N} V(x) = V(x_0). \quad (4.1) \]
Now we prove that
\[ \lim_{\varepsilon \to 0} \varepsilon a_\varepsilon = 0. \quad (4.2) \]
Suppose not, say \( |\varepsilon a_\varepsilon| \geq a_0 \) for some \( a_0 > 0 \) and an infinitesimal sequence of \( \varepsilon \)'s. Then \( |\varepsilon a_\varepsilon| \) remains bounded, otherwise, by (1.10), the limit in (4.1) would be strictly larger than \( V(x_0) \).
Accordingly, there exists an infinitesimal sequence of \( \varepsilon \)'s for which \( \varepsilon a_\varepsilon \to \alpha \), for some \( \alpha \in \mathbb{R}^N \) with \( |\alpha| \geq a_0 > 0 \). From this and (4.1), we obtain that
\[ V(x_0) = \lim_{\varepsilon \to 0} V(\varepsilon a_\varepsilon + x_0) = V(\alpha + x_0). \]
This contradicts the uniqueness of the minimal point for \( V \), and so it proves (4.2).
Now we claim that
\[ \sup_{\varepsilon} \int_{|x| \geq R} w_\varepsilon^r \, dx \to 0 \quad \text{as} \quad R \to \infty, \quad (4.3) \]
with \( r := \frac{2N}{N - 2s} \).
To see this, we can assume by contradiction that there exists \( \delta \) positive and a sequence \( R_n \to \infty \) such that
\[ \sup_{\varepsilon} \int_{|x| \geq R_n} w_\varepsilon^r \, dx \geq \delta \quad \text{as} \quad n \to \infty, \]
This implies that for a sequence of \( \varepsilon_n \to 0 \), we have
\[ \int_{|x| \geq R_n} w_\varepsilon^r \, dx \geq \delta \quad \text{as} \quad n \to \infty. \]
Because \( w_\varepsilon \) converges strongly in \( L^r \), we have (see e.g. [2, Theorem 4.9]) that there exists \( h \in L^r \) and a subsequence, still denoted by \( \varepsilon_n \) such that \( w_\varepsilon \leq h \) a.e. in \( \mathbb{R}^N \).
But then
\[ 0 < \delta \leq \int_{|x| \geq R_n} w_\varepsilon^r \, dx \leq \int_{|x| \geq R_n} h^r \, dx \to 0 \quad \text{as} \quad n \to \infty. \]
This leads to a contradiction. We thus have proved (4.3).
Next we observe that \( (-\Delta)^s w_\varepsilon - \nu(\tilde{V}_\varepsilon) w_\varepsilon^{p-1} w_\varepsilon \leq 0 \) in \( \mathbb{R}^N \). Since \( w_\varepsilon^{p-1} \in L^q_{loc} \) for some \( q > \frac{N}{2s} \), we deduce from [17, Proposition 2.6] that for any compact set \( K \), we have
\[ \max_K w_\varepsilon \leq C \int_K w_\varepsilon^r \, dx, \]
\(^2\)If \( N \leq 2s \), the above definition of \( r \) can be replaced by just fixing \( r \in (1, +\infty) \).
where $r$ is as above. We therefore conclude from (4.3) that
\[
\sup_{\varepsilon} w_\varepsilon(x) \to 0 \quad \text{as } |x| \to \infty.
\]
This together with Lemma C.2 in [16] also imply that
\[
w_\varepsilon(x) \leq \frac{\text{Const}}{1 + |x|^{N+2s}}.
\] (4.4)
By scaling back, we obtain
\[
u_\varepsilon(x) = \nu_\varepsilon \left( \frac{x - x_0}{\varepsilon} \right) = w_\varepsilon \left( \frac{x - x_0 - \varepsilon a_\varepsilon}{\varepsilon} \right) \leq \frac{\text{Const} \varepsilon^{N+2s}}{\varepsilon^{N+2s} + |x - x_0 - \varepsilon a_\varepsilon|^{N+2s}}.
\] (4.5)

It is then clear that $u_\varepsilon$ concentrates at $x_0$ in the sense of (1.9).

Now to prove the last statement of the theorem ($u_\varepsilon$ has a unique global maximum point), we observe that $u_\varepsilon \in C^2_{\text{loc}}$ and by (4.5), we have $\lim_{|x| \to \infty} u_\varepsilon(x) = 0$ for every fixed and positive $\varepsilon$. We can therefore let $u(x_\varepsilon) = \max_{R^N} u_\varepsilon$.

Then $(-\Delta)^s u_\varepsilon(x_\varepsilon) \geq 0$ and thus from (1.2) (recalling (1.4)), we deduce that
\[
u(x_\varepsilon) \geq \left( \frac{V}{\nu(V)} \right)^{\frac{1}{2s}} =: C_0.
\]
Hence by (4.5), we get
\[
C_0 \leq \frac{\text{Const} \varepsilon^{N+2s}}{\varepsilon^{N+2s} + |x_\varepsilon - x_0 - \varepsilon a_\varepsilon|^{N+2s}}.
\]
so that
\[
|x_\varepsilon - x_0 - \varepsilon a_\varepsilon| \leq C_1 \varepsilon.
\] (4.6)
From this we conclude, provided $|x - x_\varepsilon| \geq \varepsilon R \geq 2 \varepsilon C_1$, that
\[
u(x_\varepsilon) \leq \frac{\text{Const}}{1 + R - C_1} \leq \frac{\text{Const}}{1 + R/2}
\]
and this completes the proof of concentration of $u_\varepsilon$ at $x_0$.

