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Second-order subdifferential of 1- and maximum norm

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Abstract

We derive formulae for the second-order subdifferential of polyhedral norms. These formulae are fully explicit in terms of initial data. In a first step we rely on the explicit formula for the coderivative of normal cone mapping to polyhedra. Though being explicit, this formula is quite involved and difficult to apply. Therefore, we derive simple formulae for the 1-norm and – making use of a recently obtained formula for the second-order subdifferentia of the maximum function – for the maximum norm.

1 Introduction

The theory and applications of second-order generalised differentiation is a rapidly growing area of variational analysis. For major results of variational analysis we refer to the books [6, 7, 12]. There are numerous applications to problems which involve nonsmooth functions, multifunctions or sets with nonsmooth boundaries. For instance, in [4] tools of the second-order theory are used to derive stationarity conditions for equilibrium problems with equilibrium constraints (EPECs). The obtained results are applied to EPEC models of oligopolistic competition in electricity spot markets. In [8] the second-order subdifferential of the so-called *separable piecewise* $\mathcal{C}^2$ functions is applied to certain problems of continuum mechanics. In [1] convexity of piecewise linear functions and separable piecewise $\mathcal{C}^2$ functions is characterised via positive-semidefiniteness of their second-order subdifferentials. In [10, 9] and [11] second-order characterisations of tilt and full stability of local minimisers of constrained problems are derived respectively. For further applications of the second-order theory we refer to [8, 10] and the references therein.

One of the important aspects, which makes it possible to apply the second-order subdifferential construction to many nonsmooth problems, is its rich calculus. Convenient second-order calculus rules can be found in [6, 8, 10] and in references therein. Using this rules it is possible to reduce computation of second-order subdifferentials of complex functions to basic ones. Thus, along with calculus rules it is important for their efficient application to have explicit formulae for second-order subdifferentials of certain elementary functions. In [4] the second-order subdifferential of indicator functions to smooth nonpolyhedral inequality systems was calculated. For the second-order subdifferential of separable piecewise $\mathcal{C}^2$ functions we refer to [8]. In [10] the second-order subdifferential of piecewise linear-quadratic functions can be found. In [3] the second-order subdifferential of the maximum of coordinates was calculated. This result was then expanded to the extended partial second-order subdifferential of finite maxima of smooth functions by using a chain rule from [10].

In this paper we derive explicit formulae for second-order subdifferential of polyhedral norms on $\mathbb{R}^m$, i.e. the 1-norm $\|x\|_1 := \sum_{i=1}^m |x_i|$ and the ∞-norm $\|x\|_\infty := \max_{i=1,\dots,m} |x_i|$. These results generalise the simple example of the second-order subdifferential of the absolute value function on $\mathbb{R}$ given in [6].

2 Basic tools and notation

As usual, we denote by 'gr M' the graph of a multifunction M, by $\overline{\mathbb{R}}$ the extended real line, i.e. $\overline{\mathbb{R}} := [-\infty, +\infty]$, and by Z^o the polar cone of some set Z, i.e. $Z^o := \{y | \langle y, z\rangle \leq 0 \; \forall z \in Z\}$. Furthermore, we shall make use of the sign function defined as

$$\operatorname{sgn} t := \begin{cases} 1 & \text{if } t > 0, \\ 0 & \text{if } t = 0, \\ -1 & \text{if } t < 0. \end{cases}$$

We recall the following definitions (see [6]):

Definition 1. Let $C \subseteq \mathbb{R}^m$ be a closed set and $\bar{x} \in C$. The Mordukhovich normal cone to C at $\bar{x}$ is defined by

$$N_C(\bar{x}) := \{x^* | \, \exists \, (x_n, x_n^*) \to (\bar{x}, x^*) : x_n \in C, \; x_n^* \in [T_C(x_n)]^o\}.$$

Here, $[T_C(x)]^o$ refers to the Fréchet normal cone to C at x, which is the polar of the contingent cone

$$T_C(x) := \{d \in \mathbb{R}^m \, | \exists t_k \downarrow 0, d_k \to d : x + t_k d_k \in C, \forall k\}$$

to C at x. For an extended-real-valued, lower semicontinuous function $f : \mathbb{R}^m \to \overline{\mathbb{R}}$ with $|f(\bar{x})| < \infty$, the Mordukhovich normal cone induces a subdifferential via

$$\partial f(\bar{x}) := \{x^* | \, (x^*, -1) \in N_{\operatorname{epi} f}\,(\bar{x}, f(\bar{x}))\}.$$

Definition 2. Let $M : \mathbb{R}^n \rightrightarrows \mathbb{R}^m$ be a multifunction with closed graph. The Mordukhovich coderivative $D^*M(x,y) : \mathbb{R}^m \rightrightarrows \mathbb{R}^n$ of M at some $(x,y) \in \operatorname{gr} M$ is defined as

$$D^*M(x,y)(u) := \{v \in \mathbb{R}^n \, | (v, -u) \in N_{\operatorname{gr} M}(x,y)\}.$$

Definition 3. For a lower semicontinuous function $f : \mathbb{R}^n \to \overline{\mathbb{R}}$ which is finite at $x \in \mathbb{R}^n$ and for an element $s \in \partial f(x)$ the second-order subdifferential of f at x relative to s is a multifunction $\partial^2 f(x,s) : \mathbb{R}^n \rightrightarrows \mathbb{R}^n$ defined by

$$\partial^2 f(x,s)\,(u) := (D^* \partial f)\,(x,s)\,(u) \quad \forall u \in \mathbb{R}^n.$$

