Weierstraß-Institut
für Angewandte Analysis und Stochastik
Leibniz-Institut im Forschungsverbund Berlin e. V.

Preprint ISSN 2198-5855

Finite element method to fluid-solid interaction problems with unbounded periodic interfaces

Guanghui Hu[1], Andreas Rathsfeld[1], Tao Yin[2]

submitted: August 28, 2014

[1] Weierstrass Institute
Mohrenstr. 39
10117 Berlin
Germany
E-Mail: guanghui.hu@wias-berlin.de
andreas.rathsfeld@wias-berlin.de

[2] College of Mathematics and Statistics
Chongqing University
China
E-Mail: taoyin_cqu@163.com

No. 2002
Berlin 2014

2010 *Mathematics Subject Classification.* Primary 78A45 35Q74; Secondary 74F10, 35B27.

Key words and phrases. Fluid-solid interaction, periodic structure, variational approach, Helmholtz equation, Lamé system, convergence analysis, Rayleigh expansion.

The work of G. Hu is financed by the German Research Foundation (DFG) under Grant No. HU 2111/1-2. The work of T. Yin is partially supported by the China Scholarship Council and the NSFC Grant (11371385).

Edited by
Weierstraß-Institut für Angewandte Analysis und Stochastik (WIAS)
Leibniz-Institut im Forschungsverbund Berlin e. V.
Mohrenstraße 39
10117 Berlin
Germany

Fax: +49 30 20372-303
E-Mail: preprint@wias-berlin.de
World Wide Web: http://www.wias-berlin.de/

Abstract

Consider a time-harmonic acoustic plane wave incident onto a doubly periodic (biperiodic) surface from above. The medium above the surface is supposed to be filled with a homogeneous compressible inviscid fluid of constant mass density, whereas the region below is occupied by an isotropic and linearly elastic solid body characterized by its Lamé constants. This paper is concerned with a variational approach to the fluid-solid interaction problems with unbounded biperiodic Lipschitz interfaces between the domains of the acoustic and elastic waves. The existence of quasi-periodic solutions in Sobolev spaces is established at arbitrary frequency of incidence, while uniqueness is proved only for small frequencies or for all frequencies excluding a discrete set. A finite element scheme coupled with Dirichlet-to-Neumann mappings is proposed. The Dirichlet-to-Neumann mappings are approximated by truncated Rayleigh series expansions, and, finally, numerical tests in 2D are performed.

1 Introduction

Consider a time-harmonic acoustic plane wave incident onto an unbounded doubly periodic (or biperiodic) surface from above; see Figure 1. The medium above the surface is supposed to be filled with a homogeneous compressible inviscid fluid with a constant mass density, whereas the region below is occupied by an isotropic and linearly elastic solid body characterized by its Lamé constants. Due to the external incident acoustic field, an elastic wave propagating downward is incited inside the solid, while the incident acoustic wave is scattered back into the fluid. This leads to the fluid-solid interaction (FSI) problem with unbounded biperiodic interfaces separating the domains of acoustic and elastic waves. The problem has many applications in underwater acoustics, sonic and photonic crystals as well as in the field of ultrasonic non-destructive evaluation; see [7, 10, 20, 28] and the references therein. In particular, the investigation of surface (or Rayleigh) waves can be important in developing new surface acoustic wave devices and planar actuators ([10]). The application of periodic interfaces in the real world, e.g., grain structure, lamination and fiber reinforcement as well as in the manufacturing of material surfaces, motivates us to rigorously investigate FSI problems in periodic structures. Note that, so far, a vast literature has come from the engineering community only.

Since Lord Rayleigh's original work [23], grating diffraction problems have received much attention in both the physical and mathematical communities. Consequently, the scattering of pure acoustic, elastic or electromagnetic waves has been studied extensively including theoretical analysis and numerical approximation, using integral equation methods (e.g., [3, 25–27, 29]), variational methods (e.g., [1, 5, 6, 11, 12, 14, 17, 18, 21]) or the coupling scheme [2]. In particular, the variational approach appears to be well adapted to the analytical and numerical treatment of rather general two-dimensional and three-dimensional periodic diffractive structures involving complex materials and non-smooth interfaces. To investigate the FSI problem, we establish an equivalent variational formulation in a bounded periodic cell involving two nonlocal transparent boundary operators. Relying on properties of the Dirichlet-to-Neumann (DtN) maps for the Helmholtz and Navier equations, we show the existence of solutions in quasi-periodic Sobolev spaces by establishing the Fredholmness of the operator generated by the corresponding sesquilinear form. Moreover, uniqueness is proved at least for small frequencies or for all frequencies excluding a discrete set. A non-uniqueness example in Lemma 4.3 shows that uniqueness does not hold in general, even if the

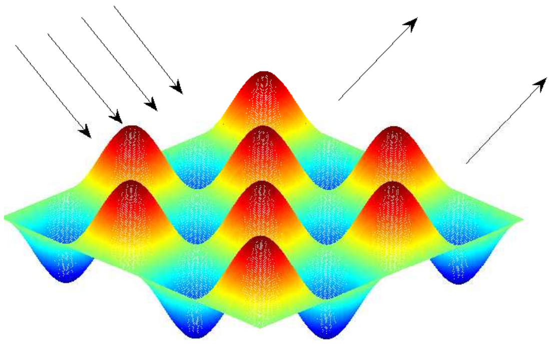


Figure 1: Scattering of plane waves from an egg-crate shaped biperiodic surface in $\mathbb{R}^3$.

interface is given by the graph of some smooth biperiodic function. This is in sharp contrast to the result in [21] for the pure Helmholtz equation and that in [13] for the pure Lamé system, where the uniqueness is proved via periodic Rellich's identities for a scattering interface given by the graph of some function. This suggests the possible existence of surface (Rayleigh or evanescent) waves in general settings, and a corresponding search for eigensolutions may help to design new surface wave devices. Based on the variational formulation, a finite element scheme with approximated Dirichlet-to-Neumann mappings in form of truncated Rayleigh series expansions is proposed. The numerical analysis is performed, and 2D examples are presented.

The paper is organized as follows. In Section 2 we rigorously formulate the interaction problems with biperiodic Lipschitz interfaces separating the domains of acoustic and elastic waves. In Section 3 we propose an equivalent variational formulation in a truncated periodic cell by introducing two non-local transparent operators. Section 4 is devoted to the solvability of the FSI problem through the variational approach. An energy balance formula will be stated in Section 5 and the numerical analysis of the finite element method is given in Section 6. In the final Sections 7 and 8 we introduce the corresponding two-dimensional setting and present numerical tests.

We end up this section by introducing some notation that will be used throughout the paper. Denote by $(\cdot)^\top$ the transpose of a vector or a matrix, and by $(\cdot)^*$ the adjoint of an operator. For $a \in \mathbb{C}$, let $|a|$ denote its modulus, and for $\mathbf{a} \in \mathbb{C}^3$, let $|\mathbf{a}|$ denote its Euclidean norm. The notation $\mathbf{a} \cdot \mathbf{b}$ stands for the inner product $\sum_{j=1}^{3} a_j b_j$ of $\mathbf{a} = (a_1, a_2, a_3)^\top$, $\mathbf{b} = (b_1, b_2, b_3)^\top \in \mathbb{C}^3$. For $x = (x_1, x_2, x_3)^\top \in \mathbb{R}^3$, we write $\tilde{x} = (x_1, x_2)^\top$ so that $x = (\tilde{x}^\top, x_3)^\top$.

2 Mathematical formulations

We assume that an acoustic wave is incident onto a biperiodic Lipschitz surface $\Gamma \subset \mathbb{R}^3$ from above. Without loss of generality we suppose that Γ is 2π-periodic in x_1 and x_2, i.e.,

$$x = (\tilde{x}^\top, x_3)^\top \in \Gamma \quad \Rightarrow \quad (\tilde{x}^\top + 2\pi\, n^\top, x_3)^\top \in \Gamma \quad \text{for all} \quad n = (n_1, n_2)^\top \in \mathbb{Z}^2.$$

Denote by Ω^+ the region above Γ, which is filled with a homogeneous compressible inviscid fluid with the constant mass density $\rho_f > 0$. The incident wave is supposed to be a time-harmonic plane wave of the form $v^{in}(x) \exp(-i\omega t)$ with frequency $\omega > 0$ and speed of sound $c_0 > 0$, where the spatially dependent function v^{in} takes the form

$$v^{in}(x) = \exp(ik\hat{\theta}\cdot x), \quad \hat{\theta} = (\sin\theta_1\cos\theta_2, \sin\theta_1\sin\theta_2, -\cos\theta_1)^\top \in \mathbb{S}^2 := \{x\in\mathbb{R}^3\colon |x|=1\}. \tag{1}$$

In (1), the vector $\hat{\theta}$ denotes the incident direction with the incident angles $\theta_1 \in (-\pi/2,\pi/2), \theta_2 \in [0,2\pi)$, and $k=\omega/c_0$ is the wave number in the fluid. We assume the region below Γ, denoted by Ω^-, is occupied by an isotropic and linearly elastic solid body characterized by the real valued constant mass density $\rho>0$ and the Lamé constants $\lambda,\mu\in\mathbb{R}$ satisfying $\mu>0$, $3\lambda+2\mu>0$.

Under the hypothesis of small amplitude oscillations both in the solid and the fluid, the direct or forward scattering problem looks for the total acoustic field $v=v^{in}+v^{sc}$ and the transmitted elastic field u generated from a known (prescribed) incident wave v^{in} such that (see e.g. [19,24,28])

$$\left\{\begin{array}{llll} (\Delta+k^2)\,v & = 0 & \text{in} & \Omega^+, \\ (\Delta^*+\omega^2\rho)\,u & = 0 & \text{in} & \Omega^-, \quad \Delta^* := \mu\Delta + (\lambda+\mu)\,\mathrm{grad}\,\mathrm{div}\,, \\ \eta\, u\cdot\nu & = \partial_\nu v & \text{on} & \Gamma, \qquad \eta := \rho_f\omega^2>0, \\ Tu & = -v\,\nu & \text{on} & \Gamma. \end{array}\right. \tag{2}$$

Here, the notation $\nu=(\nu_1,\nu_2,\nu_3)^\top\in\mathbb{S}^2$ denotes the unit normal vector on Γ pointing into Ω^- and $\partial_\nu u = \nu\cdot \mathrm{grad}\ u$. As a convention we shall use the symbol $\partial_j u$ to denote $\partial u/\partial x_j$. In (2), Tu stands for the three-dimensional *stress vector* or *traction* having the form:

$$Tu = T(\lambda,\mu)u := 2\mu\,\partial_\nu u + \lambda(\mathrm{div}\,u)\,\nu + \mu\,\nu\times \mathrm{curl}\ u \quad \text{on} \quad \Gamma. \tag{3}$$

By Betti's formula (see e.g., [22]), the role of the above stress operator in the Lamé equation is the same as that of the normal derivative in the scalar Helmholtz equation.

Throughout the paper, we write $\alpha=(\alpha_1,\alpha_2)^\top := k(\sin\theta_1\cos\theta_2, \sin\theta_1\sin\theta_2)^\top\in\mathbb{R}^2$. Obviously, the incident field v^{in} is α-quasiperiodic in the sense that $v^{in}(x)\exp(-i\alpha\cdot\tilde{x})$ is 2π-periodic with respect to x_1 and x_2. The periodicity of the structure together with the form of the incident wave implies that the solution (v,u) must also be α-quasiperiodic, i.e., for $w=v$ in Ω^+ and $w=u$ in Ω^- it holds that

$$w(\tilde{x}+2\pi\,n, x_3) = \exp(2\pi i\alpha\cdot n)\,w(x_1,x_2,x_3) \qquad \text{for all} \quad n=(n_1,n_2)^\top\in\mathbb{Z}^2. \tag{4}$$

Since the domain $\Omega^\pm$ is unbounded in the $\pm x_3$-direction, a radiation condition must be imposed at infinity to ensure well-posedness of the boundary value problem (2). Let

$$\Gamma^+ := \max_{x\in\Gamma}\{x_3\}, \quad \Gamma^- := \min_{x\in\Gamma}\{x_3\}.$$

Following [21], we require that the scattered acoustic field v^{sc} admits an upward Rayleigh expansion (see also [4,6,15])

$$v^{sc}(\tilde{x},x_3) \;=\; \sum_{n\in\mathbb{Z}^2} v_n\exp(i\alpha_n\cdot\tilde{x}+i\eta_n x_3), \quad x_3>\Gamma^+, \tag{5}$$

with the Rayleigh coefficients $v_n\in\mathbb{C}$. The parameters $\alpha_n=(\alpha_n^{(1)},\alpha_n^{(2)})^\top\in\mathbb{R}^2$ and $\eta_n\in\mathbb{C}$ in (5) are given by

$$\alpha_n=\alpha+n\in\mathbb{R}^2, \quad \eta_n=\left\{\begin{array}{ll}(k^2-|\alpha_n|^2)^{\frac12} & \text{if } |\alpha_n|\le k, \\ i(|\alpha_n|^2-k^2)^{\frac12} & \text{if } |\alpha_n|>k,\end{array}\right. \qquad \text{for } n\in\mathbb{Z}^2. \tag{6}$$

To see the corresponding expansion of the elastic field, we decompose it into the compressional and shear parts,

$$u = \frac{1}{i}\,(\operatorname{grad}\varphi + \operatorname{curl}\psi) \quad \text{with} \quad \varphi := -\frac{i}{k_p^2}\operatorname{div} u\,, \quad \psi := \frac{i}{k_s^2}\operatorname{curl} u\,, \tag{7}$$

where the scalar function φ and the vector function ψ satisfy the homogeneous Helmholtz equations

$$(\Delta + k_p^2)\,\varphi = 0 \quad \text{and} \quad (\Delta + k_s^2)\,\psi = 0 \quad \text{in} \quad \Omega^-\,, \tag{8}$$

with the compressional and shear wave numbers defined as

$$k_p := \omega\sqrt{\rho/(2\mu+\lambda)}\,, \quad k_s := \omega\sqrt{\rho/\mu}.$$

Applying the downward Rayleigh expansion for the scalar Helmholtz equation to φ and the components of ψ, i.e.,

$$\varphi(\tilde{x}, x_3) = \sum_{n\in\mathbb{Z}^2} \varphi_{p,n} \exp(i\alpha_n\cdot\tilde{x} - i\beta_n x_3), \quad \psi(\tilde{x}, x_3) = \sum_{n\in\mathbb{Z}^2} \Psi_{s,n} \exp(i\alpha_n\cdot\tilde{x} - i\gamma_n x_3)$$

with $\Psi_{s,n}\cdot(\alpha_n^\top, -\gamma_n)^\top = 0$, we finally obtain the corresponding expansion of u into downward propagating plane elastic waves

$$u(x) = \sum_{n\in\mathbb{Z}^2} \left\{ A_{p,n} \begin{pmatrix} \alpha_n \\ -\beta_n \end{pmatrix} \exp(i\alpha_n\cdot\tilde{x} - i\beta_n x_3) + \mathbf{A}_{s,n} \exp(i\alpha_n\cdot\tilde{x} - i\gamma_n x_3) \right\}, \ x_3 < \Gamma^-. \tag{9}$$

In (9), the Rayleigh coefficients are given as

$$A_{p,n} := \varphi_{p,n} \in \mathbb{C}, \quad \mathbf{A}_{s,n} := \begin{pmatrix} \alpha_n \\ -\gamma_n \end{pmatrix} \times \Psi_{s,n} \in \mathbb{C}^3.$$

In particular, we have the orthogonality

$$\mathbf{A}_{s,n}\cdot \begin{pmatrix} \alpha_n \\ -\gamma_n \end{pmatrix} = 0 \qquad \text{for all } n \in \mathbb{Z}^2. \tag{10}$$

The parameters β_n and γ_n occurring (9) are defined analogously to η_n in (6) with k replaced by k_p and k_s, respectively. By u_p and u_s we denote the compressional and shear parts of u, respectively, i.e., for $x_3 < \Gamma^-$,

$$u_p(x) = \sum_{n\in\mathbb{Z}^2} A_{p,n} \begin{pmatrix} \alpha_n \\ -\beta_n \end{pmatrix} \exp(i\alpha_n\cdot\tilde{x} - i\beta_n x_3), \quad u_s(x) = \sum_{n\in\mathbb{Z}^2} \mathbf{A}_{s,n} \exp(i\alpha_n\cdot\tilde{x} - i\gamma_n x_3).$$

Then, it is obvious that $u = u_p + u_s$ and

$$(\Delta + k_p^2)\,u_p = 0, \quad \operatorname{curl} u_p = 0, \quad (\Delta + k_s^2)\,u_s = 0, \quad \operatorname{div} u_s = 0 \quad \text{in} \quad \Omega^-.$$

Since η_n, β_n and γ_n are real for at most finitely many indices $n \in \mathbb{Z}^2$, we observe that only the finite number of plane waves in (5) corresponding to $|\eta_n| \le k$ and those in (9) corresponding to $|\beta_n| \le k_p$ and $|\gamma_n| \le k_s$ propagate into the far field. These plane waves are referred to as the upward and downward outgoing plane waves, respectively. The remaining part consists of evanescent (or surface) waves decaying exponentially as $|x_3| \to +\infty$. Thus, the Rayleigh expansion (5) converges uniformly with all derivatives in the upper half-space $\{x : x_3 > b\}$ for any $b > \Gamma^+$, while (9) converges in the lower half-space $\{x : x_3 < a\}$ for any $a < \Gamma^-$.

