Weierstraß-Institut
für Angewandte Analysis und Stochastik
Leibniz-Institut im Forschungsverbund Berlin e. V.

Preprint ISSN 2198-5855

Nonlocal isoperimetric problems

Agnese di Castro[1], Matteo Novaga[2], Berardo Ruffini[3],Enrico Valdinoci[4]

submitted: July 28, 2014

[1] [2] Dipartimento di Matematica
Universita di Pisa
Largo Bruno Pontecorvo 5,
56127 Pisa
Italy
E-Mail: dicastro@mail.dm.unipi.it
novaga@dm.unipi.it

[3] Université Joseph Fourier
BP 53
38041 Grenoble
Cedex 39
France
E-Mail: berardo.ruffini@ujf-grenoble.fr

[4] Weierstrass Institute
Mohrenstr. 39
10117 Berlin
Germany
E-Mail: Enrico.Valdinoci@wias-berlin.de

No. 1987
Berlin 2014

2010 *Mathematics Subject Classification.* 35R11, 49Q05, 53A10.

Key words and phrases. fractional perimeter, isoperimetric problem, existence, rigidity and regularity results.

Edited by
Weierstraß-Institut für Angewandte Analysis und Stochastik (WIAS)
Leibniz-Institut im Forschungsverbund Berlin e. V.
Mohrenstraße 39
10117 Berlin
Germany

Fax: +49 30 20372-303
E-Mail: preprint@wias-berlin.de
World Wide Web: http://www.wias-berlin.de/

ABSTRACT

We characterize the volume-constrained minimizers of a nonlocal free energy given by the difference of the t-perimeter and the s-perimeter, with s smaller than t. Exploiting the quantitative fractional isoperimetric inequality, we show that balls are the unique minimizers if the volume is sufficiently small, depending on $t-s$, while the existence vs. nonexistence of minimizers for large volumes remains open. We also consider the corresponding isoperimetric problem and prove existence and regularity of minimizers for all s, t. When $s = 0$ this problem reduces to the fractional isoperimetric problem, for which it is well known that balls are the only minimizers.

1. Introduction

In this paper we deal with two nonlocal isoperimetric problems, which are closely related one with the other. To introduce them, we recall the definition and some properties of the fractional perimeter. Given a number $\alpha \in (0,1)$, for a measurable set $E \subset \mathbb{R}^N$, the fractional perimeter $P_\alpha(E)$ is defined as the (squared) $H^{\alpha/2}$-seminorm of the characteristic function of E, that is,

$$P_\alpha(E) := [\chi_E]^2_{H^{\alpha/2}} = \int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \frac{|\chi_E(x)-\chi_E(y)|^2}{|x-y|^{N+\alpha}}dx\,dy = \int_E\int_{E^c} \frac{dx\,dy}{|x-y|^{N+\alpha}}.$$

The notion of fractional perimeter has been introduced in [36, 9] and it has been extensively studied in several recent papers (see for instance [24, 33, 34, 11, 18, 15] and references therein). In particular, according [10, Theorem 1] (see also [7, 14, 3]), we have that the fractional perimeter P_α, if suitably renormalized, approaches the classical perimeter P as $\alpha \nearrow 1$. More precisely, if ∂E is of class $C^{1,\gamma}$ for some $\gamma > 0$, we have

$$\lim_{\alpha\to 1^-} (1-\alpha)P_\alpha(E) = N\omega_N P(E), \tag{1.1}$$

where ω_N denotes the volume of the N-dimensional ball of radius 1. On the other hand, the fractional perimeter P_α approaches the Lebesgue measure $|\cdot|$ as $\alpha \searrow 0$, that is,

$$\lim_{\alpha\to 0^+} \alpha P_\alpha(E) = N\omega_N|E|, \tag{1.2}$$

if $P_{\bar\alpha}(E) < +\infty$ for some $\bar\alpha > 0$ (see [31] and [17, Corollary 2.6]).

In the first part of the paper we investigate the minimum problem:

$$\min_{|E|=m} \mathcal{F}_{s,t}(E) \qquad m \in (0,+\infty)\,, \tag{1.3}$$

where

$$\mathcal{F}_{s,t}(E) := \begin{cases} (1-t)P_t(E) - sP_s(E) & \text{if } 0 < s < t < 1 \\ N\omega_N P(E) - sP_s(E) & \text{if } 0 < s < t = 1 \\ (1-t)P_t(E) - N\omega_N|E| & \text{if } 0 = s < t < 1 \\ N\omega_N P(E) - N\omega_N|E| & \text{if } s = 0 \text{ and } t = 1. \end{cases}$$

Notice that thanks to (1.2) and (1.1), for all $s,t\in(0,1)$ we have

$$\mathcal{F}_{s,t}(E)\underset{t\to1}{\to}\mathcal{F}_{s,1}(E)\underset{s\to0}{\to}\mathcal{F}_{0,1}(E)\quad\text{ and }\quad\mathcal{F}_{s,t}(E)\underset{s\to0}{\to}\mathcal{F}_{0,t}(E)\underset{t\to1}{\to}\mathcal{F}_{0,1}(E)\,, \tag{1.4}$$

that is, $\mathcal{F}_{s,t}$ depends continuously on $s,t\in[0,1]$, with $s<t$.

Problem (1.3) is reminiscent of recent results about isoperimetric problems with nonlocal competing term arising in mathematical physics, where the functionals take the form

$$\mathcal{F}=P+\mathcal{NL}$$

being P the perimeter and $\mathcal{NL}$ the nonlocal term, see for instance [28, 29, 13, 26, 22, 18, 5, 27]. We mention in particular the works by Knüpfer and Muratov [28, 29] where the authors consider the case where the nonlocal term is given by a Coulombic potential.

In our framework, the energy in (1.4) presents a competing effect between the term P_t which has the tendency to "aggregate" the sets into balls, and P_s, which acts in the opposite way. We will see that, at small scales, the aggregating effect is predominant, but this does not occur at large scales.

More precisely, as a first result we show that minimizers exist and are regular at least for small volumes.

Theorem 1.1. *For any $0\leqslant s<t\leqslant1$, there exists $\bar m_0=\bar m_0(N,t-s)>0$ such that for all $m\in(0,\bar m_0)$, problem* (1.3) *has a minimizer $F\subset\mathbb{R}^N$. Moreover F is bounded with boundary of class $C^{1,\beta}$, for some $\beta=\beta(N,t-s)\in(0,1)$, outside a closed singular set of Hausdorff dimension at most $N-2$ (respectively $N-8$ if $t=1$).*

Thanks to the fractional isoperimetric inequality in a quantitative form, we also show that the the minimizer found in Theorem 1.1 is necessarily a ball, if the volume m is sufficiently small.

Theorem 1.2. *For any $0\leqslant s<t\leqslant1$ and $\bar m_0$ as in Theorem 1.1, there exists $\bar m_1=\bar m_1(N,t-s)\in(0,\bar m_0]$ such that for all $m\in(0,\bar m_1)$, the only minimizer of problem* (1.3) *is given by the ball of measure m.*

We stress that our estimates, similarly to those in [18], depend only on a lower bound on the difference $t-s$, and pass to the limit as $s\to0$ and $t\to1$ (as a matter of fact, the normalizing constants appearing in (1.4) has exactly the purpose of making our estimates stable as $s\to0$ and $t\to1$).

Moreover, as far as we know, our results are new even in the case $t=1$.

We also point out that we do not know if a minimizer exists for any volume m. However, we show that *a minimizer cannot be a ball if m is large enough* (see Theorem 6.3), so the minimization problem can be in general quite rich.

The second problem we consider is the following generalized isoperimetric problem:

$$\min_{E\subset\mathbb{R}^N}\widetilde{\mathcal{F}_{s,t}}(E)\qquad0\leqslant s<t\leqslant1\,, \tag{1.5}$$

where

$$\widetilde{\mathcal{F}_{s,t}}(E) := \begin{cases} \dfrac{((1-t)P_t(E))^{N-s}}{(sP_s(E))^{N-t}} & \text{if } 0 < s < t < 1 \\ \dfrac{(N\omega_N P(E))^{N-s}}{(sP_s(E))^{N-1}} & \text{if } 0 < s < t = 1 \\ \dfrac{(1-t)P_t(E)^N}{(N\omega_N|E|)^{N-t}} & \text{if } 0 = s < t < 1 \\ N\omega_N \frac{P(E)^N}{|E|^{N-1}} & \text{if } s = 0 \text{ and } t = 1. \end{cases}$$

Again, thanks to (1.2) and (1.1) we see that

$$\widetilde{\mathcal{F}_{s,t}}(E) \underset{t\to 1}{\to} \widetilde{\mathcal{F}_{s,1}}(E) \underset{s\to 0}{\to} \widetilde{\mathcal{F}_{0,1}}(E) \quad \text{and} \quad \widetilde{\mathcal{F}_{s,t}}(E) \underset{s\to 0}{\to} \widetilde{\mathcal{F}_{0,t}}(E) \underset{t\to 1}{\to} \widetilde{\mathcal{F}_{0,1}}(E).$$

Since, for $s = 0$ and $t = 1$, problem (1.5) reduces to the classical isoperimetric one, we can think to it as a generalized isoperimetric problem for fractional perimeters.

We now state our main result about problem (1.5).

Theorem 1.3. *For any $0 \leqslant s < t \leqslant 1$, there exists a nontrivial minimizer $E_{s,t}$ of problem (1.5). Moreover $E_{s,t}$ is bounded and has boundary of class $C^{1,\beta}$, for some $\beta = \beta(N, t-s) \in (0,1)$, outside a closed singular set of Hausdorff dimension at most $N-2$ (respectively $N-8$ if $t = 1$).*

We point out that, for $s = 0$, the problem reduces to the fractional isoperimetric problem, for which it is known that the ball is the unique minimizer [20] (see also [24] for a quantitative version of this result). However, we do not know if the ball is still a minimizer of problem (1.5) for $s > 0$.

The paper is organized as follows: in Section 2 we recall some general properties of the fractional perimeters and, more generally, of the fractional Sobolev seminorms. In Sections 3–6 we deal with problem (1.3). Section 3 contains the main tools exploited later to prove Theorems 1.1 and 1.2. The cornerstone of the section is an optimality criterion (see Proposition 3.8) which entails density estimates for minimizers (see Proposition 3.10) and the fact that minimizers must be close to a ball, if the volume is small enough (see Lemma 3.12). An elementary, but important result is then provided by Proposition 3.11, stating that any minimum must be necessary bounded and, if $t = 1$ (that is, $\mathcal{F}_{s,1} = N\omega_N P - sP_s$), also essentially connected. Section 4 contains Theorem 4.2, which solves the existence part of Theorem 1.1, while in Section 5 we prove that any minimizer has smooth boundary, out of a closed singular set. Then, in Section 6 we show that, if the volume m is below a certain threshold $\bar{m}_1 > 0$, the ball is the unique minimizer for problem (1.3). Eventually, in Section 7, we deal with problem (1.5). The main result here is given by Theorem 1.3, where we show the existence and regularity of minimizers.

2. General properties of fractional perimeters

Before starting to prove some properties of fractional perimeters it is convenient to fix some notation which will be used throughout the rest of the paper. Firstly, notice that we will denote by c_N a general positive constant depending only on the dimension N and by c_0 a positive constant depending on N and δ_0 a fixed quantity such that $0 < \delta_0 \leqslant t - s$, which will not necessarily be the same at different occurrences and which can also change from line to line;

special constants will be denoted by c_1, c_2,.... Relevant dependences on parameters will be emphasized by using parentheses.

As customary, we denote by $B(x_0, R) := \{x \in \mathbb{R}^N : |x - x_0| < R\}$ the open ball centered in $x_0 \in \mathbb{R}^N$ with radius $R > 0$. We shall use the shorter notation $B = B(0, 1)$, with $|B(0, 1)| = \omega_N$. Moreover, when not important and clear from the context, we shall denote by B_m the ball of volume m, that is of radius $R = (m/|B(0, 1)|)^{1/N}$.

Finally, as usual, given two sets E and F of $\mathbb{R}^N$, we denote the symmetric difference between E and F as $E\Delta F = (E \setminus F) \cap (F \setminus E)$.

We begin by a simple observation, whose elementary proof is omitted:

Lemma 2.1. *Let $E = E_1 \cup E_2$ a subset of $\mathbb{R}^N$ with $|E_1 \cap E_2| = 0$. Then*

$$P_\alpha(E) = P_\alpha(E_1) + P_\alpha(E_2) - 2\int_{E_1}\int_{E_2} \frac{dx\,dy}{|x-y|^{N+\alpha}}. \tag{2.1}$$

In particular

$$P_\alpha(E) \leqslant P_\alpha(E_1) + P_\alpha(E_2). \tag{2.2}$$

For further use, we also prove the following interpolation estimate (by reasoning as in [8, Proposition 4.2 and Corollary 4.4]):

Lemma 2.2. *For any $E \subset \mathbb{R}^N$ and $0 < s < t < 1$ there holds*

$$P_s(E) \leqslant c_N \frac{1}{s}\left(1 - \frac{s}{t}\right)^{-1} |E|^{1-\frac{s}{t}}(1-t)^{\frac{s}{t}} P_t(E)^{\frac{s}{t}}. \tag{2.3}$$

Proof. We reason as in [8, Prop. 4.2]. Letting $u = \chi_E$, we can write

$$\begin{aligned} P_s(E) &= \int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \frac{|u(x+h) - u(x)|}{|h|^{N+s}}\,dxdh \\ &= \int_{|h|<1}\int_{\mathbb{R}^N} \frac{|u(x+h) - u(x)|}{|h|^{N+s}}\,dxdh \\ &\quad + \int_{|h|\geqslant 1}\int_{\mathbb{R}^N} \frac{|u(x+h) - u(x)|}{|h|^{N+s}}\,dxdh \;=:\; I_1 + I_2. \end{aligned}$$

We recall that, by [8, Lemma A.1] (see also [31]), there exists a constant c_N such that

$$\int_{\mathbb{R}^N} \frac{|u(x+h) - u(x)|}{|h|^t}\,dx \leqslant c_N\,(1-t)P_t(E)\,, \tag{2.4}$$

for all $|h| > 0$. We then estimate

$$\begin{aligned} I_1 &= \int_{|h|<1}\int_{\mathbb{R}^N} \frac{|u(x+h) - u(x)|}{|h|^{N+s}}\,dxdh \\ &\leqslant c_N(1-t)P_t(E)\int_{|h|<1} \frac{1}{|h|^{N-(t-s)}}\,dh \qquad (2.5) \\ &= c_N \frac{1-t}{t-s} P_t(E), \end{aligned}$$

and

$$\begin{aligned} I_2 &= \int_{|h|\geqslant 1}\int_{\mathbb{R}^N} \frac{|u(x+h)-u(x)|}{|h|^{N+s}}\,dxdh \\ &\leqslant 2|E|\int_{|h|\geqslant 1}\frac{1}{|h|^{N+s}}\,dh \\ &= \frac{2N\omega_N}{s}|E|. \end{aligned} \tag{2.6}$$

Putting together (2.5) and (2.6) we then get

$$P_s(E) \leqslant c_N\frac{1-t}{t-s}P_t(E) + \frac{2N\omega_N}{s}|E|. \tag{2.7}$$

If we evaluate (2.7) on the set λE, with $\lambda > 0$, we obtain

$$\lambda^{N-s}P_s(E) \leqslant c_N\frac{1-t}{t-s}\lambda^{N-t}P_t(E) + \lambda^N\frac{2N\omega_N}{s}|E|,$$

that is,

$$\lambda^{t-s}P_s(E) - \lambda^t\frac{2N\omega_N|E|}{s} \leqslant \frac{c_N(1-t)}{t-s}P_t(E). \tag{2.8}$$

The expression at the left-hand side of (2.8) reaches its maximum at

$$\lambda = \left(\frac{s(t-s)P_s(E)}{2N\omega_N t|E|}\right)^{\frac{1}{s}}.$$

Substituting this value of λ into (2.8) we get (2.3). □

Remark 2.3. If we let $t \to 1^-$ in (2.3), we recover the estimate in [8, Cor. 4.4]:

$$P_s(E) \leqslant \frac{c_N}{s(1-s)}|E|^{1-s}P(E)^s. \tag{2.9}$$

Indeed the proof of Lemma 2.2 extends to the case $t = 1$, by substituting $(1-t)P_t(E)$ with $P(E)$ in the right hand side of (2.4).

