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Bifurcation results for a fractional elliptic equation with critical exponent in $\mathbb{R}^n$

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Abstract. In this paper we study some nonlinear elliptic equations in $\mathbb{R}^n$ obtained as a perturbation of the problem with the fractional critical Sobolev exponent, that is

$$(-\Delta)^s u = \varepsilon\, h\, u^q + u^p \quad \text{in } \mathbb{R}^n,$$

where $s \in (0,1)$, $n > 4s$, $\varepsilon > 0$ is a small parameter, $p = \frac{n+2s}{n-2s}$, $0 < q < p$ and h is a contionuous and compactly supported function.

To construct solutions to this equation, we use the Lyapunov-Schmidt reduction, that takes advantage of the variational structure of the problem. For this, the case $0 < q < 1$ is particularly difficult, due to the lack of regularity of the associated energy functional, and we need to introduce a new functional setting and develop an appropriate fractional elliptic regularity theory.

Contents

1. Introduction

In this paper we deal with the problem

$$(-\Delta)^s u = \varepsilon\, h\, u^q + u^p \quad \text{in } \mathbb{R}^n, \tag{1.1}$$

where $s \in (0,1)$ and $(-\Delta)^s$ is the fractional Laplacian, that is

$$(-\Delta)^s u(x) = c_{n,s}\, PV \int_{\mathbb{R}^n} \frac{u(x)-u(y)}{|x-y|^{n+2s}}\, dy \quad \text{for } x \in \mathbb{R}^n,$$

where $c_{n,s}$ is a suitable positive constant. Moreover, $n > 4s$, $\varepsilon > 0$ is a small parameter, $p = \frac{n+2s}{n-2s}$ is the fractional critical Sobolev exponent, $0 < q < p$ and h is a contionuous function that satisfies

$$\omega := supp\, h \text{ is compact} \tag{1.2}$$

$$\text{and} \qquad h_+ \not\equiv 0. \tag{1.3}$$

We will find solutions of problem (1.1) by considering it as a perturbation of the equation

$$(-\Delta)^s u = u^p \quad \text{in } \mathbb{R}^n, \tag{1.4}$$

with $p = \frac{n+2s}{n-2s}$. It is known that the minimizers of the Sobolev embedding in $\mathbb{R}^n$ are unique, up to translations and positive dilations. Namely if we set

$$z_0(x) := \alpha_{n,s} \frac{1}{(1+|x|^2)^{(n-2s)/2}}, \tag{1.5}$$

then all the minimizers of the Sobolev embedding are obtained by the formula

$$z_{\mu,\xi}(x) := \mu^{(2s-n)/2}\, z_0\left(\frac{x-\xi}{\mu}\right), \tag{1.6}$$

where $\mu > 0$, $\xi \in \mathbb{R}^n$. The normalizing constant $\alpha_{n,s}$ depends only on n and s (see [28], [35], [17] and the references therein), and the explicit value of $\alpha_{n,s}$ is not particularly relevant in our framework. Notice also that equation (1.4) is the Euler Lagrange equation of this Sobolev embedding minimization problem.

It has been showed in [17] that solutions to (1.4) of the form (1.6) are nondegenerate. Namely, setting $\partial_\mu z_{\mu,\xi}$ and $\partial_\xi z_{\mu,\xi}$ the derivative of $z_{\mu,\xi}$ with respect to the parameters μ and ξ respectively, then all bounded solutions of the linear equation

$$(-\Delta)^s\psi = p\, z_{\mu,\xi}^{p-1}\,\psi \quad \text{in } \mathbb{R}^n$$

are linear combinations of $\partial_\mu z_{\mu,\xi}$ and $\partial_\xi z_{\mu,\xi}$. We also refer to [22], where the nondegeneracy result was proved in detail for $s = 1/2$ and $n = 3$ (but the proof can be extended in higher dimensions and for fractional exponents $s \in (0, n/2)$ as well).

We set

$$[u]^2_{\dot H^s(\mathbb{R}^n)} := \iint_{\mathbb{R}^{2n}} \frac{|u(x)-u(y)|^2}{|x-y|^{n+2s}}\,dx\,dy,$$

and we define the space $\dot H^s(\mathbb{R}^n)$ as the completion of the space of smooth and rapidly deceasing functions (the so-called Schwartz space) with respect to the norm $[u]_{\dot H^s(\mathbb{R}^n)} + \|u\|_{L^{2^*}(\mathbb{R}^n)}$, where

$$2^* = \frac{2n}{n-2s}$$

is the fractional critical exponent.

We also introduce the space

$$X^s := \dot H^s(\mathbb{R}^n) \cap L^\infty(\mathbb{R}^n),$$

equipped with the norm

$$\|u\|_{X^s} := [u]_{\dot H^s(\mathbb{R}^n)} + \|u\|_{L^\infty(\mathbb{R}^n)}.$$

Given $f \in L^\beta(\mathbb{R}^n)$, where $\beta := \frac{2n}{n+2s}$, we say that $u \in X^s$ is a (weak) solution to $(-\Delta)^s u = f$ in $\mathbb{R}^n$ if

$$\iint_{\mathbb{R}^{2n}} \frac{\big(u(x)-u(y)\big)\,\big(\varphi(x)-\varphi(y)\big)}{|x-y|^{n+2s}}\,dx\,dy = \int_{\mathbb{R}^n} f\,\varphi\,dx,$$

for any $\varphi \in X^s$.

We prove the following:

Theorem 1.1. *Suppose that h is a continuous function that satisfies* (1.2) *and* (1.3)*. Then, there exist $\varepsilon_0 > 0$, $\mu_1 > 0$ and $\xi_1 \in \mathbb{R}^n$ such that problem* (1.1) *has a positive solution $u_{1,\varepsilon}$ for any $\varepsilon \in (0, \varepsilon_0)$, and $u_{1,\varepsilon} \to z_{\mu_1,\xi_1}$ in X^s as $\varepsilon \to 0$.*

Also, if h changes sign, then for any $\varepsilon \in (0, \varepsilon_0)$ there exists a second positive solution $u_{2,\varepsilon}$ to (1.1) *that, as $\varepsilon \to 0$, converges in X^s to z_{μ_2,ξ_2} with $\mu_2 > 0$, $\mu_2 \neq \mu_1$, and $\xi_2 \in \mathbb{R}^n$, $\xi_2 \neq \xi_1$.*

In order to prove Theorem 1.1 we will use a Lyapunov-Schmidt reduction, that takes advantage of the variational structure of the problem. Indeed, positive solutions to (1.1) can be found as critical points of the functional $f_\varepsilon : X^s \to \mathbb{R}$ defined by

$$
\begin{aligned}
(1.7) \qquad f_\varepsilon(u) \quad := \quad & \frac{c_{n,s}}{2} \iint_{\mathbb{R}^{2n}} \frac{|u(x)-u(y)|^2}{|x-y|^{n+2s}}\,dx\,dy \\
& - \frac{\varepsilon}{q+1} \int_{\mathbb{R}^n} h(x)\,u_+^{q+1}(x)\,dx - \frac{1}{p+1} \int_{\mathbb{R}^n} u_+^{p+1}(x)\,dx.
\end{aligned}
$$

We notice that f_ε can be written as

$$
f_\varepsilon(u) = f_0(u) - \varepsilon\, G(u), \tag{1.8}
$$

where

$$
f_0(u) := \frac{c_{n,s}}{2} \iint_{\mathbb{R}^{2n}} \frac{|u(x)-u(y)|^2}{|x-y|^{n+2s}}\,dx\,dy - \frac{1}{p+1} \int_{\mathbb{R}^n} u_+^{p+1}(x)\,dx \tag{1.9}
$$

and

$$
G(u) := \frac{1}{q+1} \int_{\mathbb{R}^n} h(x)\,u_+^{q+1}(x)\,dx. \tag{1.10}
$$

Indeed, we will use a perturbation method that allows us to find critical points of f_ε by bifurcating from a manifold of critical points of the unperturbed functional f_0 (see for instance [6] for the abstract method).

Notice that critical points of f_0 are solutions to (1.4), and so, in order to construct solutions to (1.1), we will start from functions of the form (1.6) and we will add a small error to them in such a way that we obtain solutions to the perturbed problem.

This small error will be found by means of the Implicit Function Theorem. To do this, a crucial ingredient will be the nondegeneracy condition proved in [17] for $z_{\mu,\xi}$, but the application of the linear theory in our case is non-standard and it requires a pointwise control of the functional spaces.

Roughly speaking, one additional difficulty for us is indeed that when $q < 1$ the energy functional is not smooth at the zero level set, and so the classical Implicit Function Theorem cannot be applied, unless we can avoid the singularity. For this, the classical Hilbert space framework is not enough, and we have to keep track of the pointwise behavior of the functions inside our functional analysis framework. This is for instance the main reason for which we work in the more robust space X^s rather than in the more classical space $\dot{H}^s(\mathbb{R}^n)$.

Of course, the change of functional setting produces some difficulties in the invertibility of the operators, since the Hilbert-Fredholm theory does not directly apply, and we will have to compensate it by an appropriate elliptic regularity theory.

Once these difficulties are overcome, the Lyapunov-Schmidt reduction allows us to reduce our problem to the one of finding critical points of the perturbation G, introduced in (1.10). For this, we set

$$
\Gamma(\mu,\xi) := G(z_{\mu,\xi}), \tag{1.11}
$$

where $z_{\mu,\xi}$ has been introduced in (1.6). The study of the behavior of Γ will give us the existence of critical points of G, and so the existence of solution to (1.1).

There is a huge literature concerning the search of solutions for this kind of perturbative problems in the classical case, i.e. when $s = 1$ and the fractional

Laplacian boils down to the classical Laplacian, see [1, 2, 3, 4, 5, 8, 10, 14, 15, 29, 30]. In particular, Theorem 1.1 here can be seen as the nonlocal counterpart of Theorem 1.3 in [2]. See also [25], where the concave term appears for the first time.

In the fractional case, the situation is more involved. Namely, the nonlocal Schrödinger equation has recently received a growing attention not only for the challenging mathematical difficulties that it offers, but also due to some important physical applications (see e.g. [27], the appendix in [16], and the references therein). In the subcritical case, this nonlocal Schrödinger equation can be written as

$$\varepsilon^{2s}(-\Delta)^s u + V(x)u = u^p \quad \text{in } \mathbb{R}^n,$$

with $1 < p < \frac{n+2s}{n-2s}$ and V a smooth potential. Multi-peak solutions for this type of equations were considered recently in [18]. Also in this case, a key ingredient in the proof is the uniqueness and nondegeneracy of the ground state solution of the corresponding unperturbed problem, which has been proved in [24] for any $s \in (0,1)$ and in any dimension, after previous works in dimension 1 (see [23]) and for s close to 1 (see [21]).

Moreover, given a bounded domain $\Omega \subset \mathbb{R}^n$, the Dirichlet problem

$$\begin{cases} \varepsilon^{2s}(-\Delta)^s u + u = u^p & \text{in } \Omega, \\ u = 0 & \text{in } \mathbb{R}^n \setminus \Omega, \end{cases}$$

was considered in [16], where the authors constructed solutions that concentrate at the interior of the domain.

Concentrating solutions for fractional problems involving critical or almost critical exponents were considered in [13]. See also [11] for some concentration phenomena in particular cases and [32] for the study of the soliton dynamics in related problems. See also [12] for a semilinear problem with critical power, related to the scalar curvature problem, that also exploits a Lyapunov-Schmidt reduction. It is worth pointing out that, in our case, the presence of the subcritical, possibly sublinear, power in our problem introduces extra difficulties that have required the development of certain elliptic regularity theory, and the careful analysis of the corresponding functional framework. Notice indeed that for sublinear powers q the energy functional experiences a loss of regularity, so the standard functional analysis methods are not directly available and several technical modifications are needed.

In particular, we perform here a detailed analysis of the linearized equation, that is the key ingredient to use the Lyapunov-Schmidt arguments. We think that these results are of independent interest and can be useful elsewhere.

The paper is organized as follows. In Section 2 we show some auxiliary fractional elliptic estimates needed in the subsequent sections. In Section 3 we perform the Lyapunov-Schmidt reduction, with the detailed study of the linearized equation, and the associated functional analysis theory. Section 4 is devoted to the study of the behavior of Γ, as defined in (1.11). Finally, in Section 5 we complete the proof of Theorem 1.1.

2. Fractional elliptic estimates

Here we obtain some uniform elliptic estimates on Riesz potential (though the topic is of classical flavor in harmonic analysis, we could not find in the literature a statement convenient for our purposes). These estimates will be used in Section 3 in order to obtain the continuity properties of our functionals.

We recall that

$$H^s(\mathbb{R}^n) = \{u : \mathbb{R}^n \to \mathbb{R} \text{ measurable s.t. } \|u\|_{L^2(\mathbb{R}^n)} + [u]_{\dot{H}^s(\mathbb{R}^n)} < +\infty\}.$$

To start with, we point out that the fractional Sobolev inequality holds in X^s, thanks to a simple limit procedure:

Lemma 2.1. *Let $n > 2s$. Let $f : \mathbb{R}^n \to \mathbb{R}$ be a measurable function. Suppose that there exists a sequence of functions $f_k \in H^s(\mathbb{R}^n)$ such that $f_k \to f$ in $\dot{H}^s(\mathbb{R}^n)$ and a.e. in $\mathbb{R}^n$. Then*

$$\|f\|_{L^{2^*}(\mathbb{R}^n)} \leqslant C\,[f]_{\dot{H}^s(\mathbb{R}^n)}, \tag{2.1}$$

for some $C > 0$ depending on n and s. In particular, the inequality in (2.1) *holds true for any $f \in X^s$.*

Proof. For each $k \in \mathbb{N}$, we have that $f_k \in H^s(\mathbb{R}^n)$, so we can apply the fractional Sobolev inequality (see e.g. Theorem 6.5 in [19]) and obtain

$$\|f_k\|_{L^{2^*}(\mathbb{R}^n)} \leqslant C\,[f_k]_{\dot{H}^s(\mathbb{R}^n)}. \tag{2.2}$$

Since

$$\lim_{k\to+\infty} [f_k]_{\dot{H}^s(\mathbb{R}^n)} \leqslant \lim_{k\to+\infty} [f_k - f]_{\dot{H}^s(\mathbb{R}^n)} + [f]_{\dot{H}^s(\mathbb{R}^n)} \leqslant [f]_{\dot{H}^s(\mathbb{R}^n)}$$

and, by Fatou Lemma,

$$\begin{aligned}\liminf_{k\to+\infty} \|f_k\|_{L^{2^*}(\mathbb{R}^n)} &= \left[\liminf_{k\to+\infty} \int_{\mathbb{R}^n} |f_k(x)|^{2^*}\,dx\right]^{1/2^*} \\ &\geqslant \left[\int_{\mathbb{R}^n} |f(x)|^{2^*}\,dx\right]^{1/2^*} = \|f\|_{L^{2^*}(\mathbb{R}^n)},\end{aligned}$$

we can pass to the limit in (2.2) and obtain (2.1). $\square$

Here is the fractional elliptic regularity needed for our goals:

Theorem 2.2. *Let $n > 4s$. Let $\beta := 2n/(n+2s)$ and $\psi \in L^\beta(\mathbb{R}^n)$. Let also*

$$J\psi(x) := \int_{\mathbb{R}^n} \frac{\psi(y)}{|x-y|^{n-2s}}\,dy. \tag{2.3}$$

Then:

(2.4) $J\psi \in L^{2^*}(\mathbb{R}^n)$, *and* $\|J\psi\|_{L^{2^*}(\mathbb{R}^n)} \leqslant C\,\|\psi\|_{L^\beta(\mathbb{R}^n)}$;

(2.5) $J\psi \in \dot{H}^s(\mathbb{R}^n)$, *and* $[J\psi]_{\dot{H}^s(\mathbb{R}^n)} \leqslant C\,\|\psi\|_{L^\beta(\mathbb{R}^n)}$;

(2.6) $(-\Delta)^s(J\psi) = c\psi$ *in the weak sense, i.e.*

$$\iint_{\mathbb{R}^{2n}} \frac{\big((J\psi)(x) - (J\psi)(y)\big)\big(\phi(x) - \phi(y)\big)}{|x-y|^{n+2s}}\,dx\,dy = c\int_{\mathbb{R}^n} \psi(x)\,\phi(x)\,dx$$

for any $\phi \in X^s$;

(2.7) *if, in addition, it holds that* $\psi \in L^\infty(\mathbb{R}^n)$, *then* $J\psi \in L^\infty(\mathbb{R}^n)$,

and $\|J\psi\|_{L^\infty(\mathbb{R}^n)} \leqslant C\Big(\|\psi\|_{L^\infty(\mathbb{R}^n)} + \|\psi\|_{L^\beta(\mathbb{R}^n)}\Big)$.

Here above, C and c are suitable positive constants[1] *only depending on n and s.*

Proof. The claim in (2.4) follows from an appropriate version of the Hardy-Littlewood-Sobolev inequality, namely Theorem 1 on page 119 of [34], used here with $\alpha := 2s$, $p := \beta$ and $q := 2^*$.

Now we take a sequence of smooth and rapidly decreasing functions ψ_j that converge to ψ in $L^\beta(\mathbb{R}^n)$, and we set $\Psi_j := J\psi_j$. We also set $\Psi := J\psi$. Thus, by (2.4), we have that

$$\|\Psi_j - \Psi\|_{L^{2^*}(\mathbb{R}^n)} = \|J(\psi_j - \psi)\|_{L^{2^*}(\mathbb{R}^n)} \leqslant C\|\psi_j - \psi\|_{L^\beta(\mathbb{R}^n)} \to 0$$

as $j \to +\infty$. Thus, up to a subsequence,

$$\Psi_j \to \Psi \text{ a.e. in } \mathbb{R}^n. \tag{2.8}$$

Moreover, by Lemma 2(b) in [34], we have that

$$\int_{\mathbb{R}^n} (J\psi_j)(x)\,\overline{g(x)}\,dx = c\int_{\mathbb{R}^n} \hat\psi_j(\xi)\,|\xi|^{-2s}\,\overline{\hat g(\xi)}\,d\xi, \tag{2.9}$$

for some $c > 0$, for every g that is smooth and rapidly decreasing (and possibly complex valued). As standard, we have denoted by $\hat g = \mathcal{F}g$ the Fourier transform of g.