We now prove the uniqueness of $x_\varepsilon$. Indeed, we observe that
\[
(-\Delta)^s \omega_\varepsilon = (-\Delta)^s (w_\varepsilon - (-\Delta)^s \tilde{U}(-c)) = V(\varepsilon x + \varepsilon a_\varepsilon + x_\varepsilon)[\tilde{U}(-c) - w_\varepsilon] + [V(x_\varepsilon) - V(\varepsilon x + \varepsilon a_\varepsilon + x_\varepsilon)]\tilde{U}(-c) + [\nu(V_\varepsilon) - \nu(V(x_\varepsilon))]w_\varepsilon^p + \nu(V(x_\varepsilon))[w_\varepsilon^p - \tilde{U}^p(-c)]
\]
We rewrite this as
\[
(-\Delta)^s \omega_\varepsilon + \beta_\varepsilon(x) \omega_\varepsilon = [V(x_\varepsilon) - V(\varepsilon x + \varepsilon a_\varepsilon + x_\varepsilon)]\tilde{U}(-c) + [\nu(V_\varepsilon) - \nu(V(x_\varepsilon))]w_\varepsilon^p,
\]
where we have set
\[
\beta_\varepsilon(x) = V(\varepsilon x + \varepsilon a_\varepsilon + x_\varepsilon) - \frac{\nu(V(x_\varepsilon))[w_\varepsilon^p - \tilde{U}^p(-c)]}{w_\varepsilon - \tilde{U}(-c)}.
\]
By (4.4), we have $|w_\varepsilon^p - \tilde{U}^p(-c)| \leq C|w_\varepsilon - \tilde{U}(-c)|$ and thus $|\beta_\varepsilon(x)| \leq \text{Const.}$. Applying [17, Proposition 2.6], we deduce that $\omega_\varepsilon \to 0$ in $C^0_{\text{loc}}(\mathbb{R}^N)$ for some $\alpha \in (0,1)$. Now by a bootstrap argument and using Proposition 2.1.8 in [24], we conclude that $\omega_\varepsilon = w_\varepsilon - \tilde{U}(-c) \to 0$ in $C^2_{\text{loc}}(\mathbb{R}^N)$ for some $\alpha \in (0,1)$.

We now set $\bar{w}_\varepsilon(x) = w_\varepsilon(x + \bar{x}_\varepsilon)$ with $\bar{x}_\varepsilon = \frac{\bar{x}_\varepsilon - \bar{x}_\varepsilon}{\varepsilon}$, where $0 = \nabla \bar{w}_\varepsilon(0) = \nabla \tilde{U}(\bar{x}_\varepsilon - c) + \nabla \omega_\varepsilon(\bar{x}_\varepsilon)$. Recalling that $\tilde{U}$ is symmetric decreasing with respect to the origin, that has a unique critical point and also $\omega_\varepsilon \to 0$ in $C^2_{\text{loc}}(\mathbb{R}^N)$. Therefore from (4.6) we deduce that
\[
|\bar{x}_\varepsilon - c| \to 0.
\]
It is clear that any other global maximum point of $\bar{w}_\varepsilon$ must stay in a neighborhood of $c$. We then observe that
\[
\bar{w}_\varepsilon(x) = \tilde{U}(x) + \tilde{w}_\varepsilon(x),
\]
where $\tilde{w}_\varepsilon(x) = [\tilde{U}(x + \bar{x}_\varepsilon - c) - \tilde{U}(x)] + \omega_\varepsilon(x)$. Since $\tilde{U} \in C^\infty(\mathbb{R}^N)$, we obtain $\tilde{w}_\varepsilon \to 0 \in C^2_{\text{loc}}$. Now using Lemma 4.2 in [20], we conclude that the only critical point for $\bar{w}_\varepsilon$ is the origin.
Remark 4.1. We remark that from the above proof, the minimizers $u_\varepsilon$ for $\nu_\varepsilon(V)$ has the following precise form:

$$u_\varepsilon(\varepsilon x + x_\varepsilon) = \tilde{U}(x) + \tilde{\omega}_\varepsilon(x),$$

where $\tilde{\omega}_\varepsilon \to 0$ in $H^s(\mathbb{R}^N) \cap C^{2,\alpha}_{\text{loc}}(\mathbb{R}^N) \cap L^\infty(\mathbb{R}^N)$ with $x_\varepsilon$ the unique global maximizer for $u_\varepsilon$ and $x_\varepsilon$ converges to $x_0$ which is the global minimum point for $V$. Also $\tilde{U}$ is the unique minimizer for $\nu(V(x_0))$.

5. Non-degeneracy and uniqueness: preliminaries for the proof of Theorem 1.3

Now we will deal with the functional

$$J_\varepsilon(u, \nu(V_\varepsilon)) := \frac{1}{2} \|u\|^2 - \frac{p(V_\varepsilon)}{p + 1} \int_{\mathbb{R}^N} |u|^{p+1} \, dx$$

and we will consider the scalar products that induce the norms of the fractional spaces used in this paper, namely we set

$$\langle u, v \rangle_{D,2} := \int_{\mathbb{R}^N} |\xi|^{2s} \hat{u} \hat{v} \, d\xi,$$

$$\langle u, v \rangle_{\varepsilon,V} := \int_{\mathbb{R}^N} V(\varepsilon x + x_0) u(x) v(x) \, dx,$$

and

$$\langle u, v \rangle_{\varepsilon} := \langle u, v \rangle_{D,2} + \langle u, v \rangle_{\varepsilon,V}.$$

