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Corners and edges always scatter

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Abstract

Consider time-harmonic acoustic scattering problems governed by the Helmholtz equation in two and three dimensions. We prove that bounded penetrable obstacles with corners or edges scatter every incident wave nontrivially, provided the function of refractive index is real-analytic. Moreover, if such a penetrable obstacle is a convex polyhedron or polygon, then its shape can be uniquely determined by the far-field pattern over all observation directions incited by a single incident wave. Our arguments are elementary and rely on the expansion of solutions to the Helmholtz equation.

1 Introduction and main results

Consider time-harmonic acoustic scattering from a bounded penetrable obstacle $D \subset \mathbb{R}^N$ $(N = 2, 3)$ embedded in a homogeneous medium. The incident field u^{in} is supposed to be a non-trivial solution to the Helmholtz equation

$$\Delta u^{in} + k^2 u^{in} = 0$$

in a neighborhood of D with the wavenumber $k > 0$. Thus, for example, u^{in} is allowed to be a plane wave $\exp(ikx \cdot d)$ with the incident direction $d \in \mathbb{S}^{N-1} := \{x \in \mathbb{R}^N : |x| = 1\}$, a spherical point source wave emitted from some source position located in $D^e = \mathbb{R}^N \backslash \overline{D}$, or a Herglotz wave function of the form

$$u^{in}(x) = \int_{\mathbb{S}^{N-1}} \exp(ikx \cdot d) g(d)\, ds(d), \quad g \in L^2(\mathbb{S}^{N-1}).$$

The acoustic property of the medium can be described by the refractive index function (or potential) $q(x)$. In this paper, we restrict our discussions to the following penetrable obstacles and potentials:

(a) D is a bounded polygonal domain in $\mathbb{R}^2$ or a bounded polyhedral domain in $\mathbb{R}^3$.

(b) $q(x) = 1$ for $x \in D^e$ and q is real-analytic on $\overline{D}$.

Note that the potential has been normalized to be one for $x \in D^e$ due to the homogeneity of the background medium. The wave propagation is then governed by the Helmholtz equation

$$\Delta u(x) + k^2 q(x) u(x) = 0 \tag{1.1}$$

which holds at least in a neighborhood of D. In (1.1), $u = u^{in} + u^{sc}$ denotes the total wave where u^{sc} is the scattered field satisfying

$$(\Delta + k^2) u^{sc} = 0 \quad \text{in} \quad D^e$$

and the Sommerfeld radiation condition

$$\lim_{|x|\to\infty} |x|^{\frac{N-1}{2}} \left\{ \frac{\partial u^{sc}}{\partial |x|} - iku^{sc} \right\} = 0. \tag{1.2}$$

Across the interface ∂D, we assume the continuity of the total field and its normal derivative (already implicitly contained in the formulation (1.1)), i.e.,

$$u^+ = u^-, \quad \partial_\nu u^+ = \partial_\nu u^- \quad \text{on } \partial D. \tag{1.3}$$

Here the superscripts $(\cdot)^\pm$ stand for the limits taken from outside and inside, respectively, and $\nu \in \mathbb{S}^N$ is the unit normal on ∂D pointing into D^e. The unique solvability of the scattering problem (1.1), (1.2) and (1.3) in $H^2_{loc}(\mathbb{R}^N)$ is well known (see e.g., [6, Chapter 8]). In particular, the Sommerfeld radiation condition (1.2) leads to the asymptotic expansion

$$u^{sc}(x) = \frac{e^{ik|x|}}{|x|^{(N-1)/2}}\, u^\infty(\hat{x}) + \mathcal{O}\left(\frac{1}{|x|^{N/2}}\right), \quad |x| \to +\infty, \tag{1.4}$$

uniformly in all directions $\hat{x} := x/|x|$, $x \in \mathbb{R}^N$. The function $u^\infty(\hat{x})$ is an analytic function defined on $\mathbb{S}^N$ and is referred to as the *far-field pattern* or the *scattering amplitude*. The vector $\hat{x} \in \mathbb{S}^N$ is called the observation direction of the far field. The classical inverse medium scattering problem consists of the recovery of the refractive contrast $1 - q$ or the boundary ∂D of its support from the far-field patterns corresponding to one or several incident plane waves with different incident directions or at many frequencies. We state the first result of this paper as follows.

Theorem 1.1. *Suppose that $q(O) \neq 1$ at a corner point $O \in \partial D$ in 2D, or at an edge point $O \in \partial D$ in 3D. Then, under the conditions (a) and (b), the scattered field to the scattering problem (1.1)- (1.4) cannot vanish identically.*

We note that an edge in 3D is understood as a straight line where two flat faces meet with an angle different from π. Theorem 1.1 implies that an inhomogeneous medium with a piecewise real-analytic refractive index having a corner or an edge with arbitrary angle on the boundary scatters every incident wave nontrivially, i.e., corners and edges always scatter. Under weaker smoothness conditions on the refractive index (piecewise C^∞ or Hölder continuous) but stronger assumptions on the geometry of the scatterer, this was proved in [1,22] where the acoustic scattering from a right angle corner in $\mathbb{R}^N$, a convex angle in $\mathbb{R}^2$ and a convex circular conic corner in $\mathbb{R}^3$ were studied. Our approach depends heavily on the Taylor expansion for solutions of the Helmholtz equation with analytic potentials.This differs completely from the approach in [1,22] which was based on a new construction of *Complex Geometric Optics* (CGO) solutions to the Helmholtz equation with smooth or Hölder continuous potentials. Theorem 1.1 implies the absence of so-called *non-scattering energies* or *non-scattering wavenumbers* (see [1,22]) for piecewise real-analytic potentials in corner and wedge domains, and follows straightforwardly if k^2 is not an *interior transmission eigenvalue* (ITE). Indeed, if $u^{sc} \equiv 0$, then one can derive from (1.1) and (1.3) that for u and $w = u^{in}|_D$, the interior transmission problem

$$\begin{cases} \Delta w + k^2 w = 0, & \Delta u + k^2 q u = 0 \quad \text{in} \quad D, \\ w = u, & \partial_\nu w = \partial_\nu u \quad\quad\ \ \text{on} \quad \partial D \end{cases} \tag{1.5}$$

is satisfied. If k^2 is not an ITE associated with D, then we get $w \equiv 0$ which is a contradiction. Our result reveals that u^{sc} cannot vanish even if k^2 happens to be an ITE. We refer to [2, 4, 7, 8, 21, 23] for the existence of ITEs in inverse scattering theory and to Cakoni and Haddar [5] for a recent survey on ITEs. As a consequence of the proof of Theorem 1.1 (see Section 3), we also have

Corollary 1.1. *Let $(w, u) \in H^1(D) \times H^1(D)$ be a non-trivial solution pair to (1.5). Then under the assumptions (a), (b) and the conditions of Theorem 1.1, w cannot be extended into a neighborhood of the corner or edge point O as a solution to the Helmholtz equation with the wavenumber k^2.*

Theorem 1.1 and Corollary 1.1 extend the results of [1, 22] to corners in $\mathbb{R}^2$ and edges in $\mathbb{R}^3$ with an arbitrary angle but so far only in the case of piecewise analytic potentials. Our second result concerns the uniqueness in determining the interface of a penetrable obstacle as well as the refractive index function.

Theorem 1.2. *Suppose that D_j $(j = 1, 2)$ are convex polygons or polyhedrons with the potentials q_j fulfilling the assumption (b) with $D = D_j$. Denote by $u_j^\infty(\hat{x}; k)$ the far-field patterns of the scattered fields due to the incoming wave $u^{in}(x; k)$ onto D_j.*

(i) *If $|q_j - 1| > 0$ on ∂D_j, $j = 1, 2$, then the relation $u_1^\infty(\hat{x}; k_0) = u_2^\infty(\hat{x}; k_0)$ for all $\hat{x} \in \mathbb{S}^{N-1}$ and for some fixed $k_0 > 0$ implies the coincidence of the scatterers, i.e., $D_1 = D_2 := D$.*

(ii) *If we assume additionally that $|q_1 - q_2| > 0$ on ∂D, then one cannot have*

$$u_1^\infty(\hat{x}; k) = u_2^\infty(\hat{x}; k) \quad \text{for all} \quad \hat{x} \in \mathbb{S}^{N-1},\ k \in [k_{min}, k_{max}], \tag{1.6}$$

with some $0 < k_{min} < k_{max}$.