From these two definitions results the following formula for the second-order subdifferential of the p-norm with arbitrary $1 \leq p \leq \infty$:

$$\partial^2 \left\|\cdot\right\|_p (\bar{x}, \bar{s})\,(u) = \left\{ v \,\middle|\, (v, -u) \in N_{\operatorname{gr} \partial \|\cdot\|_p}(\bar{x}, \bar{s}) \right\}, \tag{1}$$

where $(\bar{x}, \bar{s})$ is a fixed point of the $\operatorname{gr} \partial \left\|\cdot\right\|_p$. In the next proposition we give an equivalent representation of the second-order subdifferential (1).

Proposition 4. *Let $(\bar{x}, \bar{s}) \in \operatorname{gr} \partial \left\|\cdot\right\|_p$ be fixed, $1 \leq p \leq \infty$. Then it holds*

$$\partial^2 \left\|\cdot\right\|_p (\bar{x}, \bar{s})\,(u) = \left\{ v \,\middle|\, (-u, v) \in N_{\operatorname{gr} N_{\mathbb{B}_q}}(\bar{s}, \bar{x}) \right\}, \tag{2}$$

where $\mathbb{B}_q$ denotes the unit ball of the q-norm and q is defined by $\frac{1}{p} + \frac{1}{q} = 1$.

Proof. The p-norm is the support function of the unit ball $\mathbb{B}_q$:

$$\|x\|_p = \sigma_{\mathbb{B}_q}(x) := \max_{v \in \mathbb{B}_q} \langle v, x \rangle .$$

Thus we have the following equivalences

$$\begin{aligned} \bar{s} \in \partial \|\cdot\|_p (\bar{x}) &\iff \bar{s} \in \partial \sigma_{\mathbb{B}_q} (\bar{x}) \\ &\iff \bar{x} \in N_{\mathbb{B}_q} (\bar{s}) , \end{aligned}$$

where the second line follows from [12, Example 11.4]. Hence, it holds

$$(\bar{x}, \bar{s}) \in \operatorname{gr} \partial \|\cdot\|_p \iff (\bar{s}, \bar{x}) \in \operatorname{gr} N_{\mathbb{B}_q}. \tag{3}$$

With the notation $L = \left(\begin{smallmatrix} 0 & I \\ I & 0 \end{smallmatrix}\right)$, where I denotes the identity matrix, (3) can be equivalently written as $L \operatorname{gr} \partial \|\cdot\|_p = \operatorname{gr} N_{\mathbb{B}_q}$. The claim of the Proposition follows now from (1) and [12, Exercise 6.7]. □

Finally we cite the Proposition 3.2 from [5], which is a concretisation of a well-known result [2, Proof of Theorem 2]. Let $(\bar{y}, \bar{z}) \in \operatorname{gr} N_C$, where C denotes a convex polyhedron given by $C = \{y \in \mathbb{R}^n \,|\, Ay \leq b\}$, $b \in \mathbb{R}^m$ and A is a matrix of order (m, n). Let

$$I := \{j \in \{1, \dots, m\} \,|\, \langle a_j, \bar{y} \rangle = b_i\} \tag{4}$$

be the set of active indices at $\bar{y}$, where a_j and b_j refers to the rows of A and elements of b, respectively. Since $\bar{z} \in N_C(\bar{y})$, there exist $\lambda_j \geq 0$ for $j \in I$, such that

$$\bar{z} = \sum_{j \in I} \lambda_j a_j \tag{5}$$

We introduce the following subset of I:

$$J = \{j \in I \,|\, \lambda_j > 0\} \tag{6}$$

Furthermore, for each index subset $I' \in I$, we introduce the closed cone

$$M_{I'} := \{h \,|\, \langle a_j, h \rangle = 0 \,\forall j \in I',\ \langle a_j, h \rangle \leq 0 \,\forall j \in I \setminus I'\} \tag{7}$$

as well as the characteristic index set

$$\chi(I') := \{j \in I \,|\, \langle a_j, h \rangle = 0 \,\forall h \in M_{I'}\} . \tag{8}$$

Proposition 5 (Henrion, Römisch [5]). *With the notation introduced above, one has*

$$N_{\operatorname{gr} N_C}(\bar{y}, \bar{z}) = \bigcup_{J \subseteq I_1 \subseteq I_2 \subseteq I} \left\{ (u, v) \in \mathbb{R}^{2n} \,\middle|\, \begin{array}{l} \langle a_j, v \rangle = 0,\ j \in I_1, \\ \langle a_j, v \rangle \leq 0,\ j \in \chi(I_2) \setminus I_1, \\ u = \displaystyle\sum_{j \in I_1} \lambda_j a_j + \sum_{j \in \chi(I_2) \setminus I_1} \mu_j a_j, \\ \lambda_j \in \mathbb{R},\ \mu_j \in \mathbb{R}_+. \end{array} \right\} .$$

Clearly, the characteristic index set $\chi(I')$ consists of indices of all active constraints given that the constraints in I' are active. It holds $I' \subseteq \chi(I') \subseteq I$ for all $I' \subseteq I$.

