Weierstraß-Institut
für Angewandte Analysis und Stochastik
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Balanced-Viscosity solutions for multi-rate system

Alexander Mielke [1,2], Riccarda Rossi [3], Giuseppe Savaré[4]

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[1] Weierstraß-Institut
Mohrenstraße 39
10117 Berlin
Germany
E-Mail: alexander.mielke@wias-berlin.de

[2] Institut für Mathematik
Humboldt-Universität zu Berlin
Rudower Chaussee 25
12489 Berlin-Adlershof
Germany

[3] DICATAM – Sezione di Matematica
Università di Brescia
via Valotti 9
I–25133 Brescia
Italy
E-Mail: riccarda.rossi@ing.unibs.it

[4] Dipartimento di Matematica "F. Casorati"
Università di Pavia
Via Ferrata 1
27100 Pavia
Italy
E-Mail: giuseppe.savare@unipv.it

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Weierstraß-Institut für Angewandte Analysis und Stochastik (WIAS)
Leibniz-Institut im Forschungsverbund Berlin e. V.
Mohrenstraße 39
10117 Berlin
Germany

Fax: +49 30 20372-303
E-Mail: preprint@wias-berlin.de
World Wide Web: http://www.wias-berlin.de/

Abstract

Several mechanical systems are modeled by the static momentum balance for the displacement u coupled with a rate-independent flow rule for some internal variable z. We consider a class of abstract systems of ODEs which have the same structure, albeit in a finite-dimensional setting, and regularize both the static equation and the rate-independent flow rule by adding viscous dissipation terms with coefficients ε^α and ε, where $0 < \varepsilon \ll 1$ and $\alpha > 0$ is a fixed parameter. Therefore for $\alpha \neq 1$ u and z have different relaxation rates.

We address the vanishing-viscosity analysis as $\varepsilon \downarrow 0$ of the viscous system. We prove that, up to a subsequence, (reparameterized) viscous solutions converge to a parameterized curve yielding a *Balanced Viscosity* solution to the original rate-independent system, and providing an accurate description of the system behavior at jumps. We also give a reformulation of the notion of *Balanced Viscosity* solution in terms of a system of subdifferential inclusions, showing that the viscosity in u and the one in z are involved in the jump dynamics in different ways, according to whether $\alpha > 1$, $\alpha = 1$, and $\alpha \in (0, 1)$.

1. Introduction

Several mechanical systems are described by ODE or PDE systems of the type:

$$\mathrm{D}_u \mathcal{E}(t, u(t), z(t)) = 0 \qquad \text{in } \mathcal{U}^*, \qquad \text{for a.a. } t \in (0, T), \tag{1.1a}$$

$$\partial \mathcal{R}_0(z'(t)) + \mathrm{D}_z \mathcal{E}(t, u(t), z(t)) \ni 0 \qquad \text{in } \mathcal{Z}^*, \qquad \text{for a.a. } t \in (0, T), \tag{1.1b}$$

where $\mathcal{U}$, $\mathcal{Z}$ are Banach spaces, and $\mathcal{E} : [0, T] \times \mathcal{U} \times \mathcal{Z} \to \mathbb{R}$ is an energy functional. For example, within the ansatz of generalized standard materials, u is the displacement, at equilibrium, while changes in the elastic behavior due to dissipative effects are described in terms of an internal variable z in some state space $\mathcal{Z}$. In several mechanical phenomena [Mie05], dissipation due to inertia and viscosity is negligible, and the system is governed by rate-independent evolution, which means that the (convex, nondegenerate) dissipation potential $\mathcal{R}_0 : \mathcal{Z} \to [0, \infty)$ is *positively homogeneous of degree* 1. Thus system (1.1b) is invariant for time-rescalings.

It is well known that, if the map $z \mapsto \mathcal{E}(t, u, z)$ is not uniformly convex, one cannot expect the existence of absolutely continuous solutions to system (1.1). This fact has motivated the development of various weak solvability concepts for (1.1), starting with the well-established notion of *energetic solution*. The latter dates back to [MiT99] and was further developed in [MiT04] (see [DFT05], as well, in the context of crack growth), cf. also [Mie05], [Mie11] and the references therein. Despite the several good features of the energetic formulation, it is known that, in the case the energy $z \mapsto \mathcal{E}(t, u, z)$ is nonconvex, the global stability condition may lead to jumps of z as a function of time that are not motivated by, or in accord with, the mechanics of the system, cf. e.g. the discussions in [Mie03, Ex. 6.1], [KMZ08, Ex. 6.3], and [MRS09, Ex. 1].

Over the last years, an alternative selection criterion of mechanically feasible weak solution concepts for the rate-independent system (1.1) has been developed, moving from the finite-dimensional analysis in [EfM06]. It is based on the interpretation of (1.1) as originating in the vanishing-viscosity limit of the *viscous* system

$$\mathrm{D}_u \mathcal{E}(t, u(t), z(t)) = 0 \qquad \text{in } \mathcal{U}^*, \qquad \text{for a.a. } t \in (0, T), \tag{1.2a}$$

$$\partial \mathcal{R}_0(z'(t)) + \varepsilon \partial \mathcal{V}_{\mathsf{z}}(z'(t)) + \mathrm{D}_z \mathcal{E}(t, u(t), z(t)) \ni 0 \qquad \text{in } \mathcal{Z}^*, \qquad \text{for a.a. } t \in (0, T), \tag{1.2b}$$

where $\mathcal{V}_{\mathsf{z}} : \mathcal{Z} \to [0, \infty)$ is a dissipation potential with *superlinear* (for instance, quadratic) growth at infinity. Observe that the existence of solutions for the *generalized gradient system* (1.2) follows from [CoV90, Col92], cf. also [MRS13b]. This vanishing-viscosity approach leads to a notion of solution featuring a *local*, rather than *global*, stability condition for the description of rate-independent evolution, thus avoiding "too early" and "too long" jumps. Furthermore, it provides an accurate description of the energetic behavior of the system at jumps, in particular highlighting how viscosity, neglected in the limit as $\varepsilon \downarrow 0$, comes back into the picture and governs the jump dynamics. This has been demonstrated in [MRS09, MRS12, MRS13a] within the frame of abstract, finite-dimensional and infinite-dimensional, rate-independent systems, and in [MiZ14] for a wide class parabolic equations with a rate-independent term. This analysis has also been developed in several applicative contexts, ranging from crack propagation [ToZ09, KMZ08], to plasticity [DDS11, DMDS12, BFM12, FrS13], and to damage [KRZ13], among others.

In this note, we shall perform the vanishing viscosity analysis of system (1.1) by considering the viscous approximation of (1.1a), in addition to the viscous approximation of (1.1b). More precisely, we will address the asymptotic analysis as $\varepsilon \downarrow 0$ of the system

$$\varepsilon^{\alpha}\partial\mathcal{V}_{\mathsf{u}}(u'(t)) + \mathrm{D}_u\mathcal{E}(t,u(t),z(t)) = 0 \qquad \text{in } \mathcal{U}^*, \qquad \text{for a.a. } t\in(0,T), \tag{1.3a}$$

$$\partial\mathcal{R}_0(z'(t)) + \varepsilon\partial\mathcal{V}_{\mathsf{z}}(z'(t)) + \mathrm{D}_z\mathcal{E}(t,u(t),z(t)) \ni 0 \qquad \text{in } \mathcal{Z}^*, \qquad \text{for a.a. } t\in(0,T), \tag{1.3b}$$

where $\alpha > 0$ and $\mathcal{V}_{\mathsf{u}}$ a quadratic dissipation potential for the variable u. Observe that (1.3) models systems with (possibly) *different* relaxation times. In fact , the parameter $\alpha > 0$ sets which of the two variables u and z relaxes faster to *equilibrium* and *rate-independent* evolution, respectively.

Let us mention that the analysis developed in this paper is in the mainstream of a series of recent papers focused on the coupling between rate-independent and viscous systems. First and foremost, in [Rou09] a wide class of rate-independent processes in viscous solids with inertia has been tackled, while the coupling with temperature has further been considered in [Rou10]. In fact, in these systems the evolution for the internal variable z is purely rate-independent and no vanishing viscosity is added to the equation for z, viscosity and inertia only intervene in the evolution for the displacement u. For these processes, the author has proposed a notion of solution of energetic type consisting of the weakly formulated momentum equation for the displacements (and also of the weak heat equation in [Rou10]), of an energy balance, and of a *semi-stability* condition. The latter reflects the mixed *rate dependent/independent* character of the system. In [Rou09] and [Rou13] a vanishing-viscosity analysis (in the momentum equation) has been performed. As discussed in [Rou13] in the context of delamination, this approach leads to *local solutions* (cf. also [Mie11]), describing crack initiation (i.e., delamination) in a physically feasible way. In [Rac12], the vanishing-viscosity approach has also been developed in the context of a model for crack growth in the two-dimensional antiplane case, with a pre-assigned crack path, coupling a viscoelastic momentum equation with a viscous flow rule for the crack tip; again, this procedure leads to solutions jumping later than energetic solutions. With a rescaling technique, a vanishing-viscosity analysis both in the flow rule, and in the momentum equation, has been recently performed in [DaS13] for perfect plasticity, recovering energetic solutions thanks to the convexity of the energy. In [Sca14], the same analysis has led to *local solutions* for a delamination system.

With the vanishing-viscosity analysis in this paper, besides finding good *local* conditions for the limit evolution, we want to add as an additional feature a thorough description of the energetic behavior of the solutions at jumps. This shall be deduced from an *energy balance*. Moreover, in comparison to the aforementioned contributions [Rac12, DaS13, Sca14] a greater emphasis shall be put here on how the multi-rate character of system (1.3) enters in the description of the jump dynamics. In particular, we will convey that viscosity in u and viscosity z are involved in the path followed by the system at jumps in (possibly) different ways, depending on whether the parameter α is strictly bigger than, or equal to, or strictly smaller than 1.

To focus on this and to avoid overburdening the paper with technicalities, we shall keep to a simple functional analytic setting. Namely, we shall consider the *finite-dimensional* and *smooth* case

$$\mathcal{U} = \mathbb{R}^n, \qquad \mathcal{Z} = \mathbb{R}^m, \qquad \mathcal{E} \in \mathrm{C}^1([0,T]\times\mathbb{R}^n\times\mathbb{R}^m)\,. \tag{1.4}$$

Obviously, this considerably simplifies the analysis, since the difficulties attached to nonsmoothness of the energy and to infinite-dimensionality are completely avoided. Still, even within such a simple setting (where, however, we will allow for state-dependent dissipation potentials $\mathcal{R}_0$, $\mathcal{V}_{\mathsf{z}}$, and $\mathcal{V}_{\mathsf{u}}$), the key ideas of our vanishing-viscosity approach can be highlighted.

Let us briefly summarize our results, focusing on a further simplified version of (1.3). In the setting of (1.4), and with the choices

$$\mathcal{V}_{\mathsf{u}}(u') = \frac{1}{2}|u'|^2, \qquad \mathcal{V}_{\mathsf{z}}(z') = \frac{1}{2}|z'|^2,$$

system (1.3) reduces to the ODE system

$$\varepsilon^\alpha u'(t) + \mathrm{D}_u \mathcal{E}(t,u(t),z(t)) = 0 \qquad \text{in } (0,T), \tag{1.5a}$$

$$\partial \mathcal{R}_0(z'(t)) + \varepsilon z'(t) + \mathrm{D}_z \mathcal{E}(t,u(t),z(t)) \ni 0 \qquad \text{in } (0,T). \tag{1.5b}$$

First of all, following [MRS09, MRS12, MRS13a], and along the lines of the *variational* approach to gradient flows by E. De Giorgi [Amb95, AGS08], we will pass to the limit as $\varepsilon \downarrow 0$ in the *energy-dissipation* balance associated (and equivalent, by Fenchel-Moreau duality and the chain rule for $\mathcal{E}$) to (1.5), namely

$$\begin{aligned} &\mathcal{E}(t,u(t),z(t)) + \int_s^t \mathcal{R}_0(z'(r)) + \frac{\varepsilon}{2}|z'(r)|^2 + \frac{\varepsilon^\alpha}{2}|u'(r)|^2 \,\mathrm{d}r \\ &\quad + \int_s^t \frac{1}{\varepsilon}\mathcal{W}_{\mathsf{z}}^*(\mathrm{D}_z\mathcal{E}(r,u(r),z(r))) + \frac{1}{2\varepsilon^\alpha}|\mathrm{D}_u\mathcal{E}(r,u(r),z(r))|^2 \,\mathrm{d}r \\ &= \mathcal{E}(s,u(s),z(s)) + \int_s^t \partial_t \mathcal{E}(r,u(r),z(r))\,\mathrm{d}r \end{aligned} \tag{1.6}$$

for all $0 \le s \le t \le T$, where $\mathcal{W}_{\mathsf{z}}^*$ is the Legendre transform of $\mathcal{R}_0 + \mathcal{V}_{\mathsf{z}}$. As we will see in Section 4, (1.6) is well-suited to unveiling the role played by viscosity in the description of the energetic behavior of the system at jumps. Indeed, it reflect the competition between the tendency of the system to be governed by *viscous* dissipation both for the variable z and for the variable u (with different rates if $\alpha \neq 1$), and its tendency to be *locally stable* in z, and at equilibrium in u. for u, cf. also the discussion in Remark 4.4.

Secondly, to develop the analysis as $\varepsilon \downarrow 0$ for a family of curves $(u_\varepsilon, z_\varepsilon)_\varepsilon \subset H^1(0,T;\mathbb{R}^n \times \mathbb{R}^m)$ fulfilling (1.6) we will adopt a by now well-established technique from [EfM06]. Namely, to capture the viscous transition paths at jump points, we will reparameterize the curves $(u_\varepsilon, z_\varepsilon)$, for instance by their arc-length. Hence we will address the analysis as $\varepsilon \downarrow 0$ of the *parameterized curves* $(\mathsf{t}_\varepsilon, \mathsf{u}_\varepsilon, \mathsf{z}_\varepsilon)_\varepsilon$ defined on the interval $[0,S]$ with values in the extended phase space $[0,T] \times \mathbb{R}^n \times \mathbb{R}^m$, with t_ε the rescaling functions and $\mathsf{u}_\varepsilon := u_\varepsilon \circ \mathsf{t}_\varepsilon$, $\mathsf{z}_\varepsilon := z_\varepsilon \circ \mathsf{t}_\varepsilon$. Under suitable conditions it can be proved that, up to a subsequence the curves $(\mathsf{t}_\varepsilon, \mathsf{u}_\varepsilon, \mathsf{z}_\varepsilon)_\varepsilon$ converge to a triple $(\mathsf{t},\mathsf{u},\mathsf{z}) \in \mathrm{AC}([0,S];[0,T]\times\mathbb{R}^n\times\mathbb{R}^m)$. Its evolution is described by an energy-dissipation balance obtained by passing to the limit in the reparameterized version of (1.6). cf. Theorem 4.5. We will refer to $(\mathsf{t},\mathsf{u},\mathsf{z})$ as a *parameterized Balanced Viscosity* solution to the rate-independent system $(\mathbb{R}^n \times \mathbb{R}^m, \mathcal{E}, \mathcal{R}_0 + \varepsilon\mathcal{V}_{\mathsf{z}} + \varepsilon^\alpha \mathcal{V}_{\mathsf{u}})$.

The main result of this paper, Theorem 5.3, provides a more transparent reformulation of the energy-dissipation balance defining a parameterized Balanced Viscosity solution $(\mathsf{t},\mathsf{u},\mathsf{z})$. It is in terms of a system of subdifferential inclusions fulfilled by the curve $(\mathsf{t},\mathsf{u},\mathsf{z})$, namely

$$\begin{aligned} &\theta_{\mathsf{u}}(s)\mathsf{u}'(s) + (1-\theta_{\mathsf{u}}(s))\mathrm{D}_u\mathcal{E}(\mathsf{t}(s),\mathsf{u}(s),\mathsf{z}(s)) \ni 0 && \text{for a.a. } s \in (0,S), \\ &(1-\theta_{\mathsf{z}}(s))\partial\mathcal{R}_0(\mathsf{q}(s),\mathsf{z}'(s)) + \theta_{\mathsf{z}}(s)\mathsf{z}'(s) + (1-\theta_{\mathsf{z}}(s))\mathrm{D}_z\mathcal{E}(\mathsf{t}(s),\mathsf{u}(s),\mathsf{z}(s)) \ni 0 && \text{for a.a. } s \in (0,S), \end{aligned} \tag{1.7}$$

where the Borel functions θ_{u}, $\theta_{\mathsf{z}} : [0,S] \to [0,1]$ fulfill

$$\mathsf{t}'(s)\theta_{\mathsf{u}}(s) = \mathsf{t}'(s)\theta_{\mathsf{z}}(s) = 0 \qquad \text{for a.a. } s \in (0,S), \tag{1.8}$$

The latter condition reveals that the viscous terms $\mathsf{u}'(s)$ and $\mathsf{z}'(s)$ may contribute to (1.7) only at jumps of the system, corresponding to $\mathsf{t}'(s) = 0$ as the function t records the (slow) external time scale. In this respect, (1.7)–(1.8) is akin to the (parameterized) subdifferential inclusion

$$\begin{aligned} &\mathrm{D}_u\mathcal{E}(\mathsf{t}(s),\mathsf{u}(s),\mathsf{z}(s)) \ni 0 && \text{for a.a. } s \in (0,S), \\ &\partial\mathcal{R}_0(\mathsf{z}'(s)) + \theta(s)\mathsf{z}'(s) + \mathrm{D}_z\mathcal{E}(\mathsf{t}(s),\mathsf{u}(s),\mathsf{z}(s)) \ni 0 && \text{for a.a. } s \in (0,S), \end{aligned} \tag{1.9}$$

with the Borel function $\theta : [0,S] \to [0,\infty)$ fulfilling

$$\mathsf{t}'(s)\theta(s) = 0 \qquad \text{for a.a. } s \in (0,S). \tag{1.10}$$

Indeed, (1.9) is the subdifferential reformulation for the parameterized Balanced Viscosity solutions obtained by taking the limit as $\varepsilon \downarrow 0$ in (1.2), where viscosity is added only to the flow rule. However, note that (1.7) has a much more complex structure than (1.9). In addition to the switching condition (1.8), the functions

θ_{u} and θ_{z} fulfill additional constraints, cf. Theorem 5.3. They differ in the three cases $\alpha > 1$, $\alpha = 1$, and $\alpha \in (0,1)$ and show that viscosity in u and z pops back into the description of the system behavior at jumps, in a way depending on whether u relaxes faster to equilibrium than z, u and z have the same relaxation rate, or z relaxes faster to local stability than u.