The Hilbert space associated with $\langle \cdot, \cdot \rangle_{\varepsilon}$ will be denoted by $H^s_{\varepsilon}$ and, as usual, we say that $u \perp_{\varepsilon} v$ whenever $\langle u, v \rangle_{\varepsilon} = 0$. One simple, but important feature, is that the radially symmetric minimizer $U$ for $\nu(V_0)$ is perpendicular in $H^s_{0}$ (that is $H^s_{\varepsilon}$ with $\varepsilon = 0$) to its derivatives, and the derivatives themselves are perpendicular to each other, according to the following result:

Lemma 5.1. For any $i \in \{1, \ldots, N\}$, we have that

$$\langle U, \partial_i U \rangle_0 = 0$$

and

$$\int_{\mathbb{R}^N} U^p \partial_i U = 0.$$ (5.2)

Moreover, for any $i, j \in \{1, \ldots, N\}$, with $i \neq j$, we have that

$$\int_{\mathbb{R}^N} U^{p-1} \partial_i U \partial_j U = 0$$ (5.3)

and

$$\langle \partial_i U, \partial_j U \rangle_0 = 0.$$ (5.4)

Proof. By construction

$$(-\Delta)^s U + V(0) U = U^p$$ (5.5)

and so, taking derivatives,

$$(-\Delta)^s (\partial_i U) + V(0) \partial_i U = pU^{p-1} \partial_i U.$$ (5.6)

We multiply (5.5) by $\partial_i U$ and (5.6) by $U$ and integrate: we obtain, respectively,

$$\langle U, \partial_i U \rangle_0 = \int_{\mathbb{R}^N} U^p \partial_i U$$ and $$\langle U, \partial_i U \rangle_0 = p \int_{\mathbb{R}^N} U^p \partial_i U.$$ (5.7)

By comparing these two equations we obtain that

$$p \int_{\mathbb{R}^N} U^p \partial_i U = \int_{\mathbb{R}^N} U^p \partial_i U,$$

and so

$$\langle U, \partial_i U \rangle_0 = \int_{\mathbb{R}^N} U^p \partial_i U = 0,$$

that proves (5.2).

Now we use the rotational invariance of $U$ to write $U(x) = \tilde{U}(|x|)$, for some $\tilde{U} : \mathbb{R} \to \mathbb{R}$. Then we have that $\partial_i U(x) = \tilde{U}'(|x|) |x|^{-1} x_i$ and so, by symmetry

$$\int_{\mathbb{R}^N} U^{p-1}(x) \partial_i U(x) \partial_j U(x) \, dx = \int_{\mathbb{R}^N} \tilde{U}^{p-1}(|x|) |\tilde{U}'(|x|)|^2 |x|^{-2} x_i x_j \, dx = 0.$$

This establishes (5.3). Then, formula (5.4) follows multiplying (5.6) by $\partial_j U$ and integrating over $\mathbb{R}^N$. □
Our next result is of coercivity type. It is stronger than what we will need in the following of the paper we expose it here because we believe that it might be of interest.

We also mention that in the rest of the paper, the regularity assumption can be relaxed to \( V \in C^1(\mathbb{R}^N) \).

Given the radially symmetric minimizer \( U \) for \( \nu(V_0) \) and \( a \in \mathbb{R}^N \), we define \( U_a(x) := U(x-a) \) and
\[
W_\varepsilon := \left\{ v \in H^s \text{ s.t. } v \perp_\varepsilon U_a \text{ and } v \perp_\varepsilon \partial_j U_a \text{ for any } j = 1, \ldots, N \right\}. \tag{5.7}
\]
With this, we can bound the second derivative of \( J_\varepsilon(U_a, \nu(\varepsilon)) \) from below as follows:

**Lemma 5.2.** Let \( J_\varepsilon \) be as in \((5.1)\). There exists \( \varepsilon_0 > 0 \) such that for any \( \varepsilon \in (0, \varepsilon_0) \), for any \( v \in W_\varepsilon \) and for any \( a \in \mathbb{R}^N \)
\[
J''_\varepsilon(U_a, \nu(\varepsilon))[v, v] \geq \text{Const} \|v\|_\varepsilon^2.
\]
The Const above does not depend on \( a \).

