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On the solution of an optimal recovery problem and its applications in nonparametric statistics

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Abstract

We consider a non-stochastic optimization problem which arises in various non-parametric statistical settings with Hölder function classes. We prove that the solution of this problem is finite. The application of the finiteness is illustrated via regression estimation with minimax Bahadur risk. [1]

1. INTRODUCTION

The problem of optimal estimation of a function f from noisy data is widely discussed in nonparametric statistics. The optimality is considered for various minimax criteria while f belongs to some function class Σ, see Ibragimov and Khasminskii (1981). A Hölder function class $\Sigma(\beta, L)$ provides a popular example

$$\Sigma(\beta, L) = \{f : \ |f^{(m)}(t) - f^{(m)}(t_1)| \leq L|t - t_1|^{\alpha}; \ t, t_1 \in R^1\}$$

where $m = \lfloor \beta \rfloor$ is an integer such that $0 < \alpha \leq 1$, $\alpha = \beta - m$; β, L are given positive constants. In a number of minimax set-ups with the class $\Sigma(\beta, L)$ the estimators which are asymptotically optimal up to a constant turn out to be closely related to the solution of a certain optimization problem.

Korostelev (1993) found an asymptotically minimax exact constant $C_0 = C_0(\beta, L)$ for estimating of a function f, $f \in \Sigma(\beta, L)$, $0 < \beta \leq 1$, with supremum norm loss from observations

$$y_{in} = f(i/n) + \xi_{in}, \quad i = 1, 2, \ldots, n; \quad n = 1, 2, \ldots \tag{1}$$

where ξ_{in} are i.i.d. Gaussian random variables, $\xi_{in} \sim N(0, \sigma^2)$: it was proved that

$$\lim_{n \to \infty} R_n(\Sigma(\beta, L)) = w(C_0) = \ w[L^{1/(2\beta+1)}((\beta + 1)\beta^{-2})^{\beta/(2\beta+1)}]$$

where

$$R_n(\Sigma) = \inf_{f_n} \sup_{f \in \Sigma} E_f^{(n)} w(\frac{\|f_n - f\|_\infty}{\psi_n}) \ , \tag{2}$$

$$\psi_n = (\ln n/n)^{\frac{\beta}{2\beta+1}}, \ \|f\|_\infty = \sup_{0 \leq t \leq 1} |f(t)| \ ,$$

w is a loss function satisfying standard regularity conditions, $E_f^{(n)}$ is the expectation w.r.t.

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the true regression, f_n is an arbitrary estimator obtained from observations (1). The lower bound of the minimax risk (2) was found to be attained by a kernel estimator,

$$\hat{f}_n(t) = \frac{1}{n}\sum_i y_{in} K_h(t - i/n), \quad K_h(\cdot) = \frac{1}{h}K(\frac{\cdot}{h}) \tag{3}$$

with some bandwidth $h = h(n)$ and kernel K.

Donoho (1994) extended this result to the case $\beta > 1$ and to signal estimation in the white noise model, and showed the relations of stochastic problems to the optimal recovery theory (see Micchelli and Rivlin (1977)): asymptotically efficient estimators for the model (1) were proved to be of the form (3) where the optimal kernel K, as in Korostelev's result, is expressed via a solution $g^*(t) = g^*_\beta(t)$ of the following optimization problem:

$$g(0) \to \sup, \quad g \in \Sigma(\beta, 1)\ , \quad \|g\|_2 \le 1\ , \tag{4}$$

where $\|g\|_2 = (\int_{-\infty}^{\infty} g^2(t)dt\ )^{1/2}$.

We list a number of nonparametric statistical problems, different from regression estimation with sup-norm loss, where optimal estimators exploit g^*.

(a) Estimation of a probability density $f(t), t \in [0,1]$, in the uniform norm from i.i.d. observations $X_1, X_2, \ldots, X_n \sim f(t)$, with the minimax risk analogous to (2),

$$g \in \Sigma(\beta, L; b) = \{f:\ f \in \Sigma_{[0,1]}(\beta, L); \int_0^1 f(t)dt = \ 1; \ \ \min_{0\le t\le 1} f(t) \ge b > 0\}$$

$$\Sigma_{[a_1,a_2]}(\beta, L) = \{f:\ |f^{(m)}(t) - f^{(m)}(t_1)| \le L|t - t_1|^\alpha;\ a_1 \le t, t_1 \le a_2\}\ ,$$

see Korostelev, Nussbaum (1995).

(b) Adaptive estimation in the 'white noise' model:

$$dX(t) = f(t)dt + \varepsilon dW(t), t \in [0,1]\ ,$$

where $W(t)$ is a standard Wiener process, ε is a small parameter (ε corresponds to $1/\sqrt{n}$ in (1)), f belongs to $\Sigma_{[0,1]}(\beta, L)$ with unknown β, L - see Lepski (1992), Lepski and Spokoiny (1995).