Plan of the paper. In Section 2 we set up all the basic assumptions on the dissipation potentials $\mathcal{R}_0$, $\mathcal{V}_{\mathsf{u}}$, and $\mathcal{V}_{\mathsf{z}}$. Section 3 is devoted to the generalized gradient system driven by $\mathcal{E}$ and the "viscous" potential $\mathcal{R}_\varepsilon := \mathcal{R}_0 + \varepsilon\mathcal{V}_{\mathsf{z}} + \varepsilon^\alpha\mathcal{V}_{\mathsf{u}}$. In particular, we establish a series of estimates on the viscous solutions $(u_\varepsilon, z_\varepsilon)$ which will be at the core of the vanishing viscosity analysis, developed in Section 4 with Theorem 4.5. In Section 5 we will prove Theorem 5.3 and explore the mechanical interpretation of parameterized Balanced Viscosity solutions. Finally, in Section 6 we will illustrate this solution notion, focusing on how it varies in the cases $\alpha > 1$, $\alpha = 1$, $\alpha \in (0,1)$, in two different examples.

Notation. In what follows, we will denote by $\langle\cdot,\cdot\rangle$ and by $|\cdot|$ the scalar product and the norm in any Euclidean space $\mathbb{R}^d$, with $d = n$, m, $n+m$, ... Moreover, we will use the same symbol C to denote a positive constant depending on data, and possibly varying from line to line.

2. Setup

As mentioned in the introduction, we are going to address a more general version of system (1.5), where the 1-positively homogeneous dissipation potential $\mathcal{R}_0$, as well as the quadratic potentials $\mathcal{V}_{\mathsf{u}}$ and $\mathcal{V}_{\mathsf{z}}$ for u' and z', are also depending on the state variable

$$q := (u,z) \in \mathcal{Q} := \mathbb{R}^n \times \mathbb{R}^m.$$

Hence, the rate-independent system is

$$\partial_{q'}\mathcal{R}_0(q(t), z'(t)) + \mathrm{D}_q\mathcal{E}(t, q(t)) \ni 0 \qquad \text{in } (0,T), \tag{2.1}$$

namely

$$\begin{aligned} &\mathrm{D}_u\mathcal{E}(t, u(t), z(t)) = 0 && \text{for a.a. } t \in (0,T), && (2.2a)\\ &\partial\mathcal{R}_0(q(t), z'(t)) + \mathrm{D}_z\mathcal{E}(t, u(t), z(t)) \ni 0 && \text{for a.a. } t \in (0,T). && (2.2b)\end{aligned}$$

We approximate it with the following generalized gradient system

$$\partial_{q'}\mathcal{R}_\varepsilon(q(t), q'(t)) + \mathrm{D}_q\mathcal{E}(t, q(t)) \ni 0 \qquad \text{in } (0,T), \tag{2.3}$$

where the overall dissipation potential $\mathcal{R}_\varepsilon$ is of the form

$$\mathcal{R}_\varepsilon(q,q') = \mathcal{R}_\varepsilon(q,(u',z')) := \mathcal{R}_0(q,z') + \varepsilon\mathcal{V}_{\mathsf{z}}(q;z') + \varepsilon^\alpha\mathcal{V}_{\mathsf{u}}(q;u') \quad \text{with } \alpha > 0. \tag{2.4}$$

In what follows, let us specify our assumptions on the dissipation potentials $\mathcal{R}_0$, $\mathcal{V}_{\mathsf{z}}$, and $\mathcal{V}_{\mathsf{u}}$.

Dissipation: We require that

$$\begin{aligned} &\mathcal{R}_0 \in \mathrm{C}^0(\mathcal{Q}\times\mathbb{R}^m), \quad \forall\, q \in \mathcal{Q}\ \mathcal{R}_0(q,\cdot) \text{ is convex and 1-positively homogeneous, and}\\ &\exists\, C_{0,R},\, C_{1,R} > 0\ \forall\,(q,z') \in \mathcal{Q}\times\mathbb{R}^m: \qquad C_{0,R}|z'| \le \mathcal{R}_0(q,z') \le C_{1,R}|z'|, \end{aligned} \tag{$\mathrm{R_0}$}$$

$$\begin{aligned} &\mathcal{V}_{\mathsf{z}}: \mathcal{Q}\times\mathbb{R}^m \to [0,\infty) \text{ is of the form } \mathcal{V}_{\mathsf{z}}(q;z') = \frac{1}{2}\langle \mathbb{V}_{\mathsf{z}}(q)z', z'\rangle \quad \text{with}\\ &\mathbb{V}_{\mathsf{z}} \in \mathrm{C}^0(\mathcal{Q};\mathbb{R}^{m\times m}) \quad \text{and} \quad \exists\, C_{0,V},\, C_{1,V} > 0\ \forall\, q \in \mathcal{Q}: \qquad C_{0,V}|z'|^2 \le \mathcal{V}_{\mathsf{z}}(q;z') \le C_{1,V}|z'|^2, \end{aligned} \tag{$\mathrm{V_z}$}$$

$$\begin{aligned} &\mathcal{V}_{\mathsf{u}}: \mathcal{Q}\times\mathbb{R}^n \to [0,\infty) \text{ is of the form } \mathcal{V}_{\mathsf{u}}(q;u') = \frac{1}{2}\langle \mathbb{V}_{\mathsf{u}}(q)u', u'\rangle \quad \text{with}\\ &\mathbb{V}_{\mathsf{u}} \in \mathrm{C}^0(\mathcal{Q};\mathbb{R}^{n\times n}) \quad \text{and} \quad \exists\, \widetilde{C}_{0,V},\, \widetilde{C}_{1,V} > 0\ \forall\, q \in \mathcal{Q}: \qquad \widetilde{C}_{0,V}|u'|^2 \le \mathcal{V}_{\mathsf{u}}(q;u') \le \widetilde{C}_{1,V}|u'|^2. \end{aligned} \tag{$\mathrm{V_u}$}$$

For later use, let us recall that, due to the 1-homogeneity of $\mathcal{R}_0(q,\cdot)$, for every $q \in \mathcal{Q}$ the convex analysis subdifferential $\partial\mathcal{R}_0(q,\cdot) : \mathbb{R}^m \rightrightarrows \mathbb{R}^m$ is characterized by

$$\zeta \in \partial\mathcal{R}_0(q,z') \quad \text{if and only if} \quad \begin{cases} \langle \zeta, w\rangle \le \mathcal{R}_0(q,w) & \text{for all } w \in \mathbb{R}^m, \\ \langle \zeta, z'\rangle \ge \mathcal{R}_0(q,z')\,. \end{cases} \tag{2.5}$$

Furthermore, observe that ($\mathrm{V_z}$) and ($\mathrm{V_u}$) ensure that for every $q \in \mathcal{Q}$ the matrices $\mathbb{V}_{\mathsf{z}}(q) \in \mathbb{R}^{n\times n}$ and $\mathbb{V}_{\mathsf{u}}(q) \in \mathbb{R}^{m\times m}$ are positive definite, uniformly with respect to q. Furthermore, for later use we observe that the conjugate

$$\mathcal{V}_{\mathsf{u}}^*(q;\eta) = \sup_{v\in\mathbb{R}^n} \left(\langle \eta, v\rangle - \mathcal{V}_{\mathsf{u}}(q;v)\rangle\right) = \frac{1}{2}\langle \mathbb{V}_{\mathsf{u}}(q)^{-1}\eta,\eta\rangle$$

fulfills

$$\overline{C}_0|\eta|^2 \le \mathcal{V}_{\mathsf{u}}^*(q;\eta) \le \overline{C}_1|\eta|^2 \tag{2.6}$$

for some $\overline{C}_0$, $\overline{C}_1 > 0$. We have the analogous coercivity and growth properties for $\mathcal{V}_{\mathsf{z}}^*$.

Our assumptions concerning the energy functional $\mathcal{E}$, expounded below, are typical of the *variational approach* to gradient flows and generalized gradient systems. Since we are in a finite-dimensional setting, to impose *coercivity* it is sufficient to ask for boundedness of energy sublevels. The power-control condition will allow us to bound $\partial_t\mathcal{E}$ in the derivation of the basic energy estimate on system (2.3), cf. Lemma 3.1 later on. The smoothness of $\mathcal{E}$ guarantees the validity of two further, key properties, i.e. the continuity of $\mathrm{D}_q\mathcal{E}$, and the chain rule (cf. (2.10) below), which will play a crucial role for our analysis.

Later on, in Section 3, we will impose that $\mathcal{E}$ is uniformly convex with respect to u. As we will see, this condition will be at the core of the proof of an estimate for $\|u'\|_{L^1(0,T;\mathbb{R}^n)}$, uniform with respect to the parameter ε. Observe that, unlike for z' such estimate does not follow from the basic energy estimate on system (2.3), since the overall dissipation potential $\mathcal{R}_\varepsilon$ is degenerate in u' as $\varepsilon \downarrow 0$. It will require additional careful calculations.

Energy: we assume that $\mathcal{E} \in \mathrm{C}^1([0,T]\times\mathcal{Q})$ and that it is bounded from below by a positive constant (indeed by adding a constant we can always reduce to this case). Furthermore, we require that

$$\begin{aligned} &\exists\, C_{0,E}\,, \widetilde{C}_{0,E} > 0\ \forall\,(t,q) \in [0,T]\times\mathcal{Q}\ : && \mathcal{E}(t,q) \ge C_{0,E}|q|^2 - \widetilde{C}_{0,E} && \textbf{(coercivity)}, \\ &\exists\, C_{1,E} > 0\ \forall\,(t,q) \in [0,T]\times\mathcal{Q}\ : && |\partial_t\mathcal{E}(t,q)| \le C_{1,E}\mathcal{E}(t,q) && \textbf{(power control)}. \end{aligned} \tag{E}$$

In view of (2.4), ($\mathrm{V_z}$), and ($\mathrm{V_u}$), the generalized gradient system (2.3) reads

$$\varepsilon^\alpha\mathbb{V}_{\mathsf{u}}(q(t))u'(t) + \mathrm{D}_u\mathcal{E}(t,u(t),z(t)) = 0 \qquad \text{in } (0,T), \tag{2.7a}$$

$$\varepsilon\mathbb{V}_{\mathsf{z}}(q(t))z'(t) + \partial\mathcal{R}_0(z'(t)) + \mathrm{D}_z\mathcal{E}(t,u(t),z(t)) = 0 \qquad \text{in } (0,T). \tag{2.7b}$$

Existence of solutions to the generalized gradient system (2.3)**.** It follows from the results in [CoV90, MRS13b] that, under the present assumptions, for every $\varepsilon > 0$ there exists a solution $q_\varepsilon \in H^1(0,T;\mathcal{Q})$ to the Cauchy problem for (2.3). Observe that q_ε also fulfills the energy-dissipation identity

$$\mathcal{E}(t,q_\varepsilon(t)) + \int_s^t \mathcal{R}_\varepsilon(q_\varepsilon(r),q_\varepsilon'(r)) + \mathcal{R}_\varepsilon^*(q_\varepsilon(r),-\mathrm{D}_q\mathcal{E}(r,q_\varepsilon(r)))\,\mathrm{d}r = \mathcal{E}(s,q_\varepsilon(s)) + \int_s^t \partial_t\mathcal{E}(r,q_\varepsilon(r))\,\mathrm{d}r. \tag{2.8}$$

In (2.8), the dual dissipation potential $\mathcal{R}_\varepsilon^* : \mathcal{Q}\times\mathbb{R}^{n+m} \to \mathbb{R}$ is the Fenchel-Moreau conjugate of $\mathcal{R}_\varepsilon$, i.e.

$$\mathcal{R}_\varepsilon^*(q,\xi) := \sup_{v\in\mathcal{Q}} \left(\langle \xi, v\rangle - \mathcal{R}_\varepsilon(q,v)\right). \tag{2.9}$$

In fact, by the Fenchel equivalence the differential inclusion (2.3) reformulates as

$$\mathcal{R}_\varepsilon(q_\varepsilon(t),q_\varepsilon'(t)) + \mathcal{R}_\varepsilon^*(q_\varepsilon(t),-\mathrm{D}_q\mathcal{E}(t,q_\varepsilon(t))) = \langle -\mathrm{D}_q\mathcal{E}(t,q_\varepsilon(t)),q_\varepsilon'(t)\rangle \qquad \text{for a.a. } t \in (0,T).$$

Combining this with the chain rule

$$\frac{\mathrm{d}}{\mathrm{d}t}\mathcal{E}(t,q(t)) = \partial_t\mathcal{E}(t,q(t)) + \langle \mathrm{D}_q\mathcal{E}(t,q(t)),q'(t)\rangle \qquad \text{for a.a. } t \in (0,T) \tag{2.10}$$

along any curve $q \in \mathrm{AC}([0,T];\mathcal{Q})$ and integrating in time, we conclude (2.8).

The energy balance (2.8) will play a crucial role in our analysis: indeed, after deriving in Sec. 3 a series of a priori estimates, uniform with respect to the parameter $\varepsilon > 0$, we shall pass to the limit in the parameterized version of (2.8) as $\varepsilon \downarrow 0$. We will thus obtain a (parameterized) energy-dissipation identity which encodes information on the behavior of the limit system for $\varepsilon = 0$, in particular at the jumps of the limit curve q of the solutions q_ε to (2.3).

3. A priori estimates

In this section, we consider a family $(q_\varepsilon)_\varepsilon \subset H^1(0,T;\mathcal{Q})$ of solutions to the Cauchy problem for (2.3), with a converging sequence of initial data $(q_\varepsilon^0)_\varepsilon$, i.e.

$$q_\varepsilon^0 \to q^0 \tag{3.1}$$

for some $q^0 \in \mathcal{Q}$.

Our first result, Lemma 3.1, provides a series of basic estimates on the functions (q_ε), as well as a bound for $\|z_\varepsilon'\|_{L^1(0,T;\mathbb{R}^m)}$, uniform with respect to ε. It holds under conditions (R_0), (V_z), (V_u), (E), as well as (3.1).

Under a further property of the dissipation potential $\mathcal{V}_\mathsf{u}$ (cf. $(\mathrm{V}_{u,1})$ below), assuming *uniform convexity* of $\mathcal{E}$ with respect to the variable u, and requiring an additional condition the initial data $(q_\varepsilon^0)_\varepsilon$ (see (3.5)), in Proposition 3.2 we will derive the following crucial estimate, uniform with respect to ε:

$$\|q_\varepsilon'\|_{L^1(0,T;\mathbb{R}^{n+m})} \leq C. \tag{3.2}$$

We start with the following result, which does not require the above mentioned enhanced conditions.

Lemma 3.1. *Let* $\alpha > 0$. *Assume* (R_0), (V_z), (V_u), (E), *and* (3.1). *Then, there exists a constant* $C > 0$ *such that for every* $\varepsilon > 0$

$$\text{(a)} \quad \sup_{t\in[0,T]} \mathcal{E}(t, q_\varepsilon(t)) \leq C, \tag{3.3a}$$

$$\text{(b)} \quad \sup_{t\in[0,T]} |q_\varepsilon(t)| \leq C, \tag{3.3b}$$

$$\text{(c)} \quad \int_0^T |z_\varepsilon'(r)| \, \mathrm{d}r \leq C. \tag{3.3c}$$

Proof. We exploit the energy identity (2.8). Observe that $\mathcal{R}_\varepsilon^*(q,\xi) \geq 0$ for all $(q,\xi) \in \mathcal{Q} \times R^{n+m}$. Therefore, we deduce from (2.8) that

$$\mathcal{E}(t, q_\varepsilon(t)) \leq \mathcal{E}(0, q_\varepsilon(0)) + \int_0^t \partial_t \mathcal{E}(r, q_\varepsilon(r)) \, \mathrm{d}r \leq C + C_{1,E} \int_0^t \mathcal{E}(r, q_\varepsilon(r)) \, \mathrm{d}r,$$

where we have used the power control from (E) and the fact that $\mathcal{E}(0, q_\varepsilon(0)) \leq C$, since the $(q_\varepsilon(0))_\varepsilon$ is bounded. The Gronwall Lemma then yields (3.3a), and (3.3b) ensues from the coercivity of $\mathcal{E}$. Using again the power control, we ultimately infer from (2.8) that

$$\int_0^T \mathcal{R}_\varepsilon(q_\varepsilon(r), q_\varepsilon'(r)) + \mathcal{R}_\varepsilon^*(q_\varepsilon(r), -\mathrm{D}_q\mathcal{E}(r, q_\varepsilon(r))) \, \mathrm{d}r \leq C. \tag{3.4}$$

In particular, $\int_0^T \mathcal{R}_0(q_\varepsilon(r), z_\varepsilon'(r)) \, \mathrm{d}r \leq C$, whence (3.3c) by (R_0). □

The derivation of the $L^1(0,T;\mathbb{R}^n)$-estimate for $(u_\varepsilon')_\varepsilon$ similar to (3.3c) clearly does not follow from (2.8), which only yields $\int_0^T \varepsilon^\alpha |u_\varepsilon'(r)|^2 \, \mathrm{d}r \leq C$ via (3.4) and (V_u). It is indeed more involved, and, as already mentioned, it strongly relies on the uniform convexity of $\mathcal{E}$ with respect to u. Furthermore, we are able to obtain it only under the simplifying condition that the dissipation potential $\mathcal{V}_\mathsf{u}$ in fact *does not* depend on the state variable q, and under an additional well-preparedness condition on the data $(q_\varepsilon^0)_\varepsilon$, ensuring that the forces $\mathrm{D}_u\mathcal{E}(0, q_\varepsilon^0)$ tend to zero, as $\varepsilon \downarrow 0$, with rate ε^α.