Remark 6. If the vectors $\{a_j\}_{j\in I}$ are linearly independent, then $\chi(I') = I'$ for all $I' \subseteq I$.

The next example illustrates the role of the characteristic index set.

Example 7. Let C be a convex polyhedral cone with apex in $(0,0,1)$ given by $C := \{y \in \mathbb{R}^3 \,|\, Ay \le 1\}$, where

$$A := \begin{pmatrix} 1 & 1 & 1 \\ 1 & -1 & 1 \\ -1 & 1 & 1 \\ -1 & -1 & 1 \end{pmatrix}.$$

The set of active indices in the apex is given by $I = \{1,2,3,4\}$. Let $I' = \{1,4\}$, then it follows from (7) and (8) that $\chi(I') = I$. Here, the constraints in I' defines the opposite sides of the convex polyhedral cone C, which have only one common point $y = (0,0,1)$. In this point all constraints from I are active. If we set $I' = \{1,2\}$, then $\chi(I') = I'$. Here, the constraints in I' define two adjacent sides of C, which have a common edge. No further constraints from I are active in all points of this edge.

3 Special case $\bar{x} = 0$, $\bar{s} \in \operatorname{int} \partial \left\|\cdot\right\|_p (0)$

In the special case of $\bar{x} = 0$ and $\bar{s}$ is an interior point of $\partial \left\|\cdot\right\|_p (0)$, there is a simple formula which is valid for all $1 \le p \le \infty$.

Proposition 8. *Let $\bar{x} = 0$, $\bar{s} \in \operatorname{int} \partial \left\|\cdot\right\|_p (0)$. Then for all $1 \le p \le \infty$ it holds*

$$\partial^2 \left\|\cdot\right\|_p (0,\bar{s})(u) = \begin{cases} \mathbb{R}^m, & \text{if } u = 0, \\ \emptyset, & \text{if } u \neq 0. \end{cases} \tag{9}$$

Proof. Hence $\bar{s}$ is an interior point of the subdifferential $\partial \left\|\cdot\right\|_p (0)$ there exists a neighbourhood $\mathcal{U}$ of $\bar{s}$ such that $s \in \operatorname{int} \partial \left\|\cdot\right\|_p (0)$ for all $s \in \mathcal{U}$. Let $(x,s) \in \operatorname{gr} \partial \left\|\cdot\right\|_p \cap [\mathbb{R}^m \times \mathcal{U}]$. Then $s \in \partial \left\|\cdot\right\|_p (x)$. This is equivalent to $x \in N_{\mathbb{B}_q}(s)$. Due to the well-known fact that $\partial \left\|\cdot\right\|_p (0) = \mathbb{B}_q$ with $\frac{1}{p} + \frac{1}{q} = 1$, it holds $s \in \operatorname{int} \mathbb{B}_q$. Thus, $x = 0$ for all x with $(x,s) \in \operatorname{gr} \partial \left\|\cdot\right\|_p \cap [\mathbb{R}^m \times \mathcal{U}]$. This implies $\operatorname{gr} \partial \left\|\cdot\right\|_p \cap [\mathbb{R}^m \times \mathcal{U}] = \{0\} \times \mathcal{U}$. As a consequence the tangent cone to $\operatorname{gr} \partial \left\|\cdot\right\|_p$ at (x,s) is given by

$$T_{\operatorname{gr} \partial \left\|\cdot\right\|_p}(x,s) = \{0\} \times \mathbb{R}^m$$

for all $(x,s) \in \operatorname{gr} \partial \left\|\cdot\right\|_p \cap [\mathbb{R}^m \times \mathcal{U}]$. It follows, that the Fréchet normal cone has the following form

$$\widehat{N}_{\operatorname{gr} \partial \left\|\cdot\right\|_p}(x,s) = \mathbb{R}^m \times \{0\}$$

for all $(x,s) \in \operatorname{gr} \partial \left\|\cdot\right\|_p \cap [\mathbb{R}^m \times \mathcal{U}]$. Because of the fact that the Fréchet normal cone does not change in the neighbourhood of $(0,\bar{s})$, we conclude that the Mordukhovich normal cone to $\operatorname{gr} \partial \left\|\cdot\right\|_p$ at $(0,\bar{s})$ coincides with the Fréchet normal cone. The claim of the proposition follows now from (1). □

4 Maximum norm

In this section we develop a formula for the second-order subdifferential $\partial^2 \|\cdot\|_\infty(\bar{x},\bar{s})$ of the maximum norm at any arbitrary point $(\bar{x},\bar{s}) \in \operatorname{gr}\partial\|\cdot\|_\infty$. With each $x \in \mathbb{R}^m$ we associate the set of indices

$$J(x) := \{i \in \{1,\ldots,m\} \mid |x_i| = \|x\|_\infty\} \tag{10}$$

and denote by $J^c(x)$ its complement. Observe that $J(x) \neq \emptyset$.

In the next lemma we describe a local representation of the maximum norm which we will use later.

Lemma 9. *Let $\bar{x} \neq 0$. Then, there is a neighbourhood $\mathcal{U}$ of $\bar{x}$ such that*

$$\|x\|_\infty = \max_{i=1,\ldots,m} (\operatorname{sgn}\bar{x}_i)x_i \quad \forall x \in \mathcal{U}.$$

As a consequence,

$$\partial\|\cdot\|_\infty(\bar{x}) = \operatorname{conv}\{(\operatorname{sgn}\bar{x}_i)e_i | i \in J(\bar{x})\},$$

where the e_i refer to the unit vectors of $\mathbb{R}^m$.