Theorem 1.2 (i) provides an affirmative answer to the unique determination of the shape of a penetrable obstacle within the class of convex polygons or polyhedrons from a single far-field pattern. Note that this is still an overdetermined inverse problem, because only the finitely many corner points need to be identified. If the penetrable obstacle is not convex, the unique determination of its convex hull follows immediately from the proof of Theorem 1.2 (i) (see Section 3). The second assertion deals with the formally determined inverse problem of recovering a potential by using many frequencies. It follows from the discreteness of the *generalized interior transmission eigenvalues* proved by Sylvester [23], in which only the behavior of the potentials in a neighborhood of the boundary was required. If (1.6) holds, then by Theorem 1.2 (i) the supports of $1 - q_j$ coincide and by the proof of the second assertion $q_1 - q_2$ must be oscillating in a neighborhood of ∂D in D. Hence, the far-field patterns with a fixed incident direction at multi-frequencies can be used to distinguish two unknown smooth potentials whose difference does not change sign on the boundary of a convex polygon or polyhedron; see Remark 4.1 for more discussions concerning the assumption $|q_1 - q_2| > 0$ on ∂D.

Earlier uniqueness results on shape identification in inverse medium scattering were derived by sending plane waves with infinitely many directions at a fixed frequency (see e.g., [9,13,14,17,18]), which results in an overdetermined inverse problem. Intensive efforts have also been devoted to the unique determination of the variable contrast $1 - q$ from knowledge of the far-field patterns of all incident plane waves or by measuring the DtN map of the Helmholtz equation. We refer to [20, 24] and [6, Chapter 10.2] for the uniqueness in 3D and to recent results [3, 12] in 2D with certain assumptions on the regularity of q. It was shown in [11] that an inhomogeneous medium with a constant refractive index can be uniquely determined by a single far-field measurement, provided the incident wave number is small or the underlying medium is spherically symmetric.

The paper is organized as follows. In Section 2, we derive our crucial auxiliary result on the transmission problem for the Helmholtz equation in a sector (Proposition 2.1).In Sections 3 and 4, we verify Theorems 1.1 and 1.2 relying on Proposition 2.1. At the end of Section 4, we present a uniqueness result analogous to Theorem 1.2 (i) for recovering the support of a source term in an elliptic equation from a single boundary measurement.

2 Helmholtz equation with analytic potentials

In this section we study a transmission problem for the Helmholtz equation with analytic (e.g., constant) potentials in a sector. Denote by $B_R := \{x : |x| < R\}$ the ball centered at the origin O with radius $R > 0$. The following proposition is crucial in proving Theorems 1.1 and 1.2.

Proposition 2.1. *Let Π_1 and Π_2 be two closed half-lines in $\mathbb{R}^2$ originating at O or two half-planes in $\mathbb{R}^3$ intersecting at an edge passing through O, and let q_1 and q_2 be real analytic functions defined in B_R. Assume that u_1 and u_2 satisfy the Helmholtz equations*

$$\Delta u_j + q_j\, u_j = 0 \qquad \text{in} \quad B_R, \tag{2.1}$$

subject to the transmission conditions

$$u_1 = u_2, \quad \partial_\nu u_1 = \partial_\nu u_2 \quad \text{on} \quad \Pi_j \cap B_R, \quad j = 1, 2. \tag{2.2}$$

If $q_1(O) \neq q_2(O)$, then $u_1 = u_2 \equiv 0$ in B_R.

From the above proposition it follows that the Cauchy data of non-trivial solutions of two Helmholtz equations cannot coincide on two intersecting lines or planes, if the potentials involved are real-analytic, but do not coincide on the intersection. This result seems to be of independent interest. For the purpose of proving Proposition 2.1 we need the following lemma.

Lemma 2.1. *Suppose that $\alpha \in (0, 2\pi)\backslash\{\pi\}$. Then $|\sin\alpha| > |\sin(m\alpha)|/m$ for all $m \in \mathbb{N}$, $m \geq 2$.*

Proof. Clearly, it holds for $m = 2$ that $|\sin(2\alpha)|/2 = |\sin\alpha\cos\alpha| < |\sin\alpha|$. Assume the assertion is true with some $M \in \mathbb{N}$, $M \geq 2$, i.e., $|\sin(M\alpha)| < M|\sin\alpha|$. Then,

$$\begin{aligned} \frac{|\sin[(M+1)\alpha]|}{M+1} &\leq \frac{|\sin(M\alpha)|\,|\cos\alpha|}{M+1} + \frac{|\cos(M\alpha)|\,|\sin\alpha|}{M+1} \\ &< \frac{M|\sin(\alpha)|}{M+1} + \frac{|\sin\alpha|}{M+1} \\ &= |\sin\alpha|, \end{aligned}$$

that is, Lemma 2.1 is also valid with $m = M + 1$. By induction we obtain the desired result for any $m \in \mathbb{N}$, $m \geq 2$. □

The proof of Proposition 2.1 will be carried out in two and three dimensions separately, based on the Taylor expansion of real analytic functions. Similar ideas were used in [10] for justifying properties of the Navier equation with vanishing data on two perpendicular straight lines.

2.1 Proof of Proposition 2.1 in two dimensions

Let (r, φ) with $\varphi \in (-\pi, \pi]$ be the polar coordinates of $x = (x_1, x_2)$ in $\mathbb{R}^2$. For notational convenience, we set $\mathbb{N}_0 := \mathbb{N} \cup \{0\}$. From the elliptic regularity we know that u_j are real-analytic functions on B_R due to the analyticity of q_j. We begin with a lemma on the Taylor expansions of u_j in B_R.

Lemma 2.2. *Let u_j be solutions to (2.1). Then there hold the convergent expansions*

$$u_j(r,\varphi) = \sum_{n,m\in\mathbb{N}_0} r^{n+2m}\left(a^{(j)}_{n,m}\cos(n\varphi) + b^{(j)}_{n,m}\sin(n\varphi)\right), \quad r<R, \quad j=1,2, \tag{2.3}$$

with $a^{(j)}_{n,m}, b^{(j)}_{n,m} \in \mathbb{C}$, $b^{(j)}_{0,m} = 0$. Moreover, the Helmholtz equations (2.1) can be written as

$$\begin{aligned} &\sum_{n,m\in\mathbb{N}_0} 4(m+1)(m+n+1)\, r^{n+2m}\left(a^{(j)}_{n,m+1}\cos(n\varphi) + b^{(j)}_{n,m+1}\sin(n\varphi)\right) \\ &= -q_j(r,\varphi) \sum_{n,m\in\mathbb{N}_0} r^{n+2m}\left(a^{(j)}_{n,m}\cos(n\varphi) + b^{(j)}_{n,m}\sin(n\varphi)\right). \end{aligned} \tag{2.4}$$

If $q_j \in \mathbb{C}$ are constants, we then have the recurrence relations

$$c^{(j)}_{n,m+1} = -\frac{q_j}{4(m+1)(n+m+1)}\, c^{(j)}_{n,m}, \quad \forall\, n,m\in\mathbb{N}_0, \quad c^{(j)}_{n,m} := a^{(j)}_{n,m},\, b^{(j)}_{n,m}. \tag{2.5}$$

The expressions (2.3) can be obtained by passing to polar coordinates in the usual Taylor expansions of u_j at the origin, and relations (2.5) and (2.4) follow by inserting (2.3) into equation (1.1) ; see ([10, Lemma 2.2]) for the details. Without loss of generality we may assume that

$$\Pi_1 = \{(r,\varphi) : \varphi = -\varphi_0\}, \ \Pi_2 = \{(r,\varphi) : \varphi = \varphi_0\} \quad \text{for some } \varphi_0 \in (0,\pi)\backslash\{\pi/2\}.$$

To prove Proposition 2.1, we rewrite the expansions (2.3) as

$$u_j(r,\varphi) = \sum_{l\in\mathbb{N}_0} r^l\, U^{(j)}_l(\varphi), \quad U^{(j)}_l(\varphi) := \sum_{n,m\in\mathbb{N}_0: n+2m=l} \left(a^{(j)}_{n,m}\cos(n\varphi) + b^{(j)}_{n,m}\sin(n\varphi)\right). \tag{2.6}$$