Proposition 3.2. *Let $\alpha > 0$. Assume* (R_0), $(\mathrm{V_z})$, $(\mathrm{V_u})$, *and* (E). *In addition, suppose that*

$$\mathrm{D}_q \mathbb{V}_{\mathsf{u}}(q) = 0 \quad \textit{for all } q \in \mathcal{Q}, \tag{$\mathrm{V}_{u,1}$}$$

$$\begin{aligned} &\mathcal{E} \in \mathrm{C}^2([0,T] \times \mathcal{Q}) \quad \textit{and} \\ &\exists\, \mu > 0\ \forall\, (t,q) \in [0,T] \times \mathcal{Q} \ :\ \mathrm{D}_u^2 \mathcal{E}(t,q) \geq \mu \mathbb{I}_{\mathbb{R}^{n\times n}} \quad \textbf{\textit{(uniform convexity w.r.t. }} u\textbf{\textit{)}}, \end{aligned} \tag{E_1}$$

and that the initial data $(q_\varepsilon^0)_\varepsilon$ complying with (3.1) *also fulfill*

$$|\mathrm{D}_u \mathcal{E}(0, q_\varepsilon^0)| \leq C\varepsilon^\alpha. \tag{3.5}$$

Then, there exists a constant $C > 0$ such that for every $\varepsilon > 0$

$$\|u_\varepsilon'(t)\|_{L^1(0,T;\mathbb{R}^n)} \leq C. \tag{3.6}$$

Proof. It follows from $(\mathrm{V}_{u,1})$ that there exists a given matrix $\overline{\mathbb{V}}_{\mathsf{u}} \in \mathbb{R}^{n\times n}$ such that

$$\mathbb{V}_{\mathsf{u}}(q) \equiv \overline{\mathbb{V}}_{\mathsf{u}} \quad \text{for all } q \in \mathcal{Q}, \tag{3.7}$$

so that

$$\mathcal{V}_{\mathsf{u}}(q; u') = \mathcal{V}_{\mathsf{u}}(u') := \frac{1}{2}\langle \overline{\mathbb{V}}_{\mathsf{u}} u', u' \rangle. \tag{3.8}$$

Therefore (2.7a) reduces to

$$\varepsilon^\alpha \overline{\mathbb{V}}_{\mathsf{u}} u_\varepsilon'(t) + \mathrm{D}_u \mathcal{E}(t, u_\varepsilon(t), z_\varepsilon(t)) = 0 \qquad \text{for a.a. } t \in (0,T). \tag{3.9}$$

We differentiate (3.9) in time, and test the resulting equation by u_ε'. Thus we obtain for almost all $t \in (0,T)$

$$\begin{aligned} 0 &= \varepsilon^\alpha \langle \overline{\mathbb{V}}_{\mathsf{u}} u_\varepsilon''(t), u_\varepsilon'(t) \rangle + \langle \mathrm{D}_u^2 \mathcal{E}(t, u_\varepsilon(t), z_\varepsilon(t))[u_\varepsilon'(t)], u_\varepsilon'(t) \rangle + \langle \mathrm{D}_{u,z}^2 \mathcal{E}(t, u_\varepsilon(t), z_\varepsilon(t))[u_\varepsilon'(t)], z_\varepsilon'(t) \rangle \\ &\doteq S_1 + S_2 + S_3, \end{aligned} \tag{3.10}$$

where $\mathrm{D}_{u,z}^2$ denotes the second-order mixed derivative. Observe that

$$\begin{aligned} &S_1 = \frac{\varepsilon^\alpha}{2} \frac{\mathrm{d}}{\mathrm{d}t} \mathcal{V}_{\mathsf{u}}(u_\varepsilon'), && S_2 \geq \mu |u_\varepsilon'^2| \geq \tilde{\mu} \mathcal{V}_{\mathsf{u}}(u_\varepsilon'), \\ &S_3 \geq -C|u_\varepsilon'||z_\varepsilon'| \geq -C\sqrt{\mathcal{V}_{\mathsf{u}}(u_\varepsilon')}|z_\varepsilon'|. \end{aligned}$$

Indeed, to estimate S_2 we have used the uniform convexity of $\mathcal{E}(t,\cdot,z)$, and the growth of $\mathcal{V}_{\mathsf{u}}$ from $(\mathrm{V_u})$. The estimate for S_3 follows from $\sup_{t\in(0,T)} |\mathrm{D}_{u,z}^2 \mathcal{E}(t, u_\varepsilon(t), z_\varepsilon(t))| \leq C$, due to (3.3b) and the fact that $\mathrm{D}_{u,z}^2 \mathcal{E}$ is continuous on $[0,T] \times \mathcal{Q}$, and again from $(\mathrm{V_u})$. We thus infer from (3.10) that

$$\frac{\mathrm{d}}{\mathrm{d}t} \mathcal{V}_{\mathsf{u}}(u_\varepsilon'(t)) + \frac{\tilde{\mu}}{\varepsilon^\alpha} \mathcal{V}_{\mathsf{u}}(u_\varepsilon'(t)) \leq \frac{C}{\varepsilon^\alpha} \sqrt{\mathcal{V}_{\mathsf{u}}(u_\varepsilon'(t))}|z_\varepsilon'(t)| \qquad \text{for a.a. } t \in (0,T),$$

which rephrases as

$$\nu_\varepsilon(t)\nu_\varepsilon'(t) + \frac{\tilde{\mu}}{\varepsilon^\alpha}\nu_\varepsilon^2(t) \leq \frac{C}{\varepsilon^\alpha}\nu_\varepsilon(t)|z_\varepsilon'(t)|$$

where we have used the place-holder $\nu_\varepsilon(t) := \sqrt{\mathcal{V}_{\mathsf{u}}(u_\varepsilon'(t))}$. We now argue as in [Mie11] and observe that, without loss of generality, we may suppose that $\nu_\varepsilon(t) > 0$ (otherwise, we replace it by $\tilde{\nu}_\varepsilon = \sqrt{\nu_\varepsilon + \delta}$, which satisfies the same estimate, and then let $\delta \downarrow 0$), Hence, we deduce

$$\nu_\varepsilon'(t) + \frac{\tilde{\mu}}{\varepsilon^\alpha}\nu_\varepsilon(t) \leq \frac{C}{\varepsilon^\alpha}|z_\varepsilon'(t)|.$$

Applying the Gronwall lemma we obtain

$$\nu_\varepsilon(t) \leq C \exp\left(-\frac{\tilde{\mu}}{\varepsilon^\alpha}t\right)\nu_\varepsilon(0) + \frac{C}{\varepsilon^\alpha}\int_0^t \exp\left(-\frac{\tilde{\mu}}{\varepsilon^\alpha}(t-r)\right)|z_\varepsilon'(r)|\,\mathrm{d}r \doteq a_1^\varepsilon(t) + a_2^\varepsilon(t) \tag{3.11}$$

for all $t \in (0,T)$. We integrate the above estimate on $(0,T)$. Now, observe that (3.5) guarantees that $\nu_\varepsilon(0) = \sqrt{\mathcal{V}_{\mathsf{u}}(u'_\varepsilon(0))} \leq C|\overline{V}_{\mathsf{u}} u'_\varepsilon(0)| = C\varepsilon^{-\alpha}|\mathrm{D}\mathcal{E}(0,u_\varepsilon(0))| \leq C$. Hence, we find $\|a_1^\varepsilon\|_{L^1(0,T)} \leq C\nu_\varepsilon(0) \leq C_1$. In order to estimate a_2^ε we use the Young inequality for convolutions, which yields

$$\|a_2^\varepsilon\|_{L^1(0,T)} = \frac{C}{\varepsilon^\alpha}\int_0^T\int_0^t \exp\left(-\frac{\tilde{\mu}}{4\varepsilon^\alpha}(t-r)\right)|z'_\varepsilon(r)|\,\mathrm{d}r\,\mathrm{d}t \leq \frac{C}{\varepsilon^\alpha}\left(\int_0^T \exp\left(-\frac{\tilde{\mu}}{\varepsilon^\alpha}t\right)\,\mathrm{d}t\right)\left(\int_0^T |z'_\varepsilon(t)|\,\mathrm{d}t\right) \leq C_2$$

where we have exploited the a priori estimate (3.3c) for z'_ε. Thus, (3.11) implies (3.6), and we are done. □

4. Limit passage with vanishing viscosity

In this section, we assume that we are given a sequence $(q_\varepsilon)_\varepsilon \subset H^1(0,T;\mathcal{Q})$ of solutions to (2.3), satisfying the initial conditions $q_\varepsilon(0) = q_\varepsilon^0$, such that estimate (3.2) holds. As we have shown in Proposition 3.2, the well-preparedness (3.5) of the initial data $(q_\varepsilon^0)_\varepsilon$, the condition that the dissipation potential $\mathcal{V}_{\mathsf{u}}$ does not depend on the state q, and the uniform convexity (E_1) of $\mathcal{E}$ with respect to u guarantee the validity of (3.2). However, these conditions are not needed for the vanishing viscosity analysis. Therefore, hereafter we will no longer impose(3.5), we will allow for a state-dependent dissipation potential $\mathcal{V}_{\mathsf{u}} = \mathcal{V}_{\mathsf{u}}(q;u')$, and we will stay with the basic conditions (E) on $\mathcal{E}$.

The energy-dissipation balance. Following the variational approach of [MRS09, MRS12, MRS13a], we will pass to the limit in (a *parameterized* version of) the energy identity (2.8).

Preliminarily, let us explicitly calculate the convex-conjugate of the dissipation potential $\mathcal{R}_\varepsilon$ (2.4).

Lemma 4.1. *Assume* (R_0)*,* $(\mathrm{V}_{\mathsf{z}})$*, and* $(\mathrm{V}_{\mathsf{u}})$*. Then, the Fenchel-Moreau conjugate* (2.9) *of* $\mathcal{R}_\varepsilon$ *is given by*

$$\mathcal{R}_\varepsilon^*(q,\xi) = \frac{1}{\varepsilon}\mathcal{W}_{\mathsf{z}}^*(q;\zeta) + \frac{1}{\varepsilon^\alpha}\mathcal{V}_{\mathsf{u}}^*(q;\eta) \qquad \text{for all } q \in \mathcal{Q} \text{ and } \xi = (\eta,\zeta) \in \mathbb{R}^{n+m}, \tag{4.1}$$

where $\mathcal{V}_{\mathsf{u}}^*(q;\cdot)$ *is the conjugate of* $\mathcal{V}_{\mathsf{u}}(q;\cdot)$*, and*

$$\mathcal{W}_{\mathsf{z}}^*(q;\zeta) = \min_{\omega\in K(q)} \mathcal{V}_{\mathsf{z}}^*(q;\zeta-\omega) \qquad \text{with } K(q) := \partial\mathcal{R}_0(q,0), \tag{4.2}$$

$\mathcal{V}_{\mathsf{z}}^*(q;\cdot)$ *is the conjugate of* $\mathcal{V}_{\mathsf{z}}(q;\cdot)$*, while* $\mathcal{W}_{\mathsf{z}}^*$ *is the conjugate of* $\mathcal{R}_0 + \mathcal{V}_{\mathsf{z}}$*.*

Proof. Since $\mathcal{R}_\varepsilon(q,\cdot)$ is given by the sum of a contribution in the sole variable z' and another in the sole variable u', we have

$$\mathcal{R}_\varepsilon^*(q,\xi) = (\varepsilon^\alpha\mathcal{V}_{\mathsf{u}})^*(q,\eta) + \mathcal{W}_{\mathsf{z},\varepsilon}^*(q;\zeta) \qquad \text{for all } \xi = (\eta,\zeta) \in \mathbb{R}^{n+m}$$

where we have used the place-holder $\mathcal{W}_{\mathsf{z},\varepsilon}^*(q;\zeta) := (\mathcal{R}_0(q,\cdot) + \varepsilon\mathcal{V}_{\mathsf{z}}(q;\cdot))^*(\zeta)$. Now, taking into account that $\mathcal{V}_{\mathsf{u}}$ is quadratic, there holds

$$(\varepsilon^\alpha\mathcal{V}_{\mathsf{u}})^*(q,\eta) = \varepsilon^\alpha\mathcal{V}_{\mathsf{u}}^*\left(q,\frac{1}{\varepsilon^\alpha}\eta\right) = \frac{1}{\varepsilon^\alpha}\mathcal{V}_{\mathsf{u}}^*(q;\eta),$$

whereas the inf-sup convolution formula (see e.g. [IoT79]) yields $\mathcal{W}_{\mathsf{z},\varepsilon}^*(q;\zeta) = \frac{1}{\varepsilon}\mathcal{W}_{\mathsf{z}}^*(q;\zeta)$ with $\mathcal{W}_{\mathsf{z}}^*(q;\cdot)$ from (4.2). □

In view of (4.1), the energy identity (2.8) rewrites as

$$\begin{aligned}
&\mathcal{E}(t,q_\varepsilon(t)) + \int_s^t \mathcal{R}_0(q_\varepsilon(r),z'_\varepsilon(r)) + \varepsilon\mathcal{V}_{\mathsf{z}}(q_\varepsilon(r);z'_\varepsilon(r)) + \varepsilon^\alpha\mathcal{V}_{\mathsf{u}}(q_\varepsilon(r);u'_\varepsilon(r))\,\mathrm{d}r \\
&\quad + \int_s^t \frac{1}{\varepsilon}\mathcal{W}_{\mathsf{z}}^*(q_\varepsilon(r);-\mathrm{D}_z\mathcal{E}(r,q_\varepsilon(r))) + \frac{1}{\varepsilon^\alpha}\mathcal{V}_{\mathsf{u}}^*(q_\varepsilon(r);-\mathrm{D}_u\mathcal{E}(r,q_\varepsilon(r)))\,\mathrm{d}r \\
&= \mathcal{E}(s,q_\varepsilon(s)) + \int_s^t \partial_t\mathcal{E}(r,q_\varepsilon(r))\,\mathrm{d}r.
\end{aligned} \tag{4.3}$$

In fact, the second and the third integral terms on the left-hand side of (4.3) reflect the competition between the tendency of the system to be governed by *viscous* dissipation both for the variable z and for the variable u, and its tendency to fulfill the *local stability* condition

$$\mathcal{W}_{\mathsf{z}}^*(q(t); -\mathrm{D}_z\mathcal{E}(t,q(t))) = 0 \quad \text{i.e.} \quad -\mathrm{D}_z\mathcal{E}(t,q(t)) \in K(q(t)) \qquad \text{for a.a. } t \in (0,T)$$

for z, and the equilibrium condition

$$\mathcal{V}_{\mathsf{u}}^*(q(t); -\mathrm{D}_u\mathcal{E}(r,q(t))) = 0 \quad \text{i.e.} \quad -\mathrm{D}_u\mathcal{E}(t,q(t)) = 0 \qquad \text{for a.a. } t \in (0,T)$$

for u, cf. also the discussion in Remark 4.4.