Proof. By definition, it holds for all x that

$$\|x\|_\infty = \max_{i=1,\ldots,m} |x_i| = \max_{i\in J(x)} |x_i|, \tag{11}$$

where the second equality follows from $J(x) \neq \emptyset$. We have that $|\bar{x}_i| = \|\bar{x}\|_\infty > 0$ for all $i \in J(\bar{x})$. Hence, there is a neighbourhood $\mathcal{U}$ of $\bar{x}$ such that for all $x \in \mathcal{U}$ it holds $\operatorname{sgn}\bar{x}_i = \operatorname{sgn} x_i$ for $i \in J(\bar{x})$ and $J(x) \subseteq J(\bar{x})$. Furthermore, it holds $(\operatorname{sgn}\bar{x}_i)x_i \le \|x\|_\infty$ for all i. This allows us to continue (11) for all $x \in \mathcal{U}$ as

$$\begin{aligned}\|x\|_\infty &= \max\left\{\max_{i\in J(x)}|x_i|, \max_{i\in J^c(x)}(\operatorname{sgn}\bar{x}_i)x_i\right\} = \max\left\{\max_{i\in J(x)}(\operatorname{sgn} x_i)x_i, \max_{i\in J^c(x)}(\operatorname{sgn}\bar{x}_i)x_i\right\}\\ &= \max\left\{\max_{i\in J(x)}(\operatorname{sgn}\bar{x}_i)x_i, \max_{i\in J^c(x)}(\operatorname{sgn}\bar{x}_i)x_i\right\} = \max_{i=1,\ldots,m}(\operatorname{sgn}\bar{x}_i)x_i.\end{aligned}$$

□

In order to derive an explicit formula for the second-order subdifferential of the maximum norm we intend to use the Propositions 4 and 5. To be able to do this we need a representation of the unit ball $\mathbb{B}_1$ in a form which is convenient to work with. Due to the definition of the absolute value function, $\sum|v_i| \le 1$ is equivalent to the system $\langle a_j, v\rangle \le 1$, where $j \in \tilde{I} = \{1,\ldots,2^m\}$, a_j are vectors consisting of all possible combinations of 1 and -1. Thus, the unit ball $\mathbb{B}_1$ can be written as a convex polyhedron in the following form

$$\mathbb{B}_1 = \left\{v \,\middle|\, \langle a_j, v\rangle \le 1,\ j \in \tilde{I}\right\}$$

with a_j and $\tilde{I}$ described above. The set of active indices defined in (4) corresponds to

$$I = \left\{j \in \tilde{I} \,|\, \langle a_j, \bar{s}\rangle = 1\right\}. \tag{12}$$

Now by (2) we can obtain an explicit formula for the second-order subdifferential of the maximum norm, using thereby the Proposition 5 for the representation of the normal cone $N_{\operatorname{gr} N_{\mathbb{B}_1}}(\bar{s},\bar{x})$.

Theorem 10. *Let* $(\bar{x}, \bar{s}) \in \operatorname{gr} \partial \|\cdot\|_\infty$. *It holds*

$$\partial^2 \|\cdot\|_\infty (\bar{x}, \bar{s})(u) = \bigcup_{J \subseteq I_1 \subseteq I_2 \subseteq I} \left\{ v \in \mathbb{R}^m \,\middle|\, \begin{array}{l} \langle a_j, v \rangle = 0, \; j \in I_1, \\ \langle a_j, v \rangle \leq 0, \; j \in \chi(I_2) \setminus I_1, \\ u = -\sum\limits_{j \in I_1} \lambda_j a_j - \sum\limits_{j \in \chi(I_2) \setminus I_1} \mu_j a_j, \\ \lambda_j \in \mathbb{R}, \; \mu_j \in \mathbb{R}_+. \end{array} \right\}, \tag{13}$$

where I *and* J *are defined in* (12) *and* (6)*, respectively.*

Proof. The statement of this theorem follows directly from (2) and the Proposition 5. □

The above theorem yields a general formula for the second-order subdifferential of the maximum norm, which holds for arbitrary points $(\bar{x}, \bar{s}) \in \operatorname{gr} \partial \|\cdot\|_\infty$. Simpler characterisation can be achieved if we consider special points of $\operatorname{gr} \partial \|\cdot\|_\infty$. There are three cases to be distinguished: if $\bar{x} \neq 0$, then we may benefit from a formula for the second-order subdifferential of maximum functions proven in [3]; if $\bar{x} = 0$ and $\bar{s} \in \operatorname{int} \partial \|\cdot\|_\infty (0)$, then the second-order subdifferential of the maximum norm is given by the simple formula (9); the remaining case of $\bar{x} = 0$ and $\bar{s} \in \operatorname{bd} \partial \|\cdot\|_\infty (0)$ turns out to be the most delicate one. In the last case the constraints, which are active in $\bar{s}$, are not necessarily linearly independent. For an illustration see the Example 7, where C locally coincides in $(0, 0, 1)$ with the unit ball $\mathbb{B}_1$ in $\mathbb{R}^3$.

