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Thick points for Gaussian free fields with different cut-offs

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ABSTRACT. Massive and massless Gaussian free fields can be described as generalized Gaussian processes indexed by an appropriate space of functions. In this article we study various approaches to approximate these fields and look at the fractal properties of the thick points of their cut-offs. Under some sufficient conditions for a centered Gaussian process with logarithmic variance we study the set of thick points and derive their Hausdorff dimension. We prove that various cut-offs for Gaussian free fields satisfy these assumptions. We also give sufficient conditions for comparing thick points of different cut-offs.

1. INTRODUCTION

Let $D \subseteq \mathbb{R}^d$ with $d \geq 2$ be a subset of $\mathbb{R}^d$ (possibly $D = \mathbb{R}^d$). A Generalized Gaussian field (GGF, in short) X is a collection of centered Gaussian random variables indexed by a certain class of functions H, that is, the field can be written as $\{(X, f) : f \in H\}$. H is in the present paper a Hilbert space of functions on D, and more specifically a Sobolev space. Notable examples of such GGFs, and the ones especially considered in this article, are the massive and massless Gaussian free fields (GFF). The study of GFFs has received considerable attention in the context of statistical mechanics and physics, as they can be seen as multidimensional generalizations of Brownian motion (see Sheffield [23] for their construction and properties).

The two most important places (among many) where such fields have shown prominence is the construction of the Liouville Quantum Gravity measure (Duplantier and Sheffield [7]) and the theory of Gaussian multiplicative chaos (Kahane [16], Robert and Vargas [21]). In both these cases one constructs a random measure on $D \subseteq \mathbb{R}^d$, given formally by

$$m_\gamma(\mathrm{d}x) = \exp\left(\gamma X(x) - \frac{\gamma^2}{2}\mathsf{E}\left[X(x)^2\right]\right)\mathrm{d}x. \tag{1.1}$$

Since X is not defined pointwise, the notation $X(x)$ and hence (1.1) are merely formal. To avoid this discrepancy, one defines suitable approximations which converge to the desired limiting measure. These approximations, which are from now on referred to as cut-offs, form in turn a space-time Gaussian process if certain conditions of positive-definiteness are satisfied. Among many methods of approximations we would like to mention two of them: one taking into consideration the geometry of the indexing space H (for example average processes), and another its covariance structure. In this article, we will be mainly interested in the second method, namely, cut-offs created by removing the singularity of the covariance. These approaches are used extensively in the recent literature (see Duplantier et al. [8] for example) and are connected to the seminal work of Kahane [16]. We describe more explicitly some of these approximations in Subsection 2.2.

In the definition of the measure (1.1), $X(x)$ should be replaced by the cut-offs given by a space-time centered Gaussian process $\{X_\epsilon(x) : x \in D, \epsilon > 0\}$. The limit of such measures as ϵ goes to zero forms an important object in mathematical physics. In general, for log-correlated models the limiting random measure is known as Gaussian Multiplicative chaos (GMC) measure (after Kahane [16]). In particular, when the field is either a planar massless or massive free field it is known as Liouville Quantum Gravity measure. It is natural to ask whether different approximations almost surely give the same GMC. The answer in full generality is not known yet. The question was already studied in Kahane [16], Rhodes and Vargas [20] where, for certain cut-offs, the equality in law was proved. Also, in case of planar (massless) GFF, it was shown by Duplantier and Sheffield [7] that measures created by circled average process and by orthonormal basis expansion of $H_0^1(D)$ (another approach to create cut-offs) are almost surely same. To best of our knowledge this is the only almost sure result on equivalence of the measures known till now.

In this article we continue this study of almost-sure universality of cut-offs with respect to thick points (the term in this context was used in Hu et al. [14], and also referred to as multi-fractal behavior in

Kahane [16]). This is the set of points which encapsulates the extremal behavior of the field. For a cut-off $\{X_\epsilon(x) : x \in D, \epsilon > 0\}$ the thick points are defined as

$$T(a) = \left\{ x \in D : \lim_{\epsilon \to 0} \frac{X_\epsilon(x)}{\mathsf{Var}\,(X_\epsilon(x))} = a \right\}, \quad a \geq 0. \tag{1.2}$$

Their importance comes from the fact that they have full mass for the Gaussian multiplicative chaos, and give also information on the behavior of the so-called Liouville Brownian motion for the Liouville quantum gravity measure (Garban et al. [10]). The main properties of the well-known cut-offs is that $\mathsf{Var}\,(X_\epsilon(x))$ behaves like $-\log \epsilon$ as ϵ goes to 0. Under an additional Hölder type condition we show in Theorem 2.1 that the Hausdorff dimension of $T(a)$ has an upper bound of $d - a^2/2$ when $a < \sqrt{2d}$. Note that similar results were derived in Kahane [16] (for the lower bound) and with some extended conditions in Rhodes and Vargas [20]. The Hölder type condition seems to be a minimal requirement as these fields are not smooth and exhibit a fractal behavior. The condition is also satisfied by most of the cut-offs including the circle average process in $2d$ and $4d$ case.

In the second part of our main result we show that under certain assumptions (see Theorem 2.2) the Hausdorff dimension has a lower bound of $d - a^2/2$ for $a < \sqrt{2d}$. To achieve this goal we follow the approach in Kahane [16], who considers covariance kernels that can be written as series of truncated positive, positive definite kernel. These conditions were extended in the work of Robert and Vargas [21] and are are flexible enough to accommodate the covariance cut-offs in most cases. However we point out here that circle average processes do not fall under this class of sufficient conditions. Hence it still remains an open question to determine what the best conditions which can include all "reasonable" cut-offs are.

In view of the above results, one might ask whether there is a possibility of comparing the extremal behavior for different processes. We give a partial answer to this query by imposing a sufficient condition (see Theorem 2.3) on the difference of two cut-offs . In fact, we show that whenever this condition is satisfied the cut-offs have the same fractal behavior. The outline of the article is as follows. In Section 2 we review the definitions of the fields and in Subsection 2.2 of some of the well known cut-offs procedures. Then we state the main results with brief descriptions in Subsection 2.3. In Section 3 we first show that the examples considered in Subsection 2.2 satisfy the assumptions of these results. Finally, in Section 4 we provide the proofs of these results.

2. CONSTRUCTION OF FREE FIELDS AND APPROXIMATIONS

2.1. **Two examples of fields.**

2.1.1. *Massive free fields on $\mathbb{R}^d$.* Let $\mathbb{R}^d, d \geq 2$. Let $\mathcal{S}(\mathbb{R}^d)$ be the Schwartz space consisting of smooth functions whose derivatives decay faster than any polynomial. Let $\mathcal{S}'(\mathbb{R}^d)$ be the space of tempered distribution which are also the continuous linear functionals on $\mathcal{S}(\mathbb{R}^d)$. Also, $\mathcal{S}(\mathbb{R}^d)$ form a dense subset of $\mathcal{S}'(\mathbb{R}^d)$ with respect to weak*-topology. With $C_0^\infty(D)$ we denote the set of smooth and compactly supported functions on D. To avoid somehow lengthy notation, we set $L^2 = L^2(\mathbb{R}^d, \mathrm{d}x)$ dropping the reference measure. For $\xi \in \mathbb{R}^d$, let $\langle \xi \rangle_m = (m^2 + |\xi|^2)^{1/2}$ and we denote by $\langle \xi \rangle = \langle \xi \rangle_1$. To avoid confusion, we will denote in some instances the scalar product in a Hilbert space H by angle brackets with a subindex as $\langle \cdot, \cdot \rangle_H$. For $s \in \mathbb{R}$ we denote the operator $B_m^s : \mathcal{S}(\mathbb{R}^d) \mapsto \mathcal{S}(\mathbb{R}^d)$ defined by

$$B_m^s \phi(x) = \int_{\mathbb{R}^d} e^{-i\langle \xi, x \rangle_{\mathbb{R}^d}} \langle \xi \rangle_m^s \, \widehat{\phi}(\xi) \mathrm{d}\xi. \tag{2.1}$$

This corresponds to the definition of the (formal) Bessel operator $B_m^s \phi := (m^2 \mathbb{I} - \Delta)^{-s/2} \phi$. Let us denote $G_m(x) = K_0(m|x|)$, where $K_0(\cdot)$ is the modified Bessel function; it is well known (see Stein

[24]) that $\widehat{G}_d(\xi) = \langle\xi\rangle_m^{-d}$ and hence one can write

$$B_m^{-d}\phi(x) = \int_{\mathbb{R}^d} G_d(x-y)\phi(y)\mathrm{d}y.$$

We want to look at generalized massive free fields indexed by $f \in \mathcal{S}(\mathbb{R}^d)$ such that

$$\mathsf{E}\left[(X, f)(X, g)\right] = \left\langle f, B^{-d}g\right\rangle_{L^2}.$$

It can be shown that the functional

$$L(\phi) = \exp\left(-\frac{1}{2}\left\langle\phi, B_m^{-d}\phi\right\rangle_{L^2}\right),$$

is a positive definite functional and hence this induces a measure on $\mathcal{S}'(\mathbb{R}^d)$ whose characteristic functional is given by $L(\phi)$ (we provide a short proof of the fact that it satisfies the conditions of the classical Bochner-Milnos theorem in the Appendix 6). This gives us a generalized Gaussian field $\{(X, \phi), \phi \in \mathcal{S}(\mathbb{R}^d)\}$ whose covariance can be represented by

$$\mathsf{E}\left[(X, \phi)(X, \psi)\right] = \int \widehat{\phi}(\xi)\overline{\widehat{\psi}(\xi)}\langle\xi\rangle_m^{-d}\,\mathrm{d}\xi \tag{2.2}$$

(see Hida [12], Yaglom [26]). The tempered measure $\mu(\mathrm{d}x) = \langle\xi\rangle_m^{-d}\,\mathrm{d}\xi$ can be realized as the spectral measure of the covariance of this Gaussian process. We remark here that the Hilbert space associated to the GGF X is in this case the fractional Sobolev space $H^{d/2}(\mathbb{R}^d)$, that we recall being definied by

$$H^s(\mathbb{R}^d) := \{\phi \in \mathcal{S}(\mathbb{R}^d) : B^s\phi \in L^2(\mathbb{R}^d)\}, \quad s \in \mathbb{R}.$$

For details on the construction of such generalized Gaussian fields, we refer the interested readers to Gel'fand and Vilenkin [11]. See also for a white noise representation (2.5) for the massive free fields in Subsection 2.2.1.

2.1.2. *Massless planar Gaussian Free Field.*

Let $C_0^\infty(D)$ be space of smooth functions vanishing outside D, a bounded domain of $\mathbb{R}^d$. Let $H_0^1(D)$ be the Hilbert space which is closure of $C_0^\infty(D)$ under the norm

$$\|f\|_{H^1}^2 = \int_D \|\nabla f(\lambda)\|^2\mathrm{d}\lambda.$$

The dual of $H_0^1(D)$ is given by $H^{-1}(D)$ equipped with the norm

$$\|f\|_{H^{-1}} = \sup_{g \in C_0^\infty(D), \|g\|_{H^1}\leq 1} \langle f, g\rangle,$$

where $\langle,\rangle$ denotes the duality pairing. Note that for $f, g \in C_0^\infty(D)$, we have by Green's identity that $\langle f, g\rangle_{H^1} = \langle f, \Delta g\rangle_{L^2}$ and it follows that $\langle f, g\rangle_{H^{-1}} = \langle f, \Delta^{-1}g\rangle_{L^2}$, where for $g \in C_0^\infty(D)$ one denotes

$$\Delta^{-1}g(x) = \int_D G_D(x, y)g(y)\mathrm{d}y.$$

Here $G_D(x, y)$ is the Green's function for the Dirichlet problem on a planar domain and it is well known that

$$G_D(x, y) = \pi \int_0^\infty p_D(t, x, y)\mathrm{d}t. \tag{2.3}$$

$p_D(t, x, y)$ is defined in the following way. Let $\tau_D = \inf\{t \geq 0 : B_t \notin D\}$ be the first exit time from D for a Brownian motion B_t. Then for $\{B_t : 0 \leq t \leq \tau_D\}$ there exists a transition sub-density $p_D(t, x, y)$ which satisfies for all $t > 0$

$$\mathsf{P}_x\left(B_t \in A, t \leq \tau_D\right) = \int_A p_D(t, x, y)\mathrm{d}y.$$

Also recall from [19, Chapter 3] that

$$p_D(t,\,x,\,y) = p(t,\,x,\,y) - \mathrm{E}_x \left[p(t - \tau_D, B_{\tau_D}, y)\, \mathbb{1}_{\{t > \tau_D\}} \right],$$

where $p(t,\,x,\,y)$ is the transition density of the unstopped Brownian motion (also called the Gauss-Weierstrass heat kernel) given by

$$p(t,\,x,\,y) = \frac{1}{2\pi t} \exp\left(-\frac{\|x - y\|^2}{2t} \right).$$

A (massless) Gaussian free field can described as a centered Gaussian process indexed by $H^{-1}(D)$, that is, a collection $\{(\Phi, f) : f \in H^{-1}(D)\}$ such that

$$\mathrm{Cov}\left((\Phi, f)(\Phi, g)\right) = \langle f, g \rangle_{H^{-1}}.$$

One can also consider it as a centered Gaussian field indexed by $H_0^1(D)$ by duality (since $(\Phi, g)_{L^2} = (\Phi, \Delta^{-1}g)_{H^1}$ where $\Delta^{-1}g \in H_0^1(D)$ for $f \in H^{-1}(D)$). Note that the variance in both cases is the same, as $\|\Delta^{-1}g\|_{H^1}^2 = \|g\|_{H^{-1}}^2$. If we restrict ourselves to $C_0^\infty(D)$ we get from the above observations

$$\mathrm{Cov}\left((\Phi, f)(\Phi, g)\right) = \int_D \int_D f(x)g(y) G_D(x, y)\mathrm{d}x\mathrm{d}y.$$

If $D = [0, 1]^2$ the Gaussian Free Field has a formal representation as

$$\Phi = \sum_{j,\,k \in \mathbb{N}} X_{j,k} e_{j,k} \tag{2.4}$$

where $e_{j,k}$ are eigenfunctions given explicitly by

$$e_{j,k}(x, y) = 2 \frac{\sin(\pi j x) \sin(\pi k y)}{\sqrt{j^2 + k^2}},$$

which also form an orthonormal basis of $H_0^1(D)$. This Φ converges almost surely in $H^{-1}(D)$ and hence it is consistent with the above definitions. We refer the readers for a more detailed construction to Dubédat [6] and Sheffield [23].