The parameterized energy-dissipation balance. We now consider the parameterized curves $(\mathsf{t}_\varepsilon, \mathsf{q}_\varepsilon) : [0, S_\varepsilon] \to [0,T] \times \mathcal{Q}$, where for every $\varepsilon > 0$ the rescaling function $\mathsf{t}_\varepsilon : [0, S_\varepsilon] \to [0,T]$ is strictly increasing, and $\mathsf{q}_\varepsilon(s) = q_\varepsilon(\mathsf{t}_\varepsilon(s))$. We shall suppose that $\sup_{\varepsilon>0} S_\varepsilon < \infty$, and that

$$\exists\, C > 0 \quad \forall\, \varepsilon > 0 \quad \forall\, s \in [0, S_\varepsilon] : \qquad \mathsf{t}_\varepsilon'(s) + |\mathsf{q}_\varepsilon'(s)| \leq C. \tag{4.4}$$

Remark 4.2. For instance, as in [EfM06, MRS09] we might choose

$$\mathsf{t}_\varepsilon := \sigma_\varepsilon^{-1} \quad \text{ with } \sigma_\varepsilon(t) := \int_0^t (1 + |q_\varepsilon'(r)|)\, \mathrm{d}r, \tag{4.5}$$

and set $S_\varepsilon := \sigma_\varepsilon(T)$. In fact, estimate (3.2) ensures that $\sup_\varepsilon S_\varepsilon < \infty$. With the choice (4.5) for t_ε, the functions $(\mathsf{t}_\varepsilon, \mathsf{q}_\varepsilon)$ fulfill the normalization condition

$$\mathsf{t}_\varepsilon'(s) + |\mathsf{q}_\varepsilon'(s)| = 1 \qquad \text{for almost all } s \in (0, S_\varepsilon).$$

For the parameterized curves $(\mathsf{t}_\varepsilon, \mathsf{q}_\varepsilon)$, the energy-dissipation balance (4.3) reads

$$\begin{aligned} &\mathcal{E}(\mathsf{t}_\varepsilon(s_2), \mathsf{q}_\varepsilon(s_2)) + \int_{s_1}^{s_2} \mathcal{M}_\varepsilon(\mathsf{q}_\varepsilon(r), \mathsf{t}_\varepsilon'(r), \mathsf{q}_\varepsilon'(r), -\mathrm{D}_q\mathcal{E}(\mathsf{t}_\varepsilon(r), \mathsf{q}_\varepsilon(r)))\ \mathrm{d}r \\ &= \mathcal{E}(\mathsf{t}_\varepsilon(s_1), \mathsf{q}_\varepsilon(s_1)) + \int_{s_1}^{s_2} \partial_t\mathcal{E}(\mathsf{t}_\varepsilon(r), \mathsf{q}_\varepsilon(r))\mathsf{t}_\varepsilon'(r)\, \mathrm{d}r \qquad \text{for all } 0 \leq s_1 \leq s_2 \leq S, \end{aligned} \tag{4.6}$$

where we have used the dissipation functional

$$\begin{aligned} \mathcal{M}_\varepsilon(q, \tau, q', \xi) &= \mathcal{M}_\varepsilon(q, \tau, (u', z'), (\eta, \zeta)) \\ &:= \mathcal{R}_0(q, z') + \frac{\varepsilon}{\tau}\mathcal{V}_{\mathsf{z}}(q; z') + \frac{\varepsilon^\alpha}{\tau}\mathcal{V}_{\mathsf{u}}(q; u') + \frac{\tau}{\varepsilon}\mathcal{W}_{\mathsf{z}}^*(q; \zeta) + \frac{\tau}{\varepsilon^\alpha}\mathcal{V}_{\mathsf{u}}^*(q; \eta). \end{aligned} \tag{4.7}$$

The passage from (4.3) to (4.6) follows from the change of variables $t \to \mathsf{t}_\varepsilon(r)$, whence $\mathrm{d}t \to \mathsf{t}_\varepsilon'(r)\, \mathrm{d}r$, while $q_\varepsilon'(t) \to \frac{1}{\mathsf{t}_\varepsilon'(r)}\mathsf{q}_\varepsilon'(r)$. In order to pass to the limit in (4.6) as $\varepsilon \downarrow 0$, it is crucial to investigate the Γ-convergence properties of the family of functionals $(\mathcal{M}_\varepsilon)_\varepsilon$. The following result reveals that the Γ-limit of $(\mathcal{M}_\varepsilon)_\varepsilon$ depends on whether the parameter α is above, equal, or below the threshold value 1. Let us point out that, for $\alpha \in (0,1)$, setting $\delta = \varepsilon^\alpha$ we rewrite $\mathcal{M}_\varepsilon$ as

$$\mathcal{M}_\varepsilon(q, \tau, (u', z'), (\eta, \zeta)) = \mathcal{R}_0(q, z') + \frac{\delta^{1/\alpha}}{\tau}\mathcal{V}_{\mathsf{z}}(q; z') + \frac{\delta}{\tau}\mathcal{V}_{\mathsf{u}}(q; u') + \frac{\tau}{\delta^{1/\alpha}}\mathcal{W}_{\mathsf{z}}^*(q; \zeta) + \frac{\tau}{\delta}\mathcal{V}_{\mathsf{u}}^*(q; \eta) \tag{4.8}$$

with $1/\alpha > 1$. It is thus natural to expect that the upcoming results will be specular in the cases $\alpha \in (0,1)$ and $\alpha > 1$.

Proposition 4.3. *Assume* (R_0), (V_z), (V_u), *and* (E). *Then, the functionals* $(\mathcal{M}_\varepsilon)_\varepsilon$ Γ*-converge as* $\varepsilon \downarrow 0$ *to* $\mathcal{M}_0 : \mathcal{Q} \times [0,\infty) \times \mathcal{Q} \times \mathbb{R}^{n+m} \to [0,\infty]$ *defined by*

$$\mathcal{M}_0(q, \tau, (u', z'), (\eta, \zeta)) := \mathcal{R}_0(q, z') + \mathcal{M}_0^{\mathrm{red}}(q, \tau, (u', z'), (\eta, \zeta)), \tag{4.9}$$

where for $\tau > 0$ *we have*

$$\mathcal{M}_0^{\mathrm{red}}(q, \tau, (u', z'), (\eta, \zeta)) = \begin{cases} 0 & \text{if } \mathcal{W}_{\mathsf{z}}^*(q; \zeta) = \mathcal{V}_{\mathsf{u}}^*(q; \eta) = 0, \\ \infty & \text{if } \mathcal{W}_{\mathsf{z}}^*(q; \zeta) + \mathcal{V}_{\mathsf{u}}^*(q; \eta) > 0, \end{cases} \tag{4.10}$$

while for $\tau = 0$ *we have the following cases:*

- *For* $\alpha > 1$

$$\mathcal{M}_0^{\mathrm{red}}(q,0,(u',z'),(\eta,\zeta)) = \begin{cases} 2\sqrt{\mathcal{V}_{\mathsf{u}}(q;u')}\,\sqrt{\mathcal{V}_{\mathsf{u}}^*(q;\eta)} & \text{if } \mathcal{V}_{\mathsf{z}}(q;z')=0, \\ 2\sqrt{\mathcal{V}_{\mathsf{z}}(q;z')}\,\sqrt{\mathcal{W}_{\mathsf{z}}^*(q;\zeta)} & \text{if } \mathcal{V}_{\mathsf{u}}^*(q;\eta)=0, \\ \infty & \text{if } \mathcal{V}_{\mathsf{z}}(q;z')\,\mathcal{V}_{\mathsf{u}}^*(q;\eta)>0, \end{cases} \tag{4.11}$$

- *For* $\alpha = 1$

$$\mathcal{M}_0^{\mathrm{red}}(q,0,(u',z'),(\eta,\zeta)) = 2\sqrt{\mathcal{V}_{\mathsf{z}}(q;z')+\mathcal{V}_{\mathsf{u}}(q;u')}\,\sqrt{\mathcal{W}_{\mathsf{z}}^*(q;\zeta)+\mathcal{V}_{\mathsf{u}}^*(q;\eta)}, \tag{4.12}$$

- *For* $\alpha \in (0,1)$

$$\mathcal{M}_0^{\mathrm{red}}(q,0,(u',z'),(\eta,\zeta)) = \begin{cases} 2\sqrt{\mathcal{V}_{\mathsf{u}}(q;u')}\,\sqrt{\mathcal{V}_{\mathsf{u}}^*(q;\eta)} & \text{if } \mathcal{W}_{\mathsf{z}}^*(q;\zeta)=0, \\ 2\sqrt{\mathcal{V}_{\mathsf{z}}(q;z')}\,\sqrt{\mathcal{W}_{\mathsf{z}}^*(q;\zeta)} & \text{if } \mathcal{V}_{\mathsf{u}}(q;u')=0, \\ \infty & \text{if } \mathcal{V}_{\mathsf{u}}(q;u')\,\mathcal{W}_{\mathsf{z}}^*(q;\zeta)>0. \end{cases} \tag{4.13}$$

Moreover, if $(\tau_\varepsilon, q'_\varepsilon) \rightharpoonup (\tau, q')$ *in* $L^1(0,S;(0,T)\times\mathcal{Q})$ *and if* $(q_\varepsilon,\xi_\varepsilon) \to (q,\xi)$ *in* $L^1(0,S;\mathcal{Q}\times\mathbb{R}^{n+m})$*, then for every* $0 \le s_1 \le s_2 \le S$

$$\liminf_{\varepsilon\downarrow 0} \int_0^S \mathcal{M}_\varepsilon(q_\varepsilon(s),\tau_\varepsilon(s),q'_\varepsilon(s),\xi_\varepsilon(s))\,\mathrm{d}s \ge \int_0^S \mathcal{M}_0(q(s),\tau(s),q'(s),\xi(s))\,\mathrm{d}s\,. \tag{4.14}$$

Remark 4.4. Let us briefly comment on the expression (4.9) of the Γ-limit $\mathcal{M}_0$. To do so, we rephrase the constraints arising in the switching conditions for the reduced functional $\mathcal{M}_0^{\mathrm{red}}$, cf. (4.10), (4.11), and (4.13). Indeed, it follows from ($\mathrm{V_z}$) and ($\mathrm{V_u}$) (cf. (2.6)) that

$$\begin{aligned} &\mathcal{V}_{\mathsf{z}}(q;z')=0 \;\Leftrightarrow\; z'=0, && \mathcal{V}_{\mathsf{u}}(q;u')=0 \;\Leftrightarrow\; u'=0, \\ &\mathcal{V}_{\mathsf{u}}^*(q;\eta)=0 \;\Leftrightarrow\; \eta=0, && \mathcal{W}_{\mathsf{z}}^*(q;\zeta)=0 \;\Leftrightarrow\; \zeta\in K(q)=\partial\mathcal{R}_0(q,0). \end{aligned}$$

Therefore, from (4.10) we read that for $\tau>0$ the functional $\mathcal{M}_0^{\mathrm{red}}(q,\tau,\cdot,\cdot)$ is finite (and indeed equal to 0) only for η and ζ fulfilling

$$\eta = 0, \qquad \zeta \in K(q)\,.$$

For $\tau=0$, in the case $\alpha>1$, $\mathcal{M}_0^{\mathrm{red}}(q,0,\cdot,\cdot)$ is finite if and only if either $z'=0$ or $\eta=0$. As we will see when discussing the physical interpretation of our vanishing-viscosity result, this means that, at a jump (i.e. when $\tau=0$), either $z'=0$, i.e. z is frozen, or u fulfills the equilibrium condition $\eta=\mathrm{D}_u\mathcal{E}(t,u)=0$.

Also in view of (4.8), the switching conditions for $\alpha\in(0,1)$ are specular to the ones for $\alpha>1$ in a generalized sense. In fact, $\mathcal{M}_0^{\mathrm{red}}(q,0,\cdot,\cdot)$ is finite if and only if either u is frozen, or $\zeta=\mathrm{D}_z\mathcal{E}(t,z)\in K(q)$, meaning that z fulfills the *local stability* condition.

Proof. Observe that

$$\mathcal{M}_\varepsilon(q,\tau,(u',z'),(\eta,\zeta)) = \mathcal{R}_0(q,z') + \mathcal{M}_\varepsilon^{\mathrm{red}}(q,\tau,(u',z'),(\eta,\zeta))$$

with $\mathcal{M}_\varepsilon^{\mathrm{red}}(q,\tau,(u',z'),(\eta,\zeta)) := \frac{\varepsilon}{\tau}\mathcal{V}_{\mathsf{z}}(q;z')+\frac{\varepsilon^\alpha}{\tau}\mathcal{V}_{\mathsf{u}}(q;u')+\frac{\tau}{\varepsilon}\mathcal{W}_{\mathsf{z}}^*(q;\zeta)+\frac{\tau}{\varepsilon^\alpha}\mathcal{V}_{\mathsf{u}}^*(q;\eta)$. Since $\mathcal{R}_0$ is continuous with respect to both variables q and z and does not depend on ε, it is clearly sufficient to prove that the functionals $\mathcal{M}_\varepsilon^{\mathrm{red}}$ Γ-converge to $\mathcal{M}_0^{\mathrm{red}}$, namely

Γ-$\liminf$ estimate:

$$\begin{aligned} &(q_\varepsilon,\tau_\varepsilon,u'_\varepsilon,z'_\varepsilon,\eta_\varepsilon,\zeta_\varepsilon)\to(q,\tau,u',z',\eta,\zeta) \quad \text{for } \varepsilon\to 0 \\ &\quad\Longrightarrow\quad \mathcal{M}_0^{\mathrm{red}}(q,\tau,(u',z'),(\eta,\zeta)) \le \liminf_{\varepsilon\downarrow 0}\mathcal{M}_\varepsilon^{\mathrm{red}}(q_\varepsilon,\tau_\varepsilon,(u'_\varepsilon,z'_\varepsilon),(\eta_\varepsilon,\zeta_\varepsilon)), \end{aligned} \tag{4.15}$$

Γ-$\limsup$ estimate:

$$\begin{aligned} &\forall\,(q,\tau,u',z',\eta,\zeta)\;\exists\,(q_\varepsilon,\tau_\varepsilon,u'_\varepsilon,z'_\varepsilon,\eta_\varepsilon,\zeta_\varepsilon)_\varepsilon\;: \\ &\qquad\begin{cases} (q_\varepsilon,\tau_\varepsilon,u'_\varepsilon,z'_\varepsilon,\eta_\varepsilon,\zeta_\varepsilon)\to(q,\tau,u',z',\eta,\zeta) \qquad \text{and} \\ \mathcal{M}_0^{\mathrm{red}}(q,\tau,(u',z'),(\eta,\zeta)) \ge \limsup_{\varepsilon\downarrow 0}\mathcal{M}_\varepsilon^{\mathrm{red}}(q_\varepsilon,\tau_\varepsilon,(u'_\varepsilon,z'_\varepsilon),(\eta_\varepsilon,\zeta_\varepsilon)). \end{cases} \end{aligned} \tag{4.16}$$

Preliminarily, observe that minimizing with respect to τ we obtain the lower bound

$$\mathcal{M}_\varepsilon^{\mathrm{red}}(q,\tau,(u',z'),(\eta,\zeta)) \geq 2\sqrt{\varepsilon \mathcal{V}_{\mathsf{z}}(q;z') + \varepsilon^\alpha \mathcal{V}_{\mathsf{u}}(q;u')}\sqrt{\frac{1}{\varepsilon}\mathcal{W}_{\mathsf{z}}^*(q;\zeta) + \frac{1}{\varepsilon^\alpha}\mathcal{V}_{\mathsf{u}}^*(q;\eta)}. \tag{4.17}$$

In all the three cases $\alpha > 1$, $\alpha = 1$, and $\alpha \in (0,1)$, the expression (4.10) of $\mathcal{M}_0^{\mathrm{red}}$ for $\tau > 0$ can be easily checked. Indeed, for the Γ-$\liminf$ estimate, observe that it is trivial in the case $\mathcal{W}_{\mathsf{z}}^*(q;\zeta) = \mathcal{V}_{\mathsf{u}}^*(q;\eta) = 0$, as $\mathcal{M}_\varepsilon^{\mathrm{red}}$ takes positive values for all $\varepsilon > 0$. Suppose now that $\mathcal{W}_{\mathsf{z}}^*(q;\zeta) + \mathcal{V}_{\mathsf{u}}^*(q;\eta) > 0$, e.g. that $\mathcal{V}_{\mathsf{u}}^*(q;\eta) > 0$. Now, $(q_\varepsilon,\eta_\varepsilon) \to (q,\eta)$ implies that $\mathcal{V}_{\mathsf{u}}^*(q_\varepsilon;\eta_\varepsilon) \geq \bar{c} > 0$ for sufficiently small ε, and from (4.17) we deduce that

$$\liminf_{\varepsilon\downarrow 0} \mathcal{M}_\varepsilon^{\mathrm{red}}(q_\varepsilon,\tau_\varepsilon,(u_\varepsilon',z_\varepsilon'),(\eta_\varepsilon,\zeta_\varepsilon)) = \infty = \mathcal{M}_0^{\mathrm{red}}(q,\tau,(u',z'),(\eta,\zeta))\,.$$

The Γ-$\limsup$ estimate follows by taking the recovery sequence $(q_\varepsilon \tau_\varepsilon, u_\varepsilon', z_\varepsilon', \eta_\varepsilon, \zeta_\varepsilon) = (q,\tau,u',z',\eta,\zeta)$. In fact, $\mathcal{W}_{\mathsf{z}}^*(q;\zeta) + \mathcal{V}_{\mathsf{u}}^*(q;\eta) > 0$, then the $\limsup$-inequality in (4.16) is trivial. If $\mathcal{W}_{\mathsf{z}}^*(q;\zeta) = \mathcal{V}_{\mathsf{u}}^*(q;\eta) = 0$, (4.16) can be checked straightforwardly.

For $\alpha = 1$, in the case $\tau = 0$, (4.17) clearly yields the Γ-$\liminf$ estimate, whereas the Γ-$\limsup$ one can be obtained by with the recovery sequence $(q_\varepsilon, \tau_\varepsilon, u_\varepsilon', z_\varepsilon', \eta_\varepsilon, \zeta_\varepsilon) = (q, \tau_\varepsilon^*, u', z', \eta, \zeta)$ with

$$\tau_\varepsilon^* = \varepsilon \frac{\sqrt{\mathcal{V}_{\mathsf{z}}(q;z') + \mathcal{V}_{\mathsf{u}}(q;u')}}{\sqrt{\mathcal{W}_{\mathsf{z}}^*(q;\zeta) + \mathcal{V}_{\mathsf{u}}^*(q;\eta)}}.$$

For $\alpha > 1$, in the case $\tau = 0$, the Γ-$\liminf$ estimate follows taking into account that (4.17) yields

$$\mathcal{M}_\varepsilon^{\mathrm{red}}(q,\tau,(u',z'),(\eta,\zeta)) \geq \frac{2}{\sqrt{\varepsilon^{\alpha-1}}}\sqrt{\mathcal{V}_{\mathsf{z}}(q;z')\mathcal{V}_{\mathsf{u}}^*(q;\eta)}. \tag{4.18}$$

Hence, if both $\mathcal{V}_{\mathsf{z}}(q;z') > 0$ and $\mathcal{V}_{\mathsf{u}}^*(q;\eta) > 0$, then $\liminf_{\varepsilon\downarrow 0} \mathcal{M}_\varepsilon^{\mathrm{red}}(q,\tau,(u',z'),(\eta,\zeta)) = \infty$. In the case when either $\mathcal{V}_{\mathsf{z}}(q;z') = 0$ or $\mathcal{V}_{\mathsf{u}}^*(q;\eta) = 0$, we deduce the Γ-$\liminf$ estimate from (4.17). For the Γ-$\limsup$ estimate, we again take the recovery sequence $(t, q, \tau_\varepsilon^{**}, u', z', \eta, \zeta)$, where now

$$\tau_\varepsilon^{**} = \varepsilon \frac{\sqrt{\mathcal{V}_{\mathsf{z}}(q;z') + \varepsilon^{\alpha-1}\mathcal{V}_{\mathsf{u}}(q;u')}}{\sqrt{\mathcal{W}_{\mathsf{z}}^*(q;\zeta) + \frac{1}{\varepsilon^{\alpha-1}}\mathcal{V}_{\mathsf{u}}^*(q;\eta)}}.$$

The discussion of the case $\alpha \in (0,1)$ is completely analogous, also in view of (4.8).

