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# Crystal dislocations with different oriantations and collisions


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Abstract. We study a parabolic differential equation whose solution represents the atom dislocation in a crystal for a general type of Peierls-Nabarro model with possibly long range interactions and an external stress. Differently from the previous literature, we treat here the case in which such dislocation is not the superpositions of transitions all occurring with the same orientations (i.e. opposite orientations are allowed as well).

We show that, at a long time scale, and at a macroscopic space scale, the dislocations have the tendency to concentrate as pure jumps at points which evolve in time, driven by the external stress and by a singular potential. Due to differences in the dislocation orientations, these points may collide in finite time.

More precisely, we consider the evolutionary equation

$$(v_\varepsilon)_t = \frac{1}{\varepsilon}\left(\mathcal{I}_s v_\varepsilon - \frac{1}{\varepsilon^{2s}} W'(v_\varepsilon) + \sigma(t,x)\right),$$

where $v_\varepsilon = v_\varepsilon(t,x)$ is the atom dislocation fuction at time $t>0$ at the point $x\in\mathbb{R}$, $\mathcal{I}_s$ is an integro-differential operator of order $2s\in(0,2)$, $W$ is a periodic potential, $\sigma$ is an external stress and $\varepsilon>0$ is a small parameter that takes into account the small periodicity scale of the crystal.

We suppose that $v_\varepsilon(0,x)$ is the superposition of $N-K$ transition layers in the positive direction and $K$ in the negative one (with $K\in\{0,\dots,N\}$); more precisely, we fix points $x_1^0 < \cdots < x_N^0$ and we take

$$v_\varepsilon(0,x) = \frac{\varepsilon^{2s}}{W''(0)}\sigma(0,x) + \sum_{i=1}^N u\left(\zeta_i \frac{x-x_i^0}{\varepsilon}\right).$$

Here $\zeta_i$ is either $-1$ or $1$, depending on the orientation of the transition layer $u$, which in turn solves the stationary equation $\mathcal{I}_s u = W'(u)$.

We show that our problem possesses a unique solution and that, as $\varepsilon\to 0^+$, it approaches the sum of Heaviside functions $H$ with different orientations centered at points $x_i(t)$, namely

$$\sum_{i=1}^N H(\zeta_i(x-x_i(t))).$$

The point $x_i$ evolves in time from $x_i^0$, being subject to the external stress and a singular potential, which may be either attractive or repulsive, according to the different orientation of the transitions: more precisely, the speed $\dot{x}_i$ is proportional to

$$\sum_{j\neq i} \zeta_i\zeta_j \frac{x_i-x_j}{2s|x_i-x_j|^{1+2s}} - \zeta_i\sigma(t,x_i).$$

The evolution of such dynamical system may lead to collisions in finite time. We give a detailed description of such collisions when $N=2,3$ and we show that the solution itself keeps track of such collisions: indeed, at the collision time $T_c$ the two opposite dislocations have the tendency to annihilate each other and make the dislocation vanish, but only outside the collision point $x_c$, according to the formulas

$$\lim_{t\to T_c^-}\lim_{\varepsilon\to 0^+} v_\varepsilon(t,x) = 0 \quad \text{when } x\neq x_c,$$

$$\text{and} \qquad \limsup_{\substack{t\to T_c^- \\ \varepsilon\to 0^+}} v_\varepsilon(t,x_c) \geqslant 1.$$

We also study some specific cases of $N$ dislocation layers, namely when two dislocations are initially very close and when the dislocations are alternate.

To the best of our knowledge, the results obtained are new even in the model case $s=1/2$.

## 1. Introduction

The goal of this paper is to study an evolutionary partial-integro-differential equation and a system of ordinary differential equations that arise in the Peierls-Nabarro model for atoms dislocation in crystals.

We refer to [8] for a survey of the Peierls-Nabarro model. See also Section 2 in [4] for some basic physical derivation.

The main goal of the evolutionary equation associated to the Peierls-Nabarro model is to study the asymptotic behavior of the solution $v_\varepsilon$, which represents the atom dislocation function, in terms of $\varepsilon$, which in turn represents the size of the crystal scale. A suitable parabolic scaling is involved in the equation, and so the asymptotics as $\varepsilon \to 0^+$ corresponds simultaneously to the long time and macroscopic space scale behavior.

Roughly speaking, in this paper we will consider initial configurations in which the dislocation transitions occurs at some given points. Differently from the existing literature, the initial dislocations are not assumed to have all the same orientation.

We will show that, at a long time and macroscopic scale range, the solution will behave as the superposition of sharp interfaces.

These interfaces move in time according to an external stress and an interaction potential. As a main novelty with respect to the existing literature, we will show that in this case the potential has two opposite tendences, i.e. it is repulsive among dislocations with the same orientations and attractive among dislocations with opposite orientations.

In configurations in which the attractive feature of the potential prevails, the dislocation with opposite orientations may collide one with the other. Therefore we also give some explicit results about collisions in concrete cases.

Let us now formally describe the mathematical framework that we deal with. We consider the problem

$$\begin{cases} (v_\varepsilon)_t = \dfrac{1}{\varepsilon}\left(\mathcal{I}_s v_\varepsilon - \dfrac{1}{\varepsilon^{2s}} W'(v_\varepsilon) + \sigma(t,x)\right) & \text{in } (0,+\infty)\times\mathbb{R} \\ v_\varepsilon(0,\cdot) = v_\varepsilon^0 & \text{on } \mathbb{R}, \end{cases} \tag{1.1}$$

where $\varepsilon > 0$ is a small scale parameter, $W$ is a periodic potential and $\mathcal{I}_s$ is the so-called fractional Laplacian of any order $2s \in (0,2)$. Precisely, given $\varphi \in C^2(\mathbb{R}^N) \cap L^\infty(\mathbb{R}^N)$, let us define

$$\mathcal{I}_s[\varphi](x) := PV \int_{\mathbb{R}^N} \frac{\varphi(x+y)-\varphi(x)}{|y|^{N+2s}}\,dy, \tag{1.2}$$

where $PV$ stands for the principal value of the integral. We refer to [10] and [5] for a basic introduction to the fractional Laplace operator. On the potential $W$ we assume

$$\begin{cases} W \in C^{3,\alpha}(\mathbb{R}) & \text{for some } 0<\alpha<1 \\ W(v+1) = W(v) & \text{for any } v \in \mathbb{R} \\ W = 0 & \text{on } \mathbb{Z} \\ W > 0 & \text{on } \mathbb{R}\setminus\mathbb{Z} \\ W''(0) > 0. \end{cases} \tag{1.3}$$

The function $\sigma$ satisfies:

$$\begin{cases} \sigma \in BUC([0,+\infty)\times\mathbb{R}) \quad \text{and for some } M>0 \text{ and } \alpha \in (s,1) \\ \|\sigma_x\|_{L^\infty([0,+\infty)\times\mathbb{R})} + \|\sigma_t\|_{L^\infty([0,+\infty)\times\mathbb{R})} \leqslant M \\ |\sigma_x(t,x+h)-\sigma_x(t,x)| \leqslant M|h|^\alpha, \quad \text{for every } x,h \in \mathbb{R} \text{ and } t \in [0,+\infty). \end{cases} \tag{1.4}$$

We assume the initial condition in (1.1) to be a superposition of transition layers. Precisely, let us introduce the so-called basic layer solution $u$ associated to $\mathcal{I}_s$, that is the solution of

$$(1.5)\qquad \begin{cases} \mathcal{I}_s(u) = W'(u) & \text{in} \quad \mathbb{R} \\ u' > 0 & \text{in} \quad \mathbb{R} \\ \lim\limits_{x\to-\infty} u(x) = 0, \quad \lim\limits_{x\to+\infty} u(x) = 1, \quad u(0) = \dfrac{1}{2}. \end{cases}$$

The existence of a unique solution of (1.5) is proven in [1]. The name layer solution is motivated by the fact that $u$ approaches the limits 0 and 1 at $\pm\infty$. Asymptotic estimates on the decay of $u$ are proven in [9], finer estimates are given in [4] and [3] respectively when $s \in \left[\frac{1}{2}, 1\right)$ and $s \in \left(0, \frac{1}{2}\right)$. The case $s = \frac{1}{2}$ was already treated in [7].

Given $x_1^0 < x_2^0 < ... < x_N^0$, we say that the function $u\left(\frac{x-x_i^0}{\varepsilon}\right)$ is a transition layer centered at $x_i^0$ and positively oriented. Similarly, we say that the function $u\left(\frac{x_i^0-x}{\varepsilon}\right) - 1$ is a transition layer centered at $x_i^0$ and negatively oriented.

Notice that the positively oriented transition layer connects the "rest states" 0 and 1, while the negatively oriented one connects 0 with $-1$.

We consider as initial condition in (1.1) the state obtained by superposing $N$ copies of the transition layer, centered at $x_1^0, ..., x_N^0$, $N - K$ of them positively oriented and the remaining $K$ negative oriented, that is

$$(1.6)\qquad v_\varepsilon^0(x) = \frac{\varepsilon^{2s}}{\beta}\sigma(0, x) + \sum_{i=1}^{N} u\left(\zeta_i \frac{x - x_i^0}{\varepsilon}\right) - K,$$

where $\zeta_1, ..., \zeta_N \in \{-1, 1\}$, $\sum\limits_{i=1}^{N} (\zeta_i)^- = K$, $0 \leqslant K \leqslant N$ and

$$(1.7)\qquad \beta := W''(0) > 0.$$

Let us introduce the solution $(x_i(t))_{i=1,...,N}$ to the system

$$(1.8)\qquad \begin{cases} \dot{x}_i = \gamma \left( \sum\limits_{j\neq i} \zeta_i \zeta_j \dfrac{x_i - x_j}{2s|x_i - x_j|^{1+2s}} - \zeta_i \sigma(t, x_i) \right) & \text{in } (0, T_c) \\ x_i(0) = x_i^0, \end{cases}$$

where

$$(1.9)\qquad \gamma := \left( \int_{\mathbb{R}} (u'(x))^2 dx \right)^{-1},$$

with $u$ solution of (1.5) and $(0, T_c)$ is the maximal interval where the system (1.8) is well defined, i.e. where $x_i \neq x_j$ for any $i \neq j$. Therefore, $0 < T_c \leqslant +\infty$ is the first time when a collision between two particles occurs, more precisely $T_c$ is such that: there exist $i_0, j_0$ with $i_0 \neq j_0$ such that $x_{i_0}(T_c) = x_{j_0}(T_c)$ and $x_i(t) \neq x_j(t)$ for any $t \in [0, T_c)$ and any $i, j$.

We remark that (1.8) is a gradient system, i.e. it can be written as

$$\dot{x}_i(t) = -\partial_i V\big(t, x_1(t), \ldots, x_N(t)\big),$$

with

$$V(t,x_1,\dots,x_N) := V_0(x_1,\dots,x_n) + \sum_{i=1}^{N} \zeta_i \Sigma(t,x_i),$$

$$V_0(x_1,\dots,x_n) := \begin{cases} \dfrac{\gamma}{2s\,(2s-1)} \displaystyle\sum_{1\leqslant i\neq j\leqslant N} \zeta_i\zeta_j |x_j-x_i|^{1-2s} & \text{if } s\neq 1/2, \\ -\gamma \displaystyle\sum_{1\leqslant i\neq j\leqslant N} \zeta_i\zeta_j \log|x_j-x_i| & \text{if } s=1/2, \end{cases}$$

and $$\Sigma(t,r) := \gamma \int_0^r \sigma(t,y)\,dy.$$

In particular, if the external stress is independent of the time, then the potential $V = V_0$ is authonomous and the map $t \mapsto V_0\big(x_1(t),\dots,x_N(t)\big)$ is nonincreasing in time.