2.2. **The construction of cut-offs.**
There are several ways in which one can approach the question of approximating a field with infinite variance by cut-offs. We will list here only a few of those examples.

2.2.1. *White-noise cut-offs for massive free fields.* Let W be a Gaussian complex white noise with control measure $\mu(\mathrm{d}\xi) = \langle \xi \rangle_m^{-d}\, \mathrm{d}\xi$. Formally, the field X is given by the characteristic function of the white noise. That is, if $\zeta(\lambda, \xi) = e^{-i\langle \lambda, \xi \rangle_{\mathbb{R}^d}}$, one can represent the field as

$$X(\lambda) = \int_{\mathbb{R}^d} \zeta(\lambda,\,\xi) W(\mathrm{d}\xi),$$

which means that (X, ϕ) for $\phi \in \mathcal{S}(\mathbb{R}^d)$ has the stochastic integral representation

$$(X, \phi) = \int_{\mathbb{R}^d} \widehat{\phi}(\xi) W(\mathrm{d}\xi).$$

It is well-known ([18, Chapter 1]) that for any $f \in L_{\mathbb{C}}^2(\mathbb{R}^d)$ the integral above is well-posed. Note that under the control measure μ, the isometry property of the stochastic integrals again gives us the covariance of the field as (2.2). Note that since W is a complex white noise with control measure μ which is absolutely continuous with respect to the Lebesgue measure, the field can also be represented by using a standard complex white noise $\widetilde{W}$ (with control measure $\mathrm{d}\xi$) in such a way that

$$X(\lambda) = \int_{\mathbb{R}^d} \zeta(\lambda,\,\xi) \langle \xi \rangle_m^{-d} \widetilde{W}(\mathrm{d}\xi).$$

Since this directly relates to the Bessel potential operator B_m^{-d} in (2.1), these fields can be thought of as

$$X := \left(m^2 \mathbb{I} - \Delta\right)^{-d/2} \widetilde{W}. \tag{2.5}$$

The above white noise representation helps to create the first example of white-noise cut-off. Pick now an arbitrary $\epsilon > 0$. We denote the white noise cut-off as

$$X_\epsilon(x) := \frac{1}{\omega_d} \int_{B(0, 1/\epsilon)} \zeta(x, \xi) W(\mathrm{d}\xi). \tag{2.6}$$

Here $\omega_d = 2\pi^{d/2}/\Gamma(d/2)$ is the volume of the d-dimensional unit ball. Such cut-offs are also known as ultra-violet (UV) cut-offs (see Rhodes and Vargas [20]). We call this a cut-off for the field since if we denote by

$$K_\epsilon(x, y) = \mathsf{E}\left[X_\epsilon(x) X_\epsilon(y)\right] = \int_{B(0, 1/\epsilon)} \zeta(x - y, \xi) \langle \xi \rangle_m^{-d} \mathrm{d}\xi \tag{2.7}$$

then for f, g compactly supported smooth functions one has

$$\lim_{\epsilon \to 0} \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} f(x) g(y) K_\epsilon(x, y) \mathrm{d}x \mathrm{d}y = \int \widehat{f}(\xi) \overline{\widehat{g}(\xi)} \langle \xi \rangle_m^{-d} \mathrm{d}\xi,$$

and the right hand is the same as (2.2). One can also introduce many other cut-offs. One example is by taking a mollifier $\theta(\xi)$ that satisfies

1 θ is positive definite and symmetric,
2 $\int_{\mathbb{R}^d} \theta(x) \mathrm{d}x = 1$,
3 $|\theta(x)| \leq \frac{1}{1 + |x|^{d+\gamma}}$ for some $\gamma > 0$,

(an example is the Gaussian density) one can define a cut-off as

$$X_\epsilon(x) := \int_{B(0, 1/\epsilon)} \zeta(x, \xi) \sqrt{\widehat{\theta}(\epsilon \xi)} W(\mathrm{d}\xi), \tag{2.8}$$

where W is again a complex white noise with control measure $\mu(\mathrm{d}\xi) = \langle \xi \rangle_m^{-d} \mathrm{d}\xi$.

2.2.2. *Integral cut-offs.* This cut-off has been extensively used by Rhodes and Vargas [20] as it follows under the scope of the work of Kahane [16]. Consider the massive GFF on $\mathbb{R}^d$. For that one observes that $K_\epsilon(x, y) \to K_0(m\|x - y\|)$ as $\epsilon \to 0$ and $x \neq y$ (on the diagonal the modified Bessel function is infinite). For $x \neq y$ one may write

$$K_0(m\|x - y\|) = \int_1^\infty k_m(u\|x - y\|) \frac{\mathrm{d}u}{u}$$

where

$$k_m(z) = \frac{1}{2} \int_0^\infty e^{-\frac{m^2|z|^2}{2v}} e^{-v/2} \mathrm{d}v.$$

Now one denotes the integral cut-off of the covariance for $x, y \in \mathbb{R}^d$ as

$$H_\epsilon(x, y) = \int_1^{1/\epsilon} k_m(u\|x - y\|) \frac{\mathrm{d}u}{u} \tag{2.9}$$

and associates to it a centered Gaussian process (we show in the appendix that K_ϵ gives rise to a positive definite functional). Note that even when $x = y$ this is well defined and it follows that $H_\epsilon(x, x) \sim_{\epsilon \to 0} -\log \epsilon$. For the planar GFF one can define the integral cut-offs as follows. One considers for $\epsilon > 0$

$$G_{\epsilon, D}(x, y) = 2\pi \int_\epsilon^{+\infty} p_D(s, x, y) \mathrm{d}s. \tag{2.10}$$

It is well-known that $G_{\epsilon,D}$ is a positive definite kernel (for a proof see [20, Section 5.2]) and hence one can consider a Gaussian process $X_\epsilon(x)$ such that $\mathsf{E}\left[X_\epsilon(x)X_\epsilon(y)\right] = G_{\epsilon,D}(x,y)$. Note that, as before, it follows that for $f, g \in C_0^\infty(D)$ one has that

$$\lim_{\epsilon \to 0} \int_D \int_D f(x)g(y)G_{\epsilon,D}(x,y)\mathrm{d}x\mathrm{d}y = \left\langle f, \Delta^{-1}g \right\rangle_{L^2}.$$

We note that $H_\epsilon(x,y)$ in (2.9) and $K_\epsilon(x,y)$ in (2.7) are different pointwise and indeed it can be shown that there are $x, y \in \mathbb{R}^d$, such that $K_\epsilon(x,y)$ takes negative values whilst $H_\epsilon(x,y)$ is always positive.

2.2.3. *Transition semigroup cut-offs.* These approximations instead rely on the particular transition semigroup of the massless Gaussian field, which is given in terms of the transition kernel of Brownian motion and follow somehow a mixed approach between the integral cut-off (compare for example (2.10) and (2.9)) and the white-noise integration. Let us start with the planar case to illustrate the technique. Let W be a standard space-time Gaussian white noise on $D \times (0, \infty)$ with the Lebesgue measure as control measure; define the stochastic integral corresponding to the Gaussian free field as

$$X(x) = \sqrt{2\pi} \int_{D \times (0,\infty)} p_D(s/2, x, y)W(\mathrm{d}y, \mathrm{d}s).$$

Now one represents the approximating field as

$$X_\epsilon(x) := \sqrt{2\pi} \int_{D \times (\epsilon,\infty)} p_D(s/2, x, y)W(\mathrm{d}y, \mathrm{d}s).$$

It follows again that $\mathsf{E}\left[X_\epsilon(x)X_\epsilon(y)\right] = \int_\epsilon^\infty p_D(s, x, y)\mathrm{d}s$ (see Rhodes and Vargas [20]).

The very same decomposition works for the massive GFF too. Knowing that

$$B_1^{-d}u(x) = \frac{\sqrt{2\pi}}{\Gamma(d/2)} \int_0^{+\infty} \int_{\mathbb{R}^d} \mathrm{e}^t\, t^{d/2-1}p(t, x, y)u(y)\mathrm{d}t\mathrm{d}y$$

we can set

$$X_\epsilon(x) := \frac{\sqrt{2\pi}}{\sqrt{\Gamma(d/2)}} \int_{\mathbb{R}^d \times [\epsilon, +\infty)} \mathrm{e}^{t/2}\, t^{\frac{1}{2}(d/2-1)}p(t/2, x, y)W(\mathrm{d}t, \mathrm{d}y).$$

Once more we have

$$\mathsf{E}\left[X_\epsilon(x)X_\epsilon(y)\right] = \frac{\sqrt{2\pi}}{\Gamma(d/2)} \int_{[\epsilon, +\infty)} \mathrm{e}^t\, t^{s/2-1}p(t, x, y)\mathrm{d}t.$$

This decomposition extends in general to any operator whose action can be represented through the Brownian motion semigroup (as for example Hu and Zähle [13]). Being very similar to integral cut-offs such as (2.10), in the paper we do not treat these approximations as separate cases but refer to integral cut-offs for more general properties.

2.2.4. *Other cut-offs.* The properties of the aforementioned cut-offs rely on the removal of the singularity of the covariance on the diagonal. There are however other cut-offs which can be constructed starting from the geometry of the field. Here we would like to mention briefly a few of them.

Circle and sphere averages: Introduced by Duplantier and Sheffield [7], the circle average for the planar massless Gaussian Free Field is based on the solution of the boundary value problem

$$\begin{cases} -\frac{1}{2\pi}\Delta G_x^\epsilon(y) = \nu_\epsilon^x, & y \in D \\ G_\epsilon^x(y) = 0, & y \in \partial D \end{cases} \tag{2.11}$$

for all $x \in D$, where ν_ϵ^x is the uniform measure on ∂D. The circle average for the GFF Φ is then the process$\{(\Phi, G_\epsilon^x) : x \in D, \epsilon > 0\}$. This cut-off enjoys several important properties, among which being the time-change of a Brownian motion and possessing short-range correlations, in

contrast to white noise and integral cut-offs. To obtain a similar cut-off in higher dimensions is trickier due to the more complex geometry, hence more work is needed (Chen and Jakobson [4] treats the 4-dimensional case). In this case, one cannot work directly with the analogous solution of (2.11), but has to modify the field to achieve the spatial Markov property again.

Orthonormal basis expansion: Generalized Gaussian Fields always feature an orthonormal basis representation (Janson [15]), that can be described generally as follows. If $\{(X, \varphi) : \varphi \in H\}$ is a GGF associated to the Hilbert space H, and an orthonormal basis of H can be written as $(h_n)_{n \in \mathbb{N}}$, then

$$(X, \varphi) = \sum_{n \in \mathbb{N}} \langle \varphi, h_n \rangle_H h_n, \quad \varphi \in H.$$

More formally, the field is represented for every $x \in D$ as $X(x) = \sum_{n \in \mathbb{N}} \alpha_n h_n(x)$. Here $(\alpha_n)_{n \in \mathbb{N}}$ are i. i. d. standard Gaussians. In this sense, the approximation is given by

$$X_m(x) = \sum_{n \leq m} \alpha_n h_n(x), \quad m \in \mathbb{N}.$$

We have already encountered this representation in the specific case of the planar GFF in (2.4).

2.3. **Main results.** In this section we discuss the main results of this article on fractal properties of the cut-offs stated in the previous section. We give some general sufficient conditions under which the lower bound and upper bound can be proved. We also give sufficient conditions for comparing fractal properties of two different cut-offs. Later in the article we show that these sufficient conditions are satisfied by almost all of the the cut-offs described above.

Theorem 2.1. If $(X_\epsilon(x))_{\epsilon \geq 0, x \in \mathbb{R}^d}$, $d \geq 2$, is a centered Gaussian process satisfying

(A) for all $R > 0$ and for all $x, y \in B(0, R)$ and $\epsilon, \eta \geq 0$ we have

$$\mathsf{E}\left[(X_\epsilon(x) - X_\eta(y))^2\right] \leq \frac{\|x - y\| + |\eta - \epsilon|}{\eta \wedge \epsilon},$$

(B) the variance of the process satisfies

$$G(\epsilon) := \mathsf{E}\left[X_\epsilon(x)^2\right] \sim_{\epsilon \to 0} -\log \epsilon.$$

Then letting

$$T_\geq(a, R) = \left\{ x \in B(0, R) : \lim_{\epsilon \to 0} \frac{X_\epsilon(x)}{G(\epsilon)} \geq a \right\}$$

we have for $a \leq \sqrt{2d}$ that $\dim_H(T(a, R)) \leq d - \frac{a^2}{2}$ almost surely, and for $a > \sqrt{2d}$ that $T_\geq(a, R)$ is empty almost surely.

Theorem 2.1 is stated for balls of radius R, but it can be used to derive the upper bound by first covering the space with a countable number of balls and then using the countable stability property of the Hausdorff dimension, which reads as

$$\dim_H\left(\bigcup_{n \in \mathbb{N}} B_n\right) = \sup_{n \in \mathbb{N}} \dim_H(B_n) \tag{2.12}$$

for an arbitray collection of sets $(B_n)_{n \in \mathbb{N}}$. In the following Corollary we treat as a separate case the Massless GFF, both for its importance and for the slightly different proof.