Set $v := u_1 - u_2$ and $a_{n,m} := a^{(1)}_{n,m} - a^{(2)}_{n,m}$, $b_{n,m} := b^{(1)}_{n,m} - b^{(2)}_{n,m}$. Hence, for $0 \le r < R$,

$$v(r,\varphi) = \sum_{l\in\mathbb{N}_0} r^l\, V_l(\varphi), \quad V_l(\varphi) := \sum_{n,m\in\mathbb{N}_0: n+2m=l} \left(a_{n,m}\cos(n\varphi) + b_{n,m}\sin(n\varphi)\right). \tag{2.7}$$

The transmission conditions in (2.2) yield

$$v = \frac{\partial v}{\partial \varphi} = 0 \quad \text{on} \quad \varphi = \pm\varphi_0. \tag{2.8}$$

Here and henceforth, our analysis is always performed inside the ball B_R. Inserting the expansion of v into (2.8) and equating the coefficients of r^l, it follows that

$$V_l(\varphi) = \partial V_l/\partial\varphi = 0 \quad \text{on} \quad \varphi = \pm\varphi_0$$

for all $l \in \mathbb{N}_0$. Consequently,

$$\begin{cases} 0 &= \displaystyle\sum_{n,m\in\mathbb{N}_0: n+2m=l} a_{n,m}\cos(n\varphi_0) =: A^+_l, \\ 0 &= \displaystyle\sum_{n\in\mathbb{N}, m\in\mathbb{N}_0: n+2m=l} b_{n,m}\sin(n\varphi_0) =: B^+_l, \\ 0 &= \displaystyle\sum_{n\in\mathbb{N}, m\in\mathbb{N}_0: n+2m=l} a_{n,m} n\sin(n\varphi_0) =: A^-_l, \\ 0 &= \displaystyle\sum_{n\in\mathbb{N}, m\in\mathbb{N}_0: n+2m=l} b_{n,m} n\cos(n\varphi_0) =: B^-_l. \end{cases} \tag{2.9}$$

Proposition 2.1 is a direct consequence of the following lemma which will be proved by the method of induction.

Lemma 2.3. *If $q_1(O) \neq q_2(O)$, then $a_{n,m} = b_{n,m} = 0$ and $a^{(j)}_{n,m} = b^{(j)}_{n,m} = 0$ for all $n, m \in \mathbb{N}_0$, $j = 1, 2$.*

Proof. Our proof is carried out by induction on the index $l = n + 2m$, using the relations (2.4) and (2.9). Since q_j are analytic functions, there hold the relations

$$q_j(r, \varphi) = q_j(O) + \mathcal{O}(r), \quad j = 1, 2, \tag{2.10}$$

uniformly in all $(r, \varphi) \in B_R$. We divide the proof into four steps, where the third step is only presented for additional insight into the formulation of our induction hypothesis.

Step 1. We first consider the zero-order coefficient in (2.7) when $l = 0$. This corresponds to the indices $n = m = 0$ in (2.9). From $A_0^+ = 0$, we see that $a_{0,0} = 0$ and thus $a^{(1)}_{0,0} = a^{(2)}_{0,0} =: \tilde{a}_{0,0}$.

Similarly, setting $l = 1$ in (2.9) (i.e., $n = 1$ and $m = 0$) gives $a_{1,0} = b_{1,0} = 0$. This implies that $a^{(1)}_{1,0} = a^{(2)}_{1,0}$ and $b^{(1)}_{1,0} = b^{(2)}_{1,0}$. Moreover, equating the zero-order coefficients on both sides of (2.4) yields

$$4\, a^{(j)}_{0,1} = -q_j(O)\, a^{(j)}_{0,0} = -q_j(O)\, \tilde{a}_{0,0}, \quad j = 1, 2. \tag{2.11}$$

Step 2. Set $l = 2$, i.e., $n = 2, m = 0$ or $n = 0, m = 1$. It is seen from (2.9) with $l = 2$ that

$$a_{2,0}\cos(2\varphi_0) + a_{0,1} = 0, \quad a_{2,0}\sin(2\varphi_0) = 0, \quad b_{2,0}\cos(2\varphi_0) = b_{2,0}\sin(2\varphi_0) = 0,$$

from which we obtain $a_{2,0} = a_{0,1} = b_{2,0} = 0$. In particular, using (2.11) we arrive at

$$0 = 4a_{0,1} = 4(a^{(1)}_{0,1} - a^{(2)}_{0,1}) = (q_2(O) - q_1(O))\, \tilde{a}_{0,0}.$$

By the assumption $q_2(O) \neq q_1(O)$ we get $\tilde{a}_{0,0} = a^{(j)}_{0,0} = 0$, $j = 1, 2$. This in turn gives

$$a_{0,0} = 0, \qquad a^{(j)}_{0,1} = 0, \quad U^{(j)}_0(\varphi) \equiv 0, \quad j = 1, 2. \tag{2.12}$$

Further, equating the coefficients of r in (2.4) (i.e., $n = 1, m = 0$) and taking into account (2.10) and the last relation in (2.12) yields

$$8\, \left(a^{(j)}_{1,1}\cos(\varphi) + b^{(j)}_{1,1}\sin(\varphi)\right) = -q_j(O)\left(a^{(j)}_{1,0}\cos\varphi + b^{(j)}_{1,0}\sin\varphi\right)$$

for any $\varphi \in (-\pi, \pi]$. Consequently, we get the recurrence relations

$$8\, a^{(j)}_{1,1} = -q_j(O)\, a^{(j)}_{1,0}, \quad 8\, b^{(j)}_{1,1} = -q_j(O)\, b^{(j)}_{1,0}, \quad j = 1, 2, \tag{2.13}$$

in which the zero-order coefficient of q_j is involved only.

Step 3. Set $l = 3$ in (2.9), i.e., $n = 3, m = 0$ or $n = m = 1$. We then obtain the linear systems

$$0 = \begin{pmatrix} \cos(3\varphi_0) & \cos\varphi_0 \\ 3\sin(3\varphi_0) & \sin\varphi_0 \end{pmatrix}\begin{pmatrix} a_{3,0} \\ a_{1,1} \end{pmatrix} =: E_3^+(\varphi_0)\begin{pmatrix} a_{3,0} \\ a_{1,1} \end{pmatrix},$$
$$0 = \begin{pmatrix} \sin(3\varphi_0) & \sin\varphi_0 \\ 3\cos(3\varphi_0) & \cos\varphi_0 \end{pmatrix}\begin{pmatrix} b_{3,0} \\ b_{1,1} \end{pmatrix} =: E_3^-(\varphi_0)\begin{pmatrix} b_{3,0} \\ b_{1,1} \end{pmatrix}.$$

Since $\varphi_0 \in (0,\pi)\backslash\{\pi/2\}$, simple calculations and using Lemma 2.1 show that

$$\text{Det}(E_3^{\pm}(\varphi_0)) = \pm 2\sin\alpha - \sin(2\alpha) \neq 0,$$

where $\alpha := 2\varphi_0 \in (0,2\pi)\backslash\{\pi\}$ denotes the angle formed by the lines Π_1 and Π_2. Hence,

$$a_{3,0} = b_{3,0} = 0, \qquad b_{1,1} = a_{1,1} = 0. \tag{2.14}$$

Combining the second relation in (2.14) with (2.13) and making use of the fact that $a_{1,0}^{(1)} = a_{1,0}^{(2)}$, $b_{1,0}^{(1)} = b_{1,0}^{(2)}$ (see Step 1) and $q_1(O) \neq q_2(O)$, we obtain

$$a_{1,0} = a_{1,0}^{(1)} = a_{1,0}^{(2)} = 0, \quad b_{1,0} = b_{1,0}^{(1)} = b_{1,0}^{(2)} = 0, \tag{2.15}$$

implying that $U_1^{(j)}(\varphi) = 0$, $j = 1,2$. Again using (2.13), we get from (2.15) that

$$a_{1,1} = a_{1,1}^{(j)} = b_{1,1} = b_{1,1}^{(j)} = 0, \quad j = 1,2.$$

On the other hand, equating the coefficients of r^2 in (2.4) gives

$$12\,\left(a_{2,1}^{(j)}\cos(2\varphi) + b_{2,1}^{(j)}\sin(2\varphi)\right) + 16a_{0,2}^{(j)} = -q_j(O)\,\left(a_{2,0}^{(j)}\cos(2\varphi) + b_{2,0}^{(j)}\sin(2\varphi)\right) \tag{2.16}$$

for all $\varphi \in [-\varphi_0, \varphi_0]$, where we have used the fact that $U_0^{(j)}(\varphi) = U_1^{(j)}(\varphi) \equiv 0$, $j = 1,2$. Now (2.16) implies the recurrence relations

$$12\,a_{2,1}^{(j)} = -q_j(O)\,a_{2,0}^{(j)}, \quad 12\,b_{2,1}^{(j)} = -q_j(O)\,b_{2,0}^{(j)}, \quad a_{0,2}^{(j)} = 0, \quad j = 1,2.$$