(c) Estimation of a regression function at a fixed point, say $t = 0$, from observations (1), $i = 0, \pm 1, \pm 2, \ldots$, with the minimax Bahadur risk (Bahadur(1960), Ibragimov and Khasminskii (1981))

$$r_n(c) = \inf_{f_n} \sup_{f \in \Sigma(\beta, L; B)} \frac{1}{n} \log P_f^{(n)}(|f_n - f(0)| \ge c), \quad c > 0\ ,$$

where

$$\Sigma(\beta, L; B) = \{f : \, f \in \Sigma(\beta, L); \; |f(t)| \leq B, \; B > 0\} \; .$$

Korostelev (1995) proved for the Gaussian errors $\xi_{in} \sim N(0, \sigma^2)$ that for any $c > 0$

$$\lim_{n\to\infty} r_n(c) \;\; = \;\; -\sigma^{-2} c^{2+1/\beta} \; A(\beta, L) \tag{5}$$

where

$$A(\beta, L) = \; \frac{1}{2L^{1/\beta}} [g^*(0)]^{-(2+1/\beta)} \; . \tag{6}$$

Korostelev and Leonov (1995 a,b) considered the case of i.i.d. errors having a probability density $p(x)$ with finite Fisher information I_0,

$$I_0 = E[(l'(x))^2] \; < \; \infty \; , \;\; l(x) = \log p(x) \; ,$$

and showed that under general regularity assumptions on $p(x)$

$$\lim_{c\to 0} c^{-(2+1/\beta)} \lim_{n\to\infty} r_n(c) \;\; = \;\; -I_0 \; A(\beta, L) \; . \tag{7}$$

Moreover, the lower bound of the minimax Bahadur risk in (5), as well as in (7), is attained by the estimators of the form (3) where the optimal kernel K explicitly depends on g^*.

Thus, the solution g^* of (4) is of primary importance for various nonparametric problems. It is easier to study a solution $x_*(t) = x_*^\beta(t)$ of the dual optimization problem

$$\|x\|_2 \to \inf \; , \;\; x \in \Sigma(\beta, 1) \; , \;\; x(0) = 1 \; . \tag{8}$$

Indeed, using the same arguments as in Lemma 2.1 from Donoho (1994), one has:

$$x_*(t) = a g^*(bt), \;\;\; a = [g^*(0)]^{-1}, \;\;\; b = [g^*(0)]^{1/\beta}, \tag{9}$$

$$g^*(t) = a_1 x_*(b_1 t), \;\;\; a_1 = [\|x_*\|_2]^{\frac{-2\beta}{2\beta+1}}, \;\;\; b_1 = [\|x_*\|_2]^{\frac{2}{2\beta+1}} \; . \tag{10}$$

More precisely, if a solution of (4) exists, then there exists a solution of (8), these solutions being connected by (9), and vice versa.

For $\beta \leq 1$ the solution of (8) may be obtained easily - see Korostelev (1993), Donoho (1994):

$$x_*(t) = \begin{cases} 1 - |t|^\beta), & \text{if } |t| \leq 1, \\ 0, & \text{otherwise.} \end{cases}$$

The case $\beta > 1$ is much more complicated. As far as we know, neither an explicit form of the solution, nor its qualitative properties have been mentioned in the papers on nonparametric statistics. In particular, for many reasons the asymptotic behaviour of the function $x_*(t)$, as $t \to \infty$, turns out to be crucial.

A conjecture was made in (Donoho (1994)) that for $\beta = 2$ the function $x_*(t)$ does not have a compact support. However, this hypothesis is not true. Moreover, the explicit solution of this problem was obtained by Fuller (1960). The main goal of the present paper is to prove the compactness of the support of x_* for any $\beta > 1$.

The paper is organized as follows. In section 2 we refer to the case of integer β and show its relations to the optimal control problems, special attention being given to Fuller's results. The main theorems are proved in Section 3. Section 4 contains some applications to regression estimation with the minimax Bahadur risk.

2. REMARKS ON FULLER'S SOLUTION AND OPTIMAL CONTROL PROBLEMS

To start with, we reduce the problem (8) to the optimization problem on the half-line.

LEMMA 1. *If the solution of the problem (8) exists, then one is unique and symmetric.*

Proof. Let's assume that there exist two different solutions x_1, x_2 such that

$$\|x_1 - x_2\|_2 > 0 \ . \tag{11}$$

Introduce a function $\tilde{x}$, $\tilde{x}(t) = (x_1(t) + x_2(t))/2$. Since $\|x_1\|_2 = \|x_2\|_2$, we get

$$\|\tilde{x}\|_2^2 \ = \ 1/4\|x_1 + x_2\|_2^2 \ = 1/2[\ \|x_1\|_2^2 \ + (x_1, x_2)\ ] \ = [\|x_1\|_2^2 - \ 1/4\|x_1 - x_2\|_2^2\ ] \ < \|x_1\|_2^2$$

due to (11) which contradicts the optimality of x_1.

Assuming now that $x_*(t) \neq x_*(-t)$ for some $t > 0$, introducing a function $\hat{x}(t) = (x_*(t) + x_*(-t))/2$, and using the same arguments, we establish that $\|\hat{x}\|_2^2 \ < \ \|x_*\|_2^2$.□

Introduce the following notation: $n = m + 1$,

$$\tilde{m} = \begin{cases} m, & \text{for odd } m, \\ m-1, & \text{for even } m \end{cases} .$$

Let $\|\cdot\|$ (without any subscript) be a norm in the Hilbert space $L_2[0, \infty]$:

$$\|x\| \ = \ (\int_0^\infty x^2(t)dt)^{1/2} \ ,$$

and let $\|Y\|_E$, $\|A\|_E$ be Euclidean norms of a $(n \times 1)-$vector Y and a $(n \times n)-$matrix A respectively.

