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**From Sine kernel to Poisson statistics**

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## 1. INTRODUCTION AND MAIN RESULT

Although random matrices were originally introduced by John Wishart [1] in 1928 as a tool to study population dynamics in biology through principal component analysis, they became very popular much later in 1951 when Wigner [2] postulated that the fluctuations in positions of the energy levels of heavy nuclei are well described (in terms of statistical properties) by the eigenvalues of a very large Hermitian random matrix. Random matrix theory (RMT) is now an active research area in mathematics and theoretical physics with applications in statistics, biology, financial mathematics, engineering and telecommunications, number theory etc. (see [3, 4, 5, 6, 7] for a state of the art).

The classical models of Hermitian random matrices are the Gaussian orthogonal, unitary and symplectic ensembles. It is well known that the joint law of the eigenvalues of the matrices in those Gaussian ensembles is the Boltzmann-Gibbs equilibrium measure of a one-dimensional repulsive Coulomb gas confined in a harmonic well. More precisely, this joint law has a probability density  $P_\beta$  on  $\mathbb{R}^N$  ( $N$  is the dimension of the square matrices) given by

$$(1) \quad P_\beta(\lambda_1, \dots, \lambda_N) = \frac{1}{Z_N} \prod_{i < j} |\lambda_i - \lambda_j|^\beta \exp\left(-\frac{N\beta}{4} \sum_{i=1}^N \lambda_i^2\right)$$

where the inverse temperature  $\beta = 1$  for the Gaussian orthogonal ensemble, respectively  $\beta = 2, 4$  for the unitary and symplectic ensembles. The linear statistics of the point processes with joint probability density functions (jpdf)  $P_\beta, \beta = 1, 2, 4$  have been extensively studied in the literature with different methods [3, 5].

In 2002, Dumitriu and Edelman [8] came up with a new explicit ensemble of random tri-diagonal matrices whose eigenvalues are distributed according to the jpdf  $P_\beta$  for any  $\beta > 0$  (see also [9, 10] where invariant ensembles associated to general  $\beta$  were constructed).

Those tri-diagonal matrices have been very useful in the last decade, leading to important progress on the understanding of the local eigenvalues statistics in the limit of large dimension  $N$  for general  $\beta > 0$ . At the edge of the spectrum, it was first proved [11] that the largest eigenvalues converge jointly (when zooming in the edge-scaling region of width  $N^{-2/3}$  around 2) to the low lying eigenvalues of a random Schrödinger operator called the stochastic Airy operator (see also [12]). Similar results were proved for the bulk in [13] by Valkó and Virág. For  $\lambda$  belonging to the Wigner sea  $(-2, 2)$ , the authors of [13] consider the point process

$$(2) \quad \Lambda_N := (2\pi N \rho(\lambda) (\lambda_i - \lambda))_{i=1, \dots, N}$$

where  $(\lambda_1, \dots, \lambda_N)$  is distributed according to  $P_\beta$  and  $\rho(\lambda) = \frac{1}{2\pi} \sqrt{4 - \lambda^2}$  is the Wigner semi-circle density. Indeed, the mean level spacing around level  $\lambda$  for the points with law  $P_\beta$  is approximately  $1/(N\rho(\lambda))$  when  $N \gg 1$ . The mean point spacing of  $\Lambda_N$  defined in (2) is therefore of order  $2\pi$  and in this scaling, one can now investigate the limiting statistics of this point process when  $N \rightarrow \infty$ . The authors of [13] precisely answer this question proving that the point process  $\Lambda_N$  converges in law<sup>1</sup> to a point process  $\text{Sine}_\beta$  on  $\mathbb{R}$  first introduced in [13] and characterized in terms of a family  $(\alpha_\lambda)_{\lambda \in \mathbb{R}}$  of coupled one-dimensional diffusion processes,

<sup>1</sup>The convergence is with respect to vague topology for the counting measure of the point process.

the *stochastic sine equations*. As expected for the eigenvalues statistics in the bulk, the point process  $\text{Sine}_\beta$  is translation-invariant in law. The family of diffusions  $(\alpha_\lambda)_{\lambda \in \mathbb{R}}$  can be interpreted as the hyperbolic angle of the Brownian carousel with parameter  $\lambda$  and its law is characterized as follows: Given a (driving) complex Brownian motion  $(Z_t)_{t \geq 0}$ , the diffusions  $\alpha_\lambda, \lambda \in \mathbb{R}$  satisfy

$$(3) \quad d\alpha_\lambda = \lambda \frac{\beta}{4} e^{-\frac{\beta}{4}t} dt + \text{Re}((e^{-i\alpha_\lambda} - 1)dZ_t), \quad \alpha_\lambda(0) = 0.$$

Note that all the diffusions  $\alpha_\lambda, \lambda \in \mathbb{R}$  are adapted to the filtration of the Brownian motion  $(Z_t)$ . This coupling induces a strong interaction between the diffusions which makes the joint law difficult to analyse, as we shall see. A key feature shared by the processes  $\alpha_\lambda, \lambda \in \mathbb{R}$  is that they all converge almost surely as  $t \rightarrow \infty$  to a limit  $\alpha_\lambda(\infty)$  which is an integer multiple of  $2\pi$ . The characterization of the law of the  $\text{Sine}_\beta$  point process<sup>2</sup> can now be enunciated as follows

$$(4) \quad (\text{Sine}_\beta([\lambda, \lambda']))_{\lambda < \lambda'} \stackrel{(d)}{=} \left( \frac{\alpha_{\lambda'}(\infty) - \alpha_\lambda(\infty)}{2\pi} \right)_{\lambda < \lambda'}.$$

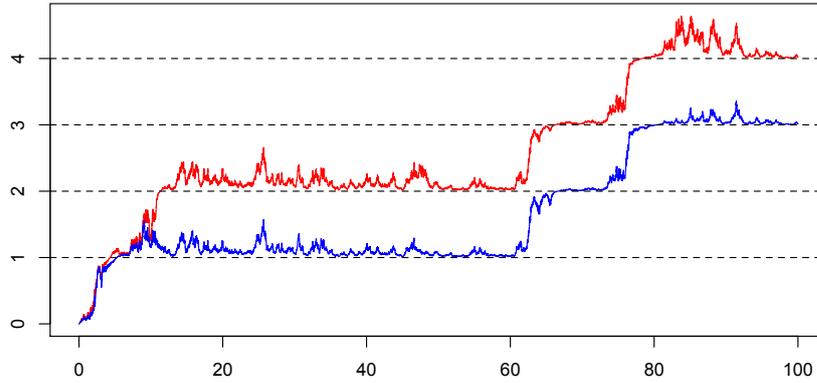


FIGURE 1. (Color online). Sample paths of two diffusions  $\alpha_\lambda/(2\pi)$  (blue curve) and  $\alpha_{\lambda'}/(2\pi)$  (red curve) for  $\lambda < \lambda'$  and for a small value of  $\beta$ . We see that whenever  $\lfloor \alpha_\lambda/(2\pi) \rfloor$  jumps,  $\lfloor \alpha_{\lambda'}/(2\pi) \rfloor$  jumps at the same time in agreement with Lemma 3.3.

In this paper, we are interested in the limiting law of the  $\text{Sine}_\beta$  process when the inverse temperature  $\beta$  goes to 0. Theorem 1.1 is the main result of this paper and gives the convergence as  $\beta \rightarrow 0$  of the  $\text{Sine}_\beta$  process towards a Poisson point process on  $\mathbb{R}$ . This convergence at the continuous ( $N = \infty$ ) level is rather natural since taking  $\beta \rightarrow 0$  amounts to decreasing the electrostatic repulsion (and hence the correlation) at the discrete level, i.e. between the  $N$  points distributed according to  $P_\beta$ . Indeed, if one sets  $\beta = 0$  abruptly for fixed  $N$ , the probability density  $P_\beta$  corresponds (up to a rescaling depending on  $\beta$ ) to the joint law of independent Gaussian variables and the point process with law  $P_0$  has Poisson statistics. Theorem 1.1 describes the statistics when  $N \rightarrow \infty$  first and then  $\beta \rightarrow 0$  and gives the precise scaling of the convergence.

<sup>2</sup>If  $A$  is a Borel set of  $\mathbb{R}$ ,  $\text{Sine}_\beta(A)$  inside  $A$  denotes the number of points inside  $A$ . In other words,  $\text{Sine}_\beta$  is the counting measure of the point process.

**Theorem 1.1.** *As  $\beta \rightarrow 0$ , the  $\text{Sine}_\beta$  point process converges weakly in the space of Radon measure (equipped with the topology of vague convergence [17]) to a Poisson point process on  $\mathbb{R}$  with intensity  $\frac{d\lambda}{2\pi}$ . In particular, we have, for any  $k \in \mathbb{N}$  and  $\lambda < \lambda'$ ,*

$$\mathbb{P}[\text{Sine}_\beta[\lambda, \lambda'] = k] \xrightarrow{\beta \rightarrow 0} \exp\left(\frac{\lambda - \lambda'}{2\pi}\right) \frac{(\lambda' - \lambda)^k}{(2\pi)^k k!},$$

*and the numbers of points of  $\text{Sine}_\beta$  inside two disjoint intervals are asymptotically independent.*

Let us compare the results stated in Theorem 1.1 with those of a previous work [14] where we consider the stochastic Airy ensemble,  $\text{Airy}_\beta$ , obtained in the scaling limit at the edge of the spectrum. In this context, we proved that the number of points  $\text{Airy}_\beta(] - \infty, \lambda])$  inside  $] - \infty, \lambda]$  displays Poisson statistics in the small  $\beta$  limit. This permitted us to obtain the limiting distributions as  $\beta \rightarrow 0$  of each of the lowest eigenvalues (individually) of the  $\text{Airy}_\beta$  ensemble. In particular, we obtained the weak convergence of the  $TW(\beta)$  distribution towards the Gumbel distribution.

Although the  $\text{Sine}_\beta$  and  $\text{Airy}_\beta$  characterizations in law (in terms of a family of coupled diffusions) look very similar, the analysis of the limiting marginal statistics of the number of points inside a finite closed interval  $\text{Airy}_\beta[\lambda, \lambda']$  for  $\lambda < \lambda'$  and the asymptotic independence when  $\beta \rightarrow 0$  of the respective numbers of points of the  $\text{Airy}_\beta$  point process into two disjoint intervals remain open even after our study [14]. In this aspect, Theorem 1.1 gives a much more powerful and complete description of the  $\text{Sine}_\beta$  process in the small  $\beta$  limit. In this case, we are able to prove the asymptotic independence between the number of points of the  $\text{Sine}_\beta$  process in two disjoint intervals. This part of the proof requires new ideas in order to obtain estimates on the relative positions between two coupled diffusions  $\alpha_\lambda$  and  $\alpha_{\lambda'}$ .

The main difference between the two models is that the  $\text{Sine}_\beta$  process is translation invariant in law while the  $\text{Airy}_\beta$  is not. The non-homogeneous intensity of the  $\text{Airy}_\beta$  process is governed by the crossover spectral density at the edge scaling of  $\beta$ -ensembles computed explicitly in [15, 16] for  $\beta = 2$  (see also [14]).

Before dressing the outline of the paper, let us mention that other related questions on the spectral statistics of random matrices at high temperature, i.e. when  $\beta \rightarrow 0$ , were investigated in [10, 18]. In [10], the authors study the empirical eigenvalue density in the limit of large dimension  $N$  for  $\beta$ -ensembles when  $\beta$  tends to 0 with  $N$  as  $\beta = 2c/N$  where  $c > 0$  is a constant. The authors compute the limiting spectral density  $\rho_c(\lambda)$  explicitly in terms of parabolic cylinder function and establish a Gauss-Wigner crossover, in the sense that the family  $\rho_c$  interpolate between the Gaussian probability distribution ( $c = 0$ ) and the Wigner semi-circle ( $c \rightarrow +\infty$ ). The case of Gaussian Wishart matrices has also been studied in [18].