Corollary 2.1. Let D be a bounded, convex regular domain. For $\delta > 0$, denote $D^{(\delta)} := \{x \in D : \mathrm{d}(x, \partial D) > \delta)\}$. If $X_\epsilon(x)$ is a planar (massless) Gaussian free field integral cut-off, satisfying assumptions (A) and (B) on $D^{(\delta)}$ for any $\delta > 0$. If we denote

$$T_\geq(a, D) := \left\{ x \in D : \lim_{\epsilon \to 0} \frac{X_\epsilon(x)}{G(\epsilon)} \geq a \right\},$$

then the conclusion of Theorem 2.1 holds with $d = 2$.

In Section 3 we will see that most of the cut-offs discussed in Subsection 2.2 satisfy the assumptions of Theorem 2.1 and Corollary 2.1. A brief sketch of the proof is as follows. The condition (A) allows us to have a modification which has nice bounds on the spatial and time variable almost surely. We use a strong version of Kolmogorov-Centsov theorem from Hu et al. [14] to derive this. Using these path properties it is possible to get an explicit cover of the space and also get good bounds for the diameters of the sets used to form the cover. The upper bound then follows easily from the definition of Hausdorff dimension.

Now we give some sufficient conditions on the cut-off for which we have a matching lower bound. We state the results for discrete time for ease of exposition, noting that it can be extended to continuous time if the processes have a continuous modification.

Theorem 2.2. *Suppose $X_n(x)$ is a centered Gaussian process with covariance kernel denoted by $p_n(x, y)$ which satisfies the following properties:*

(C) *for $x \neq y$, $p_n(x, y) \leq \frac{1}{2} \log \frac{1}{\|x-y\|} + H(x, y)$ where $\sup_{x \neq y \in [0,1]^d} H(x, y) < C < \infty$.*

(D) *There exists a sequence of positive definite covariance kernels $\tilde{p}_n(x, y)$ such that $p_n(x, y) = \sum_{k=1}^{n} \tilde{p}_k(x, y)$, and moreover $\tilde{p}_n(x, x) = 1$ for all x.*

Let $0 < a \leq \sqrt{2d}$ and consider the set of thick points on $D = [0, 1]^d$, that is,

$$T(a) := \left\{ x \in D : \lim_{n \to \infty} \frac{X_n}{n} = a \right\}. \tag{2.13}$$

Then we have $\dim_H(T(a)) \geq d - \frac{a^2}{2}$ almost surely.

If for a cut-off $X_\epsilon(x)$, $G(\epsilon) = \text{Var}(X_\epsilon(x))$ is monotone in ϵ, then one can apply the above result to the process $X_{G^{-1}(n)}(x)$ to get the lower bound for thick points from Theorem 2.2, which can now be combined with Theorem 2.1 to get the result for the Hausdorff dimension of thick points easily. The conditions in Theorem 2.2 are inspired by the works of Kahane [16]; we notice that cut-off constructed through truncating the covariance structure fall under this category, for example white noise cut-offs, integral cut-offs and semigroup cut-offs. The sphere average processes however are excluded from such assumptions.

It is also important to note that condition (C) can be relaxed to the case where the upper bound holds uniformly for n large enough. This important modification especially comes useful in dealing with white-noise cut-offs which do not have positive kernels. We point out this important modification in Remark 4.1.

The proof follows in some steps the ideas of Kahane [16] (chiefly the construction of rooted measures). The condition (C) allows one to construct a positive martingale using measures of the form (1.1) which converge for every bounded set A. Now logarithmic bounds allow one to extend this convergence to an L^2 convergence. It is then standard to construct a limiting a measure out of it. However instead of following the technical way of estimating the size of balls, we show that this measure has required finite energy almost surely using uniform bounds and weak convergence techniques. Finally, we show that the limiting measure thus obtained has finite energy and also gives full mass to the set of thick points. To show that the latetr we need to use rooted measure techniques borrowed from Kahane [16] (see also Duplantier and Sheffield [7]) and the strong law of large numbers under this measure.

As pointed out earlier, the above assumptions are not really useful when one deals with circle averages. This drawback obviously raises the question on whether we can compare covariances of two cut-offs to deduce the behavior of thick points. Our next result is in that direction.

Theorem 2.3. *Let $X_\epsilon(x)$ and $\widetilde{X}_\epsilon(x)$ be two cut-off families for the same field on D. Let $T(X, a)$ and $T(\widetilde{X}, a)$ be the set of a-thick points for $X_\epsilon(x)$ and $\widetilde{X}_\epsilon(x)$ respectively. Call $Z_\epsilon(x) := X_\epsilon(x) - \widetilde{X}_\epsilon(x)$. Suppose $Z_\epsilon(x)$ satisfies the following assumption:*

(E) *$Z_\epsilon(x)$ is symmetric in x and there exists universal constants $C > 0, C' > 0$ independent of ϵ and x such that*

$$\mathsf{E}\left[Z_\epsilon(x)^2 \right] \le C \tag{2.14}$$

and

$$\mathsf{E}\left[(Z_\epsilon(x) - Z_\epsilon(y))^2 \right] \le C' \frac{\|x - y\|}{\epsilon}. \tag{2.15}$$

Then for all $a > 0$ we have $\dim_H(T(X, a)) = \dim_H(T(\widetilde{X}, a))$ almost surely.

Again we will give an example in Section 3 where condition (E) would be satisfied. Unfortunately, we cannot compare important processes like sphere average and white-noise cut-off through this theorem. In fact, the bounded variance seems a bit restrictive and if this condition could be relaxed then one could accommodate more interesting examples.

To prove Theorem 2.3 we show first that using Sudakov-Fernique one can compare the maxima of the Gaussian process $Z_\epsilon(x)$ with a multivariate version of the Ornstein–Uhlenbeck process for which the order of expected maxima can be easily derived. To pass to the almost sure version of it one uses bounded variances and Borell's inequality. This allows one to compare the set of thick points and derive the final result.

3. EXAMPLES

In this Section we explicitly show cut-offs that satisfy the assumptions of our theorems. We will concentrate on massive and massless GFFs but the results in general can be applied to centered Gaussian process with appropriate covariance structure too. Since we could not find comprehensive proofs in the literature we try to outline the details explicitly.

3.1. **White noise cut-off for massive GFF.** We recall the cut-off (2.6)

$$K_\epsilon(x, y) = \frac{1}{\omega_d} \int_{B(0, 1/\epsilon)} \zeta(x - y, \xi) \langle \xi \rangle^{-d} \, d\xi,$$

where $\zeta(x - y, \xi) = e^{-i\langle x - y, \xi \rangle}$. We now start with proving (A)-(D).

(A)

$$
\begin{aligned}
\mathsf{E}\left[(X_{\epsilon_1}(x) - X_{\epsilon_2}(x))^2 \right] &= \int_{\mathbb{R}^d} \frac{\left(\zeta(x, \xi)\, \mathbb{1}_{B(0, 1/\epsilon_1)} - \zeta(x, \xi)\, \mathbb{1}_{B(0, 1/\epsilon_2)} \right)^2}{(\|\xi\|^2 + m^2)^{d/2}} d\xi \\
&= \int_{\mathbb{R}^d} \zeta(x, \xi)^2 \left(\mathbb{1}_{B(0, 1/\epsilon_1)} - \mathbb{1}_{B(0, 1/\epsilon_2)} \right)^2 \frac{1}{(\|\xi\|^2 + m^2)^{d/2}} d\xi \\
&= \int_{\mathbb{R}^d} |\zeta(x, \xi)|^2 (\mathbb{1}_{1/\epsilon_2 < |\xi| \le 1/\epsilon_1})^2 \frac{1}{(\|\xi\|^2 + m^2)^{d/2}} d\xi \\
&\le C \int_{1/\epsilon_2}^{1/\epsilon_1} \frac{\rho^{d-1}}{(\rho^2 + m^2)^{d/2}} d\rho \\
&\le C |\epsilon_2 - \epsilon_1| / \epsilon_1
\end{aligned}
$$

where we have used the inequality $\left|\log\left(\frac{x}{y}\right)\right| \leq \frac{|x-y|}{x\wedge y}$. For the more general case we assume (without loss of generality) $\epsilon_1 \leq \epsilon_2$ so that

$$
\begin{aligned}
\mathsf{E}\left[(X_{\epsilon_1}(x) - X_{\epsilon_2}(y))^2\right] &= \int_{\mathbb{R}^d} \frac{(\zeta(x,\xi)\,\mathbb{1}_{B(0,1/\epsilon_1)} - \zeta(y,\xi)\,\mathbb{1}_{B(0,1/\epsilon_2)})^2}{(\|\xi\|^2 + m^2)^{d/2}} d\xi \\
&= \int_{B(0,1/\epsilon_2)} (\zeta(x,\xi) - \zeta(y,\xi))^2 \frac{1}{(\|\xi\|^2 + m^2)^{d/2}} d\xi \\
&\quad + \int_{\mathbb{R}^d} \zeta(x,\xi)^2 (\mathbb{1}_{1/\epsilon_2 < |\xi| \leq 1/\epsilon_1}) \frac{1}{(\|\xi\|^2 + m^2)^{d/2}} d\xi \\
&\leq C \frac{\|x-y\|}{\epsilon_2} + \frac{|\epsilon_2 - \epsilon_1|}{\epsilon_1} \leq C \frac{\|x-y\| + |\epsilon_2 - \epsilon_1|}{\epsilon_1 \wedge \epsilon_2}.
\end{aligned}
$$

(B) It follows from the fact that,

$$
\mathsf{E}\left[X_\epsilon(x)^2\right] = \frac{1}{\omega_d} \int_{B(0,1/\epsilon)} \frac{1}{(m^2 + \|\xi\|^2)^{d/2}} d\xi \approx -\log\epsilon.
$$

(C) Recall that $G(\epsilon) = K_\epsilon(x,x) = \frac{1}{\omega_d} \int_0^{1/\epsilon} t^d (m^2 + t^2)^{-d/2} \frac{dt}{t}$. The required Gaussian process satisfying (C) is $X_{G^{-1}(n)}(x)$, so that $p_n(x,y)$ is given by

$$
p_n(x,y) = \frac{1}{2} \mathsf{E}\left[X_{G^{-1}(2n)}(x) X_{G^{-1}(2n)}(y)\right].
$$

Note that this kernel is not in general positive. We first need to compute

$$
\begin{aligned}
K_n(x,y) &= \frac{1}{\omega_d} \int_{B(0, 1/G^{-1}(n))} e^{-i\langle x-y,\xi\rangle} \langle\xi\rangle_m^{-d} d\xi \\
&= \frac{(2\pi)^{d/2}}{\omega_d m \|x-y\|^{d-2/2}} \int_0^{1/G^{-1}(n)} t^{d/2} J_{d/2-1}(m\|x-y\|t)(1+t^2)^{-d/2} dt \\
&= \frac{(2\pi)^{d/2} \Gamma(d/2)}{2\pi^{d/2}} \int_0^{m\|x-y\|/G^{-1}(n)} t^{d/2} J_{d/2-1}(t)(m^2\|x-y\|^2 + t^2)^{-d/2} dt \\
&= 2^{d/2-1} \Gamma(d/2) \int_0^{m\|x-y\|/G^{-1}(n)} t^{d/2} J_{d/2-1}(t)(m^2\|x-y\|^2 + t^2)^{-d/2} dt.
\end{aligned}
$$

Let $T_n(m\|x-y\|) := K_0(m\|x-y\|) - K_n(m\|x-y\|)$. We see that $T_n(a) \to 0$ as $n \to +\infty$, and in addition the convergence is uniform. Indeed if $d > 2$ we get

$$
|K_0(m\|x-y\|) - K_n(m\|x-y\|)| = 2^{d/2-1} \Gamma(d/2) \left|\int_{\frac{m^2\|x-y\|}{G^{-1}(n)}}^{\infty} t^{d/2} \frac{J_{d/2-1}(t)}{(m\|x-y\|^2 + t^2)^{d/2}} dt\right|
$$

$$
\leq C \int_{\frac{m^2\|x-y\|}{G^{-1}(n)}}^{\infty} \frac{t^{d/2}}{(m\|x-y\|^2 + t^2)^{d/2}} dt \leq C \int_{\frac{m\|x-y\|}{G^{-1}(n)}}^{\infty} t^{-d/2} dt \leq C(m,d) \left(G^{-1}(n)\right)^{d/2-1} \to 0
$$

as $n \to \infty$. For $d = 2$, note that we can assume n large enough so that

$$
J_0(t) = \sqrt{\frac{2}{\pi t}} \cos\left(\frac{\pi}{4} - t\right) - \frac{\left(\frac{1}{t}\right)^{3/2} \sin\left(\frac{\pi}{4} - t\right)}{4\sqrt{2\pi}} + o\left(\frac{1}{t}\right)^{5/2}.
$$

Therefore

$$|K_0(m\|x-y\|) - K_n(m\|x-y\|)| \leq \left| \int_{\frac{m\|x-y\|}{G^{-1}(n)}}^\infty \frac{t^{1/2} + o\left(t^{-1/2}\right)}{m^2\|x-y\|^2 + t^2} \mathrm{d}t \right|$$

$$\leq \int_{\frac{m\|x-y\|}{G^{-1}(n)}}^\infty \left(t^{-3/2} + o\left(t^{-5/2}\right)\right) \leq C \left(G^{-1}(n)\right)^{1/2} \to 0.$$

Hence there exists an n_0 such that for all $n \geq n_0$ uniformly in $a \in (0,\,1]$,

$$|T_n(m\|x-y\|)| \leq 1.$$

We use the triangular inequality to say that

$$K_n(x,\,y) \leq |T_n(m\|x-y\|)| + K_0(m\|x-y\|) \leq 1 + K_0(\|x-y\|)$$

$$\leq -\frac{1}{2}\log\|x-y\| + L(x,\,y),$$

where $L(x,y)$ is bounded function. It is important to stress at this point that showing Condition (C) for all n is not immediate, and hence we will explain in Remark 4.1 how the proof of the lower bound should be adapted in this case.