We also remark that the behavior of $V_0$ at infinity changes dramatically when the fractional parameter $s$ crosses the threshold $1/2$ (this is in agreement with the strongly nonlocal interactions expected when $s < 1/2$, see [3]). Nevertheless the convexity of the functions $(0,+\infty) \ni r \mapsto r^{1-2s}/(2s-1)$ (when $s \neq 1/2$) and $-\log r$ (when $s = 1/2$), which appear in the definition of $V_0$, holds for all $s \in (0,1)$.

Finally, to state our result, we recall that the (upper and lower) semi-continuous envelopes of a function $v$ are defined as

$$v^*(t,x) := \limsup_{(t',x')\to(t,x)} v(t',x')$$

and

$$v_*(t,x) := \liminf_{(t',x')\to(t,x)} v(t',x').$$

Our main result is the following:

**Theorem 1.1.** *Assume that* (1.3), (1.4) *and* (1.6) *hold, and let*

$$v_0(t,x) = \sum_{i=1}^{N} H(\zeta_i(x-x_i(t))) - K, \tag{1.10}$$

*where $H$ is the Heaviside function and $(x_i(t))_{i=1,\dots,N}$ is the solution to* (1.8).

*Then, for every $\varepsilon > 0$ there exists a unique solution $v_\varepsilon$ to* (1.1). *Furthermore, as $\varepsilon \to 0^+$, the solution $v_\varepsilon$ exhibits the following asymptotic behavior:*

$$\limsup_{\substack{(t',x')\to(t,x)\\ \varepsilon\to 0^+}} v_\varepsilon(t',x') \leqslant (v_0)^*(t,x) \tag{1.11}$$

*and*

$$\liminf_{\substack{(t',x')\to(t,x)\\ \varepsilon\to 0^+}} v_\varepsilon(t',x') \geqslant (v_0)_*(t,x), \tag{1.12}$$

*for any $(t,x) \in [0,T_c) \times \mathbb{R}$.*

We remark that equation (1.1) is not changed by adding an integer constant to the solution, so subtracting $K$ in formula (1.10) (as well as in (1.6) for consistency) is clearly unessential. We chose this normalization in order to have that

$$\lim_{x\to-\infty} v_0(t,x) = 0 \quad\text{and}\quad \lim_{x\to+\infty} v_0(t,x) = N-K.$$

That is, the dislocation function $v_0$ is normalized to start with value 0 at $-\infty$. In this way, its value at $+\infty$ is equal to the number of the dislocations that are positive oriented.

When $K = 0$ (i.e. when all the dislocation are oriented in the same direction), the result in Theorem 1.1 has been proven in [7, 4, 3], so the novelty of Theorem 1.1 consists in treating the general case in which the dislocations occur in possibly different orientation.

The long time behavior of our problem in this case is very different from the case of positive oriented transitions. Indeed, in such situation, system (1.8) is driven by a repulsive potential, i.e. the dislocations have the tendency to repell each other, and the solution of (1.8) is defined for all the times, see [6].

On the other hand, when the dislocations do not have all the same orientations, the potential in (1.8) has two types of behaviors: it acts as a *repulsive* potential for particles with the *same* orientation, and as an *attractive* potential for particles with *opposite* orientations.

This dichotomy between the repulsive and attractive properties of the potential may lead to collisions, i.e. solutions of (1.8) may cease to exist in a finite time, due to the vanishing of the denominator. As far as we know, the present literature does not offer a complete study of system (1.8) and a full description of the collision analysis is not available. Therefore we present some concrete cases in which we can detect these collisions and estimate explicitly the collision time.

The first case that we treat in the details is the one of two initial transitions with opposite orientation, i.e. $N = 2$ and $K = 1$ in (1.6). In this case, we can estimate the collision time $T_c$ when the external stress has a sign and when the initial configuration is small (in dependence of the stress), according to the following result.

**Theorem 1.2.** *Let $N = 2$ and $K = 1$. Let $\vartheta_0 := x_2^0 - x_1^0$. Then:*

- *If $\sigma(t, x) \leqslant 0$ for any $t \geqslant 0$ and any $x \in \mathbb{R}$, then*
$$T_c \leqslant \frac{s\vartheta_0^{2s+1}}{(2s+1)\gamma}.$$
- *If*
$$\vartheta_0 < \left(\frac{1}{2s\|\sigma\|_\infty}\right)^{\frac{1}{2s}}, \tag{1.13}$$
*then*
$$T_c \leqslant \frac{s\vartheta_0^{1+2s}}{\gamma\left(2s\vartheta_0^{2s}\|\sigma\|_\infty - 1\right)}.$$
- *Conversely, if* (1.13) *is violated, there are examples in which $T_c = +\infty$.*

The next case of interest is when we have three initial dislocations that have alternate orientations. In this case, we can show that the collision time is finite if no external stress is present and we can give explicit bounds on it. Also, triple collisions occur in symmetric situations.

**Theorem 1.3.** *Let*
$$C_s := \frac{2^{2s+1}}{2^{2s} - 1} > 1.$$

*Let $N=3$, $\zeta_1=\zeta_3=+1$ and $\zeta_2=-1$, and assume that $\sigma\equiv 0$.*
*Let $\vartheta_1^0:=x_2^0-x_1^0$ and $\vartheta_2^0:=x_3^0-x_2^0$. Then*

$$T_c\in[\tau_c,\ C_s\tau_c]\,,\quad \textit{with}\ \ \tau_c:=\frac{s\min\{\vartheta_1^0,\vartheta_2^0\}^{2s+1}}{(2s+1)\gamma}.$$

*Moreover, the functions $\vartheta_1(t):=x_2(t)-x_1(t)$ and $\vartheta_2(t):=x_3(t)-x_2(t)$ are order preserving in time, i.e.*

$$\textit{if } \vartheta_1^0<\vartheta_2^0 \textit{ then } \vartheta_1(t)<\vartheta_2(t) \textit{ for every } t\in[0,T_c].$$

*Furthermore, if $\vartheta_1^0=\vartheta_2^0$, then a triple collision occurs, namely $\vartheta_1(t)=\vartheta_2(t)>0$ for every $t\in[0,T_c)$, and*

$$\vartheta_1(T_c)=\vartheta_2(T_c)=0\quad \textit{with}\ \ T_c=\frac{C_s\ s\ (\vartheta_1^0)^{2s+1}}{(2s+1)\gamma}.$$

*Viceversa, if a triple collision occurs at time $T_c$, then $\vartheta_1^0=\vartheta_2^0$ and $T_c=\dfrac{C_s\ s\ (\vartheta_1^0)^{2s+1}}{(2s+1)\gamma}$.*

Next, let us go back to the case of two initial dislocations with opposite orientation, i.e. $N=2$ and $K=1$. Suppose that a collision occurs at a time $0<T_c<+\infty$, so that if $(x_1(t),x_2(t))$ is the solution of (1.8), then $x_1(T_c)=x_2(T_c)=x_c$. Then (1.11) and (1.12) imply that for any $x\neq x_c$, we have

$$\lim_{t\to T_c^-}\lim_{\varepsilon\to 0^+} v_\varepsilon(t,x)=0.$$

This can be rephrased saying that after the collision, the two dislocations cancel each other. Nevertheless, the limit of $v_\varepsilon(t,x)$ keeps memory of them, in the sence that $v_\varepsilon$ at the point of collision $x_c$ does not vanish at the limit. Indeed, we have

**Theorem 1.4.** *Assume $N=2$ and $K=1$. Let $v_\varepsilon$ be the solution to (1.1), then*

$$\limsup_{\substack{t\to T_c^-\\ \varepsilon\to 0^+}} v_\varepsilon(t,x_c)\geqslant 1. \tag{1.14}$$

In the next two results, that are Theorems 1.5 and 1.6, we deal with the case of $N$ transitions (with, in general, $N>3$). It seems that the picture in this case can be extremely rich, so we will focus on two concrete cases: when one of the initial distance between dislocations is much smaller than the others, and when the orientations of the dislocations are alternate.

For this, we assume $\sigma\equiv 0$ and, for $i=1,...,N-1$, we consider the distance between two consecutive dislocations:

$$\begin{aligned}&\vartheta_i(t):=x_{i+1}-x_i\\ \text{and}\quad &\vartheta_i^0:=x_{i+1}^0-x_i^0>0.\end{aligned}\tag{1.15}$$

Then, recalling (1.8), we have that the $\vartheta_i$'s satisfy

(1.16)

$$\begin{cases} \dot{\vartheta}_i = \dfrac{\gamma}{2s}\left(\dfrac{2\zeta_i\zeta_{i+1}}{\vartheta_i^{2s}} + \displaystyle\sum_{j=1}^{i-1}\zeta_{i+1}\zeta_j\frac{1}{(x_{i+1}-x_j)^{2s}} - \sum_{j=i+2}^{N}\zeta_{i+1}\zeta_j\frac{1}{(x_j-x_{i+1})^{2s}}\right. \\ \quad \left. - \displaystyle\sum_{j=1}^{i-1}\zeta_i\zeta_j\frac{1}{(x_i-x_j)^{2s}} + \sum_{j=i+2}^{N}\zeta_i\zeta_j\frac{1}{(x_j-x_i)^{2s}}\right) & \text{in } (0,T_c) \\ \vartheta_i(0) = \vartheta_i^0, \end{cases}$$

$i = 1, ..., N-1$. Then we show that if two transitions with opposite orientations are sufficiently close at the initial time, then a collision in finite time occurs:

**Theorem 1.5.** *Assume $N \geqslant 2$, $K \geqslant 1$ and $\sigma \equiv 0$. Then there exists $a_0 \in (0,1)$ such that, if for some $i = 1, ..., N-1$*

$$\zeta_i\zeta_{i+1} = -1 \tag{1.17}$$

$$\textit{and} \qquad \vartheta_i^0 \leqslant a_0 \min_{j\neq i}\vartheta_j^0, \tag{1.18}$$

*then*

$$\vartheta_i(t) \leqslant a_0 \min_{j\neq i}\vartheta_j(t) \quad \textit{for any } t > 0. \tag{1.19}$$

*Moreover $\vartheta_i$ goes to zero in a finite time $T_c$, with*

$$T_c \leqslant \frac{s(\vartheta_i^0)^{2s+1}}{(2s+1)\gamma[1-(N-2)a_0^{2s}]}. \tag{1.20}$$

Some observations on Theorem 1.5 are in order. First of all, condition (1.17) states that the orientations of the $i$th and $(i+1)$th dislocations have opposite signs, and (1.18) means that the initial distance between these dislocation is small (when compared with the other dislocation distances). Then, we obtain in (1.19) that this smallness and order condition on the distances is preserved in time.

Also, we remark that the estimate of the collision time obtained in (1.20) is somehow sharp, since it reduces to the one in Theorem 1.2 when $N = 2$.

Next result deals with the alternating case, i.e. the case in which after any dislocation we have a dislocation with the opposite orientation. In this case, collisions occur, and we can estimate the collision time according to the following result:

**Theorem 1.6.** *Assume $\sigma \equiv 0$ and*

$$\zeta_i\zeta_{i+1} = -1 \tag{1.21}$$

*for any $i = 1, ..., N-1$. Then a collision occurs in a finite time $T_c$, with*

$$T_c \leqslant \frac{(N-1)(x_0^N - x_0^1)^{2s+1}}{(2s+1)\gamma} \quad \textit{if } N \textit{ is odd,}$$

*and*

$$T_c \leqslant \frac{s(x_0^N - x_0^1)^{2s+1}}{(2s+1)\gamma} \quad \textit{if } N \textit{ is even.}$$

Notice that condition (1.21) says that the dislocations have an alternate orientation (i.e. if the $i$th dislocation is positive oriented, then the $(i+1)$th is negative oriented, and viceversa).