To sum up Steps 1-3, we have proved that the relations

$$\begin{cases} a_{n,m} = b_{n,m} = 0 \quad \text{for } n + 2m \leq M; \\ a_{n,m}^{(j)} = b_{n,m}^{(j)} = 0 \quad \text{for } n \leq M-2, \quad n + 2m \leq M+1,\ j = 1,2; \\ a_{n,1}^{(j)} = -\frac{q_j(O)}{4(n+1)} a_{n,0}^{(j)}, \quad b_{n,1}^{(j)} = -\frac{q_j(O)}{4(n+1)} b_{n,0}^{(j)} \quad \text{for } n \leq M-1,\ j = 1,2, \end{cases} \tag{2.17}$$

which will serve as our induction hypothesis, are valid for $M = 3$.

Step 4. We shall finish the proof by induction. Supposing that (2.17) holds with some $M \in \mathbb{N}$, we need to prove all relations in (2.17) with M replaced by $M+1$. For this purpose, it is sufficient to verify

$$\begin{cases} a_{M+1,0} = a_{M-1,1} = b_{M+1,0} = b_{M-1,1} = 0; \\ a_{M-1,0}^{(j)} = a_{M-1,1}^{(j)} = b_{M-1,0}^{(j)} = b_{M-1,1}^{(j)} = 0; \\ a_{M-2p,p+1}^{(j)} = b_{M-2p,p+1}^{(j)} = 0 \quad \text{for } p \geq 1,\ M - 2p \geq 0; \\ a_{M,1}^{(j)} = -\frac{q_j(O)}{4(M+1)} a_{M,0}^{(j)}, \quad b_{M,1}^{(j)} = -\frac{q_j(O)}{4(M+1)} b_{M,0}^{(j)}. \end{cases} \tag{2.18}$$

Note that combining (2.17) and (2.18) implies all relations in (2.17) with $M+1$ in place of M.

Equating the coefficients of r^l with $l = M+1$ in (2.9) and using the relations in the second line of (2.17), we obtain the linear systems

$$0 = E_{M+1}^{+}(\varphi_0)\begin{pmatrix} a_{M+1,0} \\ a_{M-1,1} \end{pmatrix} = E_{M+1}^{-}(\varphi_0)\begin{pmatrix} b_{M+1,0} \\ b_{M-1,1} \end{pmatrix}$$

with

$$E^{+}_{M+1}(\varphi_0) \;:=\; \begin{pmatrix} \cos((M+1)\varphi_0) & \cos((M-1)\varphi_0) \\ (M+1)\sin((M+1)\varphi_0) & (M-1)\sin((M-1)\varphi_0) \end{pmatrix}, \tag{2.19}$$

$$E^{-}_{M+1}(\varphi_0) \;:=\; \begin{pmatrix} \sin((M+1)\varphi_0) & \sin((M-1)\varphi_0) \\ (M+1)\cos((M+1)\varphi_0) & (M-1)\cos((M-1)\varphi_0) \end{pmatrix}. \tag{2.20}$$

The determinants of the linear systems are given by

$$\mathrm{Det}(E^{\pm}_{M+1}(\varphi_0)) = \pm M \sin\alpha - \sin(M\alpha), \quad \alpha := 2\varphi_0 \in (0, 2\pi)\backslash\{\pi\}. \tag{2.21}$$

Recalling from Lemma 2.1 that $|\sin\alpha| > |\sin(M\alpha)|/M$ for all $\alpha \in (0, 2\pi)\backslash\{\pi\}$, we get from the above linear systems that

$$a_{M+1,0} = a_{M-1,1} = b_{M+1,0} = b_{M-1,1} = 0, \tag{2.22}$$

which proves the relations in the first line of (2.18).

The equality $a_{M-1,0} = 0$ in the first line of (2.17) yields $a^{(1)}_{M-1,0} = a^{(2)}_{M-1,0} := \tilde{a}_{M-1,0}$. This together with (2.22) and the recurrence relations in (2.17) gives

$$0 = a_{M-1,1} = a^{(1)}_{M-1,1} - a^{(2)}_{M-1,1} = -\frac{\tilde{a}_{M-1,0}}{4M}\left(q_2(O) - q_1(O)\right),$$

from which $\tilde{a}_{M-1,0} = 0$ follows. This in turn implies that

$$a^{(j)}_{M-1,0} = a^{(j)}_{M-1,1} = 0, \quad j = 1, 2.$$

Analogously, one can deduce from $b_{M-1,1} = b_{M-1,0} = 0$ that

$$b^{(j)}_{M-1,0} = b^{(j)}_{M-1,1} = 0, \quad j = 1, 2.$$

This finishes the proof of the relations in the second line of (2.18).

In view of the definition of $U^{(j)}_{M-1}$ and the fact that (see the second line of (2.17))

$$a^{(j)}_{M-2p-1,p} = b^{(j)}_{M-2p-1,p} = 0 \quad \text{for} \quad p \geq 1, M - 2p - 1 \geq 0,$$

it follows that $U^{(j)}_{M-1}(\varphi) \equiv 0$ for all $j = 1, 2$, $\varphi \in [-\varphi_0, \varphi_0]$. To check the other relations in (2.18), we equate the coefficients of r^M in (2.4). Applying $U^{(j)}_{l} \equiv 0$ for all $l \leq M - 1$ and the results in the second line of (2.17) , we find

$$\begin{aligned} & \sum_{p\geq 1, M-2p\geq 0} 4(p+1)(M-p+1)\left(a^{(j)}_{M-2p,p+1}\cos((M-2p)\varphi)\right) \\ & + \sum_{p\geq 1, M-2p\geq 0} 4(p+1)(M-p+1)\left(b^{(j)}_{M-2p,p+1}\sin((M-2p)\varphi)\right) \\ & + 4(M+1)\left(a^{(j)}_{M,1}\cos(M\varphi) + b^{(j)}_{M,1}\sin(M\varphi)\right) \\ = \; & -q_j(O)\,\left(a^{(j)}_{M,0}\cos(M\varphi) + b^{(j)}_{M,0}\sin(M\varphi)\right), \end{aligned}$$

for all $\varphi \in [-\varphi_0, \varphi_0]$, which immediately yields the assertions in the last two lines of (2.18).

Therefore, (2.17) is valid by induction for any $M \geq 3$. In particular, all Taylor coefficients of u_j, $j = 1, 2$, are zero. This completes the proof of Lemma 2.3, from which Lemma 2.2 follows straightforwardly in the two-dimensional case. □

Remark 2.1. If $q_1 \equiv 1$, i.e., $(\Delta + k^2)u_1 = 0$ in B_R, then one can derive from the recurrence relation (2.5) that the lowest order homogeneous polynomial in the Taylor series for u_1 at the corner point is harmonic, which has already been proved in [23, Lemma 2.4]) in an alternative way. This fact together with the recurrence relation (2.5) gives rise to a simpler proof of Proposition 2.1 in two dimensions and the case of constant q_1. In the special case that q_1 and q_2 are both constants or the angle formed by Π_1 and Π_2 is $\pi/2$, the induction argument in proving Proposition 2.1 can be significantly simplified.