Now, for any ϕ smooth and rapidly decreasing and any $\delta > 0$, we take g_δ to be the inverse Fourier transform of $(|\xi|^2+\delta)^s\hat\phi$, in symbols $g_\delta := \mathcal{F}^{-1}\big((|\xi|^2+\delta)^s\hat\phi\big)$. We remark that $(|\xi|^2+\delta)^s\hat\phi$ is smooth and rapidly decreasing, hence so is g_δ. Accordingly, (2.9) implies that

$$\int_{\mathbb{R}^n} (J\psi_j)(x)\,\overline{g_\delta(x)}\,dx = c\int_{\mathbb{R}^n} \hat\psi_j(\xi)\,|\xi|^{-2s}\,\overline{(|\xi|^2+\delta)^s\hat\phi(\xi)}\,d\xi. \tag{2.10}$$

We claim that

$$g_\delta \to \mathcal{F}^{-1}(|\xi|^{2s}\hat\phi) \text{ in } L^2(\mathbb{R}^n), \text{ as } \delta \to 0. \tag{2.11}$$

To check this, we use Plancherel Theorem to compute

$$\begin{aligned} &\|g_\delta - \mathcal{F}^{-1}(|\xi|^{2s}\hat\phi)\|^2_{L^2(\mathbb{R}^n)} = \|\hat g_\delta - |\xi|^{2s}\hat\phi\|^2_{L^2(\mathbb{R}^n)} \\ &\qquad = \big\|\big[(|\xi|^2+\delta)^s - |\xi|^{2s}\big]\,\hat\phi\big\|^2_{L^2(\mathbb{R}^n)} = \int_{\mathbb{R}^n} \big|(|\xi|^2+\delta)^s - |\xi|^{2s}\big|^2\,|\hat\phi(\xi)|^2\,d\xi. \end{aligned} \tag{2.12}$$

Then we observe that, if $\delta \in (0,1)$,

$$\big|(|\xi|^2+\delta)^s - |\xi|^{2s}\big|^2 \leqslant 4(|\xi|^2+1)^{2s}$$

and the function $\xi \mapsto (|\xi|^2+1)^{2s}\,|\hat\phi(\xi)|^2$ belongs to $L^1(\mathbb{R}^n)$, since $\hat\phi$ is also rapidly decreasing, thus (2.11) follows from (2.12) and the Dominated Convergence Theorem.

Moreover, since ψ_j is rapidly decreasing, a direct computation with convolutions (see e.g. Lemma 5.1 in [16]) gives that

$$|J\psi_j(x)| \leqslant \frac{C_j}{1+|x|^{n-2s}}, \tag{2.13}$$

for some $C_j > 0$. In particular, since $n > 4s$, we have that

$$\Psi_j = J\psi_j \in L^2(\mathbb{R}^n). \tag{2.14}$$

[1]In the sequel, for simplicity we will just take $c = 1$ in (2.6). This can be accomplished simply by renaming J to $c^{-1}J$.

As a matter of fact, the derivatives of ψ_j are rapidly decreasing as well and $\nabla\Psi_j = J(\nabla\psi_j)$, thus the argument above also shows that $\nabla\Psi_j \in L^2(\mathbb{R}^n,\mathbb{R}^n)$, and so

$$\Psi_j \in H^1(\mathbb{R}^n). \tag{2.15}$$

Using (2.11), (2.14) and the Plancherel Theorem, we conclude that

$$\begin{aligned} \lim_{\delta\to 0}\int_{\mathbb{R}^n} (J\psi_j)(x)\,\overline{g_\delta(x)}\,dx &= \int_{\mathbb{R}^n} \Psi_j(x)\,\overline{\mathcal{F}^{-1}(|\xi|^{2s}\hat\phi)(x)}\,dx \\ &= \int_{\mathbb{R}^n} \hat\Psi_j(\xi)\,\overline{|\xi|^{2s}\hat\phi(\xi)}\,d\xi = \int_{\mathbb{R}^n} |\xi|^{2s}\hat\Psi_j(\xi)\,\overline{\hat\phi(\xi)}\,d\xi. \end{aligned} \tag{2.16}$$

Now we point out that, for $\delta\in(0,1)$,

$$\Big||\xi|^{-2s}\,\overline{(|\xi|^2+\delta)^s\hat\phi(\xi)}\Big| \leqslant |\xi|^{-2s}\,(|\xi|^2+1)^s|\hat\phi(\xi)|$$

and this function is in $L^1(\mathbb{R}^n)$, since $n>2s$. Accordingly, the Dominated Convergence Theorem gives that

$$\lim_{\delta\to 0}\int_{\mathbb{R}^n} \hat\psi_j(\xi)\,|\xi|^{-2s}\,\overline{(|\xi|^2+\delta)^s\hat\phi(\xi)}\,d\xi = \int_{\mathbb{R}^n} \hat\psi_j(\xi)\,\overline{\hat\phi(\xi)}\,d\xi.$$

This, (2.10) and (2.16) imply that

$$\int_{\mathbb{R}^n} |\xi|^{2s}\hat\Psi_j(\xi)\,\overline{\hat\phi(\xi)}\,d\xi = c\int_{\mathbb{R}^n} \hat\psi_j(\xi)\,\overline{\hat\phi(\xi)}\,d\xi = c\int_{\mathbb{R}^n} \psi_j(x)\,\phi(x)\,dx, \tag{2.17}$$

for any ϕ smooth and rapidly decreasing.

Now we fix $j\in\mathbb{N}$ and make use of (2.15): accordingly, by density, we find a sequence $\Psi_{j,k}$ of smooth and rapidly decreasing functions that converge to Ψ_j in $H^1(\mathbb{R}^n)$ as $k\to+\infty$.

In particular, $\Psi_{j,k}\to\Psi_j$ in $L^2(\mathbb{R}^n)$ and so, by Plancherel Theorem, also $\hat\Psi_{j,k}\to\hat\Psi_j$ in $L^2(\mathbb{R}^n)$, as $k\to+\infty$. Moreover, $|\xi|^{2s}\leqslant 1$ if $|\xi|\leqslant 1$ and $|\xi|^{2s}\leqslant|\xi|^2$ if $|\xi|\geqslant 1$, thus

$$|\xi|^{2s}\leqslant 1+|\xi|^2. \tag{2.18}$$

Consequently

$$\begin{aligned} \int_{\mathbb{R}^n} |\xi|^{2s}\big|\hat\Psi_{j,k}(\xi)-\hat\Psi_j(\xi)\big|^2\,d\xi &\leqslant \int_{\mathbb{R}^n} (1+|\xi|^2)\,\big|\mathcal{F}\big(\Psi_{j,k}(\xi)-\Psi_j(\xi)\big)\big|^2\,d\xi \\ &\leqslant C\|\Psi_{j,k}-\Psi_j\|^2_{H^1(\mathbb{R}^n)} \to 0 \end{aligned} \tag{2.19}$$

as $k\to+\infty$, and therefore

$$\lim_{k\to+\infty}\int_{\mathbb{R}^n} |\xi|^{2s}\hat\Psi_j(\xi)\,\overline{\hat\Psi_{j,k}(\xi)}\,d\xi = \int_{\mathbb{R}^n} |\xi|^{2s}|\hat\Psi_j(\xi)|^2\,d\xi.$$

Then we apply (2.17) with $\phi:=\Psi_{j,k}$; therefore we see that

$$\begin{aligned} \int_{\mathbb{R}^n} |\xi|^{2s}|\hat\Psi_j(\xi)|^2\,d\xi &= \lim_{k\to+\infty}\int_{\mathbb{R}^n} |\xi|^{2s}\hat\Psi_j(\xi)\,\overline{\hat\Psi_{j,k}(\xi)}\,d\xi \\ &= \lim_{k\to+\infty} c\int_{\mathbb{R}^n} \hat\psi_j(\xi)\,\overline{\hat\Psi_{j,k}(\xi)}\,d\xi = c\int_{\mathbb{R}^n} \hat\psi_j(\xi)\,\overline{\hat\Psi_j(\xi)}\,d\xi.. \end{aligned}$$

Thus, by the Hölder Inequality with exponents β and $2n/(n-2s)$, we obtain

$$\begin{aligned} \int_{\mathbb{R}^n} |\xi|^{2s}|\hat\Psi_j(\xi)|^2\,d\xi &= c\int_{\mathbb{R}^n} \psi_j(\xi)\,\Psi_j(\xi)\,d\xi \\ &\leqslant c\,\|\psi_j\|_{L^\beta(\mathbb{R}^n)}\,\|\Psi_j\|_{L^{2^*}(\mathbb{R}^n)} \leqslant C\,\|\psi_j\|^2_{L^\beta(\mathbb{R}^n)}, \end{aligned}$$

where (2.4) was used in the last step.

This (together with the equivalence of the seminorm in $H^s(\mathbb{R}^n)$, see Proposition 3.4 in [19]) says that

$$\iint_{\mathbb{R}^{2n}} \frac{|\Psi_j(x)-\Psi_j(y)|^2}{|x-y|^{n+2s}}\,dx\,dy \leqslant C\|\psi_j\|^2_{L^\beta(\mathbb{R}^n)}.$$

So we recall (2.8) and we take limit as $j\to+\infty$, obtaining, by Fatou Lemma and the fact that $\psi_j\to\psi$ in $L^\beta(\mathbb{R}^n)$, that

$$\iint_{\mathbb{R}^{2n}} \frac{|\Psi(x)-\Psi(y)|^2}{|x-y|^{n+2s}}\,dx\,dy \leqslant C\|\psi\|^2_{L^\beta(\mathbb{R}^n)},$$

that establishes the estimate in (2.5).

Now we show that $\Psi = J\psi \in \dot H^s(\mathbb{R}^n)$. For this, we notice that, since $\psi\in L^\beta(\mathbb{R}^n)$, there exists a sequence of smooth and rapidly decreasing functions ψ_j such that ψ_j converges to ψ in $L^\beta(\mathbb{R}^n)$ as $j\to+\infty$. So, thanks to the estimates in (2.4) and (2.5), we have that

$$\|J\psi - J\psi_j\|_{L^{2^*}(\mathbb{R}^n)} = \|J(\psi-\psi_j)\|_{L^{2^*}(\mathbb{R}^n)} \leqslant C\,\|\psi-\psi_j\|_{L^\beta(\mathbb{R}^n)} \to 0,$$

and

$$[J\psi - J\psi_j]_{\dot H^s(\mathbb{R}^n)} = [J(\psi-\psi_j)]_{\dot H^s(\mathbb{R}^n)} \leqslant C\,\|\psi-\psi_j\|_{L^\beta(\mathbb{R}^n)} \to 0,$$

as $j\to+\infty$. Therefore, setting $\Psi_j := J\psi_j$, the last two formulas say that

(2.20) Ψ_j converges to Ψ in $L^{2^*}(\mathbb{R}^n)$ and in $\dot H^s(\mathbb{R}^n)$ as $j\to+\infty$.

Moreover, we observe that, by (2.15), there exists a sequence of smooth and rapidly decreasing functions $\Psi_{j,k}$ such that $\Psi_{j,k}$ converges to Ψ_j in $H^1(\mathbb{R}^n)$ as $k\to+\infty$, and so $\Psi_{j,k}$ converges to Ψ_j in $H^s(\mathbb{R}^n)$ as $k\to+\infty$, thanks to (2.19). By the Sobolev immersion (see Theorem 6.5 in [19]), we have that $\Psi_{j,k}$ converges to Ψ_j in $L^{2^*}(\mathbb{R}^n)$ as $k\to+\infty$. Hence, using also (2.20) we obtain that $\Psi = J\psi\in \dot H^s(\mathbb{R}^n)$, and this concludes the proof of (2.5).

Now we prove (2.6). For this, we use (2.5) to see that

$$\begin{aligned}\iint_{\mathbb{R}^{2n}} \frac{\big|(\Psi_j-\Psi)(x)-(\Psi_j-\Psi)(y)\big|^2}{|x-y|^{n+2s}}\,dx\,dy &= [\Psi_j-\Psi]^2_{\dot H^s(\mathbb{R}^n)}\\ = [J(\psi_j-\psi)]^2_{\dot H^s(\mathbb{R}^n)} \leqslant C\,\|\psi-\psi_j\|^2_{L^\beta(\mathbb{R}^n)} \to 0\end{aligned}$$

as $j\to+\infty$. This says that the sequence of functions

$$M_j(x,y) := \frac{\Psi_j(x)-\Psi_j(y)}{|x-y|^{\frac{n+2s}{2}}}$$

converges to the function

$$M(x,y) := \frac{\Psi(x)-\Psi(y)}{|x-y|^{\frac{n+2s}{2}}}$$

in $L^2(\mathbb{R}^{2n})$. In particular, this implies weak convergence in $L^2(\mathbb{R}^{2n})$, that is

$$\lim_{j\to+\infty}\iint_{\mathbb{R}^{2n}} M_j(x,y)\,\gamma(x,y)\,dx\,dy = \iint_{\mathbb{R}^{2n}} M(x,y)\,\gamma(x,y)\,dx\,dy$$

for any $\gamma\in L^2(\mathbb{R}^{2n})$.

Thus, if ϕ is smooth and rapidly decreasing, we can take

$$\gamma(x,y) := \frac{\phi(x)-\phi(y)}{|x-y|^{\frac{n+2s}{2}}}$$

and obtain that

$$\begin{aligned}&\lim_{j\to+\infty}\iint_{\mathbb{R}^{2n}} \frac{\big(\Psi_j(x)-\Psi_j(y)\big)\,\big(\phi(x)-\phi(y)\big)}{|x-y|^{n+2s}}\,dx\,dy\\ &\qquad=\iint_{\mathbb{R}^{2n}} \frac{\big(\Psi(x)-\Psi(y)\big)\,\big(\phi(x)-\phi(y)\big)}{|x-y|^{n+2s}}\,dx\,dy.\end{aligned}$$

Moreover, since ψ_j converges to ψ in $L^\beta(\mathbb{R}^n)$, we have that

$$\lim_{j\to+\infty}\int_{\mathbb{R}^n}\psi_j(x)\,\phi(x)\,dx=\int_{\mathbb{R}^n}\psi(x)\,\phi(x)\,dx.$$

Consequently, we can pass to the limit (2.17) and obtain (2.6) for any ϕ which is smooth and rapidly decreasing.

It remains to establish (2.6) for any $\phi\in X^s$. For this, we fix $\phi\in X^s$ and we take a sequence ϕ_k of smooth and rapidly decreasing functions that converge to ϕ in $\dot H^s(\mathbb{R}^n)$, and so, by Lemma 2.1, also in $L^{2^*}(\mathbb{R}^n)$. Also, we know that $\Psi\in\dot H^s(\mathbb{R}^n)$, thanks to (2.5). In particular, by Cauchy-Schwarz and Hölder inequalities, we obtain that

$$\begin{aligned}&\left|\iint_{\mathbb{R}^{2n}}\frac{\big(\Psi(x)-\Psi(y)\big)\,\big((\phi-\phi_k)(x)-(\phi-\phi_k)(y)\big)}{|x-y|^{n+2s}}\,dx\,dy\right|\\ &\qquad\leqslant[\Psi]_{\dot H^s(\mathbb{R}^n)}\,[\phi-\phi_k]_{\dot H^s(\mathbb{R}^n)}\to 0\\ &\qquad\text{and }\left|\int_{\mathbb{R}^n}\psi(x)\,\Big(\phi(x)-\phi_k(x)\Big)\,dx\right|\leqslant\|\psi\|_{L^\beta(\mathbb{R}^n)}\,\|\phi-\phi_k\|_{L^{2^*}(\mathbb{R}^n)}\to 0\end{aligned}$$

as $k\to+\infty$. Therefore, we can write (2.6) for the smooth and rapidly decreasing functions ϕ_k, pass to the limit in k, and so obtain (2.6) for $\phi\in X^s$. This completes the proof of (2.6).

Now we prove (2.7). For this, we use the Hölder Inequality with exponents β and $2n/(n-2s)$ to calculate

$$\begin{aligned}|J\psi(x)| &\leqslant \int_{\mathbb{R}^n}\frac{|\psi(x-y)|}{|y|^{n-2s}}\,dy\\ &\leqslant \int_{B_1}\frac{\|\psi\|_{L^\infty(B_1)}}{|y|^{n-2s}}\,dy+\int_{\mathbb{R}^n\setminus B_1}\frac{|\psi(x-y)|}{|y|^{n-2s}}\,dy\\ &\leqslant C\,\|\psi\|_{L^\infty(B_1)}+\left[\int_{\mathbb{R}^n\setminus B_1}|\psi(x-y)|^\beta\,dy\right]^{\frac{1}{\beta}}\,\left[\int_{\mathbb{R}^n\setminus B_1}\frac{dy}{|y|^{2n}}\,dy\right]^{\frac{n-2s}{2n}}\\ &\leqslant C\,\Big(\|\psi\|_{L^\infty(\mathbb{R}^n)}+\|\psi\|_{L^\beta(\mathbb{R}^n)}\Big),\end{aligned}$$

and this establishes (2.7). □

We establish now a generalization of Theorem 8.2 in [20], that will provide us an L^∞ estimate for the solutions of some general kind of subcritical and critical problems in $\mathbb{R}^n$.

Theorem 2.3. *Let $u \in \dot{H}^s(\mathbb{R}^n)$ be a solution of*

$$(-\Delta)^s u = \sum_{i=1}^{K} h_i u^{\gamma_i} + f \quad \text{in } \mathbb{R}^n,$$

with $h_i \in L^\infty(\mathbb{R}^n)$ and $0 \leqslant \gamma_i \leqslant 2^* - 1$ for every $i = 1, \dots, K < +\infty$, and $f \in L^m(\mathbb{R}^n)$, $f \geqslant 0$, with $m \in \left(\frac{n}{2s}, +\infty\right]$. Then

$$\|u\|_{L^\infty(\mathbb{R}^n)} \leqslant C,$$

where $C > 0$ is a constant depending on n, s, $\|u\|_{L^{2^*}(\mathbb{R}^n)}$, $\|h_i\|_{L^\infty(\mathbb{R}^n)}$ and $\|f\|_{L^m(\mathbb{R}^n)}$.