It would be interesting to have a description of the crossover statistics of the  $\beta$ -ensembles obtained in the double scaling limits when  $\beta$  tends to 0 with  $N \rightarrow \infty$  (this question is briefly discussed in [14] for the statistics at the edge of the spectrum).

*Organization of the paper.* We start in section 2 by looking at the limiting marginal distributions of the random variables  $\text{Sine}_\beta[\lambda, \lambda']$  for  $\lambda < \lambda'$ . We first study a classical problem on the exit time of a diffusion trapped in the well of a stationary potential (obtained by neglecting the slow evolution with time). Then, we prove that the jump process of  $\lfloor \alpha_\lambda / (2\pi) \rfloor$  converges weakly to an (inhomogeneous) Poisson point process by first approximating  $\alpha_\lambda$  with diffusions processes

with piecewise constant drifts on a subdivision of small intervals and then by using the convergence of the exit time of the stationary well established previously. This requires estimates on the sample paths of a single diffusion, in a spirit similar to [14, 19]. In Section 3, we investigate the asymptotic independence of the numbers of points of  $\text{Sine}_\beta$  in two disjoint intervals. We prove a crucial estimate regarding the typical relative positions of two diffusions  $\alpha_\lambda$  and  $\alpha_{\lambda'}$  for  $\lambda < \lambda'$ . Loosely speaking, the main point is to use this estimate to prove that, in the limit  $\beta \rightarrow 0$ , the jumps of the process  $\lfloor \alpha_\lambda / (2\pi) \rfloor$  immediately follow those of  $\lfloor \alpha_{\lambda'} / (2\pi) \rfloor$  (see Fig. 1) while the processes  $\lfloor \alpha_\lambda / (2\pi) \rfloor$  and  $\lfloor (\alpha_{\lambda'} - \alpha_\lambda) / (2\pi) \rfloor$  never jump at the same time (see Fig. 4). The asymptotic independence follows essentially from the fact that two Poisson point processes adapted to the same filtration are independent if and only if they never jump simultaneously.

We gather in the next paragraph important properties of the family  $(\alpha_\lambda)$  already established in [Section 2.2, [13]] that we will use throughout the paper.

*First properties of the coupling of the diffusions  $\alpha_\lambda$ :*

- (i) For all  $\lambda < \lambda'$ ,  $\alpha_{\lambda'} - \alpha_\lambda$  has the same distribution as  $\alpha_{\lambda' - \lambda}$ ;
- (ii) “Increasing property”:  $\alpha_\lambda(t)$  is increasing in  $\lambda$ ;
- (iii)  $\lfloor \alpha_\lambda(t) / (2\pi) \rfloor$  is non-decreasing in  $t$ ;
- (iv)  $\mathbb{E}[\alpha_\lambda(t)] = \lambda \frac{\beta}{4} \int_0^t e^{-\beta s/4} ds$ ;
- (v)  $\alpha_\lambda(\infty) := \lim_{t \rightarrow \infty} \alpha_\lambda(t) / (2\pi)$  exists and is an integer a.s.

We will also use the following notation:

$$\lfloor x \rfloor_{2\pi} = x - 2\pi \left\lfloor \frac{x}{2\pi} \right\rfloor.$$

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## 2. LIMITING MARGINAL DISTRIBUTIONS

We are first interested in the limiting law of the random variable  $\text{Sine}_\beta[0, \lambda]$  when  $\beta \rightarrow 0$  for a single fixed  $\lambda$ . In this case, we re-write the diffusion  $\alpha_\lambda$  in a more convenient way:

$$(5) \quad d\alpha_\lambda = \lambda \frac{\beta}{4} e^{-\frac{\beta}{4}t} dt + 2 \sin\left(\frac{\alpha_\lambda}{2}\right) dB_t,$$

where  $B$  is a real standard Brownian motion. Let us introduce the change of variable  $R_\lambda := \log(\tan(\alpha_\lambda/4))$ . A straightforward computation (see [13]) shows that:

$$(6) \quad dR_\lambda = \frac{1}{2} \left( \lambda \frac{\beta}{4} e^{-\frac{\beta}{4}t} \cosh(R_\lambda) + \tanh(R_\lambda) \right) dt + dB_t, \quad R_\lambda(0) = -\infty.$$

**2.1. Trapping in the stationary potential.** In this subsection, we study an exit time problem for a Langevin diffusion  $S_\lambda$  evolving in a stationary potential  $V_\beta$  defined for  $r \in \mathbb{R}$  as

$$V_\beta(r) := -\frac{1}{2} \left( \lambda \frac{\beta}{4} \sinh(r) + \log \cosh(r) \right).$$

This problem is relevant to our study thanks to the slow variation with time of the non-stationary potential as  $\beta \rightarrow 0$  in which the diffusion  $R_\lambda$  evolves. The diffusion  $S_\lambda$  satisfies the following stochastic differential equation

$$(7) \quad dS_\lambda = -V'_\beta(S_\lambda) dt + dB_t, \quad S_\lambda(0) = -\infty.$$

In the whole paper, the diffusions  $R_\lambda$  and  $S_\lambda$  defined respectively in (6) and (7) are coupled, driven by the same Brownian motion  $B$ . Under this coupling, we have almost surely

$$R_\lambda(t) \leq S_\lambda(t)$$

for all  $t \leq \zeta$  where  $\zeta$  is the first explosion (stopping) time

$$\zeta := \inf\{t \geq 0 : S_\lambda(t) = +\infty\}.$$

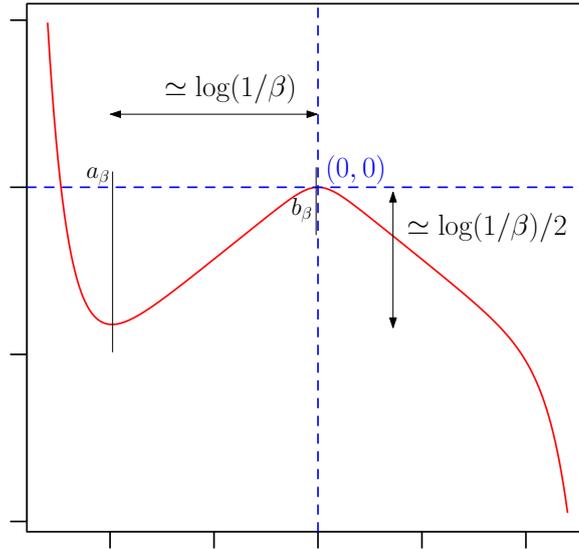


FIGURE 2. The potential  $V_\beta(r)$  as a function of  $r$ .

In this paragraph we investigate the limiting law of the stopping time  $\zeta$  through its Laplace transform defined for  $\xi > 0$  as

$$g_\xi(r) := \mathbb{E}_r[e^{-\xi\zeta}],$$

where  $r$  is the initial position of the diffusion  $S_\lambda$ .

We know from classical diffusion theory (see also [14]) that  $g_\xi$  satisfies

$$(8) \quad \frac{1}{2}g_\xi''(r) - V_\beta'(r)g_\xi'(r) = \xi g_\xi(r), \text{ with } g_\xi(r) \rightarrow 1 \text{ as } r \rightarrow +\infty.$$

Let us first examine the expectation of the first explosion time of  $S_\lambda$ . Due to the strong barrier separating the local minimum and the local maximum (see Fig. 2), it is natural to expect the asymptotic of this mean exit time not to depend on the starting point of the diffusion, when  $\beta \rightarrow 0$ , as long as it is located in the well (“memory-loss property”). This is the purpose of Proposition 2.1. We then show that this first exit time properly rescaled by its mean value converges in law to an exponential distribution (Proposition 2.2) when the starting point is in the well.

**Proposition 2.1.** *Suppose that the diffusion  $S_\lambda$  starts from  $r := r_\beta$  such that  $r_\beta \rightarrow -\infty$ . Then its expected exit time denoted  $t_\beta(r) := \mathbb{E}_r[\zeta]$  has the following equivalent when  $\beta \rightarrow 0$ :*

$$t_\beta(r_\beta) \sim \frac{8\pi}{\beta\lambda}.$$

*Proof of Proposition 2.1.* From (8), it is easy to see that the expected exit time  $t_\beta(r)$  satisfies the boundary value problem

$$(9) \quad \frac{1}{2}t_\beta''(r) - V_\beta'(r)t_\beta'(r) = -1, \quad \text{with } t_\beta(r) \rightarrow 0 \text{ as } r \rightarrow +\infty.$$

Solving (9) explicitly we obtain the integral form

$$(10) \quad t_\beta(r) = 2 \int_r^{+\infty} dx \int_{-\infty}^x \exp(2[V_\beta(x) - V_\beta(y)]) dy.$$

By extracting carefully the asymptotic behavior of this integral in the limit  $\beta \rightarrow 0$  (see Appendix A.1), we obtained the desired result.  $\square$

The Laplace transform of the exit time satisfies the fixed point equation

$$g_\xi(r) = 1 - 2\xi \int_r^{+\infty} dx \int_{-\infty}^x \exp(2[V_\beta(x) - V_\beta(y)]) g_\xi(y) dy.$$

With classical arguments similar to those of [14], we prove the following proposition in Appendix A.1.

**Proposition 2.2.** *Conditionally on  $S_\lambda(0) = r := r_\beta$  such that  $r_\beta \rightarrow -\infty$  when  $\beta \rightarrow 0$ , the first explosion time  $\zeta$  of the diffusion  $(S_\lambda(t))_{t \geq 0}$  converges weakly when rescaled by the expected exit time  $8\pi/(\beta\lambda)$  towards an exponential distribution with mean 1.*

The process  $\theta_\lambda$  such that  $S_\lambda := \log \tan(\theta_\lambda/4)$  satisfies

$$d\theta_\lambda = \lambda \frac{\beta}{4} dt + 2 \sin\left(\frac{\theta_\lambda}{2}\right) dB_t.$$

Proposition (2.2) translates into a convergence of the stopping time  $\zeta_{2\pi} := \inf\{t \geq 0 : \theta_\lambda(t) \geq 2\pi\}$ .

**Proposition 2.3.** *Conditionally on  $\theta_\lambda(0) = \theta_0 := \theta_0^\beta$  such that  $\theta_0^\beta \rightarrow 0_+$  when  $\beta \rightarrow 0$ , the stopping time  $\frac{\beta\lambda}{8\pi}\zeta_{2\pi}$  converges weakly when  $\beta \rightarrow 0$  towards an exponential distribution with mean 1.*

**2.2. Convergence of the jump process.** We consider the diffusion  $\alpha_\lambda$  defined in (3) or equivalently for a single  $\lambda$ , in (5). For  $k \in \mathbb{N}$ , let

$$(11) \quad \zeta_k := \inf\{t \geq 0 : \alpha_\lambda(t) \geq 2k\pi\}.$$

Note that those stopping times correspond to the jumps of the process  $\lfloor \alpha_\lambda / (2\pi) \rfloor$ . We denote by  $\mathcal{F}_t$  the filtration associated to the diffusion process  $\alpha_\lambda$ . In this paragraph, we will sometimes omit the subscript  $\lambda$  in  $\alpha_\lambda$  to simplify the notations. We consider the rescaled empirical measure  $\mu_\lambda^\beta$  of the  $\frac{\beta}{4}\zeta_k$  defined on  $\mathbb{R}_+$

$$(12) \quad \mu_\lambda^\beta[0, t] = \sum_{k=1}^{+\infty} \delta_{\zeta_k} \left[ 0, \frac{8\pi t}{\beta} \right].$$

We divide the time interval  $\mathbb{R}_+$  into random small intervals  $I_k := [\frac{S_k}{n} \frac{8\pi}{\beta}, \frac{S_{k+1}}{n} \frac{8\pi}{\beta}]$ ,  $k \in \mathbb{N}$ , independent of the diffusion  $\alpha$  where  $S_0 = 0$  and:

$$S_k := \sum_{i=1}^k \tau_i,$$

where the  $\tau_i$  are i.i.d. random variables with mean 1 uniformly distributed on  $[1/2, 3/2]$ .