Finally, in order to prove (4.14), we apply the Ioffe Theorem [Iof77]. For this, we introduce a functional $\overline{\mathcal{M}} : [0,\infty) \times \mathcal{Q} \times [0,\infty) \times \mathcal{Q} \times \mathbb{R}^{n+m} \to [0,\infty]$ subsuming the functionals $\mathcal{M}_\varepsilon$ and $\mathcal{M}_0$, viz.

$$\overline{\mathcal{M}}(\varepsilon; q, \tau, q', \xi) := \begin{cases} \mathcal{M}_\varepsilon(q,\tau,q',\xi) & \text{if } \varepsilon > 0, \\ \mathcal{M}_0(q,\tau,q',\xi) & \text{if } \varepsilon = 0. \end{cases}$$

Arguing in the very same way as in the proof of [MRS09, Lemma 3.1], it can be inferred that the functional $\overline{\mathcal{M}}$ is lower semicontinuous on $[0,\infty) \times \mathcal{Q} \times [0,\infty) \times \mathcal{Q} \times \mathbb{R}^{n+m}$, and that $(\tau, q') \mapsto \overline{\mathcal{M}}(\varepsilon; q, \tau, q', \xi)$ is convex for all $(\varepsilon, q, \xi) \in [0,\infty) \times \mathcal{Q} \times \mathbb{R}^{n+m}$. Hence, the Ioffe Theorem ensures that

$$\liminf_{\varepsilon\downarrow 0} \int_0^S \overline{\mathcal{M}}(\varepsilon; q_\varepsilon(s), \tau_\varepsilon(s), q_\varepsilon'(s), \xi_\varepsilon(s))\,\mathrm{d}s \geq \int_0^S \overline{\mathcal{M}}(0; q(s), \tau(s), q'(s), \xi(s))\,\mathrm{d}s,$$

whence (4.14). $\square$

Observe that the functional $\mathcal{M}_0$ (4.9) fulfills for all $(q,\tau) \in \mathcal{Q} \times [0,\infty)$

$$\mathcal{M}_0(q,\tau,q',\xi) \geq \langle q', \xi\rangle = \langle u', \eta\rangle + \langle z', \zeta\rangle \qquad \text{for all } q' = (u',z') \in \mathcal{Q} \text{ and all } \xi = (\eta,\zeta) \in \mathbb{R}^{n+m}. \tag{4.19}$$

Indeed, for $\tau > 0$, the inequality is trivial if either $\mathcal{V}_{\mathsf{u}}^*(q;\eta) > 0$ or $\mathcal{W}_{\mathsf{z}}^*(q;\zeta) > 0$. When both of them equal 0, then $\eta = 0$ and $\langle q', \xi\rangle = \langle \zeta, z'\rangle \leq \mathcal{R}_0(q,z') = \mathcal{M}_0(q,\tau,q',\xi)$. For $\tau = 0$, e.g. in the case $\alpha > 1$ we have, if $z' = 0$,

$$\langle q', \xi\rangle = \langle \eta, u'\rangle \leq \sqrt{\langle \mathbb{V}_{\mathsf{u}}(q)u', u'\rangle}\sqrt{\langle \mathbb{V}_{\mathsf{u}}(q)^{-1}\eta, \eta'\rangle} = \mathcal{M}_0^{\mathrm{red}}(q,\tau,q',\xi) + 0 = \mathcal{M}_0(q,\tau,q',\xi)$$

while, if $\eta = 0$,

$$\begin{aligned}\langle q', \xi\rangle = \langle \zeta, z'\rangle &= \langle \zeta - \omega, z'\rangle + \langle \omega, z'\rangle \\ &\leq \sqrt{\langle \mathbb{V}_{\mathsf{z}}(q) z', z'\rangle}\sqrt{\langle \mathbb{V}_{\mathsf{z}}(q)^{-1}(\zeta - \omega), (\zeta - \omega)\rangle} + \mathcal{R}_0(z') = \mathcal{M}_0(q, \tau, q', \xi)\end{aligned}$$

where we have chosen $w \in K(q)$ such that $\mathcal{W}_{\mathsf{z}}^*(q;\zeta) = \mathcal{V}_{\mathsf{z}}^*(q;\zeta - \omega) = \frac{1}{2}\langle \mathbb{V}_{\mathsf{z}}(q)^{-1}(\zeta - \omega), (\zeta - \omega)\rangle$, and from the fact that $\langle \omega, z'\rangle \leq \mathcal{R}_0(z')$.

For the ensuing discussions, the set where (4.19) holds as an equality shall play a crucial role. We postpone its precise definition right before the statement of Proposition 4.8, cf. (4.30) ahead.

The vanishing-viscosity result. Theorem 4.5 below states that, up to a subsequence the parameterized solutions $(\mathsf{t}_\varepsilon, \mathsf{q}_\varepsilon)_\varepsilon$ of the (Cauchy problems for the) viscous system (2.3), converge to a parameterized curve (t, q), complying with the analog of the energy balance (4.6), with $\mathcal{M}_0$ in place of $\mathcal{M}_\varepsilon$.

We postpone after the proof of Theorem 4.5 a thorough analysis of the notion of solution to the rate-independent system (2.2) thus obtained. Let us instead mention in advance that the line of the argument for proving the limiting parameterized energy balance (4.22) is by now quite standard, cf. the proofs of [MRS09, Thm. 3.3], [MRS12, Thm. 5.5]. In fact, the *upper energy estimate* (i.e. the inequality $\leq$ for (4.22)) shall follow from lower semicontinuity arguments, based on the application of the Ioffe Theorem [Iof77]. The *lower energy estimate* $\geq$ will instead ensue from the chain rule (2.10). We also point out that, for the compactness argument it is actually not necessary to start from parameterized curves for which estimate (4.4) holds, uniformly w.r.t. time. In fact, the uniform integrability of the sequence $(\mathsf{t}_\varepsilon', \mathsf{q}_\varepsilon')_\varepsilon$ is sufficient, cf. (4.20) below.

Theorem 4.5. *Assume* (R_0)*,* (V_z)*,* (V_u)*, and* (E)*. Let* $(q_\varepsilon)_\varepsilon \subset H^1(0,T;\mathcal{Q})$ *be a sequence of solutions to the Cauchy problem for* (2.3)*. Choose nondecreasing surjective parameterizations* $\mathsf{t}_\varepsilon : [0, S_\varepsilon] \to [0,T]$ *and set* $\mathsf{q}_\varepsilon(s) = (\mathsf{u}_\varepsilon(s), \mathsf{z}_\varepsilon(s)) := q_\varepsilon(\mathsf{t}_\varepsilon(s))$ *for* $s \in [0, S_\varepsilon]$*. Suppose that* $S_\varepsilon \to S$ *as* $\varepsilon \downarrow 0$ *up to a subsequence, and that there exist* $q_0 \in \mathcal{Q}$ *and* $m \in L^1(0,S)$ *such that* $\mathsf{q}_\varepsilon(0) \to q_0$*, and*

$$m_\varepsilon := \mathsf{t}_\varepsilon' + |\mathsf{q}_\varepsilon'| \rightharpoonup m \qquad \text{in } L^1(0,S) \text{ as } \varepsilon \downarrow 0. \tag{4.20}$$

Then, there exist a (not-relabeled) subsequence and a parameterized curve $(\mathsf{t}, \mathsf{q}) \in \mathrm{AC}([0,S];[0,T]\times\mathcal{Q})$ *such that as* $\varepsilon \downarrow 0$

$$(\mathsf{t}_\varepsilon, \mathsf{q}_\varepsilon) \to (\mathsf{t}, \mathsf{q}) \text{ in } \mathrm{C}^0([0,S];[0,T]\times\mathcal{Q}), \tag{4.21}$$

$\mathsf{t}' + |\mathsf{q}'| \leq m$ *a.e. in* $(0,S)$*, and* (t, q) *fulfills the (parameterized) energy identity*

$$\begin{aligned}&\mathcal{E}(\mathsf{t}(s_2), \mathsf{q}(s_2)) + \int_{s_1}^{s_2} \mathcal{M}_0(\mathsf{q}(r), \mathsf{t}'(r), \mathsf{q}'(r), -\mathrm{D}_q\mathcal{E}(\mathsf{t}(r), \mathsf{q}(r)))\,\mathrm{d}r \\ &= \mathcal{E}(\mathsf{t}(s_1), \mathsf{q}(s_1)) + \int_{s_1}^{s_2} \partial_t \mathcal{E}(\mathsf{t}(r), \mathsf{q}(r))\mathsf{t}'(r)\,\mathrm{d}r \qquad \text{for all } 0 \leq s_1 \leq s_2 \leq S.\end{aligned} \tag{4.22}$$

Proof. Up to a reparameterization, we may suppose that the curves $(\mathsf{t}_\varepsilon, \mathsf{q}_\varepsilon)$ are defined on the fixed time interval $[0,S]$. We split the proof is three steps.

Step 1: compactness. Observe that for every $0 \leq s_1 \leq s_2 \leq S$

$$|\mathsf{q}_\varepsilon(s_1) - \mathsf{q}_\varepsilon(s_2)| \leq \int_{s_1}^{s_2} |\mathsf{q}_\varepsilon'(s)|\,\mathrm{d}s \leq \int_{s_1}^{s_2} m_\varepsilon(s)\,\mathrm{d}s\,. \tag{4.23}$$

Since $(\mathsf{q}_\varepsilon(0))_\varepsilon$ is bounded, we deduce from (4.23) that $(\mathsf{q}_\varepsilon)_\varepsilon \subset \mathrm{C}^0([0,S];\mathcal{Q})$ is bounded as well. What is more, as the family $(m_\varepsilon)_\varepsilon$ is uniformly integrable (4.20), $(\mathsf{q}_\varepsilon)_\varepsilon$ complies with the equicontinuity condition of the Ascoli-Arzelà Theorem and so does $(\mathsf{t}_\varepsilon)_\varepsilon$, by the analog of estimate (4.23). Hence, (4.21) follows. Taking into account that $\mathcal{E} \in \mathrm{C}^1([0,T]\times\mathcal{Q})$, we immediately conclude from (4.21) that

$$\mathcal{E}(\mathsf{t}_\varepsilon, \mathsf{q}_\varepsilon) \to \mathcal{E}(\mathsf{t}, \mathsf{q}), \qquad \mathrm{D}_q\mathcal{E}(\mathsf{t}_\varepsilon, \mathsf{q}_\varepsilon) \to \mathrm{D}_q\mathcal{E}(\mathsf{t}, \mathsf{q}), \qquad \partial_t\mathcal{E}(\mathsf{t}_\varepsilon, \mathsf{q}_\varepsilon) \to \partial_t\mathcal{E}(\mathsf{t}, \mathsf{q}) \quad \text{uniformly on } [0,S]. \tag{4.24}$$

Furthermore, (4.20) also yields that the sequences $(\mathsf{t}'_\varepsilon)_\varepsilon$ and $(\mathsf{q}'_\varepsilon)_\varepsilon$ are uniformly integrable. Thus, by the Pettis Theorem, up to a further extraction we find

$$\mathsf{t}'_\varepsilon \rightharpoonup \mathsf{t}' \qquad \text{in } L^1(0,S), \qquad \mathsf{q}'_\varepsilon \rightharpoonup \mathsf{q}' \qquad \text{in } L^1(0,S;\mathcal{Q}), \tag{4.25}$$

whence $\mathsf{t}' + |\mathsf{q}'| \leq m$ a.e. in $(0,S)$.

Step 2: upper energy estimate. We now take the limit as $\varepsilon \downarrow 0$ of the (parameterized) energy-dissipation balance (4.6) for every $0 \leq s_1 \leq s_2 \leq S$:

$$\begin{aligned}
&\mathcal{E}(\mathsf{t}(s_2),\mathsf{q}(s_2)) + \int_{s_1}^{s_2} \mathcal{M}_0(\mathsf{q}(r),\mathsf{t}'(r),\mathsf{q}'(r),-\mathrm{D}_q\mathcal{E}(\mathsf{t}(r),\mathsf{q}(r)))\,\mathrm{d}r \\
&\overset{(1)}{\leq} \lim_{\varepsilon\downarrow 0} \mathcal{E}(\mathsf{t}_\varepsilon(s_2),\mathsf{q}_\varepsilon(s_2)) + \liminf_{\varepsilon\downarrow 0} \int_{s_1}^{s_2} \mathcal{M}_\varepsilon(\mathsf{q}_\varepsilon(r),\mathsf{t}'_\varepsilon(r),\mathsf{q}'_\varepsilon(r),-\mathrm{D}_q\mathcal{E}(\mathsf{t}_\varepsilon(r),\mathsf{q}_\varepsilon(r)))\;\mathrm{d}r \\
&= \lim_{\varepsilon\downarrow 0} \mathcal{E}(\mathsf{t}_\varepsilon(s_1),\mathsf{q}_\varepsilon(s_1)) + \lim_{\varepsilon\downarrow 0} \int_{s_1}^{s_2} \partial_t\mathcal{E}(\mathsf{t}_\varepsilon(r),\mathsf{q}_\varepsilon(r))\mathsf{t}'_\varepsilon(r)\,\mathrm{d}r \\
&\overset{(2)}{=} \mathcal{E}(\mathsf{t}(s_1),\mathsf{q}(s_1)) + \int_{s_1}^{s_2} \partial_t\mathcal{E}(\mathsf{t}(r),\mathsf{q}(r))\mathsf{t}'(r)\,\mathrm{d}r\,,
\end{aligned} \tag{4.26}$$

where (1) follows from the energy convergence in (4.24) and the previously proved (4.14), and (2) from (4.24), again, combined with the first of (4.25). This concludes the upper energy estimate.

Step 3: lower energy estimate. We have for all $0 \leq s_1 \leq s_2 \leq S$ that

$$\begin{aligned}
&\mathcal{E}(\mathsf{t}(s_1),\mathsf{q}(s_1)) + \int_{s_1}^{s_2} \partial_t\mathcal{E}(\mathsf{t}(r),\mathsf{q}(r))\mathsf{t}'(r)\,\mathrm{d}r \\
&\overset{(1)}{=} \mathcal{E}(\mathsf{t}(s_2),\mathsf{q}(s_2)) + \int_{s_1}^{s_2} \langle -\mathrm{D}_q\mathcal{E}(\mathsf{t}(r),\mathsf{q}(r)),\mathsf{q}'(r)\rangle\,\mathrm{d}r \\
&\overset{(2)}{\leq} \mathcal{E}(\mathsf{t}(s_2),\mathsf{q}(s_2)) + \int_{s_1}^{s_2} \mathcal{M}_0(\mathsf{q}(r),\mathsf{t}'(r),\mathsf{q}'(r),-\mathrm{D}_q\mathcal{E}(\mathsf{t}(r),\mathsf{q}(r)))\,\mathrm{d}r\,,
\end{aligned} \tag{4.27}$$

where (1) follows from the chain rule, and (2) is due to inequality (4.19). In this way, we conclude (4.22).

Finally, combining (4.26) and (4.27) it is easy to deduce that

$$\lim_{\varepsilon\downarrow 0} \int_{s_1}^{s_2} \mathcal{M}_\varepsilon(\mathsf{q}_\varepsilon(r),\mathsf{t}'_\varepsilon(r),\mathsf{q}'_\varepsilon(r),-\mathrm{D}_q\mathcal{E}(\mathsf{t}_\varepsilon(r),\mathsf{q}_\varepsilon(r)))\,\mathrm{d}r = \int_{s_1}^{s_2} \mathcal{M}_0(\mathsf{q}(r),\mathsf{t}'(r),\mathsf{q}'(r),-\mathrm{D}_q\mathcal{E}(\mathsf{t}(r),\mathsf{q}(r)))\,\mathrm{d}r$$

for all $0 \leq s_1 \leq s_2 \leq S$, whence $\int_{s_1}^{s_2} \mathcal{R}_0(\mathsf{q}_\varepsilon(r),\mathsf{z}'_\varepsilon(r))\,\mathrm{d}r \to \int_{s_1}^{s_2} \mathcal{R}_0(\mathsf{q}(r),\mathsf{z}'(r))\,\mathrm{d}r$. $\square$

Balanced Viscosity parameterized solutions. Let us now gain further insight into the notion of solution to system (1.1) arising from the vanishing-viscosity limit. First of all, we fix its definition.

Definition 4.6. *Let* $(\mathcal{R}_0,\mathcal{V}_\mathsf{z},\mathcal{V}_\mathsf{u},\mathcal{E})$ *comply with* (R_0), (V_z), (V_u), *and* (E). *A curve* $(\mathsf{t},\mathsf{q}) \in \mathrm{AC}([0,S];[0,T]\times\mathcal{Q})$ *is called a* parameterized Balanced Viscosity *(*pBV*, for short) solution to the rate-independent system* $(\mathcal{Q},\mathcal{E},\mathcal{R}_0+\varepsilon\mathcal{V}_\mathsf{z}+\varepsilon^\alpha\mathcal{V}_\mathsf{u})$ *if* $\mathsf{t}:[0,S]\to[0,T]$ *is nondecreasing, and the pair* (t,q) *complies with the energy-dissipation balance* (4.22) *for all* $0 \leq s_1 \leq s_2 \leq S$.