2.2 Proof of Proposition 2.1 in three dimensions

To extend the proof of Lemma 2.3 to the case of a wedge domain in $\mathbb{R}^3$, it is natural to employ the cylindrical coordinates (r, φ, z) of $x = (x_1, x_2, x_3) \in \mathbb{R}^3$, that is, $x_1 = r\cos\varphi$, $x_2 = r\sin\varphi$, $x_3 = z$ with $\varphi \in (-\pi, \pi]$, $r := \sqrt{x_1^2 + x_2^2}$, $z \in \mathbb{R}$. Without loss of generality, we assume the half-planes Π_j are given by

$$\Pi_1 = \{(r, \varphi, z) : \varphi = -\varphi_0\}, \quad \Pi_2 = \{(r, \varphi, z) : \varphi = \varphi_0\}$$

with some $\varphi_0 \in (0, \pi)\backslash\{\pi/2\}$. Then $\alpha := 2\varphi_0 \neq \pi$ denotes the dihedral angle formed by Π_1 and Π_2, both of which pass through the x_3-axis and contain the origin O. By the analyticity of the potentials q_j in B_R, we may expand solutions to the Helmholtz equations (2.1) into the convergent series (cf. (2.3) in the two dimensional case)

$$u_j(r, \varphi, z) = \sum_{n,m\in\mathbb{N}_0} r^{n+2m}\left(a^{(j)}_{n,m}(z)\cos(n\varphi) + b^{(j)}_{n,m}(z)\sin(n\varphi)\right) \tag{2.23}$$

uniformly in all $x \in C_{R_1} := \{(r, \varphi, z) : r < R_1, |z| < R_1\}$ for some $0 < R_1 < R$, with

$$a^{(j)}_{n,m}(z) = \sum_{\kappa\in\mathbb{N}} a^{(j)}_{n,m,\kappa}\, z^\kappa, \quad b^{(j)}_{n,m}(z) = \sum_{\kappa\in\mathbb{N}} b^{(j)}_{n,m,\kappa}\, z^\kappa, \quad a^{(j)}_{n,m,\kappa},\ b^{(j)}_{n,m,\kappa} \in \mathbb{C},\ j = 1, 2.$$

Unless otherwise stated, our analysis in this subsection is always performed inside the cylinder C_{R_1}. In (2.23), we set $b^{(j)}_{0,m,\kappa} = 0$ for all $m, \kappa \in \mathbb{N}_0$, $j = 1, 2$. Therefore, u_j can be represented as the convergent series

$$\begin{aligned} u_j(r, \varphi, z) &= \sum_{n,m,\kappa\in\mathbb{N}_0} r^{n+2m}\, z^\kappa \left(a^{(j)}_{n,m,\kappa}\cos(n\varphi) + b^{(j)}_{n,m,\kappa}\sin(n\varphi)\right) \\ &= \sum_{l,\kappa\in\mathbb{N}_0} r^l\, z^\kappa\, U^{(j)}_{l,\kappa}(\varphi), \end{aligned} \tag{2.24}$$

for $(r, \varphi, z) \in C_{R_1}$, where

$$U^{(j)}_{l,\kappa}(\varphi) := \sum_{n,m\in\mathbb{N}_0:n+2m=l} \left(a^{(j)}_{n,m,\kappa}\cos(n\varphi) + b^{(j)}_{n,m,\kappa}\sin(n\varphi)\right), \quad \varphi \in (-\pi, \pi]. \tag{2.25}$$

Moreover, the expansions (2.24) enable us to transform the Helmholtz equations (2.1) into the form (cf. (2.4))

$$
\begin{aligned}
&\sum_{n,m,\kappa\in\mathbb{N}_0} 4(m+1)(m+n+1)\ r^{n+2m}\ z^{\kappa}\ \Big(a^{(j)}_{n,m+1,\kappa}\cos(n\varphi)+b^{(j)}_{n,m+1,\kappa}\sin(n\varphi)\Big)\\
&+\sum_{n,m,\kappa\in\mathbb{N}_0} (\kappa+1)(\kappa+2)\ r^{n+2m}\ z^{\kappa}\ \Big(a^{(j)}_{n,m,\kappa+2}\cos(n\varphi)+b^{(j)}_{n,m,\kappa+2}\sin(n\varphi)\Big)\\
&=-q_j(r,\varphi,z)\sum_{n,m,\kappa\in\mathbb{N}_0} r^{n+2m}\ z^{\kappa}\ \big(a^{(j)}_{n,m,\kappa}\cos(n\varphi)+b^{(j)}_{n,m,\kappa}\sin(n\varphi)\big)\,. \qquad (2.26)
\end{aligned}
$$

As in 2D, we set $v:=u_1-u_2$, $a_{n,m,\kappa}:=a^{(1)}_{n,m,\kappa}-a^{(2)}_{n,m,\kappa}$, $b_{n,m,\kappa}:=b^{(1)}_{n,m,\kappa}-b^{(2)}_{n,m,\kappa}$ and $V_{l,\kappa}:=U^{(1)}_{l,\kappa}-U^{(2)}_{l,\kappa}$. Then we have an expansion of v analogous to u_j (see (2.24)) with the coefficients $a_{n,m,\kappa}$, $b_{n,m,\kappa}$ in place of $a^{(j)}_{n,m,\kappa}$, $b^{(j)}_{n,m,\kappa}$, respectively. From the transmission conditions (2.2) it follows that $V_{l,\kappa}(\varphi)=\partial V_{l,\kappa}/\partial\varphi=0$ on $\varphi=\pm\varphi_0$ for any fixed $l,\kappa\in\mathbb{N}_0$, leading to the relations

$$
\begin{cases}
0 &= \displaystyle\sum_{n,m\in\mathbb{N}_0:n+2m=l} a_{n,m,\kappa}\cos(n\varphi_0):=A^{+}_{l,\kappa},\\
0 &= \displaystyle\sum_{n\in\mathbb{N},m\in\mathbb{N}_0:n+2m=l} b_{n,m,\kappa}\sin(n\varphi_0):=B^{+}_{l,\kappa},\\
0 &= \displaystyle\sum_{n\in\mathbb{N},m\in\mathbb{N}_0:n+2m=l} a_{n,m,\kappa}n\cos(n\varphi_0):=A^{-}_{l,\kappa},\\
0 &= \displaystyle\sum_{n\in\mathbb{N},m\in\mathbb{N}_0:n+2m=l} b_{n,m,\kappa}n\sin(n\varphi_0):=B^{-}_{l,\kappa}.
\end{cases} \qquad (2.27)
$$

Our proof will be based on the identities (2.26) and (2.27), extending the arguments in Section 2.1 to the 3D case. The proof will proceed by induction on $l=n+2m$ and κ.

Lemma 2.4. *If $q_1(O)\neq q_2(O)$, then $a_{n,m,\kappa}=b_{n,m,\kappa}=a^{(j)}_{n,m,\kappa}=b^{(j)}_{n,m,\kappa}=0$ for all $n,m,\kappa\in\mathbb{N}_0$ and $j=1,2$.*

Proof. Using cylindrical coordinates, we may write the analytic functions q_j in the form

$$q_j(r,\varphi,z)=q_j(O)+\mathcal{O}(|r|+|z|),\quad j=1,2, \qquad (2.28)$$

which holds uniformly in all $(r,\varphi,z)\in C_{R_1}$.

Step 1. We begin by setting $l=0$ in (2.27), i.e., $n=m=0$. From $A^{+}_{0,\kappa}=0$ we get $a_{0,0,\kappa}=0$ for all $\kappa\in\mathbb{N}_0$, implying that $a^{(1)}_{0,0,\kappa}=a^{(2)}_{0,0,\kappa}=:\tilde a_{0,0,\kappa}$.