We now turn to the points $(\bar{x}, \bar{s}) \in \operatorname{gr} \partial \|\cdot\|_\infty$ with $\bar{x} \neq 0$. In this case the formula for the second-order subdifferential of the maximum norm (13) can be drastically simplified as a consequence of a result in [3, Theorem 4.2].

Let $g : \mathbb{R}^m \to \mathbb{R}^p$ be a continuously differentiable mapping and denote by

$$\varphi := \max_{j=1,\ldots,p} g_j \tag{14}$$

the associated maximum function. We fix any $\bar{x} \in \mathbb{R}^m$ and $\bar{s} \in \partial \varphi(\bar{x})$ and introduce the index set

$$\bar{I} := \{ i \in \{1, \ldots, p\} \,| g_i(\bar{x}) = \varphi(\bar{x}) \} \,. \tag{15}$$

Since $\partial \varphi(\bar{x}) = \operatorname{conv} \left\{ \nabla g_i(\bar{x}) | i \in \bar{I} \right\}$ (see [12, Exercise 8.31]), there exists a vector $v \geq 0$ such that

$$\bar{s} = \sum_{i \in \bar{I}} v_i \nabla g_i(\bar{x}). \tag{16}$$

The following theorem is a corollary to [3, Theorem 4.2]. The latter has been proven in the slightly more general context of parameter-dependent maximum functions and makes a statement on the *extended* partial second-order subdifferential, whereas here we are interested in the simpler case of non-parametric maximum functions and their conventional second-order subdifferential.

Theorem 11. *Assume that the set* $\{\nabla g_i(\bar{x})\}_{i \in \bar{I}}$ *is linearly independent and let* $v \geq 0$ *be the unique solution of (16). Denote* $L := \{ i \in \bar{I} | v_i > 0 \}$. *Then, for arbitrary* $u \in \mathbb{R}^m$*, one has that:*

$$\partial^2 \varphi(\bar{x}, \bar{s})(u) \neq \emptyset \iff \exists c \in \mathbb{R} : \langle \nabla g_i(\bar{x}), u \rangle = c \quad \forall i \in L. \tag{17}$$

In this case it holds that $\tilde{w} \in \partial^2\varphi(\bar{x}, \bar{s})(u)$ *if and only if there exists some* $w \in \mathbb{R}^m$ *such that*

$$\tilde{w} = \left[\nabla^2 \langle v, g\rangle (\bar{x})\right] u + \nabla^T g(\bar{x})w, \quad \sum_{i=1}^{p} w_i = 0, \quad w_i \geq 0 \, \forall i \in I_>, \quad w_i = 0 \, \forall i \in I_< \cup I^c. \tag{18}$$

Here, $I_> := \left\{i \in \bar{I} \,|\, \langle\nabla g_i(\bar{x}), u\rangle > c\right\}$ *and* $I_< := \left\{i \in \bar{I} \,|\, \langle\nabla g_i(\bar{x}), u\rangle < c\right\}$.

With this result we are able to prove the following theorem.

Theorem 12. *Let* $\bar{x} \neq 0$, $(\bar{x}, \bar{s}) \in \operatorname{gr} \partial \|\cdot\|_\infty$, $L(\bar{x}, \bar{s}) := \{i \in J(\bar{x}) | \bar{s}_i \neq 0\}$, *with* $J(\bar{x})$ *as in* (10). *Then for all* $u \in \mathbb{R}^m$ *one has that:*

$$\partial^2 \|\cdot\|_\infty (\bar{x}, \bar{s})(u) \neq \emptyset \iff \exists c \in \mathbb{R} : (\operatorname{sgn} \bar{x}_i)u_i = c \quad \forall i \in L(\bar{x}, \bar{s}), \tag{19}$$

In this case it holds

$$\partial^2 \|\cdot\|_\infty (\bar{x}, \bar{s})(u) = \left\{ w \in \mathbb{R}^m \,\middle|\, \sum_{i=1}^{m} (\operatorname{sgn} \bar{x}_i) w_i = 0, \; \bar{x}_i w_i \geq 0 \, \forall i \in J_>(\bar{x}), \; w_i = 0 \, \forall i \in J_<(\bar{x}) \cup J^c(\bar{x}) \right\}, \tag{20}$$

where $J_>(\bar{x}) := \{i \in J(\bar{x}) \,|(\operatorname{sgn} \bar{x}_i)u_i > c\}$ *and* $J_<(\bar{x}) := \{i \in J(\bar{x}) \,|(\operatorname{sgn} \bar{x}_i)u_i < c\}$.