We observe that the collision times obtained in Theorem 1.6 is bounded by the initial maximal dislocation distance to the power $2s+1$. This estimate is, in a sense, optimal, when compared with the explicit estimates in Theorems 1.2 and 1.3.

The rest of the paper is organized as follows. First, in Section 2 we give some general preliminary results and some heuristics which link the partial differential equation in (1.1) with the system of ordinary differential equations in (1.8).

Then, we deal with the analysis of the collisions of the dynamical system in (1.8), which has somehow an independent interest: we study the case of two, three and $N$ dislocations in Sections 3, 4 and 5, respectively. In this way, we also complete the proofs of Theorems 1.2, 1.3, 1.5 and 1.6.

Then, in Section 6 we prove Theorems 1.1 and 1.4.

## 2. Preliminary observations

2.1. **Toolbox.** In this section we recall some general auxiliary results that will be used in the rest of the paper. We recall that the existence of a unique solution of (1.5) is proven in [1], while asymptotic estimates for $u$ and $u'$ are given in [9]. Finer estimates on $u$ are shown in [4] and [3] respectively when $s \in \left[\frac{1}{2}, 1\right)$ and $s \in \left(0, \frac{1}{2}\right)$. We collect these results in the following

**Lemma 2.1.** *Assume that* (1.3) *holds, then there exists a unique solution* $u \in C^{2,\alpha}(\mathbb{R})$*. Moreover, there exists a constant* $C > 0$ *and* $\kappa > 2s$ *(only depending on* $s$*) such that*

$$\left| u(x) - H(x) + \frac{1}{2sW''(0)} \frac{x}{|x|^{2s}} \right| \leqslant \frac{C}{|x|^{\kappa}}, \quad \text{for } |x| \geqslant 1, \tag{2.1}$$

*and*

$$|u'(x)| \leqslant \frac{C}{|x|^{1+2s}} \quad \text{for } |x| \geqslant 1. \tag{2.2}$$

Next, we introduce the function $\psi$ to be the solution of

$$\begin{cases} \mathcal{I}_s\psi - W''(u)\psi = u' + \eta(W''(u) - W''(0)) & \text{in } \mathbb{R} \\ \psi(-\infty) = 0 = \psi(+\infty), \end{cases} \tag{2.3}$$

where $u$ is the solution of (1.5) and

$$\eta := \frac{1}{W''(0)} \int_{\mathbb{R}} (u'(x))^2 dx = \frac{1}{\gamma\beta}. \tag{2.4}$$

For a detailed heuristic motivation of such equation see Section 3.1 of [7]. The following results are proven in [4] and [3].

**Lemma 2.2.** *Assume that* (1.3) *holds, then there exists a unique solution* $\psi$ *to* (2.3)*. Furthermore* $\psi \in C^{1,\alpha}_{loc}(\mathbb{R}) \cap L^\infty(\mathbb{R})$ *for some* $\alpha \in (0,1)$ *and* $\psi' \in L^\infty(\mathbb{R})$*.*

2.2. **Heuristics of the dynamics.** We think that it could be useful to understand the heuristic derivation of (1.8) in the simpler setting of two particles with different orientations (i.e. $N = 2$ and $K = 1$).

For this, let $u$ be the solution of (1.5). Let us introduce the notation

$$u_{\varepsilon,1}(t,x) := u\left(\frac{x - x_1(t)}{\varepsilon}\right), \quad u_{\varepsilon,2}(t,x) := u\left(\frac{x_2(t) - x}{\varepsilon}\right) - 1,$$

and with a slight abuse of notation

$$u'_{\varepsilon,1}(t,x) := u'\left(\frac{x - x_1(t)}{\varepsilon}\right), \quad u'_{\varepsilon,2}(t,x) := u'\left(\frac{x_2(t) - x}{\varepsilon}\right).$$

Let us consider the following ansatz for $v_\varepsilon$

$$v_\varepsilon(t,x) \simeq u_{\varepsilon,1}(t,x) + u_{\varepsilon,2}(t,x) = u\left(\frac{x - x_1(t)}{\varepsilon}\right) + u\left(\frac{x_2(t) - x}{\varepsilon}\right) - 1.$$

Then, we compute

$$\begin{aligned}(v_\varepsilon)_t &= -u'\left(\frac{x - x_1(t)}{\varepsilon}\right)\frac{\dot{x}_1(t)}{\varepsilon} + u'\left(\frac{x_2(t) - x}{\varepsilon}\right)\frac{\dot{x}_2(t)}{\varepsilon} \\ &= -u'_{\varepsilon,1}(t,x)\frac{\dot{x}_1(t)}{\varepsilon} + u'_{\varepsilon,2}(t,x)\frac{\dot{x}_2(t)}{\varepsilon},\end{aligned}$$

and using the equation (1.5) and the periodicity of $W$

$$\begin{aligned}\mathcal{I}_s v_\varepsilon(t,x) &= \frac{1}{\varepsilon^{2s}}\mathcal{I}_s u\left(\frac{x - x_1(t)}{\varepsilon}\right) + \frac{1}{\varepsilon^{2s}}\mathcal{I}_s u\left(\frac{x_2(t) - x}{\varepsilon}\right) \\ &= \frac{1}{\varepsilon^{2s}}W'\left(u\left(\frac{x - x_1(t)}{\varepsilon}\right)\right) + \frac{1}{\varepsilon^{2s}}W'\left(u\left(\frac{x_2(t) - x}{\varepsilon}\right)\right) \\ &= \frac{1}{\varepsilon^{2s}}W'(u_{\varepsilon,1}(t,x)) + \frac{1}{\varepsilon^{2s}}W'(u_{\varepsilon,2}(t,x)).\end{aligned}$$

By inserting into (1.1), we obtain

$$-u'_{\varepsilon,1}\frac{\dot{x}_1}{\varepsilon} + u'_{\varepsilon,2}\frac{\dot{x}_2}{\varepsilon} = \frac{1}{\varepsilon^{2s+1}}\Big(W'(u_{\varepsilon,1}) + W'(u_{\varepsilon,2}) - W'(u_{\varepsilon,1} + u_{\varepsilon,2})\Big) + \frac{\sigma}{\varepsilon}. \tag{2.5}$$

Now we make some observations on the asymptotics of the potential $W$. First of all, we notice that the periodicity of $W$ and the asymptotic behavior of $u$ imply

$$\int_{\mathbb{R}} W'(u(x))u'(x)dx = \int_{\mathbb{R}} \frac{d}{dx}W(u(x))dx = W(1) - W(0) = 0, \tag{2.6}$$

and similarly

$$\int_{\mathbb{R}} W''(u(x))u'(x)dx = 0. \tag{2.7}$$

Next, we use estimate (2.1) and make a Taylor expansion of $W'$ at 0 to compute for $x \neq x_2$

$$\begin{aligned}
W'\left(u\left(\frac{x_2-x}{\varepsilon}\right)\right) &\simeq W'\left(H\left(\frac{x_2-x}{\varepsilon}\right)+\frac{\varepsilon^{2s}(x-x_2)}{2sW''(0)|x-x_2|^{1+2s}}\right)\\
&= W'\left(\frac{\varepsilon^{2s}(x-x_2)}{2sW''(0)|x-x_2|^{1+2s}}\right)\\
&\simeq W''(0)\frac{\varepsilon^{2s}(x-x_2)}{2sW''(0)|x-x_2|^{1+2s}}\\
&= \frac{\varepsilon^{2s}(x-x_2)}{2s|x-x_2|^{1+2s}}.
\end{aligned}$$

So, we use the substitution $y=(x-x_1)/\varepsilon$ to see that

$$\begin{aligned}
\frac{1}{\varepsilon}\int_{\mathbb{R}} W'(u_{\varepsilon,2}(t,x))u'_{\varepsilon,1}(t,x)dx &\simeq \frac{1}{\varepsilon}\int_{\mathbb{R}} \frac{\varepsilon^{2s}(x-x_2)}{2s|x-x_2|^{1+2s}}u'\left(\frac{x-x_1}{\varepsilon}\right)dx\\
&= \int_{\mathbb{R}} \frac{\varepsilon^{2s}(\varepsilon y+x_1-x_2)}{2s|\varepsilon y+x_1-x_2|^{1+2s}}u'(y)dy\\
&\simeq \frac{\varepsilon^{2s}(x_1-x_2)}{2s|x_1-x_2|^{1+2s}}\int_{\mathbb{R}} u'(y)dy\\
&= \frac{\varepsilon^{2s}(x_1-x_2)}{2s|x_1-x_2|^{1+2s}},
\end{aligned}$$

if $x_1 \neq x_2$. Hence

$$\frac{1}{\varepsilon^{2s+1}}\int_{\mathbb{R}} W'(u_{\varepsilon,2}(t,x))u'_{\varepsilon,1}(t,x)dx \simeq \frac{x_1-x_2}{2s|x_1-x_2|^{1+2s}}, \tag{2.8}$$

if $x_1 \neq x_2$. We use again the substitution $y=(x-x_1)/\varepsilon$, (2.6) and (2.7) to get

$$\begin{aligned}
&\frac{1}{\varepsilon}\int_{\mathbb{R}} W'(u_{\varepsilon,1}(t,x)+u_{\varepsilon,2}(t,x))u'_{\varepsilon,1}(t,x)dx\\
&\simeq \frac{1}{\varepsilon}\int_{\mathbb{R}} W'\left(u\left(\frac{x-x_1}{\varepsilon}\right)+H(x)+\frac{\varepsilon^{2s}(x-x_2)}{2sW''(0)|x-x_2|^{1+2s}}\right)u'\left(\frac{x-x_1}{\varepsilon}\right)dx\\
&= \int_{\mathbb{R}} W'\left(u(y)+\frac{\varepsilon^{2s}(\varepsilon y+x_1-x_2)}{2sW''(0)|\varepsilon y+x_1-x_2|^{1+2s}}\right)u'(y)dy\\
&\simeq \int_{\mathbb{R}} W'(u(y))u'(y)dy+\int_{\mathbb{R}} W''(u(y))\frac{\varepsilon^{2s}(\varepsilon y+x_1-x_2)}{2sW''(0)|\varepsilon y+x_1-x_2|^{1+2s}}u'(y)dy\\
&\simeq \frac{\varepsilon^{2s}(x_1-x_2)}{2sW''(0)|x_1-x_2|^{1+2s}}\int_{\mathbb{R}} W''(u(y))u'(y)dy\\
&= 0.
\end{aligned}$$

We deduce

$$\frac{1}{\varepsilon^{1+2s}}\int_{\mathbb{R}} W'(u_{\varepsilon,1}(t,x)+u_{\varepsilon,2}(t,x))u'_{\varepsilon,1}(t,x)dx \simeq 0. \tag{2.9}$$

Moreover, we have

$$\begin{aligned}\frac{1}{\varepsilon}\int_{\mathbb{R}} \sigma(t,x)u'_{\varepsilon,1}(t,x)dx &= \int_{\mathbb{R}} \sigma(t,\varepsilon y+x_1)u'(y)dy\\ &\simeq \sigma(t,x_1)\int_{\mathbb{R}} u'(y)dy\\ &= \sigma(t,x_1).\end{aligned} \tag{2.10}$$