Proof. Let $0 < \delta < 1$ to be chosen later, and define

$$\phi(x) := \frac{\delta^\Gamma u(x)}{\|u\|_{L^{2^*}(\mathbb{R}^n)}}, \quad x \in \mathbb{R}^n, \tag{2.21}$$

where $\Gamma := \max_{1 \leqslant i \leqslant K} \{\gamma_i\}$. Thus,

$$\|\phi\|_{L^{2^*}(\mathbb{R}^n)} = \delta^\Gamma, \tag{2.22}$$

and

$$(-\Delta)^s \phi = \sum_{i=1}^{K} \tilde{h}_i \phi^{\gamma_i} + \frac{\delta^\Gamma}{\|u\|_{L^{2^*}(\mathbb{R}^n)}} f \quad \text{in } \mathbb{R}^n, \tag{2.23}$$

where $\tilde{h}_i(x) := h_i(x) \frac{\delta^{\Gamma - \gamma_i}}{\|u\|^{1-\gamma_i}_{L^{2^*}(\mathbb{R}^n)}}$. Now, for every integer $k \in \mathbb{N}$, let us define $A_k := 1 - 2^{-k}$ and the functions

$$w_k(x) := (\phi(x) - A_k)^+, \quad \text{for every } x \in \mathbb{R}^n.$$

By construction, $w_k \in \dot{H}^s(\mathbb{R}^n)$ and $w_{k+1}(x) \leqslant w_k(x)$ a.e. in $\mathbb{R}^n$. Moreover, following [20] it can be checked that for any $k \in \mathbb{N}$,

$$\{w_{k+1} > 0\} \subseteq \{w_k > 2^{-(k+1)}\} \tag{2.24}$$

and

$$\phi(x) < 2^{k+1} w_k(x) \quad \text{for any } x \in \{w_{k+1} > 0\}. \tag{2.25}$$

Consider now

$$U_k := \|w_k\|^{2^*}_{L^{2^*}(\mathbb{R}^n)}.$$

Thus, applying (8.10) of [20] with $v := \phi - (1 - A_k)$ we obtain

$$\begin{aligned} [w_{k+1}]^2_{\dot{H}^s(\mathbb{R}^n)} &= \iint_{\mathbb{R}^{2n}} \frac{|w_{k+1}(x) - w_{k+1}(y)|^2}{|x-y|^{n+2s}} \, dx \, dy \\ &\leqslant \iint_{\mathbb{R}^{2n}} \frac{(\phi(x) - \phi(y))(w_{k+1}(x) - w_{k+1}(y))}{|x-y|^{n+2s}} \, dx \, dy, \end{aligned}$$

and thus, using w_{k+1} as a test function in (2.23) and applying (2.25) and the monotonicity of w_k we obtain

$$\begin{aligned}
[w_{k+1}]^2_{\dot{H}^s(\mathbb{R}^n)} &\leqslant \sum_{i=1}^{K} \int_{\{w_{k+1}>0\}} \tilde{h}_i \phi^{\gamma_i} w_{k+1}\,dx + \frac{\delta^\Gamma}{\|u\|_{L^{2^*}(\mathbb{R}^n)}} \int_{\{w_{k+1}>0\}} f w_{k+1}\,dx \\
&\leqslant \sum_{i=1}^{K} \|\tilde{h}_i\|_{L^\infty(\mathbb{R}^n)} 2^{\gamma_i(k+1)} \int_{\{w_{k+1}>0\}} w_k^{\gamma_i} w_{k+1}\,dx \\
&\qquad + \frac{\delta^\Gamma}{\|u\|_{L^{2^*}(\mathbb{R}^n)}} \int_{\{w_{k+1}>0\}} f w_{k+1}\,dx \\
&\leqslant \sum_{i=1}^{K} \|\tilde{h}_i\|_{L^\infty(\mathbb{R}^n)} 2^{\gamma_i(k+1)} \int_{\{w_{k+1}>0\}} w_k^{\gamma_i+1}, dx \\
&\qquad + \frac{\delta^\Gamma}{\|u\|_{L^{2^*}(\mathbb{R}^n)}} \int_{\{w_{k+1}>0\}} f w_k\,dx. \qquad (2.26)
\end{aligned}$$

On the other hand, by (2.24)

$$\begin{aligned}
U_k &= \|w_k\|^{2^*}_{L^{2^*}(\mathbb{R}^n)} \geqslant \int_{\{w_k>2^{-(k+1)}\}} w_k^{2^*}\,dx \\
&\geqslant 2^{-2^*(k+1)} |\{w_k > 2^{-(k+1)}\}| \geqslant 2^{-2^*(k+1)} |\{w_{k+1} > 0\}|,
\end{aligned}$$

and thus,

$$|\{w_{k+1} > 0\}| \leqslant 2^{2^*(k+1)} U_k. \qquad (2.27)$$

Hence, applying Hölder inequality and this estimate in (2.26), it yields

$$\begin{aligned}
[w_{k+1}]^2_{\dot{H}^s(\mathbb{R}^n)} &\leqslant \sum_{i=1}^{K} \|\tilde{h}_i\|_{L^\infty(\mathbb{R}^n)} 2^{\gamma_i(k+1)} \|w_k\|^{\gamma_i+1}_{L^{2^*}(\mathbb{R}^n)} |\{w_{k+1} > 0\}|^{1-\frac{\gamma_i+1}{2^*}} \\
&\qquad + \frac{\delta^\Gamma}{\|u\|_{L^{2^*}(\mathbb{R}^n)}} \|f\|_{L^m(\mathbb{R}^n)} \|w_k\|_{L^{2^*}(\mathbb{R}^n)} |\{w_{k+1} > 0\}|^\beta \\
&\leqslant \sum_{i=1}^{K} \|\tilde{h}_i\|_{L^\infty(\mathbb{R}^n)} 2^{(k+1)(2^*-1)} U_k \\
&\qquad + \frac{\delta^\Gamma}{\|u\|_{L^{2^*}(\mathbb{R}^n)}} \|f\|_{L^m(\mathbb{R}^n)} 2^{2^*(k+1)\beta} U_k^{1-1/m},
\end{aligned}$$

where $\beta := 1 - \frac{1}{m} - \frac{1}{2^*}$. Combining this estimate with Lemma 2.1, we get

$$\begin{aligned}
U_{k+1} &\leqslant C[w_{k+1}]^{2^*}_{\dot{H}^s(\mathbb{R}^n)} = C([w_{k+1}]^2_{\dot{H}^s(\mathbb{R}^n)})^{2^*/2} \\
&\leqslant C\left(\sum_{i=1}^{K} \|\tilde{h}_i\|_{L^\infty(\mathbb{R}^n)} 2^{(k+1)(2^*-1)} U_k + \frac{\delta^\Gamma}{\|u\|_{L^{2^*}(\mathbb{R}^n)}} \|f\|_{L^m(\mathbb{R}^n)} 2^{2^*(k+1)\beta} U_k^{1-1/m}\right)^{2^*/2} \\
&\leqslant \Bigg[\left[\left(1 + C\sum_{i=1}^{K} \|\tilde{h}_i\|_{L^\infty(\mathbb{R}^n)}\right) 2^{(2^*-1)}\right]^{k+1} U_k \\
&\qquad + \left[\left(1 + C\frac{\delta^\Gamma}{\|u\|_{L^{2^*}(\mathbb{R}^n)}} \|f\|_{L^m(\mathbb{R}^n)}\right) 2^{2^*\beta}\right]^{k+1} U_k^{1-1/m}\Bigg]^{2^*/2} \\
&\leqslant \Bigg[\left[\left(1 + C\sum_{i=1}^{K} \|h_i\|_{L^\infty(\mathbb{R}^n)} \|u\|^{\gamma_i - 1}_{L^{2^*}(\mathbb{R}^n)}\right) 2^{(2^*-1)}\right]^{k+1} U_k \\
&\qquad + \left[\left(1 + C\frac{\|f\|_{L^m(\mathbb{R}^n)}}{\|u\|_{L^{2^*}(\mathbb{R}^n)}}\right) 2^{2^*\beta}\right]^{k+1} U_k^{1-1/m}\Bigg]^{2^*/2} \\
&\leqslant \left(C_1^{k+1} U_k + C_2^{k+1} U_k^{1-1/m}\right)^{2^*/2},
\end{aligned}$$

where C_1, $C_2 > 1$ depend only on n, s, $\|u\|_{L^{2^*}(\mathbb{R}^n)}$, $\|h_i\|_{L^\infty(\mathbb{R}^n)}$ and $\|f\|_{L^m(\mathbb{R}^n)}$. We claim now that there exists $\eta \in (0, 1)$ such that

$$U_k^{2/2^*} \leqslant \delta^{2\Gamma} \eta^k, \quad \forall k \in \mathbb{N}. \tag{2.28}$$

To prove this, we proceed by induction. Indeed,

$$U_0^{2/2^*} = \left(\|w_0\|^{2^*}_{L^{2^*}(\mathbb{R}^n)}\right)^{2/2^*} \leqslant \|\phi\|^2_{L^{2^*}(\mathbb{R}^n)} = \delta^{2\Gamma},$$

thanks to (2.22). Now we suppose that the claim is true for U_k. Then,

$$\begin{aligned}
U_{k+1}^{2/2^*} &\leqslant C_1^{k+1} U_k + C_2^{k+1} U_k^{1-1/m} \\
&\leqslant C_1^{k+1} (\delta^{2\Gamma} \eta^k)^{2^*/2} + C_2^{k+1} (\delta^{2\Gamma(1-1/m)} \eta^{k(1-1/m)})^{2^*/2}.
\end{aligned}$$

Since $m > \frac{n}{2s}$ one has that $\frac{2^*}{2}\left(1 - \frac{1}{m}\right) > 1$, and thus there exist positive constants α_1 and α_2 such that

$$\frac{2^*}{2} = 1 + \alpha_1 \quad \text{and} \quad \frac{2^*}{2}\left(1 - \frac{1}{m}\right) = 1 + \alpha_2.$$

Hence,

$$\begin{aligned}
U_{k+1}^{2/2^*} &\leqslant C_1^{k+1} (\delta^{2\Gamma} \eta^k)^{1+\alpha_1} + C_2^{k+1} (\delta^{2\Gamma} \eta^k)^{1+\alpha_2} \\
&= \delta^{2\Gamma} \eta^k \left((C_1 \eta^{\alpha_1})^k C_1 \delta^{2\Gamma\alpha_1} + (C_2 \eta^{\alpha_2})^k C_2 \delta^{2\Gamma\alpha_2}\right). \qquad (2.29)
\end{aligned}$$

We set now

$$\eta := \min\left(\frac{1}{C_1^{1/\alpha_1}}, \frac{1}{C_2^{1/\alpha_2}}\right),$$

and

$$\delta^\Gamma := \min\left(\left(\frac{\eta}{2C_1}\right)^{1/2\alpha_1}, \left(\frac{\eta}{2C_2}\right)^{1/2\alpha_2}\right).$$

Notice that, since $C_1, C_2 > 1$ and $\alpha_1, \alpha_2 > 0$, we obtain that $\eta, \delta \in (0,1)$. Thus,

$$C_1 \eta^{\alpha_1} \leqslant 1, \quad C_2 \eta^{\alpha_2} \leqslant 1,$$

and

$$C_1 \delta^{2\Gamma\alpha_1} \leqslant \frac{\eta}{2}, \quad C_2 \delta^{2\Gamma\alpha_2} \leqslant \frac{\eta}{2}.$$

Substituting in (2.29) the claim follows.

Hence, taking limits in (2.28) we get

$$\lim_{k\to\infty} U_k = 0. \tag{2.30}$$

Moreover, since $0 \leqslant w_k \leqslant |\phi| \in L^{2^*}(\mathbb{R}^n)$ for any $k \in \mathbb{N}$ and $\lim_{k\to\infty} w_k = (\phi - 1)^+$ a.e. in $\mathbb{R}^n$, by the Dominated Convergence Theorem we get

$$\lim_{k\to\infty} U_k = \|(\phi-1)^+\|^{2^*}_{L^{2^*}(\mathbb{R}^n)} = 0,$$

and therefore $\phi \leqslant 1$ a.e. in $\mathbb{R}^n$. By repeating the proof with $-\phi$ instead of ϕ we conclude that $\|\phi\|_{L^\infty(\mathbb{R}^n)} \leqslant 1$. Thus, recalling the definition of ϕ in (2.21), we conclude that

$$\|u\|_{L^\infty(\mathbb{R}^n)} \leqslant \frac{\|u\|_{L^{2^*}(\mathbb{R}^n)}}{\delta^\Gamma},$$

with $\delta \in (0,1)$ fixed. This concludes the proof of Lemma 2.3. □

3. The Lyapunov-Schmidt reduction

In this section we perform the Lyapunov-Schmidt reduction. Since the argument is delicate and involves many lemmata, we prefer to develop it in different steps.

3.1. **Preliminaries on the functional setting.** Given $0 < \mu_1 < \mu_2$ and $R > 0$, we define the manifold

$$Z_0 := \{z_{\mu,\xi} \text{ s.t. } \mu_1 < \mu < \mu_2,\ |\xi| < R\}, \tag{3.1}$$

where $z_{\mu,\xi}$ was introduced in (1.6). We will perform our choice of R, μ_1 and μ_2 later on. Notice that the functions in Z_0 are critical points of f_0, as defined in (1.9).

We will often implicitly identify Z_0 with the subdomain $(\mu_1, \mu_2) \times B_R$ of $\mathbb{R}^{n+1}$ described by coordinates (μ, ξ).

In order to apply the abstract variational method discussed in the introduction, we would need in principle the functional f_ε defined in (1.7) to be C^2 on $\dot{H}^s(\mathbb{R}^n)$. Unfortunately, this is not true if $q < 1$, and therefore, in order to treat the whole set of values $q \in (0,p)$, we recall that ω is the support of the function h and we set

$$\begin{aligned} a &:= \inf\{z_{\mu,\xi}(x) \text{ s.t. } x\in\omega,\ \mu_1<\mu<\mu_2,\ |\xi|<R\}, \\ V &:= \{w \in X^s \text{ s.t } \|w\|_{X^s} < a/2\} \\ \text{and } U &:= \{u := z_{\mu,\xi} + w \text{ s.t. } z_{\mu,\xi} \in Z_0,\ w \in V\}. \end{aligned} \tag{3.2}$$

We observe that, if $u \in U$ and $x \in \omega$, then

$$u(x) = z_{\mu,\xi}(x) + w(x) \geqslant a - \|w\|_{L^\infty(\mathbb{R}^n)} \geqslant a - \|w\|_{X^s} > a - \frac{a}{2} = \frac{a}{2},$$

and so

$$u(x) > \frac{a}{2} > 0 \text{ for any } x \in \omega. \tag{3.3}$$

Therefore, recalling (1.10), we obtain that the functional G is C^2 on U. Hence, also $f_\varepsilon : U \to \mathbb{R}$ is of class C^2.

Now, we set

$$q_j := \frac{\partial z_{\mu,\xi}}{\partial \xi_j},\ j = 1, \dots, n, \quad \text{and} \quad q_{n+1} := \frac{\partial z_{\mu,\xi}}{\partial \mu}, \tag{3.4}$$

and we notice that q_j satisfies

$$(-\Delta)^s q_j = p z_{\mu,\xi}^{p-1} q_j \quad \text{in } \mathbb{R}^n \tag{3.5}$$

for every $j = 1, \dots, n+1$. We also denote by

$$T_{z_{\mu,\xi}} Z_0 := \mathit{span}\,\{q_1, \dots, q_{n+1}\}$$

the tangent space to Z_0 at $z_{\mu,\xi}$.

Moreover, $\langle \cdot, \cdot \rangle$ denotes the scalar product in $\dot{H}^s(\mathbb{R}^n)$, that is, for any $v_1, v_2 \in \dot{H}^s(\mathbb{R}^n)$,

$$\langle v_1, v_2 \rangle = \iint_{\mathbb{R}^{2n}} \frac{\big(v_1(x) - v_1(y)\big)\big(v_2(x) - v_2(y)\big)}{|x-y|^{n+2s}}\, dx\, dy.$$

We also define the notion of orthogonality with respect to such scalar product and we denote it by $\perp$. That is, we set

$$\big(T_{z_{\mu,\xi}} Z_0\big)^{\perp} := \Big\{ v \in \dot{H}^s(\mathbb{R}^n) \text{ s.t. } \langle v, \phi \rangle = 0 \text{ for all } \phi \in T_{z_{\mu,\xi}} Z_0 \Big\}.$$

In particular, we prove the following orthogonality result.

Lemma 3.1. *There exist $\lambda_i = \lambda_i(\mu, \xi)$, for $i = 1, \dots, n+1$, such that*

$$\langle q_i, q_j \rangle = \begin{cases} 0 & \text{if } i \neq j, \\ \lambda_i & \text{if } i = j, \end{cases}$$

and

$$\inf_{\substack{\mu \in (\mu_1, \mu_2) \\ |\xi| < R \\ i \in \{1, \dots, n+1\}}} \lambda_i(\mu, \xi) > 0.$$

Proof. For any $r \geqslant 0$, we write

$$\bar{z}(r) := \frac{\alpha_{n,s}}{(1+r)^{(n-2s)/2}}.$$

In this way $z_0(x) = \bar{z}(|x|^2)$ and so

$$z_{\mu,\xi}(x) = \mu^{(2s-n)/2} \bar{z}\left(\frac{|x-\xi|^2}{\mu^2} \right).$$

So we obtain that

$$\frac{\partial z_{\mu,\xi}}{\partial \xi_i}(x) = \mu^{(2s-n)/2} \bar{z}'\left(\frac{|x-\xi|^2}{\mu^2} \right) \frac{2(\xi_i - x_i)}{\mu^2}$$

and therefore

$$\frac{\partial z_{\mu,\xi}}{\partial \xi_i}(y+\xi) = \mu^{(2s-n)/2} \bar{z}'\left(\frac{|y|^2}{\mu^2} \right) \frac{2(-y_i)}{\mu^2},$$

which is odd in the variable y_i.

Similarly,

$$\frac{\partial z_{\mu,\xi}}{\partial \mu}(x) = \frac{2s-n}{2} \mu^{(2s-n-2)/2} \bar{z}\left(\frac{|x-\xi|^2}{\mu^2} \right) - \mu^{(2s-n)/2} \bar{z}'\left(\frac{|x-\xi|^2}{\mu^2} \right) \frac{2|x-\xi|^2}{\mu^3},$$

thus

$$\frac{\partial z_{\mu,\xi}}{\partial \mu}(y+\xi) = \frac{2s-n}{2} \mu^{(2s-n-2)/2} \bar{z}\left(\frac{|y|^2}{\mu^2} \right) - \mu^{(2s-n)/2} \bar{z}'\left(\frac{|y|^2}{\mu^2} \right) \frac{2|y|^2}{\mu^3}, \tag{3.6}$$

that is even in any of the variables y_i.
Notice also that

$$z_{\mu,\xi}(y+\xi) = \mu^{(2s-n)/2}\,\bar z\left(\frac{|y|^2}{\mu^2}\right),$$

which is also even in any of the variables y_i. As a consequence, using the change of variable $x = y + \xi$ we obtain that, for any i, $j \in \{1, \dots, n\}$,

$$\begin{aligned}
&\int_{\mathbb{R}^n} z_{\mu,\xi}^{p-1}(x)\,\frac{\partial z_{\mu,\xi}}{\partial \xi_i}(x)\,\frac{\partial z_{\mu,\xi}}{\partial \xi_j}(x)\,dx \\
&\qquad = \int_{\mathbb{R}^n} z_{\mu,\xi}^{p-1}(y+\xi)\,\frac{\partial z_{\mu,\xi}}{\partial \xi_i}(y+\xi)\,\frac{\partial z_{\mu,\xi}}{\partial \xi_j}(y+\xi)\,dy \\
&\qquad = \int_{\mathbb{R}^n} \mu^{(2s-n)(p+1)/2}\,\bar z^{p-1}\left(\frac{|y|^2}{\mu^2}\right)(\bar z')^2\left(\frac{|y|^2}{\mu^2}\right)\frac{4y_iy_j}{\mu^2}\,dy \\
&\qquad = \begin{cases} 0 & \text{if } i \neq j, \\ c_1 & \text{if } i = j, \end{cases}
\end{aligned} \tag{3.7}$$

for some $c_1 > 0$, which is bounded from zero uniformly.

Similarly, for any $i \in \{1, \dots, n\}$,

$$\int_{\mathbb{R}^n} z_{\mu,\xi}^{p-1}(x)\,\frac{\partial z_{\mu,\xi}}{\partial \xi_i}(x)\,\frac{\partial z_{\mu,\xi}}{\partial \mu}(x)\,dx = 0. \tag{3.8}$$

Finally, we observe that $\bar z$ is positive and decreasing, thus both $\bar z$ and $-\bar z'$ are positive: this says that the right hand side of (3.6) is positive, and indeed bounded from zero uniformly. Hence we obtain that

$$\int_{\mathbb{R}^n} z_{\mu,\xi}^{p-1}(x)\left(\frac{\partial z_{\mu,\xi}}{\partial \mu}(x)\right)^2 dx = c_2 \tag{3.9}$$

with $c_2 > 0$ and bounded from zero uniformly.