(D) We have that $p_n(x,x) = n$ and we can define

$$\widetilde{p}_k(x,\,y) = \frac{1}{2}\int_{B(0,1/G^{-1}(2k))} \mathrm{e}^{-i\langle x-y,\xi\rangle} \langle\xi\rangle^{-d} \,\mathrm{d}\xi - \int_{B(0,1/G^{-1}(2k-2))} \mathrm{e}^{-i\langle x-y,\xi\rangle} \langle\xi\rangle^{-d}\,\mathrm{d}\xi.$$

It is easy to see that $p_n(x,\,y) = \sum_{k=1}^n \widetilde{p}_k(x,\,y)$. Also note that

$$\widetilde{p}_k(x,\,x) = \frac{1}{2}\left(G(G^{-1}(2k)) - G(G^{-1}(2k-1))\right) = 1.$$

This shows condition (D) of the theorem.

3.2. **Integral cut-off for massive GFF** (2.9)**.**

(A) We have

$$\mathsf{E}\left[(X_\epsilon(x) - X_\epsilon(y))^2\right] \leq \left|\mathsf{E}\left[X_\epsilon(x)^2 - X_\epsilon(x))X_\epsilon(y)\right]\right| + \left|\mathsf{E}\left[X_\epsilon(y)^2 - X_\epsilon(x)X_\epsilon(y)\right]\right|$$

of which we can bound one summand as

$$\left|\mathsf{E}\left[X_\epsilon(x)^2 - X_\epsilon(x)X_\epsilon(y)\right]\right| = \frac{1}{2}\int_1^{1/\epsilon}\int_0^\infty \mathrm{e}^{-v/2}\left(1 - \mathrm{e}^{-\frac{\|x-y\|^2 m^2 u^2}{2v}}\right)\mathrm{d}v\frac{\mathrm{d}u}{u}$$

$$\overset{w := \frac{\|x-y\|^2 m^2 u}{\sqrt{v}}}{=} \frac{1}{2}\int_0^\infty \int_{\frac{\|x-y\|^2 m^2}{\sqrt{v}}}^{\frac{\|x-y\|^2 m^2}{\epsilon\sqrt{v}}} \mathrm{e}^{-v/2}\,\mathrm{d}v\left(1 - \mathrm{e}^{-\frac{w^2}{2}}\right)\frac{\mathrm{d}w}{w}$$

$$\overset{s := w^2}{=} \frac{1}{2}\int_0^\infty \left(\mathrm{e}^{-\frac{\|x-y\|^2 m^2}{2s}} - \mathrm{e}^{-\frac{\|x-y\|^2 m^2}{2\epsilon^2 s}}\right)\left(1 - \mathrm{e}^{-s/2}\right)\frac{\mathrm{d}s}{s}$$

$$= \frac{1}{2}\int_0^{\frac{\|x-y\|m}{\epsilon}} \left(\mathrm{e}^{-\frac{\|x-y\|^2 m^2}{2s}} - \mathrm{e}^{-\frac{\|x-y\|^2 m^2}{2\epsilon^2 s}}\right)\left(1 - \mathrm{e}^{-s/2}\right)\frac{\mathrm{d}s}{s}$$

$$+ \int_{\frac{\|x-y\|m}{\epsilon}}^\infty \left(\mathrm{e}^{-\frac{\|x-y\|^2 m^2}{2s}} - \mathrm{e}^{-\frac{\|x-y\|^2 m^2}{2\epsilon^2 s}}\right)\left(1 - \mathrm{e}^{-s/2}\right)\frac{\mathrm{d}s}{s}.$$

Now in the first integral the integrand is bounded in absolute value by $C\frac{v}{2}$, hence the whole integral is smaller than $C\frac{\|x-y\|m}{\epsilon}$. As for the second integral

$$\int_{\frac{\|x-y\|m}{\epsilon}}^{\infty} \left| \left(e^{-\frac{\|x-y\|^2 m^2}{2s}} - e^{-\frac{\|x-y\|^2 m^2}{2\epsilon^2 s}} \right) \left(1 - e^{-s/2} \right) \right| \frac{\mathrm{d}s}{s}$$

$$\leq \int_{\frac{\|x-y\|m}{\epsilon}}^{\infty} e^{-\frac{\|x-y\|^2 m^2}{2s}} \left(1 - e^{-\frac{\|x-y\|^2 m^2}{2s}(\epsilon^{-2}-1)} \right) \left(1 - e^{-s/2} \right) \frac{\mathrm{d}s}{s}$$

$$\overset{1-e^x \leq x}{\leq} \int_{\frac{\|x-y\|m}{\epsilon}}^{\infty} \frac{\|x-y\|^2 m^2}{2s} \left(\epsilon^{-2} - 1 \right) \frac{\mathrm{d}s}{s}$$

$$\leq \|x-y\|^2 m^2 \left(\frac{1-\epsilon^2}{\epsilon^2} \right) \int_{\frac{\|x-y\|m}{\epsilon}}^{\infty} \frac{\mathrm{d}s}{s^2} \leq m\frac{\|x-y\|}{\epsilon}.$$

Similarly we obtain the bound for $|\mathsf{E}\left[X_\epsilon(y)^2 - X_\epsilon(y)X_\epsilon(y)\right]|$.
(B) Using the fact that $k_m(0) = 1$ it follows that $\mathsf{E}\left[X_\epsilon(x)^2\right] = -\log\epsilon$.
(C) Note that $H_n(x,y) = \mathsf{E}\left[X_{e^{-n}}(x)X_{e^{-n}}(y)\right]$ this for $x \neq y$ satisfies the following inequality

$$H_n(x,\,y) \leq G_m(x,\,y) = \frac{1}{2}K_0(m\|x-y\|) \leq \frac{1}{2}\log\frac{1}{m\|x-y\|} + \overline{H}(x,\,y) \qquad (3.1)$$

We can derive $\sup_{x,y\in[0,1]^d}\overline{H}(x,y) \leq C$.
(D) We choose

$$\widetilde{p}_k(x,y) = \frac{1}{2}\int_{e^{2(k-1)}}^{e^{2k}} k_m(u(x-y))\frac{\mathrm{d}u}{u}, \quad k \in \mathbb{N}.$$

With this normalization one has also $\widetilde{p}_k(x,x) = 1$.

3.3. **Planar GFF semigroup cut-off** (2.10). The proof for the planar GFF is a bit more involved than for other cut-offs, and requires some preliminary lemmas and notations. We also would like to remind here that a proof tailored on the 2-d massless GFF for Theorem 2.1 is given in Corollary 2.1. We first show conditions (A) and (B) and for future references we put it as lemma.

Lemma 3.1. *Fix $\delta > 0$, and for a set D assume that $D^{(\delta)}$ is an open convex domain. There exists a constant $C = C(\delta)$ such that*

$$\mathsf{E}\left[(X_\epsilon(x) - X_\eta(y))^2\right] \leq C\frac{\|x-y\| + |\eta - \epsilon|}{\eta \wedge \epsilon}$$

holds for all $x, y \in D^{(\delta)}$ and $G(\epsilon) = \mathsf{Var}\left(X_\epsilon(x)\right) \approx -\log\epsilon$ as $\epsilon \to 0$.

Proof. We first begin by showing the second statement. Recall that

$$G(\epsilon) = 2\pi\int_\epsilon^{\infty} p_D(t,x,x)\mathrm{d}t.$$

Also note that from [17, Section 2.4] we have the following upper and lower bounds on $p_D(t,x,x)$,

$$\frac{1}{2\pi t} - \frac{1}{\pi\,e\,(\mathrm{d}(x,\partial D))^2} \leq p_D(t,x,x) \leq \frac{1}{2\pi t}. \qquad (3.2)$$

Fix $t_0 > 1$, then we ignore the part from (t_0, ∞) by using [17, Lemma 2.28], since

$$\int_{t_0}^{\infty} p_D(t,x,x)\mathrm{d}t \leq C(x,\delta)\int_{t_0}^{\infty} \frac{1}{t(\log t)^2}\mathrm{d}t < \infty.$$

Now using the fact that $x \in D^{(\delta)}$, it follows from (3.2) that,

$$\frac{2\epsilon}{e\delta^2} - \log\epsilon + \log t_0 \leq G(\epsilon) \leq -\log\epsilon + \log t_0 + C'(x, \delta).$$

The second claim is immediate after one lets $\epsilon \to 0$. Now we show the first bound. First we assume $x = y$ and $\epsilon < \epsilon'$ and use the fact that $p_D(t, x, y) \leq p(t, x, y)$.

$$\begin{aligned}
\mathsf{E}\left[(X_\epsilon(x) - X_{\epsilon'}(x))^2\right] &= \mathsf{E}\left[X_\epsilon^2(x)\right] + \mathsf{E}\left[X_{\epsilon'}^2(x)\right] - 2\mathsf{E}\left[X_\epsilon(x)X_{\epsilon'}(x)\right] \\
&= \int_\epsilon^\infty p_D(t, x, x)\mathrm{d}t - \int_{\epsilon'}^\infty p_D(t, x, x)\mathrm{d}t \\
&= \int_\epsilon^{\epsilon'} p_D(t, x, x)\mathrm{d}t \leq \int_\epsilon^{\epsilon'} p(t, x, x)\mathrm{d}t \\
&\leq c\int_\epsilon^{\epsilon'} \frac{\mathrm{d}t}{t} = \log\frac{\epsilon'}{\epsilon} \leq \frac{|\epsilon' - \epsilon|}{\epsilon \wedge \epsilon'}.
\end{aligned}$$

Now to show the Condition (A) for $x \neq y$, we use the representation from Mörters et al. [19]

$$p_D(t, x, y) = p(t, x, y) - \mathrm{E}_x\left[p(t - T_D, B_{T_D}, y)\mathbb{1}_{\{T_D < t\}}\right]$$

for E_x the law of a standard Brownian motion B with $B_0 = x$. Note that

$$\begin{aligned}
\mathsf{E}\left[(X_\epsilon(x) - X_\epsilon(y))^2\right] &\leq |\int_\epsilon^\infty p_D(t, x, x) - p_D(t, x, y)\mathrm{d}t| \\
&+ \left|\int_\epsilon^\infty p_D(t, y, y) - p_D(t, x, y)\mathrm{d}t\right|.
\end{aligned}$$

We shall show that

$$\left|\int_\epsilon^\infty (p_D(t, x, x) - p_D(t, x, y))\,\mathrm{d}t\right| \leq C\frac{\|x - y\|}{\epsilon}. \tag{3.3}$$

The other part follows similarly. So now note that (3.3) is satisfied if one replaces p_D with p. We take D to be a bounded domain, hence $\exists M > 0$ such that $\|x - y\| \leq M$ for all $x, y \in D^{(\delta)}$. So

$$\begin{aligned}
\int_\epsilon^\infty (p_D(t, x, x) - p_D(t, x, y))\,\mathrm{d}t &= \int_\epsilon^\infty \left(1 - \exp\left(-\frac{\|x - y\|}{2t}\right)\right)\frac{\mathrm{d}t}{t} \\
&\leq \frac{\|x - y\|^2}{2}\int_\epsilon^\infty \frac{\mathrm{d}t}{t^2} \leq \frac{M}{2}\frac{\|x - y\|}{\epsilon}.
\end{aligned} \tag{3.4}$$

Now we need to show the term containing the expectation has a similar bound. First note that using a multivariate version of the mean value theorem,

$$|p(t, z, x) - p(t, z, y)| \leq |\nabla p(t, z, (1 - \lambda)x + \lambda y)|\,\|x - y\|$$

with $\lambda \in [0, 1]$. We use then the notation $\xi := (1 - \lambda)x + \lambda y$ to denote a point on the line starting from x and ending at y. Observe that $\xi \in D^{(\delta)}$. From Saloff-Coste [22] we have for any $\kappa \in (0, 1)$

$$\|\nabla_\xi p(t, z, \xi)\| \leq \frac{C(\kappa)}{\sqrt{t}V(z, \sqrt{t})}\exp\left(-\frac{\|z - \xi\|^2}{4(1 - \kappa)t}\right) \tag{3.5}$$

and $V(x, r)$ is the volume of $B(x, r)$. Now using this inequality we have

$$\int_\epsilon^\infty \mathrm{E}_x\left[\left(p(t-T_D,B_{T_D},x)-p(t-T_D,B_{T_D},y)\right)\mathbb{1}_{\{t>T_D\}}\right]\mathrm{d}t$$

$$=\int_0^\infty \mathrm{E}_x\left[\left(p(t-T_D,B_{T_D},x)-p(t-T_D,B_{T_D},y)\right)\mathbb{1}_{\{t>T_D\vee\epsilon\}}\right]$$

$$\leq \mathrm{E}_x\left[\int_0^\infty |p(t,B_0,x)-p(t,B_{T_D},y)|\,\mathbb{1}_{\{t>T_D\vee\epsilon\}}\,\mathrm{d}t\right]$$

$$\leq \mathrm{E}_x\left[\int_{\tau_D}^\infty |p(t,B_0,x)-p(t,B_{T_D},y)|\,\mathrm{d}t\right]$$

$$\overset{t-\tau_D=:s}{\leq}\mathrm{E}_x\left[\int_0^\infty |p(s,B_0,x)-p(s,B_{T_D},y)|\,\mathrm{d}s\right]$$

$$\overset{(3.5)}{\leq}\|x-y\|\mathrm{E}_x\left[\int_0^\infty |p(s,B_0),x)-p(s,B_{T_D},y)|\,\mathrm{d}s\right]$$

$$\leq \|x-y\|\mathrm{E}_x\left[\int_0^\infty \frac{\exp\left(-\frac{-\|B_{T_D}-\xi\|^2}{4(1-\kappa)s}\right)}{s^{3/2}}\mathrm{d}s\right]$$

$$\leq \|x-y\|\mathrm{E}_x\left[\int_0^\infty \frac{\exp\left(-\frac{-\delta^2}{4(1-\kappa)s}\right)}{s^{3/2}}\mathrm{d}s\right]$$

$$= C(\delta,\kappa)\|x-y\|\leq C(\delta,\kappa)\frac{\|x-y\|}{\epsilon}$$

Note that here we have used the fact that $B_{T_D}\in\partial D$ and since $\xi\in D^{(\delta)}$ we have that $\|B_{T_D}-\xi\|\geq\delta$. So the above inequality combined with (3.4) shows (3.3) and hence completes the proof of the Lemma. $\square$