We show now a version of the local fractional isoperimetric inequality. For this, we recall that the fractional perimeter of a set E in a bounded set Ω is defined by

$$P_\alpha(E,\Omega) := \int_{E\cap\Omega}\int_{\mathbb{R}^N\setminus E}\frac{dx\,dy}{|x-y|^{N+\alpha}} + \int_{\Omega\setminus E}\int_{E\setminus\Omega}\frac{dx\,dy}{|x-y|^{N+\alpha}}.. \tag{2.10}$$

With this setting, we have a variant of Lemma 2.1 as follows:

Lemma 2.4. *Let Ω_1 and Ω_2 be disjoint bounded sets. Then*

$$P_\alpha(E,\Omega_1) + P_\alpha(E,\Omega_2) \leqslant P_\alpha(E,\Omega_1\cup\Omega_2) + 2\int_{\Omega_1}\int_{\Omega_2}\frac{dx\,dy}{|x-y|^{N+\alpha}}.. \tag{2.11}$$

Proof. We use (2.10) (omitting the integrands for simplicity) to compute

$$\begin{aligned}
& P_\alpha(E,\Omega_1\cup\Omega_2)-P_\alpha(E,\Omega_1)-P_\alpha(E,\Omega_2)\\
= \;& \int_{E\cap(\Omega_1\cup\Omega_2)}\int_{\mathbb{R}^N\setminus E}+\int_{(\Omega_1\cup\Omega_2)\setminus E}\int_{E\setminus(\Omega_1\cup\Omega_2)}\\
& -\int_{E\cap\Omega_1}\int_{\mathbb{R}^N\setminus E}-\int_{\Omega_1\setminus E}\int_{E\setminus\Omega_1}-\int_{E\cap\Omega_2}\int_{\mathbb{R}^N\setminus E}-\int_{\Omega_2\setminus E}\int_{E\setminus\Omega_1}\\
= \;& \int_{E\cap\Omega_1}\int_{\mathbb{R}^N\setminus E}+\int_{E\cap\Omega_2}\int_{\mathbb{R}^N\setminus E}+\int_{\Omega_1\setminus E}\int_{E\setminus(\Omega_1\cup\Omega_2)}+\int_{\Omega_2\setminus E}\int_{E\setminus(\Omega_1\cup\Omega_2)}\\
& -\int_{E\cap\Omega_1}\int_{\mathbb{R}^N\setminus E}-\int_{\Omega_1\setminus E}\int_{E\setminus\Omega_1}-\int_{E\cap\Omega_2}\int_{\mathbb{R}^N\setminus E}-\int_{\Omega_2\setminus E}\int_{E\setminus\Omega_1}\\
= \;& \int_{\Omega_1\setminus E}\int_{E\setminus(\Omega_1\cup\Omega_2)}+\int_{\Omega_2\setminus E}\int_{E\setminus(\Omega_1\cup\Omega_2)}-\int_{\Omega_1\setminus E}\int_{E\setminus\Omega_1}-\int_{\Omega_2\setminus E}\int_{E\setminus\Omega_1}\\
= \;& -\int_{\Omega_1\setminus E}\int_{(E\setminus\Omega_1)\cap\Omega_2}-\int_{\Omega_2\setminus E}\int_{(E\setminus\Omega_2)\cap\Omega_1}.
\end{aligned}$$

This implies (2.11). □

Then, we have the following local fractional isoperimetric inequality:

Lemma 2.5. *Let Ω be a open bounded set with Lipschitz boundary and let $E\subseteq\mathbb{R}^N$ such that $|E\cap\Omega|<|\Omega|/2$. Then there exists a constant $C=C(|\Omega|,N,\alpha)$ such that*

$$P_\alpha(E,\Omega)\geqslant C|E\cap\Omega|^{\frac{N-\alpha}{N}}. \tag{2.12}$$

Proof. The case $t=1$ is classical. For its proof we refer to [30, Section II.1.6]. We begin by recalling the Poincaré-type inequality for fractional Sobolev spaces (see for instance [7, Equations (2) and (3)]: for any $p\geqslant 1$ and $\alpha\in(0,1)$, given a function $f\in L^p(\Omega)$ we have that

$$\int_\Omega\int_\Omega\frac{|f(x)-f(y)|^p}{|x-y|^{N+\alpha p}}\geqslant C(N,\alpha,p,|\Omega|)\,\|f-f_\Omega\|_{L^q(\Omega)}, \tag{2.13}$$

where

$$f_\Omega=\frac{1}{\Omega}\int_\Omega|f|\,dx$$

and

$$\frac{1}{q}=\frac{1}{p}-\frac{\alpha}{N}. \tag{2.14}$$

By applying (2.13) with $p = 1$, $\alpha \in (0,1)$ and $f = \chi_E$, and by the very definition of $P_\alpha(E)$ we get that

$$\begin{aligned}
2P_\alpha(E,\Omega) &\geqslant \int_\Omega\int_\Omega \frac{|\chi_E(x)-\chi_E(y)|}{|x-y|^{N+\alpha}} \\
&\geqslant C(N,\alpha,|\Omega|)\left(\int_\Omega \left|\chi_E(x) - \frac{|E\cap\Omega|}{|\Omega|}\right|^q dx\right)^{1/q} \\
&= C(N,\alpha,|\Omega|)\left[|E\cap\Omega|\left(1-\frac{|E\cap\Omega|}{|\Omega|}\right)^q + |\Omega\setminus E|\left(\frac{|E\cap\Omega|}{|\Omega|}\right)^q\right]^{1/q} \\
&\geqslant C(N,\alpha,|\Omega|)|E\cap\Omega|^{1/q}\left(1-\frac{|E\cap\Omega|}{|\Omega|}\right) \\
&\geqslant \frac{C(N,\alpha,|\Omega|)}{2}|E\cap\Omega|^{1/q}.
\end{aligned}$$

Since, by (2.14), $q = N/(N-\alpha)$, the proof is concluded. □

Beside the local fractional isoperimetric inequality (2.12), we recall from [19] the standard (fractional) one: if $0 < t_0 \leqslant \alpha \leqslant 1$ then it holds (if $|E| < +\infty$)

$$(1-\alpha)P_\alpha(E) \;\geqslant\; c(N,t_0)|E|^{\frac{N-\alpha}{N}}, \qquad \text{where } c(N,t_0) = \frac{c_N}{t_0}, \tag{2.15}$$

We now recall some basic facts on hypersingular Riesz operators on the sphere, following [18, pp. 4-5] (see also [32, pp. 159-160]). We denote by $\mathcal{S}_k$ the space of spherical harmonics of degree k, and by $d(k)$ the dimension of $\mathcal{S}_k$. For $\alpha \in (0,1)$ we also let $\mathcal{J}_\alpha$ be the operator defined as

$$\mathcal{J}_\alpha u(x) = 2\,\text{p.v.}\int_{\partial B} \frac{u(x)-u(y)}{|x-y|^{N+\alpha}}\, d\mathcal{H}^{N-1}(y) \qquad \text{for } u \in C^2(\partial B),$$

and we let λ_k^α be the k^{th} eigenvalue of $\mathcal{J}_\alpha$, that is,

$$\mathcal{J}_\alpha Y = \lambda_k^\alpha Y \qquad \text{for any } Y \in \mathcal{S}_k.$$

We then have $\lambda_k^\alpha \to +\infty$ as $k \to +\infty$, and

$$\lambda_0^\alpha = 0 \qquad \lambda_{k+1}^\alpha > \lambda_k^\alpha \qquad \forall k \in \mathbb{N}\cup\{0\}.$$

If we let $\{Y_k^i\}_{i=1}^{d(k)}$ be an orthonormal basis of $\mathcal{S}_k$ in $L^2(\partial B)$, and denote by

$$a_k^i(u) := \int_{\partial B} u\, Y_k^i\, d\mathcal{H}^{N-1},$$

the Fourier coefficients of $u \in L^2(\partial B)$ corresponding to Y_k^i, we have

$$\begin{aligned}
[u]^2_{H^{\frac{1+\alpha}{2}}(\partial B)} &:= \int_{\partial B}\int_{\partial B} \frac{|u(x)-u(y)|^2}{|x-y|^{N+\alpha}}\, d\mathcal{H}^{N-1}(x)\, d\mathcal{H}^{N-1}(y) \\
&= \int_{\partial B} u\,\mathcal{J}_\alpha u\, d\mathcal{H}^{N-1} \\
&= \sum_{k=0}^{\infty}\sum_{i=0}^{d(k)} \lambda_k^\alpha\, a_k^i(u)^2. \qquad (2.16)
\end{aligned}$$

Proposition 2.6. ([18, Proposition 2.3]) *We have*

$$\lambda_k^\alpha \geqslant \lambda_1^\alpha = \alpha(N-\alpha)\frac{P_\alpha(B)}{P(B)} \geqslant \frac{1}{c_N(1-\alpha)}\,.$$

Proposition 2.7. *Let $u \in H^{\frac{1+t}{2}}(\partial B)$ and $0 \leqslant s \leqslant t < 1$ then the following estimate holds*

$$(1-s)[u]^2_{H^{\frac{1+s}{2}}(\partial B)} \leqslant c_N(1-t)[u]^2_{H^{\frac{1+t}{2}}(\partial B)}. \tag{2.17}$$

Proof. By (2.16) and using the estimate for λ_k established in Proposition 2.6 we get

$$\begin{aligned}
(1-s)[u]^2_{H^{\frac{1+s}{2}}(\partial B)} &= (1-s)\sum_{k=0}^{\infty}\sum_{i=0}^{d(k)} \lambda_k^s a_k^i(u)^2 = (1-s)\sum_{k=0}^{\infty}\sum_{i=0}^{d(k)} \lambda_k^{s-t}\lambda_k^t a_k^i(u)^2 \\
&\leqslant (1-s)\lambda_1^{s-t}\sum_{k=0}^{\infty}\sum_{i=0}^{d(k)} \lambda_k^t a_k^i(u)^2 \\
&\leqslant (1-s)\lambda_1^s c_N(1-t)\sum_{k=0}^{\infty}\sum_{i=0}^{d(k)} \lambda_k^t a_k^i(u)^2 \\
&= (1-s)s(N-s)\frac{P_s(B)}{P(B)} c_N(1-t)[u]^2_{H^{\frac{1+t}{2}}(\partial B)} \\
&\leqslant c_N(1-t)[u]^2_{H^{\frac{1+t}{2}}(\partial B)}.
\end{aligned}$$

□

Remark 2.8. We note that the result established in the previous proposition remains true also in the case $t = 1$. Indeed, since

$$\lim_{t\to 1^-}(1-t)[u]^2_{H^{\frac{1+t}{2}}(\partial B)} = [u]^2_{H^1(\partial B)}$$

as established in [6, Cor. 2], we can pass to the limit $t \to 1^-$ in (2.17).

3. Preliminary estimates on the energy functional

In the following we shall consider parameters $s, t \in (0,1)$ satisfying

$$t - s \geqslant \delta_0 > 0\,. \tag{3.1}$$

All the constants in this work, unless differently specified, will depend only on N and δ_0, so that it will be possible to pass to the limits in a straightforward way as $s \to 0^+$ or $t \to 1^-$.

Proposition 3.1. *There exists $c_0 = c_0(N, \delta_0)$ such that, for any $E \subset \mathbb{R}^N$ and $0 < s < t < 1$ satisfying* (3.1)*, it holds*

$$\mathcal{F}_{s,t}(E) \geqslant \frac{(1-t)P_t(E)}{2} - c_0|E|. \tag{3.2}$$

Proof. Set $m := |E|$. We apply Young inequality with exponents $\frac{t}{t-s}$ and $\frac{t}{s}$ to the right hand side of (2.3) that

$$\begin{aligned}
c_N\frac{1}{s}\left(1-\frac{s}{t}\right)^{-1}|E|^{1-\frac{s}{t}}(1-t)^{\frac{s}{t}}P_t(E)^{\frac{s}{t}} &= \left[c_N\frac{2^{\frac{s}{t}}}{s}\left(1-\frac{s}{t}\right)^{-1}m^{1-\frac{s}{t}}\right]\left[2^{-1}(1-t)P_t(E)\right]^{\frac{s}{t}} \\
&\leqslant \left[c_N\frac{2^{\frac{s}{t}}}{s}\left(1-\frac{s}{t}\right)^{-1}m^{1-\frac{s}{t}}\right]^{\frac{t}{t-s}} + \frac{(1-t)P_t(E)}{2}.
\end{aligned}$$

Thus (2.3) gives that

$$\begin{aligned}\mathcal{F}_{s,t}(E) &= (1-t)P_t(E) - sP_s(E) \geqslant (1-t)P_t(E) - c_N\left(1-\frac{s}{t}\right)^{-1}|E|^{1-\frac{s}{t}}(1-t)^{\frac{s}{t}}P_t(E)^{\frac{s}{t}} \\ &\geqslant (1-t)P_t(E) - \left[2^{\frac{s}{t}}c_N\left(1-\frac{s}{t}\right)^{-1}m^{1-\frac{s}{t}}\right]^{\frac{t}{t-s}} - \frac{(1-t)P_t(E)}{2} \\ &= \frac{(1-t)P_t(E)}{2} - \left[2^{\frac{s}{t}}c_N\frac{t}{t-s}\right]^{\frac{t}{t-s}}m\end{aligned}$$

and this concludes the proof. □

Corollary 3.2. *Let $|E| = m$. Then both $P_t(E)$ and $P_s(E)$ are bounded above by quantities only depending on m and $\mathcal{F}_{s,t}(E)$. More explicitly*

$$(1-t)P_t(E) \leqslant 2(\mathcal{F}_{s,t}(E) + c_0 m) \tag{3.3}$$

$$\text{and} \qquad sP_s(E) \leqslant c_0^{1-\frac{s}{t}}m^{1-\frac{s}{t}}(\mathcal{F}_{s,t}(E) + c_0 m)^{\frac{s}{t}}, \tag{3.4}$$

with c_0 as in Proposition 3.1.

Proof. We obtain (3.3) easily from Proposition 3.1. Then (3.4) follows from (2.3) and (3.3). □

Now we define the isovolumetric function $\phi : (0, +\infty) \to \mathbb{R}$ as

$$\phi(m) = \inf_{|E|=m} \mathcal{F}_{s,t}(E) \qquad m \in (0, +\infty).$$

A general estimate on $\phi(m)$ goes as follows:

Lemma 3.3. *We have*

$$-c_0 m \leqslant \phi(m) \leqslant c_1 m^{\frac{N-t}{N}}\left(1 - \frac{c_2}{c_1}m^{\frac{t-s}{N}}\right), \tag{3.5}$$

with c_0 as in Proposition 3.1 and

$$c_1 := \frac{(1-t)P_t(B)}{|B|^{\frac{N-t}{N}}} \quad \text{and} \quad c_2 := \frac{sP_s(B)}{|B|^{\frac{N-s}{N}}}. \tag{3.6}$$

Proof. Let us begin by proving the estimate from above of $\phi(m)$. For this, we take the unit ball B we set $\rho := (m/|B|)^{1/N}$ and we consider the ball $B(0,\rho)$ of radius ρ. Notice that $|B(0,\rho)| = \rho^N|B| = m$,

$$P_t(B(0,\rho)) = \rho^{N-t}P_t(B) = \frac{P_t(B)}{|B|^{\frac{N-t}{N}}}\,m^{\frac{N-t}{N}}$$

and

$$P_s(B(0,\rho)) = \rho^{N-s}P_s(B) = \frac{P_s(B)}{|B|^{\frac{N-s}{N}}}\,m^{\frac{N-s}{N}}.$$

By minimality, we get, with c_1 and c_2 as in (3.6),

$$\phi(m) \leqslant \mathcal{F}_{s,t}(B(0,\rho)) = (1-t)P_t(B(0,\rho)) - sP_s(B(0,\rho)) = c_1 m^{\frac{N-t}{N}}\left(1 - \frac{c_2}{c_1}m^{\frac{t-s}{N}}\right),$$

that proves (3.5).

The first inequality in (3.5) follows from Proposition 3.1. □

Remark 3.4. We recall the fractional isoperimetric inequality, which holds true for any measurable set E such that $|E| < +\infty$:

$$|E|^{\frac{N-t}{N}} \leqslant c_N t(1-t)P_t(E). \tag{3.7}$$

For the optimal constant c_N we refer to [20] (see in particular Equations (1.10) and (4.2) there).

Lemma 3.5. *There exist $m_0 = m_0(N, \delta_0)$ and $m_1 = m_1(N, \delta_0)$ such that:*

$$\text{if } m > m_1, \text{ then } \phi(m) < 0; \tag{3.8}$$

$$\text{if } m \in (0, m_0), \text{ then } \phi(m) \geqslant \tfrac{c_N}{t} m^{\frac{N-t}{N}} > 0. \tag{3.9}$$

Moreover

$$\lim_{m \to 0^+} \phi(m) = 0. \tag{3.10}$$

Proof. We have that (3.8) and (3.10) plainly follow from (3.5).