For each  $k \in \mathbb{N}$ , conditionally on  $\mathcal{F}_{\frac{S_k}{n} \frac{8\pi}{\beta}}$ , we define the two diffusion processes  $\alpha_n^+$  and  $\alpha_n^-$  such that, for  $t \in I_k$ ,

$$\begin{aligned} d\alpha_n^+ &= \lambda \frac{\beta}{4} e^{-\frac{S_k}{n} 2\pi} dt + 2 \sin\left(\frac{\alpha_n^+}{2}\right) dB_t, & \alpha_n^+\left(\frac{S_k}{n} \frac{8\pi}{\beta}\right) &= \alpha\left(\frac{S_k}{n} \frac{8\pi}{\beta}\right), \\ d\alpha_n^- &= \lambda \frac{\beta}{4} e^{-\frac{S_{k+1}}{n} 2\pi} dt + 2 \sin\left(\frac{\alpha_n^-}{2}\right) dB_t, & \alpha_n^-\left(\frac{S_k}{n} \frac{8\pi}{\beta}\right) &= \alpha\left(\frac{S_k}{n} \frac{8\pi}{\beta}\right). \end{aligned}$$

By the increasing property, it follows that for any  $k \in \mathbb{N}$  and  $t \in I_k$ , we have

$$\alpha_n^-(t) \leq \alpha(t) \leq \alpha_n^+(t).$$

**Theorem 2.4.** *As  $\beta \rightarrow 0$ , the empirical measure  $\mu_\lambda^\beta$  converges weakly in the space of Radon measure (equipped with the topology of vague convergence [17]) to a Poisson point process on  $\mathbb{R}_+$  with inhomogeneous intensity  $\frac{\lambda}{2\pi} e^{-2\pi t} dt$ . In particular, we have, for any  $k \in \mathbb{N}$ ,*

$$\mathbb{P}[\mu_\lambda^\beta[0, t] = k] \rightarrow_{\beta \rightarrow 0} \exp\left(-\frac{\lambda}{2\pi}(1 - e^{-2\pi t})\right) \frac{\left(\frac{\lambda}{2\pi}\right)^k (1 - e^{-2\pi t})^k}{k!}.$$

Thanks to the equality in law

$$\mu_\lambda^\beta(\mathbb{R}_+) \stackrel{(d)}{=} \frac{\alpha_\lambda(\infty)}{2\pi} \stackrel{(d)}{=} \text{Sine}_\beta[0, \lambda],$$

we easily deduce from Theorem 2.4 the convergence of the marginals of the  $\text{Sine}_\beta$  point process.

**Corollary 2.5.** *Let  $\lambda < \lambda'$ . The random variable  $\text{Sine}_\beta[\lambda, \lambda']$  converges weakly as  $\beta \rightarrow 0$  to a Poisson law with parameter  $\frac{\lambda' - \lambda}{2\pi}$ .*

*Proof of Theorem 2.4.*

The proof follows the same lines as the proof of [Theorem 4.1 in [14]]. The idea is to approximate the number of jumps of the diffusion  $\alpha$  by those of stationary diffusions and to use the increasing

property. To this end, we will use the subdivision of  $\mathbb{R}_+$  introduced above and the diffusions  $\alpha_n^+$  and  $\alpha_n^-$ .

From Kallenberg's theorem [17], we just need to see that, for any finite union  $I$  of disjoint and bounded intervals, we have when  $\beta \rightarrow 0$ ,

$$(13) \quad \mathbb{E}[\mu_\lambda^\beta(I)] \longrightarrow \lambda \int_I e^{-2\pi t} dt,$$

$$(14) \quad \mathbb{P}[\mu_\lambda^\beta(I) = 0] \longrightarrow \exp(-\lambda \int_I e^{-2\pi t} dt).$$

Denote by  $[t_1; t_2]$  the right most interval of  $I$ , by  $J$  the union of disjoint and bounded intervals such that  $I = J \cup [t_1; t_2]$  and by  $t_0$  the supremum of  $J$ .

Let us use the random subdivision introduced above. It is crucial to control the position of the diffusion at the starting point of the sub-intervals. As a consequence of Lemma 2.7, with large probability, the diffusion  $\alpha$  is close to 0 modulo  $2\pi$  in the beginning of each of the sub-intervals overlapping  $[0, t_2]$ .

More precisely, denote by  $\mathcal{C}_k := \{[\alpha((8\pi/\beta)S_k/n)]_{2\pi} \leq 4 \arctan(\beta^{1/4})\}$  and consider:

$$\mathcal{C} := \bigcap_{k=0}^{\lfloor 2nt_2 \rfloor + 1} \mathcal{C}_k.$$

Every sub-interval intersecting  $[0, t_2]$  is taken care of in this event as by definition  $S_k \in [k/2, 3k/2]$ . Its probability is bounded from above by

$$\begin{aligned} \mathbb{P}[\mathcal{C}^c] &\leq \sum_{k=0}^{\lfloor 2nt_2 \rfloor + 1} \mathbb{P}[[\alpha((8\pi/\beta)S_k/n)]_{2\pi} > 4 \arctan(\beta^{1/4})] \\ &\leq 2 \mathbb{E} \left[ \int_0^{3nt_2+3} 1_{\{[\alpha((8\pi/\beta)u/n)]_{2\pi} > 4 \arctan(\beta^{1/4})\}} du \right] \\ &\leq \frac{n}{4\pi} \beta T_\beta(0, 3t_2 + 1) \xrightarrow{\beta \rightarrow 0} 0 \end{aligned}$$

where we used the result of Lemma 2.7 to have the convergence towards 0 in the last line.

We now turn to the proof of 13. Note that thanks to the linearity of the expectation, we simply need to prove (13) for intervals  $I$  of the form  $I = [0, t_2]$ . The upper bound simply follows from the SDE form:

$$\mathbb{E}[\alpha(t)] = \lambda \frac{\beta}{4} \int_0^t e^{-\beta s/4} ds.$$

We immediately derive:

$$\mathbb{E}[\mu_\beta[0, t_2]] = \mathbb{E}[[\alpha((8\pi/\beta)t_2)/(2\pi)]] \leq \lambda \int_0^{t_2} e^{-2\pi s} ds.$$

For the lower bound, denote by  $N_k^+$ ,  $N_k^-$  and  $N_k$  the number of jumps of  $\alpha_n^+$ ,  $\alpha_n^-$  and  $\alpha$  in the sub-interval  $[(8\pi/\beta)S_k/n, (8\pi/\beta)S_{k+1}/n]$ .

$$\mathbb{E}[\mu_\beta[0, t_2]] \geq \sum_{k=0}^{\lfloor 2nt_2 \rfloor + 1} \mathbb{E}[N_k^- 1_{\{(S_k/n) < t_2\}} | \mathcal{C}_k] - \sum_{k=0}^{\lfloor 2nt_2 \rfloor + 1} \mathbb{E}[N_k^- 1_{\{(S_k/n) < t_2\}} | \mathcal{C}_k] \mathbb{P}[\mathcal{C}_k^c].$$

The second sum of the RHS can be simply bounded from above by:

$$\mathbb{E}[\text{number of jumps of } \theta_\lambda \text{ in } [0, 3t_2(8\pi)/\beta]] \times \sup_{k \in \{0, \dots, \lfloor 2nt_2 \rfloor + 1\}} \frac{\mathbb{P}[\mathcal{C}_k^c]}{\mathbb{P}[\mathcal{C}_k]}.$$

The first expectation is bounded from above by a number independent of  $\beta$  and the second term (bounded by  $\mathbb{P}[\mathcal{C}^c]/(1 - \mathbb{P}[\mathcal{C}^c])$ ) tends to 0 as  $\beta \rightarrow 0$ . Moreover, thanks to Proposition 2.3, we have

$$\mathbb{E}[N_k^+ | \mathcal{C}_k] \xrightarrow{\beta \rightarrow 0} \mathbb{E}[\lambda(\tau_{k+1}/n) \exp(-2\pi S_k/n)].$$

Using the convergence of the Riemann sum when  $n \rightarrow \infty$ , it gives the desired lower bound.

Let us now examine the convergence 14 for a single interval  $[t_1, t_2]$  first. By the Markov property:

$$\mathbb{P}[\mu_\beta[t_1, t_2] = 0] \leq \mathbb{E}\left[\prod_{k=0}^{\infty} \mathbb{E}[N_k^+ = 0 | \mathcal{C}_k, (\tau_i)_i] 1_{\{S_{k+1}/n \geq t_1, S_k/n \leq t_2\}}\right] + \mathbb{P}[\mathcal{C}^c].$$

Using the convergence in each of the sub-intervals (given by Proposition 2.3)

$$\mathbb{E}[N_k^+ = 0 | \mathcal{C}_k] \xrightarrow{\beta \rightarrow 0} \mathbb{E}[\exp(-\lambda(\tau_{k+1}/n) \exp(-2\pi S_k/n))],$$

we obtain:

$$\limsup_{\beta \rightarrow 0} \mathbb{P}[\mu_\beta[t_1, t_2] = 0] \leq \mathbb{E}\left[\prod_{k=0}^{\infty} \exp(-\lambda(\tau_{k+1}/n) \exp(-2\pi S_k/n)) 1_{\{S_{k+1}/n \geq t_1, S_k/n \leq t_2\}}\right].$$

Thanks to the convergence of the Riemann sum when  $n \rightarrow 0$ , we deduce the upper bound. The lower bound can be done using similar techniques than above. To generalize the result to  $I$  finite union of interval, note that thanks to the simple Markov property, we have

$$\mathbb{P}[\mu_\lambda^\beta(I) = 0] = \mathbb{E}[1_{\{\mu_\beta(J)=0\}} \mathbb{P}[\mu_\beta([t_1, t_2]) = 0 | \alpha(t_0)]]].$$

Iterating the previous argument leads to the result.  $\square$

**2.3. Estimates for a single diffusion.** We analyse in this paragraph the diffusion  $\alpha_\lambda$  and the jumps of  $\lfloor \alpha_\lambda/(2\pi) \rfloor$  in the limit  $\beta \rightarrow 0$ . The results we obtain are derived using the diffusion  $R_\lambda := \log(\tan(\alpha_\lambda/4))$  satisfying the SDE (6). We defer the proof of those technical lemmas in Appendix A.2.

Lemma 2.6 first states that when the diffusion  $\alpha_\lambda$  starts just below  $2\pi$  modulo  $2\pi$ ,  $\lfloor \alpha_\lambda/(2\pi) \rfloor$  will jump in a short time with probability going to 1.

**Lemma 2.6.** *Let  $0 < \varepsilon < 1$ . Conditionally on  $[\alpha_\lambda(0)]_{2\pi} = 2\pi - 4 \arctan(\beta^\varepsilon)$ , we define the first reaching time*

$$(15) \quad \zeta_{2\pi}^\lambda := \inf\{t \geq 0 : \alpha_\lambda(t) = 2\pi + \lfloor \alpha_\lambda(0)/(2\pi) \rfloor\}.$$

*Then, there exists a constant  $c > 0$  such that for all  $\beta > 0$  small enough,*

$$\mathbb{P}\left[\zeta_{2\pi}^\lambda < 9 \log \frac{1}{\beta}\right] \geq 1 - \beta^c.$$

On the large scale-time of the order  $1/\beta$ , we prove that the time spent by  $\alpha_\lambda$  near 0 modulo  $2\pi$  for  $\alpha_\lambda$  is negligible. This is the content of Lemma 2.7.