(C) To show this condition it is sufficient to observe that

$$G(\epsilon):=\frac{1}{2}\int_\epsilon^\infty p_D(t,x,y)\mathrm{d}t\leq\frac{1}{2}\int_0^\infty p(t,x,y)\mathrm{d}t=\frac{1}{2}G(x,y)\tag{3.6}$$

where G is the whole-plane Green's function for Brownian motion. We have also from [19, Thm. 3.34]

$$\frac{1}{2}G(x,y)\leq C+\frac{1}{2}\log\frac{1}{\|x-y\|},\quad \|x-y\|\leq 1.$$

(D) $G(\epsilon)$ is clearly decreasing in ϵ, so we define the kernels

$$\widetilde{p}_n(x,y)=\frac{1}{2}\left(\int_{G^{-1}(2n)}^\infty p_D(t,x,y)\mathrm{d}t-\int_{G^{-1}(2n+2)}^\infty p_D(t,x,y)\mathrm{d}t\right).$$

This allows us to say $\widetilde{p}_n(x,x)=\frac{1}{2}\left(G(G^{-1}(2n))-G(G^{-1}(2n+2))\right)=1$.

3.4. **Example for comparison: cut-offs** (2.6) **and** (2.8). This example illustrates the fact, that the effect of the mollifier θ in (2.8) does not affect the structure of thick points, as one might rightly expect. Indeed

let us show that Assumption (E) holds. For the cut-off $X_\epsilon(x)$ of (2.6) it holds that

$$\mathbb{E}\left[(X_\epsilon(x) - X_\epsilon(y))^2\right] \leq \int_{B(0,1/\epsilon)} \frac{(\zeta(x,\xi) - \zeta(y,\xi))^2}{(\|\xi\|^2 + m^2)^{d/2}} \mathrm{d}\xi \leq$$

$$C \int_{B(0,1/\epsilon)} \frac{1 - \cos(2\pi\langle y - x, \xi\rangle)}{(\|\xi\|^2 + m^2)^{d/2}} \mathrm{d}\xi =$$

$$C \int_{B(0,1/\epsilon)} \frac{1 - \cos(2\pi\langle y - x, \xi\rangle)}{(\|\xi\|^2 + m^2)^{d/2}} \mathrm{d}\xi \overset{1-\cos(x)\leq x,\; x\geq 0}{\leq}$$

$$C \int_{B(0,1/\epsilon)} \frac{\langle y - x, \xi\rangle}{(\|\xi\|^2 + m^2)^{d/2}} \mathrm{d}\xi \overset{Cauchy-Schwarz}{\leq}$$

$$C \|x - y\| \int_{B(0,1/\epsilon)} \frac{\|\xi\|}{(\|\xi\|^2 + m^2)^{d/2}} \mathrm{d}\xi =$$

$$C \|x - y\| \int_0^{1/\epsilon} \frac{\rho\rho^{d-1}}{(\rho^2 + m^2)^{d/2}} \mathrm{d}\rho \leq C \|x - y\| \epsilon^{-1}. \tag{3.7}$$

Then we set $Z_\epsilon(x) := X_\epsilon(x) - \widetilde{X}_\epsilon(x)$ for $\widetilde{X}_\epsilon(x)$ of (2.8). We have

$$\mathbb{E}\left[(Z_\epsilon(x))^2\right] = \mathbb{E}\left[(X_\epsilon(x) - Y_\epsilon(x))^2\right] \leq \int_{B(0,1/\epsilon)} \frac{e^{4i\pi\langle x,\xi\rangle}(1 - \hat{\theta}(\epsilon\xi))^2}{(\|\xi\|^2 + m^2)^{d/2}} \mathrm{d}\xi =$$

$$\int_{B(0,1/\epsilon)\cap\mathrm{supp}(\varphi)^c} \frac{\left|e^{4i\langle x,\xi\rangle}\right|}{(\|\xi\|^2 + m^2)^{d/2}} \mathrm{d}\xi + \int_{B(0,1/\epsilon)\cap\mathrm{supp}(\theta)} \frac{(1 - \hat{\theta}(\epsilon\xi))^2}{(\|\xi\|^2 + m^2)^{d/2}} \mathrm{d}\xi \leq$$

$$C(m,d) + \int_{\mathrm{supp}(\theta)} \frac{(1 - \hat{\theta}(\epsilon\xi))^2}{(\|\xi\|^2 + m^2)^{d/2}} \mathrm{d}\xi \leq C(m,d) \tag{3.8}$$

where we used the fact that $\widehat{\theta(\epsilon\cdot)} = \frac{1}{\epsilon}\widehat{\theta\left(\frac{\cdot}{\epsilon}\right)}$ and the bound $\left\|\widehat{\theta(\epsilon\cdot)}\right\|_\infty \leq \frac{1}{\epsilon}\left\|\theta\left(\frac{\cdot}{\epsilon}\right)\right\|_1 = 1$. This shows (2.14). To obtain (2.15) observe that

$$\mathbb{E}\left[(Z_\epsilon(x) - Z_\epsilon(y))^2\right] \leq \int_{B(0,1/\epsilon)} \frac{(\zeta(x,\xi) - \zeta(y,\xi))^2\left(1 - \hat{\theta}(\epsilon\xi)\right)^2}{(\|\xi\|^2 + m^2)^{d/2}} \mathrm{d}\xi$$

and from here one can proceed starting over again as in (3.7) to conclude the proof of the condition.

4. Proof of the main results

4.1. **Proof of Theorems 2.1 and 2.2.**

Proof of Theorem 2.1. First, we claim that by Assumption (A) of Theorem 2.1, there exists a modification $\widetilde{X}_\epsilon(x)$ of $X_\epsilon(x)$ such that for every $\gamma \in (0, 1/2)$ and $\chi, \zeta > 0$ there exists $M > 0$ such that

$$\left|\widetilde{X}_{\epsilon_1}(x) - \widetilde{X}_{\epsilon_2}(y)\right| \leq M \left(\log\frac{1}{\epsilon_2}\right)^\zeta \frac{(|(x,\epsilon_1) - (y,\epsilon_2)|)^\gamma}{\epsilon_2^{(1+\chi)\gamma}} \tag{4.1}$$

for all $x, y \in B(0, R)$ and $\epsilon_1, \epsilon_2 \in (0, 1]$ and $\epsilon_2/\epsilon_1 \in (1/2, 2]$. Indeed, by (A) we have that

$$\mathbb{E}\left[(X_{\epsilon_1}(x) - X_{\epsilon_2}(y))^\alpha\right] \leq C \left(\frac{|x - y| + |\epsilon_1 - \epsilon_2|}{\epsilon_1 \wedge \epsilon_2}\right)^{\alpha/2}.$$

We can find α and β large enough such that $\left|\frac{\beta}{\alpha} - \frac{1}{2}\right| < \delta$, and consequently by [14, Lemma C.1] there exists a modification $\widetilde{X}_\epsilon(x)$ a.s. for which (4.1) holds. Without loss of generality we now work with

this modification and with a slight abuse of notation denote it by $X_\epsilon(x)$. Now we choose some suitable parameters according to the regularity condition above. Let $\chi > 0, r \in \left(0, \frac{1}{2}\right), \zeta \in (0, 1), \widetilde{r} = (1+\chi)r$, $K = \chi^{-1}, r_n = n^{-K}$, and

$$U_R := \left\{ x \in B(0, R) : \lim_{n \to +\infty} \frac{X_{r_n}}{G(r_n)} \geq a \right\}.$$

Since for $t \in (r_{n+1}, r_n)$ we have by (4.1) and the fact that $G(r_n) = C \log n(1 + \mathrm{o}\,(1))$,

$$\left| \frac{X_t(x) - X_{r_n}(x)}{G(r_n)} \right| = \mathrm{O}\left(\frac{(\log n)^\zeta}{G(r_n)} \right) = \mathrm{o}\,(1).$$

This shows that $T_\geq(a, R) \subseteq U_R$. Let $(x_{nj})_{j=1}^{\overline{k}_n}$ be a $r_n^{1+\chi}$- net for points in $B(0, R)$. Denote

$$\mathcal{A}_n := \left\{ j : \frac{X_{r_n}(x_{nj})}{G(r_n)} \geq a - \delta(n) \right\}$$

with $\delta(n) = C(\log n)^{\zeta-1}$ (the constant C can be adjusted accordingly). Again using (4.1) it follows that, for all $N \geq 1, \bigcup_{n \geq N} \bigcup_{j \in \mathcal{A}_N} B\left(x_{nj}, r_n^{1+\chi}\right)$ covers U_R with sets having maximal diameter $2r_n^{1+\chi}$.

We first note the estimate $\mathsf{P}\,(j \in \mathcal{A}_n)$ using the following Gaussian tail bound as follows:

$$\mathsf{P}\,(j \in \mathcal{A}_n) \leq \mathsf{P}\left(\frac{X_{r_n}(x_{nj})}{\sqrt{G(r_n)}} \geq (a - \delta(n))\sqrt{G(r_n)} \right)$$

$$\leq C(\log n)^{-1/2} n^{-\frac{a^2}{2\chi}(1+\mathrm{o}(1))}$$

Furthermore

$$\mathsf{E}\,[|\mathcal{A}_n|] \leq C(\log n)^{-1/2}\overline{k}_n r_n^{-d(1+\chi)} n^{-\frac{a^2}{2\chi}(1+\mathrm{o}(1))} \leq (\log n)^{-1/2} n^{-\frac{a^2}{2\chi}+d+\frac{d}{\chi}+\mathrm{o}(1)}. \tag{4.2}$$

By denoting

$$\alpha = d - \frac{a^2}{2} + \chi\frac{d + \frac{a^2}{2}}{1 + \chi},$$

we can estimate the size of the balls in the cover as follows,

$$\mathsf{E}\left[\sum_{n \geq N} \sum_{j \in \mathcal{A}_n} \mathrm{diam}(B(x_{nj}, r_n^{1+\chi}))^\alpha \right] \leq \sum_{n \geq N} (\log n)^{-1/2} r_n^{\alpha(1+\chi)} n^{-\frac{a^2}{2\chi}+d+\frac{d}{\chi}+\mathrm{o}(1)}$$

$$\leq \sum_{n \geq N} (\log n)^{-1/2} n^{-\frac{1}{\chi}(1+\chi)\alpha-\frac{a^2}{2\chi}+d+\frac{d}{\chi}+\mathrm{o}(1)}$$

$$= C \sum_{n \geq N} (\log n)^{-1/2} n^{-d} < +\infty.$$

Therefore $\sum_{n \geq N} \sum_{j \in \mathcal{A}_n} \mathrm{diam}(B(x_{nj}, r_n^{1+\chi}))^\alpha < +\infty$ a.s. and this implies $\dim_H(T_\geq(a, R)) \leq d - \frac{a^2}{2}$ a.s. by letting $\chi \downarrow 0$.

Now we show that for every $R > 1, T_\geq(a, R)$ is empty for $a^2 > 2d$ using the above estimates. Since $a^2 > 2d$ we have that $\frac{a^2}{2\chi} - d(1 + \frac{1}{\chi}) > 1$ and hence,

$$\sum_{n \geq 1} \mathsf{P}\,(|\mathcal{A}_n| > 1) \leq \sum_{n \geq 1} \mathsf{E}\,[|\mathcal{A}_n|] \leq \sum_{n \geq 1} n^{-\left(\frac{a^2}{2\chi}-d\left(1+\frac{1}{\chi}\right)\right)} < \infty.$$

and hence by the Borel-Cantelli lemma we can conclude that, if χ becomes arbitrarily small, $|\mathcal{A}_n| = 0$ eventually and so $T_\geq(a, R)$ is empty for $a^2 > 2d$ with probability one. $\qquad\square$

Proof of Corollary 2.1. Recall that from Lemma 3.1 we have that Conditions (A) and (B) hold for the restricted domain $D^{(\delta)}$, for any $\delta > 0$. Now, since D is a regular domain we can write

$$D = \bigcup_{n \in \mathbb{N}} D^{(\delta_n)}$$

for a suitable sequence $\delta_n \downarrow 0$ and $\mathrm{d}(D^{(\delta_n)}, D^c) > 0$. Now that by repeating the arguments in the proof of Theorem 2.1 and using Lemma 3.1, we get that for all n, $\dim_H\left(T(a, D^{(\delta_n)})\right) \leq 2 - \frac{a^2}{2}$ with probability one. Hence by (2.12) we obtain that $\dim_H(T(a, D)) \leq 2 - \frac{a^2}{2}$ almost surely. $\qquad\square$

Now we provide a proof of the lower bound.