Now, to make the notation uniform, we take ζ, $\eta \in \{\xi_1, \dots, \xi_n, \mu\}$ and we consider the derivatives of $z_{\mu,\xi}$ with respect to ζ and η. Then we have that the quantity

$$\left\langle \frac{\partial z_{\mu,\xi}}{\partial \zeta}, \frac{\partial z_{\mu,\xi}}{\partial \eta} \right\rangle$$

is equal, up to dimensional constants, to

$$\begin{aligned}
&\int_{\mathbb{R}^n} (-\Delta)^{s/2}\frac{\partial z_{\mu,\xi}}{\partial \zeta}(x)\,(-\Delta)^{s/2}\frac{\partial z_{\mu,\xi}}{\partial \eta}(x)\,dx \\
=\ &\int_{\mathbb{R}^n} (-\Delta)^{s}\frac{\partial z_{\mu,\xi}}{\partial \zeta}(x)\,\frac{\partial z_{\mu,\xi}}{\partial \eta}(x)\,dx \\
=\ &\int_{\mathbb{R}^n} \frac{\partial}{\partial \zeta}(-\Delta)^{s} z_{\mu,\xi}(x)\,\frac{\partial z_{\mu,\xi}}{\partial \eta}(x)\,dx \\
=\ &\int_{\mathbb{R}^n} \frac{\partial}{\partial \zeta} z_{\mu,\xi}^{p}(x)\,\frac{\partial z_{\mu,\xi}}{\partial \eta}(x)\,dx \\
=\ &p\int_{\mathbb{R}^n} z_{\mu,\xi}^{p-1}(x)\frac{\partial z_{\mu,\xi}}{\partial \zeta}(x)\,\frac{\partial z_{\mu,\xi}}{\partial \eta}(x)\,dx,
\end{aligned}$$

hence the desired result follows from (3.7), (3.8) and (3.9). □

Concerning the statement of Lemma 3.1, we point out that the proof shows that $\lambda_1 = \dots = \lambda_n$ (while λ_{n+1} could be different), but in this paper we are not taking advantage of this additional feature.

3.2. **Solving an auxiliary equation.** Keeping the notation introduced in the previous subsection, the goal now is to solve an auxiliary equation by means of the Implicit Function Theorem to obtain the following result.

Lemma 3.2. *Let $z_{\mu,\xi} \in Z_0$. Then, for $\varepsilon > 0$ sufficiently small, there exists a unique $w = w(\varepsilon, z_{\mu,\xi}) \in \big(T_{z_{\mu,\xi}} Z_0\big)^{\perp}$ such that*

$$\begin{aligned}(3.10)\qquad &\iint_{\mathbb{R}^{2n}} \frac{\big((z_{\mu,\xi}+w)(x) - (z_{\mu,\xi}+w)(y)\big)\,\big(\varphi(x)-\varphi(y)\big)}{|x-y|^{n+2s}}\,dx\,dy \\ &= \int_{\mathbb{R}^n} \Big(\varepsilon h(x)\big(z_{\mu,\xi}(x)+w(x)\big)^q + \big(z_{\mu,\xi}(x)+w(x)\big)^p\Big)\,\varphi(x)\,dx,\end{aligned}$$

for any $\varphi \in \big(T_{z_{\mu,\xi}} Z_0\big)^{\perp} \cap X^s$.

Moreover, the function w is of class C^1 with respect to μ and ξ and there exists a constant $C > 0$ such that

$$(3.11)\qquad \|w\|_{X^s} \leqslant C\,\varepsilon, \quad \textit{and} \quad \lim_{\varepsilon\to 0} \left\|\frac{\partial w}{\partial \mu}\right\|_{X^s} + \left\|\frac{\partial w}{\partial \xi}\right\|_{X^s} = 0.$$

Indeed, recalling the definition of U given in (3.2), we can set for any $u \in U$

$$(3.12)\qquad A_\varepsilon(u) := \varepsilon\, h\, u^q + u^p.$$

We observe that $u = J(A_\varepsilon(u))$ (where J has been introduced in (2.3)) implies that u solves (up to an unessential renormalizing constant that we neglect for simplicity, recall the footnote on page 6)

$$(-\Delta)^s u = A_\varepsilon(u) \quad \text{in } \mathbb{R}^n,$$

thanks to Theorem 2.2 (see in particular (2.6)). Moreover, we have that

$$(3.13)\qquad \|J(A_\varepsilon(u))\|_{L^{2^*}(\mathbb{R}^n)} < +\infty.$$

Indeed, by (2.4) in Theorem 2.2 we get that there exists $C > 0$ such that

$$(3.14)\qquad \|J(A_\varepsilon(u))\|_{L^{2^*}(\mathbb{R}^n)} \leqslant C \|A_\varepsilon(u)\|_{L^\beta(\mathbb{R}^n)},$$

where $\beta = 2n/(n+2s)$. Now, since $u \in L^{2^*}(\mathbb{R}^n)$ and $p = (n+2s)/(n-2s)$, we have that $u^p \in L^\beta(\mathbb{R}^n)$. This and the fact that h is compactly supported imply that $\|A_\varepsilon(u)\|_{L^\beta(\mathbb{R}^n)} < +\infty$. Therefore, from (3.14) we deduce (3.13).

Analogously, making use of (2.5) and (2.7), one sees that

$$[J(A_\varepsilon(u))]_{\dot{H}^s(\mathbb{R}^n)} + \|J(A_\varepsilon(u))\|_{L^\infty(\mathbb{R}^n)} < +\infty.$$

Hence, using Theorem 2.2, we have that if $u \in U$ then $J(A_\varepsilon(u)) \in X^s$.

Now, we use the notation $U \ni u = z_{\mu,\xi} + w$, with $z_{\mu,\xi} \in Z_0$ and $w \in V$, and we recall that we are identifying the manifold Z_0 defined in (3.1) with $(\mu_1, \mu_2) \times B_R \subset \mathbb{R}^{n+1}$. We define

$$(3.15)\qquad H : (\mu_1, \mu_2) \times B_R \times V \times \mathbb{R} \times \mathbb{R}^{n+1} \to X^s \times \mathbb{R}^{n+1}$$

as $H = (H_1, H_2)$, with components

$$\begin{aligned} H_1(\mu,\xi,w,\varepsilon,\alpha) &:= z_{\mu,\xi} + w - J(A_\varepsilon(z_{\mu,\xi}+w)) - \sum_{i=1}^{n+1} \alpha_i\, q_i, \\ H_2(\mu,\xi,w,\varepsilon,\alpha) &:= \left(\langle w, q_1\rangle, \ldots, \langle w, q_{n+1}\rangle\right), \end{aligned}$$

where q_i was defined in (3.4).

Our goal is to find $w = w(\varepsilon, z_{\mu,\xi})$ (that we also think as $w = w(\varepsilon, \mu, \xi)$ with a slight abuse of notation) that solves the equation $H(\mu, \xi, w, \varepsilon, \alpha) = 0$, that is the system of equations

$$H_1(\mu, \xi, w, \varepsilon, \alpha) = 0 = H_2(\mu, \xi, w, \varepsilon, \alpha).. \tag{3.16}$$

We notice that if w satisfies (3.16) then $w \in \left(T_{z_{\mu,\xi}} Z_0\right)^{\perp}$ and $z_{\mu,\xi} + w$ is a solution of the auxiliary equation (3.10). Indeed, $H_2(\mu, \xi, w, \varepsilon, w) = 0$ implies that

$$\langle w, q_i \rangle = 0 \quad \text{for any } i = 1, \dots, n+1,$$

which means that $w \in \left(T_{z_{\mu,\xi}} Z_0\right)^{\perp}$. Moreover, $H_1(\mu, \xi, w, \varepsilon, \alpha) = 0$ gives that $z_{\mu,\xi} + w - J(A_\varepsilon(z_{\mu,\xi} + w)) \in T_{z_{\mu,\xi}} Z_0$, and so

$$\langle z_{\mu,\xi} + w - J(A_\varepsilon(z_{\mu,\xi} + w)), \varphi \rangle = 0$$

for any $\varphi \in \left(T_{z_{\mu,\xi}} Z_0\right)^{\perp} \cap X^s$. That is

$$\begin{aligned} &\iint_{\mathbb{R}^{2n}} \frac{\big((z_{\mu,\xi} + w)(x) - (z_{\mu,\xi} + w)(y)\big)\big(\varphi(x) - \varphi(y)\big)}{|x-y|^{n+2s}}\, dx\, dy \\ &\quad = \iint_{\mathbb{R}^{2n}} \frac{\big(J(A_\varepsilon(z_{\mu,\xi} + w))(x) - J(A_\varepsilon(z_{\mu,\xi} + w))(y)\big)\big(\varphi(x) - \varphi(y)\big)}{|x-y|^{n+2s}}\, dx\, dy \\ &\quad = \int_{\mathbb{R}^n} A_\varepsilon(z_{\mu,\xi} + w)(x)\, \varphi(x)\, dx, \end{aligned} \tag{3.17}$$

for any $\varphi \in \left(T_{z_{\mu,\xi}} Z_0\right)^{\perp} \cap X^s$, thanks to (2.6) in Theorem 2.2, which is (3.10).

Therefore, to prove Lemma 3.2, the strategy will be to apply the Implicit Function Theorem to find a solution of the auxiliary equation $H(\mu, \xi, w, \varepsilon, \alpha) = 0$. Since we are working in the space X^s, it is not obvious that H satisfies the hypotheses needed to apply this theorem. Indeed, the proofs of these requirements are very technically involved, so we devote the next two subsections to study in detail the behavior of the operator H.

3.2.1. *Preliminary results on H.* Consider the operator defined in (3.15). First of all, we prove some continuity property.

Lemma 3.3. *H is C^1 with respect to w.*

Proof. We first notice that H_2 depends linearly on w, and so it is C^1. Now we prove that H_1 is continuous in X^s. Indeed, for any $w_1, w_2 \in V$ we have that

$$H_1(\mu, \xi, w_1, \varepsilon, \alpha) - H_1(\mu, \xi, w_2, \varepsilon, \alpha) = w_1 - w_2 - J(A_\varepsilon(z_{\mu,\xi} + w_1)) + J(A_\varepsilon(z_{\mu,\xi} + w_1)),$$

and therefore

$$\begin{aligned} &\|H_1(\mu, \xi, w_1, \varepsilon, \alpha) - H_1(\mu, \xi, w_2, \varepsilon, \alpha)\|_{X^s} \\ &\qquad \leqslant \|w_1 - w_2\|_{X^s} + \|J(A_\varepsilon(z_{\mu,\xi} + w_1)) - J(A_\varepsilon(z_{\mu,\xi} + w_2))\|_{X^s}. \end{aligned} \tag{3.18}$$

By (2.5) and (2.7) of Theorem 2.2 and the fact that J is linear we deduce that

$$\begin{aligned} &\|J(A_\varepsilon(z_{\mu,\xi} + w_1)) - J(A_\varepsilon(z_{\mu,\xi} + w_2))\|_{X^s} \\ &\leqslant C \left(\|A_\varepsilon(z_{\mu,\xi} + w_1) - A_\varepsilon(z_{\mu,\xi} + w_2)\|_{L^\infty(\mathbb{R}^n)} + \|A_\varepsilon(z_{\mu,\xi} + w_1) - A_\varepsilon(z_{\mu,\xi} + w_2)\|_{L^\beta(\mathbb{R}^n)} \right), \end{aligned} \tag{3.19}$$

where $\beta = 2n/(n+2s)$. Now from (3.12) we deduce that

$$\begin{aligned} & A_\varepsilon(z_{\mu,\xi}+w_1) - A_\varepsilon(z_{\mu,\xi}+w_2) \\ = \;& \varepsilon\, h\, \left[(z_{\mu,\xi}+w_1)^q - (z_{\mu,\xi}+w_2)^q\right] + (z_{\mu,\xi}+w_1)^p - (z_{\mu,\xi}+w_2)^p \\ = \;& \varepsilon q\, h\, (z_{\mu,\xi}+\tilde w)^{q-1}(w_1-w_2) + p(z_{\mu,\xi}+\tilde w)^{p-1}(w_1-w_2), \end{aligned}$$

for some $\tilde w$ on the segment joining w_1 and w_2 (in particular $\tilde w \in L^{2^*}(\mathbb{R}^n)$ and $z_{\mu,\xi}+\tilde w$ satisfies (3.3)). Consequently,

$$\|A_\varepsilon(z_{\mu,\xi}+w_1) - A_\varepsilon(z_{\mu,\xi}+w_2)\|_{L^\infty(\mathbb{R}^n)} \leqslant C\,\|w_1-w_2\|_{L^\infty(\mathbb{R}^n)}. \tag{3.20}$$

Moreover, since h has compact support, we have that

$$\|\varepsilon\, h\,(z_{\mu,\xi}+\tilde w)^{q-1}(w_1-w_2)\|_{L^\beta(\mathbb{R}^n)} \leqslant C\,\|w_1-w_2\|_{L^\infty(\mathbb{R}^n)}. \tag{3.21}$$

Finally, using Hölder inequality with exponent $2^*/\beta = (n+2s)/(n-2s)$ and $\delta := (n+2s)/4s$, we get

$$\begin{aligned} & \|(z_{\mu,\xi}+\tilde w)^{p-1}(w_1-w_2)\|^\beta_{L^\beta(\mathbb{R}^n)} \\ & \quad = \int_{\mathbb{R}^n} (z_{\mu,\xi}+\tilde w)^{(p-1)\beta}(w_1-w_2)^\beta \\ & \quad \leqslant \left(\int_{\mathbb{R}^n} (z_{\mu,\xi}+\tilde w)^{(p-1)\beta\delta}\right)^{1/\delta} \left(\int_{\mathbb{R}^n} (w_1-w_2)^{2^*}\right)^{\beta/2^*} \\ & \quad = \left(\int_{\mathbb{R}^n} (z_{\mu,\xi}+\tilde w)^{2^*}\right)^{1/\delta} \left(\int_{\mathbb{R}^n} (w_1-w_2)^{2^*}\right)^{\beta/2^*} \\ & \quad \leqslant C\,\|w_1-w_2\|^\beta_{L^{2^*}(\mathbb{R}^n)} \\ & \quad \leqslant C\,[w_1-w_2]^\beta_{\dot H^s(\mathbb{R}^n)}, \end{aligned}$$

up to renaming $C>0$, where we have used Lemma 2.1 in the last line. Using this, (3.20) and (3.21) into (3.19) we obtain that

$$\|J(A_\varepsilon(z_{\mu,\xi}+w_1)) - J(A_\varepsilon(z_{\mu,\xi}+w_2))\|_{X^s} \leqslant C\,\|w_1-w_2\|_{X^s},$$

which together with (3.18) imply that

$$\|H_1(\mu,\xi,w_1,\varepsilon,\alpha) - H_1(\mu,\xi,w_2,\varepsilon,\alpha)\|_{X^s} \leqslant C\,\|w_1-w_2\|_{X^s},$$

up to renaming C. This shows the continuity of H_1 in X^s with respect to w.

Now, in order to prove that H_1 is C^1, we observe that

$$\begin{aligned} \frac{\partial H_1}{\partial w}[v] &= v - J(A'_\varepsilon(z_{\mu,\xi}+w)v) \\ &= v - J\left(q\varepsilon\, h\,(z_{\mu,\xi}+w)^{q-1}v + p(z_{\mu,\xi}+w)^{p-1}v\right). \end{aligned} \tag{3.22}$$

To see this, we take $v \in V$ and $|t|<1$ and we compute

$$\begin{aligned} & A_\varepsilon(z_{\mu,\xi}+w+tv) - A_\varepsilon(z_{\mu,\xi}+w) \\ = \;& \varepsilon\, h\, \left[(z_{\mu,\xi}+w+tv)^q - (z_{\mu,\xi}+w)^q\right] + (z_{\mu,\xi}+w+tv)^p - (z_{\mu,\xi}+w)^p \\ = \;& q\varepsilon\, h\,(z_{\mu,\xi}+w)^{q-1}tv + p(z_{\mu,\xi}+w)^{p-1}tv + O(t^2), \end{aligned}$$

and so

$$\lim_{t\to 0} \frac{A_\varepsilon(z_{\mu,\xi}+w+tv) - A_\varepsilon(z_{\mu,\xi}+w)}{t} = q\varepsilon\, h\,(z_{\mu,\xi}+w)^{q-1}v + p(z_{\mu,\xi}+w)^{p-1}v.$$

From this and the fact that J is linear we get that

$$\begin{aligned}\frac{\partial H_1}{\partial w}[v] &= \lim_{t\to 0}\frac{1}{t}\left[tv + J\left(A_\varepsilon(z_{\mu,\xi}+w+tv) - A_\varepsilon(z_{\mu,\xi}+w)\right)\right]\\ &= v - J\left(q\varepsilon\, h\,(z_{\mu,\xi}+w)^{q-1}v + p(z_{\mu,\xi}+w)^{p-1}v\right),\end{aligned}$$

which is (3.22). From (3.22) we obtain that, for any $w_1, w_2 \in V$,

$$\begin{aligned}&\left\|\frac{\partial H_1}{\partial w}(\mu,\xi,w_1,\varepsilon,\alpha) - \frac{\partial H_1}{\partial w}(\mu,\xi,w_2,\varepsilon,\alpha)\right\|_{\mathcal{L}((X^s)^*,X^s)}\\ &\quad= \sup_{\|v\|_{X^s}=1}\|J(A'_\varepsilon(z_{\mu,\xi}+w_1)v) - J(A'_\varepsilon(z_{\mu,\xi}+w_2)v)\|_{X^s}.\end{aligned}\tag{3.23}$$

Since J is linear, by (2.5) and (2.7) in Theorem 2.2 we obtain that

$$\begin{aligned}&\|J(A'_\varepsilon(z_{\mu,\xi}+w_1)v) - J(A'_\varepsilon(z_{\mu,\xi}+w_2)v)\|_{X^s}\\ &\leqslant C\left(\|A'_\varepsilon(z_{\mu,\xi}+w_1)v - A'_\varepsilon(z_{\mu,\xi}+w_2)v\|_{L^\infty(\mathbb{R}^n)} + \|A'_\varepsilon(z_{\mu,\xi}+w_1)v - A'_\varepsilon(z_{\mu,\xi}+w_2)v\|_{L^\beta(\mathbb{R}^n)}\right),\end{aligned}\tag{3.24}$$

where $\beta = 2n/(n+2s)$. We have that

$$\begin{aligned}&A'_\varepsilon(z_{\mu,\xi}+w_1)v - A'_\varepsilon(z_{\mu,\xi}+w_2)v\\ &\quad= q\,\varepsilon\, h\, v\left[(z_{\mu,\xi}+w_1)^{q-1} - (z_{\mu,\xi}+w_2)^{q-1}\right] + p\,v\,\left[(z_{\mu,\xi}+w_1)^{p-1} - (z_{\mu,\xi}+w_2)^{p-1}\right],\end{aligned}$$

and so

$$\begin{aligned}&|A'_\varepsilon(z_{\mu,\xi}+w_1)v - A'_\varepsilon(z_{\mu,\xi}+w_2)v|\\ &\quad\leqslant q|q-1|\,\varepsilon\,|h|\,|v|\,|z_{\mu,\xi}+\tilde{w}|^{q-2}|w_1-w_2| + C\,|w_1-w_2|^{p-1}|v|,\end{aligned}\tag{3.25}$$

for some $\tilde{w}$ on the segment joining w_1 and w_2. Accordingly,

$$\begin{aligned}&\|A'_\varepsilon(z_{\mu,\xi}+w_1)v - A'_\varepsilon(z_{\mu,\xi}+w_2)v\|_{L^\infty(\mathbb{R}^n)}\\ &\quad\leqslant C\left(\|w_1-w_2\|_{L^\infty(\mathbb{R}^n)} + \|w_1-w_2\|^{p-1}_{L^\infty(\mathbb{R}^n)}\right),\end{aligned}\tag{3.26}$$

since $z_{\mu,\xi}+\tilde{w}$ satisfies (3.3). Concerning the estimate for the L^β-norm, we observe that, since h is compactly supported and $v \in L^\beta_{loc}(\mathbb{R}^n)$, we have

$$\|q|q-1|\,\varepsilon\,|h|\,|v|\,|z_{\mu,\xi}+\tilde{w}|^{q-2}|w_1-w_2|\|_{L^\beta(\mathbb{R}^n)} \leqslant C\,\|w_1-w_2\|_{L^\infty(\mathbb{R}^n)}.\tag{3.27}$$