Finally

$$\frac{1}{\varepsilon}\int_{\mathbb{R}} (u'_{\varepsilon,1}(t,x))^2dx = \int_{\mathbb{R}} (u'(y))^2dy = \gamma^{-1}, \tag{2.11}$$

and using (2.2)

$$\begin{aligned}\frac{1}{\varepsilon}\int_{\mathbb{R}} u'_{\varepsilon,1}(t,x)u'_{\varepsilon,2}(t,x)dx &\simeq \frac{1}{\varepsilon}\int_{\mathbb{R}} u'\left(\frac{x-x_1}{\varepsilon}\right)\frac{\varepsilon^{1+2s}}{|x-x_2|^{1+2s}}dx\\ &= \int_{\mathbb{R}} u'(y)\frac{\varepsilon^{1+2s}}{|\varepsilon y+x_1-x_2|^{1+2s}}dy\\ &\simeq \frac{\varepsilon^{1+2s}}{|x_1-x_2|^{1+2s}}\int_{\mathbb{R}} u'(y)dy\\ &\simeq 0,\end{aligned} \tag{2.12}$$

if $x_1 \neq x_2$. Now we multiply (2.5) by $u'_{\varepsilon,1}(t,x)$, we integrate on $\mathbb{R}$ and we use (2.6), (2.8), (2.9), (2.10), (2.11) and (2.12), to get

$$-\gamma^{-1}\dot{x}_1 = \frac{x_1-x_2}{2s|x_1-x_2|^{1+2s}} + \sigma(t,x_1).$$

A similar equation is obtained if we multiply (2.5) by $u'_{\varepsilon,2}(t,x)$ and integrate on $\mathbb{R}$. Therefore we get the system

$$\begin{cases}\dot{x}_1 = -\gamma\dfrac{x_1-x_2}{2s|x_1-x_2|^{1+2s}} - \gamma\sigma(t,x_1)\\ \dot{x}_2 = -\gamma\dfrac{x_2-x_1}{2s|x_2-x_1|^{1+2s}} + \gamma\sigma(t,x_2),\end{cases} \tag{2.13}$$

which is (1.8) with $N=2$ and $K=1$. This is a heuristic justification of the link between the partial differential equation in (1.1) and the system of ordinary differential equations in (1.8).

## 3. Two transition layers: collision in finite time and proof of Theorem 1.2

Let $(x_1(t), x_2(t))$ be the solution of (2.13) with initial condition $x_1(0) = x_1^0 < x_2(0) = x_2^0$. We want to show that under some assumptions on the external force $\sigma$ the time of collision between $x_1(t)$ and $x_2(t))$ is finite and we also explicitly estimate its value. Let us denote

$$\vartheta(t) := x_2(t) - x_1(t),$$
$$\vartheta_0 := x_2^0 - x_1^0 > 0,$$

then in an interval $(0, T_c)$, $\vartheta$ is solution of

$$\begin{cases} \dot{\vartheta} = -\dfrac{\gamma}{s\vartheta^{2s}} + \gamma\sigma(t, x_1) + \gamma\sigma(t, x_2) \\ \vartheta(0) = \vartheta_0 > 0. \end{cases} \tag{3.1}$$

Let us first assume

$$\sigma \leqslant 0.$$

In this particular case, since $\vartheta$ is subsolution of

$$\begin{cases} \dot{\vartheta} = -\dfrac{\gamma}{s\vartheta^{2s}} \\ \vartheta(0) = \vartheta_0, \end{cases} \tag{3.2}$$

in the set where $\vartheta$ is positive, we have

$$\vartheta \leqslant \tilde{\vartheta},$$

where

$$\tilde{\vartheta}(t) := \left( -\frac{2s+1}{s}\gamma t + \vartheta_0^{2s+1} \right)^{\frac{1}{2s+1}}$$

is the solution of (3.2). The function $\tilde{\vartheta}(t)$ vanishes for $t = \dfrac{s\vartheta_0^{2s+1}}{(2s+1)\gamma}$, therefore also $\vartheta$ vanishes in a finite time $T_c$ with

$$T_c \leqslant \frac{s\vartheta_0^{2s+1}}{(2s+1)\gamma}. \tag{3.3}$$

This gives the first claim in Theorem 1.2.

In the general case where no sign condition is assumed on $\sigma$, $\vartheta$ is subsolution of

$$\dot{\vartheta} = -\frac{\gamma}{s\vartheta^{2s}} + 2\gamma\|\sigma\|_\infty. \tag{3.4}$$

Equation (3.4) has the stationary solution $\vartheta_s(t) := \left(\dfrac{1}{2s\|\sigma\|_\infty}\right)^{\frac{1}{2s}}$. Therefore if (1.13) is satisfied, since $\vartheta$ cannot touch $\vartheta_s$, its derivative remains negative. Hence

$$\vartheta \leqslant \vartheta_0 \quad \text{and} \quad \dot{\vartheta} < -\frac{\gamma}{s\vartheta_0^{2s}} + 2\gamma\|\sigma\|_\infty < 0.$$

As a consequence, there exists a finite time $T_c$ such that $\vartheta(T_c) = 0$. More precisely, in this case

$$\vartheta(t) \leqslant \vartheta_0 + t\left( -\frac{\gamma}{s\vartheta_0^{2s}} + 2\gamma\|\sigma\|_\infty \right)$$

and therefore

$$T_c \leqslant \frac{\vartheta_0}{2\gamma\|\sigma\|_\infty - (\gamma/s\vartheta_0^{2s})} = \frac{s\vartheta_0^{1+2s}}{\gamma\,(2s\vartheta_0^{2s}\|\sigma\|_\infty - 1)}.$$

This proves the second claim of Theorem 1.2.

We also stress that if condition (1.13) is not satisfied (i.e. if $\vartheta_0$ is not sufficiently small with respect to the external stress), then $\vartheta$ may never vanish and $T_c$ could be infinite. This is the case, for instance, when $\sigma$ is a positive constant and $\vartheta_0 = 1/(2s\sigma)^{\frac{1}{2s}}$. This completes the proof of Theorem 1.2.

## 4. Three transition layers: proof of Theorem 1.3

Suppose that we have three dislocations, two of them moving in the same direction while the central one moving in the opposite direction. Then, system (1.8) becomes

$$\begin{cases} \dot{x}_1 = \gamma\left(-\dfrac{x_1-x_2}{2s|x_1-x_2|^{1+2s}} + \dfrac{x_1-x_3}{2s|x_1-x_3|^{1+2s}} - \sigma(t,x_1)\right) \\ \dot{x}_2 = \gamma\left(-\dfrac{x_2-x_1}{2s|x_2-x_1|^{1+2s}} - \dfrac{x_2-x_3}{2s|x_2-x_3|^{1+2s}} + \sigma(t,x_2)\right) \\ \dot{x}_3 = \gamma\left(\dfrac{x_3-x_1}{2s|x_3-x_1|^{1+2s}} - \dfrac{x_3-x_2}{2s|x_3-x_2|^{1+2s}} - \sigma(t,x_3)\right) \\ x_1(0) = x_1^0 < x_2(0) = x_2^0 < x_3(0) = x_3^0. \end{cases} \tag{4.1}$$

Let $(x_1(t), x_2(t), x_3(t))$ be the solution of (4.1) and let us denote

$$\vartheta_1(t) := x_2(t) - x_1(t), \quad \vartheta_2(t) := x_3(t) - x_2(t),$$
$$\vartheta_1^0 := x_2^0 - x_1^0, \quad \vartheta_2^0 := x_3^0 - x_2^0.$$

Then in the interval $(0, T_c)$, the function $(\vartheta_1, \vartheta_2)$ is solution of

$$\begin{cases} \dot{\vartheta}_1 = \dfrac{\gamma}{s}\left(-\dfrac{1}{\vartheta_1^{2s}} + \dfrac{1}{2(\vartheta_1+\vartheta_2)^{2s}} + \dfrac{1}{2\vartheta_2^{2s}} + \sigma(t,x_1) + \sigma(t,x_2)\right) \\ \dot{\vartheta}_2 = \dfrac{\gamma}{s}\left(\dfrac{1}{2\vartheta_1^{2s}} + \dfrac{1}{2(\vartheta_1+\vartheta_2)^{2s}} - \dfrac{1}{\vartheta_2^{2s}} - \sigma(t,x_3) - \sigma(t,x_2)\right) \\ \vartheta_1(0) = \vartheta_1^0 > 0 \\ \vartheta_2(0) = \vartheta_2^0 > 0. \end{cases} \tag{4.2}$$

Remark that in the particular case $\sigma \equiv 0$ and

$$\vartheta_1^0 = \vartheta_2^0 =: \vartheta_0$$

the solution of system (4.2) is given by

$$\vartheta_1(t) = \vartheta_2(t) = \vartheta(t)$$

where $\vartheta(t)$ is the solution of

$$\begin{cases} \dot{\vartheta} = -\dfrac{\gamma}{2^{2s+1}s}\dfrac{2^{2s}-1}{\vartheta^{2s}} \\ \vartheta(0) = \vartheta_0 > 0. \end{cases} \tag{4.3}$$

Integrating (4.3), we get the following expression of $\vartheta$:

$$\vartheta(t) = \left[\vartheta_0^{2s+1} - \gamma\frac{2s+1}{s}\frac{2^{2s}-1}{2^{2s+1}}t\right]^{\frac{1}{2s+1}}.$$

We see that $\vartheta$ vanishes at time

$$T_c = \frac{2^{2s+1}}{2^{2s}-1}\frac{s\vartheta_0^{2s+1}}{(2s+1)\gamma}, \tag{4.4}$$

and we have a triple collision.

Let us next show that if $\sigma \equiv 0$, for any choice of the initial condition $(x_1^0, x_2^0, x_3^0)$ we have a collision in a finite time, and also that $\vartheta_1$ and $\vartheta_2$ are order preserving, i.e. if, for instance,

$$\vartheta_1^0 < \vartheta_2^0, \tag{4.5}$$

then

$$\vartheta_1(t) < \vartheta_2(t) \tag{4.6}$$

for any positive $t$ smaller than the collision time. Indeed, if there exists $t_0$ such that $\vartheta_1(t_0) = \vartheta_2(t_0)$, and we look at the solution $(\widetilde{\vartheta}_1(t), \widetilde{\vartheta}_2(t))$ of system (4.2) with initial condition $\vartheta_1^0 = \vartheta_2^0 = \vartheta_1(t_0)$, then by the uniqueness of the solution of the system, we have

$$(\vartheta_1(t+t_0), \vartheta_2(t+t_0)) = (\widetilde{\vartheta}_1(t), \widetilde{\vartheta}_2(t))$$

and we know that $\widetilde{\vartheta}_1(t) = \widetilde{\vartheta}_2(t)$ for any $t$ smaller than the collision time. This is in contradiction with (4.5) and it proves (4.6).