Proof of Theorem 2.2. For the proof of the lower bound we denote $D = [0, 1]^d$. Let $B = \left\{\omega : \dim_H(T(a)(\omega)) \geq d - \frac{a^2}{2}\right\}$. We first show that $\mathsf{P}(B) > 0$. For this we use the following fact from [9, Theorem 4.13]. If μ is a mass distribution on F (that is, $\mu(F^c) = 0$) with $I_\alpha(\mu) < \infty$ then $\dim_H(F) \geq \alpha$. Now we make the following claim

Claim 4.1. *There exists a random measure μ giving full mass to $T(a)$ such that $\mathsf{P}(I_\alpha(\mu) < \infty) > 0$ with $\alpha = d - \frac{a^2}{2}$.*

Note that Claim 4.1 will show

$$\mathsf{P}(B) = \mathsf{P}\left(\dim_H(T(a)) \geq d - \frac{a^2}{2}\right)$$
$$\geq \mathsf{P}(I_\alpha(\mu) < \infty) > 0.$$

To provide a proof of Claim 4.1 we break the proof into some basic steps.

Step 1: Construction of the measure. We pass now to define the approximation of the limiting measure through the Radon-Nikodym derivative

$$Q_n^{(a)}(x) := \exp\left(aX_n(x) - \frac{a^2}{2}\mathsf{E}\left[X_n(x)^2\right]\right) = \exp\left(aX_n(x) - \frac{a^2}{2}p_n(x, x)\right)$$

so that we can choose the measure on $[0, 1]^d$

$$\mu_n(\mathrm{d}x) := Q_n^{(a)}(x)\mathrm{d}x.$$

We now note down some basic properties of this measure.

$\mathsf{E}\left[\mu_n[[0, 1]^d]\right] = 1$: one has, using Fubini,

$$\mathsf{E}\left[\mu_n[[0, 1]^d]\right] = \int_{[0, 1]^d} \mathsf{E}\left[Q_n^{(a)}(x)\right]\mathrm{d}x = 1. \tag{4.3}$$

$\mathsf{E}\left[\left(\mu_n[[0,\,1]^d]\right)^2\right] < \infty$: For this notice in first place that $X_n(x) + X_n(y)$ is Gaussian with mean zero and covariance $p_n(x,\,x) + p_n(y,\,y) + 2p_n(x,\,y)$. Hence, again by means of Fubini,

$$\mathsf{E}\left[\left(\mu_n[[0,\,1]^d]\right)^2\right] = \int_{[0,\,1]^d} \int_{[0,\,1]^d} \mathsf{E}\left[Q_n^{(a)}(x)Q_n^{(a)}(y)\right] \mathrm{d}x\mathrm{d}y$$

$$= \int_{[0,\,1]^d} \int_{[0,\,1]^d} \mathsf{E}\left[\exp\left(a\left(X_n(x) + X_n(y)\right) - \frac{a^2}{2}\left(p_n(x,\,x) + p_n(y,\,y)\right)\right)\right] \mathrm{d}x\mathrm{d}y$$

$$= \int_{[0,\,1]^d} \int_{[0,\,1]^d} \mathrm{e}^{-\frac{a^2}{2}\left(p_n(x,x)+p_n(y,y)\right)} \mathsf{E}\left[\exp(a\left(X_n(x) + X_n(y)\right))\right] \mathrm{d}x\mathrm{d}y$$

$$= \int_{[0,\,1]^d} \int_{[0,\,1]^d} \mathrm{e}^{-\frac{a^2}{2}\left(p_n(x,x)+p_n(y,y)\right)} \mathrm{e}^{\frac{a^2}{2}\left(p_n(x,x)+p_n(y,y)+2p_n(x,y)\right)} \mathrm{d}x\mathrm{d}y$$

$$= \int_{[0,\,1]^d} \int_{[0,\,1]^d\cap\{y\neq x\}} \exp\left(a^2 p_n(x,\,y)\right)\mathrm{d}x\mathrm{d}y$$

$$\overset{(C)}{\leq} \sup_{x,\,y\in[0,\,1]^d,\,x\neq y} \mathrm{e}^{a^2 H(x,y)} \int_{[0,\,1]^d\cap\{y\neq x\}} \frac{1}{\|x - y\|^{\frac{a^2}{2}}}\mathrm{d}x\mathrm{d}y$$

and for integrability it suffices to have $\frac{a^2}{2} < d$.

Finite energy: Now we show that the sequence of measures μ_n has finite energy, so let $\alpha < d - \frac{a^2}{2}$. Then we have

$$\mathsf{E}\left[I_\alpha(\mu_n)\right] = \mathsf{E}\left[\int_{[0,1]^d} \int_{[0,1]^d} \frac{1}{\|x - y\|^\alpha}\mathrm{d}\mu_n(x)\mathrm{d}\mu_n(y)\right]$$

$$= \int_{[0,1]^d} \int_{[0,1]^d} \frac{1}{\|x - y\|^\alpha}\mathsf{E}\left[Q_n^{(a)}(x)Q_n^{(a)}(y)\right] \mathrm{d}x\mathrm{d}y$$

$$= \int_{[0,1]^d} \int_{[0,1]^d} \frac{1}{\|x - y\|^\alpha}e^{a^2 p_n(x,y)}\mathrm{d}x\mathrm{d}y$$

$$\leq \int_{[0,1]^d} \int_{[0,1]^d} \mathbb{1}_{\{x\neq y\}} \frac{1}{\|x - y\|^{\alpha+a^2}}e^{a^2 H(x,y)}\mathrm{d}x\mathrm{d}y$$

$$\leq \sup_{x,y\in[0,1]^d} e^{a^2 H(x,y)} \int_{[0,1]^d} \int_{[0,1]^d} \mathbb{1}_{\{x\neq y\}} \frac{1}{\|x - y\|^{\alpha+\frac{a^2}{2}}}\mathrm{d}x\mathrm{d}y < C.$$

Step 2: Limiting measure with finite energy. An easy application of Paley-Zygmund inequality shows that

$$\mathsf{P}\left(\mu_n(D) > b\mathsf{E}\left[\mu_n(D)\right]\right) \geq (1 - b)^2\frac{\mathsf{E}\left[\mu_n(D)\right]^2}{\mathsf{E}\left[\mu(D)^2\right]},$$

which implies by (4.3)

$$\mathsf{P}\left(\mu_n(D) > b\right) \geq (1 - b)^2\frac{1}{\mathsf{E}\left[\mu_n(D)^2\right]} \geq \frac{(1 - b)^2}{\sup_n \mathsf{E}\left[\mu_n(D)^2\right]}.$$

Also by Markov inequality we have that

$$\mathsf{P}\left(\mu_n(D) > b^{-1}\right) \leq \frac{\mathsf{E}\left[\mu_n(D)\right]}{b^{-1}} = b.$$

One can choose b so small that $(1 - b)^2/b > \sup_n \mathsf{E}\left[\mu_n(D)^2\right]$. Hence there exists a $v > 0$ such that

$$\mathsf{P}\left(b \leq \mu_n(D) \leq b^{-1}\right) = \mathsf{P}\left(\mu_n(D) \geq b\right) - \mathsf{P}\left(\mu_n(D) \geq b^{-1}\right) > \left(\frac{(1 - b)^2}{\sup_n \mathsf{E}\left[\mu_n(D)^2\right]} - b\right) > v > 0.$$

$$\tag{4.4}$$

Also, we get $c < \infty$ such that

$$P\left(I_\alpha(\mu_n) > c\right) \leq \frac{\sup_n \mathsf{E}\left[I_\alpha(\mu_n)\right]}{c} < v/2.$$

Now using the simple inequality $\mathsf{P}\left(A \cap B\right) \geq \mathsf{P}\left(A\right) - \mathsf{P}\left(B^c\right)$ we get

$$\mathsf{P}\left(b < \mu_n(D) < b^{-1},\, I_\alpha(\mu_n) \leq c\right) \geq \frac{v}{2} > 0.$$

Let us denote

$$C_n := \left\{\omega : \mu_n(D) \in \left[b, \frac{1}{b}\right], I_\alpha(\mu_n) \leq c\right\} \qquad \text{and} \qquad C := \limsup_{n\to\infty} C_n.$$

From (4.4) it follows immediately that $\mathsf{P}\left(C\right) = \lim_{n\to\infty}\mathsf{P}\left(\cup_{m\geq n}C_m\right) \geq \lim_{n\to\infty}P(C_n) > 0$. Let us denote by $\mathcal{M}_b$ the Borel measures such that $\mu(D) \in [b, \frac{1}{b}]$. It is well-known by Prohorov's theorem that all finite positive measures on a compact metric space form a compact metric space again. Now, if $\mu_n \in \mathcal{M}_b$ converges weakly to a measure μ then for every bounded continuous functions we have $\int_D f(x)\mathrm{d}\mu_n(x) \to \int_D f(x)\mathrm{d}\mu(x)$ and hence since the function constantly equal to 1 is a bounded continuous function on D we have that $\mu_n(D) \to \mu(D)$ and hence $\mu \in \mathcal{M}_b$. So $\mathcal{M}_b$ is a closed subset of a compact space and hence compact.

We have also that $\mu \to \mu \otimes \mu$ is continuous on $\mathcal{M}_b$. This follows easily as $[0,1]^d$ is compact; it is enough to show the convergence for products of bounded and continuous functions by the Stone-Weierstrass theorem. More precisely, if μ_n weakly converge to μ then

$$\int f(x)g(y)\mathrm{d}\mu_n\mathrm{d}\mu_n(y) \to \int f(x)g(y)\mathrm{d}\mu(x)\mathrm{d}\mu(y).$$

But since the integrals are finite the convergence follows by weak convergence and hence the continuity follows. So by continuity of the functional now we have, for any $K < \infty$,

$$I_\alpha^K(\mu) = \int\int\left(\frac{1}{\|x-y\|^\alpha} \wedge K\right)\mathrm{d}\mu(x)\mathrm{d}\mu(y)$$

is continuous on $\mathcal{M}_b$ and hence the functional $\mu \mapsto \sup_{K>0} I_\alpha^K(\mu) = I_\alpha(\mu)$ is lower semi-continuous of this subspace topology. The lower semi-continuity implies that the set $\mathcal{M}' := \{\mu \in \mathcal{M} : I_\alpha(\mu) \leq a\}$ for $a < \infty$ is compact.

Fix $\omega \in C$, hence there exist $n_k \to \infty$ such that $\omega \in C_{n_k}$ and $\mu_{n_k}(\omega, \cdot) \in \{\mu : \mu(D) \in [b, \frac{1}{b}], I_\alpha(\mu) \leq a\} = \mathcal{M}'$. Now by the previous paragraph we have that $\mathcal{M}'$ is compact, so there exists a further subsequence $\mu_{n_{k_l}}(\omega) \xrightarrow{w} \mu(\omega)$ and $\mu(\omega, D) > 0$ and $I_\alpha(\mu) < \infty$ for $\alpha \leq d - a$.

Step 3: Construction of the rooted (Peyrière) measure (Kahane [16]). Let $Y_n(x)$ be independent (over n) Gaussian variables with covariance kernel given by $\tilde{p}_n(x, y)$. Hence we have that $X_n(x)$ has the same law as $\sum_{i=1}^n Y_i(x)$. Note that for any bounded (measurable) subset $A \in [0,1]^d$, $\mu_n(A)$ is a positive martingale with respect to the filtration $\mathcal{F}_n$ generated by $\{Y_m(x) : m \leq n\}$, hence it converges almost surely. In fact, Step 1 shows that $\sup_{n\geq 1}\mathsf{E}\left[\mu_n(A)^2\right] < C < \infty$ and hence $\mu_n(A)$ is a uniformly integrable martingale and converges to some random variable C_A in L^2. The limiting random variable satisfies the following two properties:

- for A and B bounded disjoint subsets of $[0,1]^d$ we have that

$$C_{A\cup B} = C_A + C_B \qquad \text{almost surely} \, / \, L^2,$$

- for any family of bounded sets $\{A_m : A_m \downarrow \emptyset\}$ one has $C_{A_m} \to 0$ almost surely and in L^2. For the almost sure convergence, note that as $\mathsf{E}\left[C(A_m)\right] \to 0$, by Markov's inequality $C(A_m) \to 0$ with probability one.

Now by applying [5, Theorem 6.1.VI] we get that there exists a measure ν such that μ_n converges weakly to ν in L^2 and almost surely. Note that by the weak convergence we have that

$$\nu(A) \leq \liminf_{n \to \infty} \mu_n(A),$$

for A bounded open subset of $[0,1]^d$. Also, by Fatou's lemma we have that $\mathsf{E}\left[\nu(A)\right] \leq \mathsf{E}\left[\liminf_{n \to \infty} \mu_n(A)\right] \leq \liminf_{n \to \infty} \mathsf{E}\left[\mu_n(A)\right] = \mathrm{vol}(A) < \infty$. Hence we have by the L^2 convergence

$$\mathsf{E}\left[\nu(A)\right] = \lim_{n \to \infty} \mathsf{E}\left[\mu_n(A)\right] = \mathrm{vol}(A),$$

for every bounded open set A. Thus it is true that $\mathsf{E}\left[\nu([0,1]^d)\right] = 1$.