Moreover, applying Hölder inequality with exponents $\frac{2^*}{(p-1)\beta} = \frac{n+2s}{4s}$ and p we obtain that

$$\begin{aligned}\||w_1-w_2|^{p-1}|v|\|^\beta_{L^\beta(\mathbb{R}^n)} &= \int_{\mathbb{R}^n}|w_1-w_2|^{(p-1)\beta}|v|^\beta\\ &\leqslant \left(\int_{\mathbb{R}^n}|w_1-w_2|^{2^*}\right)^{4s/(n+2s)}\left(\int_{\mathbb{R}^n}|v|^{p\beta}\right)^{1/p}\\ &= \|w_1-w_2\|^{8ns/[(n+2s)(n-2s)]}_{L^{2^*}(\mathbb{R}^n)}\left(\int_{\mathbb{R}^n}|v|^{2^*}\right)^{1/p}\\ &\leqslant C\,\|w_1-w_2\|^{8ns/[(n+2s)(n-2s)]}_{L^{2^*}(\mathbb{R}^n)},\end{aligned}$$

for a suitable positive constant C. Hence, by Lemma 2.1, we have that

$$\||w_1-w_2|^{p-1}|v|\|_{L^\beta(\mathbb{R}^n)} \leqslant C\,\|w_1-w_2\|^{4s/(n-2s)}_{L^{2^*}(\mathbb{R}^n)} \leqslant C\,[w_1-w_2]^{4s/(n-2s)}_{\dot{H}^s(\mathbb{R}^n)},$$

up to relabelling C. This, (3.27) and (3.25) imply that

$$\|A'_\varepsilon(z_{\mu,\xi}+w_1)v - A'_\varepsilon(z_{\mu,\xi}+w_2)v\|_{L^\beta(\mathbb{R}^n)} \leqslant C\left(\|w_1-w_2\|_{X^s} + \|w_1-w_2\|_{X^s}^{4s/(n-2s)}\right).$$

Putting together this, (3.26), (3.24) and (3.23), we obtain that $\frac{\partial H_1}{\partial w}$ is continuous with respect to w in X^s. This implies that H_1 is C^1 with respect to w, and concludes the proof. □

Let us study now some properties of the derivative of H. In particular, consider first the operator

$$Tv := \frac{\partial H_1}{\partial w}(\mu,\xi,0,0,0)[v] = v - J(A'_0(z_{\mu,\xi})v). \tag{3.28}$$

This definition is well posed, as next result points out:

Lemma 3.4. *T is a bounded operator from $\dot H^s(\mathbb{R}^n)$ to $\dot H^s(\mathbb{R}^n)$.*

Proof. Let $\psi := A'_0(z_{\mu,\xi})v = pz_{\mu,\xi}^{p-1}v$. From (2.5), we know that

$$[J(A'_0(z_{\mu,\xi})v)]_{\dot H^s(\mathbb{R}^n)} = [J\psi]_{\dot H^s(\mathbb{R}^n)} \leqslant C\,\|\psi\|_{L^\beta(\mathbb{R}^n)} = Cp\,\|z_{\mu,\xi}^{p-1}v\|_{L^\beta(\mathbb{R}^n)},$$

with $\beta = 2n/(n+2s)$. On the other hand, using the Hölder inequality with exponents $2^*/\beta$ and $(n+2s)/4s$ we can bound the quantity $\|z_{\mu,\xi}^{p-1}v\|_{L^\beta(\mathbb{R}^n)}$ with $C\,\|v\|_{L^{2^*}(\mathbb{R}^n)}$ and thus by $C\,[v]_{\dot H^s(\mathbb{R}^n)}$, thanks to the Sobolev inequality. This gives that

$$[J(A'_0(z_{\mu,\xi})v)]_{\dot H^s(\mathbb{R}^n)} \leqslant C\,[v]_{\dot H^s(\mathbb{R}^n)},$$

which implies the desired result. □

It is important to remark that T is also a linear operator over X^s. Of course, since X^s is a subset of $\dot H^s(\mathbb{R}^n)$, the restriction operator, that we still denote by T, maps X^s continuously to $\dot H^s(\mathbb{R}^n)$. What is relevant for us is that it also maps X^s continuously to X^s, as next result explicitly states:

Lemma 3.5. *T is a bounded operator from X^s to X^s.*

Proof. Same as the one of Lemma 3.4, using (2.7) in addition to (2.5). □

As a matter of fact, T enjoys further compactness properties, as observed in the next result:

Proposition 3.6. *T is a Fredholm operator over $\dot H^s(\mathbb{R}^n)$. More explicitly, if we set $Kv := -J(A'_0(z_{\mu,\xi})v)$, we have that $T = Id_{\dot H^s(\mathbb{R}^n)} + K$, and $K : \dot H^s(\mathbb{R}^n) \to \dot H^s(\mathbb{R}^n)$ is a compact operator over $\dot H^s(\mathbb{R}^n)$.*

Proof. We already know from Lemma 3.4 that K is a bounded operator over $\dot H^s(\mathbb{R}^n)$. Now, let $\{v_k\}_{k\in\mathbb{N}}$ be a sequence such that

$$[v_k]_{\dot H^s(\mathbb{R}^n)} \leqslant 1. \tag{3.29}$$

To prove compactness, we need to see that

$$\{Kv_k\}_{k\in\mathbb{N}} \text{ contains a Cauchy subsequence in } \dot H^s(\mathbb{R}^n). \tag{3.30}$$

For this, we fix $\varepsilon > 0$ and we exploit (2.5) of Theorem 2.2 to obtain that

$$
\begin{aligned}
&[Kv_l - Kv_m]_{\dot{H}^s(\mathbb{R}^n)} \\
&\quad = [J(A_0'(z_{\mu,\xi})(v_l - v_m))]_{\dot{H}^s(\mathbb{R}^n)} \\
&\quad \leqslant C\|A_0'(z_{\mu,\xi})(v_l - v_m)\|_{L^\beta(\mathbb{R}^n)} \\
&\quad = C(\|A_0'(z_{\mu,\xi})(v_l - v_m)\|_{L^\beta(B_R)} + \|A_0'(z_{\mu,\xi})(v_l - v_m)\|_{L^\beta(\mathbb{R}^n \setminus B_R)}),
\end{aligned} \tag{3.31}
$$

where $\beta := \frac{2n}{n+2s}$, $R > 0$, and $B_R := \{x \in \mathbb{R}^n : |x| < R\}$.

Thus we notice that, for a fixed $R > 0$, the quantity $\|v_k\|_{L^2(B_R)}$ is bounded by $\|v_k\|_{L^{2^*}(B_R)}$, by Hölder inequality, and the latter quantity is in turn bounded by $[v_k]_{\dot{H}^s(\mathbb{R}^n)}$, by Sobolev inequality. These observations and (3.29) imply that

$$\|v_k\|_{W^{s,2}(B_R)} \leqslant C_R,$$

for some $C_R > 0$ that does not depend on k. Moreover, the space $W^{s,2}(B_R)$ is compactly embedded in $L^\beta(B_R)$ (see Corollary 7.2 in [19] and recall that $\beta \in (1, 2^*)$). This implies that v_k contains a Cauchy subsequence in $L^\beta(B_R)$ and so, up to a subsequence, if l and m are sufficiently large (say l, $m \geqslant N(R, \varepsilon)$, for some large $N(R, \varepsilon)$) we have that

$$\|v_l - v_m\|_{L^\beta(B_R)} \leqslant \varepsilon.$$

Notice also that

$$A_0'(z_{\mu,\xi}) = p z_{\mu,\xi}^{\frac{4s}{n-2s}} \in L^\infty(\mathbb{R}^n),$$

therefore

$$\|A_0'(z_{\mu,\xi})(v_l - v_m)\|_{L^\beta(B_R)} \leqslant \|A_0'(z_{\mu,\xi})\|_{L^\infty(\mathbb{R}^n)} \|v_l - v_m\|_{L^\beta(B_R)} \leqslant C\varepsilon \tag{3.32}$$

as long as l, $m \geqslant N(R, \varepsilon)$.

On the other hand, applying Hölder and Sobolev inequalities, and recalling (3.29) once again,

$$
\begin{aligned}
&\|A_0'(z_{\mu,\xi})(v_l - v_m)\|_{L^\beta(\mathbb{R}^n \setminus B_R)} \\
\leqslant\ & \left(\int_{\mathbb{R}^n \setminus B_R} (v_l - v_m)^{2^*} \, dx \right)^{1/2^*} \left(\int_{\mathbb{R}^n \setminus B_R} (p z_{\mu,\xi}^{\frac{4s}{n-2s}})^{\frac{n}{2s}} \, dx \right)^{2s/n} \\
\leqslant\ & C\|v_l - v_m\|_{L^{2^*}(\mathbb{R}^n)} \left(\int_{\mathbb{R}^n \setminus B_{\frac{R-|\xi|}{\mu}}} \frac{1}{|y|^{2n}} \, dy \right)^{2s/n} \\
\leqslant\ & C[v_l - v_m]_{\dot{H}^s(\mathbb{R}^n)} R^{-n} \\
\leqslant\ & CR^{-n},
\end{aligned}
$$

with $C > 0$ possibly different from line to line, but independent of R, l and m. Thus, we insert this and (3.32) into (3.31) and we deduce that

$$[Kv_l - Kv_m]_{\dot{H}^s(\mathbb{R}^n)} \leqslant C(\varepsilon + R^{-n}),$$

provided that l, $m \geqslant N(R, \varepsilon)$, possibly up to a subsequence. In particular, we can choose R depending on ε, for instance $R := \varepsilon^{-1/n}$, and define $N_\varepsilon := N(\varepsilon^{-1/n}, \varepsilon)$. So we obtain that, for l, $m \geqslant N_\varepsilon$, the quantity $[Kv_l - Kv_m]_{\dot{H}^s(\mathbb{R}^n)}$ is bounded by a constant times ε. This establishes (3.30). $\square$

Finally, for any $(v,\beta)\in \dot H^s(\mathbb{R}^n)\times\mathbb{R}^{n+1}$ we define the linear operator

$$\mathcal{T}(v,\beta) := \left(Tv - \sum_{i=1}^{n+1} \beta_i q_i, \langle v, q_1\rangle, \dots, \langle v, q_{n+1}\rangle \right), \tag{3.33}$$

with T defined in (3.28). The interest of such operator for us is that

$$\frac{\partial H}{\partial(w,\alpha)}(\mu,\xi,0,0,0)[v,\beta] = \mathcal{T}(v,\beta). \tag{3.34}$$

We have:

Proposition 3.7. *$\mathcal{T}$ is a bounded operator from $\dot H^s(\mathbb{R}^n)\times\mathbb{R}^{n+1}$ to $\dot H^s(\mathbb{R}^n)\times\mathbb{R}^{n+1}$, and from $X^s\times\mathbb{R}^{n+1}$ to $X^s\times\mathbb{R}^{n+1}$.*

Furthermore, $\mathcal{T}$ is a Fredholm operator over $\dot H^s(\mathbb{R}^n)\times\mathbb{R}^{n+1}$. More explicitly, it can be written as the identity plus a compact operator over $\dot H^s(\mathbb{R}^n)\times\mathbb{R}^{n+1}$.

Proof. Let

$$S(v,\beta) := \left(- \sum_{i=1}^{n+1} \beta_i q_i, \langle v, q_1\rangle, \dots, \langle v, q_{n+1}\rangle \right).$$

Let also $\|\cdot\|$ be either $\|\cdot\|_{\dot H^s(\mathbb{R}^n)}$ or $\|\cdot\|_{X^s}$. We have that

$$\begin{aligned} \|S(v,\beta)\| &\leqslant \sum_{i=1}^{n+1} |\beta_i|\,\|q_i\| + \sum_{i=1}^{n+1} \|v\|_{\dot H^s(\mathbb{R}^n)} \|q_i\|_{\dot H^s(\mathbb{R}^n)} \\ &\leqslant C\Big(|\beta| + \|v\|_{\dot H^s(\mathbb{R}^n)}\Big) \\ &\leqslant C\Big(|\beta| + \|v\|\Big). \end{aligned}$$

This shows that S is a bounded operator from $\dot H^s(\mathbb{R}^n)\times\mathbb{R}^{n+1}$ to $\dot H^s(\mathbb{R}^n)\times\mathbb{R}^{n+1}$, and from $X^s\times\mathbb{R}^{n+1}$ to $X^s\times\mathbb{R}^{n+1}$. Then, noticing that $\mathcal{T}=(T,0)+S$ and recalling Lemmata 3.4 and 3.5, we obtain that also $\mathcal{T}$ is a bounded operator from $\dot H^s(\mathbb{R}^n)\times\mathbb{R}^{n+1}$ to $\dot H^s(\mathbb{R}^n)\times\mathbb{R}^{n+1}$, and from $X^s\times\mathbb{R}^{n+1}$ to $X^s\times\mathbb{R}^{n+1}$.

Now we show that it is Fredholm over $\dot H^s(\mathbb{R}^n)\times\mathbb{R}^{n+1}$. For this, we set

$$\mathcal{K}(v,\beta) := \left(Kv - \sum_{i=1}^{n+1} \beta_i q_i, \langle v, q_1\rangle - \beta_1, \dots, \langle v, q_{n+1}\rangle - \beta_{n+1} \right),$$

where K is the operator in Proposition 3.6. Notice that $\mathcal{T} = Id_{\dot H^s(\mathbb{R}^n)\times\mathbb{R}^{n+1}} + \mathcal{K}$, so our goal is to show that $\mathcal{K}$ is compact over $\dot H^s(\mathbb{R}^n)\times\mathbb{R}^{n+1}$. For this, we take a sequence $(v_k,\beta_k)\in \dot H^s(\mathbb{R}^n)\times\mathbb{R}^{n+1}$ with $\|v_k\|_{\dot H^s(\mathbb{R}^n)} + \|\beta_k\|_{\mathbb{R}^{n+1}} \leqslant 1$ and we want to find a Cauchy subsequence in $\dot H^s(\mathbb{R}^n)\times\mathbb{R}^{n+1}$. To this goal, we use Proposition 3.6 to obtain a subsequence (still denoted by v_k) such that Kv_k is Cauchy in $\dot H^s(\mathbb{R}^n)$. Also, again up to subsequences, v_k is weakly convergent in $\dot H^s(\mathbb{R}^n)$, therefore $\langle v_k, q_1\rangle$ is Cauchy (and the same holds for $\langle v_k, q_2\rangle, \dots, \langle v_k, q_{n+1}\rangle$). Finally, since $\mathbb{R}^{n+1}$ is finite dimensional, up to subsequence we can assume that also β_k is Cauchy. Thanks to these considerations, and writing $\beta_k = (\beta_{k,1},\dots,\beta_{k,n+1}) \in$

$\mathbb{R}^{n+1}$, we have that

$$\begin{aligned}
& \left\| \mathcal{K}(v_k, \beta_k) - \mathcal{K}(v_m, \beta_m) \right\|_{\dot{H}^s(\mathbb{R}^n)\times\mathbb{R}^{n+1}} \\
\leqslant\ & \|Kv_k - Kv_m\|_{\dot{H}^s(\mathbb{R}^n)} + \sum_{i=1}^{n+1} |\beta_{k,i} - \beta_{m,i}|\, \|q_i\|_{\dot{H}^s(\mathbb{R}^n)} + \sum_{i=1}^{n+1} |\langle v_k - v_m,\, q_i\rangle| \\
\leqslant\ & C \left(\|Kv_k - Kv_m\|_{\dot{H}^s(\mathbb{R}^n)} + \|\beta_k - \beta_m\|_{\mathbb{R}^{n+1}} + \sum_{i=1}^{n+1} |\langle v_k - v_m,\, q_i\rangle| \right) \\
\leqslant\ & \varepsilon,
\end{aligned}$$

provided that k and m are large enough. This shows that (v_k, β_k) is Cauchy, as desired. □

3.2.2. *Invertibility issues.* Now we discuss the invertibility of the operator $\mathcal{T}$ that was introduced in (3.33). Notice that there is a subtle point here. Indeed, the operator $\mathcal{T}$ can be seen as acting over $\dot{H}^s(\mathbb{R}^n) \times \mathbb{R}^{n+1}$ or over $X^s \times \mathbb{R}^{n+1}$ (see Proposition 3.7). On the one hand, the invertibility over $\dot{H}^s(\mathbb{R}^n) \times \mathbb{R}^{n+1}$ should be expected to be easier, since the operator is Fredholm there (see the last claim in Proposition 3.7). On the other hand, since we want to obtain strong pointwise estimates to keep control of the possible singularities of our functional, it is crucial for us to invert the operator in a space that controls the functions uniformly, namely $X^s \times \mathbb{R}^{n+1}$. So our strategy will be the following: first we invert the operator in $\dot{H}^s(\mathbb{R}^n) \times \mathbb{R}^{n+1}$ (this will be accomplished using the Fredholm property in Proposition 3.7, the regularity theory in Theorem 2.3 and a nondegeneracy result in [17]). Then we will deduce from this information and a further regularity theory that $\mathcal{T}$ is actually invertible also in $X^s \times \mathbb{R}^{n+1}$.

The details of the argument go as follows. First, we recall the standard definition of invertibility:

Definition 3.8. Let X, Y Banach spaces, and let $S : X \to Y$ be a linear bounded operator. We say that S is invertible (and we write $S \in Inv(X, Y)$) if there exists a linear bounded operator $\tilde{S} : Y \to X$ such that

$$S\tilde{S} = Id_Y, \quad \tilde{S}S = Id_X.$$

Then, we show that $\mathcal{T}$ is invertible in $\dot{H}^s(\mathbb{R}^n) \times \mathbb{R}^{n+1}$:

Proposition 3.9. $\mathcal{T} \in Inv(\dot{H}^s(\mathbb{R}^n) \times \mathbb{R}^{n+1},\, \dot{H}^s(\mathbb{R}^n) \times \mathbb{R}^{n+1})$.