**Lemma 2.7.** *Let  $t > 0$  and*

$$(16) \quad T_\beta(t) := \mathbb{E} \left[ \int_0^{\frac{8\pi}{\beta}t} 1_{\{[\alpha_\lambda(u)]_{2\pi} \geq 4 \arctan(\beta^{1/4})\}} du \right].$$

*Then, there exists  $C > 0$  independent of  $\beta$  such that*

$$T_\beta(t) \leq \frac{C}{\sqrt{\beta}} \log \frac{1}{\beta}.$$

### 3. ASYMPTOTIC SPATIAL INDEPENDENCE

**3.1. Ordering of two diffusions.** For  $\lambda < \lambda'$ , we now control the expected time spent by the process  $[\alpha_{\lambda'}]_{2\pi}$  below the process  $[\alpha_\lambda]_{2\pi}$ . The following Lemma is a crucial step towards the proof of the asymptotic independence of the limiting point process on disjoint intervals and will be used for the proof of Lemmas 3.3 and 3.4.

**Lemma 3.1.** *Let  $t > 0, \lambda < \lambda'$  and*

$$\Theta_\beta(t) := \mathbb{E} \left[ \int_0^{\frac{8\pi}{\beta}t} 1_{\{[\alpha_{\lambda'}(u)]_{2\pi} \leq [\alpha_\lambda(u)]_{2\pi}\}} du \right].$$

*Then, there exists two constants  $c, C > 0$  independent of  $\beta$  such that*

$$\Theta_\beta(t) \leq \frac{C}{\beta^{1-c}}.$$

*Proof.* We set

$$\mathcal{E}_u := \{[\alpha_{\lambda'}(u)]_{2\pi} < [\alpha_\lambda(u)]_{2\pi}\}.$$

Before evaluating the probability of the event  $\mathcal{E}_u$ , we need to introduce for  $u > 0$  the last jump of the process  $[\frac{\alpha_{\lambda'}}{2\pi}]$  before time  $u$ , i.e.

$$\zeta^{(u)} := \sup\{k : \zeta_k^{\lambda'} \leq u\}.$$

The main idea is to prove that any  $\mathcal{E}_u$  is associated to a jump time of  $\alpha_{\lambda'}$  right before  $u$ .

Let us bound from above the probability of the event  $\mathcal{E}_u$  by

$$(17) \quad \begin{aligned} \mathbb{P}[\mathcal{E}_u] &\leq \mathbb{P} \left[ \mathcal{E}_u \cap \left\{ \zeta^{(u)} \in \left[ u - 9 \log \frac{1}{\beta}, u \right] \right\} \right] + \mathbb{P} \left[ \mathcal{E}_u \cap \left\{ \zeta^{(u)} < u - 9 \log \frac{1}{\beta} \right\} \right] \\ &\leq \mathbb{P} \left[ \mathcal{E}_u \cap \left\{ \zeta^{(u)} \in \left[ u - 9 \log \frac{1}{\beta}, u \right] \right\} \right] + \mathbb{P} \left[ \bigcap_{s \in [u - 9 \log \frac{1}{\beta}, u]} \mathcal{E}_s \right] \end{aligned}$$

where we have noticed the inclusion  $\mathcal{E}_u \cap \{\zeta^{(u)} < u - 9 \log \frac{1}{\beta}\} \subseteq \bigcap_{s \in [u - 9 \log \frac{1}{\beta}, u]} \mathcal{E}_s$  for the second line (see Fig. 3).<sup>3</sup>

<sup>3</sup>This inclusion follows from the definition of  $\zeta^{(u)}$  which is such that  $[\alpha_{\lambda'}(t)]_{2\pi} < [\alpha_\lambda(t)]_{2\pi}$  for all  $t \in [\zeta^{(u)}, u)$ .

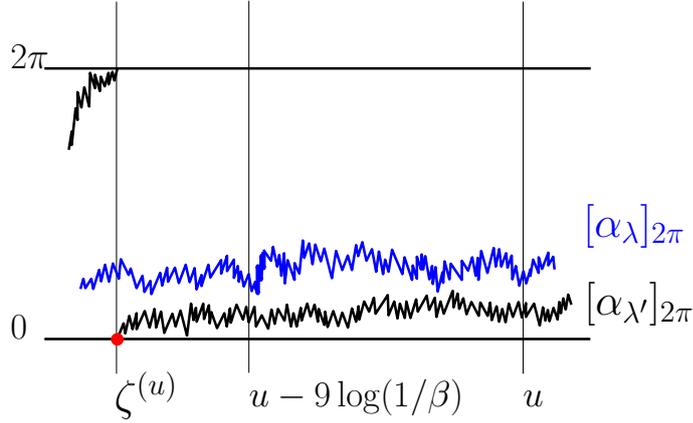


FIGURE 3. Representation of the event  $\mathcal{E}_u \cap \{\zeta^{(u)} < u - 9 \log \frac{1}{\beta}\}$ .

We now tackle the second probability of (17) using the fact that  $\alpha_\lambda$  is close to 0 modulo  $2\pi$  with large probability:

$$\mathbb{P} \left[ \bigcap_{s \in [u - 9 \log \frac{1}{\beta}, u]} \mathcal{E}_s \right] \leq \mathbb{P} \left[ \bigcap_{s \in [u - 9 \log \frac{1}{\beta}, u]} \mathcal{E}_s \cap \left\{ [\alpha_\lambda(u - 9 \log \frac{1}{\beta})]_{2\pi} \leq 4 \arctan(\beta^{1/4}) \right\} \right] + \mathbb{P} \left[ [\alpha_\lambda(u - 9 \log \frac{1}{\beta})]_{2\pi} \geq 4 \arctan(\beta^{1/4}) \right].$$

Conditionally on the event  $\mathcal{A}_{u - 9 \log \frac{1}{\beta}} := \mathcal{E}_{u - 9 \log \frac{1}{\beta}} \cap \left\{ [\alpha_\lambda(u - 9 \log \frac{1}{\beta})]_{2\pi} \leq 4 \arctan(\beta^{1/4}) \right\}$ , we introduce the stopping times

$$\begin{aligned} \zeta_{>}(u - 9 \log \frac{1}{\beta}) &:= \inf \left\{ t \geq u - 9 \log \frac{1}{\beta} : [\alpha_{\lambda'}(t)]_{2\pi} > [\alpha_\lambda(t)]_{2\pi} \right\} \\ &= \inf \left\{ t \geq u - 9 \log \frac{1}{\beta} : \lfloor \frac{\alpha_\lambda(t)}{2\pi} \rfloor = \lfloor \frac{\alpha_\lambda(u)}{2\pi} \rfloor + 1 \right\}, \\ \zeta_{=}(u - 9 \log \frac{1}{\beta}) &:= \inf \left\{ t \geq u - 9 \log \frac{1}{\beta} : [\alpha_{\lambda'}(t)]_{2\pi} = [\alpha_\lambda(t)]_{2\pi} \right\}. \end{aligned}$$

and notice that we have almost surely for all  $t \leq \zeta_{=}(u - 9 \log \frac{1}{\beta}) \wedge \zeta_{>}(u - 9 \log \frac{1}{\beta})$ ,

$$[\alpha_{\lambda'}(t)]_{2\pi} - [\alpha_\lambda(t)]_{2\pi} = [\alpha_{\lambda'}(t) - \alpha_\lambda(t)]_{2\pi}.$$

Conditionally on the event  $\mathcal{A}_{u - 9 \log \frac{1}{\beta}}$ , we define a diffusion process  $\alpha_{\lambda' - \lambda}$  and its associated first reaching time of  $2\pi$  such that

$$\begin{aligned} d\alpha_{\lambda' - \lambda} &= (\lambda' - \lambda) \frac{\beta}{4} e^{-\frac{\beta}{4}t} dt + \operatorname{Re}((e^{-i\alpha_{\lambda' - \lambda}} - 1)dZ_t), \\ \text{with } \alpha_{\lambda' - \lambda}(0) &= 2\pi - ([\alpha_\lambda(u - 9 \log \frac{1}{\beta})]_{2\pi} - [\alpha_{\lambda'}(u - 9 \log \frac{1}{\beta})]_{2\pi}), \\ \text{and } \tilde{\zeta}_{2\pi} &:= \inf \{ t \geq 0 : \alpha_{\lambda' - \lambda}(t) = 2\pi \}. \end{aligned}$$

Endowed with those definitions, we have the following upper-bound

$$\begin{aligned} \mathbb{P} \left[ \bigcap_{s \in [u - 9 \log \frac{1}{\beta}, u]} \mathcal{E}_s \mid \mathcal{A}_{u - 9 \log \frac{1}{\beta}} \right] &= \mathbb{P} \left[ \zeta_{>}(u - 9 \log \frac{1}{\beta}) \wedge \zeta_{=}(u - 9 \log \frac{1}{\beta}) \geq 9 \log \frac{1}{\beta} \mid \mathcal{A}_{u - 9 \log \frac{1}{\beta}} \right] \\ &\leq \mathbb{P} \left[ \tilde{\zeta}_{2\pi} \geq 9 \log \frac{1}{\beta} \right] \\ &\leq \beta^c \end{aligned}$$

where we have used the equality in law (18) for the second line and Lemma 2.6 (as well as the increasing property) to obtain the last line. Gathering the above inequalities, we obtain

$$\mathbb{P} [\mathcal{E}_u] \leq \mathbb{P} \left[ \zeta^{(u)} \in \left[ u - 9 \log \frac{1}{\beta}, u \right] \right] + \beta^c + \mathbb{P} \left[ [\alpha_\lambda(u - 9 \log \frac{1}{\beta})]_{2\pi} \geq 4 \arctan(\beta^{1/4}) \right].$$

Now, to conclude, we just have to integrate this latter inequality with respect to  $u$ . First notice that we have almost surely

$$\int_{9 \log(1/\beta)}^{\frac{8\pi}{\beta}t} \mathbf{1}_{\{\zeta^{(u)} \in [u - 9 \log \frac{1}{\beta}, u]\}} du \leq 9 \log \frac{1}{\beta} \mu_{\lambda'}^\beta[0, \frac{8\pi}{\beta}t].$$

Integrating the other terms as well with respect to  $u$  and taking the expectation, we get

$$\begin{aligned} \Theta_\beta(t) &\leq 9 \log \frac{1}{\beta} \left( 1 + \mathbb{E} \left[ \mu_{\lambda'}^\beta[0, \frac{8\pi}{\beta}t] \right] \right) + T_\beta(t) \\ &\leq 9 \log \frac{1}{\beta} \left( 1 + \frac{\lambda}{2\pi} \right) + 8\pi t \beta^{c-1} + T_\beta(t) \end{aligned}$$

where  $T_\beta(t)$  is defined in (16). The conclusion now follows from Lemma (2.7).  $\square$

**3.2. Limiting coupled Poisson processes.** Lemma 3.1 shall be an important tool to prove the asymptotic independence between  $\alpha_\lambda(\infty)$  and  $\alpha_{\lambda'}(\infty) - \alpha_\lambda(\infty)$  for  $\lambda < \lambda'$ .

We now work with two diffusions  $\alpha_\lambda$  and  $\alpha_{\lambda'}$  for  $\lambda < \lambda'$  coupled according to (3). We know from Theorem 2.4 that the empirical measures  $\mu_\lambda^\beta$  and  $\mu_{\lambda'}^\beta$  of the stopping time point processes  $(\zeta_k^\lambda)_{k \in \mathbb{N}}$  and  $(\zeta_k^{\lambda'})_{k \in \mathbb{N}}$  associated to the diffusions  $\alpha_\lambda$  and  $\alpha_{\lambda'}$  (see (11) and (12)) respectively converge vaguely to Poisson point processes  $(\mathcal{P}_\lambda[0, t])_{t \geq 0}$  and  $(\mathcal{P}_{\lambda'}[0, t])_{t \geq 0}$  with respective intensities  $\lambda e^{-2\pi t} dt$  and  $\lambda' e^{-2\pi t} dt$ .