Now, we can formulate our FSI problem as the following boundary value problem, in which the interface Γ is not necessarily the graph of a biperiodic function.

(BVP): Given a biperiodic Lipschitz surface $\Gamma \subset \mathbb{R}^3$ (which is 2π-periodic in x_1 and x_2 and which splits $\mathbb{R}^3$ into an upper and lower half space) and an incident field v^{in} of the form (1), find a scalar function $v = v^{in} + v^{sc} \in H^1_{loc}(\Omega^+)$ and a vector function $u \in H^1_{loc}(\Omega^-)^3$ that satisfy the equations and transmission conditions in (2), the quasi-periodic boundary condition (4) and the radiation conditions, i.e., that u and v admit the Rayleigh expansions in (5) and (9), respectively.

3 Variational formulation in a truncated domain

In this section we propose a variational formulation equivalent to (BVP), based on the approach of [15,21] and [12,13] for the scattering of acoustic and elastic waves by diffraction gratings. Thanks to the periodicity of the unbounded domains $\Omega^{\pm}$, we can restrict our discussions to one single periodic cell $\{x : 0 < x_j < 2\pi, j = 1, 2\}$ such that after a truncation in the x_3-direction the compact imbedding of Sobolev spaces can be applied. This, together with Friedrich's inequality for the Helmholtz equation and Korn's inequality for the Navier equation, enables us to justify the strong ellipticity of the sesquilinear form generated by the variational formulation.

We begin with introducing artificial boundaries

$$\Gamma_b^{\pm} := \{(x_1, x_2, \pm b) : \ 0 \le x_1, x_2 \le 2\pi\}, \quad \pm b \gtrless \Gamma^{\pm},$$

and the bounded domains

$$\Omega_b^{\pm} := \{x \in \Omega^{\pm} : \ 0 < x_1, x_2 < 2\pi, \ x_3 \lessgtr \pm b\}.$$

For simplicity we still use Γ to denote one period of the grating surface; see Figure 2. Since Γ is a Lipschitz surface, we may restrict our considerations to the case that $\Omega_b^{\pm}$ are bounded Lipschitz domains in $\mathbb{R}^3$. Let $H^1_\alpha(\Omega_b^{\pm})$ denote the Sobolev space of scalar functions on $\Omega_b^{\pm}$ which are α-quasiperiodic with respect to x_1 and x_2.

Introduce the family of product spaces (including the energy space V_1)

$$V_t = V_t(\alpha) := V_t^+ \times V_t^-, \quad V_t^+ := H^t_\alpha(\Omega_b^+), \quad V_t^- := H^t_\alpha(\Omega_b^-)^3,$$

equipped with the norm in the usual product space of $H^t(\Omega_b^+) \times H^t(\Omega_b^-)^3$. Using the transmission conditions in (2), it follows from Green's and Betti's formulas that, for $(\varphi, \psi) \in V_1$,

$$\begin{aligned} -\int_{\Omega_b^+} (\Delta + k^2) v\, \overline{\varphi}\, dx &= \int_{\Omega_b^+} \left[\operatorname{grad} v \cdot \operatorname{grad} \overline{\varphi} - k^2 v \overline{\varphi}\right] dx - \eta \int_\Gamma u \cdot \nu \overline{\varphi}\, ds - \int_{\Gamma_b^+} \partial_\nu v\, \overline{\varphi}\, ds\,, \\ -\int_{\Omega_b^-} (\Delta^* + \omega^2 \rho) u \cdot \overline{\psi}\, dx &= \int_{\Omega_b^-} \left[\mathcal{E}(u, \overline{\psi}) - \omega^2 \rho u \cdot \overline{\psi}\right] dx - \int_\Gamma v\, \nu \cdot \overline{\psi} ds - \int_{\Gamma_b^-} T u \cdot \overline{\psi}\, ds\,, \end{aligned} \tag{11}$$

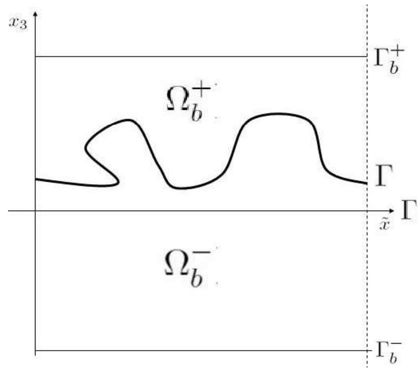


Figure 2: The geometry settings in one periodic cell. Here $\Gamma_b^{\pm} := \{(x_1, x_2, \pm b)^\top : \ 0 < x_1, x_2 < 2\pi\}$ and $\Omega_b^{\pm}$ denotes the domain between $\Gamma_b^{\pm}$ and Γ.

where the bar indicates the complex conjugate, T is the stress vector defined by (3) and

$$\mathcal{E}(u, \overline{\psi}\ ) := 2\mu \left(\sum_{i,j=1}^{3} \partial_i u_j \ \partial_i \overline{\psi}_j \right) + \lambda\, (\operatorname{div} u)(\operatorname{div} \overline{\psi}) - \mu \operatorname{curl} u \cdot \operatorname{curl} \overline{\psi}. \tag{12}$$

Now we introduce the DtN maps $\mathcal{T}^{\pm}$ on the artificial boundaries $\Gamma_b^{\pm}$.

Definition 3.1. *For any $w \in H_\alpha^s(\Gamma_b^+)$, $s > 0$, the DtN operator $\mathcal{T}^+$ applied to w is defined as $\partial_\nu v^{sc}|_{\Gamma_b^+}$, where v^{sc} is the unique α-quasiperiodic solution of the homogeneous Helmholtz equation in $x_3 > b$ which satisfies the upward radiation condition (5) and has the Dirichlet boundary value $v^{sc} = w$ on Γ_b^+.*

Analogously, for any $w \in H_\alpha^s(\Gamma_b^-)^3$, $s > 0$, the DtN operator $\mathcal{T}^-$ applied to w is defined as $Tu|_{\Gamma_b^-}$, where u is the unique α-quasiperiodic solution of the homogeneous Navier equation in $x_3 < -b$ which satisfies the downward radiation condition (9) and takes the Dirichlet boundary value $u = w$ on Γ_b^-.

In this paper we employ the following equivalent norm on $H_\alpha^s(\mathbb{R}^2)$:

$$\|w\|_{H_\alpha^s(\mathbb{R}^2)} = \Big(\sum_{n \in \mathbb{Z}^2} (1 + |n|)^{2s}\, |\hat{w}_n|^2 \Big)^{1/2}, \quad s \in \mathbb{R}, \tag{13}$$

where $\hat{w}_n \in \mathbb{C}$ are the Fourier coefficients of $\exp(-i\alpha \cdot \tilde{x})\, w(\tilde{x})$, that is,

$$w(\tilde{x}) = \sum_{n \in \mathbb{Z}^2} \hat{w}_n \exp(i\alpha_n \cdot \tilde{x}). \tag{14}$$

Letting $w \in H_\alpha^s(\Gamma_b^+)$, $s > 0$ be given as above, one can readily derive an explicit expression of the DtN map $\mathcal{T}^+$ from its definition as follows:

$$(\mathcal{T}^+ w)(\tilde{x}) = \sum_{n \in \mathbb{Z}^2} i\eta_n\, \hat{w}_n \exp(i\alpha_n \cdot \tilde{x}), \tag{15}$$

where η_n is defined as in (6). Analogously, for $w \in H_\alpha^s(\Gamma_b^-)^3$, $s > 0$ of the form (14) with $\hat{w}_n \in \mathbb{C}^3$, we can represent the DtN map $\mathcal{T}^-$ as

$$(\mathcal{T}^- w)(\tilde{x}) = \sum_{n \in \mathbb{Z}^2} iW_n\, \hat{w}_n \exp(i\alpha_n \cdot \tilde{x}), \tag{16}$$

where W_n is the 3×3 matrix given by

$$W_n = W_n(\omega, \rho, \alpha) := \frac{1}{|\alpha_n|^2 + \beta_n \gamma_n} \begin{pmatrix} a_n & b_n & -c_n \\ b_n & d_n & -e_n \\ c_n & e_n & f_n \end{pmatrix}, \tag{17}$$

with

$$\begin{aligned} a_n &:= \mu\big[(\gamma_n - \beta_n)(\alpha_n^{(2)})^2 + k_s^2 \beta_n\big], \qquad & b_n &:= -\mu \alpha_n^{(1)} \alpha_n^{(2)} (\gamma_n - \beta_n), \\ c_n &:= \big(2\mu\alpha_n^2 - \omega^2\rho + 2\mu\gamma_n\beta_n\big)\alpha_n^{(1)}, \qquad & e_n &:= \big(2\mu\alpha_n^2 - \omega^2\rho + 2\mu\gamma_n\beta_n\big)\alpha_n^{(2)}, \\ d_n &:= \mu\big[(\gamma_n - \beta_n)(\alpha_n^{(1)})^2 + k_s^2 \beta_n\big], \qquad & f_n &:= \gamma_n \omega^2 \rho. \end{aligned}$$

The expression of $\mathcal{T}^+$ is well-known (see [15,21]), whereas that of $\mathcal{T}^-$ can be derived following the way in [13] for upward propagating elastic waves. Throughout the paper we assume ω is not an exceptional frequency, i.e.,

$$\omega \notin \mathcal{D}_0 := \big\{\omega : \exists n \in \mathbb{Z}^2 \text{ s.t. } |\alpha_n(\omega)|^2 + \beta_n(\omega)\gamma_n(\omega) = 0\big\}, \tag{18}$$

so that the denominator of (17) never vanishes. The condition (18) can be guaranteed if ω is sufficiently small or if the relation $\lambda + 2\mu \le \rho c_0^2$ (equivalently $k \le k_p$) holds; see Theorem 4.4 (ii).

Remark 3.2. *The condition $\omega \notin \mathcal{D}_0$ is a technical assumption only. If $\omega \in \mathcal{D}_0$ is an exceptional frequency, then the DtN mapping is to be modified. For simplicity, we assume that the condition $|\alpha_n(\omega)|^2 + \beta_n(\omega)\gamma_n(\omega) = 0$ is satisfied if and only if $n = n_\#$ for a fixed $n_\# \in \mathbb{Z}^2$. Then we introduce the modified DtN map*

$$(\mathcal{T}_\#^- w)(\tilde{x}) = \sum_{n \in \mathbb{Z}^2:\, n \neq n_\#} i W_n\, \hat{w}_n \exp(i\alpha_n \cdot \tilde{x}).$$

The subsequent sesquilinear form A in (23) is to be modified as follows. We replace the last term $-\int_{\Gamma_b^-} \mathcal{T}^- u \cdot \overline{\psi}\, ds$ in the square bracket by

$$-\int_{\Gamma_b^-} \mathcal{T}_\#^- u \cdot \overline{\psi}\, ds - \Psi_\#(u) \ \cdot \int_{\Gamma_b^-} \phi_\# \overline{\psi}\, ds \ ,$$

where $\Psi_\# : \ V_1 \longrightarrow \mathbb{C}^3$ is a continuous linear vector-valued functional and where the function $\phi_\# : \mathbb{R}^3 \to \mathbb{C}$ is defined by $\phi_\#(\tilde{x}, x_3) := \frac{1}{2\pi}\exp(i\alpha_{n_\#} \cdot \tilde{x})$. For $\Psi_\#$ and the trace of the traction operator T, we have to require $\Psi_\#(w_j^-) = \int_{\Gamma_b^-} \overline{\phi_\#}\, T w_j^-$ and $\Psi_\#(w_j^+) = \int_{\Gamma_b^-} \overline{\phi_\#}\, w_{j+3}^+$ for $j = 1, 2, 3$, where

$$\begin{aligned} w_1^\pm(x) &:= \big(\alpha_{n_\#}^\top, \pm\beta_{n_\#}\big)^\top \exp\big(i(\alpha_{n_\#} \cdot \tilde{x} \pm \beta_{n_\#}[x_3 + b])\big), \\ w_2^\pm(x) &:= \big(\alpha_{n_\#}^{(2)}, -\alpha_{n_\#}^{(1)}, 0\big)^\top \exp\big(i(\alpha_{n_\#} \cdot \tilde{x} \pm \gamma_{n_\#}[x_3 + b])\big), \\ w_3^\pm(x) &:= \big(\alpha_{n_\#}^\top, \pm\beta_{n_\#}\big)^\top \exp\big(i(\alpha_{n_\#} \cdot \tilde{x} \pm \gamma_{n_\#}[x_3 + b])\big). \end{aligned}$$

The functions $w_j^+,\ j = 4, 5, 6$ are chosen as constant vectors multiplied by $\phi_\#$ such that the mapping $\mathbb{C}^3 \ni (\lambda_j)_{j=1}^3 \mapsto \sum_{j=1}^3 \lambda_j [T(w_j^+) - w_{j+3}^+]|_{\Gamma_b^+}$ has a trivial kernel.