Lemma 1 implies that

$$x_*^{(i)}(0) = 0, \quad i = 1, 3, \ldots, \tilde{m}.$$

Thus, the problem (8) may be reduced to the following one:

$$I(x, Y) \; \rightarrow \; \inf_{Y \in \mathcal{Y}} \; \inf_{x \in \mathcal{F}_\beta(Y)} \tag{12}$$

where

$$\begin{aligned} I(x, Y) &= \|x\|^2, \quad x \in \mathcal{F}_\beta(Y) \; , \\ \mathcal{F}_\beta(Y) &= \{x : \; x \in \Sigma_{[0,\infty]}(\beta, 1), \;\; X(0) = Y\}, \\ X(t) &= (x(t), \dot{x}(t), \ldots, x^{(n-1)}(t))', \end{aligned}$$

$$\mathcal{Y} = \{Y = (y_0, y_1, \ldots, y_{n-1})' : \; y_0 = 1; \;\; y_k = 0, \; k = 1, 3, \ldots, \tilde{m}\}.$$

So, first we need to study the properties of the solution of the optimization problem

$$I(x, Y) \; \rightarrow \; \inf_{x \in \mathcal{F}_\beta(Y)} \tag{13}$$

for an arbitrary initial condition $Y \in R^n$.

If β is integer, $\beta = n$, then (13) is equivalent to a problem which is typical in the optimal control theory and which we refer to as problem $F(n)$,

$$F(n) : \quad J(x) = \|x\|^2 \rightarrow \inf, \; x^{(n)}(t) = u(t), \; |u(t)| \leq 1, \; X(0) = Y,$$

where u is a measurable function.

For $\beta = 2$ the set $\mathcal{Y}$ consists of a single point, $\mathcal{Y} = \{(1,0)'\}$, and the general problem (12) may be reformulated as

$$F(2) : \quad J(x) = \|x\|^2 \rightarrow \inf \; , \; |x^{(2)}(t)| \leq 1 \; a.e., \; x(0) = 1, \; \dot{x}(0) = 0 \; ,$$

$\dot{x}(t)$ is an absolutely continuous function. As already mentioned, the solution of F(2) was obtained by Fuller (1960), the idea of his solution is as follows.

First, it is obvious that

$$x_*^{(2)}(t) = u_*(t) = -1, \; x_*(t) = 1 - t^2/2 \; , \; 0 \leq t \leq t_1 \; ,$$

with some $t_1 > 0$. Next, if $u_*(t) = +1$ for $t_1 \le t \le t_3$, and if t_2 provides the first local minimum of the function $x_*(t)$, then

$$x_*(t_2) = a_1 \ , \ \dot{x}_*(t_2) = 0 \ ,$$

and we have the same set-up as in the beginning but with different initial conditions. If $a_i, i \ge 0$, denote values of the local optima of the function $x_*(t)$, it turns out that

$$a_{i+1}/a_i \quad = \quad -q \ , \quad 0 \le q < 1. \tag{14}$$

By direct calculations one obtains

$$\int_0^{t_2} x_*^2(t)dt \quad = \quad \frac{|a_0|^{5/2}}{30}(23q^2 - 14q + 23)\sqrt{1+q} \ , \ a_0 = 1 \ , \tag{15}$$

therefore, (14) and (15) yield the value of the functional $J(x_*)$:

$$J(x_*) = J_q(x_*) = \frac{(23q^2 - 14q + 23)\sqrt{1+q}}{30(1 - q^{5/2})} \ .$$

The minimizer to $J_q(x_*)$ equals to

$$Q = q_{opt} = \frac{1}{16} \ (3 + \sqrt{33} - \sqrt{26 + 6\sqrt{33}}) \ ^2 \ \approx \ 0.058 \ .$$

Now we present the explicit formulas for $x_*(t)$:
let $t_{-1} = t_0 = 0, t_1 = \sqrt{1+Q}$, let for $j \ge 1$

$$t_{2j} = 2\sqrt{1+Q} \sum_{i=0}^{j-1} Q^{i/2} \ , \ t_{2j+1} = t_{2j} + \sqrt{1+Q} \ Q^{j/2} \ , \ a_j = (-1)^j Q^j \ .$$

Then

$$x_*(t) = \begin{cases} a_j + (-1)^{j+1}(t - t_{2j})^2/2 \ , & \text{if } t \in (t_{2j-1}, t_{2j+1}), \ j \ge 0, \\ 0, & \text{if } t > T^{**}, \end{cases}$$

where

$$T^{**} = \lim_{j \to \infty} t_{2j} = \ 2\sqrt{1+Q}/(1 - \sqrt{Q}) \approx 2.71 \ .$$

The optimal control u_* is defined by

$$u_*(t) = x_*^{(2)}(t) = \begin{cases} -1 \ , & \text{if } t \in (t_{2j-1}, t_{2j+1}), \ j = 2i, \\ +1 \ , & \text{if } t \in (t_{2j-1}, t_{2j+1}), \ j = 2i+1 \ , \ i \ge 0. \end{cases}$$

It is worthy to note that a time-optimal trajectory (see Pontryagin et al (1962)), i.e. the solution $\hat{x}$ to the problem

$$T \to \min,\ \ |x^{(2)}(t)| \le 1,\ \ X(0) = (1,0)',\ X(T) = (0,0)'\ ,\ \dot{x}(t) \text{ is abs. continuous,}$$

is related in above formulas to $q = 0$:

$$\hat{x}(t) = \begin{cases} 1 - t^2/2\ , & \text{if } 0 \le t \le 1, \\ (t-2)^2/2, & \text{if } 1 \le t \le 2, \\ 0, & \text{if } t \ge 2. \end{cases}$$