Now we prove (3.9). For this, we use Proposition 3.1 and the fractional isoperimetric inequality in the form (3.7) to obtain that, if $|E| = m$,

$$\mathcal{F}_{s,t}(E) \geqslant \frac{(1-t)P_t(E)}{2} - c_0 m \geqslant \frac{m^{\frac{N-t}{N}}}{2\, c_N\, t} - c_0 m = \frac{m^{\frac{N-t}{N}}}{2\, c_N\, t} \left(1 - 2c_0\, c_N\, t\, m^{\frac{t}{N}}\right).$$

In particular, if m is small enough, we have that

$$\mathcal{F}_{s,t}(E) \geqslant \frac{m^{\frac{N-t}{N}}}{4\, c_N\, t}$$

and this implies (3.9). □

Lemma 3.6. *Let m_1 be as in Lemma 3.5, and let F be a minimizer of $\mathcal{F}_{s,t}$ among sets of measure $m > m_1$. We have*

$$\frac{c_N}{t}\, m^{\frac{N-t}{N}} \leqslant (1-t)P_t(F) < c_0\, m \quad \text{and} \quad \frac{c_N}{t}\, m^{\frac{N-t}{N}} < sP_s(F) \leqslant c_0\, m, \tag{3.11}$$

for some $c_0 > 0$.

Proof. By Lemma 3.5 we know that $(1-t)P_t(F) < sP_s(F)$, hence from (2.3) and from the fractional isoperimetric inequality (3.7) we get

$$\frac{m^{\frac{N-t}{N}}}{c_N\, t} \leqslant (1-t)P_t(F) < sP_s(F) \leqslant c_0^{\frac{t-s}{t}} 2^{-\frac{s}{t}} m^{1-\frac{s}{t}} [(1-t)P_t(F)]^{\frac{s}{t}}$$

with c_0 given in Proposition 3.1. Then $(1-t)P_t(F) < c_0\, 2^{-\frac{s}{t-s}}\, m$. This and (2.3) also implies the desired bound on $sP_s(F)$. □

Remark 3.7. By inspecting the proof of the Lemma 3.5 we obtain explicit estimates for m_0 and m_1:

$$m_0 \;\geqslant\; [4c_0\, c_N\, t]^{-\frac{N}{t}} \;=\; \left[4 \left(\frac{c_N\, t\, 2^{s/t}}{t-s}\right)^{\frac{t}{t-s}} c_N\, t\right]^{-\frac{N}{t}}$$

$$m_1 \;\leqslant\; \left(\frac{c_1}{c_2}\right)^{\frac{N}{t-s}} \;=\; \left[\frac{(1-t)P_t(B)}{sP_s(B)}\right]^{\frac{N}{t-s}} |B|.$$

Moreover, the first inequality in the second formula in (3.11)

$$\frac{c_N}{t}|F|^{\frac{N-t}{N}} < sP_s(F)$$

entails that $|F| \to \infty$ as $t \to 0$ (and thus $\delta_0 \to 0$). Indeed, letting $t = s + \delta_0$, and using the fact that $sP_s(F) \to N\omega_N|F|$ as $s \to 0$, after an elementary computation we get that

$$m_1 \geqslant |F| \geqslant \left(\frac{c_N}{s+\delta_0}\right)^{\frac{N}{s+\delta_0}}.$$

which gives also a lower bound on m_1 in terms of s and δ_0. Notice that if $t \to 0$, then also s and δ_0 converge to 0 and so $m_1 \to \infty$. Also it is not a direct consequence of our investigation, we stress that it is natural to expect that also if only s converges to 0, then m_1 diverges to $+\infty$.

Proposition 3.8 (Non-optimality criterion)**.** *There exists $\varepsilon = \varepsilon(N, \delta_0)$ such that, if $F \subset \mathbb{R}^N$ can be written as $F = F_1 \cup F_2$, with $|F_1 \cap F_2| = 0$,*

$$|F_2| \leqslant \varepsilon \min(1, |F_1|), \tag{3.12}$$

$$\text{and} \qquad (1-t)[P_t(F_1) + P_t(F_2) - P_t(F)] \leqslant \frac{\mathcal{F}_{s,t}(F_2)}{2}, \tag{3.13}$$

then there exists a set G with $|G| = |F|$ and $\mathcal{F}_{s,t}(G) < \mathcal{F}_{s,t}(F)$ (i.e., F is not a minimizer).

In addition, we have that the set G is either a ball of volume m, or a dilation of the set F_1, according to the following formula:

$$G = \sqrt[N]{1 + \frac{|F_2|}{|F_1|}}\, F_1 \tag{3.14}$$

Proof. Let $m := |F|$, $m_1 := |F_1|$ and $m_2 := |F_2|$. We may suppose that $\mathcal{F}_{s,t}(F)$ is less than or equal than $\mathcal{F}_{s,t}$ of the ball of volume m, B_m, otherwise we can take G equal to such ball, decrease the energy and finish our proof. That is, we may suppose that

$$\mathcal{F}_{s,t}(F) \leqslant \mathcal{F}_{s,t}(B_m) \leqslant (1-t)P_t(B_m) \leqslant \frac{(1-t)P_t(B)}{|B|^{\frac{N-t}{N}}} m^{\frac{N-t}{N}}. \tag{3.15}$$

Let $G = \lambda F_1$, with $\lambda := \sqrt[N]{1+\gamma}$ and $\gamma = m_2/m_1$ Notice that this is in agreement with (3.14), and also $|G| = m$. Moreover, by (3.12) we have that

$$\gamma \leqslant \frac{\varepsilon \min(1, m_1)}{m_1} \leqslant \varepsilon,$$

so that $\gamma \in (0, 1)$ can be taken as small as we like.

From Bernoulli's inequality we get

$$(1+\gamma)^{\frac{N-s}{N}} \geqslant 1 + \frac{N-s}{N}\gamma,$$

and we also have

$$(1+\gamma)^{\frac{N-t}{N}} \leqslant 1 + \frac{N-s}{N}\gamma.$$

Therefore, we obtain that

$$(\lambda^{N-t} - 1)\,(1-t)P_t(F_1) - (\lambda^{N-s} - 1)\,sP_s(F_1) \leqslant \frac{N-s}{N}\,\gamma\,[(1-t)P_t(F_1) - sP_s(F_1)].$$

As a consequence, letting $\tilde\gamma := \frac{N-s}{N}\gamma$, we get

$$\begin{aligned}\mathcal{F}_{s,t}(G) &= (1-t)P_t(G) - sP_s(G)\\ &= \lambda^{N-t}(1-t)P_t(F_1) - \lambda^{N-s}sP_s(F_1)\\ &= \mathcal{F}_{s,t}(F_1) + \left[(\lambda^{N-t}-1)(1-t)P_t(F_1) - (\lambda^{N-s}-1)sP_s(F_1)\right]\\ &\leqslant \left(1+\frac{N-s}{N}\,\gamma\right)\mathcal{F}_{s,t}(F_1) = (1+\tilde\gamma)\mathcal{F}_{s,t}(F_1)\,.\end{aligned}$$

Thus we have, by (2.2) and (3.13),

$$\begin{aligned}\mathcal{F}_{s,t}(G) - \mathcal{F}_{s,t}(F) &\leqslant (1+\tilde\gamma)\mathcal{F}_{s,t}(F_1) - (1-t)P_t(F) + sP_s(F)\\ &\leqslant (1+\tilde\gamma)\mathcal{F}_{s,t}(F_1) - (1-t)P_t(F) + sP_s(F_1) + sP_s(F_2)\\ &\leqslant (1+\tilde\gamma)\mathcal{F}_{s,t}(F_1) + sP_s(F_1) + sP_s(F_2)\\ &\quad +\frac{1}{2}\mathcal{F}_{s,t}(F_2) - (1-t)P_t(F_1) - (1-t)P_t(F_2)\\ &= \tilde\gamma\mathcal{F}_{s,t}(F_1) - \frac{1}{2}\mathcal{F}_{s,t}(F_2). \qquad (3.16)\end{aligned}$$

Furthermore by (3.9), since m_2 can be chosen in $(0, m_0)$, m_0 as in Lemma 3.5 (up to decreasing the value of ε), we have

$$\mathcal{F}_{s,t}(F_2) \geqslant \phi(m_2) \geqslant \frac{c_N}{t}\, m_2^{\frac{N-t}{N}}. \qquad (3.17)$$

Also, using again (2.2) and (3.13), we have that

$$\begin{aligned}\mathcal{F}_{s,t}(F_1) &= [(1-t)P_t(F_1) + (1-t)P_t(F_2) - sP_s(F_1) - sP_s(F_2)] - \mathcal{F}_{s,t}(F_2)\\ &\leqslant \mathcal{F}_{s,t}(F) + [(1-t)P_t(F_1) + (1-t)P_t(F_2) - (1-t)P_t(F) - \mathcal{F}_{s,t}(F_2)]\\ &\leqslant \mathcal{F}_{s,t}(F) - \frac{1}{2}\mathcal{F}_{s,t}(F_2) < \mathcal{F}_{s,t}(F).\end{aligned}$$

This, (3.16) and (3.17) give that

$$\mathcal{F}_{s,t}(G) - \mathcal{F}_{s,t}(F) \leqslant \tilde\gamma\mathcal{F}_{s,t}(F) - \frac{1}{2}\mathcal{F}_{s,t}(F_2) \leqslant \tilde\gamma\mathcal{F}_{s,t}(F) - \frac{c_N}{2t}\, m_2^{\frac{N-t}{N}}.$$

Accordingly, recalling (3.15) we conclude that

$$\begin{aligned}\mathcal{F}_{s,t}(G) - \mathcal{F}_{s,t}(F) &\leqslant \frac{(1-t)P_t(B)}{|B|^{\frac{N-t}{N}}}\,\tilde\gamma(m_1+m_2)^{\frac{N-t}{N}} - \frac{c_N}{2t}\, m_2^{\frac{N-t}{N}}\\ &= m_2^{\frac{N-t}{N}}\left[\frac{(1-t)P_t(B)}{|B|^{\frac{N-t}{N}}}\,\tilde\gamma\Big(\tilde\gamma^{-1}+1\Big)^{\frac{N-t}{N}} - \frac{c_N}{2t}\right]\\ &\leqslant m_2^{\frac{N-t}{N}}\left[\frac{(1-t)P_t(B)}{|B|^{\frac{N-t}{N}}}\,\tilde\gamma\left(2\tilde\gamma^{-1}\right)^{\frac{N-t}{N}} - \frac{c_N}{2t}\right]\\ &= m_2^{\frac{N-t}{N}}\left[\frac{2^{\frac{N-t}{N}}(1-t)P_t(B)}{|B|^{\frac{N-t}{N}}}\,\tilde\gamma^{\frac{t}{N}} - \frac{c_N}{2t}\right]\end{aligned}$$

which is negative if $\tilde\gamma$ is small enough, i. e.

$$\tilde\gamma < \left[\frac{c_N}{2t}\,\frac{|B|^{\frac{N-t}{N}}}{2^{\frac{N-t}{N}}(1-t)P_t(B)}\right]^{\frac{N}{t}}.$$

The proof is concluded. □

When (3.13) does not hold, one obtains for free some interesting density bounds.

Given a measurable set E we denote by $\partial^m E$ the measure theoretic boundary of E defined as

$$\partial^m E = \{x \in \mathbb{R}^N : \ |E \cap B_r(x)| > 0 \text{ and } |E \setminus B_r(x)| > 0 \text{ for all } r > 0\}.$$

Lemma 3.9. *Let F be a set of finite t-perimeter and volume m, and let $x_0 \in \mathbb{R}^N$. Assume*

$$\text{either } F_1 := F \setminus B(x_0, r) \text{ and } F_2 := F \cap B(x_0, r), \tag{3.18}$$

$$\text{or } F_2 := F \setminus B(x_0, r) \text{ and } F_1 := F \cap B(x_0, r), \tag{3.19}$$

and suppose that $|F_2| < m_0$, with m_0 be as in Lemma 3.5, and

$$(1-t)[P_t(F_1) + P_t(F_2) - P_t(F)] \geqslant \frac{\mathcal{F}_{s,t}(F_2)}{2}. \tag{3.20}$$

Then

$$\int_{F_1}\int_{F_2} \frac{dx\,dy}{|x-y|^{N+t}} \geqslant \frac{c_N}{t(1-t)} |F_2|^{\frac{N-t}{N}} \tag{3.21}$$

If $x_0 \in \partial^m F$ and (3.20) holds for any $r \leqslant r_0$, we also have the estimate

$$|F \cap B(x_0, r)| \geqslant c_0\, r^N \qquad \text{for all } r \in (0, r_0], \tag{3.22}$$

where the constant $c_0 > 0$ depend only on N and δ_0.

Proof. Without loss of generality we can assume $x_0 = 0$. Also, using either (3.18) or (3.19), (3.20) and (2.1), we have that

$$\int_{F_1}\int_{F_2} \frac{dx\,dy}{|x-y|^{N+t}} = \frac{1-t}{2(1-t)}(P_t(F_1) + P_t(F_2) - P_t(F)) \geqslant \frac{\mathcal{F}_{s,t}(F_2)}{4(1-t)} \geqslant \frac{\phi(|F_2|)}{4(1-t)}.$$

This and (3.9) (which can be used here thanks to the fact that we are assuming $|F_2| < m_0$) imply (3.21).

Now we prove (3.22). For this, we take F_1 and F_2 as in (3.18) and we define $\mu(r) := |B(0,r) \cap F| = |F_2|$. Note that by the co-area formula

$$\mu'(r) = \mathcal{H}^{N-1}(\partial B(0,r) \cap F), \quad \text{for a. e. } r.$$

Then, by (3.21) and the fact that $F_1 := F \setminus B(0,r) \subset (B(0,r))^c$,

$$\frac{c_N}{t(1-t)}\, \mu(r)^{\frac{N-t}{N}} \leqslant \int_{F_2}\int_{F_1} \frac{dx\,dy}{|x-y|^{N+t}} \leqslant \int_{F_2}\int_{(B(0,r))^c} \frac{dx\,dy}{|x-y|^{N+t}}.$$

For any $x \in F \cap B(0,r)$, we have

$$\int_{(B(0,r))^c} \frac{dy}{|x-y|^{N+t}} \leqslant \int_{(B(x,r-|x|))^c} \frac{dy}{|x-y|^{N+t}} = \frac{N\omega_N}{t}(r-|x|)^{-t}$$

that leads to

$$\int_{F_2}\int_{(B(0,r))^c} \frac{dx\,dy}{|x-y|^{N+t}} \leqslant \frac{c_N}{t}\int_0^r \mu'(z)(r-z)^{-t}\,dz.$$

Finally we arrive at the following integro-differential inequality

$$\mu(r)^{\frac{N-t}{N}} \leqslant c_N(1-t)\int_0^r \mu'(z)(r-z)^{-t}\,dz.$$

We may integrate the last inequality in the r variable on the interval $(0,\rho)$ and get

$$\int_0^\rho \mu(r)^{\frac{N-t}{N}}\,dr \leqslant c_N(1-t)\int_0^\rho\int_0^r \mu'(z)(r-z)^{-t}\,dz\,dr,$$

interchanging the order of integration,

$$\int_0^\rho \int_0^r \mu'(z)(r-z)^{-t}\,dz\,dr = \int_0^\rho \mu'(z) \int_z^\rho (r-z)^{-t}\,dr\,dz,$$

we get

$$\int_0^\rho \mu(r)^{\frac{N-t}{N}}\,dr \leqslant c_N\,\rho^{1-t}\mu(\rho).$$

Now we arrive at the desired result, indeed, following [12] (see the end of p. 9), it is possible to prove that

$$\mu(r) \geqslant g(r) := \left[\frac{1}{2c_N(N+1-t)}\right]^{\frac{N}{t}} r^N \tag{3.23}$$

for any $r < r_0 = (m_0/\omega_N)^{1/N}$, where g satisfies

$$\int_0^\rho g(r)^{\frac{N-t}{N}}\,dr \geqslant 2c_N\,\rho^{1-t}g(\rho),$$

with the same constant c_N as in (3.23). □

The combination of Proposition 3.8 and Lemma 3.9 yield the following density estimate:

Proposition 3.10. *There exist $r_0 = r_0(m, N, \delta_0) > 0$ such that, if F is a minimizer for $\phi(m)$ and $x_0 \in \partial^m F$, there holds*

$$|B(x_0, r) \cap F| \geqslant c_0\, r^N$$

for any $r < r_0$, where c_0 is as in (3.22).