Setting $a := \frac{1}{2}(-b+\Gamma^-)$ *and* $\Gamma_a^- := \{x \in \mathbb{R}^3 :\ 0 < x_1, x_2 < 2\pi,\ x_3 = a\}$*, we can choose, e.g.,*

$$
\begin{aligned}
&\Psi_\#(u) := \\
&\quad \frac{1}{8\pi^2 \beta_{n_\#} |\alpha_{n_\#}|^2} \, \frac{1}{\exp(-i\beta_{n_\#}(a+b)) - \exp(-i\gamma_{n_\#}(a+b))} \bigg\{ \int_{\Gamma_a^-} \overline{\phi_\#}\, u \cdot (\beta_{n_\#} \alpha_{n_\#}^\top, -|\alpha_{n_\#}|^2)^\top \\
&\qquad\qquad - \exp(-i\gamma_{n_\#}(a+b)) \int_{\Gamma_b^-} \overline{\phi_\#}\, u \cdot (\beta_{n_\#} \alpha_{n_\#}^\top, -|\alpha_{n_\#}|^2)^\top \bigg\} \int_{\Gamma_b^-} \overline{\phi_\#}\, T w_1^- \\
&\quad + \frac{1}{4\pi^2 |\alpha_{n_\#}|^2} \, \frac{1}{\exp(-i\gamma_{n_\#}(a+b)) - \exp(i\gamma_{n_\#}(a+b))} \bigg\{ \int_{\Gamma_a^-} \overline{\phi_\#}\, u \cdot (\alpha_{n_\#}^{(2)}, -\alpha_{n_\#}^{(1)}, 0)^\top \\
&\qquad\qquad - \exp(i\gamma_{n_\#}(a+b)) \int_{\Gamma_b^-} \overline{\phi_\#}\, u \cdot (\alpha_{n_\#}^{(2)}, -\alpha_{n_\#}^{(1)}, 0)^\top \bigg\} \int_{\Gamma_b^-} \overline{\phi_\#}\, T w_2^- \\
&\quad + \frac{1}{8\pi^2 \beta_{n_\#} |\alpha_{n_\#}|^2} \, \frac{1}{\exp(-i\gamma_{n_\#}(a+b)) - \exp(-i\beta_{n_\#}(a+b))} \bigg\{ \int_{\Gamma_a^-} \overline{\phi_\#}\, u \cdot (\beta_{n_\#} \alpha_{n_\#}^\top, -|\alpha_{n_\#}|^2)^\top \\
&\qquad\qquad - \exp(-i\beta_{n_\#}(a+b)) \int_{\Gamma_b^-} \overline{\phi_\#}\, u \cdot (\beta_{n_\#} \alpha_{n_\#}^\top, -|\alpha_{n_\#}|^2)^\top \bigg\} \int_{\Gamma_b^-} \overline{\phi_\#}\, T w_3^- \\
&\quad + \frac{1}{8\pi^2 \beta_{n_\#} |\alpha_{n_\#}|^2} \, \frac{1}{\exp(i\beta_{n_\#}(a+b)) - \exp(i\gamma_{n_\#}(a+b))} \bigg\{ \int_{\Gamma_a^-} \overline{\phi_\#}\, u \cdot (\beta_{n_\#} \alpha_{n_\#}^\top, |\alpha_{n_\#}|^2)^\top \\
&\qquad\qquad - \exp(i\gamma_{n_\#}(a+b)) \int_{\Gamma_b^-} \overline{\phi_\#}\, u \cdot (\beta_{n_\#} \alpha_{n_\#}^\top, |\alpha_{n_\#}|^2)^\top \bigg\} \int_{\Gamma_b^-} \overline{\phi_\#}\, w_4^+ \\
&\quad + \frac{1}{4\pi^2 |\alpha_{n_\#}|^2} \, \frac{1}{\exp(i\gamma_{n_\#}(a+b)) - \exp(-i\gamma_{n_\#}(a+b))} \bigg\{ \int_{\Gamma_a^-} \overline{\phi_\#}\, u \cdot (\alpha_{n_\#}^{(2)}, -\alpha_{n_\#}^{(1)}, 0)^\top \\
&\qquad\qquad - \exp(-i\gamma_{n_\#}(a+b)) \int_{\Gamma_b^-} \overline{\phi_\#}\, u \cdot (\alpha_{n_\#}^{(2)}, -\alpha_{n_\#}^{(1)}, 0)^\top \bigg\} \int_{\Gamma_b^-} \overline{\phi_\#}\, w_5^+ \\
&\quad + \frac{1}{8\pi^2 \beta_{n_\#} |\alpha_{n_\#}|^2} \, \frac{1}{\exp(i\gamma_{n_\#}(a+b)) - \exp(i\beta_{n_\#}(a+b))} \bigg\{ \int_{\Gamma_a^-} \overline{\phi_\#}\, u \cdot (\beta_{n_\#} \alpha_{n_\#}^\top, |\alpha_{n_\#}|^2)^\top \\
&\qquad\qquad - \exp(i\beta_{n_\#}(a+b)) \int_{\Gamma_b^-} \overline{\phi_\#}\, u \cdot (\beta_{n_\#} \alpha_{n_\#}^\top, |\alpha_{n_\#}|^2)^\top \bigg\} \int_{\Gamma_b^-} \overline{\phi_\#}\, w_6^+ .
\end{aligned}
$$

Remark 3.3. *Suppose that* w *satisfies the upward* α*-quasiperiodic Rayleigh expansion*

$$
w(x) = \sum_{n \in \mathbb{Z}^2} \left\{ A_{p,n} \begin{pmatrix} \alpha_n \\ \beta_n \end{pmatrix} \exp(i\alpha_n \cdot \tilde{x} + i\beta_n x_3) + \boldsymbol{A}_{s,n} \exp(i\alpha_n \cdot \tilde{x} + i\gamma_n x_3) \right\}, \ x_3 > \Gamma^+,
$$

as a solution to the Navier equation in Ω^+*, with the Rayleigh coefficients* $A_{p,n} \in \mathbb{C}$, $\boldsymbol{A}_{s,n} \in \mathbb{C}^3$ *such that* $\boldsymbol{A}_{s,n} \cdot (\alpha_n^\top, \gamma_n)^\top = 0$*. Then one can prove that (see [13, Lemma 1])*

$$
(Tw)|_{\Gamma_b^+} = \sum_{n \in \mathbb{Z}^2} i\, W_n^\top\, \hat{w}_n \exp(i\alpha_n \cdot \tilde{x}),
$$

where $\hat{w}_n$ *denotes the Fourier coefficient of* $\exp(-i\alpha \cdot \tilde{x})\, w(\tilde{x}, b)$ *of order* n*. Hence, the matrix in (17) differs from that in [13] only in the signs before* c_n *and* e_n*.*

Making use of the norm (13) and the asymptotic behavior

$$\eta_n, \beta_n, \gamma_n \sim i|n|, \quad |\beta_n - \gamma_n| \sim \frac{1}{|n|^2}\frac{k_s^2 - k_p^2}{2}, \quad |\alpha_n|^2 + \beta_n\gamma_n \sim \frac{k_p^2 + k_s^2}{2} \quad \text{as } |n| \to \infty,$$

one can straightforwardly verify that

$$\mathcal{T}^+ : H_\alpha^s(\mathbb{R}^2) \to H_\alpha^{s-1}(\mathbb{R}^2), \quad \mathcal{T}^- : H_\alpha^s(\mathbb{R}^2)^3 \to H_\alpha^{s-1}(\mathbb{R}^2)^3, \quad s > 0$$

are both bounded operators. Moreover, the operator $-\mathrm{Re}\,\mathcal{T}^+$ is positive semidefinite over $H_\alpha^s(\Gamma_b^+)$, i.e.,

$$-\mathrm{Re}\int_{\Gamma_b^+} \mathcal{T}^+ w\,\overline{w}\,ds = 4\pi^2 \sum_{|\alpha_n|\geq k} |\eta_n|\,|\hat{w}_n|^2 \geq 0 \quad \text{for all} \quad w \in H_\alpha^s(\Gamma_b^+). \tag{19}$$

Unfortunately, the positive semidefiniteness of $-\mathrm{Re}\,\mathcal{T}^-$ over $H_\alpha^s(\Gamma_b^-)^3$ does not hold in general (see [12,13]). With the definitions of $\mathcal{T}^\pm$, we can reformulate the terms $\partial_\nu v$ and Tu on the right hand sides of (11) as

$$(\partial_\nu v)|_{\Gamma_b^+} = f_0 + \mathcal{T}^+(v|_{\Gamma_b^+}), \quad (Tu)|_{\Gamma_b^-} = \mathcal{T}^-(u|_{\Gamma_b^-}), \tag{20}$$

with

$$f_0 := (\partial_\nu v^{in})|_{\Gamma_b^+} - \mathcal{T}^+(v^{in}|_{\Gamma_b^+}), \quad f_0(\tilde{x}) = -2i\eta_0 \exp(i\alpha\cdot\tilde{x} - i\eta_0 b) \in H_\alpha^{-1/2}(\Gamma_b^+), \tag{21}$$

which follows from the expression of v^{in} in (1). Combining (20) and (11), we obtain the following variational formulation of (BVP): Find $(v,u) \in V_1$ such that

$$A\big((v,u),(\varphi,\psi)\big) = \int_{\Gamma_b^+} f_0\,\overline{\varphi}\,ds \quad \text{for all } (\varphi,\psi) \in V_1, \tag{22}$$

where the sesquilinear form $A : V_1 \times V_1 \to \mathbb{C}$ is defined as

$$\begin{aligned} A\big((v,u),(\varphi,\psi)\big) \;:=\; & \int_{\Omega_b^+} \left[\mathrm{grad}\,v\cdot\mathrm{grad}\,\overline{\varphi} - k^2 v\overline{\varphi}\,\right] dx - \eta\int_\Gamma u\cdot\nu\,\overline{\varphi}\,ds - \int_{\Gamma_b^+} \mathcal{T}^+ v\,\overline{\varphi}\,ds \\ & +\eta\left[\int_{\Omega_b^-} \left[\mathcal{E}(u,\overline{\psi}) - \omega^2\rho u\cdot\overline{\psi}\,\right] dx - \int_\Gamma v\,\nu\cdot\overline{\psi}ds - \int_{\Gamma_b^-} \mathcal{T}^- u\cdot\overline{\psi}ds\right] \end{aligned} \tag{23}$$

for all $(\varphi,\psi) \in V_1$. The above sesquilinear form obviously generates a continuous linear operator $\mathcal{A} : V_1 \to V_1'$ such that

$$A\big((v,u),(\varphi,\psi)\big) = \langle \mathcal{A}(v,u),(\varphi,\psi)\rangle \quad \text{for all } (\varphi,\psi) \in V_1. \tag{24}$$

Here V_1' denotes the dual space of V_1 with respect to the duality $\langle\cdot,\cdot\rangle$ extending the product in $L^2(\Omega_b^+) \times L^2(\Omega_b^-)^3$.

4 Solvability results

Having established the equivalent variational formulation in a truncated domain in Section 3, the purpose of this section is to derive uniqueness and existence of weak solutions to the variational equation (24). We first prove the strong ellipticity of the sesquilinear form A.

Lemma 4.1. *The sesquilinear form A defined in (23) is strongly elliptic over V_1, and the operator $\mathcal{A}$ defined by (24) is always a Fredholm operator with index zero.*

Proof. Since the matrix $-\mathrm{Re}\,(iW_n^\top)$ is positive for large $|n|$ (see [13, Lemma 2]), the operator $-\mathrm{Re}\,(\mathcal{T}^-)$ can be decomposed into the sum of a positive definite operator $\mathcal{T}_1$ and a finite rank operator $\mathcal{T}_2$ from $H_\alpha^{1/2}(\Gamma_b^-)^3$ to $H_\alpha^{-1/2}(\Gamma_b^-)^3$. We split the sesquilinear form A into the sum $A = A_1 + A_2$, where

$$
\begin{aligned}
A_1\big((v,u),(\varphi,\psi)\big) &:= \int_{\Omega_b^+} [\mathrm{grad}\, v\cdot \mathrm{grad}\,\overline{\varphi} + v\overline{\varphi}\,]\,dx - \int_{\Gamma_b^+} \mathcal{T}^+ v\,\overline{\varphi}\,ds \\
&\quad +\eta\left[\int_{\Omega_b^-} \big[\mathcal{E}(u,\overline{\psi}) + u\cdot\overline{\psi}\,\big]\,dx + \int_{\Gamma_b^-} \mathcal{T}_1\, u\cdot\overline{\psi} ds\right], \\
A_2\big((v,u),(\varphi,\psi)\big) &:= -\int_{\Omega_b^+} \big[(1+k^2)v\overline{\varphi}\,\big]\,dx - \eta\int_\Gamma u\cdot\nu\,\overline{\varphi}\,ds \\
&\quad +\eta\left[\int_{\Omega_b^-} \big[-(1+\omega^2\rho)u\cdot\overline{\psi}\,\big]\,dx - \int_\Gamma v\,\nu\cdot\overline{\psi}\,ds + \int_{\Gamma_b^-} \mathcal{T}_2 u\cdot\overline{\psi}\,ds\right].
\end{aligned}
$$

Recalling (19) and Korn's inequality (see, e.g., [19, Chap. 5.4] or [12]), we have

$$
\mathrm{Re}\,A_1\big((v,u),(v,u)\big) \ge c_1\left(||v||^2_{V_1^+} + ||u||^2_{V_1^-}\right) \quad \text{for all } (v,u)\in V_1,
$$

with some constant $c_1 > 0$. Moreover, applying the Cauchy-Schwarz inequality yields

$$
\begin{aligned}
\mathrm{Re}\,A_2\big((v,u),(v,u)\big) \;&\ge\; -c_2\left(||v||^2_{L^2(\Omega_b^+)} + ||v||^2_{L^2(\Gamma)} + ||u||^2_{L^2(\Omega_b^-)^3} + ||u||^2_{L^2(\Gamma)^3}\right) \\
&\quad +\eta\,\mathrm{Re}\,(\mathcal{T}_2 u, u)_{L^2(\Gamma_b^-)^3},
\end{aligned}
$$

for some constant $c_2 > 0$. From the compact imbeddings $H^1(\Omega_b^\pm) \hookrightarrow L^2(\Omega_b^\pm)$, $H^{1/2}(\Gamma) \hookrightarrow L^2(\Gamma)$ and the compactness of $\mathcal{T}_2$, we conclude that the sesquilinear form A is strongly elliptic over $V_1 \times V_1$. Consequently, the operator $\mathcal{A}$ defined by (24) is always a Fredholm operator with index zero. □

From Lemma 4.1 and the Fredholm alternative, it follows that the variational formulation (22) is uniquely solvable provided the homogeneous operator equation $\mathcal{A}(v,u) = 0$ has only the trivial solutions $v = 0, u = 0$. However, uniqueness cannot be proved in the general case. It will be shown below that only the upward outgoing modes of v^{sc} and the downward outgoing modes of u can be uniquely determined, whereas the other evanescent modes maybe non-unique.

Lemma 4.2. *Assume $(v^{sc}, u) \in V_1$ is a solution pair to the variational problem (22) with $v^{in} = 0$ (or equivalently, $f_0 = 0$). Then there holds*

$$
v_n = 0 \quad \textit{for} \quad |\alpha_n| < k, \quad A_{p,n} = 0 \quad \textit{for} \quad |\alpha_n| < k_p, \quad |\boldsymbol{A}_{s,n}| = 0 \quad \textit{for} \quad |\alpha_n| < k_s,
$$

where v_n, $A_{p,n}$ and $\boldsymbol{A}_{s,n}$ denote the Rayleigh coefficients of v^{sc} and u (see (5) and (9)).