Take $l=1$, i.e., $n=1$, $m=0$. Again by using (2.27) we get $a_{1,0,\kappa}=b_{1,0,\kappa}=0$. Comparing the constants on both sides of (2.26) gives

$$4\,a^{(j)}_{0,1,0}+2\,a^{(j)}_{0,0,2}=-q_j(O)\ a^{(j)}_{0,0,0},\quad \text{i.e.,}\quad 4\,a^{(j)}_{0,1,0}+2\,\tilde a_{0,0,2}=-q_j(O)\,\tilde a_{0,0,0}.$$

Thus,

$$a_{0,1,0}=a^{(1)}_{0,1,0}-a^{(2)}_{0,1,0}=[q_2(O)-q_1(O)]\,\tilde a_{0,0,0}/4. \qquad (2.29)$$

Step 2. Set $n+2m=l=2$. This implies that $n=2$, $m=0$ or $n=0$, $m=1$. Arguing analogously to the second step in the proof of Lemma 2.3, one can derive from (2.27) with $l=2$ that

$$a_{2,0,\kappa}=b_{2,0,\kappa}=0, \quad a_{0,1,\kappa}=b_{0,1,\kappa}=0 \quad \text{for all} \quad \kappa\in\mathbb{N}_0. \tag{2.30}$$

In particular, the relation $a_{0,1,0}=0$ together with (2.29) and the fact that $q_1(O)\neq q_2(O)$ yields $\tilde{a}_{0,0,0}=0$, which in turn results in

$$a^{(j)}_{0,0,0}=0, \quad U^{(j)}_{0,0}\equiv 0, \quad a^{(j)}_{0,1,0}=-\tilde{a}_{0,0,2}/2=:\tilde{a}_{0,1,0} \quad \text{for} \quad j=1,2.$$

Equating the coefficients of z (i.e., $n+2m=l=0$, $\kappa=1$) on both sides of (2.26) and using $U^{(j)}_{0,0}\equiv 0$ yields

$$4\,a^{(j)}_{0,1,1}+6\,\tilde{a}_{0,0,3}=-q_j(O)\,\tilde{a}_{0,0,1},$$

leading to

$$a_{0,1,1}=a^{(1)}_{0,1,1}-a^{(2)}_{0,1,1}=[q_2(O)-q_1(O)]\,\tilde{a}_{0,0,1}/4.$$

In view of the second relation in (2.30) with $\kappa=1$ we get

$$a^{(j)}_{0,0,1}=0, \quad U^{(j)}_{0,1}\equiv 0, \quad a^{(j)}_{0,1,1}=-3/2\tilde{a}_{0,0,3},$$

In summary, we have proved for $K=1$ that

$$a^{(j)}_{0,0,\kappa}=0, \quad U^{(j)}_{0,\kappa}\equiv 0, \quad a^{(j)}_{0,1,\kappa}=-\frac{(\kappa+1)(\kappa+2)}{4}\,\tilde{a}_{0,0,\kappa+2}, \quad 0\leq\kappa\leq K. \tag{2.31}$$

Now assume that (2.31) holds with some $K\in\mathbb{N}$, $K>1$. Below we show the validity of (2.31) with $\kappa=K+1$. Equating the coefficients of z^{K+1} in (2.26) and taking into account $U^{(j)}_{0,\kappa}\equiv 0$ for all $\kappa\leq K$ yields

$$4\,a^{(j)}_{0,1,K+1}+(K+2)(K+3)\,\tilde{a}_{0,0,K+3}=-q_j(O)\,\tilde{a}_{0,0,K+1}. \tag{2.32}$$

In view of the last relation in (2.30) with $\kappa=K+1$, we infer from (2.32) that

$$0=a_{0,1,K+1}=a^{(1)}_{0,1,K+1}-a^{(2)}_{0,1,K+1}=[q_2(O)-q_1(O)]\,\tilde{a}_{0,0,K+1}/4,$$

from which the relation $\tilde{a}_{0,0,K+1}=a^{(j)}_{0,0,K+1}=0$ follows. Together with (2.32), this also leads to the last relation in (2.31) with $\kappa=K+1$. Since $b^{(j)}_{0,0,K+1}=0$, by the definition (2.25) we further have that $U^{(j)}_{0,K+1}\equiv 0$. Hence, the relations in (2.31) are valid by induction for any $\kappa\in\mathbb{N}_0$. Moreover, there holds $a^{(j)}_{0,1,\kappa}=0$ for all $\kappa\in\mathbb{N}_0$, $j=1,2$.

Step 3. This step continues the induction on $l=2$ and leads to the validity of the induction assumption for $n+2m=2$. We proceed by equating the coefficients of rz^κ in (2.26) (i.e., $n=1, m=0$). Simple calculations show that

$$\begin{aligned} & 8\left(a^{(j)}_{1,1,\kappa}\cos\varphi+b^{(j)}_{1,1,\kappa}\sin\varphi\right)+(\kappa+1)(\kappa+2)\left(a^{(j)}_{1,0,\kappa+2}\cos\varphi+b^{(j)}_{1,0,\kappa+2}\sin\varphi\right) \\ = \;& -q_j(O)\;\left(a^{(j)}_{1,0,\kappa}\cos\varphi+b^{(j)}_{1,0,\kappa}\sin\varphi\right), \end{aligned}$$

for all $\varphi \in [-\varphi_0, \varphi_0]$. In the last step we have used the fact that $U_{0,\kappa}^{(j)} \equiv 0$ for all $\kappa \in \mathbb{N}_0$ (see (2.31)). This implies that for $j = 1, 2$ and $\kappa \in \mathbb{N}_0$,

$$\begin{aligned} 8\, a_{1,1,\kappa}^{(j)} + (\kappa+1)(\kappa+2) a_{1,0,\kappa+2}^{(j)} &= -q_j(O)\, a_{1,0,\kappa}^{(j)}, \\ 8\, b_{1,1,\kappa}^{(j)} + (\kappa+1)(\kappa+2) b_{1,0,\kappa+2}^{(j)} &= -q_j(O)\, b_{1,0,\kappa}^{(j)}. \end{aligned} \tag{2.33}$$

In view of Step 1, we may set $a_{1,0,\kappa}^{(j)} = \tilde{a}_{1,0,\kappa}$, $b_{1,0,\kappa}^{(j)} = \tilde{b}_{1,0,\kappa}$ for $j = 1, 2$, and thus by (2.33)

$$a_{1,1,\kappa} = [q_2(O) - q_1(O)]\tilde{a}_{1,0,\kappa}/8, \quad b_{1,1,\kappa} = [q_2(O) - q_1(O)]\tilde{b}_{1,0,\kappa}/8, \quad \kappa \in \mathbb{N}_0. \tag{2.34}$$

To sum up Steps 1-3 we have obtained that

$$a_{1,0,\kappa} = a_{2,0,\kappa} = b_{1,0,\kappa} = b_{2,0,\kappa} = 0,\ a_{0,0,\kappa}^{(j)} = a_{0,1,\kappa}^{(j)} = b_{0,0,\kappa}^{(j)} = b_{0,1,\kappa}^{(j)} = 0. \tag{2.35}$$

Combining (2.34), (2.35) and the second relation in (2.31) yields that the induction hypothesis (cf. (2.17) in the 2D case)

$$\left\{ \begin{array}{l} a_{n,m,\kappa} = b_{n,m,\kappa} = 0, \quad n + 2m \le M; \\ a_{n,m,\kappa}^{(j)} = b_{n,m,\kappa}^{(j)} = 0, \quad n \le M-2, n + 2m \le M+1; \\ a_{n,1,\kappa} = \frac{q_2(O)-q_1(O)}{4(n+1)}\, a_{n,0,\kappa}^{(j)},\ b_{n,1,\kappa} = \frac{q_2(O)-q_1(O)}{4(n+1)}\, b_{n,0,\kappa}^{(j)},\ n \le M-1 \end{array} \right. \tag{2.36}$$

holds for $M = 2$, with $j = 1, 2$, $\kappa \in \mathbb{N}_0$.

Step 4. In this step we will prove (2.36) for all $M \in \mathbb{N}$, $M \ge 2$. Assume that (2.36) holds for some $M \in \mathbb{N}$. We need to justify (2.36) with M replaced by $M + 1$. For this purpose it is sufficient to verify additionally that

$$\left\{ \begin{array}{l} a_{M+1,0,\kappa} = a_{M-1,1,\kappa} = b_{M+1,0,\kappa} = b_{M-1,1,\kappa} = 0; \\ a_{M-1,0,\kappa}^{(j)} = a_{M-1,1,\kappa}^{(j)} = b_{M-1,0,\kappa}^{(j)} = b_{M-1,1,\kappa}^{(j)} = 0; \\ a_{M-2p,p+1,\kappa}^{(j)} = b_{M-2p,p+1,\kappa}^{(j)} = 0, \quad p \ge 1, M - 2p \ge 0; \\ a_{M,1,\kappa} = \frac{q_2(O)-q_1(O)}{4(M+1)}\, a_{M,0,\kappa}^{(j)},\ b_{M,1,\kappa} = \frac{q_2(O)-q_1(O)}{4(M+1)}\, b_{M,0,\kappa}^{(j)} \end{array} \right. \tag{2.37}$$

for all $j = 1, 2$, $\kappa \in \mathbb{N}_0$.