Proof. Let F_1 and F_2 be as in (3.18). Up to choosing r_0 small enough, that is,

$$\omega_N r_0^N \leqslant \varepsilon(N, \delta_0)\min(1, m)\,,$$

we can suppose that F_1 and F_2 satisfy the hypotheses of Proposition 3.8. Thus, since F is a minimum, we obtain that (3.13) cannot hold true. Hence (3.20) is satisfied, and so we can apply (3.22) in Lemma 3.9 and obtain the desired result. □

Proposition 3.11. *Let F be a minimum for $\phi(m)$. Then F is essentially bounded. Moreover, if $t = 1$, for any $s < t$, $s \in (0, 1)$, F is also essentially connected in the sense of* [2]*, that is, it cannot be decomposed into two disjoint sets F_1 and F_2 of positive measure such that $P(F) = P(F_1) + P(F_2)$.*

Proof. Let F be a minimum. First we prove that it is bounded. By contradiction, if not, there exists a sequence $x_k \in \partial^m F$ such that $|x_k| \to \infty$ as $k \to \infty$. In particular, up to a subsequence, we may suppose that all the balls $B(x_k, 1)$ are disjoint, hence so are the balls $B(x_k, r)$ when $r \in (0, 1)$. Hence

$$m = |F| \geqslant \sum_k |B(x_k, r) \cap F|.$$

On the other hand, by Proposition 3.10, we know that $|B(x_k, r) \cap F| \geqslant c_0 r^N$ if r is small enough, hence we obtain that

$$m \geqslant \sum_k c_0 r^N = +\infty,$$

which is clearly not possible.

This proves that F is bounded. Now we show that, if $t = 1$, F is also essentially connected. Suppose, by contradiction, that F can be decomposed into two disjoint sets F_1 and F_2 of positive measure such that

$$P(F) = P(F_1) + P(F_2). \tag{3.24}$$

Since F is bounded, so are F_1 and F_2, say F_1, $F_2 \subseteq B(0,R)$, for some $R > 0$. Hence, we consider the translation $F_{2,k} := F_2 + (k, 0, \dots, 0)$ and we observe that if $x \in F_1$ and $y \in F_{2,k}$ we have that

$$|x-y| \geqslant |y| - |x| \geqslant k - 2R \geqslant \frac{k}{2}$$

if k is large enough. Accordingly, we have that

$$\int_{F_1}\int_{F_{2,k}} \frac{dx\,dy}{|x-y|^{N+s}} \leqslant \int_{B(0,R)}\int_{B(0,R)+(k,0,\dots,0)} \frac{dx\,dy}{(k/2)^{N+s}} = \frac{c_N R^{2N}}{k^{N+s}}$$

and so

$$\lim_{k\to+\infty} \int_{F_1}\int_{F_{2,k}} \frac{dx\,dy}{|x-y|^{N+s}} = 0. \tag{3.25}$$

Notice also that, if $G_k := F_1 \cup F_{2,k}$ we have that $|G_k| = |F_1| + |F_{2,k}| = |F_1| + |F_2| = |F|$, for k large, and so, by the minimality of F, (2.1), (2.2) and (3.24) we have that

$$\begin{aligned}
& N\omega_N P(F_1) + N\omega_N P(F_2) - sP_s(F_1) - sP_s(F_2) + 2s\int_{F_1}\int_{F_2} \frac{dx\,dy}{|x-y|^{N+s}} \\
= \;& N\omega_N P(F) - sP_s(F) \\
= \;& \mathcal{F}_{s,1}(F) \\
\leqslant \;& \mathcal{F}_{s,1}(G_k) \\
= \;& N\omega_N P(G_k) - sP_s(G_k) \\
\leqslant \;& N\omega_N P(F_1) + N\omega_N P(F_{2,k}) - sP_s(F_1) - sP_s(F_{2,k}) + 2s\int_{F_1}\int_{F_{2,k}} \frac{dx\,dy}{|x-y|^{N+s}} \\
= \;& N\omega_N P(F_1) + N\omega_N P(F_2) - sP_s(F_1) - sP_s(F_2) + 2s\int_{F_1}\int_{F_{2,k}} \frac{dx\,dy}{|x-y|^{N+s}}.
\end{aligned}$$

Therefore, taking the limit as $k \to +\infty$ and using (3.25), we obtain that

$$2s\int_{F_1}\int_{F_2} \frac{dx\,dy}{|x-y|^{N+s}} \leqslant 0.$$

This says that either F_1 or F_2 must have zero measure, against our assumptions. □

We conclude the section with the following estimate on the fractional isoperimetric deficit, which will be important to localize minimizing sequences.

Lemma 3.12. *There exists $m_2 = m_2(N, \delta_0)$ such that for any $m \in (0, m_2)$ the following statement holds true.*

Let $F \subset \mathbb{R}^N$ be a set of finite perimeter. Assume that $\mathcal{F}_{s,t}(F) \leqslant \mathcal{F}_{s,t}(B_m)$. Then there exists $c_0 > 0$ such that

$$D_t(F) := \frac{P_t(F) - P_t(B_m)}{P_t(B_m)} \leqslant c_0\, m^{\frac{t-s}{N}}. \tag{3.26}$$

In addition, there exists a translation of F (still denoted by F for simplicity) such that

$$|F \,\Delta\, B_m| \leqslant c_0\, m^{1+\frac{t-s}{2N}}. \tag{3.27}$$

Proof. First recall that

$$P_t(B_m) = \frac{P_t(B)}{|B|^{\frac{N-t}{N}}}\, m^{\frac{N-t}{N}}. \tag{3.28}$$

Also, by our assumptions,

$$(1-t)P_t(F) - sP_s(F) = \mathcal{F}_{s,t}(F) \leqslant \mathcal{F}_{s,t}(B_m) \leqslant (1-t)P_t(B_m). \tag{3.29}$$

Using (3.4) we have that

$$\begin{aligned} sP_s(F) &\leqslant c_0^{1-\frac{s}{t}} m^{1-\frac{s}{t}} [(1-t)P_t(B_m) + c_0\, m]^{\frac{s}{t}} \\ &\leqslant c_0^{1-\frac{s}{t}} m^{1-\frac{s}{t}} \left[\frac{(1-t)P_t(B)}{|B|^{\frac{N-t}{N}}}\, m^{\frac{N-t}{N}} + c_0\, m \right]^{\frac{s}{t}} \\ &\leqslant c_0^{1-\frac{s}{t}} \left[\frac{(1-t)P_t(B)}{|B|^{\frac{N-t}{N}}} + c_0 \right]^{\frac{s}{t}} m^{\frac{N-s}{N}}, \end{aligned}$$

for small m. From this and (3.29), we have that

$$\frac{P_t(F) - P_t(B_m)}{P_t(B_m)} \leqslant c_0^{1-\frac{s}{t}} \left[\frac{(1-t)P_t(B)}{|B|^{\frac{N-t}{N}}} + c_0 \right]^{\frac{s}{t}} \frac{|B|^{\frac{N-t}{N}}}{(1-t)P_t(B)}\, m^{\frac{N-s}{N} - \frac{N-t}{N}} \leqslant c_0 m^{\frac{t-s}{N}}.$$

This proves (3.26).

To prove (3.27) it is sufficient to use (3.26) and the estimate

$$c_0\, D_t(F) \geqslant \frac{|F \,\Delta\, B_m|^2}{|B_m|^2},$$

which was proved in [18, Theorem 1.1] for any $t \geqslant \delta_0 > 0$. Together with (3.26) and possibly increasing the constant c_0, this implies (3.27). □

4. Existence of minimizers

In order to prove the first statement in Theorem 1.1, and for further use as well, we prove a general result on integro-differential equations:

Lemma 4.1. *Let $m, t \in (0,1)$. Let $c, \bar\rho \geqslant 0$ be such that*

$$c \geqslant (1-t) m^{\frac{t}{N}}, \tag{4.1}$$

and let $\mu : [0, +\infty) \to [0, m]$ be a non-increasing function such that

$$-\int_\rho^\infty \mu'(z)(z-\rho)^{-t}\, dz \geqslant \frac{3c}{1-t} \mu(\rho)^{\frac{N-t}{N}} \qquad \text{for all } \rho \geqslant \bar\rho\,. \tag{4.2}$$

Then, there holds

$$\mu\left(\bar\rho + \frac{(2m)^{\frac{t}{N}} N}{ct} \right) = 0\,. \tag{4.3}$$

Proof. Integrating (4.2) between $R \geqslant \bar\rho$ and $+\infty$, we obtain

$$-\int_R^\infty \left(\int_\rho^\infty \mu'(z)(z-\rho)^{-t}\, dz \right) d\rho \geqslant \frac{3c}{1-t} \int_R^\infty \mu(\rho)^{\frac{N-t}{N}}\, d\rho. \tag{4.4}$$

Also, if $z \in [R, R+1]$ we have that $z - R \leqslant 1$ and so, since $\mu' \leqslant 0$ a. e., we get that

$$-\int_R^{R+1} \mu'(z)(z-R)^{1-t}\,dz \leqslant -\int_R^{R+1} \mu'(z)\,dz = \mu(R) - \mu(R+1).$$

Therefore, interchanging the order of integration in (4.4), integrating by parts and using that $\mu \in [0, m]$ and (4.1), we see that

$$\begin{aligned}
-\int_R^\infty \left(\int_\rho^\infty \mu'(z)(z-\rho)^{-t}\,dz\right) d\rho &= -\int_R^\infty \left(\int_R^z \mu'(z)(z-\rho)^{-t}\,d\rho\right) dz \\
&= -\frac{1}{1-t}\int_R^\infty \mu'(z)(z-R)^{1-t}\,dz \\
&\leqslant \frac{\mu(R)-\mu(R+1)}{1-t} - \frac{1}{1-t}\int_{R+1}^\infty \mu'(z)(z-R)^{1-t}\,dz \\
&= \frac{\mu(R)}{1-t} + \int_{R+1}^\infty \mu(z)(z-R)^{-t}\,dz \\
&\leqslant \frac{\mu(R)}{1-t} + \int_{R+1}^\infty \mu(z)\,dz \\
&\leqslant \frac{\mu(R)}{1-t} + m^{\frac{t}{N}}\int_R^\infty \mu(z)^{\frac{N-t}{N}}\,dz \\
&\leqslant \frac{1}{1-t}\left(\mu(R) + c\int_R^\infty \mu(z)^{\frac{N-t}{N}}\,dz\right).
\end{aligned}$$

Recalling (4.4), this gives the integro-differential inequality

$$\mu(\rho) \geqslant 2c\int_\rho^\infty \mu(z)^{\frac{N-t}{N}}\,dz \qquad \text{for all } \rho \geqslant \bar\rho\,. \tag{4.5}$$

Let now

$$g(\rho) := \begin{cases} \left[(2\mu(\bar\rho))^{\frac{t}{N}} - \dfrac{ct}{N}(\rho-\bar\rho)\right]^{\frac{N}{t}} & \text{if } \rho \in \left[\bar\rho, \bar\rho + \dfrac{(2\mu(\bar\rho))^{\frac{t}{N}}N}{ct}\right] \\ \\ 0 & \text{if } \rho > \bar\rho + \dfrac{(2\mu(\bar\rho))^{\frac{t}{N}}N}{ct}\,. \end{cases}$$

Notice that g is continuous and it satisfies

$$2c\int_\rho^\infty g(z)^{\frac{N-t}{N}}\,dz = 2g(\rho) \qquad \text{for all } \rho \in \left[\bar\rho, \bar\rho + \frac{(2\mu(\bar\rho))^{\frac{t}{N}}N}{ct}\right]. \tag{4.6}$$

We now claim that

$$g(\rho) \geqslant \mu(\rho) \qquad \text{for all } \rho \in \left[\bar\rho, \bar\rho + \frac{(2\mu(\bar\rho))^{\frac{t}{N}}N}{ct}\right]. \tag{4.7}$$

Indeed, we consider the set $S := \inf\{\rho > \bar\rho : \ \mu(z) \geqslant g(z) \text{ for all } z \geqslant \rho\}$. By construction, $S \subseteq [\bar\rho, +\infty)$. Furthermore, if $z \geqslant \bar\rho + [(2\mu(\bar\rho))^{\frac{t}{N}}N]/ct$ then $g(z) = 0 \leqslant \mu(z)$, therefore $\bar\rho + [(2\mu(\bar\rho))^{\frac{t}{N}}N]/ct \in I$. As a consequence, we can define $R_* := \inf I$, and we have that

$$R_* \in [\bar\rho,\ \bar\rho + [(2\mu(\bar\rho))^{\frac{t}{N}}N]/ct]. \tag{4.8}$$

By definition of R_*, there exists a sequence $R_n \to R_*$, with $R_n \leqslant R_*$, such that $g(R_n) > \mu(R_n)$. Then, recalling (4.5) and (4.6), we have

$$\begin{aligned} g(R_n) &> \mu(R_n) \\ &\geqslant 2c \int_{R_n}^{\infty} \mu(z)^{\frac{N-t}{N}} dz \\ &\geqslant 2c \int_{R_n}^{R_*} \mu(z)^{\frac{N-t}{N}} dz + 2c \int_{R_*}^{\infty} g(z)^{\frac{N-t}{N}} dz \\ &= 2c \int_{R_n}^{R_*} \mu(z)^{\frac{N-t}{N}} dz + 2g(R_*). \end{aligned} \tag{4.9}$$

Passing to the limit in (4.9) as $n \to +\infty$ we get $g(R_*) \geqslant 2g(R_*)$, which means $g(R_*) = 0$. This implies that $R_* \geqslant \bar\rho + [(2\mu(\bar\rho))^{\frac{t}{N}} N]/ct$.

This information, combined with (4.8), gives that $R_* = \bar\rho + [(2\mu(\bar\rho))^{\frac{t}{N}} N]/ct$, and this in turn implies (4.7).

Then, we evaluate (4.7) at $\rho = \bar\rho + [(2\mu(\bar\rho))^{\frac{t}{N}} N]/ct$ and we obtain (4.3).

□

With the above result, we are in the position to prove the first statement in Theorem 1.1, concerning the existence of minimizers for small volumes.

Theorem 4.2. *For any $0 \leqslant s < t \leqslant 1$, $t - s \geqslant \delta_0 > 0$, there exists $\bar m_0 = \bar m_0(N, \delta_0) > 0$ such that for all $m \in (0, \bar m_0)$, problem (1.3) has a minimizer $F \subset \mathbb{R}^N$.*

Proof. Suppose $0 < s < t < 1$. We use the direct method of the calculus of variations. Let us consider a minimizing sequence $\{F_k\} \subset \mathbb{R}^N$, that is a sequence of sets of finite t-perimeter F_k with $|F_k| = m$ such that

$$\lim_{k\to\infty} \mathcal{F}_{s,t}(F_k) = \phi(m). \tag{4.10}$$

Let also set $r_m := (m/\omega_N)^{1/N} > 0$, so that $|B(0, r_m)| = m$. Our goal is to show that we can reduce ourselves to the case in which F_k lies in a large ball, independent of k. More precisely, we claim that there exist $\rho_* > 0$ and sets G_k, with $|G_k| = m$, such that

$$G_k \subseteq B(0, \rho_*) \quad \text{and} \quad \mathcal{F}_{s,t}(G_k) \leqslant \mathcal{F}_{s,t}(F_k). \tag{4.11}$$

To prove it, we take $\rho \geqslant r_m$ and we set

$$X_k^\rho := F_k \cap B(0, \rho) \text{ and } Y_k^\rho := F_k \setminus B(0, \rho). \tag{4.12}$$

We distinguish two cases:

$$\text{either for any } \rho \geqslant r_m \text{ we have} \qquad (1-t)[P_t(X_k^\rho) + P_t(Y_k^\rho) - P_t(F_k)] \geqslant \frac{\mathcal{F}_{s,t}(Y_k^\rho)}{2} \tag{4.13}$$

$$\text{or there exists } \rho \geqslant r_m \text{ such that} \qquad (1-t)[P_t(X_k^\rho) + P_t(Y_k^\rho) - P_t(F_k)] \leqslant \frac{\mathcal{F}_{s,t}(Y_k^\rho)}{2}. \tag{4.14}$$

Let us first deal with (4.13). Then, thanks to (4.13), we can apply Lemma 3.9 using the setting in (3.19): accordingly, from (3.21) we see that

$$\int_{X_k^\rho} \int_{Y_k^\rho} \frac{dx\, dy}{|x-y|^{N+t}} \geqslant \frac{c_N}{t(1-t)} |Y_k^\rho|^{\frac{N-t}{N}}.$$

Let us define the non-increasing function $\eta(\rho) := |F_k \setminus B(0,\rho)| = |Y_k^\rho|$. Note that by the co-area formula

$$\eta'(\rho) = -\mathcal{H}^{N-1}(\partial B(0,\rho) \cap F), \quad \text{for a. e. } \rho > 0.$$

Proceeding in a similar way to the Lemma 3.9, we have

$$\begin{aligned}
\int_{Y_k^\rho}\int_{X_k^\rho} \frac{dx\,dy}{|x-y|^{N+t}} &\leqslant \int_{Y_k^\rho}\int_{B(0,\rho)} \frac{dx\,dy}{|x-y|^{N+t}} \\
&\leqslant \int_{Y_k^\rho}\left(\int_{(B(y,|y|-\rho))^c} \frac{dx}{|x-y|^{N+t}}\right) dy \\
&\leqslant -\frac{N\omega_N}{t}\int_\rho^\infty \eta'(z)(z-\rho)^{-t}\,dz,
\end{aligned}$$

whence

$$-\int_\rho^\infty \eta'(z)(z-\rho)^{-t}\,dz \geqslant \frac{c_N}{1-t}\eta(\rho)^{\frac{N-t}{N}},$$

which is (4.2). We now apply Lemma 4.1 with $\mu = \eta$, $c = c_N/3$ and $\bar\rho = r_m$. Notice that, possibly reducing $\bar m_0$, we can ensure that condition (4.1) is satisfied. From (4.3) we conclude that

$$\eta\left(r_m + \frac{3(2m)^{\frac{t}{N}}N}{c_N t}\right) = 0\,,$$

that is,

$$F_k \subseteq B\left(0, r_m + \frac{3(2m)^{\frac{t}{N}}N}{c_N t}\right).$$

This proves (4.11) for such

$$\rho_* := r_m + \frac{3(2m)^{\frac{t}{N}}N}{c_N t}$$

in case (4.13) (here one can take $G_k := F_k$).