The value of the functional J on this trajectory equals to $J(\hat{x}) = 23/30$, thus

$$J(x_*)/J(\hat{x}) \approx 0.9965.$$

The problem F(2) played an important role in the optimal control theory since it provided an example of a linear system with the optimal control having an infinite number of switchings on a finite time interval. This phenomenon is often called a 'Fuller's phenomenon' while corresponding optimal control is called 'chattering', see Marchal (1973). A large bibliography on this topic as well as a number of new results are presented in Zelikin, Borisov (1991). In particular, a problem that we call $C(n)$ is studied in the cited above paper:

$$C(n): \qquad \int_0^\infty r(X(t))dt \ \to \ \inf \quad ,$$

$$\dot{X}(t) \ = \ AX(t) \ + \ bu(t), \ \ X(0) = Y \in R^n\ , \tag{16}$$

where r is a continuously differentiable function,

$$r:\ R^n \ \to \ R^1,\ r(0) = 0,\ r(X) \ge 0\ ;$$

$X(t) = (x_1(t), x_2(t), \dots, x_n(t))'$ is an absolutely continuous vector function,
$u(t)$ is a measurable function, $|u(t)| \le 1$;
A, b are constant $(n \times n)$-matrix and $(n \times 1)$-vector respectively.

Introduce the following assumptions.

A1. Vectors $b, Ab, \dots, A^{n-1}b$ are linearly independent.

A2. The function $r(X)$ is strictly convex on variables $x_1, \dots, x_k$; $r(X)$ does not depend on $x_{k+1}, \dots, x_n$, $k \le n$.

A3. Let $G = \{x \in R^n : x_1 = \dots = x_k = 0\ ;\ \ x_j \ne 0$ for some $j > k$ $\}$, then

$$\min_{|u|\le 1}\ \max_{1\le j\le k} |(Ax + bu)_j| \ > \ 0 \ \ if \ \ x \in G\ ,$$

The following result is proved in Zelikin, Borisov (1991).

Theorem ZB. *Under Assumptions A1-A3 for an arbitrary fixed integer* n *and for an arbitrary fixed* $Y \in R^n$ *there exists a unique solution* $\tilde{X}(t) = \tilde{X}_{n,Y}(t)$ *of the problem* $C(n)$*, this solution having a compact support: for some* T, $T = T(n, Y) < \infty$,

$$\|X(t)\|_E = 0 \ , t \geq T.$$

It is obvious that the problem $F(n)$ is the special case of the problem $C(n)$ with

$$x_1 = x, \ x_2 = \dot{x}, \ldots, x_n = \dot{x}_{n-1} \ ; \quad r(X) = x_1^2 \ ,$$

$$A = \begin{pmatrix} 0 & 1 & 0 & 0 & \ldots & 0 & 0 \\ 0 & 0 & 1 & 0 & \ldots & 0 & 0 \\ 0 & 0 & 0 & 1 & \ldots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & \ldots & 1 & 0 \\ 0 & 0 & 0 & 0 & \ldots & 0 & 1 \\ 0 & 0 & 0 & 0 & \ldots & 0 & 0 \end{pmatrix} , \qquad b = \begin{pmatrix} 0 \\ 0 \\ 0 \\ \vdots \\ 0 \\ 0 \\ 1 \end{pmatrix} \tag{17}$$

It may be easily verified that Assumptions A1, A2 are valid for $F(n)$, and since $k = 1$, $(Ax + bu)_1 = x_2$, Assumption A3 may be rewritten in the form

A3'. $|x_2| > 0$ for $x \in G$.

But $\tilde{X} = (0, 0, 1, 0, \ldots, 0)' \in G$, therefore, Assumption A3 is violated and Theorem ZB cannot be applied directly to $F(n)$, $n \geq 3$.

Nevertheless, the approach used in Zelikin, Borisov (1991) may be applied for the analysis of the problem $F(n)$. Moreover, in the next section we prove the result analogous to Theorem ZB for the solution of (13) with arbitrary β. The final step is to establish the finiteness of the solution of the general problem (12). It should be emphasized that Lemma 6 from the next section which plays a key role in the proof of the compactness follows the ideas of Lemma 16 from Zelikin, Borisov (1991).

3. MAIN RESULTS

First, we sketch the basic steps of our proof. Let β be an arbitrary fixed number, $\beta > 1$.

(A) We show that for any $Y \in R^n$ there exists an 'admissible' function from the set

$\mathcal{F}_\beta(Y)$, i.e. a function with finite $L_2[0,\infty]$-norm (Lemma 2).

(B) Since the functional $I(x,Y)$ is non-negative, it follows from step (A) that there exists a value $I^*(Y)$,

$$I^*(Y) = I^*_\beta(Y) = \inf_{x \in \mathcal{F}_\beta(Y)} I(x,Y) \ , \quad 0 \le I^*(Y) < \infty \ , \tag{18}$$

and we prove that the infimum is attained by some $z(Y) = z_\beta(Y) \in \mathcal{F}_\beta(Y)$,

$$I^*(Y) = I(z(Y), Y)$$

(Theorem 1).

(C) We establish the finiteness of the function $z(Y)$ for any Y (Lemmas 3-6, Theorem 2).

(D) We prove the finiteness of the solution of the general problem (12) (Theorem 3).