**Proof.** Up to translations, we can suppose that \( x_0 = 0 \). We consider \( \chi \in C^\infty_c(\mathbb{R}^N, (0,2)) \) such that \( \chi = 1 \) in \( B_1 \) and \( \chi = 0 \) in \( \mathbb{R}^N \setminus B_2 \). Also we take \( R > 1 \), to be chosen suitably large in the sequel. We define
\[
\chi_R(x) = \chi\left(a + \frac{x}{R}\right),
\]
\[
\bar{\chi}_R := 1 - \chi_R,
\]
\[
v_1 := \chi_R v,
\]
\[
v_2 := \bar{\chi}_R v,
\]
and
\[
I_1 := \int_{\mathbb{R}^{2N}} \frac{\chi_R(x) \bar{\chi}_R(x) (v(x) - v(y))^2}{|x-y|^{N+2s}} \, dx \, dy.
\]
First we prove that
\[
|\langle v_1, v_2 \rangle_{D^{s,2}}| \leq \eta_R(v), \tag{5.8}
\]
with \( \eta_R(v) \) not depending on \( \varepsilon \) and such that
\[
\lim_{R \to +\infty} \eta_R(v) = 0, \tag{5.9}
\]
for \( v \) fixed. To this goal, we compute
\[
(v_1(x) - v_1(y))(v_2(x) - v_2(y))
\]
\[
= \left(v(x) \chi_R(x) - v(y) \chi_R(y)\right) \left(v(x) \bar{\chi}_R(x) - v(y) \bar{\chi}_R(y)\right)
\]
\[
= \left(v(x) \chi_R(x) - \chi_R(y)\right) \left(v(x) - v(y)\right) + \chi_R(y) \left(v(x) - v(y)\right)
\]
\[
\cdot \left(\bar{\chi}_R(x) \left(v(x) - v(y)\right) + v(y) \left(\bar{\chi}_R(x) - \bar{\chi}_R(y)\right)\right)
\]
\[
= \left(v(x) \left(v(x) - \chi_R(y)\right) + \chi_R(y) \left(v(x) - v(y)\right)\right)
\]
\[
\cdot \left(\bar{\chi}_R(x) \left(v(x) - v(y)\right) - v(y) \left(\chi_R(x) - \chi_R(y)\right)\right)
\]
\[
= -v(x) v(y) \left(\chi_R(x) - \chi_R(y)\right)^2 + \chi_R(x) \bar{\chi}_R(x) \left(v(x) - v(y)\right)^2
\]
\[
+ v(x) \bar{\chi}_R(x) \left(\chi_R(x) - \chi_R(y)\right) \left(v(x) - v(y)\right)
\]
\[
- v(y) \chi_R(y) \left(v(x) - v(y)\right) \left(\chi_R(x) - \chi_R(y)\right).
\]
Therefore
\[
|\langle v_1, v_2 \rangle_{D^{s,2}}| \leq I_1 + \text{Const} (J_1 + J_2) \tag{5.10}
\]
with
\[
J_1 := \int_{\mathbb{R}^{2N}} |v(x)| |v(y)| \left(\chi_R(x) - \chi_R(y)\right)^2 \, d\mu(x, y),
\]
\[
J_2 := \int_{\mathbb{R}^{2N}} |v(x)| |v(x) - v(y)| \left|\chi_R(x) - \chi_R(y)\right| \, d\mu(x, y), \tag{5.11}
\]
and \( d\mu(x, y) := |x - y|^{-N-2s} \, dx \, dy \).
Now we observe that $\|\nabla \chi_R\|_{L^\infty} \leq \text{Const} R^{-1}$, and so
\[
|\chi_R(x) - \chi_R(y)| \leq \text{Const} \min\{1, R^{-1}|x-y|\}. 
\] (5.12)

Therefore, for any $x \in \mathbb{R}^N$,\[
\int_{\mathbb{R}^N} \frac{|\chi_R(x) - \chi_R(y)|^2}{|x-y|^{N+2s}} dy \leq \text{Const} \left[ \int_{B(0,R)} \frac{R^{-2}|x-y|^2}{|x-y|^{N+2s}} dy + \int_{\mathbb{R}^N \setminus B(0,R)} \frac{1}{|x-y|^{N+2s}} dy \right] = \text{Const} R^{-2s}.
\]

Using this and the Hölder inequality we obtain\[
J_2 \leq \sqrt{\int_{\mathbb{R}^{2N}} |v(x)|^2 |\chi_R(x) - \chi_R(y)|^2 d\mu(x,y)} \cdot \sqrt{\int_{\mathbb{R}^{2N}} |v(x) - v(y)|^2 d\mu(x,y)} \leq \sqrt{\text{Const} R^{-2s} \int_{\mathbb{R}^N} |v(x)|^2 dx \cdot \|v\|_{D^{s,2}}} \leq \text{Const} R^{-s} \|v\|_{L^2} \cdot \|v\|_{D^{s,2}} \leq \text{Const} R^{-s} \|v\|_{L^2}^2.
\] (5.13)

Now we define \[
\mathbb{R}^{2N}_R := \{(x, y) \in \mathbb{R}^{2N} \text{ s.t. } |x-y| < R\}
\] and \[
\mathcal{V} := \{(x, y) \in \mathbb{R}^{2N} \text{ s.t. } |v(x)| \geq |v(y)|\}.
\]

By symmetry\[
\int_{\mathbb{R}^{2N}_R} |v(x)| |v(y)| |x-y|^2 d\mu(x,y) \leq 2 \int_{\mathbb{R}^{2N}_R \cap \mathcal{V}} |v(x)|^2 |x-y|^2 d\mu(x,y) \leq 2 \int_{\mathbb{R}^N} |v(x)|^2 |x-y|^2 d\mu(x,y) \]
\[
\leq 2 \int_{\mathbb{R}^N} |v(x)|^2 \left[ \int_{B(x,R)} |x-y|^{2-N-2s} dy \right] dx \leq \text{Const} R^{2-2s} \int_{\mathbb{R}^N} |v(x)|^2 dx = \text{Const} R^{2-2s} \|v\|_{L^2}^2.
\] (5.14)

Similarly,\[
\int_{(\mathbb{R}^{2N}_{-R})^c} |v(x)| |v(y)| d\mu(x,y) \leq 2 \int_{(\mathbb{R}^{2N}_{-R})^c \cap \mathcal{V}} |v(x)|^2 d\mu(x,y) \leq 2 \int_{\mathbb{R}^N} |v(x)|^2 \left[ \int_{\mathbb{R}^N \setminus B(0,R)} |x-y|^{-N-2s} dy \right] dx \leq \text{Const} R^{-2s} \int_{\mathbb{R}^N} |v(x)|^2 dx = \text{Const} R^{-2s} \|v\|_{L^2}^2.
\]