So, we now deal with case (4.14). In this case, we use (3.27) and we obtain (up to a translation of F_k that is still denoted by F_k) that

$$|F_k \setminus B(0,r_m)| + |B(0,r_m)\setminus F_k| = |F_k\,\Delta\,B(0,r_m)| \leqslant c_0\, m^{1+\frac{t-s}{2N}},$$

c_0 as in Lemma 3.12. In particular, if $\rho \geqslant r_m$ is the one given by (4.14) we have that

$$\begin{aligned}
|F_k \cap B(0,\rho)| &\geqslant |F_k \cap B(0,r_m)| \\
&= |B(0,r_m)| - |B(0,r_m)\setminus F_k| \\
&\geqslant m - c_0\, m^{1+\frac{t-s}{2N}} \\
&\geqslant \frac{m}{2}
\end{aligned}$$

if m is small enough, i. e.

$$m \leqslant \left[\frac{1}{2c_0}\right]^{\frac{2N}{t-s}} \tag{4.15}$$

and moreover

$$|F_k \setminus B(0,\rho)| \leqslant |F_k \setminus B(0,r_m)| \leqslant c_0\, m^{1+\frac{t-s}{2N}}.$$

Therefore, for small m, recalling (4.12) we see that

$$2c_0\, m^{\frac{t-s}{2N}}\ \min\big(1,|X_k^\rho|\big) = 2c_0\, m^{\frac{t-s}{2N}}\ \min\big(1,|F_k\cap B(0,\rho)|\big) \geqslant 2c_0\, m^{\frac{t-s}{2N}}\ \frac{m}{2}$$
$$= c_0\, m^{1+\frac{t-s}{2N}} \geqslant |F_k\setminus B(0,\rho)| = |Y_k^\rho|.$$

Thanks to this and (4.14), we can apply Proposition 3.8, with $\varepsilon := 2c_0\, m^{\frac{t-s}{2N}}$, $F_1 := X_k^\rho$ and $F_2 := Y_k^\rho$.

Hence, from Proposition 3.8, we find G_k such that $\mathcal{F}_{s,t}(G_k) \leqslant \mathcal{F}_{s,t}(F_k)$; notice also that, in light of (3.14), we know that G_k is either a ball or a dilation of X_k^ρ, which is contained in $B(0,2\rho)$. Thus also G_k is contained in a ball of universal radius, and this establishes (4.11) also in case (4.14).

Thus, by (4.11), we have constructed a minimizing sequence G_k that is uniformly contained in a fixed ball. By Proposition 3.1, we also obtain that

$$(1-t)P_t(G_k) \leqslant 2[\mathcal{F}_{s,t}(G_k)+c_0 m] \leqslant 2[\mathcal{F}_{s,t}(B(0,r_m))+c_0 m],$$

hence the t-perimeter of G_k is bounded uniformly in k.

By the compact embedding of $H^{\frac{t}{2}}$ into $H^{\frac{s}{2}}$ (see [16, Section 7]), up to extracting a subsequence, the sets G_k converge in $W^{s,1}$ (hence also in L^1) to a limit set G, and it holds

$$\lim_{k\to+\infty} P_s(G_k) = P_s(G).$$

The lower semicontinuity of the t-perimeter yields that

$$\liminf_{k\to+\infty} P_t(G_k) \geqslant P_t(G)$$

Hence, by (4.10) and (4.11),

$$\begin{aligned}\mathcal{F}_{s,t}(G) &= (1-t)P_t(G) - sP_s(G) \leqslant \liminf_{k\to+\infty}[(1-t)P_t(G_k) - sP_s(G_k)]\\ &= \liminf_{k\to+\infty}\mathcal{F}_{s,t}(G_k) \leqslant \liminf_{k\to+\infty}\mathcal{F}_{s,t}(F_k) \leqslant \phi(m),\end{aligned}$$

hence $\mathcal{F}_{s,t}(G)=\phi(m)$ and so $F:=G$ is the desired minimizer.

In the case $0=s<t\leqslant 1$, our problem reduces to the (fractional) isoperimetric problem, hence it is well known that there exists a minimizer F for (1.3) and it is a ball of volume m, for any $m>0$.

When $0<s<t=1$ the previous arguments can be easily adapted, including the analog of Lemma 4.1 which becomes an ordinary differential inequality, and the only difference is that one needs to use the compact embedding of BV into $H^{\frac{s}{2}}$ for $0<s<1$. □

5. Regularity of minimizers

The aim of this section is to prove the regularity and rigidity theory necessary to prove the second statement in Theorem 1.1 and Theorem 1.2. We begin with a simple observation.

Lemma 5.1. *Let ϕ be the function describing problem (1.3). Then F is a minimizer of $\phi(m)$ if and only if $F/m^{1/N}$ is a minimizer of problem*

$$\min\{(1-t)P_t(U) - m^{\frac{t-s}{N}} sP_s(U) : |U|=1\}.$$

Proof. Let $F \subseteq \mathbb{R}^N$ such that $|F| = m$ and let $U = F/m^{1/N}$. Then

$$\begin{aligned}(1-t)P_t(F) - sP_s(F) &= (1-t)P_t(m^{1/N}U) - sP_s(m^{1/N}U)\\ &= m^{\frac{N-t}{N}}\left[(1-t)P_t(U) - m^{\frac{t-s}{N}}sP_s(U)\right],\end{aligned}$$

which gives the desired result. □

The previous lemma allow us to consider, in what follows, the functional

$$\mathcal{F}^{\varepsilon}_{s,t} = (1-t)P_t - \varepsilon s P_s,$$

where we set $\varepsilon = m^{(t-s)/N}$. Indeed, the behavior of a minimizer of $\phi(m)$ is the same, up to a rescaling, to that of

$$\min\left\{\mathcal{F}^{\varepsilon}_{s,t}(E) : |E| = \omega_N\right\}. \tag{5.1}$$

Indeed

$$\begin{aligned}&F \text{ is a minimizer for problem (5.1) if and only if}\\ &\left(\tfrac{m}{\omega_N}\right)^{\frac{1}{N}} F \text{ is a minimizer for problem (1.3) with } \varepsilon = \left(\tfrac{m}{\omega_N}\right)^{\frac{t-s}{N}}.\end{aligned} \tag{5.2}$$

The next lemma allows us to say that if F is a set of $\mathbb{R}^N$ such that $||F| - \omega_N|$ is small enough than the volume constraint can be dropped. Let us consider the following problem:

$$\min\left\{\mathcal{G}_{\varepsilon,\Lambda}(E) : ||E| - \omega_N| < \Lambda^{-1}\right\}, \tag{5.3}$$

for some $\Lambda > 0$, where

$$\mathcal{G}_{\varepsilon,\Lambda}(E) = (1-t)P_t(E) - \varepsilon s P_s(E) + \Lambda||E| - \omega_N|.$$

Letting

$$\varepsilon_0 := \left(\frac{\bar{m}_0}{\omega_N}\right)^{\frac{t-s}{N}}, \tag{5.4}$$

with $\bar{m}_0$ as in Theorem 4.2, we have the following result:

Lemma 5.2. *There exists $\Lambda_0 = \Lambda_0(N, \delta_0) > 0$ such that F_ε is a volume constrained minimizer of problem (5.1), with $\varepsilon < \varepsilon_0$, if and only if F_ε is a minimizer of problem (5.3), for any $\Lambda \geqslant \Lambda_0(1+\varepsilon_0)$.*

Proof. First, let F_ε be a minimizer of problem (5.3) with $|F_\varepsilon| = \omega_N$. Then, for any set G with $|G| = \omega_N$, we have that

$$\mathcal{F}^{\varepsilon}_{s,t}(G) = \mathcal{G}_{\varepsilon,\Lambda}(G) \geqslant \mathcal{G}_{\varepsilon,\Lambda}(F_\varepsilon) = \mathcal{F}^{\varepsilon}_{s,t}(F_\varepsilon),$$

which shows that F_ε is a minimizer of problem (5.1).

Viceversa, we prove that a volume constrained minimizer F_ε of problem (5.1), with $\varepsilon < \varepsilon_0$, is also a minimizer of (5.3) for any Λ sufficiently large. For this, we argue by contradiction and we assume that there exist a sequence $\Lambda_n \to +\infty$, and sets $E_n \subset \mathbb{R}^N$ such that, letting $\mathcal{G}_n := \mathcal{G}_{\varepsilon,\Lambda_n}$, we have

$$\mathcal{G}_n(E_n) < \mathcal{G}_n(F_\varepsilon) = \mathcal{F}^{\varepsilon}_{s,t}(F_\varepsilon). \tag{5.5}$$

Notice that for all $n \in \mathbb{N}$ there holds

$$\sigma_n := \big||E_n| - \omega_N\big| > 0. \tag{5.6}$$

Indeed, if by contradiction we suppose that $\sigma_n = 0$ for some $n \in \mathbb{N}$, we would have that $|E_n| = \omega_N$, thus

$$\mathcal{G}_n(E_n) = \mathcal{F}^\varepsilon_{s,t}(E_n) \geqslant \mathcal{F}^\varepsilon_{s,t}(F_\varepsilon),$$

due to the minimality of F_ε. This would be in contradiction with (5.5), and so (5.6) is proved. We also claim that there exists a constant $c_0 > 0$ independent of n, such that

$$(1-t)P_t(E_n) \leqslant c_0 \qquad \text{and} \qquad sP_s(E_n) \leqslant c_0 \qquad \text{for all } n \in \mathbb{N}. \tag{5.7}$$

To show this, proceeding as in Proposition 3.1 and (5.5), we see that, for $\Lambda_n \geqslant c_0\, \varepsilon_0^{\frac{t}{t-s}}$, we have

$$\begin{aligned}
(1-t)P_t(E_n) &\leqslant 2\left[\mathcal{F}^\varepsilon_{s,t}(E_n) + c_0\varepsilon_0^{\frac{t}{t-s}}|E_n|\right] \\
&\leqslant 2\left[\mathcal{F}^\varepsilon_{s,t}(E_n) + c_0\,\varepsilon_0^{\frac{t}{t-s}}\big||E_n| - \omega_N\big| + \omega_N c_0 \varepsilon_0^{\frac{t}{t-s}}\right] \\
&\leqslant 2\left[\mathcal{G}_n(E_n) + \omega_N c_0\,\varepsilon_0^{\frac{t}{t-s}}\right] \\
&\leqslant 2\left[\mathcal{F}^\varepsilon_{s,t}(F_\varepsilon) + \omega_N c_0\,\varepsilon_0^{\frac{t}{t-s}}\right] \\
&\leqslant 2\left[\mathcal{F}^\varepsilon_{s,t}(B) + \omega_N c_0\,\varepsilon_0^{\frac{t}{t-s}}\right] \\
&\leqslant 2\left[(1-t)P_t(B) + \omega_N c_0\,\varepsilon_0^{\frac{t}{t-s}}\right],
\end{aligned}$$

recalling that B denotes the ball centered in 0 and radius 1, with $|B(0,1)| = \omega_N$. This gives the bound for $(1-t)P_t(E_n)$, and then the bound on $sP_s(E_n)$ follows from (2.3). This proves (5.7).

From (5.5) and (5.7) it follows that $\Lambda_n\sigma_n$ is also uniformly bounded, that is,

$$\sigma_n \leqslant \frac{c_0}{\Lambda_n} \to 0 \qquad \text{as } n \to +\infty.$$

Moreover, for $\sigma_n \leqslant 1/2$ we have, supposing $\sigma_n = |E_n| - \omega_N > 0$ (the other case can be treated in a similar way),

$$\left(\frac{|E_n|}{\omega_N}\right)^{-\frac{N-s}{N}} \geqslant \left(1 + \frac{\sigma_n}{\omega_N}\right)^{-\frac{N-s}{N}} \geqslant 1 - \frac{N-s}{N}\frac{\sigma_n}{\omega_N}\,, \tag{5.8}$$

and similarly

$$\left(\frac{|E_n|}{\omega_N}\right)^{-\frac{N-t}{N}} \leqslant \left(1 - \frac{\sigma_n}{\omega_N}\right)^{-\frac{N-t}{N}} = 1 + \frac{N-t}{N}\frac{\sigma_n}{\omega_N}\,. \tag{5.9}$$

We now define

$$\tilde{E}_n = \left(\frac{|E_n|}{\omega_N}\right)^{-\frac{1}{N}} E_n\,,$$

and we use (5.7), (5.8) and (5.9) to obtain

$$sP_s(\tilde{E}_n) = \left(\frac{|E_n|}{\omega_N}\right)^{-\frac{N-s}{N}} sP_s(E_n) \geqslant \left(1 - \frac{N-s}{N}\frac{\sigma_n}{\omega_N}\right) sP_s(E_n) \geqslant sP_s(E_n) - c_0\sigma_n,$$

$$\text{and} \qquad (1-t)P_t(\tilde{E}_n) = \left(\frac{|E_n|}{\omega_N}\right)^{-\frac{N-t}{N}} (1-t)P_t(E_n) \leqslant (1-t)P_t(E_n) + c_0\sigma_n\,,$$

where the constant c_0 may differ from line to line.