Proof. Taking the imaginary part of (22) with $\varphi = v^{sc}, \psi = u, v^{in} = 0$ and using the fact that $\eta > 0$, we get

$$-\mathrm{Im}\left(\mathcal{T}^+ v^{sc}, v^{sc}\right)_{L^2(\Gamma_b^+)} - \eta\,\mathrm{Im}\left(\mathcal{T}^- u, u\right)_{L^2(\Gamma_b^-)^3} = 0. \tag{25}$$

From the explicit expressions for $\mathcal{T}^+$ and $\mathcal{T}^-$, we can derive that

$$\begin{aligned}
\mathrm{Im}\left(\mathcal{T}^+ v^{sc}, v^{sc}\right)_{L^2(\Gamma_b^+)} &= 4\pi^2 \sum_{n:|\alpha_n|<k} \eta_n\,|v_n|^2,\\
\mathrm{Im}\left(\mathcal{T}^- u, u\right)_{L^2(\Gamma_b^-)} &= 4\pi^2\Big(\sum_{n:|\alpha_n|<k_p} \beta_n\,|A_{p,n}|^2\,\omega^2\rho + \sum_{n:|\alpha_n|<k_s} \gamma_n\,|\mathbf{A}_{s,n}|^2\mu\Big),
\end{aligned} \tag{26}$$

where the second equality follows from the arguments in proving [13, Lemma 3]. Since $\eta_n > 0$ for $|\alpha_n| < k$, $\beta_n > 0$ for $|\alpha_n| < k_p$ and $\gamma_n > 0$ for $|\alpha_n| < k_s$, we complete the proof of Lemma 4.2 by combining (25) and (26). □

Using the arguments of the above proof, we cannot prove uniqueness of solutions to (22) for general biperiodic Lipschitz interfaces separating domains of the fluid and solid. Moreover, uniqueness does not hold in general, even if Γ is the graph of a smooth biperiodic function. To see this, we construct a non-uniqueness example where Γ is a flat surface parallel to the x_1x_2-plane.

Lemma 4.3. *Assume that $\Gamma = \Gamma_0 := \{x\colon x_3 = 0\}$ is a flat interface, the incident angle $\theta_2 = 0$ and that $k = k_p = k\sin\theta_1 + m_0$ for some $m_0 \in \mathbb{Z}$. Then there exists at least one non-trivial solution pair $(v^{sc}, u) \in V_1$ to the homogeneous variational problem $A((v^{sc}, u), (\varphi, \psi)) = 0$ for all $(\varphi, \psi) \in V$.*

Proof. Observing that the interface Γ_0 is invariant in x_2 and the incident direction $\hat{\theta} = (\sin\theta_1, 0, -\cos\theta_1)$ is orthogonal to the x_2-axis, the original three-dimensional scattering problem reduces to a two-dimensional problem in the x_1x_3-plane. Consequently, we look for upward and downward Rayleigh expansion solutions v^{sc} and u of the special form

$$\begin{aligned}
v^{sc}(x) &= \sum_{m\in\mathbb{Z}} v_m\, e^{i(\tilde{\alpha}_m x_1 + \eta_m x_3)}, \quad x_3 > 0,\\
u(x) &= \sum_{m\in\mathbb{Z}} \left(A_{p,m}\begin{pmatrix}\tilde{\alpha}_m\\ 0\\ -\beta_m\end{pmatrix} e^{i(\tilde{\alpha}_m x_1 - \beta_m x_3)} + A_{s,m}\begin{pmatrix}\gamma_m\\ 0\\ \tilde{\alpha}_m\end{pmatrix} e^{i(\tilde{\alpha}_m x_1 - \gamma_m x_3)}\right), \quad x_3 < 0,
\end{aligned}$$

with $v_m, A_{p,m}, A_{s,m} \in \mathbb{C}$, $\tilde{\alpha}_m := \alpha_1 + m = \alpha_n^{(1)}$ for $n = (m, 0)$. Here, $\alpha_1 = k\sin\theta_1$ due to the assumption that $\theta_2 = 0$. The parameters $\eta_m, \beta_m, \gamma_m$ for $m \in \mathbb{Z}$ are defined in the same way as $\eta_n, \beta_n, \gamma_n$ (see (6)) with $n = (m, 0)$ and $\alpha = (\alpha_1, 0)^\top$. Note that the solution pair (v^{sc}, u) does not depend on x_2.

Elementary calculations show that, using $\nu = (0, 0, -1)$ on Γ_0,

$$\begin{aligned}
(Tu)(x)|_{\Gamma_0} &= i\sum_{m\in\mathbb{Z}} \begin{pmatrix} 2\mu\tilde{\alpha}_m\beta_m & \omega^2\rho - 2\mu\tilde{\alpha}_m^2\\ 2\mu\tilde{\alpha}_m^2 - \omega^2\rho & 2\mu\tilde{\alpha}_m\gamma_m\end{pmatrix}\begin{pmatrix}A_{p,m}\\ A_{s,m}\end{pmatrix} e^{i\tilde{\alpha}_m x_1},\\
\nu\cdot u(x)|_{\Gamma_0} &= \sum_{m\in\mathbb{Z}} (A_{p,m}\beta_m - A_{s,m}\tilde{\alpha}_m)\, e^{i\tilde{\alpha}_m x_1},\\
(\partial_\nu v^{sc})(x)|_{\Gamma_0} &= \sum_{m\in\mathbb{Z}} -i\, v_m \eta_m e^{i\tilde{\alpha}_m x_1}.
\end{aligned}$$

Hence, the coupling conditions between $v = v^{sc}$ and u on Γ_0 are equivalent to the algebraic equations

$$D_m \begin{pmatrix} v_m \\ iA_{p,m} \\ iA_{s,m} \end{pmatrix} = 0, \quad D_m := \begin{pmatrix} 0 & 2\mu\tilde{\alpha}_m\beta_m & \omega^2\rho - 2\mu\tilde{\alpha}_m^2 \\ -1 & 2\mu\tilde{\alpha}_m^2 - \omega^2\rho & 2\mu\tilde{\alpha}_m\gamma_m \\ -\eta_m/(\rho_f\omega^2) & \beta_m & -\tilde{\alpha}_m \end{pmatrix} \tag{27}$$

The determinant of D_m is given by

$$\mathrm{Det}(D_m) = -\frac{\eta_m}{\rho_f\omega^2}\begin{vmatrix} 2\mu\tilde{\alpha}_m\beta_m & \omega^2\rho - 2\mu\tilde{\alpha}_m^2 \\ 2\mu\tilde{\alpha}_m^2 - \omega^2\rho & 2\mu\tilde{\alpha}_m\gamma_m \end{vmatrix} - \omega^2\rho\,\beta_m.$$

Under the assumption that $k = k_p$ and $k = k\sin\theta_1 + m_0 = \tilde{\alpha}_{m_0}$ for some $m_0 \in \mathbb{Z}$, we have $\eta_{m_0} = \beta_{m_0} = 0$. Thus, the linear system (27) has the non-trivial solution $(v_{m_0}, A_{p,m_0}, A_{s,m_0})$, if this vector satisfies the relation

$$v_{m_0} + i\lambda k^2 A_{p,m_0} = 0, \quad A_{s,m_0} = 0.$$

This implies that, one of the non-trivial solutions (v^{sc}, u) is of the form

$$v^{sc}(x) = c\,e^{ikx_1} \quad \text{in} \quad x_3 > 0, \qquad u(x) = -ic/(\lambda k^2)\,(k,0,0)^\top e^{ikx_1} \qquad \text{in} \quad x_3 < 0,$$

for a constant $c \in \mathbb{C}$. □

Below we show the existence of Jones frequencies for the FSI problem in periodic structures. The frequency $\omega \in \mathbb{R}_+$ is called a Jones frequency with the quasi-periodic parameter $\alpha = (\alpha_1, \alpha_2)^\top \in \mathbb{R}^2$, if there exists at least one non-trivial α-quasiperiodic solution to the boundary value problem

$$(\Delta^* + \omega^2\rho)u = 0 \quad \text{in} \quad \Omega^-, \quad Tu = 0, \nu\cdot u = 0 \quad \text{on} \quad \Gamma, \quad u \text{ admits an expansion (9).} \tag{28}$$

Obviously, the solution $(0, u)$ solves the homogeneous transmission problem (2) with $v^{in} = 0$, provided u is a solution of (28). This implies that the FSI problem is not uniquely solvable at Jones frequencies. To construct a non-trivial solution to (28), we suppose that $\gamma_n = \sqrt{k_s^2 - |\alpha_n|^2} = 0$ for some $n \in \mathbb{Z}^2$ and that $\Gamma := \{x:\ x_3 = 0\}$ is a flat surface. Then the following α-quasiperiodic function is a solution of (28):

$$u(x) = \begin{pmatrix} \alpha_n^\perp \\ 0 \end{pmatrix} e^{i\alpha_n\cdot\tilde{x}} = \begin{pmatrix} -\alpha_n^{(2)} \\ \alpha_n^{(1)} \\ 0 \end{pmatrix} e^{i(\alpha_n^{(1)}x_1 + \alpha_n^{(2)}x_2)}.$$

Although there is no uniqueness in general, we can verify the existence of solutions to (BVP) at any frequency $\omega \in \mathbb{R}$ and the unique solvability for all frequencies excluding possibly a discrete set. This exceptional set does not necessarily include the values $\omega \in \mathcal{D}_0$ for which there is an $n \in \mathbb{Z}^2$ with $|\alpha_n|^2 + \beta_n\gamma_n \neq 0$ (cf. Remark 3.2). The main results of this section are stated in the following theorem, where the number c_0 denotes the speed of sound in the fluid.

Theorem 4.4. **(i)** *For the incident plane wave v^{in} of the form (1), there always exists a solution $(v, u) \in V_1$ to the variational problem (22) and hence to (BVP).*

(ii) *Assume $\lambda + 2\mu \leq \rho c_0^2$. There exists a small frequency $\omega_0 > 0$ such that for all $\omega \in (0, \omega_0]$ the solution to (22) is unique. Moreover, the variational problem (22) admits a unique solution for all frequencies excluding a discrete set $\mathcal{D}$ with the only possible accumulation point at infinity.*

Proof. (i) The variational problem (22) can be formulated as the equivalent operator equation $\mathcal{A}(v,u) = \mathcal{F}_0$, where $\mathcal{F}_0 \in V_1'$ is defined as the right hand side of (22). By the Fredholm alternative and Lemma 4.2, this operator equation (22) is solvable provided $\mathcal{F}_0$ is orthogonal to all solutions $(\tilde{v},\tilde{u})$ of the homogeneous adjoint equation $\mathcal{A}^*(\tilde{v},\tilde{u}) = 0$, i.e., $\langle \mathcal{F}_0, (\tilde{v},\tilde{u})\rangle = 0$. Note that the $\tilde{v}$ of such a pair can always be extended to a solution of the Helmholtz equation in the unbounded domain Ω^+ by setting

$$\tilde{v}(x) \;=\; \sum_{n\in\mathbb{Z}^2} \tilde{v}_n \exp(i\,\alpha_n\cdot\tilde{x}\,-i\overline{\eta}_n[x_3-b]), \quad x_3>b,$$

where the Rayleigh coefficients $\tilde{v}_n$ are determined as the n-th Fourier coefficient of $(e^{-i\alpha\cdot\tilde{x}}\tilde{v})|_{\Gamma_b^+}$. The above $\tilde{v}$ has a finite number of incoming plane waves that propagate downward, while the others terms in the sum are exponentially growing modes as $x_3\to\infty$. On the other hand, by arguing as in the proof of Lemma 4.2, it can be derived by taking the imaginary part of the equation

$$0 = \langle \mathcal{A}^*(\tilde{v},\tilde{u}),(\varphi,\psi)\rangle = \langle (\tilde{v},\tilde{u}),\mathcal{A}(\varphi,\psi)\rangle = \overline{A\big((\varphi,\psi),(\tilde{v},\tilde{u})\big)}$$

with $(\varphi,\psi) = (\tilde{v},\tilde{u})$ that $\tilde{v}$ has vanishing Rayleigh coefficients of the incoming modes, i.e., $\tilde{v}_n = 0$ for $|\alpha_n| < k$. In particular, we have $\tilde{v}_0 = 0$ and hence

$$\langle \mathcal{F}_0, (\tilde{v},\tilde{u})\rangle = \int_{\Gamma_b^+} f_0\,\overline{\tilde{v}}ds = \int_{\Gamma_b^+} f_0\,\overline{\tilde{v}_0}\exp(-i\,\alpha_0\cdot\tilde{x})ds(\tilde{x}) = 0,$$

with f_0 given in (21). Applying the Fredholm alternative yields the existence of a solution to (BVP).

(ii) We first prove uniqueness for small frequencies. The assumption $\lambda+2\mu \le \rho c_0^2$ implies that $k\le k_p$. If $\mathcal{A}(v^{sc},u) = 0$ for some $(v^{sc},u)\in V$, we conclude from $k\le k_p$ and Lemma 4.2 that the zero-order Rayleigh coefficients of v^{sc} and u vanish, i.e., $v_0 = 0, A_{p,0} = 0$ and $\mathbf{A}_{s,0} = 0$. This together with the asymptotic behavior

$$|\eta_n| \ge C_0\,(1+|n|^2)^{1/2}, \quad |n|\ne 0, \qquad \text{as} \quad k=\omega/c_0\to 0^+,$$

for some constant $C_0>0$, leads to the estimate (see (19))

$$\begin{aligned}
\mathrm{Re}\left\{-\int_{\Gamma_b^+}\overline{v}^{sc}\,\mathcal{T}^+v^{sc}ds\right\} &= 4\pi^2\sum_{|n|\ne 0}|\eta_n|\,|v_n\,e^{i\eta_n b}|^2\\
&= 4\pi^2\sum_{n\in\mathbb{Z}^2}|\eta_n|\,|v_n\,e^{i\eta_n b}|^2\\
&\ge C_1\,||v^{sc}||^2_{H_\alpha^{1/2}(\Gamma_b^+)},
\end{aligned} \tag{29}$$

for some $C_1>0$ and $\omega\in(0,\omega_1]$ with $\omega_1>0$ being sufficiently small. In a completely similar manner, from the asymptotic properties of the matrix W_n as $\omega\to 0^+$ (see [12, Lemma 2]) we obtain

$$\mathrm{Re}\left\{-\int_{\Gamma_b^-}\overline{u}\cdot\mathcal{T}^-u\,ds\right\} \ge C_2\,||u||^2_{H_\alpha^{1/2}(\Gamma_b^-)^3}. \tag{30}$$

Inserting (29) into (22) and setting $(\varphi,\psi) = (v^{sc},0)$, $v^{in}=0$, we arrive at

$$\begin{aligned}
0 &= \mathrm{Re}\,A\big((v^{sc},u),(v^{sc},0)\big)\\
&\ge ||\mathrm{grad}\,v^{sc}||^2_{L^2(\Omega_b^+)} + C_1\,||v^{sc}||^2_{H_\alpha^{1/2}(\Gamma_b^+)} - \omega^2/c_0^2\,||v^{sc}||^2_{L^2(\Omega_b^+)} - \omega^2\,\rho_f\int_\Gamma u\cdot\nu\overline{v^{sc}}\,ds.
\end{aligned}$$

Applying Friedrich's and the Cauchy-Schwarz inequalities, it follows that

$$0 \geq C_3||v^{sc}||^2_{H^1_\alpha(\Omega_b^+)} - C_4\,\omega^2||u||^2_{L^2(\Gamma)^3}, \quad \omega \in (0, \omega_1], \tag{31}$$

for some constants $C_3, C_4 > 0$ uniformly in all $\omega \in (0, \omega_1]$. Similarly, inserting (30) into (22) with $(\varphi, \psi) = (0, u)$ and $f_0 = 0$ and applying Korn's inequality (see e.g., [19, Chap. 5.4] or [12]), we obtain

$$0 = \operatorname{Re} A\big((v^{sc}, u), (0, u)\big) \geq C_5||u||^2_{H^1_\alpha(\Omega_b^-)^3} - C_6\,||v^{sc}||^2_{L^2(\Gamma)}, \quad \omega \in (0, \omega_1], \tag{32}$$

where $C_5, C_6 > 0$ are independent of $\omega \in (0, \omega_1]$. Now, combining (31), (32) and using the trace lemma we arrive at $v^{sc} = 0, u = 0$ for all $\omega \in (0, \omega_0]$ with some small frequency $\omega_0 > 0$. The existence follows directly from uniqueness by the Fredholm alternative.