Furthermore, (t,q) *is called*

- non-degenerate*, if*

$$\mathsf{t}'(s) + |\mathsf{q}'(s)| > 0 \qquad \text{for a.a. } s \in (0,S); \tag{4.28}$$

- surjective*, if* $\mathsf{t}:[0,S]\to[0,T]$ *is surjective.*

Remark 4.7. Observe that, even in the case when the function m in (4.20) is a.e. strictly positive, Theorem 4.5 does not guarantee the existence of non-degenerate pBV solutions. However, any degenerate pBV solution (t,q) can be reparameterized to a non-degenerate one $(\tilde{\mathsf{t}},\tilde{\mathsf{q}}):[0,\tilde{S}]\to[0,T]\times\mathcal{Q}$, even fulfilling the *normalization condition*

$$\tilde{\mathsf{t}}'(\sigma) + \tilde{\mathsf{q}}'(\sigma) = 1 \qquad \text{for a.a. } \sigma \in (0,\tilde{S})\,. \tag{4.29}$$

Indeed, following [MRS09, Rmk. 2], starting from a (possibly degenerate) solution (t,q), we set

$$\sigma(s) := \int_0^s \mathsf{t}'(r) + |\mathsf{q}'(r)|\,\mathrm{d}r \qquad \text{and } \tilde{S} := \sigma(S),$$

and define $(\tilde{\mathsf{t}}(\sigma), \tilde{\mathsf{q}}(\sigma)) := (\mathsf{t}(s), \mathsf{q}(s))$ if $\sigma = \sigma(s)$. Then, the very same calculations as in [MRS09, Rmk. 2] lead to (4.29).

We conclude this section with a characterization of pBV solutions in the same spirit as [MRS09, Prop. 2] and [MRS12, Prop. 5.3], [MRS13a, Cor. 4.5]. We show that the energy identity (4.22) defining the concept of pBV solutions is equivalent to the corresponding energy inequality on the interval $[0, S]$, and to the energy inequality in a differential form. Finally, (4.31) below provides a further reformulation of this solution concept which involves the *contact set* (cf. [MRS12, MRS13a])

$$\Sigma(q) := \{(\tau, q', \xi) \in [0,\infty) \times \mathcal{Q} \times \mathbb{R}^{n+m} \,:\, \mathcal{M}_0(q, \tau, q', \xi) = \langle q', \xi\rangle\} \tag{4.30}$$

Observe that for all $q \in \mathcal{Q}$ the set $\Sigma(q)$ is closed, as the functional $\mathcal{M}_0(q,\cdot,\cdot,\cdot)$ is lower semicontinuous. In Proposition we will provide 5.1 the explicit representation of $\Sigma(q)$. This and (4.31) we will be at the core of the reformulation of pBV solutions in terms of subdifferential inclusions, which we will discuss in Sec. 5.

Proposition 4.8. *Let* $(\mathcal{R}_0, \mathcal{V}_{\mathsf{z}}, \mathcal{V}_{\mathsf{u}}, \mathcal{E})$ *comply with* (R_0), $(\mathrm{V}_{\mathrm{z}})$, $(\mathrm{V}_{\mathrm{u}})$, *and* (E). *A curve* $(\mathsf{t},\mathsf{q}) \in \mathrm{AC}([0,S];[0,T]\times \mathcal{Q})$*, with* t *nondecreasing, is a* pBV *solution to the rate-independent system* $(\mathcal{Q}, \mathcal{E}, \mathcal{R}_0 + \varepsilon\mathcal{V}_{\mathsf{z}} + \varepsilon^\alpha \mathcal{V}_{\mathsf{u}})$ *if and only if one of the following equivalent conditions is satisfied:*

(1) (4.22) *holds as an inequality on* $(0,S)$*, i.e.*

$$\begin{aligned}&\mathcal{E}(\mathsf{t}(S),\mathsf{q}(S)) + \int_0^S \mathcal{M}_0(\mathsf{q}(r),\mathsf{t}'(r),\mathsf{q}'(r),-\mathrm{D}_q\mathcal{E}(\mathsf{t}(r),\mathsf{q}(r)))\ \mathrm{d}r\\ &\leq \mathcal{E}(\mathsf{t}(0),\mathsf{q}(0)) + \int_0^S \partial_t \mathcal{E}(\mathsf{t}(r),\mathsf{q}(r))\mathsf{t}'(r)\,\mathrm{d}r;\end{aligned}$$

(2) the above energy inequality holds in the differential form $\frac{\mathrm{d}}{\mathrm{d}s}\mathcal{E}(\mathsf{t},\mathsf{q}) + \mathcal{M}_0(\mathsf{q},\mathsf{t}',\mathsf{q}',-\mathrm{D}_q\mathcal{E}(\mathsf{t},\mathsf{q})) \leq \partial_t\mathcal{E}(\mathsf{t},\mathsf{q})\mathsf{t}'$ *a.e. in* $(0,S)$;

(3) the triple $(\mathsf{t}',\mathsf{q}',-\mathrm{D}_q\mathcal{E}(\mathsf{t},\mathsf{q}))$ *belongs to the contact set, i.e.*

$$(\mathsf{t}'(s),\mathsf{q}'(s),-\mathrm{D}_q\mathcal{E}(\mathsf{t}(s),\mathsf{q}(s))) \in \Sigma(\mathsf{q}(s)) \qquad \textit{for a.a. } s \in (0,S). \tag{4.31}$$

The proof of Proposition 4.8 is omitted: it follows by exploiting the chain rule (2.10), with arguments akin to those in the proof of Theorem 4.5, see also [MRS09, Prop. 2] and [MRS12, Prop. 5.3], [MRS13a, Cor. 4.5].

5. Physical interpretation

The following result provides a thorough description of the (closed) contact set $\Sigma(q)$, cf. (4.30). As we will see, the representation of $\Sigma(q)$ substantially different in the three cases $\alpha > 1$, $\alpha = 1$, and $\alpha \in (0,1)$. That is why, in Proposition 5.1 below we will use the notation $\Sigma_{\alpha>1}(q)$, $\Sigma_{\alpha=1}(q)$, and $\Sigma_{\alpha\in(0,1)}(q)$. We will prove that these sets are given by the union of subsets describing the various evolution regimes for the variables u and z. The notation for these subsets will be of the form

$$\mathrm{A}_{\mathsf{r}}\mathrm{B}_{\mathsf{s}} \qquad \text{with } \mathrm{A},\mathrm{B} \in \{\mathrm{E},\mathrm{R},\mathrm{V},\mathrm{B}\} \text{ and } \mathsf{r},\mathsf{s} \in \{\mathsf{u},\mathsf{z}\}.$$

The letters E, R, V, B stand for *Equilibrated*, *Rate-independent*, *Viscous*, and *Blocked*, respectively. For instance, $\mathrm{E}_{\mathsf{u}}\mathrm{R}_{\mathsf{z}}$ is the set of (τ, q', ξ) corresponding to equilibrium for u and rate-independent evolution for z, cf. (5.2) below; we postpone more comments after the statement of Proposition 5.1. Observe that all of these sets depend on the state variable q, as does $\Sigma(q)$. However, for simplicity we will not highlight this in their notation. In their description we shall always refer to the representation $q' = (u', z')$ for the velocity variable, and $\xi = (\eta, \zeta)$ for the force variable.

Proposition 5.1. *Assume* (R_0), $(\mathrm{V}_{\mathrm{z}})$, $(\mathrm{V}_{\mathrm{u}})$, *and* (E). *Then, for*

$\underline{\alpha > 1}$**:** *the contact set is given by*

$$\Sigma_{\alpha>1}(q) = \mathrm{E}_{\mathsf{u}}\mathrm{R}_{\mathsf{z}} \cup \mathrm{V}_{\mathsf{u}}\mathrm{B}_{\mathsf{z}} \cup \mathrm{E}_{\mathsf{u}}\mathrm{V}_{\mathsf{z}} \tag{5.1}$$

where

$$\mathrm{E}_{\mathsf{u}}\mathrm{R}_{\mathsf{z}} := \{(\tau, q', \xi) \,:\, \tau > 0, \quad q' = (u', z'),\ \xi = (0, \zeta) \ \textit{and}\ \partial\mathcal{R}_0(q, z') \ni \zeta\}, \tag{5.2}$$

$$\mathrm{V}_{\mathsf{u}}\mathrm{B}_{\mathsf{z}} := \{(\tau, q', \xi) \,:\, (\tau, q', \xi) = (0, (u', 0), (\eta, \zeta)) \ \textit{and}\ \exists\, \theta_{\mathsf{u}} \in [0,1] \,:\, \theta_{\mathsf{u}}\mathbb{V}_{\mathsf{u}}(q)u' = (1-\theta_{\mathsf{u}})\eta\}, \tag{5.3}$$

$$\begin{aligned} \mathrm{E}_{\mathsf{u}}\mathrm{V}_{\mathsf{z}} := \{(\tau, q', \xi) \,:\, & \tau = 0,\ q' = (u', z'),\ \xi = (0, \zeta)\ \textit{and} \\ & \exists\, \theta_{\mathsf{z}} \in [0,1] \,:\, (1-\theta_{\mathsf{z}})\partial\mathcal{R}_0(q, z') + \theta_{\mathsf{z}}\mathbb{V}_{\mathsf{z}}(q)z' \ni (1-\theta_{\mathsf{z}})\zeta\}. \end{aligned} \tag{5.4}$$

$\underline{\alpha = 1}$**:** *the contact set is given by*

$$\Sigma_{\alpha=1}(q) = \mathrm{E}_{\mathsf{u}}\mathrm{R}_{\mathsf{z}} \cup \mathrm{V}_{\mathsf{u}}\mathrm{V}_{\mathsf{z}} \tag{5.5}$$

where

$$\mathrm{V}_{\mathsf{u}}\mathrm{V}_{\mathsf{z}} := \left\{(\tau, q', \xi) \,:\, \tau = 0,\ \textit{and}\ \exists\, \theta \in [0,1] \,:\, \left\{ \begin{array}{l} \theta\mathbb{V}_{\mathsf{u}}(q)u' = (1-\theta)\eta, \\ (1-\theta)\partial\mathcal{R}_0(q, z') + \theta\mathbb{V}_{\mathsf{z}}(q)z' \ni (1-\theta)\zeta \end{array} \right. \right\}. \tag{5.6}$$

$\underline{\alpha \in (0,1)}$**:** *the contact set is given by*

$$\Sigma_{\alpha\in(0,1)}(q) = \mathrm{E}_{\mathsf{u}}\mathrm{R}_{\mathsf{z}} \cup \mathrm{B}_{\mathsf{u}}\mathrm{V}_{\mathsf{z}} \cup \mathrm{V}_{\mathsf{u}}\mathrm{R}_{\mathsf{z}} \tag{5.7}$$

with

$$\begin{aligned} \mathrm{B}_{\mathsf{u}}\mathrm{V}_{\mathsf{z}} := \{(\tau, q', \xi) \,:\, & \tau = 0,\ q' = (0, z'),\ \xi = (\eta, \zeta)\ \textit{and} \\ & \exists\, \theta_{\mathsf{z}} \in [0,1] \,:\, (1-\theta_{\mathsf{z}})\partial\mathcal{R}_0(q, z') + \theta_{\mathsf{z}}\mathbb{V}_{\mathsf{z}}(q)z' \ni (1-\theta_{\mathsf{z}})\zeta\}, \end{aligned} \tag{5.8}$$

$$\mathrm{V}_{\mathsf{u}}\mathrm{R}_{\mathsf{z}} := \left\{(\tau, q', \xi) \,:\, (\tau, q', \xi) = (0, (u', z'), (\eta, \zeta))\ \textit{and} \left\{ \begin{array}{l} \exists\, \theta_{\mathsf{u}} \in [0,1] \,:\, \theta_{\mathsf{u}}\mathbb{V}_{\mathsf{u}}(q)u' = (1-\theta_{\mathsf{u}})\eta, \\ \partial\mathcal{R}_0(q, z') \ni \zeta \end{array} \right. \right\}. \tag{5.9}$$

As (4.31) reveals, the contact set encompasses all the relevant information on the evolution of a *parameterized Balanced Viscosity* solution. The form of the sets $\mathrm{E}_{\mathsf{u}}\mathrm{R}_{\mathsf{z}}$, $\mathrm{V}_{\mathsf{u}}\mathrm{B}_{\mathsf{z}}$ ... which constitute it is strictly related to the mechanical interpretation of pBV solutions which shall be explored at the end of this section. Let us just explain here that

- the set $\mathrm{E}_{\mathsf{u}}\mathrm{R}_{\mathsf{z}}$ corresponds to equilibrium for the variable u (as $\eta = 0$), and a *stick-slip* regime for z, which evolves rate-independently as expressed by $\partial\mathcal{R}_0(q, z') \ni \zeta$. Observe that the stationary state $u' = z' = 0$ is also encompassed.
- The set $\mathrm{V}_{\mathsf{u}}\mathrm{B}_{\mathsf{z}}$ corresponds to the case in which the variable u still has to relax to an equilibrium and thus is governed by a *fast* dynamics at a jump $\tau = 0$, while z is "blocked by viscosity" and thus stays constant ($z' = 0$).
- The set $\mathrm{E}_{\mathsf{u}}\mathrm{V}_{\mathsf{z}}$ corresponds to the regime in which z evolves according to viscosity at a jump $\tau = 0$, and u follows z in such a way that it is at an equilibrium ($\eta = 0$).
- The set $\mathrm{V}_{\mathsf{u}}\mathrm{V}_{\mathsf{z}}$ corresponds to the case where the evolution of the system at a jump $\tau = 0$ is governed by viscosity both in u and in z.
- The set $\mathrm{B}_{\mathsf{u}}\mathrm{V}_{\mathsf{z}}$ encompasses the case in which the variable z at a jump $\tau = 0$ evolves according to viscosity, while u is blocked by viscosity ($u' = 0$).
- The set $\mathrm{V}_{\mathsf{u}}\mathrm{R}_{\mathsf{z}}$ describes viscous evolution for u and rate-independent evolution for z.

Remark 5.2. Let us stress once more that, as mentioned in advance, in the vanishing-viscosity limit the evolution regimes for $\alpha > 1$ and $\alpha \in (0,1)$ mirror each other. Indeed, formulae (5.1) and (5.7) are specular, up to observing that the analog of the equilibrium regime E_{u} is indeed the rate-independent regime R_{z}, see also Figure 5.1.

Proof of Proposition 5.1. In all the three cases $\alpha > 1$, $\alpha = 1$, and $\alpha \in (0,1)$, for $\tau > 0$ the contact condition $\mathcal{M}_0(q,\tau,q',\xi) = \langle \xi, q'\rangle$ can hold only if the constraints $\eta = 0$ and $\zeta \in K(q)$ are satisfied. Then, $\mathcal{M}_0(q,\tau,q',\xi) = \langle \xi, q'\rangle$ reduces to $\mathcal{R}_0(q,z') = \langle \zeta, z'\rangle$. Since $\zeta \in K(q)$, this is equivalent to $\zeta \in \partial\mathcal{R}_0(q,z')$ by (2.5). This gives the set $\mathrm{E_u R_z}$, which contributes to the contact set $\Sigma(q)$ in the three cases $\alpha > 1$, $\alpha = 1$, and $\alpha \in (0,1)$.

For $\alpha = 1$, observe that in the case $\tau = 0$ the contact condition is

$$\mathcal{R}_0(z') + 2\sqrt{\mathcal{V}_\mathsf{z}(q;z') + \mathcal{V}_\mathsf{u}(q;u')}\sqrt{\mathcal{W}^*_\mathsf{z}(q;\zeta) + \mathcal{V}^*_\mathsf{u}(q;\eta)} = \langle \zeta, z'\rangle + \langle \eta, u'\rangle. \tag{5.10}$$

Let us first address the case in which $\sigma_1 := \sqrt{\mathcal{V}_\mathsf{z}(q;z') + \mathcal{V}_\mathsf{u}(q;u')} = 0$ or $\sigma_2 := \sqrt{\mathcal{W}^*_\mathsf{z}(q;\zeta) + \mathcal{V}^*_\mathsf{u}(q;\eta)} = 0$. The former case corresponds to the stationary state $u' = z' = 0$, which means $\theta = 1$ in (5.6). The latter to $\mathcal{W}^*_\mathsf{z}(q;\zeta) = 0$ (if and only if $\zeta \in K(q)$) and $\eta = 0$ Hence (5.10) becomes $\mathcal{R}_0(z') = \langle \zeta, z'\rangle$, whence $\zeta \in \partial\mathcal{R}_0(q,z')$ by (2.5), again. This corresponds to $\theta = 0$ in (5.6). If $\sigma_1\sigma_2 > 0$, then we rewrite $2\sigma_1\sigma_2$ as $\lambda\sigma_1^2 + \frac{1}{\lambda}\sigma_2^2$, with $\lambda > 0$ given by $\lambda = \frac{\sigma_2}{\sigma_1}$. With such λ (5.10) rewrites as

$$\mathcal{R}_0(z') + \lambda(\mathcal{V}_\mathsf{z}(q;z') + \mathcal{V}_\mathsf{u}(q;u')) + \frac{1}{\lambda}(\mathcal{W}^*_\mathsf{z}(q;\zeta) + \mathcal{V}^*_\mathsf{u}(q;\eta)) = \langle \zeta, z'\rangle + \langle \eta, u'\rangle.$$

Upon multiplying both sides by λ, using that $\mathcal{V}_\mathsf{z}$ and $\mathcal{V}_\mathsf{u}$ are positively homogeneous of degree 2, and rearranging terms, we get

$$\mathcal{R}_0(z') + \mathcal{V}_\mathsf{z}(q;\lambda z') + \mathcal{W}^*_\mathsf{z}(q;\zeta) - \langle \zeta, \lambda z'\rangle = \langle \eta, \lambda u'\rangle - \mathcal{V}_\mathsf{u}(q;\lambda u') - \mathcal{V}^*_\mathsf{u}(q;\eta).$$

By the Fenchel-Moreau equivalence, this gives

$$\begin{aligned} &\mathbb{V}_\mathsf{u}(q)(\lambda u') = \eta, \\ &\partial\mathcal{R}_0(q,\lambda z') + \mathbb{V}_\mathsf{z}(q)(\lambda z') \ni \zeta \end{aligned}$$

with $\lambda > 0$. Then, (5.6) follows with $\theta \in (0,1)$ such that $\lambda = \frac{\theta}{1-\theta}$. All in all, for $\alpha = 1$ we have proved that, if $(\tau,q',\xi) \in \Sigma_{\alpha=1}(q)$, then either $(\tau,q',\xi) \in \mathrm{E_u R_z}$, or $(\tau,q',\xi) \in \mathrm{V_u V_z}$. This concludes the proof of (5.5) for $\Sigma_{\alpha=1}(q)$.