Proof. By Proposition 3.7 and the theory of Fredholm operators (see e.g. [9], pages 168-169, for a very brief summary, and Chapter IV, Section 5, of [26], or [31], for a detailed analysis), it is enough to show that $\mathcal{T}$ is injective over $\dot{H}^s(\mathbb{R}^n) \times \mathbb{R}^{n+1}$. For this, let us take $(v, \beta) \in \dot{H}^s(\mathbb{R}^n) \times \mathbb{R}^{n+1}$ such that $\mathcal{T}(v, \beta) = 0$, that is, by (3.33),

$$\begin{aligned}
& Tv = \sum_{i=1}^{n+1} \beta_i q_i, \\
& \langle v, q_1\rangle = \cdots = \langle v, q_{n+1}\rangle = 0.
\end{aligned} \tag{3.35}$$

Fixed $j \in \{1, \ldots, n+1\}$, using (3.28), (2.6) and (3.5), we observe that

$$
\begin{aligned}
\langle Tv, q_j \rangle &= \langle v - pJ(z_{\mu,\xi}^{p-1} v),\, q_j \rangle \\
&= \langle v,\, q_j \rangle - p \int_{\mathbb{R}^n} (-\Delta)^s J(z_{\mu,\xi}^{p-1} v)\, q_j \\
&= \langle v,\, q_j \rangle - p \int_{\mathbb{R}^n} z_{\mu,\xi}^{p-1} v\, q_j \\
&= \langle v,\, q_j \rangle - \int_{\mathbb{R}^n} v\, (-\Delta)^s q_j \\
&= \langle v,\, q_j \rangle - \langle v,\, q_j \rangle \\
&= 0.
\end{aligned}
\tag{3.36}
$$

This, (3.35) and Lemma 3.1 give that

$$
0 = \langle Tv, q_j \rangle = \sum_{i=1}^{n+1} \beta_i \langle q_i, q_j \rangle = \lambda_j \beta_j,
$$

and so

$$
\beta_j = 0 \text{ for every } j \in \{1, \ldots, n+1\}. \tag{3.37}
$$

Therefore, $v \in \dot{H}^s(\mathbb{R}^n)$ is a weak solution of $Tv = 0$, that is, by (3.28) and (2.6), the equation $(-\Delta)^s v = p z_{\mu,\xi}^{p-1} v$. Accordingly, by Theorem 2.3, we obtain that $v \in L^\infty(\mathbb{R}^n)$.

Thanks to this, we can apply the nondegeneracy result in [17], that gives that v must be a linear combination of $q_1, \ldots, q_{n+1}$. So we write

$$
v = \sum_{i=1}^{n+1} c_i q_i \tag{3.38}
$$

for some $c_i \in \mathbb{R}$, we recall (3.35) and once again Lemma 3.1, and we compute

$$
0 = \langle v, q_j \rangle = \sum_{i=1}^{n+1} c_i \langle q_i, q_j \rangle = c_j \lambda_j,
$$

that gives $c_j = 0$ for every $j \in \{1, \ldots, n+1\}$. By plugging this information into (3.38), we conclude that $v = 0$. This and (3.37) give that $(v, \beta) = 0$ and so $\mathcal{T}$ is injective on $\dot{H}^s(\mathbb{R}^n) \times \mathbb{R}^{n+1}$. □

Next, we aim to prove that $\mathcal{T} \in Inv(X^s \times \mathbb{R}^{n+1},\, X^s \times \mathbb{R}^{n+1})$. For this scope, we need an improved regularity theory result, which goes as follows:

Lemma 3.10. *Let $C_o > 0$. For any $u \in X^s$, $(\alpha, \beta) \in \mathbb{R}^{n+1} \times \mathbb{R}^{n+1}$ and any $\psi \in \dot{H}^s(\mathbb{R}^n)$ which is a weak solution of*

$$
(-\Delta)^s \psi = p \sum_{i=1}^{n+1} \alpha_i z_{\mu,\xi}^{p-1} q_i + p z_{\mu,\xi}^{p-1} \psi + p z_{\mu,\xi}^{p-1} u \tag{3.39}
$$

with

$$
[\psi]_{\dot{H}^s(\mathbb{R}^n)} \leqslant C_o \Big(\|u\|_{X^s} + \|\beta\|_{\mathbb{R}^{n+1}} \Big), \tag{3.40}
$$

we have that $\psi \in L^\infty(\mathbb{R}^n)$ and

$$
\|\psi\|_{L^\infty(\mathbb{R}^n)} \leqslant C \Big(\|u\|_{X^s} + \|\alpha\|_{\mathbb{R}^{n+1}} + \|\beta\|_{\mathbb{R}^{n+1}} \Big) \tag{3.41}
$$

for some $C > 0$.

Proof. The core of the proof is that the equation is linear in the triplet (ψ, u, α), so we get the desired result by a careful scaling argument. The rigorous argument goes as follows. First, we use Theorem 2.3 to get that $\psi \in L^\infty(\mathbb{R}^n)$, so we focus on the proof of (3.41). Suppose, by contradiction, that (3.41) is false. Then, for any k there exists a quadruplet $(\psi_k, u_k, \alpha_k, \beta_k) \in \dot{H}^s(\mathbb{R}^n) \times X^s \times \mathbb{R}^{n+1} \times \mathbb{R}^{n+1}$ such that

$$(-\Delta)^s \psi_k = p \sum_{i=1}^{n+1} \alpha_{k,i} z_{\mu,\xi}^{p-1} q_i + p z_{\mu,\xi}^{p-1} \psi_k + p z_{\mu,\xi}^{p-1} u_k, \tag{3.42}$$

$$\|\psi_k\|_{L^\infty(\mathbb{R}^n)} > k \Big(\|u_k\|_{X^s} + \|\alpha_k\|_{\mathbb{R}^{n+1}} + \|\beta_k\|_{\mathbb{R}^{n+1}} \Big) \tag{3.43}$$

and

$$[\psi_k]_{\dot{H}^s(\mathbb{R}^n)} \leqslant C_o \Big(\|u_k\|_{X^s} + \|\beta_k\|_{\mathbb{R}^{n+1}} \Big). \tag{3.44}$$

We remark that $\|\psi_k\|_{L^\infty(\mathbb{R}^n)} < +\infty$, since $\psi_k \in L^\infty(\mathbb{R}^n)$, and $\|\psi_k\|_{L^\infty(\mathbb{R}^n)} > 0$, due to (3.43). Thus, we can define

$$\tilde{\psi}_k := \frac{\psi_k}{\|\psi_k\|_{L^\infty(\mathbb{R}^n)}}, \qquad \tilde{u}_k := \frac{u_k}{\|\psi_k\|_{L^\infty(\mathbb{R}^n)}},$$

$$\tilde{\alpha}_k := \frac{\alpha_k}{\|\psi_k\|_{L^\infty(\mathbb{R}^n)}} \qquad \text{and} \qquad \tilde{\beta}_k := \frac{\beta_k}{\|\psi_k\|_{L^\infty(\mathbb{R}^n)}}.$$

Notice that

$$\begin{aligned} & \|\tilde{\psi}_k\|_{L^\infty(\mathbb{R}^n)} = 1 \\ \text{and} \quad & \|\tilde{u}_k\|_{X^s} + \|\tilde{\alpha}_k\|_{\mathbb{R}^{n+1}} + \|\tilde{\beta}_k\|_{\mathbb{R}^{n+1}} = \frac{\|u_k\|_{X^s} + \|\alpha_k\|_{\mathbb{R}^{n+1}} + \|\beta_k\|_{\mathbb{R}^{n+1}}}{\|\psi_k\|_{L^\infty(\mathbb{R}^n)}} \leqslant \frac{1}{k}, \end{aligned} \tag{3.45}$$

thanks to (3.43).

Also, by linearity, equation (3.42) becomes

$$(-\Delta)^s \tilde{\psi}_k = p \sum_{i=1}^{n+1} \tilde{\alpha}_{k,i} z_{\mu,\xi}^{p-1} q_i + p z_{\mu,\xi}^{p-1} \tilde{\psi}_k + p z_{\mu,\xi}^{p-1} \tilde{u}_k.$$

The right hand side of this equation is bounded uniformly in $L^\infty(\mathbb{R}^n)$, thanks to (3.45) and the fact that $z_{\mu,\xi} \in L^\infty(\mathbb{R}^n)$.

Thus, by Proposition 5 in [33], we know that for every $x \in \mathbb{R}^n$, there exists a constant $C > 0$ and $a \in (0, 1)$ such that

$$\|\tilde{\psi}_k\|_{C^a(B_{1/4}(x))} \leqslant C.$$

We remark that C and a are independent of k and x, therefore

$$\|\tilde{\psi}_k\|_{C^a(\mathbb{R}^n)} \leqslant C. \tag{3.46}$$

From (3.45), we know that there exists a point $x_k \in \mathbb{R}^n$ such that $\tilde{\psi}_k(x_k) \geqslant 1/2$. By (3.46), there exists $\rho > 0$, which is independent of k, such that $\tilde{\psi}_k \geqslant 1/4$ in $B_\rho(x_k)$. As a consequence,

$$\|\tilde{\psi}_k\|_{L^{2^*}(\mathbb{R}^n)} \geqslant \left(\int_{B_\rho(x_k)} \left(\frac{1}{4}\right)^{2^*} dx \right)^{1/2^*} \geqslant c_o,$$

with $c_o > 0$ independent of k. Thus, by Sobolev inequality,

$$[\tilde{\psi}_k]_{\dot{H}^s(\mathbb{R}^n)} \geqslant c_o, \tag{3.47}$$

up to renaming c_o. On the other hand, by (3.44) and (3.43), we have that

$$[\tilde{\psi}_k]_{\dot{H}^s(\mathbb{R}^n)} = \frac{[\psi_k]_{\dot{H}^s(\mathbb{R}^n)}}{\|\psi_k\|_{L^\infty(\mathbb{R}^n)}} \leqslant \frac{C_o\Big(\|u_k\|_{X^s} + \|\beta_k\|_{\mathbb{R}^{n+1}}\Big)}{\|\psi_k\|_{L^\infty(\mathbb{R}^n)}} \leqslant \frac{C_o}{k}.$$

This is in contradiction with (3.47) when k is large, and therefore the desired result is established. $\square$

Finally, we show that $\mathcal{T}$ is invertible in $X^s(\mathbb{R}^n) \times \mathbb{R}^{n+1}$:

Proposition 3.11. $\mathcal{T} \in Inv(X^s \times \mathbb{R}^{n+1},\, X^s \times \mathbb{R}^{n+1})$.

Proof. By Proposition 3.9, we know that $\mathcal{T} \in Inv(\dot{H}^s(\mathbb{R}^n) \times \mathbb{R}^{n+1}, \dot{H}^s(\mathbb{R}^n) \times \mathbb{R}^{n+1})$. Therefore, there exists an operator

$$\tilde{\mathcal{T}} : \dot{H}^s(\mathbb{R}^n) \times \mathbb{R}^{n+1} \to \dot{H}^s(\mathbb{R}^n) \times \mathbb{R}^{n+1}$$

that is linear and bounded and such that $\mathcal{T}\tilde{\mathcal{T}} = \tilde{\mathcal{T}}\mathcal{T} = Id_{\dot{H}^s(\mathbb{R}^n)\times\mathbb{R}^{n+1}}$. The boundedness of $\tilde{\mathcal{T}}$ as an operator acting over $\dot{H}^s(\mathbb{R}^n) \times \mathbb{R}^{n+1}$ can be explicitly written as

$$\|\tilde{\mathcal{T}}(u,\beta)\|_{\dot{H}^s(\mathbb{R}^n)\times\mathbb{R}^{n+1}} \leqslant C\,\|(u,\beta)\|_{\dot{H}^s(\mathbb{R}^n)\times\mathbb{R}^{n+1}}. \tag{3.48}$$

Now, since X^s is a subset of $\dot{H}^s(\mathbb{R}^n)$, we can consider the restriction operator of $\tilde{\mathcal{T}}$ acting on $X^s \times \mathbb{R}^{n+1}$ (this restriction operator will be denoted by $\tilde{\mathcal{T}}$ as well). We observe that, for any $u \in X^s$, we have that $u \in \dot{H}^s(\mathbb{R}^n)$, therefore, for any $\beta \in \mathbb{R}^{n+1}$,

$$\mathcal{T}\tilde{\mathcal{T}}(u,\beta) = Id_{\dot{H}^s(\mathbb{R}^n)\times\mathbb{R}^{n+1}}(u,\beta) = (u,\beta).$$

Furthermore, if $u \in X^s$ and $\beta \in \mathbb{R}^{n+1}$, then $\mathcal{T}(u,\beta) \in X^s \times \mathbb{R}^{n+1}$, due to Proposition 3.7. Hence the restriction of $\tilde{\mathcal{T}}$ over $X^s \times \mathbb{R}^{n+1}$ may act on $\mathcal{T}(u,\beta)$, for any $(u,\beta) \in X^s \times \mathbb{R}^{n+1}$, and we obtain that

$$\tilde{\mathcal{T}}\mathcal{T}(u,\beta) = Id_{\dot{H}^s(\mathbb{R}^n)\times\mathbb{R}^{n+1}}(u,\beta) = (u,\beta).$$

It remains to prove that

$$\|\tilde{\mathcal{T}}(u,\beta)\|_{X^s\times\mathbb{R}^{n+1}} \leqslant C\Big(\|u\|_{X^s} + \|\beta\|_{\mathbb{R}^{n+1}}\Big). \tag{3.49}$$

To prove it, we first use (3.48) to bound $\|\tilde{\mathcal{T}}(u,\beta)\|_{\dot{H}^s(\mathbb{R}^n)\times\mathbb{R}^{n+1}}$ with $[u]_{\dot{H}^s(\mathbb{R}^n)} + \|\beta\|_{\mathbb{R}^{n+1}}$, and then we observe that the latter quantity is in turn bounded by $\|u\|_{X^s} + \|\beta\|_{\mathbb{R}^{n+1}}$. Thus, in order to show that $\mathcal{T}$ is bounded as an operator over $X^s \times \mathbb{R}^{n+1}$, we only have to bound $\|\tilde{\mathcal{T}}(u,\beta)\|_{L^\infty(\mathbb{R}^n)\times\mathbb{R}^{n+1}}$.

That is to say that the desired result is proved if we show that, for any $u \in X^s$ and any $\beta \in \mathbb{R}^{n+1}$ we have that

$$\|\tilde{\mathcal{T}}(u,\beta)\|_{L^\infty(\mathbb{R}^n)\times\mathbb{R}^{n+1}} \leqslant C\Big(\|u\|_{X^s} + \|\beta\|_{\mathbb{R}^{n+1}}\Big). \tag{3.50}$$

To prove this, we fix $u \in X^s$ and $\beta \in \mathbb{R}^{n+1}$ and we set $(v,\alpha) := \tilde{\mathcal{T}}(u,\beta) \in \dot{H}^s(\mathbb{R}^n) \times \mathbb{R}^{n+1}$. Thus, by (3.33),

$$X^s \times \mathbb{R}^{n+1} \ni (u,\beta) = \mathcal{T}(v,\alpha) = \left(Tv - \sum_{i=1}^{n+1} \alpha_i q_i, \langle v, q_1\rangle, \dots, \langle v, q_{n+1}\rangle\right). \tag{3.51}$$

Taking the first coordinate and using (3.36), we obtain that, for any $j \in \{1, \dots, n+1\}$,

$$\langle u, q_j \rangle = \langle Tv - \sum_{i=1}^{n+1} \alpha_i q_i,\, q_j \rangle = -\sum_{i=1}^{n+1} \alpha_i \langle q_i,\, q_j \rangle.$$

Thus, by Lemma 3.1, we have that $\langle u, q_j \rangle = -\alpha_j \lambda_j$ and therefore

$$|\alpha_j| \leqslant C\, [u]_{\dot{H}^s(\mathbb{R}^n)}.$$

Accordingly

$$\|\alpha\|_{\mathbb{R}^{n+1}} \leqslant C\, \|u\|_{X^s}. \tag{3.52}$$

Now we set $\psi := v - u$. Notice that $\psi \in \dot{H}^s(\mathbb{R}^n)$, since so are u and v. Moreover, taking the first coordinate in (3.51) and using (3.28) and (2.6), we see that ψ is a weak solution of

$$\begin{aligned}
(-\Delta)^s \psi &= (-\Delta)^s v - (-\Delta)^s u \\
&= (-\Delta)^s v - (-\Delta)^s T v + \sum_{i=1}^{n+1} \alpha_i (-\Delta)^s q_i \\
&= (-\Delta)^s J(A_0'(z_{\mu,\xi}) v) + \sum_{i=1}^{n+1} \alpha_i (-\Delta)^s q_i \\
&= p z_{\mu,\xi}^{p-1} v + p \sum_{i=1}^{n+1} \alpha_i z_{\mu,\xi}^{p-1} q_i \\
&= p z_{\mu,\xi}^{p-1} \psi + p z_{\mu,\xi}^{p-1} u + p \sum_{i=1}^{n+1} \alpha_i z_{\mu,\xi}^{p-1} q_i.
\end{aligned}$$

The reader may check that this agrees with (3.39). Furthermore, by (3.48),

$$\begin{aligned}
[v]_{\dot{H}^s(\mathbb{R}^n)} &\leqslant \|(v, \alpha)\|_{\dot{H}^s(\mathbb{R}^n) \times \mathbb{R}^{n+1}} \\
&= \|\tilde{\mathcal{T}}(u, \beta)\|_{\dot{H}^s(\mathbb{R}^n) \times \mathbb{R}^{n+1}} \\
&\leqslant C\Big([u]_{\dot{H}^s(\mathbb{R}^n)} + \|\beta\|_{\mathbb{R}^{n+1}} \Big).
\end{aligned}$$

Consequently,

$$[\psi]_{\dot{H}^s(\mathbb{R}^n)} \leqslant [u]_{\dot{H}^s(\mathbb{R}^n)} + [v]_{\dot{H}^s(\mathbb{R}^n)} \leqslant C\Big([u]_{\dot{H}^s(\mathbb{R}^n)} + \|\beta\|_{\mathbb{R}^{n+1}} \Big),$$

up to renaming constants. The reader may check that this implies (3.40). Accordingly the assumptions of Lemma 3.10 are satisfied, and we deduce from it that

$$\|\psi\|_{L^\infty(\mathbb{R}^n)} \leqslant C\Big(\|u\|_{X^s} + \|\alpha\|_{\mathbb{R}^{n+1}} + \|\beta\|_{\mathbb{R}^{n+1}} \Big).$$

Consequently, using (3.52), we obtain that

$$\begin{aligned}
\|v\|_{L^\infty(\mathbb{R}^n)} &\leqslant \|u\|_{L^\infty(\mathbb{R}^n)} + \|\psi\|_{L^\infty(\mathbb{R}^n)} \\
&\leqslant C\Big(\|u\|_{X^s} + \|\alpha\|_{\mathbb{R}^{n+1}} + \|\beta\|_{\mathbb{R}^{n+1}} \Big) \\
&\leqslant C\Big(\|u\|_{X^s} + \|\beta\|_{\mathbb{R}^{n+1}} \Big),
\end{aligned}$$

up to renaming constants. Using this and once again (3.52), we obtain that

$$\begin{aligned}\|\tilde{\mathfrak{T}}(u,\beta)\|_{L^\infty(\mathbb{R}^n)\times\mathbb{R}^{n+1}} &= \|(v,\alpha)\|_{L^\infty(\mathbb{R}^n)\times\mathbb{R}^{n+1}}\\ &= \|v\|_{L^\infty(\mathbb{R}^n)} + \|\alpha\|_{\mathbb{R}^{n+1}} \leqslant C\Big(\|u\|_{X^s} + \|\beta\|_{\mathbb{R}^{n+1}}\Big).\end{aligned}$$

This establishes (3.50) and in turn (3.49), and so it completes the proof of the desired result. □

3.2.3. *Proof of Lemma 3.2.* Once we have studied in detail the operator H, we can prove Lemma 3.2. As we pointed out at the beginning of this subsection, the idea is to do it by means of the Implicit Function Theorem. For the sake of completeness, we write here the precise statement of this theorem that we will use (see Theorem 2.3, page 38, of [7]).