Due to the equality in law

$$(18) \quad \alpha_{\lambda'} - \alpha_\lambda \stackrel{(d)}{=} \alpha_{\lambda' - \lambda},$$

Theorem 2.4 also implies the convergence of the (positive) random measure  $\mu_{\lambda' - \lambda}^\beta$  such that for all  $t \geq 0$ ,

$$\mu_{\lambda' - \lambda}^\beta[0, t] := \left\lfloor \frac{\alpha_{\lambda'}(t) - \alpha_\lambda(t)}{2\pi} \right\rfloor$$

towards a Poisson point process  $\mathcal{P}_{\lambda' - \lambda}$  with intensity  $(\lambda' - \lambda) e^{-2\pi t} dt$ .

**Lemma 3.2.** *Let  $\lambda < \lambda' < \lambda''$  and  $\mathcal{F} := (\mathcal{F}_t)_{t \geq 0}$  be the natural filtration associated to the process  $(\mathcal{P}_\lambda, \mathcal{P}_{\lambda'}, \mathcal{P}_{\lambda''-\lambda'})$  i.e. such that*

$$\mathcal{F}_t := \sigma(\mathcal{P}_\lambda(s), \mathcal{P}_{\lambda'}(s), \mathcal{P}_{\lambda''-\lambda'}(s), 0 \leq s \leq t).$$

*Then, the processes  $\mathcal{P}_\lambda$  and  $\mathcal{P}_{\lambda'}$  and  $\mathcal{P}_{\lambda''-\lambda'}$  are  $(\mathcal{F}_t)$ -Poisson point processes with respective intensities  $\lambda e^{-2\pi t} dt$  and  $\lambda' e^{-2\pi t} dt$  and  $(\lambda'' - \lambda') e^{-2\pi t} dt$ .*

This lemma is a straightforward consequence of the coupling family of diffusions  $(\alpha_\lambda)_{\lambda \in \mathbb{R}}$ . The crucial point is that all the diffusions are measurable with respect to the same driving (complex) Brownian motion  $(Z_t)$ . A detailed proof can be found in Appendix A.3.

To fix notations, we will denote by  $(\xi_i^\lambda)_{i \in \mathbb{N}}$  resp.  $(\xi_i^{\lambda'})_{i \in \mathbb{N}}$ ,  $(\xi_i^{\lambda''-\lambda'})_{i \in \mathbb{N}}$  the points associated to the Poisson processes  $\mathcal{P}_\lambda$  and  $\mathcal{P}_{\lambda'}$  and  $\mathcal{P}_{\lambda''-\lambda'}$  such that

$$\mathcal{P}_\lambda[0, t] = \sum_i \delta_{\xi_i^\lambda}[0, t], \quad \mathcal{P}_{\lambda'}[0, t] = \sum_i \delta_{\xi_i^{\lambda'}}[0, t], \quad \mathcal{P}_{\lambda''-\lambda'}[0, t] = \sum_i \delta_{\xi_i^{\lambda''-\lambda'}}[0, t].$$

For  $\beta > 0$ , we also recall the notations

$$\mu_\lambda^\beta[0, t] = \sum_i \delta_{\xi_i^\lambda}[0, \frac{8\pi t}{\beta}], \quad \mu_{\lambda'}^\beta[0, t] = \sum_i \delta_{\xi_i^{\lambda'}}[0, \frac{8\pi t}{\beta}], \quad \mu_{\lambda''-\lambda'}^\beta[0, t] = \sum_i \delta_{\xi_i^{\lambda''-\lambda'}}[0, \frac{8\pi t}{\beta}].$$

**Lemma 3.3.** *Let  $\lambda < \lambda'$ . Then, we have almost surely*

$$\mathcal{P}_\lambda \subseteq \mathcal{P}_{\lambda'}$$

*i.e. for all  $i \in \mathbb{N}$ , there exists  $j \geq i$  such that  $\xi_i^\lambda = \xi_j^{\lambda'}$ .*

*Proof.* We have to prove that for any  $t > 0$ ,

$$(19) \quad \mathbb{P}[\exists i : |\xi_i^\lambda| < t, \xi_i^\lambda \notin \mathcal{P}_{\lambda'}] = 0.$$

The probability (19) is the increasing limit of the sequence  $(p_n)_{n \in \mathbb{N}}$  defined as

$$(20) \quad p_n := \mathbb{P}\left[\exists i : |\xi_i^\lambda| < t, \forall j \geq i, |\xi_i^\lambda - \xi_j^{\lambda'}| > \frac{1}{2n}\right].$$

It suffices to prove that  $p_n = 0$  for any  $n$ . Let us introduce the probability

$$(21) \quad p_n^\beta := \mathbb{P}\left[\exists i : |\zeta_i^\lambda| < \frac{8\pi}{\beta}t, \forall j \geq i, |\zeta_i^\lambda - \zeta_j^{\lambda'}| > \frac{8\pi}{\beta} \frac{1}{2n}\right].$$

If  $n$  is fixed, we have  $p_n \leq \liminf_{\beta \rightarrow 0} p_n^\beta$ <sup>4</sup> so that we just need to check that

$$\limsup_{\beta \rightarrow 0} p_n^\beta = 0.$$

We need to work with a random subdivision of the interval  $[0, \frac{8\pi t}{\beta}]$ . As before, we consider a sequence  $(\tau_k)_{k \in \mathbb{N}}$  of i.i.d. positive random variables distributed uniformly in  $[\frac{1}{2}, \frac{3}{2}]$  and form the sum  $S_k = \sum_{i=1}^k \tau_i$ .

<sup>4</sup>Using the convergence of the points of the measures in a compact set towards the limit points in the topology of vague convergence, see [21].

Noting that for all  $x, y$  such that  $|x - y| \leq 1/(2n)$ , there exists  $k \in \mathbb{N}$  such that  $x, y \in [S_k/n, S_k/n + 2/n]$ , we obtain

$$p_n^\beta \leq \mathbb{P} \left[ \exists k \leq \lfloor 2nt \rfloor + 1 : \lfloor \frac{\alpha_\lambda}{2\pi} \rfloor \text{ jumps on the interval } \frac{8\pi}{\beta} \left[ \frac{S_k}{n}, \left( \frac{S_k}{n} + \frac{2}{n} \right) \right] \text{ but not } \lfloor \frac{\alpha_{\lambda'}}{2\pi} \rfloor \right].$$

Due to the increasing property, the event inside the probability can not happen if the process  $[\alpha_\lambda]_{2\pi}$  starts below  $[\alpha_{\lambda'}]_{2\pi}$  at the beginning of the interval. Therefore,

$$p_n^\beta \leq \sum_{k=1}^{\lfloor 2nt \rfloor + 1} \mathbb{P} \left[ [\alpha_{\lambda'} \left( \frac{8\pi S_k}{\beta n} \right)]_{2\pi} \leq [\alpha_\lambda \left( \frac{8\pi S_k}{\beta n} \right)]_{2\pi} \right].$$

which can in turn be upper-bounded as follows

$$\begin{aligned} \sum_{k=1}^{\lfloor 2nt \rfloor + 1} \mathbb{P} \left[ [\alpha_{\lambda'} \left( \frac{8\pi S_k}{\beta n} \right)]_{2\pi} \leq [\alpha_\lambda \left( \frac{8\pi S_k}{\beta n} \right)]_{2\pi} \right] &\leq \frac{\beta n}{4\pi} \mathbb{E} \left[ \int_0^{\frac{8\pi}{\beta}(3t+1)} 1_{\{[\alpha_{\lambda'}(u)]_{2\pi} \leq [\alpha_\lambda(u)]_{2\pi}\}} du \right] \\ (22) \qquad \qquad \qquad &\leq \frac{\beta n}{4\pi} \Theta_\beta(3t+1) \leq \beta^c \end{aligned}$$

where we have used Lemma 3.1 to obtain the last inequality which holds for  $\beta$  small enough ( $c$  is a constant which does not depend on  $\beta$ ).

□

**Lemma 3.4.** *Let  $\lambda < \lambda'$ . Then the  $(\mathcal{F}_t)$ -Poisson point processes  $\mathcal{P}_\lambda$  and  $\mathcal{P}_{\lambda'-\lambda}$  are independent.*

*Proof of Lemma 3.4.* From a classical result [20, 21] on Poisson processes, we know that it suffices to prove that the two  $(\mathcal{F}_t)$ -Poisson processes  $\mathcal{P}_\lambda$  and  $\mathcal{P}_{\lambda'-\lambda}$  do not jump simultaneously, i.e. that for  $t > 0$ ,

$$(23) \qquad \mathbb{P} \left[ \exists i, j \in \mathbb{N} : |\xi_i^\lambda| < t, |\xi_j^{\lambda'-\lambda}| < t, \xi_i^\lambda = \xi_j^{\lambda'-\lambda} \right] = 0.$$

For  $n \in \mathbb{N}$ , we consider the probability

$$(24) \qquad p_n^\beta := \mathbb{P} \left[ \exists i, j \in \mathbb{N} : |\zeta_i^\lambda| < \frac{8\pi t}{\beta}, |\zeta_j^{\lambda'-\lambda}| < \frac{8\pi t}{\beta}, |\zeta_i^\lambda - \zeta_j^{\lambda'-\lambda}| < \frac{8\pi}{\beta} \frac{1}{2n} \right].$$

If  $n$  is fixed, then we have the convergence (as the studied set is open as for Lemma 3.3)

$$\liminf_{\beta \rightarrow 0} p_n^\beta \geq p_n := \mathbb{P} \left[ \exists i, j \in \mathbb{N} : |\xi_i^\lambda| < t, |\xi_j^{\lambda'-\lambda}| < t, |\xi_i^\lambda - \xi_j^{\lambda'-\lambda}| < \frac{1}{2n} \right].$$

To prove (23), it therefore suffices to prove that

$$(25) \qquad \limsup_{n \rightarrow \infty} \limsup_{\beta \rightarrow 0} p_n^\beta = 0.$$

For this, we need to work with a random subdivision of the interval  $[0, \frac{8\pi t}{\beta}]$ . As before, we consider a sequence  $(\tau_k)_{k \in \mathbb{N}}$  of i.i.d. positive random variables uniformly distributed in  $[\frac{1}{2}, \frac{3}{2}]$  and independent of the processes and form the sum  $S_k = \sum_{i=1}^k \tau_i$  ( $S_0 := 0$ ).

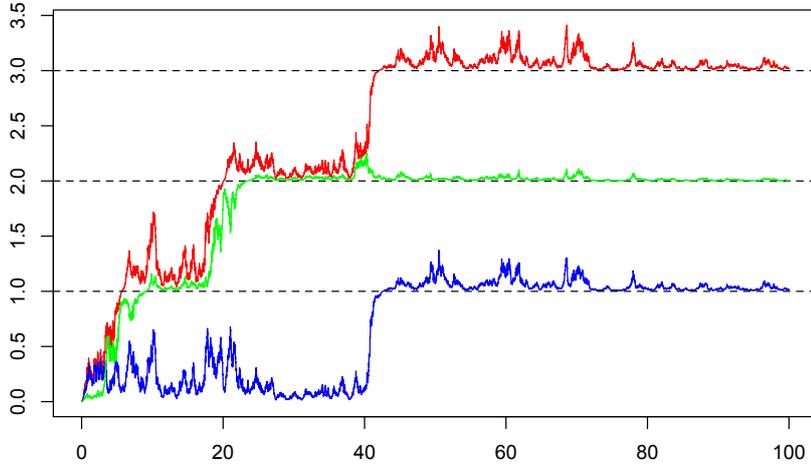


FIGURE 4. (Color online). Sample paths of the diffusions  $\alpha_\lambda/(2\pi)$  (blue curve) and  $\alpha_{\lambda'}/(2\pi)$  (red curve) together with  $(\alpha_{\lambda'} - \alpha_\lambda)/(2\pi)$  (green curve) for  $\lambda < \lambda'$  and for a small value of  $\beta$ . Again we see that  $\lfloor \alpha_\lambda/(2\pi) \rfloor$  and  $\lfloor (\alpha_{\lambda'} - \alpha_\lambda)/(2\pi) \rfloor$  never jump simultaneously while  $\lfloor \alpha_{\lambda'}/(2\pi) \rfloor$  and  $\lfloor (\alpha_{\lambda'} - \alpha_\lambda)/(2\pi) \rfloor$  always jump at the same times in agreement with Lemma 3.4, Lemma 3.3 and Remark 3.5.