Let $\mathcal{F}'_\beta(Y) = \{x : \ x \in \mathcal{F}_\beta(Y) \ , \ \|x\| < \infty\}$.

Lemma 2. *The set* $\mathcal{F}'_\beta(Y)$ *is non-empty for an arbitrary fixed* $Y \in R^n$.

Proof. As above, use the notation

$$\beta = m + \alpha \ , \ 0 < \alpha \le 1 \ , \ n = m + 1 \ .$$

Let T be a fixed positive number, let $X(t)$ be a solution of the system (16) where A, b are defined by (17), $X = (x, \dot{x}, \ldots, x^{(m)})'$, and

$$u(t) = -b' e^{-A't} D^{-1} Y \ , \tag{19}$$

here A', b' denote transposed matrices,

$$D = D(T) = \int_0^T \ e^{-As} \ b \ b' \ e^{-A's} ds \ .$$

It is well known, see Krasovsky (1968), that the matrix D is non-singular for any $T > 0$, $X(T) = 0$, and for $T \ge 1$ the following estimate is valid:

$$\max_{0 \le t \le T} \|e^{-A't} \ D^{-1}\|_E \ \le \ d/T \ , \tag{20}$$

$d = d(n)$ is a constant depending on n and not depending on T. (19) and (20) entail for arbitrary $t, t_1 \in [0,T]$:

$$|x^{(m)}(t) - x^{(m)}(t_1)| \le \max_{0 \le t \le T} |u(t)| \ |t - t_1|.$$

$$\leq d\|Y\|_E \, T^{-1} \, |t-t_1|^\alpha \, |t-t_1|^{1-\alpha} \leq d\|Y\|_E \, T^{-\alpha} \, |t-t_1|^\alpha \tag{21}$$

Therefore, if we define $x_1(t)$ by

$$x_1(t) = \begin{cases} x(t) \, , & 0 \leq t \leq T, \\ 0 \, , & t > T; \end{cases} \qquad T = \begin{cases} 1 \, , & d\|Y\|_E \leq 1, \\ (d\|Y\|_E)^{1/\alpha} \, , & d\|Y\|_E > 1 \, , \end{cases}$$

then (21) implies that $x_1 \in \mathcal{F}'_\beta(Y)$.□

THEOREM 1. *For an arbitrary fixed $Y \in R^n$ there exists a unique solution of the problem (13).*

Proof. Let

$$\hat{F} = \mathcal{F}_\beta(Y) \bigcap \{x: \ \|x\|^2 \leq 2I^*(Y)\} \, ,$$

where $I^*(Y)$ is defined by (18). Lemma 2 entails that the set $\hat{F}$ is non-empty. Note that $\hat{F}$ is a convex set while $\|x\|^2$ is a continuous strongly convex functional in the Hilbert space $L_2[0,\infty]$. Thus, if we prove that $\hat{F}$ is closed with respect to the strong convergence in $L_2[0,\infty]$, then the statement of the theorem is a mere consequence of fundamental results in the optimization theory (see, for example, Vasiljev (1981), Theorem 1.3.8).

So, let $x_n \in \hat{F}$ and for some x

$$\|x_n - x\| \to 0 \quad as \quad n \to \infty \, . \tag{22}$$

Since the space $L_2[0,\infty]$ is complete, $\|x\| < \infty$. Next, strong convergence implies weak convergence as well, thus, from the identity

$$\|x_n - x\|^2 = \|x_n\|^2 \ + \ \|x\|^2 \ - \ 2(x_n, x) \, ,$$

applying (22), we immediately get:

$$\lim_{n\to\infty} \|x_n\| = \|x\|, \quad \|x\| \leq 2I^*(Y) \, . \tag{23}$$

To prove that $x \in \mathcal{F}_\beta(Y)$, first remark that the definition of the set $\mathcal{F}_\beta(Y)$ entails

$$|x^{(k)}(t) - x^{(k)}(t_1) \ - \ \sum_{i=1}^{m-k} \frac{(t-t_1)^i}{i!} x^{(k+i)}(t_1)| \ \leq \ \frac{|t-t_1|^{\beta-k}}{(\beta-k)!} \tag{24}$$

$$k = 0, 1, \ldots, m-1 \; ; \quad (\beta - k)! = (\beta-k)(\beta-k-1)\ldots(\beta-m+1).$$

Now let T be an arbitrary fixed positive constant and let

$$\mathcal{F}_\beta^{[0,T]}(Y) = \{x : \; x \in \Sigma_{[0,T]}(\beta, 1), \; X(0) = Y \}.$$

(24) yields that for an arbitrary $x \in \mathcal{F}_\beta^{[0,T]}(Y)$

$$\sup_{0 \le k \le m} \; \sup_{0 \le t \le T} |x^{(k)}(t)| \; \le \; C_1 \; ,$$

where $C_1 = C_1(T, Y, \beta)$ is a constant not depending on x. Therefore, the functional class $\mathcal{F}_\beta^{[0,T]}(Y)$ is uniformly bounded and equicontinuous which entails that the sequence $\{x_n\}$ converges to x uniformly in the interval $[0, T]$ and that $x \in \mathcal{F}_\beta^{[0,T]}(Y)$ (see, for example, Flett (1966), Theorem 8.3.3, or Ilyin and Poznyak (1980), Theorem 1.9). Since T is arbitrary, we get that $x \in \mathcal{F}_\beta(Y)$, which together with (23) proves the theorem. □

Let $z(t, Y) = z_\beta(t, Y)$ be a solution of (13) and

$$Z(t, Y) \; = \; (z(t, Y), \; \frac{\partial z(t, Y)}{\partial t}, \ldots, \; \frac{\partial^m z(t, Y)}{\partial t^m} \;)' \; ,$$

$$G_\mu(Y) \; = \; (\; \mu y_0, \; \mu^{1-1/\beta} y_1, \ldots, \; \mu^{1-m/\beta} y_m)' \; , \quad \mu > 0.$$

Next lemma is an analog of Lemma 14 from Zelikin, Borisov (1991), see also Bershanskii (1979), transformation P3; Donoho (1994), Lemma 2.1.