Now we construct the rooted measure using this ν. Let us define a measure on $[0,1]^d \times \Omega$ as

$$M(\mathrm{d}x, \, \mathrm{d}\omega) = \nu^\omega(\mathrm{d}x)\mathsf{P}\left(\mathrm{d}\omega\right).$$

Note that by the previous observation $M([0,1]^d \times \Omega) = 1$ and hence M is a probability measure. We observe that for any g compactly supported (possibly random) it holds that

$$\int_{[0,1]^d} g(x)\mu_k(\mathrm{d}x) \xrightarrow{d} \int_{[0,1]^d} g(x)\nu(\mathrm{d}x)$$

by L^2 convergence, from which also L^1 convergence follow:

$$\mathsf{E}\left[\int_{[0,1]^d} g(x)\nu(\mathrm{d}x)\right] = \lim_{k \to \infty} \mathsf{E}\left[\int_{[0,1]^d} g(x)\mu_k(\mathrm{d}x)\right]. \tag{4.5}$$

Step 4: $Y_n(x)$ are independent under M. First note that $X_n(x) \stackrel{d}{=} \sum_{i=1}^n Y_i(x)$. Let us denote by

$$P_n^{(a)}(x) = \exp\left(aY_n(x) - \frac{a^2}{2}\tilde{p}_n(x,x)\right).$$

This will help us to show that for all bounded positive continuous functions and $N \geq 1$

$$\int \prod_{n=1}^N f_n(Y_n(x))\mathrm{d}M = \prod_{n=1}^N \int f_n(Y_n(x))\mathrm{d}M. \tag{4.6}$$

Note that $(g_N(x) := \prod_{n=1}^N f_n(Y_n(x)))$

$$\int \prod_{n=1}^N f_n(Y_n(x))\mathrm{d}M = \mathsf{E}\left[\int_{[0,1]^d} \prod_{n=1}^N f_n(Y_k(x))\nu(\mathrm{d}x)\right]$$

$$= \mathsf{E}\left[\int_{[0,1]^d} g_N(x)\nu(\mathrm{d}x)\right]$$

$$\stackrel{(4.5)}{=} \lim_{k \to \infty} \mathsf{E}\left[\int_{[0,1]^d} g_N(x)\mu_k(\mathrm{d}x)\right].$$

Choose $k > N$ and using the fact that $Q_k(x) \stackrel{d}{=} P_1^{(a)}(x) \cdots P_N^{(a)}(x) \cdots P_k^{(a)}(x)$ we have that

$$\mathsf{E}\left[\int_{[0,1]^d} g_N(x)\mu_k(\mathrm{d}x)\right] = \mathsf{E}\left[\int_{[0,1]^d} g_N(x)P_1^{(a)}(x) \cdots P_N^{(a)}(x) \cdots P_k^{(a)}(x)\mathrm{d}x\right]$$

$$= \int_{[0,1]^d} \mathsf{E}\left[\prod_{n=1}^N f_n(Y_n(x))P_n^{(a)}(x)\right]\mathrm{d}x.$$

The last line follows since $\mathsf{E}\left[P_m^{(a)}(x)\right] = 1$ for $m \geq 1$. Now we claim that $\{\mathsf{E}\left[f_n(Y_n(x))P_n(x)\right]\}$ are independent random variables in x. Let us define a new probability measure as

$$\mathsf{P}'(\mathrm{d}\omega) = P_n^{(a)}(x)\mathsf{P}\left(\mathrm{d}\omega\right).$$

Now note that the law of $Y_n(x)$ under P' is given as follows

$$\mathsf{P}'(Y_n(x) \in \mathrm{d}y) = \exp\left(ay - \frac{a^2}{2}\tilde{p}_n(x,x)\right)\mathsf{P}\left(Y_n(x) \in \mathrm{d}y\right)$$

$$= \frac{1}{\sqrt{2\pi\tilde{p}_n(x,x)}}\exp\left(ay - \frac{a^2}{2}\tilde{p}_n(x,x) - \frac{y^2}{2\tilde{p}_n(x,x)}\right)\mathrm{d}y$$

$$= \frac{1}{\sqrt{2\pi\tilde{p}_n(x,x)}}\exp\left(-\frac{(y - a\tilde{p}_n(x,x))^2}{2\tilde{p}_n(x,x)}\right)\mathrm{d}y$$

Therefore $\mathsf{E}\left[f_n(Y_n(x))P_n^{(a)}(x)\right] = \mathsf{E}'[f(Z_n)]$ where $Z_n \sim N(a\tilde{p}_n(x,x), \tilde{p}_n(x,x))$. So this gives us

$$\int \prod_{n=1}^N f_n(Y_n(x))\mathrm{d}M \overset{(4.6)}{=} \prod_{n=1}^N \mathsf{E}_M\left[f_n(Y_n(x))P_n^{(a)}(x)\right].$$

Finally we note that, by repeating the calculations above,

$$\prod_{n=1}^N \int f_n(Y_n(x))\mathrm{d}M = \prod_{n=1}^N \lim_{k\to\infty}\mathsf{E}\left[\int_{[0,1]^d} f_n(Y_n(x))\mathrm{d}\mu_k(x)\right] = \prod_{n=1}^N \mathsf{E}\left[f_n(Y_n(x))P_n^{(a)}(x)\right].$$

Step 5: Law of Y_n under rooted measure. To recap, from the previous step we have that for a compactly supported function

$$\int f(Y_n(x))\mathrm{d}M = \mathsf{E}\left[f(Y_n(x))P_n^{(a)}(x)\right] = \mathsf{E}'[f(Z)],$$

where Z is a random variable independent of x and n and is distributed like $\mathcal{N}(a\tilde{p}_n(x,x), \tilde{p}_n(x,x))$ under P'. In particular, by taking smooth approximations to the identity function we get that

$$\int Y_n(x)\mathrm{d}M = a.$$

Set

$$A(x,\omega) := \{\omega : x \in T(a)^c\}.$$

By the strong law of large numbers, $A(x,\omega)$ has M-probability zero, and hence $\nu(T(a)^c) = 0$ P-almost surely. Hence ν is a mass distribution on $T(a)$.

Step 6: Uniqueness of the measures: Recall from Step 2 that we can find a non-null random set C where we can find a limit measure $\mu(\omega, \cdot)$ with finite α-energy for $\alpha < d - \frac{a^2}{2}$. Also we have along a subsequence

$$\mathsf{P}\left(\omega \in C : \int_D f(x)\mu_{n_m}(\omega, \mathrm{d}x) \to \int_D f(x)\mu(\omega, \mathrm{d}x), \forall f \in C_b(D)\right) > 0.$$

On the other hand Step 5 yields a full-measure set S and a measure $\nu(\omega, \cdot)$ for which $T(a)^c$ is P-almost surely a null set:

$$\mathsf{P}\left(\omega \in S : \int_D f(x)\mu_{n_m}(\omega, \mathrm{d}x) \to \int_D f(x)\nu(\omega, \mathrm{d}x), \forall f \in C_b(D)\right) = 1$$

So we have that for an appropriate subset of a probability space

$$\mathsf{P}\left(\omega : \nu(\omega, \cdot) = \mu(\omega, \cdot)\right) \geq \mathsf{P}\left(C \cap S\right) > 0.$$

This in particular shows that $\mathsf{P}\left(\omega : I_\alpha(\mu) < \infty, \; \mu(T(a)^c) = 0\right) > 0$ and hence this shows Claim 4.1. We now complete the proof using a 0-1 law to prove $\{I_\alpha(\mu) < \infty, \; \mu(T(a)^c) = 0\}$ is a tail event.

Step 7: 0-1 law. Recall that $\{Y_k(x) : x \in [0,1]^d\}$ is a centered Gaussian process with covariance given by $\widetilde{p}_k(x, y)$. Also, we know that $\{X_n(x), x \in [0,1]^d, \; n \in \mathbb{N}\}$ has the law of $\{\sum_{j=0}^n Y_j(x), \; x \in [0,1]^d, \; n \in \mathbb{N}\}$. So

$$\left\{x \in [0,1]^d : \lim_{n \to \infty} \frac{X_n(x)}{n} = a\right\} \stackrel{d}{=} \left\{x \in [0,1]^d : \lim_{n \to +\infty} \frac{\sum_{j=0}^n Y_j(x)}{n} = a\right\}.$$

Let us denote the sigma field generated by the process $\{Y_n(x) : x \in [0,1]^d\}$ by $\mathcal{F}_n$. Let $\mathcal{T}_n = \sigma(\bigcup_{j \geq n} \mathcal{F}_j)$. Note that the sigma fields $\mathcal{F}_j$ are independent. Denote the tail sigma field by $\mathcal{T} = \bigcap_{n>0} \mathcal{T}_n$. We claim that $B \in \mathcal{T}$. In fact, let

$$A_j := \bigcup_{N \geq j} \bigcap_{m \geq n \geq N} \left\{\omega : \left|\sum_{j \leq m} \frac{Y_j(x)}{m} - \sum_{j \leq n} \frac{Y_j(x)}{n} - a\right| < \frac{1}{j}\right\}.$$

We see that $A_j \in \mathcal{T}_j$ and $A_j \in \mathcal{T}_\ell$ for any $\ell \leq j$. Since we have

$$\bigcap_j A_j = \left\{\lim_{n \to +\infty} \sum_{j \leq n} \frac{Y_j(x)}{n} = a\right\}$$

we can see that $\left\{\lim_{n \to +\infty} \sum_{j \leq n} \frac{Y_j(x)}{n} = a\right\} \in \mathcal{T}_j$ for any natural number j. So it follows by Kolmogorov's 0-1 law that $\mathsf{P}(B) \in \{0, 1\}$. Since $\mathsf{P}(B) > 0$ by Steps 5 and 6, we have that $\mathsf{P}(B) = 1$ and this completes the proof.

$\square$

Remark 4.1. The proof of Theorem 2.2 requires a different argument for white noise cut-offs. Step 1 in fact can be said to hold eventually for n large enough, as Condition (C) holds from a certain n_0 onwards. The limiting event C described in Step 2 depends on the tail behavior of the measures μ_n and hence remains the same. What one should be careful about is the L^2 convergence of the martingale $\mu_n(A)$ in Step 3 which is ensured by Doob's martingale inequality, since

$$\sup_{n \leq n_0} \mathsf{E}\left[\mu_n(A)^2\right] \leq C\mathsf{E}\left[\mu_{n_0}(A)^2\right] \leq C.$$

4.2. **Proof of Theorem 2.3.** Before we start the proof of Theorem 2.3, we state a useful claim which we implement in the proof.

Claim 4.2. *Let $\{G_\epsilon(x), x \in B(0,R), \epsilon \in (0,1)\}$ be a centered Gaussian process, such that for some positive constant C*

$$\mathsf{E}\left[(G_\epsilon(x) - G_\epsilon(y))^2\right] \leq C\frac{\|x - y\|}{\epsilon}. \tag{4.7}$$

Then there exists constants C_1 (depending only on C, d and R) such that

$$\mathsf{E}\left[\sup_{x \in B(0,R)} G_\epsilon(x)\right] \leq C_1\sqrt{-\log \epsilon}.$$

Proof of Claim 4.2. Without loss of generality let us take $R = 1$, $D = B(0,1)$ and let $T(x)$ be a continuous, stationary, centered Gaussian process (indexed by $x \in D$) with

$$\mathrm{Cov}(T(x), T(y)) = \frac{\sigma}{\rho}\exp(-\rho\|x - y\|),$$

where $\sigma = 2C$ and ρ is some positive constant less than $\epsilon/2$. Such a Gaussian process exists, see for example [2, Lemma 2.1]. Using the fact that $1 - e^{-x} \geq (x \wedge 1)/2$ we have that

$$
\begin{aligned}
\mathsf{E}\left[(T(x/\epsilon) - T(y/\epsilon))^2\right] &= \frac{\sigma}{\rho} - 2\mathrm{Cov}(T(x/\epsilon), T(y/\epsilon)) \\
&= \frac{\sigma}{\rho}\left(1 - \exp\left(-\rho\left\|x - y\right\|\epsilon^{-1}\right)\right) \\
&\geq C\frac{\left\|x - y\right\|}{\epsilon}.
\end{aligned}
$$

This shows that $\mathsf{E}\left[(G_\epsilon(x) - G_\epsilon(y))^2\right] \leq \mathsf{E}\left[(T(x/\epsilon) - T(y/\epsilon))^2\right]$. Hence by Sudakov-Fernique's inequality ([1, Theorem 2.9]), we have that

$$
\mathsf{E}\left[\sup_{x\in D} G_\epsilon(x)\right] \leq \mathsf{E}\left[\sup_{x\in D} T(x/\epsilon)\right] = \mathsf{E}\left[\sup_{x\in B(0,\epsilon^{-1})} T(x)\right]. \tag{4.8}
$$

Now we can can apply Lemma 11.2 of Chatterjee [3] to conclude that

$$
\mathsf{E}\left[\sup_{x\in B(0,\epsilon^{-1})} T(x)\right] \leq C(d)\sqrt{\log N(B(0, 1/\epsilon))},
$$

where, for $A \subseteq \mathbb{R}^d$, $N(A)$ denotes the 1-packing number. Since it is bounded by the 1-covering number of $B(0, 1/\epsilon)$, it is easy to see that $N(B(0, 1/\epsilon))$ is bounded from above by ϵ^{-d} and hence the claim now follows from (4.8). $\qquad\square$

Now using Claim 4.2 we derive a proof of Theorem 2.3.

Proof of Theorem 2.3. First observe that Assumption (E) implies we can apply the modified Kolmogorov-Centsov theorem as in Theorem 2.1 and derive that, for $x \in D = B(0, R)$ and $\epsilon \in (0, 1]$, there exists a modification $\widetilde{Z}_\epsilon(x)$ of $Z_\epsilon(x)$ such that for every $\gamma \in (0, 1/2)$ and $a, b > 0$ there exists $M > 0$ such that

$$
\left|\widetilde{Z}_{\epsilon_1}(x) - \widetilde{Z}_{\epsilon_2}(y)\right| \leq M\left(\log\frac{1}{\epsilon_2}\right)^b \frac{(\left\|x - y\right\| + |\epsilon_1 - \epsilon_2|)^\gamma}{\epsilon_2^{(1+a)\gamma}} \tag{4.9}
$$

for all $x, y \in B(0, R)$ and $\epsilon_1, \epsilon_2 \in (0, 1]$ and $\epsilon_2/\epsilon_1 \in (1/2, 2]$.