The probability (24) can be upper-bounded as follows

$$\begin{aligned} & \mathbb{P} \left[ \exists i, j \in \mathbb{N}, k \leq \lfloor 2nt \rfloor + 1 : \frac{8\pi S_k}{\beta n} \leq \zeta_i^\lambda, \zeta_j^{\lambda' - \lambda} \leq \frac{8\pi}{\beta} \left( \frac{S_k}{n} + \frac{2}{n} \right) \right] \\ &= \mathbb{P} \left[ \exists k \leq \lfloor 2nt \rfloor + 1 : \lfloor \frac{\alpha_{\lambda'} - \alpha_\lambda}{2\pi} \rfloor \text{ and } \lfloor \frac{\alpha_\lambda}{2\pi} \rfloor \text{ both jump on the interval } \frac{8\pi}{\beta} \left[ \frac{S_k}{n}, \left( \frac{S_k}{n} + \frac{2}{n} \right) \right] \right] \end{aligned}$$

(26)

$$\leq \sum_{k=1}^{\lfloor 2nt \rfloor + 1} \mathbb{P} \left[ \lfloor \alpha_{\lambda'} \left( \frac{8\pi S_k}{\beta n} \right) \rfloor_{2\pi} \leq \lfloor \alpha_\lambda \left( \frac{8\pi S_k}{\beta n} \right) \rfloor_{2\pi} \right]$$

(27)

$$+ \sum_{k=1}^{\lfloor 2nt \rfloor + 1} \mathbb{P} \left[ \lfloor \frac{\alpha_{\lambda'}}{2\pi} \rfloor \text{ jumps two times during the interval } \frac{8\pi}{\beta} \left[ \frac{S_k}{n}, \left( \frac{S_k}{n} + \frac{2}{n} \right) \right] \right].$$

For this bound, we have used the fact that, conditionally on

$$(28) \quad \lfloor \alpha_\lambda \left( \frac{8\pi S_k}{\beta n} \right) \rfloor_{2\pi} \leq \lfloor \alpha_{\lambda'} \left( \frac{8\pi S_k}{\beta n} \right) \rfloor_{2\pi},$$

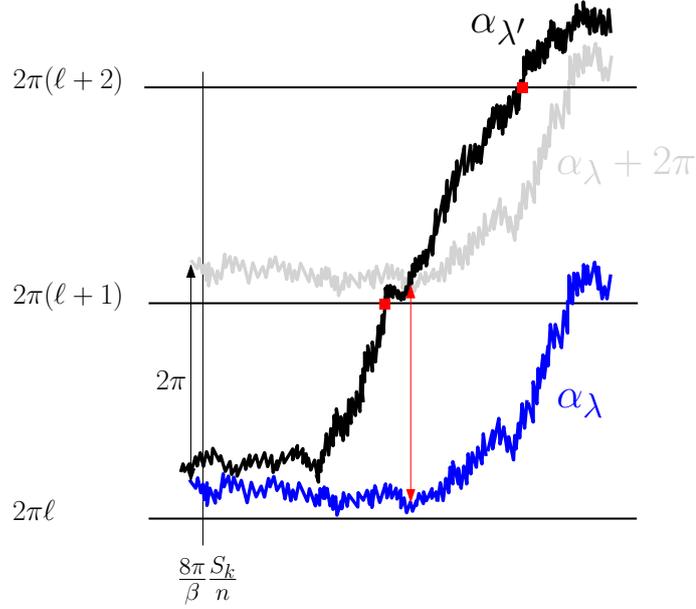


FIGURE 5. Illustration for inclusion (29).

the increasing property and the equality in law (18) impose

$$(29) \quad \left\{ \lfloor \frac{\alpha_{\lambda'} - \alpha_{\lambda}}{2\pi} \rfloor \text{ and } \lfloor \frac{\alpha_{\lambda}}{2\pi} \rfloor \text{ both jump on the interval } \frac{8\pi}{\beta} \left[ \frac{S_k}{n}, \left( \frac{S_k}{n} + \frac{2}{n} \right) \right] \right\} \\ \subseteq \left\{ \lfloor \frac{\alpha_{\lambda'}}{2\pi} \rfloor \text{ jumps two times during the interval } \frac{8\pi}{\beta} \left[ \frac{S_k}{n}, \left( \frac{S_k}{n} + \frac{2}{n} \right) \right] \right\}.$$

Indeed, the equality in law (18) implies that  $\lfloor \frac{\alpha_{\lambda'}(t) - \alpha_{\lambda}(t)}{2\pi} \rfloor$  is increasing with respect to  $t$  (once the difference  $\alpha_{\lambda'} - \alpha_{\lambda}$  has reached the value  $2k\pi$  where  $k \in \mathbb{N}$ , it remains forever above this value). This fact and the increasing property imply that under the event (29), the process  $\lfloor \frac{\alpha_{\lambda'}}{2\pi} \rfloor$  has to jump two times on the interval  $\frac{8\pi}{\beta} \left[ \frac{S_k}{n}, \left( \frac{S_k}{n} + \frac{2}{n} \right) \right]$  (see Fig. 5). We now estimate the two sums in (26). The first one was already tackled in (22).

For the other sum in (26), we write

$$\sum_{k=1}^{\lfloor 2nt \rfloor + 1} \mathbb{P} \left[ \lfloor \frac{\alpha_{\lambda'}}{2\pi} \rfloor \text{ jumps two times during the interval } \frac{8\pi}{\beta} \left[ \frac{S_k}{n}, \left( \frac{S_k}{n} + \frac{2}{n} \right) \right] \right] \\ = \sum_{k=1}^{\lfloor 2nt \rfloor + 1} \mathbb{P} \left[ \mu_{\lambda'}^{\beta} \left[ \frac{8\pi}{\beta} \left[ \frac{S_k}{n}, \left( \frac{S_k}{n} + \frac{2}{n} \right) \right] \right] = 2 \right].$$

If  $n$  is fixed and  $\beta \rightarrow 0$ , Theorem 2.4 gives the following convergence

$$\begin{aligned}
& \sum_{k=1}^{\lfloor 2nt \rfloor + 1} \mathbb{P} \left[ \mu_{\lambda'}^\beta \left[ \frac{8\pi S_k}{\beta n} \left[ 1, 1 + \frac{2}{n} \right] \right] = 2 \right] \\
& \xrightarrow{\beta \rightarrow 0} \frac{\lambda^2}{16\pi^2} \sum_{k=1}^{\lfloor 2nt \rfloor + 1} \exp \left( -\frac{\lambda}{2\pi} e^{-2\pi \frac{S_k}{n}} (1 - e^{-2\pi \frac{2}{n}}) \right) e^{-4\pi \frac{S_k}{n}} (1 - e^{-2\pi \frac{2}{n}})^2 \\
(30) \quad & = O(n^{-1}).
\end{aligned}$$

The convergence (25) now follows from the two estimates (22) and (30) and the independence between the two Poisson processes  $\mathcal{P}_\lambda$  and  $\mathcal{P}_{\lambda' - \lambda}$  is proved.  $\square$

**Remark 3.5.** Using similar arguments as in the proof of Lemma 3.3, one could also show that for all  $\lambda < \lambda'$ ,  $\mathcal{P}_{\lambda' - \lambda} \subseteq \mathcal{P}_{\lambda'}$ . Using then Lemma 3.4, it implies the almost-sure equality  $\mathcal{P}_{\lambda'} = \mathcal{P}_\lambda + \mathcal{P}_{\lambda' - \lambda}$ <sup>5</sup>.

Nevertheless, we do not need this stronger result as we only use for the  $\text{Sine}_\beta$  case the previous equality for the total number of points in  $\mathbb{R}_+$  i.e.  $\alpha_{\lambda'}(\infty)/(2\pi) = \alpha_\lambda(\infty)/(2\pi) + (\alpha_{\lambda'}(\infty) - \alpha_\lambda(\infty))/(2\pi)$

As an immediate consequence of Lemma 3.3 and Lemma 3.4, we obtain the following result using the characterization of independent Poisson processes mentioned above.

**Lemma 3.6.** Let  $\lambda_0 < \lambda_1 < \lambda_2$ . Then, the  $(\mathcal{F}_t)$ -Poisson processes  $\mathcal{P}_{\lambda_0}$  and  $\mathcal{P}_{\lambda_2 - \lambda_1}$  are independent.

**3.3. Proof of Theorem 1.1.** From Kallenberg's theorem [17], we simply need to check that, for any finite union  $I$  of disjoint and bounded intervals, we have when  $\beta \rightarrow 0$ ,

$$(31) \quad \mathbb{E}[\text{Sine}_\beta(I)] \longrightarrow \frac{1}{2\pi} \int_I d\lambda,$$

$$(32) \quad \mathbb{P}[\text{Sine}_\beta(I) = 0] \longrightarrow \exp\left(-\frac{1}{2\pi} \int_I d\lambda\right).$$

The convergence (31) follows from Corollary 2.5.

Towards (32), we consider first a union of two disjoint interval of the form

$$I := [\lambda_0, \lambda_1] \cup [\lambda_2, \lambda_3],$$

where  $\lambda_0 < \lambda_1 < \lambda_2 < \lambda_3$ . By translation invariance of the  $\text{Sine}_\beta$  point process, we can suppose without loss of generality that  $\lambda_0 = 0$ . By definition of  $\text{Sine}_\beta$ , we have the following equality in law

$$(\text{Sine}_\beta[0, \lambda_1], \text{Sine}_\beta[\lambda_2, \lambda_3], ) \stackrel{(d)}{=} \left( \frac{\alpha_{\lambda_1}(\infty)}{2\pi}, \frac{\alpha_{\lambda_3}(\infty) - \alpha_{\lambda_2}(\infty)}{2\pi} \right).$$

<sup>5</sup>Note that this equality is obviously wrong when  $\beta > 0$  i.e we don't have the equality between  $\lfloor \alpha_{\lambda'}(t)/(2\pi) \rfloor$  and  $\lfloor \alpha_{\lambda'}(t)/(2\pi) \rfloor + \lfloor (\alpha_{\lambda'}(t) - \alpha_\lambda(t))/(2\pi) \rfloor$ .

Using Lemma 3.6, we deduce that

$$\begin{aligned} & \mathbb{P} [\text{Sine}_\beta[0, \lambda_1] = 0, \text{Sine}_\beta[\lambda_2, \lambda_3] = 0] \\ & \rightarrow_{\beta \rightarrow 0} \mathbb{P} [\mathcal{P}_{\lambda_1}(\infty) = 0] \mathbb{P} [\mathcal{P}_{\lambda_3 - \lambda_2}(\infty) = 0] = \exp\left(-\frac{\lambda_1}{2\pi}\right) \exp\left(-\frac{\lambda_3 - \lambda_2}{2\pi}\right). \end{aligned}$$

Now, if  $k \geq 3$  and  $I := [\lambda_0, \lambda_1] \cup \dots \cup [\lambda_{k-1}, \lambda_k]$  where  $\lambda_0 \leq \dots \leq \lambda_k$ , the limiting Poisson point processes  $(\mathcal{P}_{\lambda_i - \lambda_{i-1}})_{i=1, \dots, k}$  obtained as the limit of the processes  $\lfloor (\alpha_{\lambda_i} - \alpha_{\lambda_{i-1}})/(2\pi) \rfloor$  never jump simultaneously (there are pairwise independent). Therefore, it is a family of independent processes [Prop. 1, Chap. 0.4 of [20]].