In turn, inequality (4.6) implies that $\vartheta_1(t)$ is subsolution of the equation (4.3) with initial condition $\vartheta_1(0) = \vartheta_1^0$. Therefore we have

$$\vartheta_1(t) \leqslant \overline{\vartheta}_1(t) := \left[(\vartheta_1^0)^{2s+1} - \gamma\frac{2s+1}{s}\frac{2^{2s}-1}{2^{2s+1}}t\right]^{\frac{1}{2s+1}}.$$

In particular, the collision time $T_c$ of the system (4.2) is finite and

$$T_c \leqslant \frac{2^{2s+1}}{2^{2s}-1}\frac{s(\vartheta_1^0)^{2s+1}}{(2s+1)\gamma}.$$

Next, since $\vartheta_1(t)$ is supersolution of the equation (3.2), we have

$$\vartheta_1(t) \geqslant \underline{\vartheta}_1(t) := \left[(\vartheta_1^0)^{2s+1} - \frac{2s+1}{s}\gamma t\right]^{\frac{1}{2s+1}}$$

and therefore

$$T_c \geqslant \frac{s\,(\vartheta_1^0)^{2s+1}}{(2s+1)\gamma}.$$

Finally, suppose that a triple collision occurs at some time $T_c$. We want to show that $\vartheta_1(t) = \vartheta_2(t)$ for all $t \in [0, T_c)$ and determine $T_c$. For this, suppose, by contradiction, that

$$\vartheta_1(t_0) < \vartheta_2(t_0). \tag{4.7}$$

Then, by considering $t_0$ the initial time of the flow, we deduce from (4.6) that $\vartheta_1(t) < \vartheta_2(t)$ for every $t \in [t_0, T_c)$. Using this and (4.2), we see that

$$\dot{\vartheta}_2 - \dot{\vartheta}_1 = \frac{\gamma}{s}\left(\frac{3}{2\vartheta_1^{2s}} - \frac{3}{2\vartheta_2^{2s}}\right) > 0$$

for every $t \in [t_0, T_c)$. As a consequence, for any fixed $a \in (0, T_c)$,

$$(\vartheta_2 - \vartheta_1)(T_c - a) = (\vartheta_2 - \vartheta_1)(t_0) + \int_{t_0}^{T_c - a} (\dot{\vartheta}_2 - \dot{\vartheta}_1)(t)\, dt > (\vartheta_2 - \vartheta_1)(t_0).$$

This and (4.7) are in contradiction with the fact that

$$\lim_{a \to 0^+} (\vartheta_2 - \vartheta_1)(T_c - a) = 0,$$

and so we have proved that $\vartheta_1(t) = \vartheta_2(t)$ for all $t \in [0, T_c)$. In particular, we have that $\vartheta_1^0 = \vartheta_2^0$ and so the collision time is determined by (4.4). This completes the proof of Theorem 1.3.

**Remark 4.1.** If the three dislocations are not alternated, i.e., $x_1$ and $x_2$ move in the same direction, while $x_3$ in the opposite one, then $x_2$ and $x_3$ collide in a finite time $T_c$ satisfying (3.3). Indeed, in this case the repulsion between $x_1$ and $x_2$ and the attraction between $x_2$ and $x_3$ contribute positively to the collision.

## 5. $N$ transition layers: some special cases and proof of Theorems 1.5 and 1.6

Now we deal with the case of $N$ transition layers. Since the general picture can be very rich to describe, we focus on the cases of small initial configuration and alternate orientations, and we prove Theorems 1.5 and 1.6.

5.1. **Proof of Theorem 1.5.** We fix $a_0 > 0$ small enough such that

$$-1 + (N-2)a_0^{2s} + (N-1)a_0^{2s+1} < 0. \tag{5.1}$$

Let us denote

$$\vartheta_{\rm m}(t) := \min_{j \neq i} \vartheta_j(t).$$

Of course, no confusion arises between the subscript m, that denotes this minimization and the indices $i$ and $j$. Also, by (1.18) and (5.1), we have that

$$-1 + (N-2)\frac{\vartheta_i^{2s}(0)}{\vartheta_{\rm m}^{2s}(0)} \leqslant -1 + (N-2)a_0^{2s} < 0. \tag{5.2}$$

We want to show that for any $t > 0$

$$\frac{\vartheta_i(t)}{\vartheta_{\rm m}(t)} \leqslant a_0. \tag{5.3}$$

From system (1.16), we infer that $\vartheta_i$ satisfies

$$\dot{\vartheta}_i(t) \leqslant \frac{\gamma}{s}\left(-\frac{1}{\vartheta_i^{2s}} + \frac{N-2}{\vartheta_{\rm m}^{2s}}\right) = \frac{\gamma}{s\vartheta_i^{2s}}\left(-1 + (N-2)\frac{\vartheta_i^{2s}}{\vartheta_{\rm m}^{2s}}\right), \tag{5.4}$$

while for any $j \neq i$

$$\dot{\vartheta}_j(t) \geqslant -\frac{\gamma(N-1)}{s\vartheta_{\rm m}^{2s}}.$$

From (5.2) and (5.4) we deduce that there exists $T > 0$, that we choose maximal, such that

$$\dot{\vartheta}_i(t) \leqslant 0 \text{ for any } t \in (0, T). \tag{5.5}$$

Moreover, in $(0,T)$ we have that

$$\begin{aligned}\frac{d}{dt}\left(\frac{\vartheta_i}{\vartheta_j}\right) &= \frac{\dot\vartheta_i\vartheta_j - \vartheta_i\dot\vartheta_j}{\vartheta_j^2}\\ &\leqslant \frac{\dot\vartheta_i\vartheta_{\rm m} - \vartheta_i\dot\vartheta_j}{\vartheta_j^2}\\ &\leqslant \frac{\gamma}{s\vartheta_j^2}\left(-\frac{\vartheta_{\rm m}}{\vartheta_i^{2s}} + \frac{(N-2)\vartheta_{\rm m}}{\vartheta_{\rm m}^{2s}} + \frac{(N-1)\vartheta_i}{\vartheta_{\rm m}^{2s}}\right)\\ &= \frac{\gamma\vartheta_{\rm m}}{s\vartheta_j^2\vartheta_i^{2s}}\left(-1+(N-2)\frac{\vartheta_i^{2s}}{\vartheta_{\rm m}^{2s}} + (N-1)\frac{\vartheta_i^{2s+1}}{\vartheta_{\rm m}^{2s+1}}\right).\end{aligned}$$

Integrating in $(0,t)$ and passing to the minimum on $j$, we infer that for any $t\in(0,T)$

$$\frac{\vartheta_i(t)}{\vartheta_{\rm m}(t)} \leqslant a_0 + \min_j \int_0^t \frac{\gamma\vartheta_{\rm m}(\tau)}{s\vartheta_j^2(\tau)\vartheta_i^{2s}(\tau)}\left(-1+(N-2)\frac{\vartheta_i^{2s}(\tau)}{\vartheta_{\rm m}^{2s}(\tau)} + (N-1)\frac{\vartheta_i^{2s+1}(\tau)}{\vartheta_{\rm m}^{2s+1}(\tau)}\right)d\tau. \tag{5.6}$$

Let us call

$$g(\tau) := -1+(N-2)\frac{\vartheta_i^{2s}(\tau)}{\vartheta_{\rm m}^{2s}(\tau)} + (N-1)\frac{\vartheta_i^{2s+1}(\tau)}{\vartheta_{\rm m}^{2s+1}(\tau)}.$$

We observe that $g(0)<0$, thanks to (5.1). Thus, we want to show that

$$g(\tau)<0 \text{ for any } \tau\in(0,T). \tag{5.7}$$

Assume by contradiction that this is not true. Then there exists $t_0\in(0,T)$ such that

$$g(\tau)<0 \text{ for } \tau\in(0,t_0) \tag{5.8}$$

and $g(t_0)=0$. Then $\frac{\vartheta_i(t_0)}{\vartheta_{\rm m}(t_0)}=k$ with

$$-1+(N-2)k^{2s}+(N-1)k^{2s+1}=0. \tag{5.9}$$

On the other hand, by (5.4) and (5.8), we see that

$$\dot\vartheta_i < \frac{\gamma}{s\vartheta_i^{2s}}g<0$$

in $(0,t_0)$, and therefore, recalling (5.5), we conclude that $t_0<T$. In particular, we can use (5.6) with $t:=t_0$.

Thus, from (5.6) and (5.8) we infer that

$$k=\frac{\vartheta_i(t_0)}{\vartheta_{\rm m}(t_0)} \leqslant a_0+\min_j\int_0^t \frac{\gamma\vartheta_{\rm m}(\tau)}{s\vartheta_j^2(\tau)\vartheta_i^{2s}(\tau)}\,g(\tau)\,d\tau < a_0.$$

This and (5.9) give that

$$0=-1+(N-2)k^{2s}+(N-1)k^{2s+1} < -1+(N-2)a_0^{2s}+(N-1)a_0^{2s+1}$$

and this is in contradiction with (5.1). Therefore we have completed the proof of (5.7).

In turn, we see that (5.6) and (5.7) imply (5.3), and thus (1.19).

Finally (5.3) and (5.4) yield that

$$\dot\vartheta_i(t)\leqslant -\frac{\gamma[1-(N-2)a_0^{2s}]}{s\vartheta_i^{2s}},$$

and therefore $\vartheta_i$ goes to zero in a time $T_c$ satisfying (1.20). Thus the proof of Theorem 1.5 is complete.

5.2. **Proof of Theorem 1.6.** Without loss of generality, we can assume $\zeta_1 = 1$. Let us first assume $N$ odd. Then $\zeta_N = 1$, and from (1.8) and (1.15) we get

$$\begin{aligned}\dot{x}_N - \dot{x}_1 &= \frac{\gamma}{2s}\sum_{j=1}^{N-1}\frac{\zeta_j}{(x_N - x_j)^{2s}} + \frac{\gamma}{2s}\sum_{j=2}^{N}\frac{\zeta_j}{(x_j - x_1)^{2s}} \\ &= \frac{\gamma}{2s}\Bigg[\frac{1}{(\vartheta_1 + \cdots + \vartheta_{N-1})^{2s}} - \frac{1}{(\vartheta_2 + \cdots + \vartheta_{N-1})^{2s}} + \cdots + \frac{1}{(\vartheta_{N-2} + \vartheta_{N-1})^{2s}} - \frac{1}{\vartheta_{N-1}^{2s}} \\ &\quad - \frac{1}{\vartheta_1^{2s}} + \frac{1}{(\vartheta_1 + \vartheta_2)^{2s}} - \cdots - \frac{1}{(\vartheta_1 + \cdots + \vartheta_{N-2})^{2s}} + \frac{1}{(\vartheta_1 + \cdots + \vartheta_{N-1})^{2s}}\Bigg].\end{aligned}$$

So, for every $\ell \in \{1, \ldots, N-2\}$, we introduce the notation

$$\begin{aligned}\alpha_\ell &:= \frac{1}{(\vartheta_{\ell+1} + \cdots + \vartheta_{N-1})^{2s}} - \frac{1}{(\vartheta_\ell + \cdots + \vartheta_{N-1})^{2s}} \\ \text{and} \qquad \beta_\ell &:= \frac{1}{(\vartheta_1 + \cdots + \vartheta_\ell)^{2s}} - \frac{1}{(\vartheta_1 + \cdots + \vartheta_{\ell+1})^{2s}}.\end{aligned}$$

In this way, we have that $\alpha_\ell$, $\beta_\ell \geqslant 0$ and

$$\dot{x}_N - \dot{x}_1 = -\frac{\gamma}{2s}\sum_{\ell=1}^{N-2}(\alpha_\ell + \beta_\ell). \tag{5.10}$$

Moreover, for any $a$, $b \geqslant 0$ and any $\xi \in [0,1]$ we have that

$$(a+b)\cdot\frac{(\xi a + b)^{2s-1}}{b^{2s}} \geqslant (\xi a + b)\cdot\frac{(\xi a + b)^{2s-1}}{b^{2s}} = \frac{(\xi a + b)^{2s}}{b^{2s}} \geqslant 1. \tag{5.11}$$

Thus, using a Taylor expansion we see that there exists $\xi_\ell \in [0,1]$ such that

$$\begin{aligned}\alpha_\ell &= \frac{(\vartheta_\ell + \cdots + \vartheta_{N-1})^{2s} - (\vartheta_{\ell+1} + \cdots + \vartheta_{N-1})^{2s}}{(\vartheta_{\ell+1} + \cdots + \vartheta_{N-1})^{2s}(\vartheta_\ell + \cdots + \vartheta_{N-1})^{2s}} \\ &= \frac{2s(\xi_\ell\vartheta_\ell + \vartheta_{\ell+1} + \cdots + \vartheta_{N-1})^{2s-1}\vartheta_\ell}{(\vartheta_{\ell+1} + \cdots + \vartheta_{N-1})^{2s}(\vartheta_\ell + \cdots + \vartheta_{N-1})^{2s}} \\ &\geqslant \frac{2s\vartheta_\ell}{(\vartheta_\ell + \cdots + \vartheta_{N-1})^{1+2s}},\end{aligned} \tag{5.12}$$

where we have used (5.11) here with $\xi := \xi_\ell$, $a := \vartheta_\ell$ and $b := \vartheta_{\ell+1} + \cdots + \vartheta_N$.