LEMMA 3. $\quad Z(t, G_\mu(Y)) \; = \; G_\mu(Z(\frac{t}{\mu^{1/\beta}}, Y)) \; , \quad I^*(G_\mu(Y)) = \mu^{2+1/\beta} I^*(Y) \; .$

Proof. Note that if $z_1(t) = \mu z(\frac{t}{\mu^{1/\beta}}, Y)$, then $z_1 \in \mathcal{F}_\beta(G_\mu(Y))$, thus,

$$I^*(G_\mu(Y)) \; \le \; I(G_\mu(Y), z_1) \; \le \; \mu^2 \int_0^\infty [z(\frac{t}{\mu^{1/\beta}}, Y) \;]^2 dt \; = \; \mu^{2+1/\beta} I^*(Y) \; , \qquad (25)$$

On the other hand, if $z_2(t) = \mu^{-1} z(\mu^{1/\beta} t, \; G_\mu(Y))$, then $z_2 \in \mathcal{F}_\beta(Y)$, and

$$I^*(Y) \; \le \; I(Y, z_2) \; \le \; \mu^{-2} \int_0^\infty [z(\mu^{1/\beta} t, G_\mu(Y)) \;]^2 dt \; = \; \mu^{-(2+1/\beta)} I^*(G_\mu(Y)) \; . \qquad (26)$$

The statement of the lemma now follows from (25),(26). □

LEMMA 4. $\quad I^*(Y)$ *is a continuous strictly convex function in* R^n.

Proof. Let $z_1(t) = z(t, Y_1), \quad z_2(t) = z(t, Y_2)$,

$$\tilde{z}(t) = \alpha z_1(t) + (1 - \alpha) z_2(t) \; , \; 0 \le \alpha \le 1 \; .$$

Note that the functional $\|z\|^2$ is strongly convex and, hence, strictly convex. Since $\tilde{z}(0) = \alpha Y_1 + (1-\alpha)Y_2$, then

$$I^*(\alpha Y_1 + (1-\alpha)Y_2) \;\leq\; \|\tilde{z}\|^2 \;<\; \alpha\|z_1\|^2 + (1-\alpha)\|z_2\|^2 \;=\; \alpha I^*(Y_1) + (1-\alpha)I^*(Y_2) \; .$$

Next, since $I^*(Y) \geq 0$, this function is proper convex and, hence, continuous - see Rockafellar (1972), Theorem 10.4 (compare also with Zelikin, Borisov (1991), Lemma 13). □

Next lemma verifies the continuity of the function $z(t,Y)$ in the second argument, see also Zelikin, Borisov (1991), Corollary 2.

LEMMA 5. *Let*

$$z_n(t) = z(t, Y_n) \; , \quad z_0(t) = z(t,Y), \quad \lim_{n\to\infty} Y_n = Y \; ,$$

and let $T > 0$ be an arbitrary fixed constant. Then the sequence $\{z_n\}$ converges to z_0 uniformly in $[0,T]$.

Proof. Using the same arguments as in the proof of Theorem 1, we obtain that there exists a subsequence $\{z_{n_k}\}$ converging to some function $\tilde{z}$ uniformly in $[0,T]$. Applying Fatou lemma and lemma 4 leads to :

$$\int_0^\infty \tilde{z}^2(t)dt \;\leq\; \liminf_{k\to\infty} \int_0^\infty z_{n_k}^2(t)dt \;=\; \lim_{k\to\infty} I^*(Y_{n_k}) = \int_0^\infty z_0^2(t)dt \; . \tag{27}$$

Since the solution of (13) is unique, (27) implies that $\tilde{z}(t) = z_0(t)$ and that the subsequence $\{n_k\}$ coincides with the original sequence. □

Let w_c be a level surface of the Bellman function $I^*(Y)$,

$$w_c = \{Y: \; I^*(Y) = c\} \; , \; c > 0.$$

Introduce functions

$$t_c(Y) = \{t: \; I^*(z(t,Y)) = c/2, \; Y \in w_c\},$$

$$\tau(c) = \sup_{Y \in w_c} t_c(Y).$$

LEMMA 6. $\tau(c/2) = v^{1/\beta}\tau(c)$ *with* $v = 2^{-[1/(2+1/\beta)]}$.