We use the latter inequality together with (5.12) and (5.14) to conclude that\[
J_1 \leq \text{Const} \left[ R^{-2} \int_{\mathbb{R}^N} |v(x)| |v(y)| |x-y|^2 d\mu(x,y) + \int_{(\mathbb{R}^{2N}_{-R})^c} |v(x)| |v(y)| d\mu(x,y) \right] \leq \text{Const} R^{-2s} \|v\|_{L^2}^2.
\]

Hence, by (5.10) and (5.13),\[
\langle v_1, v_2 \rangle_{D^{s,2}} \leq I_1 + \text{Const} (R^{-2s} + R^{-s}) \|v\|_{L^2}^2.
\] (5.15)

Now we estimate $I_1$. For this we observe that the function $\chi_R \bar{\chi}_R$ is supported in $B(a,2R) \setminus B(a,R)$, hence\[
I_1 \leq \int_{(B(a,2R) \setminus B(a,R)) \times \mathbb{R}^N} \frac{(v(x) - v(y))^2}{|x-y|^{N+2s}} dx dy.
\]
Since $v$ is a fixed function of $H^s$, we have that
\[
\lim_{R \to +\infty} \int_{(B(a,2R) \setminus B(a,R)) \times \mathbb{R}^N} \frac{(v(x) - v(y))^2}{|x - y|^{N+2s}} \, dx \, dy = 0.
\]
These considerations and (5.15) imply (5.8), as desired.
From (5.8), we obtain that
\[
\|v\|_{D^{s,2}}^2 = \|v_1 + v_2\|_{D^{s,2}}^2 = \|v_1\|_{D^{s,2}}^2 + \|v_2\|_{D^{s,2}}^2 + 2\langle v_1, v_2 \rangle_{D^{s,2}}
\]
\[
\leq \|v_1\|_{D^{s,2}} + \|v_2\|_{D^{s,2}} + 2\eta_R(v).
\]
Moreover
\[
\|v\|_{n,V}^2 = \|v_1\|_{n,V}^2 + \|v_2\|_{n,V}^2 + \int_{\mathbb{R}^N} V(\varepsilon x) v_1(x) v_2(x) \, dx
\]
\[
\leq \text{Const} (\|v_1\|_{n,V}^2 + \|v_2\|_{n,V}^2).
\]
This and (5.16) yield that
\[
\|v\|_\varepsilon^2 \leq \text{Const} (\|v_1\|_\varepsilon^2 + \|v_2\|_\varepsilon^2 + \eta_R(v)).
\]
(5.17)
On the other hand, $v_1 v_2 = \chi_{R \tilde{X}} v_1 v_2$, therefore $v_1 v_2 \geq 0$ and it is supported in $B(a, 2R) \setminus B(a, R)$. In this domain $U_a$ is of the order $R^{-(N+2s)}$, therefore
\[
\int_{\mathbb{R}^N} U_a^{p-1} v_1 v_2 \leq \text{Const} R^{-(p-1)(N+2s)} \int_{B(a,2R) \setminus B(a,R)} |v|^2 \leq \text{Const} R^{-(p-1)(N+2s)} \|v\|_{L^2}^2.
\]
From this and (5.8) we infer that
\[
J_\varepsilon''(U_a, \nu(V_\varepsilon))[v_1, v_2] = \langle v_1, v_2 \rangle_{D^{s,2}} + \int_{\mathbb{R}^N} V(\varepsilon x) v_1 v_2 - p \nu(v_\varepsilon) \int_{\mathbb{R}^N} U_a^{p-1} v_1 v_2
\]
\[
\geq - \text{Const} R^{-s} \|v\|_\varepsilon^2 + 0 - \text{Const} R^{-(p-1)(N+2s)} \|v\|_{L^2}^2
\]
\[
\geq - \text{Const} R^{-s} \|v\|_\varepsilon^2,
\]
(5.18)
up to renaming constants, where $\gamma := \min\{s, (p - 1)(N + 2s)\} > 0$ (here we have also used Lemma 3.1 to bound $\nu(V_\varepsilon)$ uniformly in $\varepsilon$). Similarly, $v_2$ is supported outside $B(0, R)$, hence
\[
\int_{\mathbb{R}^N} U_a^{p-1} v_2^2 \leq \text{Const} R^{-(p-1)(N+2s)} \int_{\mathbb{R}^N} v_2^2,
\]
and therefore
\[
J_\varepsilon''(U_a, \nu(V_\varepsilon))[v_2, v_2] = \|v_2\|_{p-1}^2 - p \nu(v_\varepsilon) \int_{\mathbb{R}^N} U_a^{p-1} v_2^2 \geq \|v_2\|_{\varepsilon}^2 \geq - \text{Const} R^{-(p-1)(N+2s)} \|v\|_{L^2}^2.
\]
(5.19)
Next we estimate $J_\varepsilon''(U_a, \nu(V_\varepsilon))[v_1, v_1]$. To this goal, we project $v_1$ along the space spanned by $U_a$ and its derivatives, i.e. we set
\[
\psi := \frac{1}{\|U_a\|_0^2} (v_1, U_a) U_a + \frac{1}{\|\partial_i U_a\|_0^2} (v_1, \partial_i U_a) \partial_i U_a,
\]
where the repeated indices convention is used, and $w := v_1 - \psi$. Therefore
\[
J_\varepsilon''(U_a, \nu(V_\varepsilon))[v_1, v_1] = J_\varepsilon''(U_a, \nu(V_\varepsilon))[w, w] + J_\varepsilon''(U_a, \nu(V_\varepsilon))[\psi, \psi] + 2J_\varepsilon''(U_a, \nu(V_\varepsilon))[w, \psi].
\]
(5.20)
We observe that the norms $\|\cdot\|_0$ and $\|\cdot\|_\varepsilon$ are comparable, thanks to (1.4). Therefore
\[
|\psi| \leq \frac{\|v_1\|_0}{\|U_a\|_0} U_a + \frac{\|v_1\|_0}{\|\partial_i U_a\|_0} |\partial_i U_a| \leq \text{Const} \|v_1\|_\varepsilon \left(U_a + |\partial_i U_a|\right),
\]
(5.21)
and hence
\[
\int_{\mathbb{R}^N} (1 + |x|) \psi^2 \leq \text{Const} \|v_1\|_{\varepsilon}^2.
\]
Using this, the fact that \( |V(\varepsilon x) - V(0)| \leq \text{Const} \varepsilon |x| \), and that \( v_1 \) is supported in \( B(a, R) \), we conclude that