Similarly, using (5.11) with $a := \vartheta_{\ell+1}$ and $b := \vartheta_1 + \cdots + \vartheta_\ell$, we see that

$$\beta_\ell \geqslant \frac{2s\vartheta_{\ell+1}}{(\vartheta_1 + \cdots + \vartheta_{\ell+1})^{1+2s}}. \tag{5.13}$$

From (5.12) and (5.13) we obtain that

$$\alpha_\ell + \beta_\ell \geqslant \frac{2s\,(\vartheta_\ell + \vartheta_{\ell+1})}{(\vartheta_1 + \cdots + \vartheta_{N-1})^{1+2s}}. \tag{5.14}$$

Now, for any fix $t > 0$ let $j(t) \in \{1, \ldots, N-1\}$ be such that

$$\vartheta_{j(t)}(t) = \max_{j=1,\ldots,N-1}\vartheta_j(t).$$

Then, at time $t$ we have that

$$\vartheta_1 + \cdots + \vartheta_{N-1} \leqslant (N-1)\vartheta_{j(t)}$$

and so (5.14) implies that

$$\alpha_\ell + \beta_\ell \geqslant \frac{2s\,(\vartheta_\ell + \vartheta_{\ell+1})}{(N-1)\,\vartheta_{j(t)}\,(\vartheta_1 + \cdots + \vartheta_{N-1})^{2s}},$$

for every $\ell \in \{1, \ldots, N-2\}$. In particular, we can choose either $\ell(t) := j(t)$ (if $j(t) \neq N-1$) or $\ell(t) := j(t) - 1$ (if $j(t) = N-1$) and obtain that

$$\begin{aligned} \alpha_{\ell(t)} + \beta_{\ell(t)} &\geqslant \frac{2s\,\vartheta_{j(t)}}{(N-1)\,\vartheta_{j(t)}\,(\vartheta_1 + \cdots + \vartheta_{N-1})^{2s}} \\ &= \frac{2s}{(N-1)\,(\vartheta_1 + \cdots + \vartheta_{N-1})^{2s}}. \end{aligned}$$

This and (5.10) yield that, for any time $t$ before collisions, we have

$$\dot{x}_N - \dot{x}_1 \leqslant -\frac{\gamma}{(N-1)(x_N - x_1)^{2s}}.$$

Since the solution of

$$\begin{cases} \dot{\vartheta} = -\dfrac{\gamma}{(N-1)\vartheta^{2s}} \\ \vartheta(0) = \vartheta_0 > 0 \end{cases}$$

vanishes at the time $t = \frac{(N-1)\vartheta_0^{2s+1}}{(2s+1)\gamma}$, we can conclude that a collision ocurs at some time $T_c$ with

$$T_c \leqslant \frac{(N-1)(x_0^N - x_0^1)^{2s+1}}{(2s+1)\gamma}.$$

The case $N$ even is simpler, thanks to direct cancellations. Indeed in this case, from (1.8) and (1.15), we have

$$\begin{aligned} \dot{x}_N - \dot{x}_1 &= -\frac{\gamma}{2s}\sum_{j=1}^{N-1}\frac{\zeta_j}{(x_N - x_j)^{2s}} + \frac{\gamma}{2s}\sum_{j=2}^{N}\frac{\zeta_j}{(x_j - x_1)^{2s}} \\ &= \frac{\gamma}{2s}\Bigg[-\frac{1}{(\vartheta_1 + \cdots + \vartheta_{N-1})^{2s}} + \cdots + \frac{1}{(\vartheta_{N-2} + \vartheta_{N-1})^{2s}} - \frac{1}{\vartheta_{N-1}^{2s}} \\ &\quad -\frac{1}{\vartheta_1^{2s}} + \frac{1}{(\vartheta_1 + \vartheta_2)^{2s}} - \cdots - \frac{1}{(\vartheta_1 + \cdots + \vartheta_{N-1})^{2s}}\Bigg] \\ &\leqslant -\frac{\gamma}{s(\vartheta_1 + \cdots + \vartheta_{N-1})^{2s}} \\ &= -\frac{\gamma}{s(x_N - x_1)^{2s}}. \end{aligned}$$

Therefore, a collision occurs in a time $T_c$ with

$$T_c \leqslant \frac{s(x_0^N - x_0^1)^{2s+1}}{(2s+1)\gamma},$$

which completes the proof of Theorem 1.6.

## 6. Proof of Theorems 1.1 and 1.4

6.1. **Proof of Theorem 1.1.** As in [7, 4, 3], the proof of Theorem 1.1 relies on the construction of suitable barriers that allow the use of Perron's method. Since in our case the different transitions not need to be all oriented in the same direction, some care is needed in order to take into account the cancellations arising from the different signs of the $\zeta_i$'s.

More concretely, to prove the asymptotic behavior of $v_\varepsilon$, namely inequalities (1.11) and (1.12), we construct suitable sub and supersolutions of (1.1). We consider an auxiliary small parameter $\delta > 0$ and define $(\overline{x}_1(t), ..., \overline{x}_N(t))$ to be the solution of the system

$$\begin{cases} \dot{\overline{x}}_i = \gamma \left( \sum_{j \neq i} \zeta_i \zeta_j \frac{\overline{x}_i - \overline{x}_j}{2s|\overline{x}_i - \overline{x}_j|^{1+2s}} - \zeta_i \sigma(t, \overline{x}_i) - \zeta_i \delta \right) & \text{in } (0, T_c - t_\delta) \\ \overline{x}_i(0) = x_i^0 - \zeta_i \delta, \end{cases} \tag{6.1}$$

$i = 1, ..., N$. Here $T_c$ is the collision time of the system (1.8). If we call $T_c^\delta$ the collision time of the perturbed system (6.1), then

$$\liminf_{\delta \to 0^+} T_c^\delta \geqslant T_c. \tag{6.2}$$

To check this, fix $a \in (0, T_c)$, to be taken arbitrarily small in the sequel. Then the solution of system (1.8) satisfies

$$m_a := \min_{\substack{t \in [0, T_c - a] \\ 1 \leqslant i \neq j \leqslant N}} |x_i(t) - x_j(t)| > 0.$$

Accordingly the right hand side of the equation in (1.8) (together with its derivatives) is bounded when $t \in [0, T_c - a]$ by a quantity that depends on $a$. Therefore, we are in the position to apply the continuity result of the solution with respect to the parameter $\delta$: we obtain that there exists $\delta_a > 0$ such that, when $\delta \in (0, \delta_a)$ the trajectories of (6.1) lie in a $(m_a/2)$-neighborhood of the trajectories of (1.8). In particular, for any $\delta \in (0, \delta_a)$, we have that

$$\min_{\substack{t \in [0, T_c - a] \\ 1 \leqslant i \neq j \leqslant N}} |\overline{x}_i(t) - \overline{x}_j(t)| \geqslant \frac{m_a}{2}$$

and so the corresponding collision time cannot occur before $T_c - a$. That is $T_c^\delta \geqslant T_c - a$ for all $\delta \in (0, \delta_a)$, and so

$$\liminf_{\delta \to 0^+} T_c^\delta \geqslant T_c - a.$$

By taking $a$ as close as we wish to 0, we obtain (6.2).

In light of (6.2), for $\delta$ small enough, we have that (6.1) is well defined in $(0, T_c - t_\delta)$ where $t_\delta \to 0^+$ as $\delta \to 0^+$. Next, we set

$$\overline{c}_i(t) := \dot{\overline{x}}_i(t), \quad i = 1, ..., N \tag{6.3}$$

and

$$\overline{\sigma} := \frac{\sigma + \delta}{W''(0)}. \tag{6.4}$$

Let $u$ and $\psi$ be respectively the solution of (1.5) and (2.3). We define

$$\overline{v}_\varepsilon(t, x) := \varepsilon^{2s} \overline{\sigma}(t, x) - K + \sum_{i=1}^N u\left(\zeta_i \frac{x - \overline{x}_i(t)}{\varepsilon}\right) - \sum_{i=1}^N \zeta_i \varepsilon^{2s} \overline{c}_i(t) \psi\left(\zeta_i \frac{x - \overline{x}_i(t)}{\varepsilon}\right). \tag{6.5}$$

In order to simplify the notation, we set, for $i = 1, ..., N$

$$\tilde{u}_i(t,x) := u\left(\zeta_i \frac{x - \overline{x}_i(t)}{\varepsilon}\right) - H\left(\zeta_i \frac{x - \overline{x}_i(t)}{\varepsilon}\right), \tag{6.6}$$

and

$$\psi_i(t,x) := \psi\left(\zeta_i \frac{x - \overline{x}_i(t)}{\varepsilon}\right).$$

Finally, let

$$I_\varepsilon := \varepsilon(\overline{v}_\varepsilon)_t + \frac{1}{\varepsilon^{2s}}(W'(\overline{v}_\varepsilon) - \varepsilon^{2s}\mathcal{I}_s \overline{v}_\varepsilon - \varepsilon^{2s}\sigma). \tag{6.7}$$

The next two propositions show that $\overline{v}_\varepsilon$ is a supersolution of (1.1).

**Proposition 6.1.** *For any $T < T_c - t_\delta$, there exists $\varepsilon_0 > 0$ such that for $0 < \varepsilon \leqslant \varepsilon_0$, we have*

$$(\overline{v}_\varepsilon)_t \geqslant \frac{1}{\varepsilon}\left(\mathcal{I}_s \overline{v}_\varepsilon - \frac{1}{\varepsilon^{2s}} W'(\overline{v}_\varepsilon) + \sigma(t,x)\right) \quad \textit{in } (0,T) \times \mathbb{R}.$$

**Proposition 6.2.** *There exists $\delta_0 > 0$ such that, for every $0 < \delta \leqslant \delta_0$, we have*

$$\overline{v}_\varepsilon(0,x) \geqslant v^0_\varepsilon(x) \quad \textit{for any } x \in \mathbb{R}.$$

We have the following asymptotic behavior for $\overline{v}_\varepsilon$:

**Lemma 6.3.** *For any $(t,x) \in [0,T_c) \times \mathbb{R}$, we have that*

$$\lim_{\delta \to 0^+} \limsup_{\substack{(t',x') \to (t,x) \\ \varepsilon \to 0^+}} \overline{v}_\varepsilon(t',x') \leqslant (v_0)^*(t,x).$$

The proof of Proposition 6.1 is postponed to the next Section 6.3, to avoid interruptions in the flow of the main arguments, while for the proofs of Lemma 6.3 and Proposition 6.2 we refer respectively to the proofs of Lemma 8.1 and Proposition 8.2 in [4].

Let us now conclude the proof of Theorem 1.1. First remark that, for $\varepsilon$ sufficiently small, the initial condition $v^0_\varepsilon$ given in (1.6) satisfies

$$-(N+1) \leqslant v^0_\varepsilon \leqslant N+1.$$

Moreover the functions

$$\underline{u}_\varepsilon(t,x) := -(N+1) - K_\varepsilon t \quad \text{and} \quad \overline{u}_\varepsilon(t,x) := N+1+K_\varepsilon t$$

where

$$K_\varepsilon := \frac{1}{\varepsilon^{1+2s}}\|W'\|_{L^\infty(\mathbb{R})} + \frac{1}{\varepsilon}\|\sigma\|_{L^\infty(\mathbb{R})},$$

are respectively sub and supersolution of (1.1). Hence, the existence of a unique, continuous solution $v_\varepsilon$ of (1.1) is guaranteed by the Perron's method and the comparison principle.