Setting $l = M + 1$ in (2.27) and using the relations in the second line of (2.36), one can get the linear systems

$$0 = E_{M+1}^{+}(\varphi_0) \begin{pmatrix} a_{M+1,0,\kappa} \\ a_{M-1,1,\kappa} \end{pmatrix} = E_{M+1}^{-}(\varphi_0) \begin{pmatrix} b_{M+1,0,\kappa} \\ b_{M-1,1,\kappa} \end{pmatrix} \quad \text{for all} \quad \kappa \in \mathbb{N}_0,$$

with the 2×2 matrices $E_{M+1}^{\pm}$ defined in the same way as in (2.19) and (2.20). Recalling from (2.21) and Lemma 2.1 that $\mathrm{Det}(E_{M+1}^{\pm}(\varphi_0)) \ne 0$, we obtain the vanishing of the coefficients $a_{M+1,0,\kappa}$, $a_{M-1,1,\kappa}$, $b_{M+1,0,\kappa}$ and $b_{M-1,1,\kappa}$. In particular, the equalities $a_{M-1,1,\kappa} = b_{M-1,1,\kappa} = 0$ in combination with the last line of (2.36) with $n = M - 1$ and the fact that $q_1(O) \ne q_2(O)$ yield

$$a_{M-1,0,\kappa}^{(j)} = b_{M-1,0,\kappa}^{(j)} = 0 \quad \text{for all} \quad \kappa \in \mathbb{N}_0, \quad j = 1, 2. \tag{2.38}$$

By definition, the relation $U^{(j)}_{M-1,\kappa} \equiv 0$ follows from (2.38) and the equalities in the second line of (2.36). Consequently, equating the coefficients of $r^{M-1}\, z^{\kappa}$ in (2.26) yields $a^{(j)}_{M-1,1,\kappa} = b^{(j)}_{M-1,1,\kappa} = 0$ for all $\kappa \in \mathbb{N}_0$, $j = 1, 2$. Hence, the first two lines of (2.37) have been verified.

We now consider the coefficients of $r^M z^{\kappa}$ on both sides of (2.26). Observing that $U^{(j)}_{n,\kappa} \equiv 0$ for all $n \leq M-1$ and applying the relations in the second line of (2.36), we obtain for $\varphi \in [\varphi_0, \varphi_0]$ that

$$
\begin{aligned}
& 4\,(M+1)\left(a^{(j)}_{M,1,\kappa}\cos(M\varphi) + b^{(j)}_{M,1,\kappa}\sin(M\varphi)\right) \\
& +(\kappa+1)(\kappa+2)\left(a^{(j)}_{M,0,\kappa+2}\cos(M\varphi) + b^{(j)}_{M,0,\kappa+2}\sin(M\varphi)\right) \\
& + \sum_{p\geq 1, M-2p\geq 0} 4(p+1)(M-p+1)\, a^{(j)}_{M-2p,p+1,\kappa}\cos((M-2p)\varphi) \\
& + \sum_{p\geq 1, M-2p\geq 0} 4(p+1)(M-p+1)\, b^{(j)}_{M-2p,p+1,\kappa}\sin((M-2p)\varphi) \\
= \;\; & -q_j(O)\,\left(a^{(j)}_{M,0,\kappa}\cos(M\varphi) + b^{(j)}_{M,0,\kappa}\sin(M\varphi)\right).
\end{aligned}
$$

By the arbitrariness of φ, we have

$$
\begin{aligned}
4\,(M+1)\, a^{(j)}_{M,1,\kappa} + (\kappa+1)(\kappa+2) a^{(j)}_{M,0,\kappa+2} = -q_j(O)\, a^{(j)}_{M,0,\kappa}, \\
4\,(M+1)\, b^{(j)}_{M,1,\kappa} + (\kappa+1)(\kappa+2) b^{(j)}_{M,0,\kappa+2} = -q_j(O)\, b^{(j)}_{M,0,\kappa},
\end{aligned}
\tag{2.39}
$$

and

$$
a^{(j)}_{M-2p,p+1,\kappa} = b^{(j)}_{M-2p,p+1,\kappa} = 0 \quad \text{for } p \geq 1, M-2p \geq 0, j = 1, 2.
$$

This proves the third line of (2.37). From the first line of (2.36), it follows that $a^{(1)}_{M,0,\kappa} = a^{(2)}_{M,0,\kappa}$, $b^{(1)}_{M,0,\kappa} = b^{(2)}_{M,0,\kappa}$ for all $\kappa \in \mathbb{N}_0$. Then the equalities in the last line of (2.37) immediately follow from (2.39), so that we have justified all relations in (2.37).

By induction, (2.36) holds for any $M \geq 2$. This leads to the vanishing of all Taylor coefficients of u_j, $j = 1, 2$, which proves Lemma 2.4. □

Since u_j are analytic functions, applying Lemma 2.4 yields $u_1 = u_2 \equiv 0$. Proposition 2.1 is thus proven in three dimensions.

3 Corners and edges always scatter

In this section, we shall prove that the scattered field generated by an inhomogeneous medium with piecewise analytic refractive indices having a corner or an edge on the boundary cannot vanish identically. Let (r, φ, z) and (r, φ) again denote the cylindrical and polar coordinates in $\mathbb{R}^3$ and $\mathbb{R}^2$, respectively. For $R > 0$ and $0 < \varphi_0 < \pi$, $\varphi_0 \neq \pi/2$, define the sector or wedge domains

$$
S_R := \{|\varphi| < \varphi_0\} \cap B_R, \quad S^e_R := B_R \backslash \overline{S}_R \quad \text{in} \quad \mathbb{R}^N,
$$

and set $\Gamma_R := \{|\varphi| = \varphi_0\} \cap B_R$. Recall that B_R denotes the open disk or ball with radius $R > 0$ centered at the origin O. Before proving Theorems 1.1 and 1.2, we first verify a result analogous to Proposition 2.1 for the transmission problem between the Helmholtz equations with analytic and piecewise analytic potentials.

Lemma 3.1. *Suppose that the potential q_2 is analytic on $\overline{S}_R$ and satisfies $q_2 \equiv 1$ in S_R^e, and that q_1 is analytic in B_R. Let $v_1, v_2 \in H^2(B_R)$ be solutions to*

$$\Delta v_1(x) + k^2 q_1(x) v_1(x) = 0, \qquad \Delta v_2(x) + k^2 q_2(x) v_2(x) = 0 \quad \textit{in} \quad B_R$$

subject to the transmission conditions

$$v_1 = v_2, \quad \partial_\nu v_1 = \partial_\nu v_2 \quad \textit{on} \quad \Gamma_R. \tag{3.1}$$

Then we have $v_1 = v_2 \equiv 0$ on B_R if $q(O) \neq 1$.

Proof. The analyticity of q_2 on $\overline{S}_R$ enables us to extend it to a real-analytic function $\tilde{q}_2$ defined in a neighborhood of O. Since v_1 is analytic in B_R, by (3.1) the Cauchy data of v_2 are also analytic on Γ_R. Applying the Cauchy-Kowalewski theorem, one can always find a solution $\tilde{v}_2$ to the elliptic Cauchy problem

$$\begin{cases} \Delta \tilde{v}_2(x) + k^2 \tilde{q}_2(x) \tilde{v}_2(x) = 0 & \text{in} \quad S_\epsilon^e, \\ \tilde{v}_2 = v_2, \quad \partial_\nu \tilde{v}_2 = \partial_\nu v_2 & \text{on} \quad \Gamma_\epsilon, \end{cases}$$

for some $0 < \epsilon < R$. Setting $w_2 := v_2$ in S_ϵ and $w_2 := \tilde{v}_2$ in S_ϵ^e, one can readily check that w_2 is an H^2-solution to $\Delta w_2(x) + k^2 \tilde{q}(x) w_2(x) = 0$ and thus analytic in B_ϵ. Now, applying Proposition 2.1 to $u_1 = v_1$, $u_2 = w_2$ in the ball B_ϵ yields $v_1 = w_2 \equiv 0$ in B_ϵ. This together with the unique continuation leads to $v_1 = v_2 \equiv 0$ in B_R. □

Theorem 1.1 and Corollary 1.1 can be proved straightforwardly by applying Lemma 3.1.