In view of the analytic Fredholm theory (see e.g. [8, Theorem 8.26] or [16, Theorem I. 5. 1]) and the unique solvability of (BVP) at small frequencies, we obtain uniqueness and existence for all frequencies $\omega \in \mathbb{R}^+ \backslash \mathcal{D}$, where $\mathcal{D}$ is a discrete set including the set $\mathcal{D}_0$ of exceptional frequencies (see (18)). Note that the DtN maps $\mathcal{T}^\pm$ are not analytic at $\omega \in \mathcal{D}_0$. Moreover, we conclude from the arguments in [12, Theorem 6] or [15, Theorem 3.3] that $\mathcal{D}$ cannot have a finite accumulation point. The proof is completed. $\square$

Remark 4.5. *Theorem 4.4 (i) remains valid for a broad class of incident waves of the form*

$$v^{in}(x) = \sum_{n \in \mathbb{Z}^2 : |\alpha_n| < k} q_n \exp(i\alpha_n \cdot \tilde{x} - i\eta_n x_3), \quad q_n \in \mathbb{C}.$$

5 Energy balance

The energy balance in the FSI problem asserts that the sum of the reflected energy in the fluid and the transmitted energy in the solid should be equal to the energy of the incident wave. Let the incident plane wave $v^{in} = \exp(i\alpha \cdot x' - \eta_0 x_3)$ be given by (1), with $\eta_0 = k\cos\theta_1$. Define the efficiency of the reflected acoustic wave of order n as

$$E_n^+ := \frac{\eta_n}{\eta_0}\,|v_n|^2.$$

This is the ratio of the energy flux of the reflected mode of order n over the energy flux of the incoming mode. The energy flux is measured over a unit of time period on a unit square parallel to the x_1x_2-plane. In the FSI problem, the efficiencies of the transmitted compressional and shear elastic waves in the fluid are defined as

$$E_{p,n}^- := \frac{\beta_n}{\eta_0}\,|A_{p,n}|^2\,\omega^2 \rho\,\eta, \quad E_{s,n}^- := \frac{\gamma_n}{\eta_0}\,|A_{s,n}|^2\,\mu\,\eta,$$

respectively. The energy balance formula which can be used as an indicator of the validity of the numerical solution is formulated as follows.

Theorem 5.1. *It holds that*

$$1 = \sum_{n \in \mathbb{Z}^2 : \eta_n > 0} E_n^+ + \sum_{n \in \mathbb{Z}^2 : \beta_n > 0} E_{p,n}^- + \sum_{n \in \mathbb{Z}^2 : \gamma_n > 0} E_{s,n}^-.$$

Proof. It follows from (11) that

$$\begin{aligned}0 &= \int_{\Omega_b^+}\left[\operatorname{grad} v\cdot\operatorname{grad}\overline{\varphi}-k^2 v\overline{\varphi}\right]dx-\eta\int_{\Gamma}u\cdot\nu\overline{\varphi}\,ds-\int_{\Gamma_b^+}\partial_\nu v\,\overline{\varphi}\,ds\\ &\quad+\eta\left[\int_{\Omega_b^-}\left[\mathcal{E}(u,\overline{\psi})-\omega^2\rho u\cdot\overline{\psi}\right]dx-\int_{\Gamma}v\,\nu\cdot\overline{\psi}ds-\int_{\Gamma_b^-}\mathcal{T}^-u\cdot\overline{\psi}ds\right],\end{aligned}$$

for all $(\varphi,\psi)\in H^1(\Omega_b^+)\times H^1(\Omega_b^-)^3$, where $v=v^{in}+v^{sc}$ denotes the total acoustic field in the fluid. Choosing $(\varphi,\psi)=(v,u)$ and taking the imaginary part of the above expression yields (cf. (25))

$$\operatorname{Im}\left(\partial_\nu v,v\right)_{L^2(\Gamma_b^+)}+\eta\operatorname{Im}\left(\mathcal{T}^-u,u\right)_{L^2(\Gamma_b^-)^3}=0, \tag{33}$$

It can be readily checked that

$$\operatorname{Im}\left(\partial_\nu v,v\right)_{L^2(\Gamma_b^+)}=\operatorname{Im}\left(\partial_\nu v^{in},v^{in}\right)_{L^2(\Gamma_b^+)}+\operatorname{Im}\left(\mathcal{T}^+v^{sc},v^{sc}\right)_{L^2(\Gamma_b^+)}. \tag{34}$$

Indeed, by the definition of $\mathcal{T}^+$ (see (15)) we observe

$$\begin{aligned}&\operatorname{Im}\left[\left(\partial_\nu v^{in},v^{sc}\right)_{L^2(\Gamma_b^+)}+\left(\mathcal{T}^+v^{sc},v^{in}\right)_{L^2(\Gamma_b^+)}\right]\\ &\qquad= 4\pi^2\operatorname{Im}\left[-i\eta_0\overline{v}_0\,e^{-2i\eta_0 b}+i\eta_0 v_0 e^{2i\eta_0 b}\right] = 0\end{aligned}$$

where v_0 denotes the zero-th order Rayleigh coefficient of v^{sc} (see (5)). On the other hand,

$$\operatorname{Im}\left(\partial_\nu v^{in},v^{in}\right)_{L^2(\Gamma_b^+)}=-4\pi^2\,\eta_0. \tag{35}$$

Inserting (35), (26) and (34) into (33) yields the desired result of the lemma. □

Remark 5.2. *If the Rayleigh expansion of u takes the following form equivalent to (9):*

$$u(x)=\sum_{n\in\mathbb{Z}^2}\left\{A_{p,n}\begin{pmatrix}\alpha_n\\-\beta_n\end{pmatrix}\exp(i\alpha_n\cdot\tilde{x}-i\beta_n x_3)+\begin{pmatrix}\alpha_n\\-\gamma_n\end{pmatrix}\times\widetilde{\boldsymbol{A}}_{s,n}\exp(i\alpha_n\cdot\tilde{x}-i\gamma_n x_3)\right\}, \tag{36}$$

for $x_3<\Gamma^-$ with $\widetilde{\boldsymbol{A}}_{s,n}\in\mathbb{C}^3$ such that $\widetilde{\boldsymbol{A}}_{s,n}\cdot(\alpha_n,-\gamma_n)^\top=0$, then it holds that (cf. (26))

$$\operatorname{Im}\int_{\Gamma_b^-}\overline{u}\cdot\mathcal{T}^-u\,ds=4\pi^2\omega^2\rho\left(\sum_{n:|\alpha_n|<k_p}\beta_n\,|A_{p,n}|^2+\sum_{n:|\alpha_n|<k_s}\gamma_n\,|\widetilde{\boldsymbol{A}}_{s,n}|^2\right). \tag{37}$$

In this case, the definition of the efficiency $E_{s,n}^-$ in Theorem 5.1 should be replaced by

$$E_{s,n}^-:=\frac{\gamma_n}{\eta_0}\,|\widetilde{A}_{s,n}|^2\,\omega^2\rho\,\eta.$$

The quantity in (37) denotes the energy flux through Γ_b^- for the transmitted elastic wave of the form (36).

6 Discretization via truncated DtN mappings and finite element method (FEM)

6.1 Truncation of DtN mappings

Clearly, for the numerical treatment of the infinite number of terms in the definition of the DtN maps (15) and (16), we have to truncate the sums. We choose an integer $N > 0$ and introduce the truncated DtN maps

$$(\mathcal{T}_N^+ w)(\tilde{x}) \;:=\; \sum_{n\in\mathbb{Z}^2:\,|n|\le N} i\eta_n\,\hat{w}_n \exp(i\alpha_n\cdot\tilde{x}), \tag{38}$$

$$(\mathcal{T}_N^- w)(\tilde{x}) \;:=\; \sum_{n\in\mathbb{Z}^2:\,|n|\le N} iW_n\,\hat{w}_n \exp(i\alpha_n\cdot\tilde{x}). \tag{39}$$

We suppose that N is sufficiently large that all the propagating plane wave modes have indices with $|n| \le N$. Replacing the DtN maps in (23), we arrive at the approximate sesquilinear form

$$\begin{aligned} A_N\big((v,u),(\varphi,\psi)\big) \;:=\; & \int_{\Omega_b^+}\big[\operatorname{grad} v\cdot\operatorname{grad}\overline{\varphi} - k^2 v\overline{\varphi}\,\big]\,dx - \eta\int_\Gamma u\cdot\nu\,\overline{\varphi}\,ds - \int_{\Gamma_b^+}\mathcal{T}_N^+ v\,\overline{\varphi}\,ds \\ & +\eta\left[\int_{\Omega_b^-}\big[\mathcal{E}(u,\overline{\psi}) - \omega^2\rho u\cdot\overline{\psi}\,\big]\,dx - \int_\Gamma v\,\nu\cdot\overline{\psi}ds - \int_{\Gamma_b^-}\mathcal{T}_N^- u\cdot\overline{\psi}ds\right]. \end{aligned} \tag{40}$$

Using this, the equation (22) turns to

$$A_N\big((v_N,u_N),(\varphi,\psi)\big) = \mathcal{F}_0\big((\varphi,\psi)\big) := \int_{\Gamma_b^+} f_0\,\overline{\varphi}\,ds \quad \text{for all } (\varphi,\psi)\in V_1, \tag{41}$$

which is equivalent to the operator equation $\mathcal{A}_N(v_N,u_N) = \mathcal{F}_0$. Here $\mathcal{A}_N\colon\, V_1 \to V_1'$ is the approximate operator of $\mathcal{A}$ appearing in the operator equation $\mathcal{A}(v,u) = \mathcal{F}_0$ corresponding to (22). Now the exponential decay of the Rayleigh coefficients imply the following truncation error estimate.

Lemma 6.1. *i) Suppose $(v,u)\in V_1$ is the solution of $\mathcal{A}(v,u) = \mathcal{F}_0$ with $\mathcal{F}_0$ as in (41), then the Rayleigh coefficients of (v,u) satisfy*

$$v(x) \;=\; \sum_{n\in\mathbb{Z}^2} v_n^+ \exp(i\alpha_n\cdot\tilde{x}+i\eta_n[x_3-b]) + v^{in}(x),\;\; x_3>\Gamma^+,\quad |v_n^+| \le c\|v\|_{H^1_\alpha(\Omega_b^+)}q^{|n|}, \tag{42}$$

$$\begin{aligned} u(x) \;=\; & \sum_{n\in\mathbb{Z}^2}\Big\{u_{p,n}^-\begin{pmatrix}\alpha_n\\ \beta_n\end{pmatrix}\exp(i\alpha_n\cdot\tilde{x}-i\beta_n[x_3+b]) + \boldsymbol{u}_{s,n}^-\exp(i\alpha_n\cdot\tilde{x}-i\gamma_n[x_3+b])\Big\},\;\; x_3<\Gamma^-, \\ & |u_{p,n}^-| \le c\|u\|_{H^1_\alpha(\Omega_b^-)^3}q^{|n|},\;\; |\boldsymbol{u}_{s,n}^-| \le c\|u\|_{H^1_\alpha(\Omega_b^-)^3}q^{|n|}, \end{aligned} \tag{43}$$

for any n. Here c and q are constants independent of N and (u,v) such that $c>0$ and $0<q<1$. Recall that $\boldsymbol{u}_{s,n}^-\cdot(\alpha_n^\top,-\gamma_n)^\top = 0$.
ii) Suppose $(v_N,u_N)\in V_1$ is the solution of $\mathcal{A}_N(v,u) = \mathcal{F}_0$ with $\mathcal{F}_0$ as in (41), then the Rayleigh

coefficients of (v_N,u_N) *satisfy*[1]

$$v_N(x) \;=\; \sum_{n\in\mathbb{Z}^2}\Big\{v^+_{N,n}\exp(i\alpha_n\cdot\tilde{x}+i\eta_n[x_3-b])+v^-_{N,n}\exp(i\alpha_n\cdot\tilde{x}-i\eta_n[x_3-b])\Big\}+v^{in}(x), \tag{44}$$

$$x_3>\Gamma^+,$$

$$v^-_{N,n}=0 \text{ if } |n|\le N,\quad v^-_{N,n}=v^+_{N,n} \text{ if } |n|>N,\quad |v^\pm_{N,n}|\le c\|v_N\|_{H^1_\alpha(\Omega_b^+)}q^{|n|}, \tag{45}$$

$$u_N(x) \;=\; \sum_{n\in\mathbb{Z}^2}\Big\{u^+_{N,p,n}\begin{pmatrix}\alpha_n\\ \beta_n\end{pmatrix}\exp(i\alpha_n\cdot\tilde{x}+i\beta_n[x_3+b])+\boldsymbol{u}^+_{N,s,n}\exp(i\alpha_n\cdot\tilde{x}+i\gamma_n[x_3+b])$$

$$+u^-_{N,p,n}\begin{pmatrix}\alpha_n\\ -\beta_n\end{pmatrix}\exp(i\alpha_n\cdot\tilde{x}-i\beta_n[x_3+b])+\boldsymbol{u}^-_{N,s,n}\exp(i\alpha_n\cdot\tilde{x}-i\gamma_n[x_3+b])\Big\},$$

$$x_3<\Gamma^-,$$

$$u^+_{N,p,n}=0,\ \boldsymbol{u}^+_{N,p,n}=0 \text{ if } |n|\le N,\quad u^+_{N,p,n}=u^-_{N,p,n},\ \boldsymbol{u}^+_{N,p,n}=\boldsymbol{u}^-_{N,p,n} \text{ if } |n|>N,$$

$$|u^\pm_{N,p,n}|\le c\|u_N\|_{H^1_\alpha(\Omega_b^-)^3}q^{|n|},\ |\boldsymbol{u}^\pm_{N,s,n}|\le c\|u_N\|_{H^1_\alpha(\Omega_b^-)^3}q^{|n|}, \tag{46}$$

for any n. *Here* c *and* q *are constants independent of* N *and* (u,v) *such that* $c>0$ *and* $0<q<1$. *Note that* $\boldsymbol{u}^\pm_{N,s,n}\cdot(\alpha_n^\top,\pm\gamma_n)^\top=0$.
iii) Suppose the operator $\mathcal{A}:\ V_1\to V_1'$ *is invertible. Then, of course, the problem (BVP) is uniquely solvable. Moreover, there is an integer* $N_0>0$ *s.t.* $\mathcal{A}_N:\ V_1\to V_1'$ *is invertible for* $N\ge N_0$ *and*

$$\sup_{N\ge N_0}\|\mathcal{A}_N^{-1}\|<\infty. \tag{47}$$

For $(v,u)\in V_1$ *the solution of* $\mathcal{A}(v,u)=\mathcal{F}_0$ *with* $\mathcal{F}_0$ *as in (41) and for* $(v_N,u_N)\in V_1$ *the solution of* $\mathcal{A}_N(v,u)=\mathcal{F}_0$ *with the same* $\mathcal{F}_0$, *we obtain the estimate*

$$\|(v,u)-(v_N,u_N)\|_{V_1}\le c\|(v,u)\|_{V_1}q^N, \tag{48}$$

for any N. *Here* c *and* q *are constants independent of* N *and* (v,u) *such that* $c>0$ *and* $0<q<1$.