Next, from Propositions 6.1 and 6.2, and the comparison principle, for any $T < T_c$ there exist $\delta_0$ and $\varepsilon_0$ such that for $0 < \delta \leqslant \delta_0$ and $0 < \varepsilon \leqslant \varepsilon_0$, we have

$$v_\varepsilon(t,x) \leqslant \overline{v}_\varepsilon(t,x) \quad \text{for any } (t,x) \in (0,T) \times \mathbb{R}. \tag{6.8}$$

Passing to the limit as $\varepsilon \to 0^+$, recalling Lemma 6.3 and taking $\delta$ as small as we wish in the end, we get (1.11) for any $(t,x) \in [0,T_c) \times \mathbb{R}$.

Similarly, to prove (1.12), for $\delta > 0$ small, we define $(\underline{x}_1(t), ..., \underline{x}_N(t))$ to be the solution of the system

$$\begin{cases} \dot{\underline{x}}_i = \gamma \left( \sum_{j\neq i} \zeta_i\zeta_j \frac{\underline{x}_i - \underline{x}_j}{2s|\underline{x}_i - \underline{x}_j|^{1+2s}} - \zeta_i\sigma(t, \underline{x}_i) + \zeta_i\delta \right) & \text{in } (0, T_c - t_\delta) \\ \underline{x}_i(0) = x_i^0 + \zeta_i\delta, \end{cases} \tag{6.9}$$

$i = 1, ..., N$, and

$$\underline{v}_\varepsilon(t, x) := \varepsilon^{2s} \frac{\sigma(t, x) - \delta}{W''(0)} - K + \sum_{i=1}^{N} u\left(\zeta_i \frac{x - \underline{x}_i(t)}{\varepsilon}\right) - \sum_{i=1}^{N} \zeta_i\varepsilon^{2s}\dot{\underline{x}}_i(t)\psi\left(\zeta_i \frac{x - \underline{x}_i(t)}{\varepsilon}\right).$$

Then, one can prove that $\underline{v}_\varepsilon$ is a subsolution of (1.1) and therefore

$$v_\varepsilon(t, x) \geqslant \underline{v}_\varepsilon(t, x) \quad \text{for any } (t, x) \in (0, T) \times \mathbb{R}, \tag{6.10}$$

and any $T < T_c$, and any $\delta$ and $\varepsilon$ small enough. Passing to the limit as $\varepsilon \to 0^+$ and then letting $\delta \to 0^+$, we get (1.12), thus ending the proof of Theorem 1.1.

6.2. **Proof of Theorem 1.4.** Let us take a sequence $(T_k)_k$ such that $T_k \to T_c^-$ as $k \to +\infty$. Then, from (6.10) with $N = 2$ and $K = 1$, there exist $\delta_k^0$ and $\varepsilon_k^0$ such that for any $\delta \in (0, \delta_k^0]$ and $\varepsilon \in (0, \varepsilon_k^0]$ we have

$$\begin{aligned} v_\varepsilon(T_k, x_c) \geqslant\; & O(\varepsilon^{2s}) + u\left(\frac{x_c - \underline{x}_1(T_k)}{\varepsilon}\right) + u\left(\frac{\underline{x}_2(T_k) - x_c}{\varepsilon}\right) - 1 \\ & - \varepsilon^{2s}\dot{\underline{x}}_1(T_k)\psi\left(\frac{x_c - \underline{x}_1(T_k)}{\varepsilon}\right) + \varepsilon^{2s}\dot{\underline{x}}_2(T_k)\psi\left(\frac{\underline{x}_2(T_k) - x_c}{\varepsilon}\right). \end{aligned} \tag{6.11}$$

We remark that $x_1(t) < x_2(t)$ for any $t \in (0, T_c)$, and that both $x_1(t)$ and $x_2(t)$ approach $x_c$ as $t \to T_c^-$. Consequently, by (1.8), we see that

$$\dot{x}_1 \geqslant \gamma \left( \frac{1}{2s(x_2 - x_1)|^{2s}} - \|\sigma_x\|_{L^\infty([0,+\infty)\times\mathbb{R})} \right) \to +\infty$$

as $t \to T_c^-$. Similarly, we have that $\dot{x}_2 \to -\infty$ as $\to T_c^-$.

We deduce that $x_1$ is definitely incrasing in time, and $x_2$ definitely decreasing. In particular, we have that $x_1(t) < x_c < x_2(t)$ when $t$ is close enough to $T_c$ (and so for $t = T_k$ and $k$ large enough).

Therefore, we can take $\delta = \delta_k > 0$ sufficiently small that

$$\underline{x}_1(T_k) < x_c < \underline{x}_2(T_k).$$

As a consequence, we can choose $\varepsilon = \varepsilon_k > 0$ so small that

$$\frac{x_c - \underline{x}_1(T_k)}{\varepsilon_k}, \; \frac{\underline{x}_2(T_k) - x_c}{\varepsilon_k} \to +\infty \quad \text{as } k \to +\infty.$$

Then, by (6.9), we have that

$$\varepsilon_k^{2s}\dot{\underline{x}}_1(T_k) = \frac{\gamma}{2s\left(\frac{\underline{x}_2(T_k) - x_c}{\varepsilon_k} + \frac{x_c - \underline{x}_1(T_k)}{\varepsilon_k}\right)^{2s}} + O(\varepsilon_k^{2s}) \to 0 \quad \text{as } k \to +\infty.$$

Similarly

$$\varepsilon_k^{2s}\dot{\underline{x}}_2(T_k) \to 0 \quad \text{as } k \to +\infty.$$

Thus, recalling (6.11), we infer that

$$\limsup_{k\to+\infty} v_{\varepsilon_k}(T_k, x_c) \geqslant 1.$$

This implies that

$$\limsup_{\substack{t\to T_c^- \\ \varepsilon\to 0^+}} v_\varepsilon(t, x_c) \geqslant 1,$$

which concludes the proof of Theorem 1.4.

6.3. **Proof of Proposition 6.1.** Let us start with the following

**Lemma 6.4.** *For any $T < T_c - t_\delta$ in $(0,T)\times\mathbb{R}$ we have, for $i = 1, ..., N$*

$$\begin{aligned} I_\varepsilon &= O(\tilde{u}_i)(\varepsilon^{-2s}\sum_{j\neq i}\tilde{u}_j + \overline{\sigma} + \zeta_i\overline{c}_i\eta) + \delta \\ &\quad + \sum_{j\neq i}\big(O(\psi_j) + O(\tilde{u}_j) + O(\varepsilon^{-2s}\tilde{u}_j^2)\big) + O(\varepsilon^{2s}), \end{aligned} \tag{6.12}$$

*where $O(\varepsilon^{2s})$ depends on $T$ and $\delta$.*

*Proof.* Fix $i = 1, ..., N$. We have

$$\begin{aligned} \varepsilon(\overline{v}_\varepsilon)_t &= \varepsilon^{2s+1}\overline{\sigma}_t - \sum_{j=1}^N \zeta_j\overline{c}_j u'\left(\zeta_j\frac{x-\overline{x}_j}{\varepsilon}\right) \\ &\quad + \sum_{j=1}^N\left(-\zeta_j\varepsilon^{2s+1}\dot{\overline{c}}_j\psi\left(\zeta_j\frac{x-\overline{x}_j}{\varepsilon}\right) + \zeta_j\varepsilon^{2s}\overline{c}_j^2\psi'\left(\zeta_j\frac{x-\overline{x}_j}{\varepsilon}\right)\right) \\ &= -\sum_{j=1}^N \zeta_j\overline{c}_j u'\left(\zeta_j\frac{x-\overline{x}_j}{\varepsilon}\right) + O(\varepsilon^{2s}). \end{aligned} \tag{6.13}$$

Next, using the periodicity of $W$ and a Taylor expansion of $W'$ at $\tilde{u}_i$, we compute:

$$\begin{aligned} \varepsilon^{-2s}W'(\overline{v}_\varepsilon) &= \varepsilon^{-2s}W'\left(\varepsilon^{2s}\overline{\sigma} + \sum_{j\neq i}\tilde{u}_j + \tilde{u}_i - \sum_{j\neq i}\zeta_j\varepsilon^{2s}\overline{c}_j\psi_j - \zeta_i\varepsilon^{2s}\overline{c}_i\psi_i\right) \\ &= \varepsilon^{-2s}W'(\tilde{u}_i) + \varepsilon^{-2s}W''(\tilde{u}_i)(\varepsilon^{2s}\overline{\sigma} + \sum_{j\neq i}\tilde{u}_j - \sum_{j\neq i}\zeta_j\varepsilon^{2s}\overline{c}_j\psi_j - \zeta_i\varepsilon^{2s}\overline{c}_i\psi_i) \\ &\quad + \sum_{j\neq i}O(\varepsilon^{-2s}\tilde{u}_j^2) + O(\varepsilon^{2s}). \end{aligned} \tag{6.14}$$

Finally, we evaluate

$$
\begin{aligned}
\mathcal{I}_s\overline{v}_\varepsilon &= \varepsilon^{2s}\mathcal{I}_s\overline{\sigma} + \varepsilon^{-2s}\sum_{j\neq i}\mathcal{I}_s u\left(\zeta_j\frac{x-\overline{x}_j}{\varepsilon}\right) + \varepsilon^{-2s}\mathcal{I}_s u\left(\zeta_i\frac{x-\overline{x}_i}{\varepsilon}\right)\\
&\quad - \sum_{j\neq i}\zeta_j\overline{c}_j\mathcal{I}_s\psi\left(\zeta_j\frac{x-\overline{x}_j}{\varepsilon}\right) - \zeta_i\overline{c}_i\mathcal{I}_s\psi\left(\zeta_i\frac{x-\overline{x}_i}{\varepsilon}\right)\\
&= O(\varepsilon^{2s}) + \varepsilon^{-2s}\sum_{j\neq i}W'(\tilde{u}_j) + \varepsilon^{-2s}W'(\tilde{u}_i)\\
&\quad - \sum_{j\neq i}\zeta_j\overline{c}_j\left[W''(\tilde{u}_j)\psi_j + u'\left(\zeta_j\frac{x-\overline{x}_j}{\varepsilon}\right) + \eta(W''(\tilde{u}_j)-W''(0))\right]\\
&\quad - \zeta_i\overline{c}_i\left[W''(\tilde{u}_i)\psi_i + u'\left(\zeta_i\frac{x-\overline{x}_i}{\varepsilon}\right) + \eta(W''(\tilde{u}_i)-W''(0))\right].
\end{aligned}
\tag{6.15}
$$

Summing (6.13), (6.14) and (6.15), and noticing that the terms involving $u'$, and the term

$$\varepsilon^{-2s}W'(\tilde{u}_i) - \zeta_i\overline{c}_iW''(\tilde{u}_i)\psi_i$$

appearing in both (6.14) and (6.15), cancel, we get

$$
\begin{aligned}
I_\varepsilon &= \varepsilon(\overline{v}_\varepsilon)_t + \varepsilon^{-2s}W'(\overline{v}_\varepsilon) - \mathcal{I}_s\overline{v}_\varepsilon - \sigma\\
&= -\varepsilon^{-2s}\sum_{j\neq i}W'(\tilde{u}_j) + W''(\tilde{u}_i)\left(\overline{\sigma} + \varepsilon^{-2s}\sum_{j\neq i}\tilde{u}_j\right) + \sum_{j\neq i}\zeta_j\overline{c}_j(W''(\tilde{u}_j)-W''(\tilde{u}_i))\psi_j\\
&\quad + \sum_{j\neq i}\zeta_j\overline{c}_j\eta(W''(\tilde{u}_j)-W''(0)) + \zeta_i\overline{c}_i\eta(W''(\tilde{u}_i)-W''(0)) - \sigma + \sum_{j\neq i}O(\varepsilon^{-2s}\tilde{u}_j^2) + O(\varepsilon^{2s}).
\end{aligned}
$$