Proof of Theorem 1.1. Assume in Theorem 1.1 that the scattered field vanishes identically in $\mathbb{R}^N$. Applying Lemma 3.1 to a neighborhood of the underlying corner or edge point O with $v_1 := u^{in}$, $q_1 \equiv 1$ and $v_2 = u$ yields $u^{in} \equiv 0$ near O. By the unique continuation we get $u^{in} \equiv 0$ in a neighborhood of D, which is a contradiction. □

Proof of Corollary 1.1. To prove Corollary 1.1, we assume the function w can be extended to an analytic function in neigh(O), an open neighborhood of the corner or edge point $O \in \partial D$, implying that the Cauchy data of u are analytic on neigh$(O) \cap \partial D$. Then, applying the Cauchy-Kowalewski theorem as in the proof of Lemma 3.1, the function u can also be extended analytically into neigh(O). Applying Lemma 3.1 to $v_1 := w$, $q_1 \equiv 1$ and $v_2 = u$ in neigh(O) and recalling the unique continuation gives $u = w \equiv 0$ in D, which is impossible. Corollary 1.1 is thus proven. □

4 Uniqueness in inverse transmission problems and other applications

As a by-product of Lemma 3.1, we verify global uniqueness with a single incident wave in recovering the shape of a penetrable medium within convex hulls of polygons or polyhedrons, i.e., Theorem 1.2 (i).

The second assertion of Theorem 1.2 follows from the discreteness of a *generalized interior transmission eigenvalue problem* studied in [23] which requires only the behavior of the potentials in a neighborhood of the boundary.

Proof of Theorem 1.2. (i) Let u_j and u_j^{sc} ($j = 1, 2$) denote the total and scattered fields corresponding to D_j. Since $u_1^\infty(\hat{x}; k_0) = u_2^\infty(\hat{x}; k_0)$ for all $\hat{x} \in \mathbb{S}^{N-1}$, applying Rellich's lemma we know

$$u_1(x) = u_2(x) \tag{4.1}$$

for all $x \in \mathbb{R}^N \backslash \{\overline{D}_1 \cup \overline{D}_2\}$ lying in a small neighborhood of $\partial(D_1 \cup D_2)$. If $\partial D_1 \neq \partial D_2$, without loss of generality we may assume there exists a corner $O \in \mathbb{R}^N$ of ∂D_2 such that $O \notin \overline{D}_1$. Notice that this step cannot be achieved if D_1 and D_2 are not convex. Since this corner stays away from D_1, the function u_1 satisfies the Helmholtz equation with the wave number k^2 around O, while u_2 fulfills the Helmholtz equation with the variable potential $k^2 q_2$. The transmission conditions between u_1 and u_2 on ∂D_2 in a neighborhood of O follow from those of u_2 across ∂D_2 and the relation (4.1). Now, applying Lemma 3.1 to an open neighborhood of O in $D_2 \backslash \overline{D}_1$ with $v_1 = u_1$, $q_1 \equiv 1$ and $v_2 = u_2$, we obtain the vanishing of u_1 near O and thus $u_1 \equiv 0$ (or equivalently, $u_1^{sc} = -u^{in}$) in a neighborhood of D_1 by the unique continuation. This implies that the radiating solution u_1^{sc} is also an entire function in $\mathbb{R}^N$. Hence, $u_1^{sc} \equiv 0$ in $\mathbb{R}^N$ (see e.g., [6, Chapter 2.2]) and $u^{in} = u_1 - u_1^{sc} \equiv 0$ in a neighborhood of D_1, contradicting our assumption on the incident wave. Thus $\partial D_1 = \partial D_2$.

(ii) Assume on the contrary that the relation (1.6) holds. Again using Rellich's lemma, we see that $u_1 = u_2$ at least in a neighborhood of ∂D in $\mathbb{R}^N \backslash \overline{D}$. From the transmission conditions for u_j across ∂D, it follows that $(u_1|_D, u_2|_D) \in H^2(D)^2$ is a solution to the following *generalized interior transmission eigenvalue problem* (GITEP):

$$\begin{cases} \Delta u_1 + k^2 q_1 u_1 = 0 & \text{in} \quad D, \\ \Delta u_2 + k^2 q_2 u_2 = 0 & \text{in} \quad D, \\ u_1 = u_2, \quad \partial_\nu u_1 = \partial_\nu u_2 & \text{on} \quad \partial D, \end{cases}$$

for all $k \in [k_{\min}, k_{max}]$. Note that in the classical *transmission eigenvalue problem*, one of the potentials, q_1 or q_2, is identical to one. Since $q_j \in C^\infty(\overline{D})$ and $|q_1 - q_2| > 0$ on ∂D, there exists a positive number δ such that $|q_1 - q_2| > \delta$ in a neighborhood of ∂D in D. By [23], the above GITEP has at most a countable number of transmission eigenvalues in $\mathbb{R}$. Hence, there exists at least one wave number $k_0 \in [k_{\min}, k_{max}]$ for which $u_1(x; k_0) = u_2(x; k_0) \equiv 0$ in D. Arguing as in the proof of the first assertion, the scattered field can be analytically extended into the whole space, leading to the vanishing of u^{in} in a neighborhood of D. This contradiction implies that the relation (1.6) cannot hold for all frequencies in a finite interval. $\square$

Remark 4.1. If the condition $|q_1 - q_2| > 0$ on ∂D can be removed in Theorem 1.2 (ii), then one can get global uniqueness in determining a piecewise analytic potential q from the far-field patterns at multi-frequencies, provided the support of $1 - q$ is a convex polygon or polyhedron. However, when $q_1 - 1$ (where we assume $q_2 \equiv 1$) changes sign in a neighborhood of the boundary, the resulting variational formulation in the natural H^1 framework corresponding to the classical ITEP may lose the Fredholm property (see [2]). Thus one cannot expect to apply the analytic Fredholm theorem in order to prove discreteness of transmission eigenvalues. A new functional framework was proposed in [2] for recovering the Fredholmness, which however does not apply to general configurations. To the best of our knowledge, it still remains open how to prove uniqueness of a penetrable obstacle from the scattering data at one incident direction and many frequencies; see [15] for a discussion.

As another application of Lemma 3.1, we consider an inverse problem arising from the metal-to-semiconductor contact in electric devices. Let $D \subset \Omega$ be a bounded convex polygon or polyhedron, and let χ_D be the characteristic function of the subdomain D. Suppose that Ω is a bounded Lipschitz domain. Consider the boundary value problem

$$-\Delta u + q(x)\chi_D u = 0 \quad \text{in} \quad D, \qquad u = f \quad \text{on} \quad \partial D, \tag{4.2}$$

where $f \in H^{1/2}(\partial\Omega)$. Following the proof of Theorem 1.2 (i), we can show a uniqueness result in recovering ∂D by using a single boundary measurement taken on $\partial\Omega$.

Theorem 4.1. *Suppose that q is real-analytic on $\overline{D}$ and $q \neq 0$ for all $x \in \partial D$, and let $u \in H^1(D)$ be a solution to (4.2). Then the Cauchy data $(f, \partial_\nu u|_{\partial\Omega})$ uniquely determine the shape of D.*

The above theorem has removed the positivity assumption in [19] on the input f and on the potential q, and it is valid in both two and three dimensions. However, we require q to be real-analytic, whereas a less restrictive C^2-smoothness assumption was sufficient in [19].

We think that all results of this paper carry over to the case of piecewise smooth potentials. In fact, Lemma 3.1 may be extended to the case of a potential q_2 that is infinitely smooth on $\overline{S}_R$, provided the C^∞-smoothness near corners and edges for solutions to the inhomogeneous Helmholtz equation can be proved for vanishing Cauchy data and smooth right hand sides. Future efforts will also be made to extend the results to corner and wedge domains with arbitrary angles in higher dimensions, including circular conic corners in $\mathbb{R}^N$ ($N \geq 3$).

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