Theorem 1.1 is proved.  $\square$

## APPENDIX A. PROOF OF AUXILIARY RESULTS.

**A.1. Asymptotic of integrals and Laplace transform.** *Asymptotic of the integral (10).* We denote by  $a_\beta$  (respectively  $b_\beta$ ) the local minimum (resp. maximum) of the potential  $V_\beta(r)$  and derive asymptotic expansions for  $a_\beta$  and  $b_\beta$  in the limit  $\beta \rightarrow 0$  using

$$V'_\beta(r) = 0 \Leftrightarrow \frac{\lambda\beta}{8} = \frac{e^{-r} - e^r}{(e^r - e^{-r})^2}.$$

We deduce that

$$\begin{aligned} a_\beta &= \log \beta + \log(\lambda) - 3 \log 2 + O(\beta^2), \\ b_\beta &= -\frac{\lambda\beta}{4} + o(\beta^2), \\ V_\beta(a_\beta) &= \frac{1}{2} \log \beta + \frac{1}{2} \log \lambda - \log 2 + \frac{1}{2} + O(\beta^2), \\ V_\beta(b_\beta) &= \frac{\lambda^2}{64} \beta^2 + O(\beta^2) = O(\beta^2). \end{aligned}$$

More generally, we have for  $x, y \in \mathbb{R}$

$$\begin{aligned} V_\beta(a_\beta + y) &= \frac{1}{2} \log \frac{\beta\lambda}{4} + \frac{y + e^{-y}}{2} - \frac{1}{2} \left(\frac{\lambda\beta}{8}\right)^2 e^y - \frac{1}{2} \log(1 + \left(\frac{\lambda\beta}{8}\right)^2 e^{2y}) + O(\beta^2), \\ V_\beta(b_\beta + x) &= -\frac{1}{2} \log \frac{e^x + e^{-x}}{2} \left(1 - \frac{\lambda\beta}{8} + O(\beta^2)\right) - \frac{\lambda\beta}{8} \sinh(x) + \frac{\lambda^2 \beta^2}{64} \cosh(x) + O(\beta^2). \end{aligned}$$

Those computations permit us to find the asymptotic behaviour of  $t_\beta(r_\beta)$  (using also the bounded convergence theorem),

$$(33) \quad t_\beta(r_\beta) \sim \frac{8}{\beta\lambda} \int_{r_\beta - b_\beta}^{+\infty} \frac{dx}{\cosh(x)} \int_{-\infty}^{x+a_\beta} e^{-y} e^{-e^{-y}} dy \sim \frac{8}{\beta\lambda} \int_{r_\beta - b_\beta}^{+\infty} \frac{dx}{\cosh(x)} \sim \frac{8\pi}{\beta\lambda}.$$

$\square$

*Proof of Proposition 2.2.*

We consider the Laplace transform  $g_{\beta\lambda/(8\pi)\zeta}(r)$  of the rescaled exit time  $\beta\lambda/(8\pi)\zeta$  of the diffusion starting from  $r$ . It suffices to prove that, if  $(r_\beta)_{\beta>0}$  is such that  $r_\beta \rightarrow -\infty$  as  $\beta \rightarrow 0$ ,

we have for any  $0 < \xi < 1$ ,  $g_{\beta\lambda/(8\pi)\xi}(r_\beta) \rightarrow 1/(1 + \xi)$  (Laplace transform of an exponential distribution with parameter 1). To simplify notations, we will just write  $g_\beta(r)$  for  $g_{\beta\lambda/(8\pi)\xi}(r)$ .

Let us recall the fixed point equation

$$(34) \quad g_\beta(r) = 1 - 2\frac{\beta\lambda}{8\pi}\xi \int_r^{+\infty} dx \int_{-\infty}^x \exp(2[V_\beta(x) - V_\beta(y)]) g_\beta(y) dy.$$

Noting that  $g_\beta(r) \leq 1$ , we obtain the lower bound

$$(35) \quad 1 - \xi\frac{\beta\lambda}{8\pi}t_\beta(r) \leq g_\beta(r).$$

Using the asymptotic (33), we obtain a lower bound

$$(36) \quad 1 - \xi \leq \liminf_{\beta \rightarrow 0} g_\beta(r_\beta).$$

Plugging (36) into (34), we get an upper bound

$$g_\beta(r) \leq 1 - \xi\frac{\beta\lambda}{8\pi}t_\beta(r) + 2\xi^2\left(\frac{\beta\lambda}{8\pi}\right)^2 \int_r^{+\infty} dx \int_{-\infty}^x \exp(2[V_\beta(x) - V_\beta(y)]) t_\beta(y) dy.$$

After the derivation of the asymptotic quadruple integral (similar to the one of (33)), we obtain

$$\limsup_{\beta \rightarrow 0} g_\beta(r_\beta) \leq 1 - \xi + \xi^2.$$

Iterating this argument (the multiple integrals are always such that the integration ranges permit to catch the maximum value of  $V_\beta(x) - V_\beta(y)$  inside the exponential as in (33)), we get the result.  $\square$

$\square$

**A.2. Proof of the estimates for a single diffusion.** Recall the definition of  $R_\lambda := \log(\tan(\alpha_\lambda/4))$  and its differential equation (6):

$$dR_\lambda = \frac{1}{2} \left( \lambda \frac{\beta}{4} e^{-\frac{\beta}{4}t} \cosh(R_\lambda) + \tanh(R_\lambda) \right) dt + dB_t, \quad R_\lambda(0) = -\infty,$$

and denote by  $T_r$  the first passage time to  $r$  i.e.  $T_r := \inf\{t \geq 0 : R_\lambda(t) = r\}$ . In this sub-section, we first study the diffusion  $R_\lambda$  and then translate the estimates to the diffusion  $\alpha_\lambda$ . We will denote by  $\mathbb{P}_{r,t}$  the law of the diffusion  $R_\lambda$  starting from position  $r$  at time  $t$ . As before, we omit the subscript  $t$  if  $t = 0$ .

Our first lemma A.1 shows that when the diffusion  $R_\lambda$  is outside of the well the probability that it explodes in a short time tends to 1.

**Lemma A.1.** *Let  $0 < \varepsilon < 1$ . Then, there exists a constant  $c > 0$  such that for all  $\beta > 0$  small enough,*

$$\mathbb{P}_{\varepsilon \log \frac{1}{\beta}} \left[ T_{+\infty} < 9 \log \frac{1}{\beta} \right] \geq 1 - \beta^c.$$

Note that it immediately gives the analogous result for  $\alpha_\lambda$  of Lemma 2.6.

*Proof of Lemma A.1.* We have

$$\mathbb{P}_{\varepsilon \log \frac{1}{\beta}} \left[ T_{+\infty} < 9 \log \frac{1}{\beta} \right] \geq \mathbb{P}_{\varepsilon \log \frac{1}{\beta}} \left[ T_{2 \log \frac{1}{\beta}} < 8 \log \frac{1}{\beta} \wedge T_{\frac{\varepsilon}{2} \log \frac{1}{\beta}} \right] \mathbb{P}_{2 \log \frac{1}{\beta}, 8 \log \frac{1}{\beta}} \left[ T_{+\infty} < \log \frac{1}{\beta} \right].$$

If  $\frac{\varepsilon}{2} \ln \frac{1}{\beta} < r < 2 \log \frac{1}{\beta}$ , the drift term  $-V'_\beta(r) \geq \frac{1}{2} \frac{e^r - e^{-r}}{e^r + e^{-r}} \geq \frac{1}{4}$  for  $\beta$  small enough. Thus, for  $t < T_{\frac{\varepsilon}{2} \log \frac{1}{\beta}}$ , we have  $R_\lambda(t) \geq B_t + t/4$ . Therefore,

$$\begin{aligned} & \mathbb{P}_{\varepsilon \log \frac{1}{\beta}} \left[ T_{2 \log \frac{1}{\beta}} < 8 \log \frac{1}{\beta} \wedge T_{\frac{\varepsilon}{2} \log \frac{1}{\beta}} \right] \\ & \geq \mathbb{P}_{\varepsilon \log \frac{1}{\beta}} \left[ \inf_{0 \leq t \leq 8 \log \frac{1}{\beta}} B_t > \frac{\varepsilon}{2} \log \frac{1}{\beta}, B_{8 \log \frac{1}{\beta}} > 0 \right]. \end{aligned}$$

This latter probability is easily computed with the reflection principle for Brownian motion.

$$\begin{aligned} & \mathbb{P}_{\varepsilon \log \frac{1}{\beta}} \left[ \inf_{0 \leq t \leq 8 \log \frac{1}{\beta}} B_t > \frac{\varepsilon}{2} \log \frac{1}{\beta}, B_{8 \log \frac{1}{\beta}} > 0 \right] = \mathbb{P}_0 \left[ \sup_{0 \leq t \leq 8 \log \frac{1}{\beta}} B_t \leq \frac{\varepsilon}{2} \log \frac{1}{\beta}, B_{8 \log \frac{1}{\beta}} \leq \varepsilon \log \frac{1}{\beta} \right] \\ & = \mathbb{P}_0 \left[ -\frac{3\varepsilon}{2} \log \frac{1}{\beta} \leq B_{8 \log \frac{1}{\beta}} \leq \varepsilon \log \frac{1}{\beta} \right] = 1 - O(\exp(-c \log \frac{1}{\beta})). \end{aligned}$$

We now consider the probability

$$\mathbb{P}_{2 \log \frac{1}{\beta}, 8 \log \frac{1}{\beta}} \left[ T_{+\infty} < \log \frac{1}{\beta} \right]$$

We define  $G_\lambda(t) := R_\lambda(t) - B_t$  such that  $G_\lambda(0) = 2 \log \frac{1}{\beta}$ . The function  $G_\lambda$  satisfies the following random *ordinary* differential equation

$$G'_\lambda(t) = \frac{1}{2} \left( \frac{\lambda\beta}{8} e^{-\frac{\beta}{4}t} \frac{e^{G_\lambda(t)+B_t} + e^{-G_\lambda(t)-B_t}}{2} + \frac{e^{G_\lambda(t)+B_t} - e^{-G_\lambda(t)-B_t}}{e^{G_\lambda(t)+B_t} + e^{-G_\lambda(t)-B_t}} \right).$$

On the event

$$\mathcal{E}_\beta := \left\{ -\frac{1}{2} \log \frac{1}{\beta} \leq \sup_{0 \leq t \leq \log \frac{1}{\beta}} |B_t| \leq \frac{1}{2} \log \frac{1}{\beta} \right\},$$

which occurs with probability  $1 - \exp(-c \log(\frac{1}{\beta}))$  where  $c > 0$  is a constant independent of  $\beta$ , we can easily check that

$$G'_\lambda(t) \geq \frac{\lambda\beta^{3/2}}{32} e^{G_\lambda(t)} - \frac{1}{2}.$$

This leads us to study the Cauchy problem

$$T'_\lambda(t) = \frac{\lambda\beta^{3/2}}{32} e^{H_\lambda(t)} - \frac{1}{2}, \quad H_\lambda(0) = 2 \log \frac{1}{\beta}.$$

This ordinary differential equation can be solved explicitly. We find

$$e^{-H_\lambda(t)} e^{-\frac{t}{2}} = \beta^2 - \frac{\lambda\beta^{3/2}}{16} (1 - e^{-\frac{t}{2}}).$$

It is easy to see that the exploding time of  $H_\lambda$  is of order  $\sqrt{\beta}$  as  $\beta \rightarrow 0$ . On the event  $\mathcal{E}_\beta$ , we have  $H_\lambda(t) \leq G_\lambda(t)$  for all  $t \geq 0$ .