Proof. i) The solution v is analytic in the layer $\{x:\ \Gamma^+<x_3<b\}$ and admits the Rayleigh expansion (42). In particular, v restricted to $\{x:\ \Gamma^+<x_3<\Gamma^++2\varepsilon\}$ with a small $\varepsilon>0$ is a smooth function. Setting $\Gamma^+_\varepsilon:=\{x:\ \ x_3=\Gamma^++\varepsilon\}$, each Sobolev norm $\|v|_{\Gamma^+_\varepsilon}\|_{H^s(\Gamma^+_\varepsilon)}$ is bounded by a constant multiple of the $H^{1/2}$ norms of the restrictions to the curves $\{x:\ \ x_3=\Gamma^+\}$ and $\{x:\ \ x_3=\Gamma^++2\varepsilon\}$, i.e., bounded by constant times $\|v\|_{H^1_\alpha(\Omega_b^+)}$. Thus the Fourier coefficients $v_n^+\exp(i\eta_n[\Gamma^++\varepsilon-b])$ of v restricted to Γ^+_ε satisfy

$$\begin{aligned}\big|v_n^+\exp(i\eta_n[\Gamma^++\varepsilon-b])\big| &\le c\|v\|_{H^1(\Omega_b^+)},\\ |v_n^+| &\le c\|v\|_{H^1_\alpha(\Omega_b^+)}q^{|n|},\quad q:=\exp(\Gamma^++\varepsilon-b),\end{aligned}$$

where we have used $\eta_n\sim i|n|$ for $|n|\to\infty$. The assertions for u follow analogously.
ii) According to the integral $\int_{\Gamma_b^+}\mathcal{T}_N^+v\,\varphi$ in the variational form (40), the solution v_N satisfies the boundary

[1] For the simplicity of the formulas, we assume $\eta_n\neq 0$. The at most finite number of terms with $\eta_n=0$ do not affect the asymptotics. Note that, for $\eta_n=0$, the modes $x\mapsto\exp(i\alpha_n\cdot\tilde{x}\pm\gamma_n x_3)$ are to be replaced by $x\mapsto\exp(i\alpha_n\cdot\tilde{x})(1\pm x_3)$. Moreover, for the simplicity of the formulas, we assume $\gamma_n\neq 0$ and $\beta_n\neq 0$. Again, the at most finite number of exceptional terms do not affect the asymptotics.

condition $\partial_3 v_N|_{\Gamma_b^+} = \mathcal{T}_N^+(v_N|_{\Gamma_b^+})$, i.e., by (44) we conclude

$$\eta_n v_{N,n}^+ - \eta_n v_{N,n}^- = \begin{cases} \eta_n(v_{N,n}^+ + v_{N,n}^-) & \text{if } |n| \le N \\ 0 & \text{if } |n| > N. \end{cases} \tag{49}$$

Hence, $v_{N,n}^- = 0$ for $|n| \le N$ and $v_{N,n}^+ = v_{N,n}^-$ if $|n| > N$. The proof of the remaining assertions for v_N is analogous to that of part i). Note e.g. that the boundedness of the Rayleigh coefficients over $\{x: \ x_3 = \Gamma^+ + \varepsilon\}$ follows from the boundedness of the Fourier coefficients over the planes $\{x: \ x_3 = \Gamma^+ + p\varepsilon\}$ for $p = 0.5, 0.75, 1.25, 1.5$. The assertions for u_N follow analogously.
iii) In accordance with Lemma 4.1 the operator $\mathcal{A} : \ V_1 \to V_1'$ is strongly elliptic. Due to the proof of this lemma, $\mathcal{A}_N : \ V_1 \to V_1'$ is strongly elliptic too. Indeed, the only N dependent parts of $\mathcal{A}_N$ are the integrals over $\Gamma_b^\pm$. The truncated operator $-\mathrm{Re}\,\mathcal{T}_N$ is positive semidefinite (cf. (19)) and its quadratic form can be estimated from below by zero too. Similarly, we can treat the truncation $\mathcal{T}_{1,N}$ of $\mathcal{T}_1$. The truncation $\mathcal{T}_{2,N}$ of the compact operator $\mathcal{T}_2$, however, tends to zero in operator norm as $N \to \infty$. Thus $\mathcal{A}_N : \ V_1 \to V_1'$ is strongly elliptic at least for sufficiently large N.

Moreover, the above mentioned proof of Lemma 4.1 implies the uniform strong ellipticity estimate

$$\mathrm{Re}\,\langle \mathcal{A}_N(v,u),(v,u)\rangle \ge c\|(v,u)\|_{V_1}^2 - \mathrm{Re}\,\langle \mathcal{U}(v,u),(v,u)\rangle$$

with constant c and compact operator $\mathcal{U}$ independent of N. We define $\mathcal{B}_N := \mathcal{A}_N + \mathrm{Re}\mathcal{U}$ and $\mathcal{B} := \mathcal{A} + \mathrm{Re}\mathcal{U}$. Then the uniform strong ellipticity of the $\mathcal{A}_N$ and $\mathcal{A}$ implies that $\mathrm{Re}\,\mathcal{B}_N$ and $\mathrm{Re}\,\mathcal{B}$ are coercive, i.e., the $\mathcal{B}_N^{-1}$ are uniformly bounded and $\mathcal{B}_N^{-1}$ converges to $\mathcal{B}^{-1}$ strongly. From

$$\begin{aligned} \mathcal{A}_N &= \mathcal{B}_N(I - \mathcal{B}_N^{-1}\mathrm{Re}\mathcal{U}) = \mathcal{B}_N(I - \mathcal{B}^{-1}\mathrm{Re}\mathcal{U}) - \mathcal{B}_N(\mathcal{B}_N^{-1} - \mathcal{B}^{-1})\mathrm{Re}\mathcal{U} \\ &= \mathcal{B}_N\mathcal{B}^{-1}(\mathcal{B} - \mathrm{Re}\mathcal{U}) - \mathcal{B}_N(\mathcal{B}_N^{-1} - \mathcal{B}^{-1})\mathrm{Re}\mathcal{U} \\ &= \Big(\mathcal{A}^{-1}\mathcal{B}\mathcal{B}_N^{-1}\Big)^{-1} - \mathcal{B}_N(\mathcal{B}_N^{-1} - \mathcal{B}^{-1})\mathrm{Re}\mathcal{U}, \quad \|(\mathcal{B}_N^{-1} - \mathcal{B}^{-1})\mathrm{Re}\mathcal{U}\| \to 0, \end{aligned}$$

we conclude that $\mathcal{A}_N^{-1}$ is uniformly bounded. Using this fact and the exponential decay of the Rayleigh coefficients in the parts i) and ii) of the lemma, the estimate (47) is a simple consequence of

$$\begin{aligned} (v,u) - (v_N,u_N) &= \mathcal{A}^{-1}\mathcal{F}_0 - \mathcal{A}_N^{-1}\mathcal{F}_0 \ = \ \mathcal{A}_N^{-1}(\mathcal{A}_N - \mathcal{A})\mathcal{A}^{-1}\mathcal{F}_0, \\ \|(v,u) - (v_N,u_N)\|_{V_1} &\le c\|(\mathcal{A}_N - \mathcal{A})(v,u)\|_{V_1}. \end{aligned}$$

□

6.2 FEM

Now we consider the classical FEM. We introduce FE meshes over the domains $\Omega_b^\pm$ and denote the meshsize, i.e., the maximal diameter of the simplex subdomains by h. Using this h, we denote the space of piecewise linear functions in V_1, which are linear over each subdomain of the mesh, by V_h. Note that, for the sake of simplicity, we restrict ourselves to the linear case. Higher order elements can be treated analogously and are useful especially for large wavenumbers. For a given truncation number N and a given mesh of meshsize h, we compute the approximate solution $(v_{N,h}, u_{N,h}) \in V_h$ as the solution of the finite-element system

$$A_N\big((v_{N,h},u_{N,h}),(\varphi_h,\psi_h)\big) = \mathcal{F}_0(\varphi_h), \tag{50}$$

for all $(\varphi_h, \psi_h) \in V_h$.

To get convergence estimates for this FEM, we need the following two assumptions on the regularity of the solution. Suppose the Sobolev space index s_1, s_2 are fixed in the intervals $(1,2]$ and $[0,1)$, respectively.

(RA1) For given v_0 and u_0, consider the boundary value problem of quasi-periodic functions $(v,u) \in H^1_\alpha(\Omega_a^+) \times H^1_\alpha(\Omega_a^-)^3$ defined by

$$\begin{cases} (\Delta + k^2)\, v &= 0 \quad \text{in } \Omega_a^+ := \{x \in \Omega_b^+ \;:\; x_3 < \frac{1}{2}(b + \Gamma^+)\}, \\ (\Delta^* + \omega^2\rho)\, u &= 0 \quad \text{in } \Omega_a^- := \{x \in \Omega_b^- \;:\; \frac{1}{2}(-b + \Gamma^-) < x_3\}, \\ \eta\, u \cdot \nu &= \partial_\nu v \quad \text{on } \Gamma, \\ Tu &= -v\,\nu \quad \text{on } \Gamma, \\ v &= v_0 \quad \text{on } \Gamma_a^+ := \{x\colon\; 0 < x_1, x_2 < 2\pi, x_3 = \frac{1}{2}(b + \Gamma^+)\}, \\ u &= u_0 \quad \text{on } \Gamma_a^- := \{x\colon\; 0 < x_1, x_2 < 2\pi, x_3 = \frac{1}{2}(-b + \Gamma^-)\}. \end{cases} \tag{51}$$

Suppose that any solution (v,u) of the variational formulation corresponding to (51) with $v_0 \in H^{s_1-1/2}(\Gamma_a^+)$ and $u_0 \in H^{s_1-1/2}(\Gamma_a^-)^3$ has the regularity $v \in H^{s_1}(\Omega_a^+)$ and $u \in H^{s_1}(\Omega_a^-)^3$.

(RA2) Consider the sesquilinear form corresponding to (51)

$$\begin{aligned} C\big((v,u),(\varphi,\psi)\big) \;&:=\; \int_{\Omega_a^+} \big[\operatorname{grad} v \cdot \operatorname{grad} \overline{\varphi} - k^2 v \overline{\varphi}\,\big]\, dx - \eta \int_\Gamma u \cdot \nu\, \overline{\varphi}\, ds \\ &\quad + \eta \left[\int_{\Omega_a^-} \big[\mathcal{E}(u, \overline{\psi}) - \omega^2 \rho u \cdot \overline{\psi}\,\big]\, dx - \int_\Gamma v\,\nu \cdot \overline{\psi} ds \right]. \end{aligned} \tag{52}$$

Clearly, for any functional $\mathcal{F} \in V_1'$, the solution (φ, ψ) of the adjoint variational equation $C((v,u),(\varphi,\psi)) = \mathcal{F}(v,u), \forall (v,u) \in V_1$ is in V_1. We suppose that, for $\mathcal{F}(v,u) := \langle v, f_v \rangle + \eta \langle u, f_u \rangle$ with functions $f_v \in H^{-s_2}(\Omega_a^+)$ and $f_u \in H^{-s_2}(\Omega_a^-)^3$, the solution (φ,ψ) is in $H^{2-s_2}(\Omega_a^+) \times H^{2-s_2}(\Omega_a^-)^3$ and satisfies the estimate

$$\|\varphi\|_{H^{2-s_2}(\Omega_a^+)} + \|\psi\|_{H^{2-s_2}(\Omega_a^-)^3} \le c\big\{\|f_v\|_{H^{-s_2}(\Omega_a^+)} + \|f_u\|_{H^{-s_2}(\Omega_a^-)^3}\big\}, \tag{53}$$

where c is independent of f_v and f_u.

Remark 6.2. *The assumptions (RA1) and (RA2) are fulfilled for smooth boundaries Γ. If Γ is piecewise linear, then the assumptions hold if the singularities at the vertices and edges are sufficiently mild (cf. e.g. [9]).*

Theorem 6.3. *Suppose the operator $\mathcal{A} \,:\, V_1 \to V_1'$ is invertible, i.e., the variational equation (22) is uniquely solvable for any right hand from V_1'.*
i) There exist $N_0 > 0$ and $h_0 > 0$ such that, for any $N > N_0$ and $h < h_0$, the FEM system (50) has a unique solution $(v_{N,h}, u_{N,h}) \in V_h$. For $N \to \infty$ and $h \to 0$, the FEM solutions $(v_{N,h}, u_{N,h})$ converge in the norm of V_1 to the solution $(u,v) \in V_1$ of (22).
ii) Suppose the right hand $\mathcal{F}_0$ is defined as in (41), i.e., in accordance to the plane wave incidence in the

scattering problem (BVP). Furthermore, suppose regularity assumption (RA1) is satisfied with $1<s_1\leq 2$. Then there exist constants c and q with $c>0$ and $0<q<1$ such that, for any $N>N_0$ and $h<h_0$,

$$\|(v_{N,h},u_{N,h})-(v,u)\|_{V_1}\leq c\|(v,u)\|_{H^{s_1}(\Omega_b^+)\times H^{s_1}(\Omega_b^-)^3}\{h^{s_1-1}+q^N\}. \tag{54}$$

iii) Suppose the right hand $\mathcal{F}_0$ is defined as in (41). Furthermore, suppose the regularity assumptions (RA1) with $1<s_1\leq 2$ and (RA2) with $0\leq s_2<1$ are satisfied. Then there exist constants c and q with $c>0$ and $0<q<1$ such that, for any $N>N_0$ and $h<h_0$,

$$\|(v_{N,h},u_{N,h})-(v,u)\|_{H^{s_2}(\Omega_b^+)\times H^{s_2}(\Omega_b^-)^3}\leq c\|(v,u)\|_{H^{s_1}(\Omega_b^+)\times H^{s_1}(\Omega_b^-)^3}\{h^{s_1-s_2}+q^N\}. \tag{55}$$

Proof. i) Clearly, $(v_{N,h},u_{N,h})-(v,u)=[(v_{N,h},u_{N,h})-(v_N,u_N)]+[(v_N,u_N)-(v,u)]$. In view of Lemma 6.1, it remains to analyze the convergence $[(v_{N,h},u_{N,h})-(v_N,u_N)]\to 0$. However, all estimates for this FEM must be shown uniformly w.r.t. N. We denote the L^2 orthogonal projection of V_1 onto the spline space V_h by P_h. From the proof of Lemma 6.1, we recall $\mathcal{A}_N=\mathcal{B}_N-\mathrm{Re}\,\mathcal{U}$ with a compact operator $\mathcal{U}$, the uniform coercivity $\mathrm{Re}\,\langle\mathcal{B}_N(v,u),(v,u)\rangle\geq c\|(v,u)\|^2$ and the strong convergence $\mathcal{A}_N^{-1}\to\mathcal{A}^{-1}$. In accordance with the proof of [18, Lemma 5.5], the uniform stability follows if we can show that the operator norm of $(P_h-I)\mathcal{A}_N^{-1}\mathrm{Re}\,\mathcal{U}:\ V_1\to V_1$ is smaller than any prescribed threshold for h sufficiently small (compare the operator $(P_h-I)B^{-1}T$ in [18, Lemma 5.5]). However, this is true since

$$(P_h-I)\mathcal{A}_N^{-1}\mathrm{Re}\,\mathcal{U}=(P_h-I)[\mathcal{A}^{-1}\mathrm{Re}\,\mathcal{U}]+(P_h-I)[\mathcal{A}_N^{-1}-\mathcal{A}^{-1}]\mathrm{Re}\,\mathcal{U},$$

since $[\mathcal{A}^{-1}\mathrm{Re}\,\mathcal{U}]$ and $\mathcal{U}$ are compact, and since $P_h\to I$ as well as $\mathcal{A}_N^{-1}\to\mathcal{A}^{-1}$. Now the uniform stability implies

$$\|(v_N,u_N)-(v_{N,h},u_{N,h})\|_{V_1}\leq c\inf_{(\varphi_h,\psi_h)\in V_h}\|(v_N,u_N)-(\varphi_h,\psi_h)\|_{V_1}. \tag{56}$$

The uniform convergence of the FEM in the norm of V_1 follows since the discrete set $\{(v_N,u_N):\ N=0,1,\ldots\}$ is precompact due to $(v_N,u_N)\to(v,u)$.