Proof. Thanks to Lemma 9, around $\bar{x}$ the local representation $\|x\|_\infty = \max_{i=1,\dots,m} (\operatorname{sgn} \bar{x}_i)x_i$ holds true. Hence,

$$\partial^2 \|\cdot\|_\infty (\bar{x}, \bar{s}) = \partial^2\varphi(\bar{x}, \bar{s})$$

with φ as defined in (14) and linear functions $g_i(x) := (\operatorname{sgn} \bar{x}_i)x_i$ for $i = 1, \dots, m$. From (15) and (10) we derive that

$$\bar{I} = \{i \in \{1, \dots, m\} \,|(\operatorname{sgn} \bar{x}_i)\bar{x}_i = \|\bar{x}\|_\infty\} = \{i \in \{1, \dots, m\} \,||\bar{x}_i| = \|\bar{x}\|_\infty\} = J(\bar{x}).$$

Therefore,

$$\{\nabla g_i(\bar{x})\}_{i \in \bar{I}} = \{(\operatorname{sgn} \bar{x}_i)e_i\}_{i \in J(\bar{x})} .$$

Observing that $i \in J(\bar{x})$ implies $|\bar{x}_i| = \|\bar{x}\|_\infty > 0$ and, hence, $\operatorname{sgn} \bar{x}_i \neq 0$, it follows that the set $\{\nabla g_i(\bar{x})\}_{i \in \bar{I}}$ is linearly independent. This allows us to invoke Theorem 11. First, note that (16) implies $\bar{s}_i = (\operatorname{sgn} \bar{x}_i)v_i$ for $i \in \bar{I}$. By virtue of $\operatorname{sgn} \bar{x}_i \neq 0$ and $v_i \geq 0$, this allows to identify the index set L from Theorem 11 to be the same as the index set $L(\bar{x}, \bar{s})$ introduced in the statement of this Theorem. Hence, (17) entails (19). Similarly, the index sets $I_>$ and $I_<$ from Theorem 11 are easily seen to coincide with the index sets $J_>(\bar{x})$ and $J_<(\bar{x})$, respectively, introduced in the statement of this Theorem. Now, (18) entails that $\tilde{w} \in \partial^2 \|\cdot\|_\infty (\bar{x}, \bar{s})(u)$ if and only if there exists some $w \in \mathbb{R}^m$ such that

$$\tilde{w}_i = (\operatorname{sgn} \bar{x}_i)w_i \, \forall i = 1, \dots, m, \quad \sum_{i=1}^{m} w_i = 0, \; w_i \geq 0 \, \forall i \in J_>(\bar{x}), \; w_i = 0 \, \forall i \in J_<(\bar{x}) \cup J^c(\bar{x}).$$

Here, we exploited the fact that the second-order term in (18) vanishes due to the linearity of g. Recalling that $\operatorname{sgn} \bar{x}_i \neq 0$ and, hence, $(\operatorname{sgn} \bar{x}_i)^2 = 1$ for $i \in J(\bar{x})$, it is easily seen that the set of relations above is equivalent to

$$\sum_{i=1}^{m} (\operatorname{sgn} \bar{x}_i)\tilde{w}_i = 0, \quad (\operatorname{sgn} \bar{x}_i)\tilde{w}_i \geq 0 \, \forall i \in J_>(\bar{x}), \quad \tilde{w}_i = 0 \, \forall i \in J_<(\bar{x}) \cup J^c(\bar{x})).$$

This amounts to (20). □

Example 13. Let $m = 7$, $\bar{x} = (1, -1, 1, -1, 1, 0, 0)$ and $\bar{s} = (\frac{1}{3}, -\frac{2}{3}, 0, 0, 0, 0, 0)$, then $J(\bar{x}) = \{1, 2, 3, 4, 5\}$, $J^c(\bar{x}) = \{6, 7\}$ and $L(\bar{x}, \bar{s}) = \{1, 2\}$. Due to the Lemma 9, $\bar{s} \in \partial \|\cdot\|_\infty (\bar{x})$. Let $u = (2, -2, 2, -3, 0, -1, 4)$, then $(\operatorname{sgn} \bar{x}_i)u_i = c$ for all $i \in L(\bar{x}, \bar{s})$ with $c = 2$. Thus, $\partial^2 \|\cdot\|_\infty (\bar{x}, \bar{s})(u) \neq \emptyset$. $J_>(\bar{x}) = \{4\}$, $J_<(\bar{x}) = \{5\}$. It holds

$$\partial^2 \|\cdot\|_\infty (\bar{x}, \bar{s})(u) = \left\{ w \in \mathbb{R}^7 \mid w_1 - w_2 + w_3 - w_4 = 0,\ w_4 \leq 0,\ w_5 = w_6 = w_7 = 0 \right\}.$$

In the remaining case $\bar{x} = 0$, $\bar{s} \in \operatorname{bd} \partial \|\cdot\|_\infty (0)$ no simplification of (13) is obvious. In the points, where the active constraints are not linearly independent, we have to compute the characteristic index set $\chi(\cdot)$ and rely on (13). But certain specification are still possible.

Proposition 14. *Let $\bar{x} = 0$ and $\bar{s} \in \operatorname{bd} \partial \|\cdot\|_\infty (0)$. Then the index set J defined in* (6) *and used in* (13) *is empty. Furthermore, $(a_j)_i = \operatorname{sgn} \bar{s}_i$ for all $j \in I$ and all $i \in L(0, \bar{s})$, where I and $L(\bar{x}, \bar{s})$ are defined in* (12) *and the Theorem 12 respectively.*

Proof. Since $\bar{s} \in \operatorname{bd} \partial \|\cdot\|_\infty (0)$ it follows by (3) that $0 \in N_{\mathbb{B}_1}(\bar{s})$. Thus, there exist multipliers $\lambda_j \geq 0$ such that $\sum_{j \in I} \lambda_j a_j = 0$. As a consequence it holds

$$0 = \sum_{j \in I} \lambda_j \langle a_j, \bar{s} \rangle = \sum_{j \in I} \lambda_j.$$

Thus, $\lambda_j = 0$ for all $j \in I$ and hence, $J = \emptyset$.