\[
\int_{\mathbb{R}^N} |V(\varepsilon x) - V(0)| w^2 \\
= \int_{B(a, R)} |V(\varepsilon x) - V(0)| v_1^2 + \int_{\mathbb{R}^N} |V(\varepsilon x) - V(0)| \psi^2 - 2 \int_{\mathbb{R}^N} |V(\varepsilon x) - V(0)| v_1 \psi \\
\leq \text{Const} \left[ \int_{B(a, R)} |V(\varepsilon x) - V(0)| v_1^2 + \int_{\mathbb{R}^N} |V(\varepsilon x) - V(0)| \psi^2 \right] \\
\leq \text{Const} \varepsilon (R + |a|) \|v_1\|_{L^2}^2 .
\]

(5.22)

Now we remark that

\[
2|\langle v_1, \psi \rangle| = 2|\langle v_1/2, 2\psi \rangle| \leq \|v_1\|_{\varepsilon} + 4\|\psi\|_{\varepsilon}
\]

and so, since the two norms are comparable,

\[
\text{We deduce from (5.24) and (5.25) that}
\]

\[
\text{In a similar way, since also}
\]

\[
|\langle v_1, \varepsilon U_a \rangle| \leq \text{Const} \varepsilon (R + |a|) \|v\|_{L^2} + R^{-(p-1)(N+2s)} \|v\|_{L^2}.
\]

(5.24)

In a similar way, since also \( v \perp \partial U_a \), we have that

\[
|\langle v_1, \partial U_a \rangle| \leq \text{Const} \varepsilon (R + |a|) \|v\|_{L^2} + R^{-(p-1)(N+2s)} \|v\|_{L^2}.
\]

(5.25)

We deduce from (5.24) and (5.25) that

\[
\|\psi\|_0 \leq \text{Const} \left( |\langle v_1, U_a \rangle| + |\langle v_1, \partial U_a \rangle| \right) \leq \text{Const} \varepsilon (R + |a|) \|v\|_{L^2} + R^{-(p-1)(N+2s)} \|v\|_{L^2}
\]

and so, since the two norms are comparable,

\[
\|\psi\|_{\varepsilon} \leq \text{Const} \varepsilon (R + |a|) \|v\|_{L^2} + R^{-(p-1)(N+2s)} \|v\|_{L^2}.
\]

So, we use the fact that

\[
2|\langle v_1, \varepsilon \rangle| = 2|\langle v_1/2, 2\varepsilon \rangle| \leq \|v_1\|_{\varepsilon} + 4\|\psi\|_{\varepsilon}
\]

\[
\]
to conclude that
\[
\|w\|_2^2 = \|v_1\|_2^2 + \|\psi\|_2^2 - 2\langle v_1, \psi \rangle_x
\geq \frac{3}{4}\|v_1\|_2^2 - \text{Const} (\varepsilon (R + |a|) + R^{-(p-1)(N+2s)})^2 \|v\|_{L^2}^2.
\]
Exploiting this and (5.23) we obtain
\[
J''_\varepsilon(U, \nu(V_\varepsilon))[w, w] \geq \text{Const} \|v_1\|_2^2 - \text{Const} (\varepsilon (R + |a|) + R^{-(p-1)(N+2s)})^2 \|v\|_{L^2}^2
- \text{Const} \varepsilon (R + |a|) \|v_1\|_2^2 - \text{Const} \nu(V_\varepsilon) - \nu(V_0) \|v_1\|_x.
\]
Notice now that
\[
J''_\varepsilon(U, \nu(V_\varepsilon))[v, v] = J''_\varepsilon(U, \nu(V_\varepsilon))[v_1, v_1] + J''_\varepsilon(U, \nu(V_\varepsilon))[v_2, v_2] + 2J''_\varepsilon(U, \nu(V_\varepsilon))[v_1, v_2].
\]
Thus, by collecting (5.18), (5.19) and (5.26), we obtain
\[
J''_\varepsilon(U, \nu(V_\varepsilon))[v, v] \geq \text{Const} (\|v_1\|_2^2 + \|v_2\|_2^2)
- \text{Const} (\varepsilon (R + |a|) + R^{-\gamma}) \|v\|_2^2
- \text{Const} R^{-(p-1)(N+2s)} \|v\|_{L^2}^2
- \text{Const} \varepsilon (R + |a|) \|v_1\|_2^2 - \text{Const} \nu(V_\varepsilon) - \nu(V_0) \|v_1\|_x.
\]
Now, recalling (5.17) and (5.9), and sending first \(\varepsilon \to 0\) and then \(R \to +\infty\), we get the desired result.