In the case $\alpha > 1$ and $\tau = 0$, $\mathcal{M}_0(q,\tau,q',\xi)$ is finite if and only if either $z' = 0$, or $\eta = 0$. In the former case, the contact condition reduces to $\sqrt{\langle \mathbb{V}_\mathsf{u}(q)u', u'\rangle}\sqrt{\langle \mathbb{V}_\mathsf{u}(q)^{-1}\eta, \eta\rangle} = \langle \eta, u'\rangle$, which is equivalent to the fact that there exists $\theta_\mathsf{u} \in [0,1]$ with $\theta_\mathsf{u}\mathbb{V}_\mathsf{u}(q)u' = (1-\theta_\mathsf{u})\eta$. This yields the set $\mathrm{V_u B_z}$. In the latter case, the contact condition rephrases as

$$\mathcal{R}_0(q,z') + \sqrt{\langle \mathbb{V}_\mathsf{z}(q)z', z'\rangle}\sqrt{\langle \mathbb{V}_\mathsf{z}(q)^{-1}(\zeta-\omega), \zeta-\omega\rangle} = \langle \zeta, z'\rangle = \langle \omega, z'\rangle + \langle \zeta-\omega, z'\rangle,$$

with $\omega \in K(q)$ such that $\mathcal{W}^*_\mathsf{z}(q;\zeta) = \frac{1}{2}\langle \mathbb{V}_\mathsf{z}(q)^{-1}(\zeta-\omega), \zeta-\omega\rangle$. It is immediate to check that the above chain of equalities implies

$$\begin{cases} \omega \in \partial\mathcal{R}_0(q,z'), \\ (1-\theta_\mathsf{z})(\zeta-\omega) = \theta_\mathsf{z}\mathbb{V}_\mathsf{z}(q)z' \quad \text{for some } \theta_\mathsf{z} \in [0,1]. \end{cases}$$

This yields the set $\mathrm{E_u V_z}$. All in all, in the case $\alpha > 1$ we have proved that, if $(\tau,q',\xi) \in \Sigma_{\alpha>1}(q)$, then either $(\tau,q',\xi) \in \mathrm{E_u R_z}$, or $(\tau,q',\xi) \in \mathrm{V_u B_z}$, or $(\tau,q',\xi) \in \mathrm{E_u V_z}$. This concludes (5.1).

The proof of (5.7) follows the very same lines and is thus omitted. □

The **main result** of this paper is the following theorem, which is in fact a direct consequence of the characterization (4.31) of pBV solutions in terms of the contact set, and of Proposition 5.1. Observe that, we confine ourselves to *non-degenerate* pBV solutions only. This is not restrictive, in view of Remark 4.7.

Theorem 5.3 (Reformulation as a system of subdifferential inclusions)**.** *Assume* $(\mathrm{R_0})$*,* $(\mathrm{V_z})$*,* $(\mathrm{V_u})$*, and* (E)*. A curve* $(\mathsf{t},\mathsf{q}) \in \mathrm{AC}([0,S];[0,T]\times\mathcal{Q})$ *with nondecreasing* t *is a* non-degenerate *parameterized Balanced Viscosity solution to the rate-independent system* $(\mathcal{Q},\mathcal{E},\mathcal{R}_0 + \varepsilon\mathcal{V}_\mathsf{z} + \varepsilon^\alpha\mathcal{V}_\mathsf{u})$ *if and only if* $\mathsf{t}' + |\mathsf{q}'| > 0$ *a.e. in* $(0,S)$ *and there exist two Borel functions* θ_u, $\theta_\mathsf{z} : [0,S] \to [0,1]$ *such that the pair* (t,q) *with* $\mathsf{q} = (\mathsf{u},\mathsf{z})$ *satisfies the system of equations for a.a.* $s \in (0,S)$*:*

$$\begin{aligned} &\theta_\mathsf{u}(s)\,\mathbb{V}_\mathsf{u}(\mathsf{q}(s))\mathsf{u}'(s) + (1-\theta_\mathsf{u}(s))\,\mathrm{D}_u\mathcal{E}(\mathsf{t}(s),\mathsf{u}(s),\mathsf{z}(s)) \ni 0, \\ &(1-\theta_\mathsf{z}(s))\,\partial\mathcal{R}_0(\mathsf{q}(s),\mathsf{z}'(s)) + \theta_\mathsf{z}(s)\,\mathbb{V}_\mathsf{z}(\mathsf{q}(s))\mathsf{z}'(s) + (1-\theta_\mathsf{z}(s))\,\mathrm{D}_z\mathcal{E}(\mathsf{t}(s),\mathsf{u}(s),\mathsf{z}(s)) \ni 0, \end{aligned} \tag{5.11}$$

with

$$\mathsf{t}'(s)\,\theta_{\mathsf{u}}(s) = \mathsf{t}'(s)\,\theta_{\mathsf{z}}(s) = 0 \tag{5.12}$$

and the following additional conditions depending on α*:*

$$\underline{\alpha > 1:} \qquad \theta_{\mathsf{u}}(s)\,(1-\theta_{\mathsf{z}}(s)) = 0; \tag{5.13}$$

$$\underline{\alpha = 1:} \qquad \theta_{\mathsf{u}}(s) = \theta_{\mathsf{z}}(s); \tag{5.14}$$

$$\underline{\alpha \in (0,1):} \qquad \theta_{\mathsf{z}}(s)\,(1-\theta_{\mathsf{u}}(s)) = 0. \tag{5.15}$$

Figure 5.1 displays the structure of the allowed values for the parameters $(t', \theta_{\mathsf{u}}, \theta_{\mathsf{z}})$ depending on α.

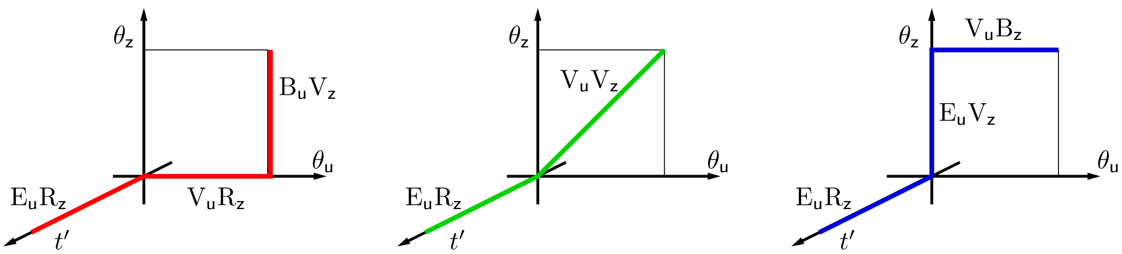

FIGURE 5.1. The switching between the different regimes, depending on the cases $\alpha < 1$, $\alpha = 1$, and $\alpha > 1$, are displayed via the allowed combinations of the triples $(t', \theta_{\mathsf{u}}, \theta_{\mathsf{z}})$.

Remark 5.4. Observe that the conditions (5.13) and (5.15) are specular (cf. Remark 5.2), revealing once more that the evolution regimes for $\alpha > 1$ and $\alpha < 1$ reflect each other. Nonetheless, a major difference occurs in that, under suitable conditions, for $\alpha > 1$ the regime $\mathrm{V_u B_z}$ only occurs at the beginning, when u relaxes fast to equilibrium, cf. Proposition 5.5.

Finally, let us get further insight into the mechanical interpretation of system (5.11), with the constraints (5.12) and (5.13)–(5.15). Preliminarily, let us point out that, as in the case of parameterized solutions to the rate-independent system

$$\partial\mathcal{R}_0(z(t), z'(t)) + \mathrm{D}_q\mathcal{I}(t, z(t)) \ni 0 \qquad \text{in } (0,T), \tag{5.16}$$

in the *sole* variable z, $\mathsf{t}'(s) = 0$ if and only if the system is jumping in the (slow) external time scale. Therefore, from (5.12) we gather that, in all of the three cases $\alpha > 1$, $\alpha = 1$, and $\alpha \in (0,1)$, when the system does not jump, then it is either in the *sticking* regime (i.e. $\mathsf{u}' = \mathsf{z}' = 0$), or in the *sliding regime*, namely the evolution of z is purely rate-independent (i.e. $\partial\mathcal{R}_0(\mathsf{q}, \mathsf{z}') + \mathrm{D}_z\mathcal{E}(\mathsf{t}, \mathsf{q}) \ni 0$), and u follows z in such a way that it is at an equilibrium (i.e. $-\mathrm{D}_u\mathcal{E}(\mathsf{t}, \mathsf{q}) = 0$). It is the description of the system behavior at jumps that significantly differs for $\alpha > 1$, $\alpha = 1$, and $\alpha \in (0,1)$.

$\underline{\text{Case } \alpha > 1:}$ fast relaxation of u. Here u relaxes faster to equilibrium than z. With (5.12) and (5.13) we are imposing at a jump that either $\mathsf{z}' = 0$ (which follows from $\theta_{\mathsf{z}} = 1$, i.e. $\mathrm{V_u B_z}$) or u is at equilibrium (corresponding to $\theta_{\mathsf{u}} = 0$, i.e. $\mathrm{E_u V_z}$). In fact, z cannot change until u has relaxed to equilibrium. When u has reached the equilibrium, then z may have either a *sliding jump* (i.e. $\theta_{\mathsf{z}} = 0$), or a *viscous jump* ($\theta_{\mathsf{z}} \in (0,1)$).

Our next result shows that, in fact, under the condition that the energy $\mathcal{E}$ is uniformly convex with respect to the variable u (cf. Proposition 3.2), after an initial phase in which z is constant and u relaxes to an equilibrium evolving by viscosity (i.e. the solution is in regime $\mathrm{V_u B_z}$), u never leaves the equilibrium afterwards. In that case the evolution of the system is completely described by z, which turns out to be a parameterized Balanced Viscosity solution to the rate-independent system driven by the *reduced energy functional* obtained minimizing out the variable u.

Proposition 5.5. *Assume* (R_0), (V_z), (V_u), *and* (E)*. Additionally, suppose that* $\mathcal{E}$ *complies with* (E_1)*, and denote by* $u = M(t,z)$ *the unique solution of* $\mathrm{D}_u\mathcal{E}(t,u,z) = 0$*, i.e. the minimizer of* $\mathcal{E}(t,\cdot,z)$*. Let* $(\mathsf{t},\mathsf{q}) \in \mathrm{AC}([0,S];[0,T]\times\mathcal{Q})$ *be a parameterized Balanced Viscosity solution to the rate-independent system* $(\mathcal{Q},\mathcal{E},\mathcal{R}_0+\varepsilon\mathcal{V}_\mathsf{z}+\varepsilon^\alpha\mathcal{V}_\mathsf{u})$ *with* $\alpha > 1$*. Set*

$$\mathfrak{S} := \{s \in [0,S] \,:\, \mathrm{D}_u\mathcal{E}(\mathsf{t}(s),\mathsf{q}(s)) = 0\}. \tag{5.17}$$

Then, $\mathfrak{S}$ *is either empty or it has the form* $[s_*,S]$ *for some* $s_* \in [0,S]$*.*

(a) Assume $s_* > 0$*, then for* $s \in [0,s_*) = [0,S]\setminus\mathfrak{S}$ *we have* $\mathsf{t}(s) = \mathsf{t}(0)$ *and* $\mathsf{z}(s) = \mathsf{z}(0)$*, whereas* u *is a solution to the reparameterized the gradient flow for* $(\mathbb{R}^n,\mathcal{E}(\mathsf{t}(0),\cdot,\mathsf{z}(0)),\mathbb{V}_\mathsf{u})$ *(regime* $\mathrm{V}_\mathsf{u}\mathrm{B}_\mathsf{z}$*), namely*

$$0 = \theta_\mathsf{u}(s)\,\mathbb{V}_\mathsf{u}(\mathsf{u}(s),\mathsf{z}(0))\dot{\mathsf{u}}(s) + (1-\theta_\mathsf{u}(s))\,\mathrm{D}_u\mathcal{E}(\mathsf{t}(0),\mathsf{u}(s),\mathsf{z}(0)) \quad \text{with } \mathsf{u}(0) \neq M(\mathsf{t}(0),\mathsf{z}(0)). \tag{5.18}$$

(b) Assume $\mathfrak{S} = [s_*,S]$ *with* $s_* < S$*, then for* $s \in [s_*,S]$ *we have* $\mathsf{u}(s) = M(\mathsf{t}(s),\mathsf{z}(s))$ *whereas the pair* (t,z) *is a parameterized Balanced Viscosity solution to the reduced rate-independent system* $(\mathbb{R}^m,\mathcal{J},\mathcal{R}_0+\varepsilon\mathcal{V}_\mathsf{z})$ *with the* reduced energy functional $\mathcal{J} : [0,T]\times\mathbb{R}^m \to \mathbb{R}; (t,z) \mapsto \min_{u\in\mathbb{R}^n}\mathcal{E}(t,u,z) = \mathcal{E}(t,M(t,z),z)$*, which corresponds to the regimes* $\mathrm{E}_\mathsf{u}\mathrm{V}_\mathsf{z}$ *and* $\mathrm{E}_\mathsf{u}\mathrm{R}_\mathsf{z}$*.*

Proof. To avoid overloaded notation we will often omit the state-dependence of the functions $\mathbb{V}_\mathsf{u}$ and $\mathbb{V}_\mathsf{z}$. For easy reference we repeat all the conditions for a BV solution (t,q) (cf. Theorem 5.3), in the case $\alpha > 1$:

$$\text{(i)}\quad 0 = \theta_\mathsf{u}\mathbb{V}_\mathsf{u}\mathsf{u}' + (1-\theta_\mathsf{u})\mathrm{D}_u\mathcal{E}(\mathsf{t},\mathsf{u},\mathsf{z}), \qquad \text{(ii)}\quad 0 \in (1-\theta_\mathsf{z})\partial\mathcal{R}_0(\mathsf{q},\mathsf{z}') + \theta_\mathsf{z}\mathbb{V}_\mathsf{z}\mathsf{z}' + (1-\theta_\mathsf{z})\mathrm{D}_z\mathcal{E}(\mathsf{t},\mathsf{u},\mathsf{z}),$$

$$\text{(iii)}\quad \mathsf{t}'\theta_\mathsf{u} = 0, \qquad \text{(iv)}\quad \mathsf{t}'\theta_\mathsf{z} = 0, \qquad \text{(v)}\quad \theta_\mathsf{u}\,(1-\theta_\mathsf{z}) = 0, \qquad \text{(vi)}\quad \mathsf{t}' + |\mathsf{u}'| + |\mathsf{z}'| > 0,$$

which have to hold for a.a. $s \in (0,S)$.

Step 1: By the continuity of (t,z) and $\mathrm{D}_u\mathcal{E}$ the set $\mathfrak{S}$ is closed, hence its complement is relatively open. Consider an interval (s_1,s_2) not intersecting with $\mathfrak{S}$. Using (i) we find $\theta_\mathsf{u} > 0$ a.e. in (s_1,s_2). Hence, (iii) implies $\mathsf{t}' = 0$ a.e., and we obtain $\mathsf{t}(s) = \mathsf{t}(s_1)$ for $s \in [s_1,s_2]$. By (v) we find $\theta_\mathsf{z} = 1$ a.e. Now, (ii) implies $z' = 0$ a.e., which implies $\mathsf{z}(s) = \mathsf{z}(s_1)$ for $s \in [s_1,s_2]$. From (vi) we conclude $\mathsf{u}' \neq 0$ a.e. Thus, we summarize

$$\mathsf{t}(s) = \mathsf{t}(s_1), \quad \mathsf{z}(s) = \mathsf{z}(s_1), \quad 0 = \mathbb{V}_u(\mathsf{u}(s),\mathsf{z}(s_1))\mathsf{u}'(s) + \lambda(s)\mathrm{D}_u\mathcal{E}(\mathsf{t}(s_1),\mathsf{u}(s),\mathsf{z}(s_1)),$$

where $\lambda(s) = (1-\theta_\mathsf{u}(s))/\theta_\mathsf{u}(s) \in (0,\infty)$ a.e. In particular, u satisfies (5.18). From $u \in \mathrm{AC}([0,S];\mathbb{R}^m)$ and (i) we obtain $\lambda \in L^1(s_1,s_2)$. Setting $\tau(s) = \int_{s_1}^s \lambda(\sigma)\,\mathrm{d}\sigma$ and defining the inverse $\hat{s}$ via $s = \hat{s}(\tau)$ we find $\hat{s}'(\tau) > 0$ and $\hat{s} \in W^{1,1}(0,\tau(s_2))$. Moreover, the function $\hat{u} : \tau \mapsto \mathsf{u}(\hat{s}(\tau))$ is a solution of the gradient flow

$$0 = \mathbb{V}_u(\hat{u}(\tau),\mathsf{z}(s_1))\hat{u}'(\tau) + \mathrm{D}_u\mathcal{E}(\mathsf{t}(s_1),\hat{u}(\tau),\mathsf{z}(s_1)). \tag{5.19}$$

Furthermore, we see that $s \mapsto \mathcal{E}(\mathsf{t}(s_1),\mathsf{u}(s),\mathsf{z}(s_1))$ is strictly decreasing on $[s_1,s_2]$, since its time derivative is given by $-\langle\mathsf{u}'(s),\mathbb{V}_u\mathsf{u}'(s)\rangle/\lambda(s)$ which is negative a.e.