Let $\bar{s} \in \operatorname{bd} \partial \|\cdot\|_\infty (0) = \operatorname{bd} \mathbb{B}_1$. Then on the one hand

$$\sum_{i=1}^m |\bar{s}_i| = \sum_{i \in L(0,\bar{s})} |\bar{s}_i| = 1$$

and on the other hand

$$\langle a_j, \bar{s} \rangle = \sum_{i=1}^m (a_j)_i \bar{s}_i = \sum_{i \in L(0,\bar{s})} (a_j)_i \bar{s}_i = 1.$$

for all $j \in I$. By virtue of $s_i = \operatorname{sgn} s_i |s_i|$ we have

$$\sum_{i \in L(0,\bar{s})} \left((a_j)_i \operatorname{sgn} \bar{s}_i - 1 \right) |\bar{s}_i| = 0.$$

Thus, $(a_j)_i \operatorname{sgn} \bar{s}_i = 1$ and, consequently, $(a_j)_i = \operatorname{sgn} \bar{s}_i$ for all $j \in I$ and all $i \in L(0, \bar{s})$. □

5 1-norm

In the next theorem we present an explicit formula for the second-order subdifferential $\partial^2 \|\cdot\|_1 (\bar{x}, \bar{s})$ of the 1-norm at any arbitrary point $(\bar{x}, \bar{s}) \in \operatorname{gr} \partial \|\cdot\|_1$.

Theorem 15. *For any fixed $(\bar{x}, \bar{s}) \in \operatorname{gr} \partial \|\cdot\|_1$ and any fixed $u \in \mathbb{R}^m$ the second-order subdifferential $\partial^2 \|\cdot\|_1 (\bar{x}, \bar{s})(u)$ is nonempty if and only if $u_j = 0$ for all $j \in I^c$, where $I := \{j \in \{1, \ldots, m\} \mid |\bar{s}_j| = 1\}$. In this case it holds*

$$\partial^2 \|\cdot\|_1 (\bar{x}, \bar{s})(u) = \left\{ v \in \mathbb{R}^m \,\middle|\, \begin{array}{l} v_j = 0,\ j \in K \\ \bar{s}_j v_j \leq 0,\ j \in L \setminus K \end{array} \right\}, \tag{21}$$

where $K := \{j \in I \mid u_j \neq 0, \bar{s}_j u_j > 0\} \cup J$, $L := \{j \in I \mid u_j \neq 0\} \cup J$ and $J := \{j \in I \mid \bar{x}_j \neq 0\}$.

Proof. In this proof we will, once again, make use of the Propositions 4 and 5. Thus, we begin with the statement, that the unit ball of the maximum norm is a convex polyhedron, given by

$$\mathbb{B}_\infty = \{v \in \mathbb{R}^m \mid |v_i| \leq 1\}.$$

It holds

$$\bar{s} \in \mathbb{B}_\infty \iff |\bar{s}_i| = \langle (\operatorname{sgn} \bar{s}_i) e_i, \bar{s} \rangle \leq 1, \; i = 1, \ldots, m.$$

Thus, if we set

$$a_j := (\operatorname{sgn} s_j)\, e_j, \; b_j := 1, \; j = 1, \ldots, m \tag{22}$$

the index set I introduced in the statement of this theorem coincides with the same index set defined in (4). Furthermore, by (3) and (22) it holds

$$\bar{x} = \sum_{j \in I} \lambda_j a_j = \sum_{j \in I} \lambda_j \, (\operatorname{sgn} \bar{s}_j)\, e_j$$

with $\lambda_j \geq 0$. Hence, $\bar{x}_j = \lambda_j (\operatorname{sgn} \bar{s}_j)$ for $j \in I$. Recalling that $\operatorname{sgn} \bar{s}_j \neq 0$ for $j \in I$ it follows $\lambda_j > 0 \Leftrightarrow \bar{x}_j \neq 0$ for $j \in I$. Consequently, the index set J from this theorem coincide with the same index set defined in (6).

The vectors $\{(\operatorname{sgn} \bar{s}_j) e_j\}_{j \in I}$ are linearly independent, therefore it holds $\chi(I') = I'$ for all $I' \subseteq I$.

Due to (22) it holds $\langle a_j, v \rangle = \operatorname{sgn} \bar{s}_j v_j$. Together with $\bar{s}_j \neq 0$ for $j \in I$ this implies

$$\langle a_j, v \rangle = 0, \; j \in I_1 \iff v_j = 0, \; j \in I_1 \tag{23}$$

$$\langle a_j, v \rangle \leq 0, \; j \in I_2 \setminus I_1 \iff \bar{s}_j v_j \leq 0, \; j \in I_2 \setminus I_1 \tag{24}$$

for all $I_1 \subseteq I_2 \subseteq I$.