Now, since $W'(0)=0$, we use a Taylor expansion of $W'$ around 0, to see that

$$\varepsilon^{-2s}\sum_{j\neq i}W'(\tilde{u}_j) = \varepsilon^{-2s}\sum_{j\neq i}W''(0)\tilde{u}_j + \sum_{j\neq i}O(\varepsilon^{-2s}\tilde{u}_j^2),$$

so that

$$
\begin{aligned}
I_\varepsilon &= -\varepsilon^{-2s}\sum_{j\neq i}W''(0)\tilde{u}_j + W''(\tilde{u}_i)\left(\overline{\sigma} + \varepsilon^{-2s}\sum_{j\neq i}\tilde{u}_j\right) + \sum_{j\neq i}\zeta_j\overline{c}_j(W''(\tilde{u}_j)-W''(\tilde{u}_i))\psi_j\\
&\quad + \sum_{j\neq i}\zeta_j\overline{c}_j\eta(W''(\tilde{u}_j)-W''(0)) + \zeta_i\overline{c}_i\eta(W''(\tilde{u}_i)-W''(0)) - \sigma + \sum_{j\neq i}O(\varepsilon^{-2s}\tilde{u}_j^2) + O(\varepsilon^{2s}).
\end{aligned}
$$

Next, we add and subtract the term $W''(0)\overline{\sigma}$ to get

$$
\begin{aligned}
I_\varepsilon &= \varepsilon^{-2s}\sum_{j\neq i}(W''(\tilde{u}_i)-W''(0))\tilde{u}_j + (W''(\tilde{u}_i)-W''(0))\overline{\sigma} + \sum_{j\neq i}\zeta_j\overline{c}_j(W''(\tilde{u}_j)-W''(\tilde{u}_i))\psi_j\\
&\quad + \sum_{j\neq i}\zeta_j\overline{c}_j\eta(W''(\tilde{u}_j)-W''(0)) + \zeta_i\overline{c}_i\eta(W''(\tilde{u}_i)-W''(0)) + W''(0)\overline{\sigma} - \sigma\\
&\quad + \sum_{j\neq i}O(\varepsilon^{-2s}\tilde{u}_j^2) + O(\varepsilon^{2s}).
\end{aligned}
$$

Now, clearly

$$\zeta_j\overline{c}_j\eta(W''(\tilde{u}_j)-W''(0)) = O(\tilde{u}_j), \quad W''(\tilde{u}_i)-W''(0) = O(\tilde{u}_i)$$

and

$$\zeta_j \bar{c}_j (W''(\tilde{u}_j) - W''(\tilde{u}_i))\psi_j = O(\psi_j).$$

Therefore, we conclude that

$$\begin{aligned} I_\varepsilon = {} & O(\tilde{u}_i)(\varepsilon^{-2s} \sum_{j \neq i} \tilde{u}_j + \overline{\sigma} + \zeta_i \overline{c}_i \eta) + W''(0)\overline{\sigma} - \sigma \\ & + \sum_{j \neq i} \big( O(\psi_j) + O(\tilde{u}_j) + O(\varepsilon^{-2s} \tilde{u}_j^2) \big) + O(\varepsilon^{2s}). \end{aligned}$$

By (6.4), we finally obtain (6.12). □

Let us now conclude the proof of Proposition 6.1. Recalling (6.7), it suffices to show that for any $x \in \mathbb{R}$ and $t < T$

$$I_\varepsilon \geqslant 0 \tag{6.16}$$

for $\delta$ and $\varepsilon$ small enough.

*Case 1.* Suppose that there exists an index $i = 1, ..., N$ such that $x$ is close to $\overline{x}_i(t)$ more than $\varepsilon^\gamma$:

$$|x - \overline{x}_i(t)| \leqslant \varepsilon^\gamma \quad \text{with } 0 < \gamma < \frac{\kappa - 2s}{\kappa}, \tag{6.17}$$

where $\kappa$ is given in Lemma 2.1.

Since the $\overline{x}_j$'s are separated for $t < T$, we have for $j \neq i$

$$|x - \overline{x}_j(t)| \geqslant |\overline{x}_i(t) - \overline{x}_j(t)| - |x - \overline{x}_i(t)| \geqslant |\overline{x}_i(t) - \overline{x}_j(t)| - \varepsilon^\gamma \geqslant \overline{\vartheta} > 0,$$

for $\varepsilon$ sufficiently small, where $\overline{\vartheta}$ is independent of $\varepsilon$. Hence, from (2.1) and (6.6), we get for $j \neq i$

$$\begin{aligned} & \left| \frac{\tilde{u}_j(t,x)}{\varepsilon^{2s}} + \frac{\zeta_j}{2sW''(0)} \frac{x - \overline{x}_j(t)}{|x - \overline{x}_j(t)|^{1+2s}} \right| \\ & = \frac{1}{\varepsilon^{2s}} \left| u\left(\zeta_j \frac{x - \overline{x}_j(t)}{\varepsilon}\right) - H\left(\zeta_j \frac{x - \overline{x}_j(t)}{\varepsilon}\right) + \zeta_j \frac{\varepsilon^{2s}}{2sW''(0)} \frac{x - \overline{x}_j(t)}{|x - \overline{x}_j(t)|^{1+2s}} \right| \\ & \leqslant C \frac{\varepsilon^\kappa}{\varepsilon^{2s}} \frac{1}{|x - \overline{x}_j(t)|^\kappa} \\ & \leqslant C \varepsilon^{\kappa - 2s}. \end{aligned}$$

Next, a Taylor expansion of the function $\dfrac{x - \overline{x}_j(t)}{|x - \overline{x}_j(t)|^{1+2s}}$ around $\overline{x}_i(t)$, gives

$$\left| \frac{x - \overline{x}_j(t)}{|x - \overline{x}_j(t)|^{1+2s}} - \frac{\overline{x}_i(t) - \overline{x}_j(t)}{|\overline{x}_i(t) - \overline{x}_j(t)|^{1+2s}} \right| \leqslant \frac{2s}{|\xi - \overline{x}_j(t)|^{1+2s}} |x - \overline{x}_i(t)| \leqslant C\varepsilon^\gamma,$$

where $\xi$ is a suitable point lying on the segment joining $x$ to $\overline{x}_i(t)$.

The last two inequalities imply for $j \neq i$

$$\left| \frac{\tilde{u}_j}{\varepsilon^{2s}} + \frac{\zeta_j}{2sW''(0)} \frac{\overline{x}_i(t) - \overline{x}_j(t)}{|\overline{x}_i(t) - \overline{x}_j(t)|^{1+2s}} \right| \leqslant C(\varepsilon^\gamma + \varepsilon^{\kappa - 2s}).$$

Therefore, from (6.12), we get that

$$I_\varepsilon = O(\tilde{u}_i)\left(\sum_{j\neq i}\frac{-\zeta_j}{2sW''(0)}\frac{\overline{x}_i(t)-\overline{x}_j(t)}{|\overline{x}_i(t)-\overline{x}_j(t)|^{1+2s}}+\overline{\sigma}+\zeta_i\overline{c}_i\eta\right)+\delta$$
$$+\sum_{j\neq i}\left(O(\psi_j)+O(\tilde{u}_j)+O(\varepsilon^{-2s}\tilde{u}_j^2)\right)+O(\varepsilon^{2s})+O(\varepsilon^{\gamma})+O(\varepsilon^{\kappa-2s}). \tag{6.18}$$

Now, we compute the term between parenthesis. From the definitions of $\overline{c}_i$, $\eta$ and $\overline{\sigma}$ given respectively in (6.3), (2.4) and (6.4), and the system of ODE's (6.1), we obtain

$$\begin{aligned}\sum_{j\neq i}\frac{-\zeta_j}{2sW''(0)}\frac{\overline{x}_i(t)-\overline{x}_j(t)}{|\overline{x}_i(t)-\overline{x}_j(t)|^{1+2s}}+\overline{\sigma}+\zeta_i\overline{c}_i\eta &= \frac{\sigma(t,x)-\sigma(t,\overline{x}_i(t))}{W''(0)}\\ &= O(|x-\overline{x}_i(t)|)\\ &= O(\varepsilon^{\gamma}).\end{aligned} \tag{6.19}$$

Finally, from the estimates (2.1) and the fact that $\lim_{|x|\to\pm\infty}\psi(x)=0$, we have for $j\neq i$

$$\tilde{u}_j,\ \varepsilon^{-2s}\tilde{u}_j^2 = O(\varepsilon^{2s}), \quad \text{and} \quad \psi_j = O(1), \tag{6.20}$$

as $\varepsilon\to 0$. From (6.18), (6.19) and (6.20), we get that for $\varepsilon$ small enough

$$I_\varepsilon \geqslant \frac{\delta}{2},$$

which implies (6.16).

*Case 2.* Suppose that for $i=1,...,N$ we have

$$|x-\overline{x}_i(t)|\geqslant \varepsilon^{\gamma}.$$

In this case, the estimate in (2.1) on $u$ implies for $j=1,...,N$

$$\left|\frac{\tilde{u}_j}{\varepsilon^{2s}}+\frac{\zeta_j}{2sW''(0)}\frac{x-\overline{x}_j(t)}{|x-\overline{x}_j(t)|^{1+2s}}\right|\leqslant C\frac{\varepsilon^{\kappa}}{\varepsilon^{2s}}\frac{1}{|x-\overline{x}_j(t)|^{\kappa}}\leqslant C\varepsilon^{\kappa-2s-\gamma\kappa}.$$

Moreover

$$\frac{1}{|x-\overline{x}_j(t)|^{2s}}\leqslant \varepsilon^{-2\gamma s}.$$

As a consequence, recalling (2.4), (6.4) and (6.1)

$$\begin{aligned}\varepsilon^{-2s}\sum_{j\neq i}\tilde{u}_j+\overline{\sigma}+\zeta_i\overline{c}_i\eta &= \sum_{j\neq i}\frac{\zeta_j}{2sW''(0)}\frac{x-\overline{x}_j(t)}{|x-\overline{x}_j(t)|^{1+2s}}+O(1)\\ &= O(\varepsilon^{-2\gamma s}).\end{aligned}$$

Therefore, from (6.12), we have

$$I_\varepsilon = \delta + O(\tilde{u}_i)O(\varepsilon^{-2\gamma s})+\sum_{j\neq i}\left(O(\psi_j)+O(\tilde{u}_j)+O(\varepsilon^{-2s}\tilde{u}_j^2)\right)+O(\varepsilon^{2s}).$$

Now, we observe that again from (2.1), for $i=1,...,N$

$$\tilde{u}_i = O\left(\frac{\varepsilon^{2s}}{|x-\overline{x}_i|^{2s}}\right)=O\left(\frac{\varepsilon^{2s}}{\varepsilon^{2\gamma s}}\right)=O\left(\varepsilon^{2s(1-\gamma)}\right).$$

As a consequence

$$O(\varepsilon^{-2s}\tilde{u}_j^2)=O\left(\varepsilon^{2s(1-2\gamma)}\right), \quad \text{and} \quad O(\tilde{u}_i)O(\varepsilon^{-2\gamma s})=O\left(\varepsilon^{2s(1-2\gamma)}\right).$$

Again the asymptotic behavior of $\psi$ implies

$$\psi_i = O(1).$$

We conclude that

$$I_\varepsilon = \delta + O(1).$$

Hence for $\varepsilon$ small enough, we have

$$I_\varepsilon \geqslant \frac{\delta}{2},$$

which again implies (6.16).

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