Step 2: Since $\mathfrak{S}$ is closed the complement is an at most countable disjoint union of intervals of the form $(s_1,S]$, (s_2,s_3), $[0,s_4)$, or $[0,S]$ which are maximal in the sense that they cannot be extended without meeting $\mathfrak{S}$. Thus, for the "open" sides s_j this means $s_j \in \mathfrak{S}$. In the first two cases this means $\mathsf{u}(s_j) = M(\mathsf{t}(s_j),\mathsf{z}(s_j))$, i.e. we start a gradient flow with initial condition in the global minimizer. Hence, the solution stays constant for all future times, i.e. $\mathsf{u}(s) = \mathsf{u}(s_{1,2})$ for $s \in (s_1,S]$ or (s_2,s_3), respectively. But this contradicts the fact that $s \mapsto \mathcal{E}(\mathsf{t}(s_j),\mathsf{u}(s),\mathsf{z}(s_j))$ is strictly decreasing (cf. Step 1). Hence, the first two cases cannot occur, and we conclude $\mathfrak{S} = [s_*,S]$ with $s_* = s_4$ or $\mathfrak{S} = \emptyset$. In particular, assertion (a) is established.

Step 3: To show (b) assume $s \in \mathfrak{S} = [s_*,S]$, then $\mathsf{u}(s) = M(\mathsf{t}(s),\mathsf{z}(s))$ by the definition of $\mathfrak{S}$. Observe that $\mathrm{D}_z\mathcal{J}(t,z) = \mathrm{D}_z\mathcal{E}(t,M(t,z),z) + \mathrm{D}_zM(t,z)^T\mathrm{D}_u\mathcal{E}(t,M(t,z),z) = \mathrm{D}_z\mathcal{E}(t,M(t,z),z) + 0$. Thus, (t,z) solves

$$\text{(ii)'}\quad 0 \in (1-\theta_\mathsf{z})\partial\mathcal{R}_0(\mathsf{z},\mathsf{z}') + \theta_\mathsf{z}\mathbb{V}_\mathsf{z}\mathsf{z}' + (1-\theta_\mathsf{z})\mathrm{D}_z\mathcal{J}(\mathsf{t},\mathsf{z}), \qquad \text{(iv)'}\quad \mathsf{t}'\theta_\mathsf{z} = 0, \qquad \text{(vi)'}\quad \mathsf{t}' + |\mathsf{z}'| > 0,$$

which proves that (t,z) is a BV solution of the reduced system. For the latter relation note that $\mathsf{t}'(s)+|\mathsf{z}'(s)| = 0$ implies $\mathsf{u}'(s) = \frac{\mathrm{d}}{\mathrm{d}s}M(\mathsf{t}(s),\mathsf{z}(s)) = 0$ so that (vi)' follows from (vi). ☐

Our approach in Step 1 of the above proof uses the qualitative ideas from [Zan07, ARS14], but our reduction to the simpler convex case makes the analysis much easier.

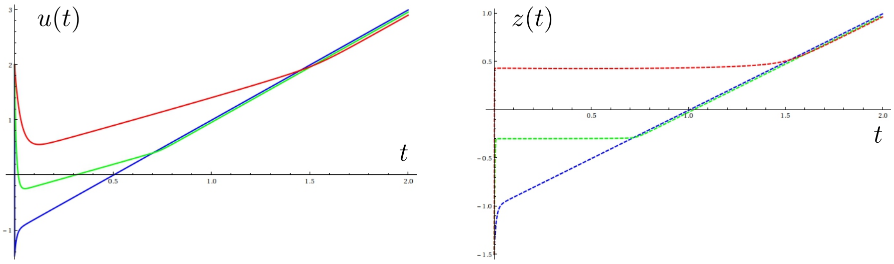

FIGURE 6.1. Solutions for (6.1) for the three cases $\alpha = 2$ (blue), $\alpha = 1$ (green), and $\alpha = 1/2$ (red).

Case $\alpha = 1$: comparable relaxation times, Here u and z relax at the same rate. At a jump, the system may switch to the viscous regime $\mathrm{V_u V_z}$, where *both* in the evolution of u, and in the evolution for z, viscous dissipation intervenes, modulated by the same coefficient $\theta = \theta_\mathsf{u} = \theta_\mathsf{z}$.

Case $\alpha \in (0, 1)$: fast relaxation of z. Here z relaxes faster than u, and jumps in the z-component are faster than jumps in the u-component. If z jumps (possibly governed by viscous dissipation), than u stays fixed, i.e. u is blocked while z moves viscously (regime $\mathrm{B_u V_z}$). But then u has still to relax to equilibrium, and it will do it on a faster scale than the rate-independent motion of z, if z stays in locally stable states (regime $\mathrm{V_u R_z}$). Finally, full rate-independent behavior in the regime $\mathrm{E_u R_z}$ will occur, where $\mathsf{t}'(s) > 0$. Unlike in the case $\alpha > 1$, all three regimes may occur more than once in the evolution of the system, see Section 6.2 for an example.

6. EXAMPLES

To illustrate the difference between the three limit models (namely for $\alpha > 1$, $\alpha = 1$, and $\alpha \in (0, 1)$), we discuss two examples. The first one treats a quadratic energy and emphasizes the different initial behavior before the solution converges to a truly rate-independent regime. In the second example we show that solutions that start in a rate-independent regime and coincide for the three different limit models may separate if viscous jumps start, leading to different rate-independent behavior afterwards.

6.1. Initial relaxation for a system with quadratic energy. We consider the energy functional $\mathcal{E}(t, u, z) = \frac{1}{2}(u - z)^2 + \frac{1}{2}z^2 - tu$ and trivial viscous energies leading to the ODE system

$$\begin{cases} 0 = \varepsilon^\alpha \dot{u} + u - z - t, \\ 0 \in \mathrm{Sign}(\dot{z}) + \varepsilon \dot{z} + 2z - u \end{cases} \qquad \text{with } (u(0), z(0)) = (2, -3/2). \tag{6.1}$$

We show simulations for the three cases $\alpha = 2$ (blue), $\alpha = 1$ (green), and $\alpha = 1/2$ (red) with sufficiently small ε (typically $0.001 \dots 0.03$). The components u and z as functions of time are depicted in Figure 6.1.

However, to detect different jump behavior at $t \approx 0$ it is advantageous to look at the parameterized solutions, which are depicted in Figure 6.2, showing (t, q) for the three different cases. The parameterization was calculated using $\dot{s}(t) = \max\{0.5, |\dot{u}(t)|, \dot{z}(t)|\}$. In the parameterized form we fully see the structure of the jump for $t \approx 0$. For $\alpha = 2$ we obtain first a jump from the initial datum $(u, z) = (2, -1.5)$ to $(u, z) = (-1.5, -1.5)$ on the timescale ε^2, which is the regime $\mathrm{V_u B_z}$. Then, u is equilibrated, and a jump to $(-1, -1)$ along the diagonal $u = z$ occurs on the timescale ε, which is the regime $\mathrm{E_u V_z}$. Finally, the solution finds the rate-independent regime $\mathrm{E_u R_z}$ with $(u(t), z(t)) = q_{\mathrm{ri}}(t) := (2t{-}1, t{-}1)$.

For $\alpha = 1/2$ the solution first jumps to $(2, 0.5)$ on the time scale ε, which is the regime $\mathrm{B_u V_z}$. Next, and then there is a jump to $(0.5, 0.5)$ in the time scale $\varepsilon^{1/2}$, which is regime $\mathrm{V_u R_z}$. Then, the rate-independent regime $\mathrm{E_u R_z}$ starts, namely via $(u(t), z(t)) = (t{-}0.5, 0.5)$ for $t \in]0, 1.5]$ and q_{ri} for $t > 1.5$.

The behavior for $\alpha = 1$ is intermediate: the jump occurs along a nonlinear curve in regime $\mathrm{V_u V_z}$, and q_{ri} is joined for $t \geq t_* \approx 0.7$, which is regime $\mathrm{E_u R_z}$.

The different behavior and the different regimes are also nicely seen by plotting the trajectories in the (u, z)-plane, see Figure 6.3, where the three different cases for α are depicted again.

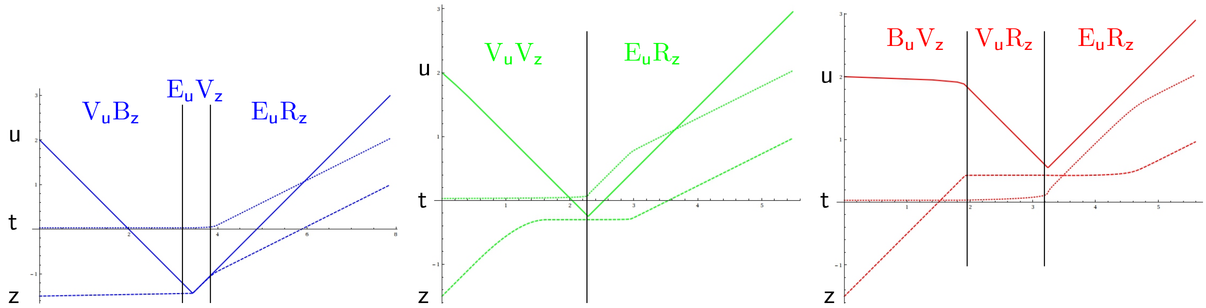

FIGURE 6.2. Solutions $(\mathsf{t}, \mathsf{u}, \mathsf{z})$ for (6.1) with dotted t, full u, and dashed z. Left $\alpha = 2$, middle $\alpha = 1$, right $\alpha = 1/2$.

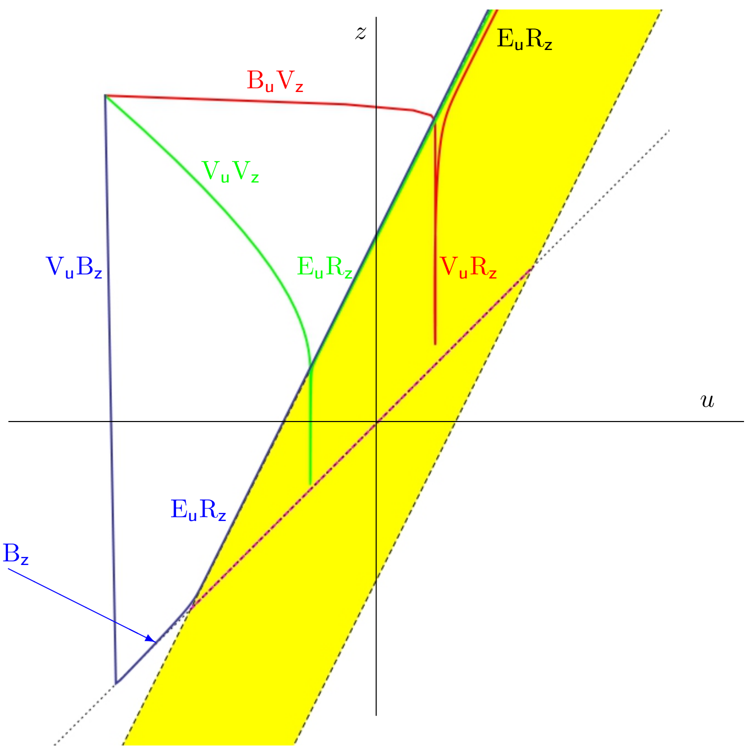

FIGURE 6.3. Solutions $(z(t), u(t))$ for (6.1). The dotted line is the diagonal $u = z$, while the yellow area is the locally stable region $|2z{-}u| \le 1$.

6.2. **Different jumps starting from the rate-independent regime.** Finally we provide an example where the jumps start out of a rate-independent motion, i.e. we first have the regime E$_u$R$_z$, and then the system becomes unstable and develops a jump. For this purpose we use the nonconvex energy

$$\mathcal{E}(t,u,z) = \frac{1}{2}(u{-}g(z))^2 + F(z) - tu \quad \text{with } g(z) = 4z^3 - 4z$$
$$\text{and } F'(z) = -1 + (z{+}1)^2\big({-}40 + 10(z{+}1)^2 + 38\mathrm{e}^{-10(z+0.5)^2}\big).$$

Using the standard viscous potentials as above, the ODE system reads

$$\begin{cases} 0 = \varepsilon^\alpha \dot u + u - g(z) - t, \\ 0 \in \mathrm{Sign}(\dot z) + \varepsilon \dot z + F'(z) + g'(z)(g(z){-}u) \end{cases} \quad \text{with } (u(-0.2), z(-0.2)) = (-2.4, -1.2). \tag{6.2}$$

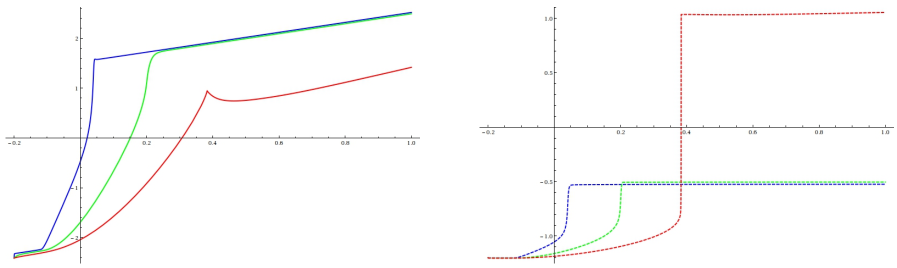

FIGURE 6.4. Solutions for (6.2): left $u(t)$ and right $z(t)$

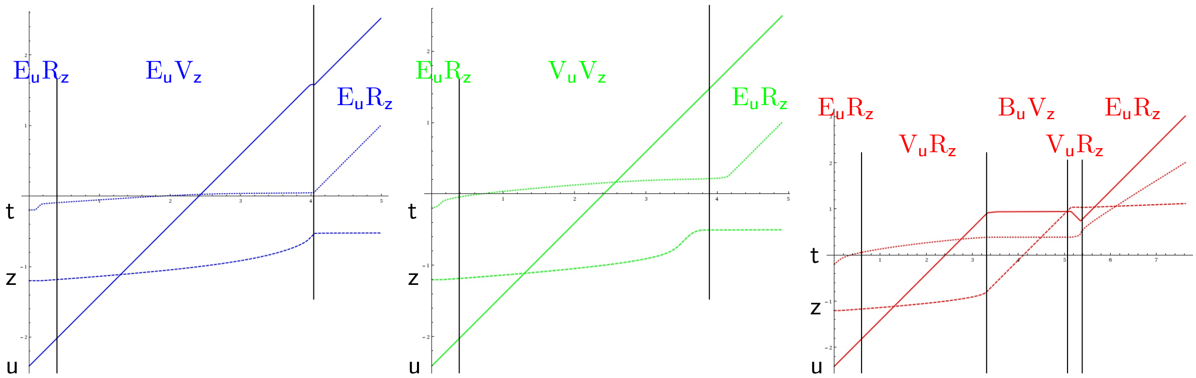

FIGURE 6.5. Solutions $(\mathsf{t}, \mathsf{u}, \mathsf{z})$ for (6.2) with dotted t, full u, and dashed z. Left $\alpha = 2$, middle $\alpha = 1$, right $\alpha = 1/2$.

Figure 6.4 shows simulation results of $u(t)$ and $z(t)$ for the three cases $\alpha = 2$ (blue), $\alpha = 1$ (green), and $\alpha = 1/2$ (red) with sufficiently small ε. We see that the solutions stay together for $t \in [-0.2, -0.1]$, which is exactly the time they stay in regime $\mathrm{E_u R_z}$. Then, in all three cases a jump develops, but this is quite different for different α. In Figure 6.5 we provide graphics of the same solutions, but now in the reparameterized form $(\mathsf{t}, \mathsf{u}, \mathsf{z})$ for the three α-values 2, 1, and 1/2, where again the parameterization s is chosen such that $\dot{s}(t) = \max\{0.5, |\dot{u}(t)|, \dot{z}(t)|\}$. However, for this example numerical instabilities prevented us from taking ε small enough to have a better separation of time scale. Even in the viscous regimes we still see $\mathsf{t}' > 0$ but small. Nevertheless, Figure 6.5 clearly shows the different regimes.

Figure 6.6 shows the trajectories in the (z, u)-plane.

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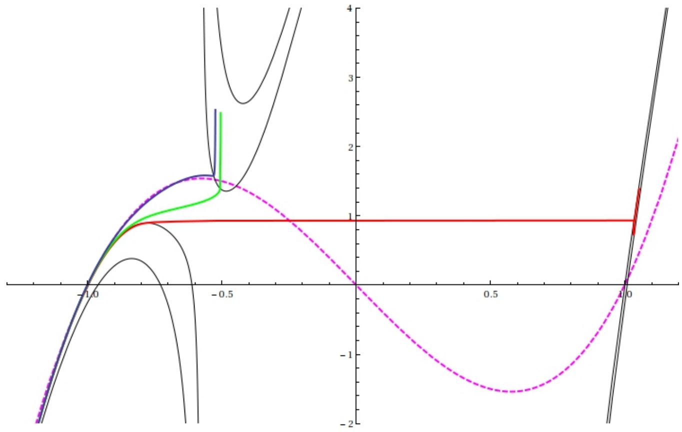

FIGURE 6.6. Solutions $(z(t), u(t))$ for (6.2). The dashed magenta line is $u = g(z)$, while the black curves display the boundaries of the locally stable domain $|F'(z) + g'(z)(g(z)-u)| \leq 1$.

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