Proof. Let $Y \in R^n$, $I^*(Y) = c$. First, remark that

$$\lim_{t\to\infty} Z(t,Y) = 0 \; , \tag{28}$$

otherwise it easily follows that $I^*(Y)=\infty$ (see also Zelikin, Borisov (1991), problem 7). Next, we prove that

$$\tau(c)<\infty. \tag{29}$$

On the contrary, let us assume that there exists a sequence $\{Y_n, Y_n\in w_c\}$ such that

$$I^*(z(t_n,Y_n))=c/2\ ,\ \ \lim_{n\to\infty} t_n=\infty\ ,\ \ t_n=t_c(Y_n)\ .$$

Assume w.l.g. that $\{t_n\}$ is non-decreasing and $\lim_{n\to\infty} Y_n=Y_0$. (28) entails that for an arbitrary positive R there exists $T=T(R)>0$ such that

$$|z(t,Y_0)|\le R/4\ ,\ \ t\ge T\ . \tag{30}$$

It follows from lemma 4 that there exists $R=R(c)$ such that

$$\|Y\|_E\ge R\ \ if\ \ w(Y)\ge c/2\ . \tag{31}$$

Put $T=T(R)$ and let $t_n\ge T$. From Lemma 5 we get for all $m\ge M,\ M=M(t_n,R)$,

$$|z(t_n,Y_m)-z(t_n,Y_0)|\le R/2. \tag{32}$$

Now remark that due to the monotonicity of $\{t_n\}$ (31) implies:

$$|z(t_n,Y_m)|\ge R,\quad m\ge n.$$

Therefore, it follows from (32) that $|z(t_n,Y_0)|>R/2$, which contradicts (30) and proves (29).

Let $Y_1\in w_c,\ t_1=t_c(Y_1)$ and

$$t_1\ge\tau(c)-\varepsilon, \tag{33}$$

ε is an arbitrary small positive. Lemma 3 entails that

$$G_v(Y_1)\in w_{c/2}\ ,\ \ Z(v^{1/\beta}t_1,G_v(Y_1))\ =\ G_v(z(t_1,Y_1)),$$

$$I^*(Z(v^{1/\beta}t_1,G_v(Y_1)))\ =\ v^{2+1/\beta}I^*(z(t_1,Y_1))\ =\ c/4\ ,$$

thus,

$$\tau(c/2)\ge t_{c/2}(G_v(Y_1))=v^{1/\beta}t_1. \tag{34}$$

Now let $Y_2\in w_{c/2},\ t_2=t_{c/2}(Y_2)$ and

$$t_2\ge\tau(c/2)-\varepsilon \tag{35}$$

If $Y_3 = G_{v^{-1}}(Y_2)$, then $Y_3 \in w_c$, and, similar to (34), we obtain from (35) and Lemma 3

$$\tau(c/2) - \varepsilon \le t_2 \;=\; t_{c/2}(G_v(Y_3)) \;=\; v^{1/\beta} t_c(Y_3) \;,$$

thus,

$$\tau(c/2) - \varepsilon \le v^{1/\beta}\tau(c) \tag{36}$$

The statement of the lemma follows from (33), (34), (36). □

Let $T^*(Y) = T^*_\beta(Y) = \inf\{T :\; z(t,Y) = 0 \;,\;\; t \ge T \,\}$.

THEOREM 2. *$T^*(Y) < \infty$ for any $Y \in R^n$.*
Proof. Let $I^*(Y) = c$. Then due to Lemma 6

$$T^*(Y) \;\le\; \tau(c) + \tau(c/2) + \ldots + \tau(c/2^k) + \ldots \;=\; \tau(c)[1 + v^{1/\beta} + v^{2/\beta} + \ldots] \;=$$

$$=\; \tau(c)/(1 - v^{1/\beta}) \;\; < \infty \;.□$$

THEOREM 3. *For an arbitrary $\beta > 1$ there exists a unique solution $x_* = x_*^\beta$ of the optimization problem (12); $x_*(t) = 0$ for all $t \ge T^{**}$, $\;T^{**} = T^{**}_\beta \;<\; \infty$.*
Proof. Lemma 4 implies that

$$I^*(Y) \to \infty \;\; as \;\; \|Y\|_E \to \infty \;,$$

therefore, since I^* is strictly convex, there exists a unique $\;Y^* = Y^*_\beta$, $\|Y^*\|_E < \infty$, such that

$$\inf_{Y \in \mathcal{Y}} I^*(Y) \;=\; I^*(Y^*)$$

and

$$x_*(t) = z(t, Y^*) \;,\;\; T^{**} = T^*(Y^*) \;.\; □$$

Remark 1. For the integer case, $\beta = n$, general results of the optimal control theory (see Pontryagin et al (1962), Afanasyev et al (1990)) imply that if $t < T^*(Y)$, then

$$|z^{(n)}(t,Y)| = 1 \;,\;\; 0 \le t < T^*(Y) \;.$$

Thus, the solution of (13) is a spline: introduce the notation

$$g_1(t) = g_1(t,Y) = \sum_{i=0}^{n-1} y_i \frac{t^i}{i!} \pm \frac{t^n}{n!} \;, 0 \le t \le t_1 \;,$$

$$g_2(t) = g_1(t) \mp 2\frac{(t-t_1)^n}{n!} \ , \ t_1 \le t \le t_2 \ ,$$

$$g_3(t) = g_2(t) \pm 2\frac{(t-t_2)^n}{n!} \ , \ t_2 \le t \le t_3 \ ,$$

$$\dots$$

$$g_{k+1}(t) = g_k(t) \pm (-1)^k \ 2\frac{(t-t_k)^n}{n!} \ , \ t_k \le t \le t_{k+1} \ , etc.$$