We work with a modification of the process and also use the same notation for the process and its modification. First we show that

$$
\limsup_{\epsilon\to 0} \frac{Z_\epsilon(x)}{-\log\epsilon} = 0. \tag{4.10}
$$

From Claim 4.2 we have that $\mathsf{E}\left[\sup_{x\in D} Z_\epsilon(x)\right] \leq C\sqrt{-\log\epsilon}$. By Borell's inequality ([1, Chapter 5.1]),

$$
\mathbb{P}\left(\left|\sup_{x\in D} Z_\epsilon(x) - \mathsf{E}\left[\sup_{x\in D} Z_\epsilon(x)\right]\right| \geq r\right) \leq Ce^{-cr^2/2}, \tag{4.11}
$$

where $c = (\sup_{x\in D} \mathsf{E}\left[Z_\epsilon(x)^2\right])^{-1}$. Let $a > 0$ and if we choose $\epsilon_n := n^{-1/a}$, $r_n = \sqrt{\log\epsilon_n^{-3a/c}}$ then it follows that

$$
\sum_{n=1}^\infty P\left(\left|\sup_{x\in D} Z_{\epsilon_n}(x) - \mathsf{E}\left[\sup_{x\in D} Z_{\epsilon_n}(x)\right]\right| \geq r_n\right) \leq C\sum_{n=1}^\infty \frac{1}{n^{3/2}} < +\infty.
$$

Now by an easy application of Borel-Cantelli we have that $\sup_{x\in D} Z_{\epsilon_n}(x) = \mathrm{o}(-\log\epsilon_n)$ almost surely, since $\frac{r_n}{-\log\epsilon_n} \to 0$ as $n \to +\infty$. Now we claim that due to continuity we can move from the discrete

sequence to the continuous sequence. We plug in (4.9) the choice of $\epsilon_n = n^{-1/a}$ and let $\epsilon \in (\epsilon_{n+1}, \epsilon_n)$ in order to have

$$\left| \sup_{x \in D} Z_\epsilon(x) - \sup_{x \in D} Z_{\epsilon_n}(x) \right| \leq \left(\log \frac{1}{\epsilon_n} \right)^b \frac{|\epsilon - \epsilon_n|^\gamma}{\epsilon_n^{(1+a)\gamma}} \leq C(\log n)^b = o(\log n).$$

This implies that $\frac{|\sup_{x \in D} Z_\epsilon(x) - \sup_{x \in D} Z_{\epsilon_n}(x)|}{-\log \epsilon_n} \to 0$ and hence, using $Z_\epsilon(x) = Z_\epsilon(x) - Z_{\epsilon_n}(x) + Z_{\epsilon_n}(x)$ we get that $\limsup_{\epsilon \to 0} \frac{Z_\epsilon(x)}{-\log \epsilon} = 0$ almost surely. What is left to show is the equality of the set of thick points, and we begin with the inclusion $T(X, a) \subseteq T(\tilde{X}, a)$ almost surely. The other follows similarly. Let $x \in T(X, a)$, then as a consequence of (4.10) it holds that $\limsup_{\epsilon \to 0} \frac{\tilde{X}_\epsilon(x)}{-\log \epsilon} \geq a$. Since $Z_\epsilon(x)$ is a symmetric process in x, we have

$$\liminf_{\epsilon \to 0} \frac{\tilde{X}_\epsilon(x)}{-\log \epsilon} \geq \liminf_{\epsilon \to 0} \left(\inf_{x \in D} \frac{-Z_\epsilon(x)}{-\log \epsilon} \right) + \liminf_{\epsilon \to 0} \frac{X_\epsilon(x)}{-\log \epsilon}.$$

Hence using $\liminf_n x_n = -\limsup_n(-x_n)$ we have

$$\liminf_{\epsilon \to 0} \frac{\tilde{X}_\epsilon(x)}{-\log \epsilon} \geq \liminf_{\epsilon \to 0} \frac{X_\epsilon(x)}{-\log \epsilon} = a.$$

This completes the proof of the fact that $T(X, a) \subseteq T(\tilde{X}, a)$; reversing the roles of $X_\epsilon(x)$ and $\tilde{X}_\epsilon(x)$ we get the other inclusion to complete the proof. $\qquad \square$

5. ACKNOWLEDGMENTS

6. APPENDIX

Lemma 6.1. *If B_m is a symmetric operator on $\mathbb{R}^d$ with Fourier multiplier $m(\cdot)$ such that $\sup_{\xi \in \mathbb{R}^d} m(\xi) = c' < \infty$, then the characteristic functional $L(\phi) := \exp\left(-\frac{1}{2} \langle \phi, B_m^{-d} \phi \rangle_{L^2}\right)$ is a positive definite functional and continuous in the Frechét topology of $\mathcal{S}(\mathbb{R}^d)$.*

For our proof we will follow the ideas contained in Sun and Wu [25].

Proof. Let $\phi \in \mathcal{S}(\mathbb{R}^d)$ and $s \geq 0$, and define

$$L(\phi) = \exp\left(-\frac{1}{2} \langle \phi, B^{-d} \phi \rangle_{L^2}\right) = \exp\left(-\frac{1}{2} \langle B^{-d/2} \phi, B^{-d/2} \phi \rangle\right) = \exp\left(-\frac{1}{2} (\langle \phi, \phi \rangle_{H^{-d/2}})\right). \tag{6.1}$$

One clearly has $L(0) = 1$. The positive definiteness can be shown by verifying the condition

$$\sum_{j=1}^n \sum_{k=1}^n z_j \bar{z}_k L(\phi_j - \phi_k) \geq 0$$

for all $n \in \mathbb{N}$, $z_1, \ldots, z_n \in \mathbb{C}$ and $\phi_1, \ldots, \phi_n \in \mathcal{S}(\mathbb{R}^d)$. Let then μ be a Gaussian measure on $V = \mathrm{span}(\phi_1, \ldots, \phi_n)$, with covariance matrix given by $\left(\langle B^{-d/2} \phi_k, B^{-d/2} \phi_j \rangle_{L^2} \right)_{i,j=1,\ldots,n}$ and hence

$$\int_V e^{i \langle B^{-d/2} \phi, B^{-d/2} t \rangle} \mu(\mathrm{d}t) = L(\phi)$$

for all $\phi \in \mathcal{S}(\mathbb{R}^d)$. Now ($\mathcal{F}$ here denotes the Fourier transform)

$$\sum_{j=1}^{n}\sum_{k=1}^{n} z_j \bar{z}_k L(\phi_j - \phi_k) = \sum_{j,k=1}^{n} \int_V z_j \bar{z}_k e^{i\left\langle B^{-d/2}(\phi_j - \phi_k), B^{-d/2}t\right\rangle} \mu(\mathrm{d}t)$$

$$= \int_V \left| \sum_{j=1}^{n} z_j e^{i\left(\left\langle B^{-d/2}\phi_j, B^{-d/2}t\right\rangle\right)} \right|^2 \mu(\mathrm{d}t) \geq 0.$$

Using the inequality that $|e^{-x} - e^{-y}| \leq \|x - y\|$ we have that

$$|L(\phi) - L(\psi)| \leq |\left\langle B^{-d/2}\phi, B^{-d/2}\phi\right\rangle - \left\langle B^{-d/2}\psi, B^{-d/2}\psi\right\rangle|$$

$$= |\left\langle \phi, B^{-d}\phi\right\rangle - \left\langle \psi, B^{-d}\psi\right\rangle|$$

$$= |\left\langle \mathcal{F}\phi, \mathcal{F}B^{-d}\phi\right\rangle - \left\langle \mathcal{F}\psi, \mathcal{F}B^{-d}\psi\right\rangle|$$

$$= |\left\langle \mathcal{F}\phi, m(\cdot)\mathcal{F}\phi\right\rangle - \left\langle \mathcal{F}\psi, m(\cdot)\mathcal{F}\psi\right\rangle|$$

$$= \left| \int_{\mathbb{R}^d} \left(|\widehat{\phi}(\xi)|^2 - |\widehat{\psi}(\xi)|^2 \right) m(\xi)\mathrm{d}\xi \right|$$

$$\leq c \left| \|\widehat{\phi}\|_2^2 - \|\widehat{\psi}\|_2^2 \right|.$$

Now it is easy to see that the map $\phi \mapsto \|\phi\|_2$ is continuous in the Fréchet topology of $\mathcal{S}(\mathbb{R}^d)$. $\qquad \square$

Remark 6.1. Note that the Fourier multiplier for the Bessel operator B_m^s is $(m - 4\pi\|\xi\|^2)^{-s/2}$ and thus satisfies the above assumptions.

Lemma 6.2. *The function*

$$H_\epsilon(x, y) = \int_1^{1/\epsilon} k_m(u\|x - y\|) \frac{\mathrm{d}u}{u},$$

where $k_m(z) = \frac{1}{2}\int_0^\infty e^{-\frac{m^2|z|^2}{2v}} e^{-v/2}\mathrm{d}v$, is positive definite for each $\epsilon > 0$.

Proof. In full generality, we will show that $f_k(x, y) = \int_a^b \frac{k_m(u(x-y))}{u}\mathrm{d}u$ is a positive definite functional for any $0 \leq a < b < \infty$. To this purpose we consider ϕ in $C_c^\infty(\mathbb{R}^d)$. To show that

$$\int_{\mathbb{R}^d}\int_{\mathbb{R}^d} \phi(x)\phi(y)f_k(x, y)\mathrm{d}x\mathrm{d}y \geq 0$$

we expand the expression as follows:

$$\int_{\mathbb{R}^d}\int_{\mathbb{R}^d} \phi(x)\phi(y)f_k(x, y)\mathrm{d}x\mathrm{d}y$$

$$= \int_{\mathbb{R}^d \times \mathbb{R}^d} \phi(x)\phi(y)\left(\int_a^b \left(\frac{1}{2}\int_0^\infty e^{-\frac{u^2m^2\|x-y\|^2}{2t}} e^{-t/2}\mathrm{d}t \right) \frac{\mathrm{d}u}{u} \right)\mathrm{d}x\mathrm{d}y$$

$$= \frac{(2\pi)^{d/2}}{2}\int_{\mathbb{R}^d \times \mathbb{R}^d} \phi(x)\phi(y)\left(\int_a^b \left(\int_0^\infty \frac{e^{-\frac{u^2m^2\|x-y\|^2}{2t}}}{(2\pi t)^{d/2}}u^d\frac{t^{d/2}}{u^d} e^{-t/2}\mathrm{d}t \right) \frac{\mathrm{d}u}{u} \right)\mathrm{d}x\mathrm{d}y$$

$$\overset{\exists C(m)>0}{=} C(m)\int_{\mathbb{R}^d \times \mathbb{R}^d}\int_a^b\int_0^\infty \phi(x)\phi(y)p\left(\frac{t}{m^2u^2}, x, y \right)\frac{t^{d/2}}{u^d}e^{-t/2}\mathrm{d}t\frac{\mathrm{d}u}{u}\mathrm{d}x\mathrm{d}y$$

The semi-group property of the Gauss-Weierstrass heat kernel

$$p(t, x, y) = \int_{\mathbb{R}^d} p(t/2, x, z)p(t/2, z, y)\mathrm{d}z$$

is well-known. Hence we have

$$\int_{\mathbb{R}^d \times \mathbb{R}^d} \phi(x)\phi(y) f_k(x,y) \mathrm{d}x \mathrm{d}y$$

$$= C(m) \int_{\mathbb{R}^d \times \mathbb{R}^d} \left\{ \int_a^b \left(\int_0^\infty \phi(x)\phi(y) p\left(\frac{t}{m^2 u^2}, x, y \right) \frac{t^{d/2}}{u^d} \, \mathrm{e}^{-t/2} \, \mathrm{d}t \right) \frac{\mathrm{d}u}{u} \right\} \mathrm{d}x \mathrm{d}y$$

$$= C(m) \int_{\mathbb{R}^d \times \mathbb{R}^d} \left\{ \int_a^b \left[\int_0^\infty \left(\int_{\mathbb{R}^d} \phi(x)\phi(y) p\left(\frac{t}{2m^2 u^2}, x, z \right) p\left(\frac{t}{2m^2 u^2}, y, z \right) \mathrm{d}z \right) \right. \right.$$

$$\left. \left. \frac{t^{d/2}}{u^d} \, \mathrm{e}^{-t/2} \, \mathrm{d}t \right] \frac{\mathrm{d}u}{u} \right\} \mathrm{d}x \mathrm{d}y$$

$$= C(m) \int_{\mathbb{R}^d} \left\{ \int_a^b \left[\int_0^\infty \left(\int_{\mathbb{R}^d} \phi(x) p\left(\frac{t}{2m^2 u^2}, x, z \right) \mathrm{d}x \right) \left(\int_{\mathbb{R}^d} \phi(y) p\left(\frac{t}{2m^2 u^2}, y, z \right) \mathrm{d}y \right) \right. \right.$$

$$\left. \left. \frac{t^{d/2}}{u^d} \, \mathrm{e}^{-t/2} \, \mathrm{d}t \right] \frac{\mathrm{d}u}{u} \right\} \mathrm{d}z$$

$$= C(m) \int_{\mathbb{R}^d} \left\{ \int_a^b \left[\int_0^\infty \left(\int_{\mathbb{R}^d} \phi(x) p\left(\frac{t}{2m^2 u^2}, x, z \right) \mathrm{d}x \right)^2 \frac{t^{d/2}}{u^d} \, \mathrm{e}^{-t/2} \, \mathrm{d}t \right] \frac{\mathrm{d}u}{u} \right\} \mathrm{d}z \geq 0.$$

$\square$

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