Therefore, since $|\tilde{E}_n| = \omega_N$, the minimality of F_ε gives

$$\begin{aligned}\mathcal{F}^\varepsilon_{s,t}(F_\varepsilon) &\leqslant \mathcal{F}^\varepsilon_{s,t}(\tilde{E}_n) = (1-t)P_t(\tilde{E}_n) - \varepsilon s P_s(\tilde{E}_n)\\ &\leqslant (1-t)P_t(E_n) - \varepsilon s P_s(E_n) + c_0(1+\varepsilon_0)\sigma_n\\ &= \mathcal{F}^\varepsilon_{s,t}(E_n) + c_0(1+\varepsilon_0)\sigma_n.\end{aligned}$$

By plugging this into (5.5) we find that

$$\begin{aligned}\mathcal{G}_\varepsilon(E_n) &< \mathcal{F}^\varepsilon_{s,t}(F_\varepsilon) \leqslant \mathcal{F}^\varepsilon_{s,t}(E_n) + c_0(1+\varepsilon_0)\sigma_n\\ &= \mathcal{G}_\varepsilon(E_n) - \Lambda_n\sigma_n + c_0(1+\varepsilon_0)\sigma_n.\end{aligned}$$

We simplify the term $\mathcal{G}_\varepsilon(E_n)$ and we divide by σ_n, which is possible thanks to (5.6), we conclude that

$$0 < -\Lambda_n + c_0(1+\varepsilon_0).$$

This gives a contradiction for Λ_n large enough, and proves that F_ε is a minimizer for problem (5.3). □

Lemma 5.3. *Let F_ε be a minimizer of problem* (5.3) *with $\varepsilon < \varepsilon_0$ and $\Lambda \geqslant \Lambda_0$, ε_0 and Λ_0 as in Lemma 5.2, and let E_ε be a set of finite perimeter with $\big||E_\varepsilon| - \omega_N\big| < 1/\Lambda$. Then,*

$$\begin{aligned}(1-t)P_t(F_\varepsilon) \leqslant\ & (1-t)P_t(E_\varepsilon) + \varepsilon\, c_N \left(1-\frac{s}{t}\right)^{-1} |F_\varepsilon \Delta E_\varepsilon|^{1-\frac{s}{t}} [(1-t)P_t(F_\varepsilon\Delta E_\varepsilon)]^{\frac{s}{t}}\\ &+\Lambda\big||E_\varepsilon| - \omega_N\big|.\end{aligned} \tag{5.10}$$

Proof. Notice that, denoting by $\int_U = \int_U f$ for a non-negative function f, the following computation holds

$$\int_{F_\varepsilon}\int_{F_\varepsilon^c} = \int_{F_\varepsilon\setminus E_\varepsilon}\int_{(F_\varepsilon\cup E_\varepsilon)^c} + \int_{F_\varepsilon\setminus E_\varepsilon}\int_{E_\varepsilon\setminus F_\varepsilon} + \int_{F_\varepsilon\cap E_\varepsilon}\int_{(F_\varepsilon\cup E_\varepsilon)^c} + \int_{F_\varepsilon\cap E_\varepsilon}\int_{E_\varepsilon\setminus F_\varepsilon}.$$

By interchanging the roles of F_ε and E_ε, and setting $f(x,y) = |x-y|^{-(s+N)}$ we get

$$\begin{aligned}P_s(F_\varepsilon) - P_s(E_\varepsilon) &= \int_{F_\varepsilon\setminus E_\varepsilon}\int_{(F_\varepsilon\cup E_\varepsilon)^c} - \int_{E_\varepsilon\setminus F_\varepsilon}\int_{(F_\varepsilon\cup E_\varepsilon)^c} + \int_{F_\varepsilon\cap E_\varepsilon}\int_{E_\varepsilon\setminus F_\varepsilon} - \int_{E_\varepsilon\cap F_\varepsilon}\int_{F_\varepsilon\setminus E_\varepsilon}\\ &\leqslant \int_{F_\varepsilon\setminus E_\varepsilon}\int_{(F_\varepsilon\cup E_\varepsilon)^c} + \int_{E_\varepsilon\setminus F_\varepsilon}\int_{F_\varepsilon\cap E_\varepsilon} \leqslant P_s(F_\varepsilon\Delta E_\varepsilon).\end{aligned} \tag{5.11}$$

Therefore, by the minimality of F_ε we get

$$\begin{aligned}(1-t)P_t(F_\varepsilon) &\leqslant (1-t)P_t(E_\varepsilon) + \varepsilon\,[sP_s(F_\varepsilon) - sP_s(E_\varepsilon)] + \Lambda\Big(\big||E_\varepsilon| - \omega_N\big| - \big||F_\varepsilon| - \omega_N\big|\Big)\\ &\leqslant (1-t)P_t(E_\varepsilon) + \varepsilon s P_s(F_\varepsilon\Delta E_\varepsilon) + \Lambda\big||E_\varepsilon| - \omega_N\big|.\end{aligned}$$

Hence the desired result follows from (2.3). □

We point out that from Lemma 5.3 it follows that F_ε is a multiplicative ω-minimizer for the t-perimeter. In the sequel, as customary, the fractional perimeter of a set E in a ball $B(x,R)$ will be denoted by $P_t(E, B(x,R))$.

Corollary 5.4. *Let ε_0 and Λ_0 be as in Lemma 5.2. Let F_ε be a minimizer of* (5.1) *with $\varepsilon < \varepsilon_0$, let $x \in \partial^m F_\varepsilon$, and let E_ε be a set of finite t-perimeter with*

$$F_\varepsilon \Delta E_\varepsilon \subset B(x,R). \tag{5.12}$$

There holds

$$P_t(F_\varepsilon, B(x,R)) \leqslant \frac{1+c_0R^{t-s}}{1-c_0R^{t-s}}\, P_t(E_\varepsilon, B(x,R)) \tag{5.13}$$

for some $c_0 > 0$ and for any $R < R_0 = R_0(N, \delta_0)$.

Proof. We observe that, by direct calculations, from (5.12), follows

$$\begin{aligned} &P_t(F_\varepsilon) - P_t(E_\varepsilon) = P_t(F_\varepsilon, B(x,R)) - P_t(E_\varepsilon, B(x,R)) \\ \text{and} \qquad &P_t(F_\varepsilon \Delta E_\varepsilon) \leqslant P_t(F_\varepsilon, B(x,R)) + P_t(E_\varepsilon, B(x,R)). \end{aligned} \tag{5.14}$$

Furthermore, thanks to Lemma 5.2 we know that F_ε is also a minimizer of (5.3), with $\Lambda = \Lambda_0$. From (5.10) and the fractional isoperimetric inequality (3.7), we then get

$$\begin{aligned} (1-t)P_t(F_\varepsilon, B(x,R)) \;\leqslant\; & (1-t)P_t(E_\varepsilon, B(x,R)) \\ & + \varepsilon_0\, c_N \left(1-\frac{s}{t}\right)^{-1} [c_N\, t]^{1-\frac{s}{t}} |F_\varepsilon \Delta E_\varepsilon|^{\frac{t-s}{N}} (1-t)P_t(F_\varepsilon \Delta E_\varepsilon) \\ & + \Lambda_0 \big| |E_\varepsilon| - \omega_N \big|. \end{aligned} \tag{5.15}$$

Moreover, again from the fractional isoperimetric inequality and using (5.12),

$$\begin{aligned} \Lambda_0 \big| |E_\varepsilon| - \omega_N \big| &= \Lambda_0 \Big| \big| |E_\varepsilon| - \omega_N \big| - \big| |F_\varepsilon| - \omega_N \big| \Big| \\ &\leqslant \Lambda_0 |F_\varepsilon \Delta E_\varepsilon|^{\frac{N-t}{N}} |F_\varepsilon \Delta E_\varepsilon|^{\frac{t}{N}} \\ &\leqslant c_N\, \Lambda_0\, t\, (1-t) P_t(F_\varepsilon \Delta E_\varepsilon)\, |F_\varepsilon \Delta E_\varepsilon|^{\frac{t}{N}} \\ &\leqslant c_N\, \Lambda_0\, t\, (1-t) P_t(F_\varepsilon \Delta E_\varepsilon) R^t. \end{aligned}$$

From this, (5.14) and (5.15) we arrive at

$$\begin{aligned} (1-t)P_t(F_\varepsilon, B(x,R)) \;\leqslant\; & (1-t)P_t(E_\varepsilon, B(x,R)) \\ & + \varepsilon_0\, c_N \left(1-\frac{s}{t}\right)^{-1} [c_N\, t]^{1-\frac{s}{t}} R^{t-s} (1-t)P_t(F_\varepsilon \Delta E_\varepsilon) \\ & + \Lambda_0\, c_N\, t R^t\, (1-t)P_t(F_\varepsilon \Delta E_\varepsilon) \\ \leqslant\; & (1-t)P_t(E_\varepsilon, B(x,R)) \\ & + c_0 R^{t-s} (1-t) \left[P_t(F_\varepsilon, B(x,R)) + P_t(E_\varepsilon, B(x,R)) \right] \end{aligned}$$

which gives (5.13), if $R < \min\{1, 1/c_0^{\frac{1}{t-s}}\} =: R_0$, with

$$c_0 := \varepsilon_0\, c_N \left(1-\frac{s}{t}\right)^{-1} [c_N\, t]^{1-\frac{s}{t}} + \Lambda_0\, c_N\, t.$$

□

Lemma 5.5. *There exists $\Theta = \Theta(N, \delta_0) > 0$ and $R_0 = R_0(N, \delta_0) > 0$ such that, for any $x \in \partial^m F_\varepsilon$ and $R < R_0$, there holds*

$$(1-t)P_t(F_\varepsilon, B(x,R)) \leqslant \Theta\, R^{N-t}. \tag{5.16}$$

Proof. Let $E_\varepsilon = F_\varepsilon \setminus B(x,R)$, and observe that $P_t(E_\varepsilon, B(x,R)) \leqslant P_t(B(x,R))$. From (5.13), possibly reducing R_0, we then get

$$(1-t)P_t(F_\varepsilon, B(x,R)) \leqslant (1+c_0\, R^{t-s})(1-t)P_t(B(x,R)) \leqslant \Theta\, R^{N-t}.$$

□

From Lemma 5.5 it follows that F_ε is also an additive ω-minimizer for the t-perimeter.

Corollary 5.6. *Let ε_0 be as in Lemma 5.2. Let F_ε be a minimizer of (5.1) with $\varepsilon < \varepsilon_0$, let $x \in \partial^m F_\varepsilon$, and let E_ε be a set of finite t-perimeter with*

$$F_\varepsilon \Delta E_\varepsilon \subset B(x, R). \tag{5.17}$$

There holds

$$(1-t)P_t(F_\varepsilon, B(x,R)) \leqslant (1-t)P_t(E_\varepsilon, B(x,R)) + c_0\, R^{N-s} \tag{5.18}$$

for any $R < R_0$, with R_0, c_0 depending only on N, δ_0.

Proof. By (5.13) and (5.16), possibly increasing the constant c_0 we have

$$(1-t)P_t(E_\varepsilon, B(x,R)) \geqslant \left(1 - c_0 R^{t-s}\right)(1-t)P_t(F_\varepsilon, B(x,R)) \geqslant (1-t)P_t(F_\varepsilon, B(x,R)) - c_0\Theta\, R^{N-s}$$

for any $R < R_0$. □

The following result can be proved exactly as in [18, Theorem 3.4].

Proposition 5.7. *Let $t_0 \in (0,1)$, there exist $\tau, \delta, q \in (0,1)$, depending only on N, t_0, such that if F is an additive ω-minimizer of P_t for any $t \in [t_0, 1]$, with $0 \in \partial^m F$ and*

$$\partial^m F \cap B(0,1) \subset \left\{ y \in \mathbb{R}^N : \ |(y-x)\cdot e| < \tau \right\},$$

for some $e \in S^{N-1}$, then there exists $e_0 \in S^{N-1}$ such that

$$\partial^m F \cap B(0,\eta) \subset \left\{ y \in \mathbb{R}^N : \ |(y-x)\cdot e_0| < q\delta\tau \right\}.$$

From Corollary 5.6 and Proposition 5.7 we derive the $C^{1,\beta}$ regularity minimizer of (5.1) following standard arguments that can be found in [12, Theorem 1] (see also [18, Corollary 3.5]).

Corollary 5.8. *There exists $\beta = \beta(N, \delta_0) < 1$ such that any minimizer F_ε of (5.1), with $\varepsilon < \varepsilon_0$, as in Lemma 5.2, has boundary of class $C^{1,\beta}$ outside of a closed singular set of Hausdorff dimension at most $N-2$.*

Remark 5.9. If $t = 1$, by the general regularity theory for ω-minimizers of the classical perimeter developed in [4, 35] we have that F_ε has boundary of class $C^{1,\beta}$ outside of a closed singular set of Hausdorff dimension at most $N-8$.

We are in the position of completing the proof of Theorem 1.1.

Proof of Theorem 1.1. The existence follows from Theorem 4.2. The regularity of ∂F follows from Corollary 5.4 and Corollary 5.8. □

6. Rigidity of minimizers for small volumes

We now develop the rigidity theory needed to prove Theorem 1.2.

Theorem 6.1. *For any $\eta > 0$ there exists $\bar\varepsilon = \bar\varepsilon(\eta, N, \delta_0) > 0$ such that any minimizer F_ε of (5.1), with $\varepsilon < \bar\varepsilon$, can be written as*

$$\partial F_\varepsilon = \{(1 + u_\varepsilon(x))x : x \in \partial B\}, \tag{6.1}$$

where B is the ball of radius 1 having the same barycenter of F_ε, and $u_\varepsilon : \partial B \to \mathbb{R}$ satisfies

$$\|u_\varepsilon\|_{C^1(\partial B)} \leqslant \eta.$$

Proof. From Lemma 3.12, putting $m = \varepsilon^{\frac{N}{t-s}}\omega_N$ there, it follows that $|F_\varepsilon \Delta B| \to 0$ as $\varepsilon \to 0$. From the density lower bound proved in Proposition 3.10 it then follows that $\partial F_\varepsilon \to \partial B$ in the Hausdorff topology. The result now follows via a standard argument based on the ω-minimality of F_ε and on the regularity of the limit set B (see [18, Corollary 3.6] and, for $t = 1$, [35, Theorem 1] and [30, Theorem 26.6]). □

Theorem 6.2. *There exist $\tau_0, c_1, c_2 > 0$ depending only on N, with $c_1 < c_2$, with the following property. Suppose that E_τ is such that, for $\tau \in [0, \tau_0]$, ∂E_τ takes the form*

$$\partial E_\tau = \{(1 + \tau u(x))x : x \in \partial B\},$$

where $u : \partial B \to \mathbb{R}$ satisfies

$$\|u\|_{C^1(\partial B)} \leqslant 1/2.$$

Suppose moreover that the barycenter of E_τ is the same of that of B, say 0, and that $|E_\tau| = |B|$. Then, for all $\alpha \in (0, 1)$ it holds true that

$$c_1\, \tau^2 \left([u]^2_{H^{\frac{1+\alpha}{2}}(\partial B)} + \alpha P_\alpha(B) \|u\|^2_{L^2(\partial B)} \right) \leqslant P_\alpha(E_\tau) - P_\alpha(B) \leqslant c_2\, \tau^2 [u]^2_{H^{\frac{1+\alpha}{2}}(\partial B)}. \tag{6.2}$$

Proof. The first inequality in (6.2) has been proved in [18, Theorem 2.1]. It remains to prove the second inequality.

As in [18, Formula (2.20)], after some calculations we get that

$$P_\alpha(E_\tau) = \frac{\tau^2}{2} g(\tau) + \frac{P_\alpha(B)}{P(B)} h(\tau), \tag{6.3}$$

where we set

$$h(\tau) := \int_{\partial B} (1 + \tau u(x))^{N-\alpha}\, d\mathcal{H}^{N-1}(x),$$

$$\text{and} \qquad g(\tau) := \int_{\partial B} \int_{\partial B} \left(\int_{u(y)}^{u(x)} \int_{u(y)}^{u(x)} f_{|x-y|}(1 + \tau r, 1 + \tau \rho)\, dr\, d\rho \right) d\mathcal{H}^{N-1}(x)\, d\mathcal{H}^{N-1}(y),$$

being

$$f_\theta(a, b) := \frac{a^{N-1} b^{N-1}}{(|a - b|^2 + ab\theta^2)^{\frac{N+\alpha}{2}}}. \tag{6.4}$$

We observe that r and ρ in the definition of g range in $[-\|u\|_{L^\infty(\partial B)}, \|u\|_{L^\infty(\partial B)}] \subseteq [-1, 1]$, since $\|u\|_{L^\infty(\partial B)} \leqslant 1$. Hence, comparing with the definition of g, we notice that a and b in (6.4) range in $[1 - \tau, 1 + \tau]$, and therefore they are bounded and bounded away from zero. As a consequence, we get

$$f_\theta(a, b) \leqslant \frac{C_1}{(C_2 + C_3\theta^2)^{\frac{N+\alpha}{2}}} \leqslant \frac{C_1}{(C_3\theta^2)^{\frac{N+\alpha}{2}}} = \frac{C_4}{\theta^{N+\alpha}},$$

for suitable constants $C_1, \dots, C_4 > 0$. Therefore, up to renaming the constants, we have

$$g(\tau) \leqslant \int_{\partial B} \int_{\partial B} \left(\int_{u(y)}^{u(x)} \int_{u(y)}^{u(x)} \frac{c_N}{|x - y|^{N+\alpha}}\, dr\, d\rho \right) d\mathcal{H}^{N-1}(x)\, d\mathcal{H}^{N-1}(y) = c_N\, [u]^2_{H^{\frac{1+\alpha}{2}}(\partial B)}.$$

Thus, since $h(0) = P(B)$, by (6.3) we get

$$P_\alpha(E_\tau) - P_\alpha(B) \leqslant c_N\, \tau^2 [u]^2_{H^{\frac{1+\alpha}{2}}(\partial B)} + \frac{P_\alpha(B)}{P(B)} (h(\tau) - h(0)). \tag{6.5}$$