ii) This part follows from (56) and the approximation property of finite-element functions if we can prove $\|(v_N,u_N)\|_{H^{s_1}(\Omega_b^+)\times H^{s_1}(\Omega_b^-)^3}<c$. However, $\|(v_N,u_N)\|_{V_1}<c$ and the proof to Lemma 6.1 i) and ii) implies that the H^{s_1} norms over $\Omega_b^\pm\setminus\Omega_a^\pm$ are uniformly bounded. Consequently, we conclude $v|_{\Gamma_a^+}\in H^{s_1-1/2}(\Gamma_a^+)$ and $u|_{\Gamma_a^-}\in H^{s_1-1/2}(\Gamma_a^-)^3$ such that assumption (RA1) yields

$$\|(v_N,u_N)\|_{H^{s_1}(\Omega_b^+)\times H^{s_1}(\Omega_b^-)^3}<c.$$

iii) The estimate in Sobolev norms of order less than 1 follows from Nitsche's trick, from part ii) of the Lemma and from the approximation property. It remains only to show that the operators $\mathcal{A}_N^*$: $H^{2-s_2}(\Omega_b^+)\times H^{2-s_2}(\Omega_b^-)^3\to H^{-s_2}(\Omega_b^+)\times H^{-s_2}(\Omega_b^-)^3$ are invertible with uniformly bounded inverse operators. More precisely, for given $g_v\in H^{-s_2}(\Omega_b^+)$ and $g_u\in H^{-s_2}(\Omega_b^+)^3$, we have to show that the solution $(\varphi,\psi)=[\mathcal{A}_N^*]^{-1}(g_u,g_v)\in V_1$ satisfies

$$\begin{aligned}\|\varphi\|_{H^{2-s_2}(\Omega_b^+)} &\leq c\left\{\|g_u\|_{H^{-s_2}(\Omega_b^+)}+\|g_v\|_{H^{-s_2}(\Omega_b^+)^3}\right\}\\ \|\psi\|_{H^{2-s_2}(\Omega_b^+)^3} &\leq c\left\{\|g_u\|_{H^{-s_2}(\Omega_b^+)}+\|g_v\|_{H^{-s_2}(\Omega_b^+)^3}\right\}.\end{aligned}$$

We choose a partition of unity $1 = \sum_{j=1}^{3} \chi_j(x_3)$ with smooth functions χ_j such that

$$\begin{aligned}
\left[\frac{1}{4}\Gamma^+ + \frac{3}{4}b, b\right] &\subseteq \{x_3 : \ \chi_1(x_3) = 1)\} \subseteq \ \operatorname{supp}\chi_1 \ \subseteq \left[\frac{1}{2}(\Gamma^+ + b), b\right], \\
\left[\frac{3}{4}\Gamma^- - \frac{1}{4}b, \frac{3}{4}\Gamma^+ + \frac{1}{4}b\right] &\subseteq \{x_3 : \ \chi_2(x_3) = 1)\} \subseteq \ \operatorname{supp}\chi_2 \ \subseteq \left[\frac{1}{2}(\Gamma^- - b), \frac{1}{2}(\Gamma^+ + b)\right], \\
\left[-b, \frac{1}{4}\Gamma^- - \frac{3}{4}b\right] &\subseteq \{x_3 : \ \chi_3(x_3) = 1)\} \subseteq \ \operatorname{supp}\chi_3 \ \subseteq \left[-b, \frac{1}{2}(\Gamma^- - b)\right].
\end{aligned}$$

So it is sufficient to prove the regularity estimates for the functions $[\chi_1\varphi]$, $[\chi_2\varphi]$, $[\chi_2\psi]$ and $[\chi_3\psi]$ instead of φ and ψ.

The functions $[\chi_2\varphi]$ and $[\chi_2\varphi]$, however, are solutions of the boundary value problem appearing in the assumption (RA2) with H^{-s_2} bounded right-hand side. Thus $[\chi_2\psi]$ and $[\chi_3\psi]$ have bounded H^{2-s_2} norms according to assumption (RA2). The function $[\chi_1\varphi]$ is a solution of the Helmholtz equation with inhomogeneous H^{-s_2} bounded right-hand side and the boundary condition $\partial_3[\chi_1\varphi]|_{\Gamma_b^+} = \mathcal{T}_N^*([\chi_1\varphi]|_{\Gamma_b^+})$. Now we take a quasi-periodic H^{-s_2} extension of the right-hand side of the Helmholtz equation which has a bounded support in x_3-direction. Using a volume potential based on a quasi-periodic Green's function satisfying the radiation condition for the lower half plane, we can construct a quasi-periodic solution φ_0 of the inhomogeneous Helmholtz equation with the just extended right-hand side. Since this is H^{2-s_2} bounded, it remains to estimate the H^{2-s_2} norm of $\varphi_{00} := [\chi_1\varphi] - \varphi_o$. This function, however, is a solution of the homogeneous Helmholtz equation in $\{x : \ 0 < x_1, x_2 < 2\pi, x_3 < b\}$ satisfying the radiation condition and the inhomogeneous boundary condition $\partial_3\varphi_{00}|_{\Gamma_b^+} - \mathcal{T}_N^*(\varphi_{00}|_{\Gamma_b^+}) = \partial_3\varphi_0|_{\Gamma_b^+} - \mathcal{T}_N^*(\varphi_0|_{\Gamma_b^+})$. The uniform H^{2-s_2} bound of the solution of the latter problem can be derived easily by Rayleigh expansions. Finally, the estimate for $[\chi_3\psi]$ is analogous to that for $[\chi_1\varphi]$. $\square$

7 Variational formulation in two dimensions

In this section, we interchange the second and third components of the points in $\mathbb{R}^3$ and assume that the biperiodic surface Γ is invariant in the x_3-direction. The cross-section of Γ in the (x_1, x_2)-plane will be represented by a curve Λ which is 2π-periodic in x_1. All elastic waves are assumed to be propagating perpendicular to the x_3-axis, so that the problem can be treated as a problem of plane elasticity. This implies that the incident plane wave is of the form

$$v^{in}(x_1, x_2) = \exp(i\alpha x_1 - i\eta_0 x_2), \quad \alpha := k\sin\theta, \ \eta_0 := k\cos\theta, \tag{57}$$

where $\theta \in (-\pi/2, \pi/2)$ denotes the incident angle.

The boundary value problem for finding α-quasiperiodic solutions $v = v(x_1, x_2)$ and $u = u(x_1, x_2) = (u_1(x_1, x_2), u_2(x_1, x_2))^\top$ can be formulated analogously to (2) with the two-dimensional *traction* operator having the form

$$Tu = 2\mu\, \partial_{\mathbf{n}} u + \lambda \operatorname{div} u\, \mathbf{n} + \mu \begin{pmatrix} n_2\,(\partial_1 u_2 - \partial_2 u_1) \\ n_1\,(\partial_2 u_1 - \partial_1 u_2) \end{pmatrix} \quad \text{on} \quad \Lambda\,, \tag{58}$$

where $\mathbf{n} = (n_1, n_2)^\top$ denotes the exterior unit normal on Λ. As done in 3D, we will confine ourselves to a single periodic cell by setting

$$\Lambda_b^\pm := \left\{(x_1, \pm b)^\top\colon\ 0 \leq x_1 \leq 2\pi\right\}, \quad \Omega_b^\pm := \left\{(x_1, x_2)^\top\colon\ \exists x \in \Omega^\pm \text{ s.t. } 0 < x_1 < 2\pi,\ x_2 \lessgtr \pm b\right\}.$$

The upward and downward Rayleigh expansions for v^{sc} and u can be expressed as

$$\begin{aligned} v^{sc}(x) &= \sum_{n\in\mathbb{Z}} v_n \exp(i\alpha_n x_1 + i\eta_n x_2), \quad x_2 > \Lambda^+, \\ u(x) &= \sum_{n\in\mathbb{Z}} \left\{ A_{p,n} \begin{pmatrix} \alpha_n \\ -\beta_n \end{pmatrix} \exp(i\alpha_n x_1 - i\beta_n x_2) - A_{s,n} \begin{pmatrix} \gamma_n \\ \alpha_n \end{pmatrix} \exp(i\alpha_n x_1 - i\gamma_n x_2) \right\},\ x_2 < \Lambda^-, \end{aligned} \tag{59}$$

with α_n, η_n, β_n and γ_n defined analogously to the 3D case. The DtN maps $\mathcal{T}^\pm$ can be represented as

$$(\mathcal{T}^+ w)(x) := \sum_{n\in\mathbb{Z}} i\eta_n\, \hat{w}_n \exp(i\alpha_n x_1) \quad \text{for } w = \sum_{n\in\mathbb{Z}} \hat{w}_n \exp(i\alpha_n x_1) \in H_\alpha^s(\Lambda_b^+), s \geq 1/2, \tag{60}$$

$$(\mathcal{T}^- w)(x) := \sum_{n\in\mathbb{Z}} i\, W_n\, \hat{w}_n \exp(i\alpha_n x_1) \quad \text{for } w = \sum_{n\in\mathbb{Z}} \hat{w}_n \exp(i\alpha_n x_1) \in H_\alpha^s(\Lambda_b^-)^2, s \geq 1/2, \tag{61}$$

where W_n is the 2×2 matrix

$$W_n := \begin{pmatrix} \omega^2\beta_n/d_n & -2\mu\alpha_n + \omega^2\alpha_n/d_n \\ 2\mu\alpha_n - \omega^2\alpha_n/d_n & \omega^2\gamma_n/d_n \end{pmatrix}, \quad d_n := \alpha_n^2 + \beta_n\gamma_n. \tag{62}$$

The expression (62) follows from the arguments of [12] and differs from the matrix corresponding to upward propagating elastic waves only in the signs of the off-diagonal terms. We state the variational formulation for the FSI problem in the two-dimensional setting as follows: Find $(v, u) \in V_1 := H_\alpha^1(\Omega_b^+) \times H_\alpha^1(\Omega_b^-)^2$ such that

$$A\big((v, u), (\varphi, \psi)\big) = \int_{\Lambda_b^+} f_0\, \overline{\varphi}\, ds \quad \text{for all } (\varphi, \psi) \in V_1, \tag{63}$$

where the sesquilinear form $A : V_1 \times V_1 \to \mathbb{C}$ is defined analogously to (23) with Λ, $\Lambda_b^\pm$, $\mathbf{n}$ in place of Γ, $\Gamma_b^\pm$ and ν, and

$$\begin{aligned} \mathcal{E}(u, \overline{\varphi}) \;&=\; (2\mu + \lambda)\,(\partial_1 u_1\, \partial_1\overline{\varphi}_1 + \partial_2 u_2\, \partial_2\overline{\varphi}_2) + \mu\,(\partial_2 u_1\, \partial_2\overline{\varphi}_1 + \partial_1 u_2\, \partial_1\overline{\varphi}_2) \\ &\quad + \lambda\,(\partial_1 u_1\, \partial_2\overline{\varphi}_2 + \partial_2 u_2\, \partial_1\overline{\varphi}_1) + \mu\,(\partial_2 u_1\, \partial_1\overline{\varphi}_2 + \partial_1 u_2\, \partial_2\overline{\varphi}_1). \end{aligned}$$

The function $f_0 \in H_\alpha^{-1/2}(\Lambda_b^+)$ on the right hand of (63) is given by

$$f_0(x_1) := -2i\eta_0 \exp(i\alpha x_1 - i\eta_0 b).$$

All the uniqueness, existence and non-uniqueness results in Section 4 carry over to the 2D case. Moreover, there holds the energy balance formula

$$1 = \sum_{n\in\mathbb{Z}:\eta_n>0} \frac{\eta_n}{\eta_0}\,|v_n|^2 + \omega^2\rho\,\eta \left(\sum_{n\in\mathbb{Z}:\beta_n>0} \frac{\beta_n}{\eta_0}\,|A_{p,n}|^2 + \sum_{n\in\mathbb{Z}:\gamma_n>0} \frac{\gamma_n}{\eta_0}\,|A_{s,n}|^2 \right). \tag{64}$$

The variational formulations (63) and (22) are convenient for theoretical justifications. However, in numerical implementations we prefer the following formulation equivalent to (63):

$$A\big((v^{sc},u),(\varphi,\psi)\big) = \int_\Gamma \left(\partial_{\mathbf{n}} v^{in}\overline{\varphi} - \eta \mathbf{n} v^{in}\cdot\overline{\psi}\right)\, ds \quad \text{for all } (\varphi,\psi)\in V_1. \tag{65}$$

In other words, we compute the scattered field $v^{sc} = v - v^{in}$ instead of the total field v over the domain Ω_b^+.

The truncation of the DtN mappings and the FEM can be defined analogously to the 3D case. With a straightforward generalization of the conditions (RA1) and (RA2), Theorem 6.3 remains true.

8 Numerical examples

In this section, we present several numerical tests to confirm our theoretical results in 2D. We take the parameters

$$\omega = 1,\ \mu = 1,\ \lambda = 1,\ \rho_f = 2,\ \rho = 1.$$

The computational domains $\Omega_b^\pm$ are discretized by quasi-uniform triangular elements. A direct solver is employed for computing solutions of the resulting linear system. In our numerical tests, the energy function is defined by

$$\begin{aligned}
E_{N,h} &:= \sum_{n=-N:\eta_n>0}^{N} \frac{\eta_n}{\eta_0}\,|v_n^{N,h}|^2 + \omega^2\rho\eta\left(\sum_{n=-N:\beta_n>0}^{N}\frac{\beta_n}{\eta_0}\,|A_{p,n}^{N,h}|^2 + \sum_{n=-N:\gamma_n>0}^{N}\frac{\gamma_n}{\eta_0}\,|A_{s,n}^{N,h}|^2\right),\\
v_n^{N,h} &:= \frac{1}{2\pi}\int_{\Gamma_b^+} v_{N,h}(x_1,b)\exp\big(-i(\alpha_n x_1+\eta_n b)\big)dx_1,\\
A_{p,n}^{N,h} &:= \frac{1}{2\pi}\,\frac{1}{\alpha^2+\beta_n\gamma_n}\int_{\Gamma_b^+} u_{N,h}(x_1,b)\cdot(\alpha_n,-\gamma_n)^\top \exp\big(-i(\alpha_n x_1+\beta_n b)\big)dx_1,\\
A_{s,n}^{N,h} &:= \frac{1}{2\pi}\,\frac{1}{\alpha^2+\beta_n\gamma_n}\int_{\Gamma_b^+} u_{N,h}(x_1,b)\cdot(-\beta_n,-\alpha_n)^\top \exp\big(-i(\alpha_n x_1+\gamma_n b)\big)dx_1,
\end{aligned}$$

where $N = 20$ is the truncation number of the Rayleigh series. Note that the exact value of the energy function is $E_{\infty,0} = 1$ (cf. (64)).