6. Uniqueness of radial solutions

In this section we assume that \(V\) is radial and we consider the functional in (5.1). We denote by \(H^s\) the space of \(H^s\) of radially symmetric function. We will make use of the minimizer \(U\) for \(\nu(V_0)\), normalized with \(\|U\|_{L^{p+1}} = 1\), which is a solution of
\[
\langle U, v \rangle_{D^{s,2}} + V(0) \langle U, v \rangle_{L^2} = \nu(V_0) \int_{\mathbb{R}^N} U^p(x) v(x) \, dx,
\]
for every \(v \in H^s\).

We also define \(I_\varepsilon\) as the restriction of \(u \mapsto J_\varepsilon(u, \nu(V_\varepsilon))\) on \(H^s_\varepsilon\). Next, we define the operator \(\Phi_\varepsilon : H^s_\varepsilon \to H^s_\varepsilon\) by
\[
\Phi_\varepsilon(\omega) := I_\varepsilon(U + \omega).
\]
By (6.2), we mean: for all \(w \in H^s_\varepsilon\)
\[
\langle \Phi_\varepsilon(\omega), w \rangle = I'_\varepsilon(U + \omega)[w].
\]

Lemma 6.1. There exists \(\delta > 0\) sufficiently small such that: if \(\Phi_\varepsilon(w_1) = \Phi_\varepsilon(w_2)\) for some \(w_1, w_2 \in H^s_\varepsilon\) with \(\|w_1\|_\varepsilon + \|w_2\|_\varepsilon \leq \delta\), then \(w_1 = w_2\).

Proof. The proof is a consequence of Lemma 5.2. The details go as follows. First we fix the following notation: given \(f \in H^s_\varepsilon\), we define
\[
c_f := \frac{\langle f, U \rangle_x}{\|U\|_2^2} \quad \text{and} \quad \tilde{f} := f - c_f U.
\]
Notice that \(\tilde{f}\) is radial, since so are \(f\) and \(U\), and that \(\langle \tilde{f}, U \rangle_x = 0\). As a matter of fact, since both \(\tilde{f}\) and \(U\) are radial, a direct computation based on odd symmetry shows that also \(\langle f, \partial_t U \rangle_x = 0\), that is
\[
\tilde{f} \in W_\varepsilon,
\]
according to the definition in (5.7).

Notice that \(f = \tilde{f} + c_f U\). We also consider the reflection of \(f\) with respect to \(U\), namely
\[
f^* := \tilde{f} - c_f U.
\]
Now we observe that
\[
J''_0(U, \nu(V_0))[U, v] = (1 - p) \nu(V_0) \int_{\mathbb{R}^N} U^{p-1}(x) U(x) v(x) \, dx
\]
for any \(v \in H^s_\varepsilon\). Indeed, for any \(v \in H^s\),
\[
J''_0(U, \nu(V_0))[U, v] = \langle U, v \rangle_{D^{s,2}} + V(0) \langle U, v \rangle_{L^2} - p\nu(V_0) \int_{\mathbb{R}^N} U^{p-1}(x) U(x) v(x) \, dx
\]
\[
- (1 - p) \nu(V_0) \int_{\mathbb{R}^N} U^p(x) v(x) \, dx.
\]
thanks to (6.1), and this establishes (6.6).

Furthermore
\[ J''_\varepsilon(U, \nu(V_\varepsilon))[U, v] - J''_0(U, \nu(V_0))[U, v] = \int_{\mathbb{R}^N} (V(\varepsilon x) - V(0)) U(x) v(x) \, dx - p(\nu(V_\varepsilon) - \nu(V_0)) \int_{\mathbb{R}^N} U^p(x) v(x) \, dx. \]

This, combined with (6.6) gives that
\[ J''_\varepsilon(U, \nu(V_\varepsilon))[U, v] = \int_{\mathbb{R}^N} (V(\varepsilon x) - V(0)) U(x) v(x) \, dx - c_\varepsilon \int_{\mathbb{R}^N} U^p(x) v(x) \, dx, \]

where
\[ c_\varepsilon := (1 - p) \nu(V_0) - p(\nu(V_\varepsilon) - \nu(V_0)). \]

Notice that \( c_\varepsilon \to (1 - p) \nu(V_0) > 0 \) as \( \varepsilon \to 0 \), due to Lemma 3.1. In particular
\[ J''_\varepsilon(U, \nu(V_\varepsilon))[U, U] = \eta_\varepsilon - c_\varepsilon, \]

with
\[ \eta_\varepsilon := \int_{\mathbb{R}^N} (V(\varepsilon x) - V(0)) U^2(x) \, dx \to 0, \]
as \( \varepsilon \to 0 \), by dominated convergence theorem. We conclude that
\[ J''_\varepsilon(U, \nu(V_\varepsilon))[U, U] \leq -\frac{c_\varepsilon}{2} \tag{6.7} \]
for small \( \varepsilon \). Now, for any \( v, w \in H_*^s \), we set
\[
\mathcal{N}_\varepsilon(v)[w] := \Phi'_\varepsilon(v)[w] - \Phi'_\varepsilon(0)[w] - \langle \Phi'_\varepsilon(0)[v], w \rangle
\]
\[ = I'_\varepsilon(U + v)[w] - I'_\varepsilon(U)[w] - I''\varepsilon(U)[v, w], \]
\[ = \nu(V_\varepsilon) \left( -\int_{\mathbb{R}^N} |U + v|^p w \, dx + \int_{\mathbb{R}^N} U^p w \, dx + p \int_{\mathbb{R}^N} U^{p-1} w \, dx \right). \]