Now we want to estimate $h(\tau) - h(0)$. Since $|E_\tau| = |B|$, using polar coordinates, we get

$$\int_{\partial B} (1+\tau u)^N \, d\mathcal{H}^{N-1} = N|E_\tau| = N|B| = P(B). \tag{6.6}$$

Thus

$$h(\tau) - h(0) = \int_{\partial B} (1+\tau u)^{N-\alpha} \, d\mathcal{H}^{N-1} - P(B) = \int_{\partial B} (1+\tau u)^N ((1+\tau u)^{-\alpha} - 1) \, d\mathcal{H}^{N-1}. \tag{6.7}$$

By a Taylor expansion, we know that for any $x \geqslant 0$ small enough, it holds

$$\begin{aligned} &\big((1+x)^{-\alpha} - 1)(1+x)^N\big) \\ &\quad = \left(-\alpha x + \frac{\alpha(\alpha+1)}{2} x^2 + \alpha\beta(x)\right)\left(1 + Nx + \frac{N(N-1)}{2} x^2 + \gamma(x)\right), \end{aligned}$$

with $|\beta(x)| + |\gamma(x)| \leqslant c_N x^3$, so that

$$\big((1+x)^{-\alpha} - 1)(1+x)^N\big) \leqslant -\alpha x + \left(\frac{\alpha(\alpha+1)}{2} - N\alpha\right) x^2 + \alpha \, c_N \, x^3.$$

By applying such an inequality to (6.7), and using the fact that $\|u\|_{L^\infty(\partial B)} < 1$, we get

$$h(\tau) - h(0) \leqslant -\alpha \int_{\partial B} \left[\tau u + \left(N - \frac{\alpha+1}{2}\right) \tau^2 u^2\right] \, d\mathcal{H}^{N-1} + \alpha \, c_N \, \tau^3 \|u\|^2_{L^2(\partial B)}. \tag{6.8}$$

Also, from (6.6), we have

$$0 = \int_{\partial B} ((1+\tau u)^N - 1) \, d\mathcal{H}^{N-1} \leqslant \int_{\partial B} (N\tau u + N(N-1)\tau^2 u^2 + c_N \, \tau^3 u^3) \, d\mathcal{H}^{N-1}.$$

Hence, since $\|u\|_{L^\infty(\partial B)} < 1$, we obtain

$$-\int_{\partial B} \tau u \, d\mathcal{H}^{N-1} \leqslant \frac{N-1}{2} \, \tau^2 \|u\|^2_{L^2(\partial B)} + c_N \, \tau^3 \|u\|^2_{L^2(\partial B)},$$

so that (6.8) gives

$$h(\tau) - h(0) \leqslant -\frac{\tau^2}{2} \alpha \, (N-\alpha) \|u\|^2_{L^2(\partial B)} + \alpha \, c_N \, \tau^3 \|u\|^2_{L^2(\partial B)} \leqslant 0$$

for $\tau \leqslant \tau_0(N)$. By inserting this into (6.5) we obtain the second inequality in (6.2). □

We now complete the proof of Theorem 1.2.

Proof of Theorem 1.2. We have to show that there exists $\varepsilon_1 = \varepsilon_1(N, \delta_0) \in (0, \varepsilon_0]$, ε_0 as in (5.4), and so $\bar{m}_1 = \bar{m}_1(N, \delta_0) \in (0, \bar{m}_0]$, such that the ball B is the only minimizer of problem (5.1) for $\varepsilon < \varepsilon_1$. Let $\varepsilon < \varepsilon_1$ and let F_ε be a minimum of problem (5.1), which exists by Theorem 1.1. By the minimality of F_ε we have

$$(1-t)P_t(F_\varepsilon) - (1-t)P_t(B) \leqslant \varepsilon \, (sP_s(F_\varepsilon) - sP_s(B)) \tag{6.9}$$

where B has the same barycenter of F_ε. Possibly reducing ε we can assume that ∂F_ε can be written as in (6.1), with $\|u_\varepsilon\|_{C^1(\partial B)} \leqslant \tau_0/2$, where τ_0 is as in Theorem 6.2. Then, from (6.9)

and (6.2) it follows

$$\begin{aligned} c_1(1-t)[u_\varepsilon]^2_{H^{\frac{1+t}{2}}(\partial B)} &\leqslant c_1(1-t)\left([u_\varepsilon]^2_{H^{\frac{1+t}{2}}(\partial B)} + tP_t(B)\|u_\varepsilon\|^2_{L^2(\partial B)}\right) \\ &\leqslant \big((1-t)P_t(F_\varepsilon) - (1-t)P_t(B)\big) \\ &\leqslant \varepsilon\,(sP_s(F_\varepsilon) - sP_s(B)) \\ &\leqslant \varepsilon\, s\, c_2\,[u_\varepsilon]^2_{H^{\frac{1+s}{2}}(\partial B)}. \end{aligned} \tag{6.10}$$

From (2.17) it then follows

$$c_1(1-t)[u_\varepsilon]^2_{H^{\frac{1+t}{2}}(\partial B)} \leqslant c_N \frac{\varepsilon s}{(1-s)}\,(1-t)[u_\varepsilon]^2_{H^{\frac{1+t}{2}}(\partial B)}$$

which implies $u_\varepsilon = 0$, that is $F_\varepsilon = B$, whenever ε is sufficiently small. □

We conclude the section with the following counterpart to Theorem 1.2.

Theorem 6.3. *For all $0 < s < t \leqslant 1$, there exists a volume $\bar m_2 = \bar m_2(N,s,t) \geqslant \bar m_1$ such that, for $m > m_2$, the ball is not a local minimizer of problem* (1.3).

Proof. We have to show that there exists $\varepsilon_2 \geqslant \varepsilon_1$ such that the ball B is not a local minimizer of problem 5.1 for $\varepsilon > \varepsilon_2$. We look for a competitor $F_\varepsilon \neq B$ which can be written as in (6.1), with $u \not\equiv 0$ and and $\|u\|_{C^1(\partial B)} \leqslant \tau_0/2$, where τ_0 is as in Theorem 6.2. As above, from (6.2) it follows

$$\begin{aligned} \big((1-t)P_t(F_\varepsilon) - (1-t)P_t(B)\big) &\leqslant c_2(1-t)[u]^2_{H^{\frac{1+t}{2}}(\partial B)} \\ &< \varepsilon c_1 s[u]^2_{H^{\frac{1+s}{2}}(\partial B)} \\ &\leqslant \varepsilon\,(sP_s(F_\varepsilon) - sP_s(B)), \end{aligned} \tag{6.11}$$

as soon as

$$\varepsilon > \varepsilon_2 := \frac{c_2(1-t)[u]^2_{H^{\frac{1+t}{2}}(\partial B)}}{c_1 s[u]^2_{H^{\frac{1+s}{2}}(\partial B)}}.$$

This shows that F_ε has lower energy than B, so that the ball cannot be a local minimizer of problem (1.3). □

Notice that $\lim_{s\to 0}\bar m_2(N,s,t) = +\infty$ for all $t \in (0,1]$, which is consistent with the fact that the ball is the unique minimizer of the t-perimeter, with volume constraint.

7. A fractional isoperimetric problem

Consider the functional $\widetilde{\mathcal{F}_{s,t}}$ defined by

$$\widetilde{\mathcal{F}_{s,t}}(E) = \begin{cases} \frac{((1-t)P_t(E))^{N-s}}{(sP_s(E))^{N-t}} & \text{if } 0 < s < t < 1 \\ \frac{(N\omega_N P(E))^{N-s}}{(sP_s(E))^{N-1}} & \text{if } 0 < s < t = 1 \\ \frac{(1-t)P_t(E)^N}{(N\omega_N|E|)^{N-t}} & \text{if } 0 = s < t < 1 \\ N\omega_N \frac{P(E)^N}{|E|^{N-1}} & \text{if } s = 0 \text{ and } t = 1. \end{cases}$$

In this section we consider the generalized isoperimetric problem

$$\min_{E\subset\mathbb{R}^N} \widetilde{\mathcal{F}_{s,t}}(E), \qquad 0\leqslant s<t\leqslant 1\,. \tag{7.1}$$

Remark 7.1. Notice that the quantity in (7.1) is scale invariant, hence without loss of generality we can look for minimizers E satisfying a volume constraint $|E|=\omega_N$.

The main aim of this section is the following existence theorem.

Theorem 7.2. *There exists a minimizer of problem* (7.1).

To prove it, we need some preliminary results, namely a suitable version of the isoperimetric inequality (Lemma 7.4) and an existence result with uniform estimates for a constrained minimization problem (Lemma 7.5).

Remark 7.3. In what follows, with a slight abuse of notation, we extend the functionals $(1-t)P_t(\cdot)$ and $sP_s(\cdot)$ to $t=1$ and $s=0$ respectively, meaning that for $t=1$ it equals $N\omega_N P(\cdot)$, while for $s=0$ it equals $N\omega_N|\cdot|$.

Lemma 7.4. *Let $s<t\in[0,1]$ satisfy* (3.1). *For any $E\subset\mathbb{R}^N$ there holds*

$$\frac{((1-t)P_t(E))^{\frac{N-s}{N-t}}}{(sP_s(E))}\geqslant c \tag{7.2}$$

for some $c=c(N,\delta_0)>0$.

Proof. Let $s<t\in[0,1]$ and let $\delta_0=t-s$. Notice that

$$\frac{t}{t-s}=1+\frac{s}{t-s}\leqslant 1+\frac{1}{\delta_0}. \tag{7.3}$$

Then from (2.15), and since $\delta_0<t$, it follows

$$|E|^{1-\frac{s}{t}}\leqslant C(N,\delta_0)\,((1-t)P_t(E))^{\frac{N(t-s)}{(N-t)t}}\,.$$

Plugging this estimate into (2.3) (or (2.9) if $t=1$) we get

$$sP_s(E)\ \leqslant\ C(N,\delta_0)\,\frac{t}{t-s}\,((1-t)P_t(E))^{\frac{N-s}{N-t}}\ ,$$

which, together with (7.3) gives (7.2). □

We notice that, if $s=0$, the claim is an immediate consequence of the the fractional isoperimetric inequality (2.15).

Lemma 7.5. *Let $s<t\in[0,1]$ satisfy* (3.1). *For $R>1$ let $Q_R=[-R,R]^N$. Then, there exists a minimizer E_R of the problem*

$$\min_{E\subset Q_R\,|E|=m}\frac{((1-t)P_t(E))^{N-s}}{(sP_s(E))^{N-t}}\,. \tag{7.4}$$

Moreover

$$(1-t)P_t(E_R)\leqslant C \tag{7.5}$$

where C is independent of R.

Proof. We recall that, thanks to the notation introduced in Remark 7.3 we can deal at once with the cases $t < 1$ and $t = 1$. By Lemma 7.4 we know that

$$C(R) = \inf_{E \subset Q_R\, |E|=m} \frac{((1-t)P_t(E))^{\frac{N-s}{N-t}}}{(sP_s(E))}$$

is a strictly positive quantity. Clearly the map $R \mapsto C(R)$ is non-increasing. Let $C = C(1) + 1$ and let E_n be a minimizing sequence for (7.4), so that for n big enough it holds $(1-t)P_t(E_n) \leqslant C(sP_s(E_n))^{(N-t)/(N-s)}$. Possibly increasing the constant C, from (2.3) (or (2.9) if $t = 1$) it follows

$$(1-t)P_t(E_n) \leqslant C((1-t)P_t(E_n))^{\frac{s(N-t)}{t(N-s)}}$$

which gives

$$(1-t)P_t(E_n) \leqslant C \qquad \text{for all } n. \tag{7.6}$$

The existence of a minimizer now follows by the direct method the calculus of variations, since the compact embedding of $L^1(Q_R)$ and $H^s(Q_R)$ into $H^t(Q_R)$ and the estimate (7.5) directly follows from (7.6). □

We now prove Theorem 7.2.

Proof of Theorem 7.2. If $s = 0$ then the claim of the theorem is equivalent to that of the isoperimetric inequality (the fractional isoperimetric inequality if $t < 1$). Thus we consider just the case $s > 0$. Again, we shall always write $(t-1)P_t$ meaning that such a functional is equivalent to the classical perimeter if $t = 1$ (see Remark 7.3).
Let E_n be a minimizer of (7.4) with $R = n \in \mathbb{N}$ and $m = 1/2$. We divide Q_n into $(2n)^N$ unit cubes with vertices in $\mathbb{Z}^N$, and we let $\{Q_{i,n}\}_{i=1}^{I_n}$ be the unit cubes with non-negligible intersection with E_n, that is, $x_{i,n} = |E_n \cap Q_{i,n}| \in (0, 1/2]$ for all $i \in \{1, \dots, I_n\}$, for some $I_n \in \{1, \dots, (2n)^N\}$.

We remark that, from (2.10) (and omitting the integrands for simplicity), we have that

$$\sum_{i=1}^{\infty} P_t(E_n, Q_{i,n}) = \sum_{i=1}^{\infty} \int_{E_n \cap Q_{i,n}} \int_{\mathbb{R}^N \setminus E_n} + \int_{Q_{i,n} \setminus E_n} \int_{E_n \setminus Q_{i,n}} \leqslant \int_{E_n \cap Q_n} \int_{\mathbb{R}^N \setminus E_n} + \int_{Q_n \setminus E_n} \int_{E_n},$$

which implies that

$$\sum_{i=1}^{\infty} P_t(E_n, Q_{i,n}) \leqslant 2 \int_{E_n} \int_{\mathbb{R}^N \setminus E_n} = 2P_t(E_n). \tag{7.7}$$

Now, up to reordering the cubes $Q_{i,n}$ we can assume that the sequence $\{x_{i,n}\}_{i=1}^{I_n}$ is non-increasing in i, and we set $x_{i,n} := 0$ for $i > I_n$. We have that

$$\sum_{i=1}^{\infty} x_{i,n} = \frac{1}{2} \tag{7.8}$$

and, recalling (2.12), (7.5) and (7.7), and the fact that $x_{i,n} \leqslant |E_n| = 1/2 = |Q_{i,n}|/2$, we get

$$\sum_{i=1}^{\infty} x_{i,n}^{\frac{N-t}{N}} \leqslant C \sum_{i=1}^{\infty} (1-t)P_t(E_n, Q_{i,n}) \leqslant 2C\,(1-t)P_t(E_n) \leqslant C, \tag{7.9}$$

up to renaming C. As in [25, Lemma 4.2], from (7.8) and (7.9) it follows that

$$\sum_{i=k}^{\infty} x_{i,n} \;\leqslant\; C\, k^{-\frac{1}{N}} \tag{7.10}$$

for all $k \in \mathbb{N}$, where C depends only on (N, s, t).

Up to extracting a subsequence (using either a diagonal process or Tychonoff Theorem), we can suppose that $x_{i,n} \to \alpha_i \in [0,1/2]$ as $n \to +\infty$ for every $i \in \mathbb{N}$, so that by (7.8) and (7.10) we have

$$\sum_i \alpha_i = \frac{1}{2}\,. \tag{7.11}$$

Fix now $z_{i,n} \in Q_{i,n}$. Up to extracting a further subsequence, we can suppose that $d(z_{i,n}, z_{j,n}) \to c_{ij} \in [0,+\infty]$, and (recalling (7.5)) that there exists $G_i \subseteq \mathbb{R}^N$ such that

$$(E_n - z_{i,n}) \to G_i \quad \text{in the } L^1_{\rm loc}\text{-convergence} \tag{7.12}$$

for every $i \in \mathbb{N}$. We say that $i \sim j$ if $c_{ij} < +\infty$ and we denote by $[i]$ the equivalence class of i. Notice that G_i equals G_j up to a translation, if $i \sim j$. Let $\mathcal{A} := \{[i] : i \in \mathbb{N}\}$. We claim that

$$\sum_{[i]\in\mathcal{A}} P_t(G_i) \leqslant \liminf_{n\to+\infty} P_t(E_n) \qquad \text{and} \qquad \sum_{[i]\in\mathcal{A}} P_s(G_i) \leqslant \liminf_{n\to+\infty} P_s(E_n)\,. \tag{7.13}$$

To prove it, we first fix $M \in \mathbb{N}$ and $R > 0$. We take different equivalent classes $i_1, \dots, i_M$ and we notice that if $i_k \neq i_j$ then the set $z_{i_k,n} + Q_R$ is drifting far apart from $z_{i_j,n} + Q_R$, and so

$$\lim_{n\to+\infty} \int_{z_{i_k,n}+Q_R} \int_{z_{i_j,n}+Q_R} \frac{dx\,dy}{|x-y|^{N+t}} = 0.$$

Accordingly, by (2.11), (7.12) and the lower semicontinuity of the perimeter,

$$\begin{aligned}
\sum_{i=1}^M P_t(G_i, Q_R) &\leqslant \liminf_{n\to+\infty} \sum_{k=1}^M P_t(E_n, (z_{i_k,n} + Q_R)) \\
&\leqslant \liminf_{n\to+\infty} P_t\left(E_n, \bigcup_{k=1}^N (z_{i_k,n} + Q_R)\right) + 2 \sum_{\substack{1\leqslant k,j\leqslant M \\ i_k\neq i_j}} \int_{z_{i_k,n}+Q_R} \int_{z_{i_j,n}+Q_R} \frac{dx\,dy}{|x-y|^{N+t}} \\
&\leqslant \liminf_{n\to+\infty} P_t(E_n).
\end{aligned}$$

By sending first $R \to +\infty$ and then $M \to +\infty$, this yields (7.13).