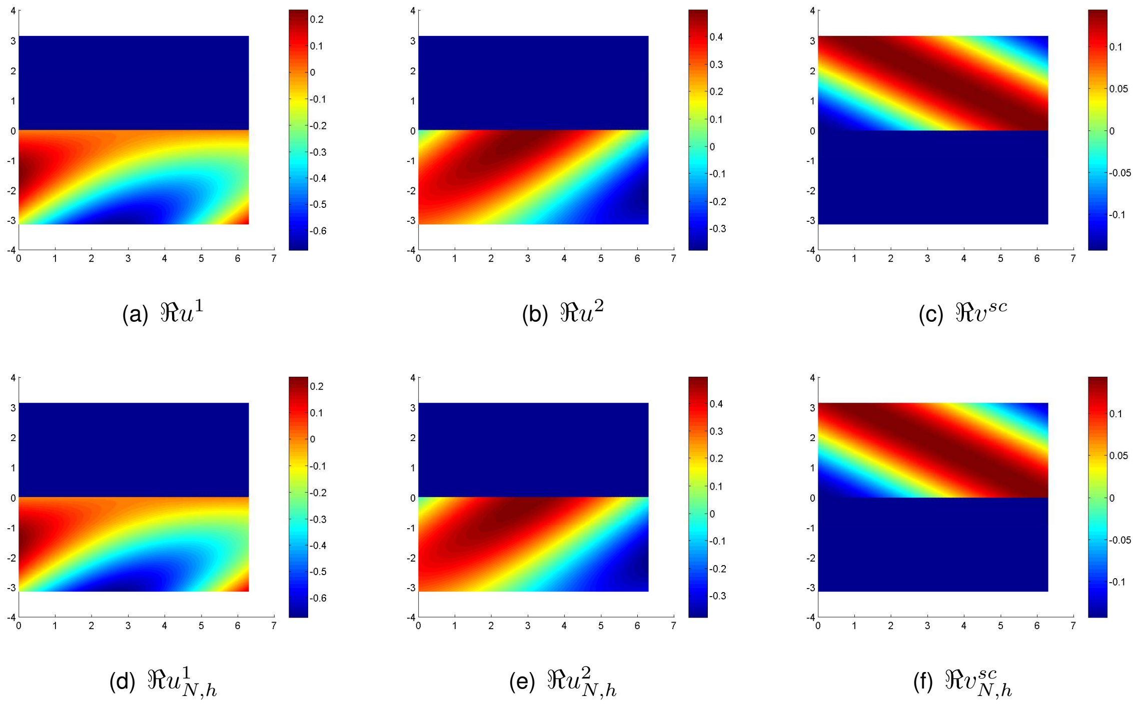
We first introduce a model problem with analytical solutions so that the accuracy of the numerical solutions can be evaluated. Assume that the scattering interface Γ is the straight line $\Gamma_0 := \{(x_1,x_2):\ x_2 = 0\}$. Recall the incident plane wave (57). Then the unique solution of (2) takes the form

$$v^{sc}(x) = a_1\exp(i\alpha x_1 + i\eta_0 x_2),\quad x\in\Omega^+, \tag{66}$$

$$u(x) = a_2\begin{pmatrix}\alpha\\ -\beta_0\end{pmatrix}\exp(i\alpha x_1 - i\beta_0 x_2) + a_3\begin{pmatrix}\gamma_0\\ \alpha\end{pmatrix}\exp(i\alpha x_1 - i\gamma_0 x_2),\ x\in\Omega^-, \tag{67}$$

where the coefficients $a_j,\ j=1,2,3$ can be obtained by solving the linear system

$$\begin{pmatrix} i\eta_0 & \rho_f\omega^2\beta_0 & -\rho_f\omega^2\alpha\\ 0 & 2i\mu\alpha\beta_0 & 2i\mu\gamma_0^2 - i\mu k_s^2\\ 1 & 2i\mu\beta_0^2 + i\lambda k_p^2 & -2i\mu\alpha\gamma_0\end{pmatrix}\begin{pmatrix}a_1\\ a_2\\ a_3\end{pmatrix} = \begin{pmatrix}i\eta_0\\ 0\\ -1\end{pmatrix}. \tag{68}$$



(a) $\Re u^1$ (b) $\Re u^2$ (c) $\Re v^{sc}$

(d) $\Re u^1_{N,h}$ (e) $\Re u^2_{N,h}$ (f) $\Re v^{sc}_{N,h}$

Figure 3: Exact and numerical solutions with $k = 1$ and meshsize $h = 0.0245$ for straight line interface.

This system is easily derived from the transmission conditions.

For this first example, we consider the model problem and set the height of the computational domain above the interface to $b = \pi$. Figure 3 shows the exact and numerical solutions of the elastic displacement in the solid and the scattered acoustic field in the fluid, where we have taken $k = 1$ and meshsize $h = 0.0245$. In Figure 4 we present the numerical error $\|(v^{sc}, u) - (v^{sc}_{N,h}, u_{N,h})\|$ in the spaces

$$V_0 := L^2_\alpha(\Omega_b^+) \times L^2_\alpha(\Omega_b^-)^2, \quad V_1 := H^1_\alpha(\Omega_b^+) \times H^1_\alpha(\Omega_b^-)^2$$

with respect to $1/h$ for $k = 1, 3$ and 5. We can obviously observe that

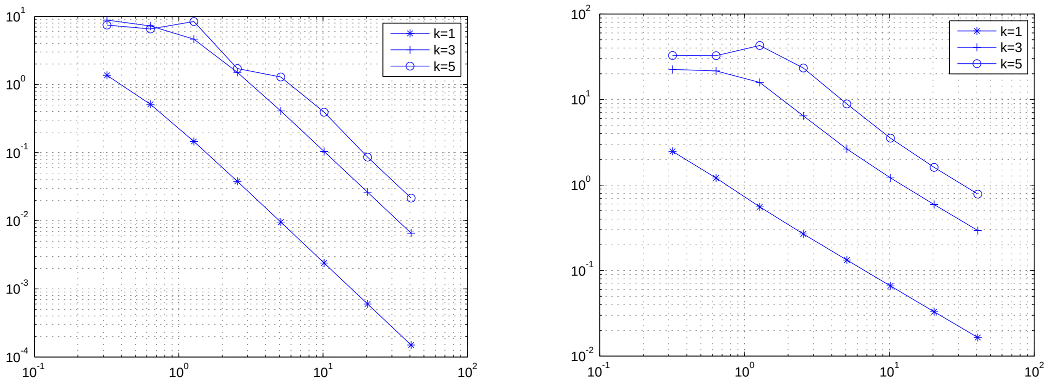
$$||(v^{sc}, u) - (v^{sc}_{N,h}, u_{N,h})||_{V_0} = O(h^2), \quad ||(v^{sc}, u) - (v^{sc}_{N,h}, u_{N,h})||_{V_1} = O(h). \tag{69}$$

Next, we compute the Rayleigh coefficients $v_n^{N,h}$ from the values of the numerical solution $v^{sc}_{N,h}$ taken on Γ_b. In Figure 5, we show the values of $v_n^{N,h} \exp(i\eta_n\pi)$ and $v_n \exp(i\eta_n\pi)$ for $n = -20, \ldots, 20$. Note that, for the exact solution, only the Rayleigh coefficients of order zero do not vanish. The exact and the computed Rayleigh coefficients together with the numerical energy function are listed in Table 1.

In the second example, we consider the two smooth grating profiles $\Gamma = \Gamma_f := \{(x_1, f(x_1)) : 0 < x_1 < 2\pi\}$ with f defined as (see Figure 6):

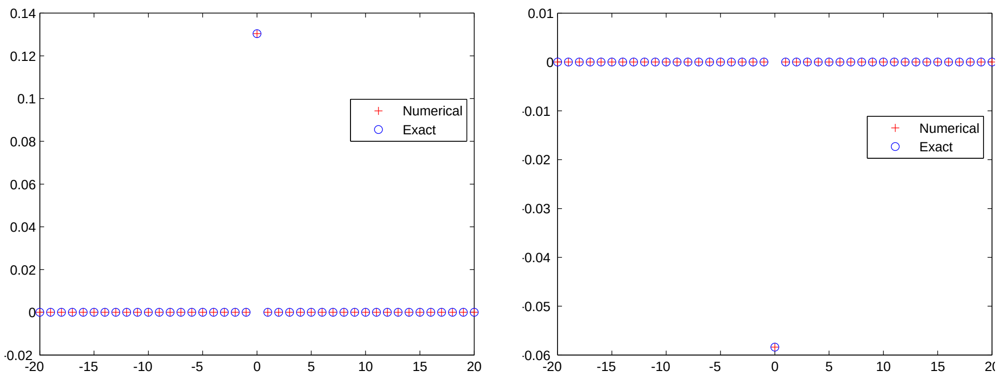
(1) $f(x_1) = 0.4\sin(x_1)$,
(2) $f(x_1) = 0.3\sin(x_1) + 0.2\sin(2x_1)$

and plot the corresponding numerical energy functions with respect to $1/h$ in Figure 7. In these cases the wavenumber in the fluid is taken as $k = 9$ and we have set $b = 3$. The numerical solutions are consistent with the proposed energy balance formula and thus support our theoretical results.



(a) $||(v^{sc}, u) - (v^{sc}_{N,h}, u_{N,h})||_{V_0}$ (b) $||(v^{sc}, u) - (v^{sc}_{N,h}, u_{N,h})||_{V_1}$

Figure 4: Log-log plot of errors vs. $1/h$. Errors in V_0-norm (left) and V_1-norm (right).



(a) Real part (b) Imaginary part

Figure 5: Exact and numerical values $v_n \exp(i\eta_n b)$ and $v_n^{N,h} \exp(i\eta_n b)$ for $n = -20, \cdots, 20$ with $k = 1$, $b = \pi$ and meshsize $h = 0.0245$.

k	$\lvert v_0\rvert$	$\lvert v_0^{N,h}\rvert$	$\lvert A_{p,0}\rvert$	$\lvert A_{p,0}^{N,h}\rvert$	$\lvert A_{s,0}\rvert$	$\lvert A_{s,0}^{N,h}\rvert$	$E_{N,h}$
1	0.1429	0.1428	0.8571	0.8571	0.4949	4.4949	0.9999
3	1.0000	0.9997	3.5175	3.5074	4.1741	4.1643	0.9995
5	1.0000	0.9996	3.2065	3.1781	3.3911	3.3636	0.9993
7	1.0000	0.9988	3.1041	3.0389	3.1919	3.1278	0.9757

Table 1: Exact and numerical Rayleigh coefficients of order zero and numerical energy function with meshsize $h = 0.0491$.

	b=0.7 (h=0.3181)	b=1 (h=0.3385)	b=2 (h=0.3083)	b=3 (h=0.3621)	N_0
$k=1$	4	3	2	2	2
$k=3$	5	4	4	4	5
$k=5$	7	7	7	9	8

Table 2: The values of N_0 compared with the N_τ depending on k and b.

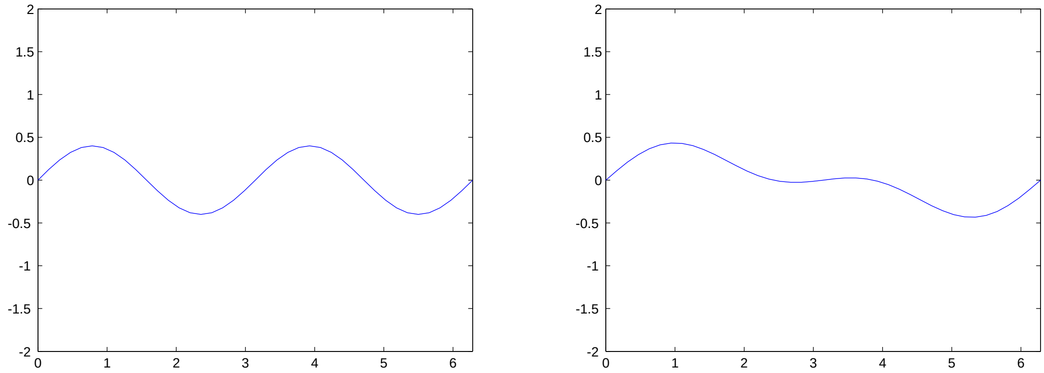
In the third example, we consider system (2) with inhomogeneous right-hand side over the interface $g \in H^{-1/2}(\Gamma)$ and $h \in H^{-1/2}(\Gamma)^2$, that is,

$$\begin{cases} (\Delta + k^2)\, v &= 0 \quad \text{in} \quad \Omega^+, \\ (\Delta^* + \omega^2\rho)\, u &= 0 \quad \text{in} \quad \Omega^-, \\ \eta\, u\cdot\nu - \partial_\nu v &= g \quad \text{on} \quad \Gamma, \\ Tu + v\,\nu &= h \quad \text{on} \quad \Gamma. \end{cases} \tag{70}$$

The investigations presented in Sections 2-7 are still true for (70). Now we choose g and h such that the exact solutions of (70) take the forms (66) and (67) with $a_1 = 1$, $a_2 = 2$ and $a_3 = -1$. In Figure 8 we present the numerical errors for grating 1 in the spaces V_0 and V_1 with respect to $1/h$ for $k = 1, 3$ and 5. Again, we observe that (69) holds. Next we consider the problem (70) for grating 2 with homogeneous data $f = g = 0$ and the corresponding approximate solution for a fixed mesh. We define $k_0 := \max\{k, k_p, k_s\}$ and set $N_0 := \max\{|n|:\ |\alpha_n| \le k_0 \text{ or } |\alpha_{-n}| \le k_0\}$ and

$$N_\tau := \min\left\{N:\ \frac{||(v^{sc}_{N,h}, u_{N,h}) - (v^{sc}_{20,h}, u_{20,h})||_{V_0}}{||(v^{sc}_{20,h}, u_{20,h})||_{V_0}} \le 0.001\right\}.$$

Table 2 exhibits the numbers N_τ and N_0 depending on the wave number k and the x_2-coordinates $\pm b$ of the truncation boundaries $\Gamma_b^\pm$. The truncation number N can be chosen relatively small. In our example, we even do not need to choose N_τ larger than N_0.



(a) Grating 1 (b) Grating 2

Figure 6: One-dimensional periodic interfaces.

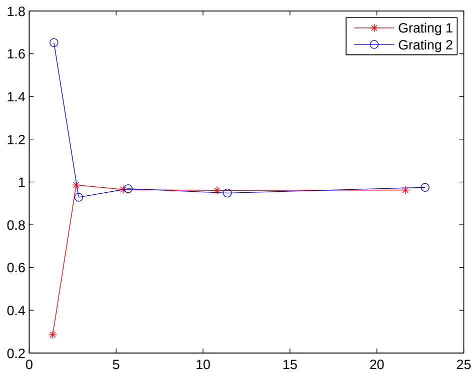
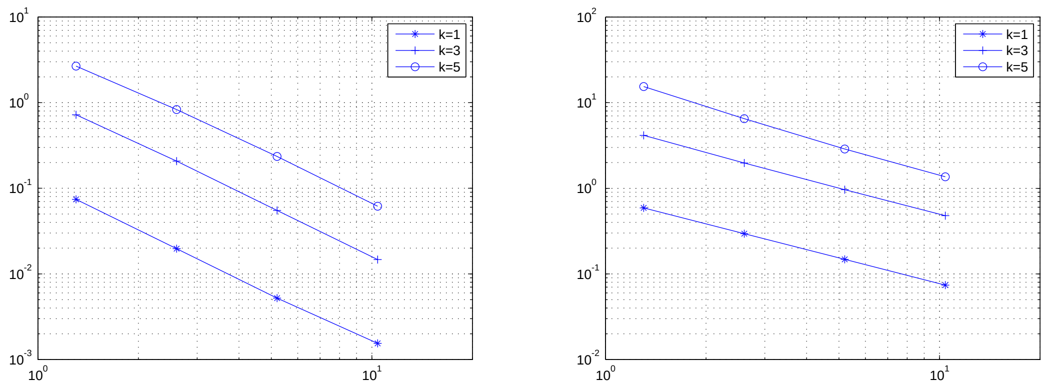


Figure 7: Numerical energy function $E_{N,h}$ vs. $1/h$ for the one-dimensional periodic interfaces.



(a) $||(v^{sc}, u) - (v^{sc}_{N,h}, u_{N,h})||_{V_0}$ (b) $||(v^{sc}, u) - (v^{sc}_{N,h}, u_{N,h})||_{V_1}$

Figure 8: Log-log plot of errors vs. $1/h$. Errors in V_0-norm (left) and V_1-norm (right).

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