Referring to page 128 in [1], we obtain
\[ \|\mathcal{N}_\varepsilon(v_1) - \mathcal{N}_\varepsilon(v_2)\| \leq \text{Const} (\|v_1\|_\varepsilon + \|v_1\|_{p-1} + \|v_2\|_\varepsilon + \|v_2\|_{p-1})\|v_1 - v_2\|_\varepsilon. \tag{6.8} \]

Now we take \( w := w_1 - w_2 \) and we use the notation in (6.5) and the assumption that \( \Phi_\varepsilon(w_1) = \Phi_\varepsilon(w_2) \) to compute:
\[
0 = \Phi_\varepsilon(w_1)[w^*] - \Phi_\varepsilon(w_2)[w^*]
\]
\[ = \mathcal{N}_\varepsilon(w_1)[w^*] + \mathcal{N}_\varepsilon(w_2)[w^*] - \langle \Phi'_\varepsilon(0)[w_1], w^* \rangle - \langle \Phi'_\varepsilon(0)[w_2], w^* \rangle
\]
\[ = \langle \Phi'_\varepsilon(0)[w_1], w^* \rangle - \langle \Phi'_\varepsilon(0)[w_2], w^* \rangle + \mathcal{N}_\varepsilon(w_1)[w^*] - \mathcal{N}_\varepsilon(w_2)[w^*] \]
\[ = J''_\varepsilon(U, \nu(V_\varepsilon))[w, w^*] + \mathcal{N}_\varepsilon(w_1)[w^*] - \mathcal{N}_\varepsilon(w_2)[w^*]. \]

Thus, we write \( w = \bar{w} + c_w U \) and \( w^* = \bar{w} - c_w U \), and we exploit (6.8) and (6.7), to see that
\[
0 \geq J''_\varepsilon(U, \nu(V_\varepsilon))[\bar{w}, \bar{w}] - c_w^2 J''_\varepsilon(U, \nu(V_\varepsilon))[U, U] - \text{Const} \delta^{\min(1, p-1)} \|w\|_{\varepsilon} \|w^*\|_{\varepsilon}
\]
\[
\geq J''_\varepsilon(U, \nu(V_\varepsilon))[\bar{w}, \bar{w}] + \frac{c_w^2}{2} - \text{Const} \delta^{\min(1, p-1)} \|w\|_{\varepsilon} \|w^*\|_{\varepsilon}. \]

Now, thanks to (6.4), we can make use of Lemma 5.2 and write that \( J''_\varepsilon(U, \nu(V_\varepsilon))[\bar{w}, \bar{w}] \geq \text{Const} \|\bar{w}\|_{\varepsilon}^2 \). So we obtain that
\[ 0 \geq \text{Const} \left( \|\bar{w}\|_{\varepsilon}^2 + c_w^2 \right) - \text{Const} \delta^{\min(1, p-1)} \|w\|_{\varepsilon} \|w^*\|_{\varepsilon}. \tag{6.9} \]

Also, by (6.4),
\[ \|w\|_{\varepsilon}^2 = \|\bar{w}\|_{\varepsilon}^2 + c_w^2 \|U\|_{\varepsilon}^2 + 2 (\bar{w}, U)_\varepsilon = \|\bar{w}\|_{\varepsilon}^2 + c_w^2 \|U\|_{\varepsilon}^2 \]
and, similarly,
\[ \|\bar{w}\|_{\varepsilon}^2 = \|\bar{w}\|_{\varepsilon}^2 + c_w^2 \|U\|_{\varepsilon}^2. \]

In particular, \( \|w\|_{\varepsilon}^2 \leq \text{Const} \|\bar{w}\|_{\varepsilon}^2 + c_w^2 \) and (6.9) becomes
\[
0 \geq \text{Const} \left( \|\bar{w}\|_{\varepsilon}^2 + c_w^2 \right) - \text{Const} \delta^{\min(1, p-1)} \|w\|_{\varepsilon}^2
\]
\[
\geq \left( \text{Const} - \text{Const} \delta^{\min(1, p-1)} \right) \|w\|_{\varepsilon}^2, \]
which implies that \( \|w\|_{\varepsilon} = 0 \) if \( \delta \) is small enough.
7. Completeness of the proof of Theorem 1.3

Now we complete the proof of Theorem 1.3. For this, let $v^i_\varepsilon$ be a minimizer for $\nu(V^i_\varepsilon)$, with $i = 1, 2$. Since $V$ is radial then using symmetric decreasing arguments of the moving plane argument, we have that $v^i_\varepsilon$ is radial (or using the minimization in the space $H^1_i$ and the compactness in [18]). Then by Lemma 3.1, provided $\varepsilon$ is sufficiently small, we have

$$v^i_\varepsilon(x) = U + w^i_\varepsilon$$

with $\|w^i_\varepsilon\|_\varepsilon \to 0$ as $\varepsilon \to 0$.

It turns out that $\Phi_\varepsilon^{1/2}(w^i_\varepsilon) = I^i_\varepsilon(v^i_\varepsilon) = 0$, so we conclude that $w^1_\varepsilon = w^2_\varepsilon$, due to Lemma 6.1.

\[\Box\]

References