Theorem 3.12 (Implicit Function Theorem)**.** *Let X, Y, Z be Banach spaces, and let Λ and U be open sets of X and Y respectively. Let $H \in C^1(\Lambda \times U, Z)$ and suppose that $H(\lambda^*, u^*) = 0$ and $\dfrac{\partial H}{\partial u}(\lambda^*, u^*) \in Inv(Y, Z)$.*

Then there exist neighborhoods Θ of λ^* in X and U^* of u^* in Y, and a map $g \in C^1(\Theta, Y)$ such that

a) *$H(\lambda, g(\lambda)) = 0$, for all $\lambda \in \Theta$.*
b) *$H(\lambda, u) = 0$, with $(\lambda, u) \in \Theta \times U^*$, implies $u = g(\lambda)$.*
c) *$g'(\lambda) = -\left(\dfrac{\partial H}{\partial u}(p)\right)^{-1} \circ \dfrac{\partial H}{\partial \lambda}(p)$, where $p = (\lambda, g(\lambda))$ and $\lambda \in \Theta$.*

Now we conclude the proof of Lemma 3.2.

Proof of Lemma 3.2. Consider H defined in (3.15). First we observe that H is C^1 with respect to μ and ξ. Indeed, $z_{\mu,\xi}$ is C^1 with respect to μ and ξ. Moreover, J is linear and $A_\varepsilon(z_{\mu,\xi} + w)$ is C^1 with respect to $z_{\mu,\xi}$ since $z_{\mu,\xi} + w$ is bounded from zero on the support of h (recall (3.3)), therefore H_1 is C^1 with respect to $z_{\mu,\xi}$.

Also, H is C^1 with respect to ε and α, since it depends linearly on these variables (recall that J is linear and A_ε is linear with respect to ε). Finally, H is C^1 with respect to w thanks to Lemma 3.3.

Now we use the Implicit Function Theorem. Indeed, we notice that

$$H_1(\mu, \xi, 0, 0, 0) = z_{\mu,\xi} - J(A_0(z_{\mu,\xi})) = z_{\mu,\xi} - J(z^p_{\mu,\xi}) = 0, \tag{3.53}$$

since $z_{\mu,\xi}$ is a solution to (1.4) (recall also (2.6)). Moreover,

$$H_2(\mu, \xi, 0, 0, 0) = 0. \tag{3.54}$$

In order to follow the notation of Theorem 3.12, we set

$$X := \mathbb{R} \times \mathbb{R}^n \times \mathbb{R}, \quad Y := X^s \times \mathbb{R}^{n+1}, \quad Z := X^s \times \mathbb{R}^{n+1},$$
$$\Lambda := (\mu_1, \mu_2) \times B_R \times \mathbb{R}, \quad U := V \times \mathbb{R}^{n+1},$$

and

$$\lambda^* := (\mu, \xi, 0), \quad u^* := (0, 0), \quad u := (w, \alpha).$$

Thus, we have proved that

(i) $H \in C^1(\Lambda \times U, Z)$, by the linear dependance of the variables and Lemma 3.3;
(ii) $H(\lambda^*, u^*) = 0$, by (3.53) and (3.54);
(iii) $\dfrac{\partial H}{\partial u}(\lambda^*, u^*) \in Inv(Y, Z)$, by (3.33), (3.34) and Proposition 3.11.

Notice here that, since V was defined as

$$V := \{w \in X^s \text{ s.t } \|w\|_{X^s} < a/2\},$$

it is an open subset of X^s. Therefore, all the hypotheses of the Implicit Function Theorem are satisfied, and we conclude the existence of $w \in X^s$ solution to (3.16), that is, there exists $w \in X^s \cap \big(T_{z_{\mu,\xi}} Z_0\big)^{\perp}$ that solves the auxiliary equation in (3.10). Furthermore, since H is of class C^1 with respect to ε, μ and ξ in X^s, we deduce that so is w.

Now we focus on the proof of (3.11). We observe that

$$\left\| \frac{\partial(w,\alpha)}{\partial\varepsilon} \right\|_{X^s\times\mathbb{R}^{n+1}} \leqslant C. \tag{3.55}$$

Indeed, we write

$$H\big(\mu,\, \xi,\, w(\varepsilon, z_{\mu,\xi}),\, \varepsilon,\, \alpha(\varepsilon, z_{\mu,\xi})\big) = 0, \tag{3.56}$$

we differentiate with respect to ε and we set $\varepsilon := 0$. Since

$$w(0, z_{\mu,\xi}) = 0 \text{ and } \alpha(0, z_{\mu,\xi}) = 0, \tag{3.57}$$

we obtain that

$$\frac{\partial H}{\partial\varepsilon}(\mu,\xi,0,0,0) + \frac{\partial H}{\partial(w,\alpha)}(\mu,\xi,0,0,0)\frac{\partial(w,\alpha)}{\partial\varepsilon}(0,z_{\mu,\xi}) = 0.$$

Therefore, using the invertibility assumption, we get that

$$\frac{\partial(w,\alpha)}{\partial\varepsilon}(0,z_{\mu,\xi}) = -\left(\frac{\partial H}{\partial(w,\alpha)}(\mu,\xi,0,0,0)\right)^{-1}\frac{\partial H}{\partial\varepsilon}(\mu,\xi,0,0,0),$$

and so, since H is C^1 with respect to X^s,

$$\left\| \frac{\partial(w,\alpha)}{\partial\varepsilon}(0,z_{\mu,\xi}) \right\|_{X^s\times\mathbb{R}^{n+1}} \leqslant C.$$

Then, since (w, α) is C^1 in ε, in virtue of the Implicit Function Theorem, we obtain (3.55).

From (3.55) and (3.57) we obtain that

$$\|(w,\alpha)\|_{X^s\times\mathbb{R}^{n+1}} \leqslant C\varepsilon,$$

and this implies the first estimate in (3.11).

Now we prove the second and third estimates in (3.11). In this case, we will see that the roles of μ and ξ are basically the same: for this, we write $\varpi \in \mathbb{R}$ for any of the variables $(\mu, \xi) \in \mathbb{R}^{n+1}$ and we use the linearized equation to see that

$$(-\Delta)^s\frac{\partial z_{\mu,\xi}}{\partial\varpi} = p z_{\mu,\xi}^{p-1}\frac{\partial z_{\mu,\xi}}{\partial\varpi}.$$

This information can be written as

$$\frac{\partial H}{\partial\varpi}(\mu,\xi,0,0,0) = 0.$$

Now we take derivatives of (3.56) with respect to ϖ and we set $\varepsilon := 0$. Recalling (3.57) we obtain that

$$\begin{aligned} 0 &= \frac{\partial H}{\partial \varpi}(\mu,\xi,0,0,0) + \frac{\partial H}{\partial (w,\alpha)}(\mu,\xi,0,0,0)\frac{\partial (w,\alpha)}{\partial \varpi}(0,z_{\mu,\xi}) \\ &= \frac{\partial H}{\partial (w,\alpha)}(\mu,\xi,0,0,0)\frac{\partial (w,\alpha)}{\partial \varpi}(0,z_{\mu,\xi}). \end{aligned}$$

Hence, from the invertibility condition, we conclude that

$$\frac{\partial (w,\alpha)}{\partial \varpi}(0,z_{\mu,\xi}) = 0.$$

Since (w,α) are C^1 in ε, we obtain that

$$\lim_{\varepsilon\to 0} \left\| \frac{\partial (w,\alpha)}{\partial \varpi}(\varepsilon,z_{\mu,\xi}) \right\|_{X^s\times\mathbb{R}^{n+1}} = 0.$$

This gives the second and third claim in (3.11) and completes the proof of Lemma 3.2. □

3.3. **Finite-dimensional reduction.** Up to this point, we have found a function w so that $z_{\mu,\xi}+w$ satisfies our problem in the weak sense, when we test with functions $\varphi \in (T_{z_{\mu,\xi}}Z_0)^\perp \cap X^s$. The following result states that actually the equation is satisfied for every test function in X^s, i.e. that $z_{\mu,\xi}+w$ is a solution to (1.1).

Indeed, consider the reduced functional $\Phi_\varepsilon : Z_0 \to \mathbb{R}$, defined by

$$\Phi_\varepsilon(z) := f_\varepsilon(z+w),$$

where $w = w(\varepsilon, z)$ is provided by Lemma 3.2.

Proposition 3.13. *Suppose that Φ_ε has a critical point $z_{\mu^\varepsilon,\xi^\varepsilon} \in Z_0$ for ε small enough. Thus, $z_{\mu^\varepsilon,\xi^\varepsilon} + w$ is a critical point of f_ε, where $w = w(\varepsilon, z_{\mu^\varepsilon,\varepsilon^\varepsilon}) \in (T_{z_{\mu^\varepsilon,\xi^\varepsilon}}Z_0)^\perp$ is provided by Lemma 3.2.*

Proof. For simplicity, we will denote $\mu := \mu^\varepsilon$ and $\xi := \xi^\varepsilon$, and thus $z_{\mu,\xi} := z_{\mu^\varepsilon,\xi^\varepsilon}$. Since $z_{\mu,\xi}$ is a critical point of Φ_ε, we know that there exists $\varepsilon_0 > 0$ such that for every $0 < \varepsilon < \varepsilon_0$ it holds

$$\frac{d}{dt}\Phi_\varepsilon(z_{\mu,\xi}+t\varphi)\Big|_{t=0} = 0 \text{ for every } \varphi \in (T_{z_{\mu,\xi}}Z_0)\cap X^s. \tag{3.58}$$

Recalling the definition of Φ_ε, we observe that

$$\begin{aligned} &\frac{d}{dt}\Phi_\varepsilon(z_{\mu,\xi}+t\varphi)\Big|_{t=0} = \lim_{t\to 0}\frac{\Phi_\varepsilon(z_{\mu,\xi}+t\varphi)-\Phi_\varepsilon(z_{\mu,\xi})}{t} \\ &\quad = \lim_{t\to 0}\frac{f_\varepsilon(z_{\mu,\xi}+t\varphi+w(\varepsilon,z_{\mu,\xi}+t\varphi)) - f_\varepsilon(z_{\mu,\xi}+w(\varepsilon,z_{\mu,\xi}))}{t} \\ &\quad = \lim_{t\to 0}\frac{f_\varepsilon(z_{\mu,\xi}+t\varphi+w(\varepsilon,z_{\mu,\xi})+t\frac{\partial w}{\partial z_{\mu,\xi}}\varphi+o(t)) - f_\varepsilon(z_{\mu,\xi}+w(\varepsilon,z_{\mu,\xi}))}{t} \\ &\quad = \frac{d}{dt}f_\varepsilon\left(z_{\mu,\xi}+w(\varepsilon,z_{\mu,\xi})+t\left[\varphi+\frac{\partial w}{\partial z_{\mu,\xi}}\varphi\right]\right)\Big|_{t=0}, \end{aligned}$$

and hence (3.58) is equivalent to

(3.59)
$$\iint_{\mathbb{R}^{2n}} \frac{\big((z_{\mu,\xi}+w)(x)-(z_{\mu,\xi}+w)(y)\big)\,\big((\varphi+\frac{\partial w}{\partial z_{\mu,\xi}}\varphi)(x)-(\varphi+\frac{\partial w}{\partial z_{\mu,\xi}}\varphi)(y)\big)}{|x-y|^{n+2s}}\,dx\,dy$$
$$= \int_{\mathbb{R}^n} \Big(\varepsilon h(x)\big(z_{\mu,\xi}(x)+w(x)\big)^q + \big(z_{\mu,\xi}(x)+w(x)\big)^p\Big)\,(\varphi+\frac{\partial w}{\partial z_{\mu,\xi}}\varphi)(x)\,dx,$$

for any $\varphi \in \big(T_{z_{\mu,\xi}}Z_0\big)\cap X^s$.

Moreover, since w solves (3.16), $H_1(\mu,\xi,w,\varepsilon,\alpha)=0$ is equivalent to affirm that

$$\begin{aligned}
&\iint_{\mathbb{R}^{2n}} \frac{\big((z_{\mu,\xi}+w)(x)-(z_{\mu,\xi}+w)(y)\big)\,\big(\phi(x)-\phi(y)\big)}{|x-y|^{n+2s}}\,dx\,dy\\
&\quad-\int_{\mathbb{R}^n} \Big(\varepsilon h(x)\big(z_{\mu,\xi}(x)+w(x)\big)^q + \big(z_{\mu,\xi}(x)+w(x)\big)^p\Big)\,\phi(x)\,dx,\\
&\quad=\sum_{i=1}^{n+1}\alpha_i \iint_{\mathbb{R}^{2n}} \frac{\big(q_i(x)-q_i(y)\big)\,\big(\phi(x)-\phi(y)\big)}{|x-y|^{n+2s}}\,dx\,dy,
\end{aligned}\tag{3.60}$$

for any $\phi\in X^s$.

Consider now $q_j \in T_{z_{\mu,\xi}}Z_0$ defined in (3.4). Thus, taking $\varphi := q_j$ in (3.59) and applying (3.60) with $\phi := q_j + \frac{\partial w}{\partial z_{\mu,\xi}} q_j$ we obtain

(3.61)
$$\begin{aligned}
0 &= \sum_{i=1}^{n+1}\alpha_i \iint_{\mathbb{R}^{2n}} \frac{\big(q_i(x)-q_i(y)\big)\,\big((q_j+\frac{\partial w}{\partial z_{\mu,\xi}}q_j)(x)-(q_j+\frac{\partial w}{\partial z_{\mu,\xi}}q_j)(y)\big)}{|x-y|^{n+2s}}\,dx\,dy\\
&= \sum_{i=1}^{n+1}\alpha_i\langle q_i,q_j\rangle + \sum_{i=1}^{n+1}\alpha_i\langle q_i,\frac{\partial w}{\partial z_{\mu,\xi}}q_j\rangle\\
&= \lambda_j\alpha_j + \sum_{i=1}^{n+1}\alpha_i\langle q_i,\frac{\partial w}{\partial z_{\mu,\xi}}q_j\rangle,
\end{aligned}$$

where Lemma 3.1 was also used in the last line.

Set now the $(n+1)\times(n+1)$ matrix $B^\varepsilon = (b^\varepsilon_{ij})$, defined as

$$\begin{aligned}
b^\varepsilon_{ij} &:= \langle q_i, \frac{\partial w}{\partial \xi_j}\rangle, \quad i=1,\dots,n+1,\ j=1,\dots,n,\\
b^\varepsilon_{i,n+1} &:= \langle q_i, \frac{\partial w}{\partial \mu}\rangle, \quad i=1,\dots,n+1.
\end{aligned}$$

By Cauchy-Schwartz inequality and (3.11) one has

$$\lim_{\varepsilon\to0}\langle q_i,\frac{\partial w}{\partial \xi_j}\rangle = \lim_{\varepsilon\to0}\langle q_i,\frac{\partial w}{\partial \mu}\rangle = 0, \quad i=1,\dots,n+1,\ j=1,\dots,n,\tag{3.62}$$

and thus $\lim\limits_{\varepsilon\to0}\|B^\varepsilon\|=0$. Recalling that

$$\frac{\partial w}{\partial z_{\mu,\xi}}q_j = \frac{\partial w}{\partial z_{\mu,\xi}}\frac{\partial z_{\mu,\xi}}{\partial \xi_j} = \frac{\partial}{\partial \xi_j}w(\varepsilon,z_{\mu,\xi}) = \frac{\partial w}{\partial \xi_j} \text{ for } j=1,\dots,n$$

and

$$\frac{\partial w}{\partial z_{\mu,\xi}}q_{n+1} = \frac{\partial w}{\partial z_{\mu,\xi}}\frac{\partial z_{\mu,\xi}}{\partial \mu} = \frac{\partial}{\partial \mu}w(\varepsilon,z_{\mu,\xi}) = \frac{\partial w}{\partial \mu},$$

equation (3.61) becomes

$$\lambda_j \alpha_j + \sum_{i=1}^{n+1} \alpha_i b_{ij}^{\varepsilon} = 0, \quad i,j = 1,\dots,n+1,$$

that is nothing but a $(n+1)\times(n+1)$ linear system with associated matrix $\lambda\, Id_{\mathbb{R}^{n+1}} + B^{\varepsilon}$, whose entries are $\lambda_j \delta_{ij} + b_{ij}^{\varepsilon}$, where $\delta_{jj} = 1$ and $\delta_{ij} = 0$ whether $i \neq j$. Thus, since $\lim\limits_{\varepsilon\to 0} \|B^{\varepsilon}\| = 0$, there exists $\varepsilon_1 > 0$ such that for $\varepsilon < \varepsilon_1$ the matrix $\lambda\, Id_{\mathbb{R}^{n+1}} + B^{\varepsilon}$ is invertible, and therefore $\alpha_i = 0$ for every $i = 1,\dots,n+1$. Hence, coming back to (3.60), we get

$$\begin{aligned}&\iint_{\mathbb{R}^{2n}} \frac{\big((z_{\mu,\xi}+w)(x) - (z_{\mu,\xi}+w)(y)\big)\big(\phi(x)-\phi(y)\big)}{|x-y|^{n+2s}}\, dx\, dy \\ &\qquad = \int_{\mathbb{R}^n} \Big(\varepsilon h(x)\big(z_{\mu,\xi}(x)+w(x)\big)^q + \big(z_{\mu,\xi}(x)+w(x)\big)^p\Big)\, \phi(x)\, dx,\end{aligned}$$

for every $\phi \in X^s$, that is, $z_{\mu,\xi} + w$ is a critical point of f_{ε}. □

4. Study of the behavior of Γ

At this point, we have reduced our original problem to a finite-dimensional one. Indeed, we define the perturbed manifold

$$Z_{\varepsilon} := \{u := z_{\mu,\xi} + w(\varepsilon, z_{\mu,\xi}) \text{ s.t. } z_{\mu,\xi} \in Z_0\},$$

which is a natural constraint for the functional f_{ε}.