We can conclude that

$$\mathbb{P}_{2 \log \frac{1}{\beta}, 8 \log \frac{1}{\beta}} \left[ T_{+\infty} < \log \frac{1}{\beta} \right] \geq \mathbb{P}[\mathcal{E}_\beta] = 1 - \exp(-c \log(\frac{1}{\beta})).$$

□

**Lemma A.2.** For all  $\beta > 0$  small enough,

$$(37) \quad \mathbb{P}_{-\frac{1}{4} \log \frac{1}{\beta}} \left[ T_{+\infty} < 10 \log \frac{1}{\beta} \right] \geq \sqrt{\beta}.$$

*Proof of Lemma A.2.* The probability under consideration can be bounded below by

$$\mathbb{P}_{-\frac{1}{4} \log \frac{1}{\beta}} \left[ T_{\frac{1}{4} \log \frac{1}{\beta}} < \log \frac{1}{\beta} \right] \mathbb{P}_{\frac{1}{4} \log \frac{1}{\beta}, \log \frac{1}{\beta}} \left[ T_{+\infty} < 9 \log \frac{1}{\beta} \right].$$

For the first probability, we note that  $R_\lambda(t) \geq -\frac{t}{2} + B_t$  and thus

$$\begin{aligned} \mathbb{P}_{-\frac{1}{4} \log \frac{1}{\beta}} \left[ T_{\frac{1}{4} \log \frac{1}{\beta}} < \log \frac{1}{\beta} \right] &\geq \mathbb{P}_0 \left[ B(\log \frac{1}{\beta}) - \frac{1}{2} \log \frac{1}{\beta} \geq \frac{1}{2} \log \frac{1}{\beta} \right] \\ &= \mathbb{P}_0 \left[ B_1 \geq \sqrt{\log \frac{1}{\beta}} \right] \sim \frac{\sqrt{\beta}}{2\sqrt{\varepsilon \log \frac{1}{\beta}}}. \end{aligned}$$

From Lemma A.1, the second probability is bigger than  $1 - \beta^c$  for some positive constant  $c > 0$ . □

On the large scale-time of the order  $1/\beta$ , we prove that the time spent by  $R_\lambda$  in the well of the potential  $V$  (near 0 modulo  $2\pi$  for  $\alpha_\lambda$ ) is negligible, i.e. small compared to the typical time  $1/\beta$  between two jumps. This is Lemma A.3:

**Lemma A.3.** Let  $t > 0$  and

$$T_\beta(t) := \mathbb{E} \left[ \int_0^{\frac{8\pi}{\beta} t} 1_{\{R_\lambda(u) \geq -\frac{1}{4} \log \frac{1}{\beta}\}} du \right].$$

Then, there exists  $C > 0$  independent of  $\beta$  such that

$$T_\beta(t) \leq \frac{C}{\sqrt{\beta}} \log \frac{1}{\beta}.$$

Lemma A.3 can be translated for  $\alpha_\lambda$ , it gives Lemma 2.7.

*Proof of Lemma A.3.* The key estimate is the lower-bound (37) given by Lemma A.2 on the probability for the diffusion  $R_\lambda$  starting from the position  $-\frac{1}{4} \log \frac{1}{\beta}$  to blow up in a short time. The idea is then to relate the time spent by the diffusion  $R_\lambda$  above the level  $-\frac{1}{4} \log \frac{1}{\beta}$  with the number of explosions in the interval  $[0, \frac{8\pi t}{\beta}]$  which is of order  $O(1)$  (the typical time between two explosions is of order  $1/\beta$  from Lemma 2.2). For any  $u \in \mathbb{R}$ , we have

$$\begin{aligned} &\mathbb{P} \left[ R_\lambda(u) \geq -\frac{1}{4} \log \frac{1}{\beta} \right] \\ &\leq \mathbb{P} \left[ R_\lambda(u) \geq -\frac{1}{4} \log \frac{1}{\beta}, T_{+\infty} < 10 \log \frac{1}{\beta} \right] + \mathbb{P} \left[ R_\lambda(u) \geq -\frac{1}{4} \log \frac{1}{\beta}, T_{+\infty} \geq 10 \log \frac{1}{\beta} \right] \\ &\leq \mathbb{P} \left[ R_\lambda(u) \geq -\frac{1}{4} \log \frac{1}{\beta}, T_{+\infty} < 10 \log \frac{1}{\beta} \right] + \left(1 - \sqrt{\beta}\right) \mathbb{P} \left[ R_\lambda(u) \geq -\frac{1}{4} \log \frac{1}{\beta} \right], \end{aligned}$$

where we have used the simple Markov property, the inequality (37) as well as the increasing property in the third line. We deduce that

$$(38) \quad \begin{aligned} & \mathbb{P} \left[ R_\lambda(u) \geq -\frac{1}{4} \log \frac{1}{\beta} \right] \leq \frac{1}{\sqrt{\beta}} \mathbb{P} \left[ R_\lambda(u) \geq -\frac{1}{4} \log \frac{1}{\beta}, T_{+\infty} < 10 \log \frac{1}{\beta} \right] \\ & \leq \frac{1}{\sqrt{\beta}} \mathbb{P} \left[ \text{The interval } [u, u + 10 \log \frac{1}{\beta}] \text{ contains at least one explosion} \right]. \end{aligned}$$

Denoting by  $k = \mu_\lambda^\beta[0, t]$  the total number of explosions of the diffusion  $R_\lambda$  in the interval  $[0, \frac{8\pi t}{\beta}]$  and by  $0 < \zeta_1 < \dots < \zeta_k < \frac{8\pi t}{\beta}$  the explosion times, we easily see that almost surely

$$\int_0^{\frac{8\pi t}{\beta}} 1_{\{\exists i: \zeta_i \in [u, u + 10 \log \frac{1}{\beta}]\}} du \leq 10 \log \frac{1}{\beta} \mu_\lambda^\beta[0, t].$$

Using this inequality and integrating (38) with respect to  $u$ , we finally obtain

$$T_\beta(t) \leq \mathbb{E}[\mu_\lambda^\beta[0, t]] \frac{10}{\sqrt{\beta}} \log \frac{1}{\beta} \leq \frac{\lambda}{2\pi} \frac{10}{\sqrt{\beta}} \log \frac{1}{\beta}.$$

□

**A.3. Proof of Lemma 3.2.** Theorem 2.4 gives the weak convergence of the random measures  $\mu_\lambda^\beta$  and  $\mu_{\lambda'}^\beta$  and  $\mu_{\lambda''-\lambda'}^\beta$  in the space of measures on  $\mathbb{R}_+$  (equipped with the topology of vague convergence). From [21, Lemma 11.1.XI], it is equivalent to the convergence of each of the cumulative processes in the Skorokhod càdlàg space. We conclude that the family of (random) càdlàg processes

$$\left\{ \left( \mu_\lambda^\beta[0, t], \mu_{\lambda'}^\beta[0, t], \mu_{\lambda''-\lambda'}^\beta[0, t] \right)_{t \geq 0}, \beta > 0 \right\}$$

is tight. Prokhorov's theorem gives the existence of a sequence  $\beta_k \rightarrow 0$  when  $k \rightarrow \infty$  such that

$$(39) \quad \left( \mu_{\lambda}^{\beta_k}[0, t], \mu_{\lambda'}^{\beta_k}[0, t], \mu_{\lambda''-\lambda'}^{\beta_k}[0, t] \right)_{t \geq 0}$$

converges weakly in the Skorokhod space when  $k \rightarrow \infty$  to some limiting càdlàg process whose marginal distributions are given respectively by the law of the Poisson point processes  $(\mathcal{P}_\lambda(t))_{t \geq 0}$  and  $(\mathcal{P}_{\lambda'}(t))_{t \geq 0}$  and  $(\mathcal{P}_{\lambda''-\lambda'}(t))_{t \geq 0}$ .

Denote  $\mathcal{C}_0(\mathbb{R}_+)$  the space of continuous functions  $f : \mathbb{R}_+ \rightarrow \mathbb{R}_+$  such that  $f(t) \rightarrow 0$  as  $t \rightarrow +\infty$ . To prove that  $\mathcal{P}_\lambda$  is indeed a  $(\mathcal{F}_t)$ -Poisson process with the correct intensity, it suffices to check that its Laplace functional satisfies

$$(40) \quad \mathbb{E} \left[ \exp \left( - \int_s^t f(u) \mathcal{P}_\lambda(du) \right) \middle| \mathcal{F}_s \right] = \exp \left( - \int_s^t (1 - e^{-f(u)}) \lambda e^{-2\pi u} du \right).$$

We have to check that the filtration  $(\mathcal{F}_t)$  does not contain too much information compared to the natural filtration denoted  $(\mathcal{F}_t^\lambda)$  associated to the process  $\mathcal{P}_\lambda$  only, for (40) to remain valid.

We denote by  $(\mathcal{G}_t)$  the filtration associated to the complex Brownian motion  $(Z_t)$  which drives both processes  $\alpha_\lambda, \alpha_{\lambda'}$  and  $\alpha_{\lambda''}$  according to (3).

The conclusion of Theorem 2.4 is equivalent to the convergence of the Laplace functional of the point process  $\mu_\lambda^\beta$  to the Laplace functional of the Poisson process  $\mathcal{P}_\lambda$ . More precisely, for any

$\varphi \in \mathcal{C}_0(\mathbb{R}_+)$  and any  $0 < s < t$ , we have

$$(41) \quad \mathbb{E} \left[ \exp \left( - \int_s^t \varphi(u) \mu_\lambda^\beta(du) \right) \middle| \mathcal{G}_{\frac{8\pi}{\beta}s} \right] \rightarrow_{\beta \rightarrow 0} \exp \left( - \int_s^t (1 - e^{-\varphi(u)}) \lambda e^{-2\pi u} du \right).$$

The main point is that the sigma field  $\mathcal{G}_{\frac{8\pi}{\beta}s}$  already contains the information on the two diffusions  $(\alpha_\lambda(u))_{0 \leq u \leq s}$ ,  $(\alpha_{\lambda'}(u))_{0 \leq u \leq s}$  and  $(\alpha_{\lambda''}(u))_{0 \leq u \leq s}$  up to time  $s$  (they are deterministic function of the driving Brownian motion  $(Z_u)_{0 \leq u \leq s}$ ). From the joint convergence of the triplet (39) along the sequence  $(\beta_k)$ , we can deduce (using again [21, Lemma 11.1.XI]) that, for  $\varphi, f, g, h \in \mathcal{C}_0(\mathbb{R}_+)$ ,

$$(42) \quad \lim_{k \rightarrow \infty} \mathbb{E} \left[ \exp \left( - \int_s^t \varphi(u) \mu_\lambda^{\beta_k}(du) \right) \exp \left( - \int_0^s (f(u) \mu_\lambda^{\beta_k}(du) + g(u) \mu_{\lambda'}^{\beta_k}(du) + h(u) \mu_{\lambda''-\lambda'}^{\beta_k}(du)) \right) \right] \\ = \mathbb{E} \left[ \exp \left( - \int_s^t \varphi(u) \mathcal{P}_\lambda(du) \right) \exp \left( - \int_0^s (f(u) \mathcal{P}_\lambda(du) + g(u) \mathcal{P}_{\lambda'}(du) + h(u) \mathcal{P}_{\lambda''-\lambda'}(du)) \right) \right].$$

On the other hand, using in addition (41), we can check that

$$(43) \quad \lim_{k \rightarrow \infty} \mathbb{E} \left[ \exp \left( - \int_s^t \varphi(u) \mu_\lambda^{\beta_k}(du) \right) \exp \left( - \int_0^s (f(u) \mu_\lambda^{\beta_k}(du) + g(u) \mu_{\lambda'}^{\beta_k}(du) + h(u) \mu_{\lambda''-\lambda'}^{\beta_k}(du)) \right) \right] \\ = \exp \left( - \int_s^t (1 - e^{-\varphi(u)}) \lambda e^{-2\pi u} du \right) \mathbb{E} \left[ \exp \left( - \int_0^s (f(u) \mathcal{P}_\lambda(du) + g(u) \mathcal{P}_{\lambda'}(du) + h(u) \mathcal{P}_{\lambda''-\lambda'}(du)) \right) \right].$$

Gathering (42) and (43), we obtain (40). □

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