Then

$$z(t,Y) = \begin{cases} g_i(t), & \text{if } t \in [t_i, t_{i+1}], \ t_0 = 0 \\ 0, & t > T^*(Y) \end{cases} \quad , \ \lim_{i\to\infty} t_i = T^*(Y) \ .$$

Remark 2. Communications with A.Afanasyev show that the results of Afanasyev et al (1990) may be generalized to the non-integer case as well, which leads to the following: let

$$\tilde{g}_1(t) = \tilde{g}_1(t,Y) = \sum_{i=0}^{m} y_i \frac{t^i}{i!} \pm \frac{t^\beta}{\beta!} \ , \ 0 \le t \le t_1 \ , \ \ \beta! = \beta(\beta-1)\dots,(\beta-m+1) \ ,$$

Next, let for $t \in [t_1, t_2]$

$$\tilde{g}_2^{(m)}(t) = \tilde{g}_1^{(m)}(t_1) \mp (t-t_1)^\alpha \ = \ y_m \pm t_1^\alpha \ \mp (t-t_1)^\alpha \ ,$$

$$\tilde{g}_2^{(m-1)}(t) = \tilde{g}_1^{(m-1)}(t_1) + \int_{t_1}^{t} \tilde{g}_2^{(m)}(v)dv \ ,$$

$$\dots$$

$$\tilde{g}_2(t) = \tilde{g}_1(t_1) + \int_{t_1}^{t} \tilde{g}_2^{(1)}(v)dv \ ,$$

For $t \in [t_k, t_{k+1}]$

$$\tilde{g}_{k+1}^{(m)}(t) = \tilde{g}_k^{(m)}(t_k) \ \pm (-1)^k \ (t-t_k)^\alpha \ ,$$

$$\tilde{g}_{k+1}^{(m-1)}(t) = \tilde{g}_k^{(m-1)}(t_k) + \int_{t_k}^{t} \tilde{g}_{k+1}^{(m)}(v)dv \ ,$$

$$\dots$$

$$\tilde{g}_{k+1}(t) = \tilde{g}_k(t_k) + \int_{t_k}^{t} \tilde{g}_{k+1}^{(1)}(v)dv \ , \ etc.$$

and

$$z(t,Y) = \begin{cases} \tilde{g}_i(t), & \text{if } t \in [t_i, t_{i+1}], \ t_0 = 0 \\ 0, & t > T^*(Y) \end{cases} \quad , \ \lim_{i\to\infty} t_i = T^*(Y) \ .$$

(note that $\tilde{g}_i = g_i$ for $\beta = n$).

Thus, the function $z(t,Y)$ is on the boundary of the set $\Sigma(\beta,1)$: if $\tilde{t} = t_k, t_k < t < t_{k+1}$, then

$$|z^{(m)}(t,Y) - z^{(m)}(\tilde{t},Y)| \;=\; (t-\tilde{t})^{\alpha} \;,$$

hence, the equality is attained in Hölder inequality.

Though it might be hardly expected to obtain the explicit solution for $\beta \neq 2$, we hope to find estimates for the sequence of switching moments $\{t_k\}$ and for the value of its limit, $T^*(Y)$.

4. APPLICATIONS TO REGRESSION ESTIMATION WITH BAHADUR RISK

In this section we mention possible applications of the finiteness of the function x_* to regression estimation with the minimax Bahadur risk.

Regression estimation at a fixed point. In Korostelev and Leonov (1995a) the double limiting equality (7) was proved without any assumption on the finiteness of x_* for $\beta > 1$. Therefore, it took considerably more technical efforts to obtain lower and upper bounds of the minimax risk $r_n(c)$ for $\beta > 1$, if compared to the case $\beta < 1$. These cases were studied separately - see Theorem 2 from the cited paper, remarks on pp.15,18,19. Now, since for $\beta > 1$ the function x_* does have a compact support due to Theorem 3 of the previous section, there is absolutely no difference between the cases mentioned above. Therefore, the complicated proof for $\beta > 1$ may be omitted.

Regression estimation with supremum norm loss. Consider a model

$$y_{in} = f(d_{in}) + \xi_{in}, \quad i = 1,2,\ldots,n; \quad n = 1,2,\ldots \tag{37}$$

$D_n = \{d_{in} \in [0,1],\ i = 1,\ldots,n\}$ is an arbitrary deterministic design and

$$\tilde{r}_n(c;D_n) = \inf_{f_n(D_n)} \sup_{f\in\Sigma(\beta,L;B)} \frac{1}{n}\log P_f^{(n)}(\|f_n(D_n) - f\|_\infty \geq c), \quad c > 0 \;,$$

where $\|f\|_\infty = \sup_{0\leq t\leq 1}|f(t)|$, $f_n(D_n)$ is an arbitrary estimator obtained from observations (37) with the design D_n.

Correspondence with A.Korostelev show that applying Theorem 3 for the model (37) with the Gaussian errors $\xi_{in} \sim N(0,\sigma^2)$ is expected to lead to the following analog of (5):

$$\lim_{n\to\infty}\inf_{D_n}\tilde{r}_n(c;D_n) \;=\; -\sigma^{-2}c^{2+1/\beta}\;A(\beta,L)\;[1+o(1)]\;, \tag{38}$$

where $o(1) \to 0$ as $c \to 0$; $A(\beta, L)$ is defined by (6). Hence, the exact constants appearing on the right-hand side of equalities (5) and (38) coincide. We also expect to verify the analog of (7) for the risk $\tilde{r}_n(c; D_n)$.

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