We recall (1.10)and (3.2) and we give the following

Definition 4.1. We say that $u \in U$ is a proper local maximum (or minimum, respectively) of G if there exists a neighborhood $\mathcal{U}$ of u such that

$$G(u) \geqslant G(v) \ \ \forall\, v \in \mathcal{U} \qquad (G(u) \leqslant G(v) \ \ \forall\, v \in \mathcal{U}, \text{ respectively}),$$

and

$$G(u) > \sup_{v\in\partial\mathcal{U}} G(v) \qquad (G(u) < \inf_{v\in\partial\mathcal{U}} G(v), \text{ respectively}).$$

With this, one can prove that:

Proposition 4.2. *Suppose that $z_{\mu,\xi} \in Z_0$ is a proper local maximum or minimum of G. Then, for $\varepsilon > 0$ sufficiently small, $u_{\varepsilon} := z_{\mu,\xi} + w(\varepsilon, z_{\mu,\xi}) \in Z_{\varepsilon}$ is a critical point of f_{ε}.*

The proof of this can be found for instance in [6] (see in particular Theorem 2.16 there). A simple explanation goes as follows. First we notice that, for any $z_{\mu,\xi} \in Z_0$,

$$f_0'(z_{\mu,\xi}) = 0, \tag{4.1}$$

where f_0 is defined in (1.9). Indeed, $z_{\mu,\xi}$ is a critical point of f_0, being a solution to (1.4). Now, recalling (1.8) and using Taylor expansion in the vicinity of $z_{\mu,\xi}$, we have

$$\begin{aligned} f_{\varepsilon}(z_{\mu,\xi}+w) &= f_0(z_{\mu,\xi}+w) - \varepsilon\, G(z_{\mu,\xi}+w) \\ &= f_0(z_{\mu,\xi}) + f_0'(z_{\mu,\xi})\, w + o(|w|) - \varepsilon\, G(z_{\mu,\xi}) - \varepsilon\, G'(z_{\mu,\xi})\, w + o(\varepsilon) \\ &= f_0(z_{\mu,\xi}) - \varepsilon\, G(z_{\mu,\xi}) + o(\varepsilon) \\ &= f_0(z_0) - \varepsilon\, G(z_{\mu,\xi}) + o(\varepsilon), \end{aligned}$$

where we have used (4.1) and (3.11), and the translation and dilation invariance of f_0.

Therefore, we have reduced our problem to find critical points of G. For this, we set

$$\Gamma(\mu,\xi) := G(z_{\mu,\xi}) = \frac{\mu^{-\gamma_s}}{q+1}\int_{\mathbb{R}^n} h(x) z_0^{q+1}\left(\frac{x-\xi}{\mu}\right)dx, \tag{4.2}$$

where

$$\gamma_s := \frac{(n-2s)(q+1)}{2}. \tag{4.3}$$

Now we prove some lemma concerning the behavior of Γ. In the first one we compute the limit of Γ as μ tends to zero.

Lemma 4.3. *Let* Γ *be as in* (4.2). *Then*

$$\lim_{\mu\to 0}\Gamma(\mu,\xi) = 0 \quad \textit{uniformly in } \xi.$$

Proof. Thanks to (1.2), there exists $r>1$ such that

$$\omega = supp\, h \subset B_r. \tag{4.4}$$

We first suppose that $\xi\in\mathbb{R}^n$ is such that $|\xi|\geqslant 2r$. Therefore, if $|y|<r$ then

$$|\xi+y| \geqslant |\xi|-|y| > r,$$

and so $\xi+y\in B_r^c\subset\omega^c$. This implies that

$$h(y+\xi)=0 \ \text{ if } |\xi|\geqslant 2r \text{ and } |y|<r. \tag{4.5}$$

Now, we observe that, using the change of variable $y=x-\xi$, Γ can be written as

$$\Gamma(\mu,\xi) = \frac{\mu^{-\gamma_s}}{q+1}\int_{\mathbb{R}^n} h(y+\xi)\, z_0^{q+1}\left(\frac{y}{\mu}\right)dy.$$

Hence, using (4.5) we have that, if $|\xi|\geqslant 2r$,

$$\begin{aligned}\Gamma(\mu,\xi) &= \frac{\mu^{-\gamma_s}}{q+1}\int_{|y|\geqslant r} h(y+\xi)\, z_0^{q+1}\left(\frac{y}{\mu}\right)dy\\ &\leqslant \frac{\mu^{-\gamma_s}}{q+1}\max_{|y|\geqslant r} z_0^{q+1}\left(\frac{y}{\mu}\right)\int_{|y|\geqslant r} h(y+\xi)\, dy.\end{aligned}$$

This implies that

$$|\Gamma(\mu,\xi)| \leqslant \frac{\mu^{-\gamma_s}}{q+1}\max_{|y|\geqslant r} z_0^{q+1}\left(\frac{y}{\mu}\right)\|h\|_{L^1(\mathbb{R}^n)}. \tag{4.6}$$

Now, recalling (1.5), we obtain that

$$z_0^{q+1}\left(\frac{y}{\mu}\right) = \alpha_{n,s}^{q+1}\frac{\mu^{(n-2s)(q+1)}}{(\mu^2+|y|^2)^{\frac{(n-2s)(q+1)}{2}}},$$

and so

$$\max_{|y|\geqslant r} z_0^{q+1}\left(\frac{y}{\mu}\right) = \mu^{(n-2s)(q+1)}\max_{|y|\geqslant r}\frac{\alpha_{n,s}^{q+1}}{(\mu^2+|y|^2)^{\frac{(n-2s)(q+1)}{2}}} \leqslant C\,\mu^{(n-2s)(q+1)},$$

for a suitable constant $C>0$ independent on μ. Using this in (4.6) and recalling (4.3), (1.2) and the fact that h is continuous, we get (up to renaming C)

$$|\Gamma(\mu,\xi)| \leqslant C\,\mu^{\frac{(n-2s)(q+1)}{2}},$$

which tends to zero as $\mu \to 0$. This concludes the proof in the case $|\xi| \geqslant 2r$.

If instead $|\xi| < 2r$ then one has

$$\begin{aligned}\int_{\mathbb{R}^n} h(x)\, z_0^{q+1}\left(\frac{x-\xi}{\mu}\right) dx &\leqslant \int_{|x|<r} h(x)\, z_0^{q+1}\left(\frac{x-\xi}{\mu}\right) dx \\ &\leqslant \|h\|_{L^\infty(\mathbb{R}^n)} \int_{|x|<r} z_0^{q+1}\left(\frac{x-\xi}{\mu}\right) dx,\end{aligned} \tag{4.7}$$

thanks to (4.4), (1.2) and the fact that h is continuous.

We claim that

$$\int_{|x|<r} z_0^{q+1}\left(\frac{x-\xi}{\mu}\right) dx \leqslant C\, \mu^{\min\{n,(n-2s)(q+1)\}}, \tag{4.8}$$

for some positive constant C independent of μ (possibly depending on r). To prove this, we recall (1.5) and we get

$$\begin{aligned}&\int_{|x|<r} z_0^{q+1}\left(\frac{x-\xi}{\mu}\right) dx \\ = \;& \alpha_{n,s}^{q+1} \int_{|x|<r} \frac{dx}{\left(1+\frac{|x-\xi|^2}{\mu^2}\right)^{\frac{(n-2s)(q+1)}{2}}} \\ \leqslant \;& \alpha_{n,s}^{q+1}\left(\int_{|x-\xi|\leqslant\mu} dx + \int_{\mu<|x-\xi|<3r} \frac{\mu^{(n-2s)(q+1)}}{|x-\xi|^{(n-2s)(q+1)}}\, dx\right) \\ \leqslant \;& C\left(\mu^n + \mu^{(n-2s)(q+1)} \int_\mu^{3r} \rho^{n-1-(n-2s)(q+1)}\, d\rho\right) \\ \leqslant \;& C\left(\mu^n + \mu^{(n-2s)(q+1)}\, \mu^{-[(n-2s)(q+1)-n]_+}\right) \\ \leqslant \;& C\left(\mu^n + \mu^{\min\{n,(n-2s)(q+1)\}}\right) \\ \leqslant \;& C\, \mu^{\min\{n,(n-2s)(q+1)\}},\end{aligned}$$

up to changing C from line to line, and this shows (4.8). Therefore, by (4.2), (4.3) and (4.7) we have that

$$|\Gamma(\mu,\xi)| \leqslant C\, \mu^{-\frac{(n-2s)(q+1)}{2}}\, \mu^{\min\{n,(n-2s)(q+1)\}}.$$

Hence, if $(n-2s)(q+1) \leqslant n$ we get that

$$|\Gamma(\mu,\xi)| \leqslant C\, \mu^{(n-2s)(q+1)},$$

which implies that $\Gamma(\mu,\xi)$ tends to zero as $\mu \to 0$. If instead $n < (n-2s)(q+1)$ we obtain that

$$|\Gamma(\mu,\xi)| \leqslant C\, \mu^{n-\frac{(n-2s)(q+1)}{2}}.$$

In this case, we observe that, since $q \in (0,p)$ with $p = \frac{n+2s}{n-2s}$, then $q+1 < \frac{2n}{n-2s}$, and so

$$n - \frac{(n-2s)(q+1)}{2} > n - \frac{n-2s}{2}\,\frac{2n}{n-2s} = 0.$$

This implies that also in this case $\Gamma(\mu,\xi)$ tends to zero as $\mu \to 0$. This concludes the proof of Lemma 4.3. □

Now we compute the limit of Γ as $\mu + |\xi|$ tends to $+\infty$.

Lemma 4.4. *Let* Γ *be as in* (4.2). *Then*

$$\lim_{\mu+|\xi|\to+\infty} \Gamma(\mu,\xi) = 0.$$

Proof. Suppose that $\mu \to +\infty$. Then recalling (1.2), the fact that h is continuous and (1.5) we have

$$|\Gamma(\mu,\xi)| \leqslant C\,\mu^{-\gamma_s}\,\|h\|_{L^1(\mathbb{R}^n)},$$

for some positive constant C independent on μ. Therefore $\Gamma(\mu,\xi)$ tends to zero as $\mu \to +\infty$.

Now suppose that $\mu \to \bar\mu$ for some $\bar\mu \in [0,+\infty)$, therefore $|\xi| \to +\infty$. If $\bar\mu = 0$, then we can use Lemma 4.3 and we get the desired result. Hence, we can suppose that $\bar\mu \in (0,+\infty)$. In this case, we make the change of variable $y = x - \xi$ and we write Γ as

$$\Gamma(\mu,\xi) = \frac{\mu^{-\gamma_s}}{q+1}\int_{\mathbb{R}^n} h(y+\xi)\, z_0^{q+1}\left(\frac{y}{\mu}\right) dy. \tag{4.9}$$

Since h has compact support (recall (1.2)), there exists $r > 0$ such that $\omega = \mathit{supp}\, h \subset B_r$ and so (4.9) becomes

$$\Gamma(\mu,\xi) = \frac{\mu^{-\gamma_s}}{q+1}\int_{|y+\xi|\leqslant r} h(y+\xi)\, z_0^{q+1}\left(\frac{y}{\mu}\right) dy. \tag{4.10}$$

We also notice that, since $|\xi| \to +\infty$, we can suppose that $|\xi| > 2r$. Therefore, if $y \in B_r(-\xi)$, then $|y+\xi| \leqslant r < |\xi|/2$, which implies that

$$|y| \geqslant |\xi| - |y+\xi| \geqslant |\xi| - \frac{|\xi|}{2} = \frac{|\xi|}{2}.$$

Hence, recalling (1.5), we obtain that if $y \in B_r(-\xi)$

$$\begin{aligned} z_0^{q+1}\left(\frac{y}{\mu}\right) &= \frac{\alpha_{n,s}^{q+1}\,\mu^{(n-2s)(q+1)}}{(\mu^2+|y|^2)^{\frac{(n-2s)(q+1)}{2}}} \\ &\leqslant \frac{\alpha_{n,s}^{q+1}\,\mu^{(n-2s)(q+1)}}{|y|^{(n-2s)(q+1)}} \\ &\leqslant \frac{2^{(n-2s)(q+1)}\,\alpha_{n,s}^{q+1}\,\mu^{(n-2s)(q+1)}}{|\xi|^{(n-2s)(q+1)}}. \end{aligned}$$

Using this, (1.2) and the fact that h is continuous into (4.10), we have that

$$|\Gamma(\mu,\xi)| \leqslant C\,\mu^{\gamma_s}\,\frac{1}{|\xi|^{(n-2s)(q+1)}}\,\|h\|_{L^1(\mathbb{R}^n)},$$

for some constant independent on μ and ξ. Since $\mu \to \bar\mu \in (0,+\infty)$, this implies that

$$\Gamma(\mu,\xi) \to 0 \quad \text{as } |\xi| \to +\infty,$$

thus concluding the proof of Lemma 4.4. □

Finally we show the following:

Lemma 4.5. *Let* Γ *be as in* (4.2). *Suppose that there exists* $\xi_0 \in \mathbb{R}^n$ *such that* $h(\xi_0) > 0$ *(*$h(\xi_0) < 0$ *respectively). Then*

$$\lim_{\mu\to 0} \frac{\Gamma(\mu,\xi_0)}{\mu^{n-\gamma_s}} = A,$$

for some $A > 0$*, possibly* $A = +\infty$ *(*$A < 0$*, possibly* $A = -\infty$*, respectively).*

Proof. We prove the lemma only in the case $h(\xi_0) > 0$, since the other case is analogous. We notice that, by using the change of variable $y = (x - \xi)/\mu$, we can rewrite Γ as

$$\Gamma(\mu, \xi) = \frac{\mu^{n-\gamma_s}}{q+1} \int_{\mathbb{R}^n} h(\mu y + \xi)\, z_0^{q+1}(y)\, dy. \tag{4.11}$$

Now, suppose first that $\frac{2s}{n-2s} < q < p$. In this case, we have that z_0 defined in (1.5) satisfies

$$z_0^{q+1} \in L^1(\mathbb{R}^n). \tag{4.12}$$

Then, from (4.11) we obtain

$$\frac{\Gamma(\mu, \xi_0)}{\mu^{n-\gamma_s}} = \frac{1}{q+1} \int_{\mathbb{R}^n} h(\mu y + \xi_0)\, z_0^{q+1}(y)\, dy.$$

We observe that

$$h(\mu y + \xi_0)\, z_0^{q+1}(y) \to h(\xi_0)\, z_0^{q+1}(y) \quad \text{as } \mu \to 0.$$

Moreover, thanks to (1.2), the fact that h is continuous and (4.12), we have that

$$h(\mu y + \xi_0)\, z_0^{q+1}(y) \leqslant \|h\|_{L^\infty(\mathbb{R}^n)}\, z_0^{q+1}(y) \in L^1(\mathbb{R}^n),$$

and so from the Dominated Convergence Theorem, we get

$$\frac{\Gamma(\mu, \xi_0)}{\mu^{n-\gamma_s}} \to \frac{h(\xi_0)}{q+1} \int_{\mathbb{R}^n} z_0^{q+1}(y)\, dy \quad \text{as } \mu \to 0,$$

showing the lemma in the case $\frac{2s}{n-2s} < q < p$. Notice that in this case

$$A := \frac{h(\xi_0)}{q+1} \int_{\mathbb{R}^n} z_0^{q+1}(y)\, dy$$

is strictly positive and bounded.

If instead $z_0^{q+1} \notin L^1(\mathbb{R}^n)$, then we use Fatou's Lemma to get

$$\liminf_{\mu \to 0} \int_{\mathbb{R}^n} h(\mu y + \xi_0)\, z_0^{q+1}(y)\, dy \geqslant h(\xi_0) \int_{\mathbb{R}^n} z_0^{q+1}(y)\, dy,$$

which implies that in this case $A := +\infty$. This concludes the proof of Lemma 4.5. $\square$

5. Proof of Theorem 1.1

Now we are ready to complete the proof of Theorem 1.1.

We observe that, thanks to (1.3) and Lemma 4.5, there exist $\mu_0 > 0$ as small as we want and $\xi_0 \in \mathbb{R}^n$ such that

$$\Gamma(\mu_0, \xi_0) \geqslant \frac{\mu_0^{n-\gamma_s}}{2} \min\{A, 1\} =: B. \tag{5.1}$$

Now, we use Lemma 4.3 to say that if $\mu > 0$ is sufficiently small, then

$$\Gamma(\mu, \xi) < \frac{B}{2} \text{ for any } \xi \in \mathbb{R}^n.$$

In particular, if $\mu_1 := \mu_0/2$, then

$$\Gamma(\mu_1, \xi) < \frac{B}{2} \text{ for any } \xi \in \mathbb{R}^n. \tag{5.2}$$

Moreover, from Lemma 4.4 we deduce that there exists $R_* > 0$ such that if $\mu+|\xi| > R_*$ we have

$$\Gamma(\mu, \xi) < \frac{B}{2}.$$

In particular, we can take $\mu_2 = R_2 = R_* + \mu_0 + |\xi_0| + 1$ and we have that

$$\Gamma(\mu, \xi) < \frac{B}{2} \text{ if either } \mu = \mu_2 \text{ and } |\xi| \leqslant R_2 \text{ or } \mu \leqslant \mu_2 \text{ and } |\xi| = R_2. \tag{5.3}$$

Now we perform our choice of R, μ_1 and μ_2 in (3.1): we take μ_1 and μ_2 such that (5.2) and (5.3) are satisfied, and $R = R_2$.

Also, we set

$$S := \{\mu_1 \leqslant \mu \leqslant \mu_2 \text{ and } |\xi| \leqslant R\},$$

and we notice that Γ admits a maximum in S, since Γ is continuous and S is a compact set. Moreover, thanks to (5.2) and (5.3) we have that

$$\Gamma(\mu, \xi) < \frac{B}{2} \text{ if } (\mu, \xi) \in \partial S. \tag{5.4}$$

On the other hand,

$$|\xi_0| < R_2 \text{ and } \mu_1 < \mu_0 < \mu_2,$$

which implies that $(\mu_0, \xi_0) \in S$. Therefore, (5.1) and (5.4) imply that the maximum of Γ is achieved at some point (μ_*, ξ_*) in the interior of S.

Now, we go back to the functional G, and recalling (4.2) we obtain that G admits a maximum z_{μ_*,ξ_*} in the critical manifold Z_0 defined in (3.1). Hence, we can apply Proposition 4.2 and we obtain the existence of a critical point of f_ε, that is a solution to (1.1), given by

$$u_{1,\varepsilon} := z_{\mu_*,\xi_*} + w(\varepsilon, z_{\mu_*,\xi_*}).$$

Also, $u_{1,\varepsilon}$ is positive thanks to (3.11).

Furthermore, if h changes sign, then there exists $\tilde{\xi}_0 \in \mathbb{R}^n$ such that $h(\tilde{\xi}_0) < 0$, and so we can use Lemma 4.5 to say that

$$\Gamma(\tilde{\mu}_0, \tilde{\xi}_0) \leqslant \frac{\tilde{\mu}_0^{n-\gamma_s}}{2} \max\{A, -1\},$$

for some $\tilde{\mu}_0 > 0$. Then we can repeat all the above arguments (with suitable modifications) to find a local minimum of Γ, and so a a local minimum of G. Then, again from Proposition 4.2 we obtain the existence of a second positive solution. This concludes the proof of Theorem 1.1.

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