Next we set

$$u := -\sum_{j \in I_1} \lambda_j a_j - \sum_{j \in I_2 \setminus I_1} \mu_j a_j, \tag{25}$$

where $\lambda_j \in \mathbb{R}$ and $\mu_j \in \mathbb{R}_+$. It holds

$$u_j = \begin{cases} -\lambda_j \operatorname{sgn} \bar{s}_j & \text{if } j \in I_1, \\ -\mu_j \operatorname{sgn} \bar{s}_j & \text{if } j \in I_2 \setminus I_1, \\ 0 & \text{if } j \in I_2^c. \end{cases}$$

Thus, due to $\bar{s}_j \neq 0$ for $j \in I$, (25) is equivalent to

$$\begin{cases} u_j \in \mathbb{R}, \; j \in I_1, \\ \bar{s}_j u_j \leq 0, \; j \in I_2 \setminus I_1, \\ u_j = 0, \; j \in I_2^c. \end{cases} \tag{26}$$

Now, from (23), (24), (26) and the Propositions 4 and 5 it follows

$$\partial^2 \|\cdot\|_1 (\bar{x}, \bar{s})(u) = \bigcup_{J \subseteq I_1 \subseteq I_2 \subseteq I} A_{I_1, I_2}, \tag{27}$$

where

$$A_{I_1,I_2} = \left\{ v \in \mathbb{R}^m \,\middle|\, \begin{array}{l} v_j = 0,\ j \in I_1 \\ \bar{s}_j v_j \le 0,\ j \in I_2 \setminus I_1 \\ \bar{s}_j u_j \le 0,\ j \in I_2 \setminus I_1 \\ u_j = 0,\ j \in I_2^c \end{array} \right\}.$$

It holds

$$A_{I_1,I_2} \neq \emptyset \iff \begin{cases} u_j = 0,\ j \in I_2^c \\ \bar{s}_j u_j \le 0,\ j \in I_2 \setminus I_1 \end{cases} \iff \begin{cases} u_j = 0,\ j \in I^c \\ \{j \in I \,|u_j \neq 0\} \subseteq I_2 \\ \{j \in I \,|u_j \neq 0, \bar{s}_j u_j > 0\} \subseteq I_1 \end{cases}$$

With the notation

$$M := \left\{ (I_1, I_2) \,\middle|\, \begin{array}{l} \{j \in I \,|u_j \neq 0, \bar{s}_j u_j > 0\} \cup J \subseteq I_1 \\ \{j \in I \,|u_j \neq 0\} \cup J \subseteq I_2 \end{array} \right\} \tag{28}$$

(27) can be equivalently written as

$$\partial^2 \left\|\cdot\right\|_1 (\bar{x}, \bar{s})(u) = \begin{cases} \bigcup_{(I_1,I_2)\in M} B_{I_1,I_2} & \text{if } u_j = 0\ \forall j \in I^c, \\ \emptyset & \text{otherwise,} \end{cases} \tag{29}$$

where

$$B_{I_1,I_2} = \left\{ v \in \mathbb{R}^m \,\middle|\, \begin{array}{l} v_j = 0,\ j \in I_1 \\ \bar{s}_j v_j \le 0,\ j \in I_2 \setminus I_1 \end{array} \right\}. \tag{30}$$

We show that B_{I_1,I_2} is a decreasing family of sets. Let for this sake $I_1^a \subseteq I_1^b \subseteq I_2$, $(I_1^a, I_2) \in M$, $\left(I_1^b, I_2\right) \in M$. We claim that $B_{I_1^a,I_2} \supseteq B_{I_1^b,I_2}$. For arbitrarily given $v \in B_{I_1^b,I_2}$ it holds

$$v_j = 0 \; j \in I_1^b, \;\; \bar{s}_j v_j \le 0 \; j \in I_2 \setminus I_1^b. \tag{31}$$

Since $I_1^a \subseteq I_2^b$ and $I_2 \setminus I_1^a = \left(I_2 \setminus I_1^b\right) \cup \left(I_1^b \setminus I_1^a\right)$, it follows from (31) that $v \in B_{I_1^a,I_2}$. Next we consider $I_1 \subseteq I_2^a \subseteq I_2^b$, $(I_1, I_2^a) \in M$, $\left(I_1, I_2^b\right) \in M$. Since $I_2^a \setminus I_1 \subseteq I_2^b \setminus I_1$ and due to (30) it holds $B_{I_1,I_2^a} \supseteq B_{I_1,I_2^b}$.

The claim of the theorem follows now from (28), (29), (30) and the fact that B_{I_1,I_2} is a decreasing family of sets. □

Example 16. Let $\bar{x} = (1, -2, 0, 0)$ and $\bar{s} = (1, -1, -1, 0.5)$. Then $I = \{1, 2, 3\}$ and $J = \{1, 2\}$. It holds $\bar{x} = \sum_{j\in I} \lambda_j a_j$ with $\lambda_j \ge 0$ and a_j defined in (22). Thus, $\bar{x} \in N_{\mathbb{B}_\infty}(\bar{s})$ and as a consequence $\bar{s} \in \partial\| \cdot \|_1(\bar{x})$. Let $u = (1, 0, -1, 0)$. Then $L = \{1, 2, 3\}$, $K = \{1, 2\}$ and $u_j = 0$ for all $j \in I^c = \{4\}$. Thus, $\partial^2 \left\|\cdot\right\|_1 (\bar{x}, \bar{s})(u) \neq \emptyset$ and it holds

$$\partial^2 \left\|\cdot\right\|_1 (\bar{x}, \bar{s})(u) = \left\{ v \in \mathbb{R}^4 \mid v_1 = v_2 = 0,\ v_3 \ge 0 \right\}.$$

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