Now we claim that

$$\sum_{[i]\in\mathcal{A}} |G_i| = \frac{1}{2}. \tag{7.14}$$

Indeed, for every $i \in \mathbb{N}$ and $R > 0$ we have

$$|G_i| \geqslant |G_i \cap Q_R| = \lim_{n\to+\infty} |(E_n - z_{i,n}) \cap Q_R|.$$

If j is such that $j \sim i$ and $c_{ij} \leqslant \frac{R}{2}$, possibly enlarging R we have $Q_{j,n} - z_{i,n} \subset Q_R$ for all $n \in \mathbb{N}$, so that

$$\begin{aligned}
|(E_n - z_{i,n}) \cap Q_R| &= \sum_{j=1}^{I_n} |(E_n - z_{i,n}) \cap Q_R \cap (Q_{j,n} - z_{i,n})| \\
&\geqslant \sum_{j:\, c_{ij}\leqslant\frac{R}{2}} |(E_n - z_{i,n}) \cap Q_R \cap (Q_{j,n} - z_{i,n})| = \sum_{j:\, c_{ij}\leqslant\frac{R}{2}} |(E_n - z_{i,n}) \cap (Q_{j,n} - z_{i,n})| \\
&= \sum_{j:\, c_{ij}\leqslant\frac{R}{2}} |E_n \cap Q_{j,n}|,
\end{aligned}$$

and so

$$|G_i| \geqslant \lim_{n\to+\infty} |(E_n - z_{i,n}) \cap Q_R| \geqslant \lim_{n\to+\infty} \sum_{j:\, c_{ij}\leqslant \frac{R}{2}} |E_n \cap Q_{j,n}| = \sum_{j:\, c_{ij}\leqslant \frac{R}{2}} \alpha_j.$$

Letting $R \to +\infty$ we then have

$$|G_i| \geqslant \sum_{j:\, i\sim j} \alpha_j = \sum_{j\in[i]} \alpha_j\,,$$

hence, recalling (7.11),

$$\sum_{[i]\in\mathcal{A}} |G_i| \geqslant \frac{1}{2},$$

thus proving (7.14) (since the other inequality is trivial).

We now claim that

$$\sum_{[i]\in\mathcal{A}} P_s(G_i) \geqslant \limsup_{n\to+\infty} P_s(E_n). \tag{7.15}$$

Indeed, by (7.14) we have that for any $\varepsilon > 0$ there exist R, ℓ such that there exist ℓ distinct equivalence classes $[i_1], \dots, [i_\ell] \in \mathcal{A}$ such that

$$\frac{1}{2} - \varepsilon \leqslant \sum_{k=1}^{\ell} |G_{i_k} \cap B_R| = \lim_{n\to+\infty} \sum_{k=1}^{\ell} |(E_n - z_{i_k,n}) \cap B_R|\,. \tag{7.16}$$

For $\rho > 0$ we let

$$E^\rho_{n,1} = E_n \cap \bigcup_{k=1}^{\ell} (z_{i_k,n} + B_\rho) \qquad E^\rho_{n,2} = E_n \setminus E_{n,1}\,.$$

For n sufficiently large we have that the balls $z_{i_k,n} + B_R$ are disjoint (since the $z_{i_k,n}$ are drifting far away from each other, being each i_k in a different equivalence class). Therefore (7.16) gives that

$$|E^R_{n,1}| \geqslant \frac{1}{2} - 2\varepsilon \qquad \text{and} \qquad |E^R_{n,2}| \leqslant 2\varepsilon \tag{7.17}$$

if n is large enough. We claim that

$$\int_{E^{\bar\rho}_{n,1}} \int_{E^{\bar\rho}_{n,2}} \frac{dx\,dy}{|x-y|^{N+s}} \leqslant \frac{c_N}{s(1-s)} |E^{\bar\rho}_{n,2}|^{\frac{N-s}{N}} \qquad \text{for some } \bar\rho \in \left[R, R + (2\delta)^{-\frac{1}{N}}\right], \tag{7.18}$$

where the constants c_N, δ depend only on N.

Indeed, if this is not the case, we would have that

$$\begin{aligned} &|E^\rho_{n,2}| > 0 \ \text{ and} \\ &\int_{E^{\rho}_{n,1}} \int_{E^{\rho}_{n,2}} \frac{dx\,dy}{|x-y|^{N+s}} > \frac{c_N}{s(1-s)} |E^{\rho}_{n,2}|^{\frac{N-s}{N}} \qquad \text{for every } \rho \in \left[R, R + (2\delta)^{-\frac{1}{N}}\right]. \end{aligned} \tag{7.19}$$

So we let

$$\mu(\rho) := |E^\rho_{n,1}| = |E_n| - |E^\rho_{n,2}| = \frac{1}{2} - |E^\rho_{n,2}|$$

and we obtain

$$\begin{aligned}\frac{c_N}{s(1-s)}\left(\frac12-\mu(\rho)\right)^{\frac{N-s}{N}} &< \int_{E^\rho_{n,1}}\int_{E^\rho_{n,2}}\frac{dx\,dy}{|x-y|^{N+s}}\\ &\leqslant \int_{E^\rho_{n,1}}\int_{\mathbb{R}^N\setminus\bigcup_{k=1}^{\ell}(z_{i_k,n}+B_\rho)}\frac{dx\,dy}{|x-y|^{N+s}}\\ &\leqslant \frac{N\omega_N}{s}\int_{E^\rho_{n,1}}\frac{dx}{(\rho-|x-z_{i_{k(x)},n}|)^s}\\ &= \frac{N\omega_N}{s}\int_0^\rho\frac{\mu'(z)}{(\rho-z)^s}\,dz\end{aligned}\tag{7.20}$$

for all $\rho\in\left[R,R+(2\delta)^{-\frac1N}\right]$, where $k(x)\in\mathbb{N}$ is such that $x\in z_{i_{k(x)},n}+B_\rho$.

From (7.20) and Lemma 4.1 (used here with $m:=1/2$ and $\bar\rho:=R$), we obtain that $\mu(\rho)=1/2$ (and so $|E^\rho_{n,2}|=0$) for $\rho=R+(2\delta)^{-\frac1N}$, which leads to a contradiction with (7.19). We thus proved (7.18). Notice that inequality (7.18) holds also with t instead of s. So, by (7.18) and the fact that $|E^{\bar\rho}_{n,2}|\leqslant 2\varepsilon$ (recall (7.17)), we obtain that

$$\int_{E^{\bar\rho}_{n,1}}\int_{E^{\bar\rho}_{n,2}}\frac{dx\,dy}{|x-y|^{N+s}}\leqslant C\varepsilon^{\frac{N-s}{N}}\quad\text{and}\quad\int_{E^{\bar\rho}_{n,1}}\int_{E^{\bar\rho}_{n,2}}\frac{dx\,dy}{|x-y|^{N+t}}\leqslant C\varepsilon^{\frac{N-t}{N}},\tag{7.21}$$

for some $C>0$, possibly depending on n, s and t.

From this, (2.1) and (7.5) we obtain

$$\begin{aligned}P_t(E^{\bar\rho}_{n,1})+P_t(E^{\bar\rho}_{n,2}) &= P_t(E_n)+2\int_{E^{\bar\rho}_{n,1}}\int_{E^{\bar\rho}_{n,2}}\frac{dx\,dy}{|x-y|^{N+t}}\\ &\leqslant P_t(E_n)+C\varepsilon^{\frac{N-t}{N}}\leqslant C\,.\end{aligned}\tag{7.22}$$

Now, by (2.3), we get

$$P_s(E^{\bar\rho}_{n,2})\leqslant C\,|E^{\bar\rho}_{n,2}|^{1-\frac st}P_t(E^{\bar\rho}_{n,2})^{\frac st}\,,$$

up to renaming C. Using this, (7.22) and then (7.17) once more, and possibly renaming C again, we conclude that

$$\begin{aligned}P_s(E^{\bar\rho}_{n,2}) &\leqslant C\,|E^{\bar\rho}_{n,2}|^{1-\frac st}\Big(P_t(E^{\bar\rho}_{n,1})+P_t(E^{\bar\rho}_{n,2})\Big)^{\frac st}\\ &\leqslant C\,|E^{\bar\rho}_{n,2}|^{1-\frac st}\\ &\leqslant C\varepsilon^{1-\frac st}\,.\end{aligned}$$

Consequently, using (2.1) and (7.21), we conclude that

$$\begin{aligned}P_s(E^{\bar\rho}_{n,1}) &= P_s(E_n)-P_s(E^{\bar\rho}_{n,2})+2\int_{E^{\bar\rho}_{n,1}}\int_{E^{\bar\rho}_{n,2}}\frac{dx\,dy}{|x-y|^{N+s}}\\ &\geqslant P_s(E_n)-C\varepsilon^{1-\frac st}.\end{aligned}\tag{7.23}$$

Also, from (7.5), (7.12) and the compact embedding of $H^{\frac t2}$ into $H^{\frac s2}$ (see [16, Section 7]), we see that

$$\lim_{n\to+\infty}P_s\left((E_n-z_{i_k,n})\cap B_{\bar\rho}\right)=P_s(G_{i_k}\cap B_{\bar\rho}).\tag{7.24}$$

Now we recall that if K is a convex set, then $P(E\cap K)\leqslant P_t(E)$ (see for instance [18, Lemma B.1]). Together with (2.2) and (7.23), this implies

$$\begin{aligned}\sum_{[i]\in\mathcal{A}} P_s(G_i) &\geqslant \sum_{k=1}^{\ell} P_s(G_{i_k}\cap B_{\bar\rho})\\ &= \lim_{n\to+\infty}\sum_{k=1}^{\ell} P_s\left((E_n-z_{i_k,n})\cap B_{\bar\rho}\right)\\ &\geqslant \lim_{n\to+\infty} P_s(E^{\bar\rho}_{n,1})\\ &\geqslant \limsup_{n\to+\infty} P_s(E_n) - C(N,s,t)\varepsilon^{\frac{t-s}{t}},\end{aligned}$$

which gives (7.15) by letting $\varepsilon\to 0^+$.

From (7.13) and (7.15) we obtain that

$$\frac{\sum_{[i]\in\mathcal{A}}(1-t)P_t(G_i)}{\left(s\sum_{[i]\in\mathcal{A}}P_s(G_i)\right)^{\frac{N-t}{N-s}}} \leqslant \liminf_{n\to+\infty}\frac{(1-t)P_t(E_n)}{(sP_s(E_n))^{\frac{N-t}{N-s}}}. \tag{7.25}$$

Let us now prove that the there exists j such that

$$\frac{(1-t)P_t(G_j)}{(sP_s(G_j))^{\frac{N-t}{N-s}}}\leqslant \frac{\sum_{[i]\in\mathcal{A}}(1-t)P_t(G_i)}{\left(s\sum_{[i]\in\mathcal{A}}P_s(G_i)\right)^{\frac{N-t}{N-s}}} =: S. \tag{7.26}$$

Indeed, if it is not the case, we get

$$\begin{aligned}S &= \frac{\sum_{[i]\in\mathcal{A}}(1-t)P_t(G_i)}{\left(s\sum_{[i]\in\mathcal{A}}P_s(G_i)\right)^{\frac{N-t}{N-s}}} = \frac{\sum_{[i]\in\mathcal{A}}\left(\frac{(1-t)P_t(G_i)}{(sP_s(G_i))^{\frac{N-t}{N-s}}}\right)(sP_s(G_i))^{\frac{N-t}{N-s}}}{\left(s\sum_{[i]\in\mathcal{A}}P_s(G_i)\right)^{\frac{N-t}{N-s}}}\\ &> S\frac{\sum_{[i]\in\mathcal{A}}(sP_s(G_i))^{\frac{N-t}{N-s}}}{\left(s\sum_{[i]\in\mathcal{A}}P_s(G_i)\right)^{\frac{N-t}{N-s}}}\geqslant S,\end{aligned}$$

which is impossible. To get the last estimate we used the elementary inequality $(\sum_i c_i)^\alpha\leqslant\sum_i c_i^\alpha$ which holds true for $c_i\geqslant 0$ and $\alpha\in(0,1)$.

Now, let j be the index satisfying (7.26). Then, by (7.25) we get

$$\frac{(1-t)P_t(G_j)}{(sP_s(G_j))^{\frac{N-t}{N-s}}}\leqslant\liminf_{n\to\infty}\frac{(1-t)P_t(E_n)}{(sP_s(E_n))^{\frac{N-t}{N-s}}}. \tag{7.27}$$

Then, given any set E, fixed any $\epsilon>0$, we intersecate E with a big ball B_{R_ϵ} in such a way that

$$\frac{(1-t)P_t(E\cap B_{R_\epsilon})}{(sP_s(E\cap B_{R_\epsilon}))^{\frac{N-t}{N-s}}}\leqslant\frac{(1-t)P_t(E)}{(sP_s(E))^{\frac{N-t}{N-s}}}+\epsilon.$$

Then, by the minimality of E_n,

$$\frac{(1-t)P_t(E\cap B_{R_\epsilon})}{(sP_s(E\cap B_{R_\epsilon}))^{\frac{N-t}{N-s}}}\geqslant\frac{(1-t)P_t(E_n)}{(sP_s(E_n))^{\frac{N-t}{N-s}}}$$

for any $n \geqslant n_\epsilon$. Thus, by (7.27),

$$\frac{(1-t)P_t(E)}{(sP_s(E))^{\frac{N-t}{N-s}}} + \epsilon \geqslant \liminf_{n\to\infty} \frac{(1-t)P_t(E_n)}{(sP_s(E_n))^{\frac{N-t}{N-s}}} \geqslant \frac{(1-t)P_t(G_j)}{(sP_s(G_j))^{\frac{N-t}{N-s}}}.$$

By sending $\epsilon \searrow 0$ we see that G_j is the desired minimizer, which concludes the proof. □

Proposition 7.6. *Let F be a minimizer of (7.1). Then F is a multiplicative ω-minimizer of the t-perimeter, that is, for any set E such that $F\Delta E \subset B(x,R)$, there holds*

$$P_t(F, B(x,R)) \leqslant (1 + CR^{t-s})\, P_t(E, B(x,R)) \qquad \text{for any } R < R_0\,,$$

where R_0, C depend only on N, δ_0 and $|F|$.

Proof. First, if $\alpha \in (0,1)$, by graphic the functions, one sees that, for any $r \geqslant 0$,

$$1 - r^\alpha \leqslant |1-r|. \tag{7.28}$$

Also, from (5.11), we know that

$$P_s(F) - P_s(E) \leqslant P_s(F\Delta E),$$

for any sets E and F, and so, by possibly exchanging the roles of E and F we obtain

$$|P_s(E) - P_s(F)| \leqslant P_s(F\Delta E) \tag{7.29}$$

Now, letting E be such that $F\Delta E \subset B(x,R)$, using the minimality of F, (7.28) and (7.29) we see that

$$\begin{aligned}
P_t(E) &\geqslant P_s(E)^{\frac{N-t}{N-s}} \frac{P_t(F)}{P_s(F)^{\frac{N-t}{N-s}}} \\
&= P_t(F) + \left(\frac{P_s(E)^{\frac{N-t}{N-s}}}{P_s(F)^{\frac{N-t}{N-s}}} - 1 \right) P_t(F) \\
&\geqslant P_t(F) - \left| \frac{P_s(E)}{P_s(F)} - 1 \right| P_t(F) \\
&\geqslant P_t(F) - \frac{P_t(F)}{P_s(F)} |P_s(E) - P_s(F)| \\
&\geqslant P_t(F) - \frac{P_t(F)}{P_s(F)} P_s(F\Delta E).
\end{aligned}$$

Hence, by applying the fractional isoperimetric inequality (2.15) to $P_s(F)$, we obtain that

$$P_t(E) \geqslant P_t(F) - C(N,\delta_0)|F|^{-\frac{N-s}{N}} P_s(F\Delta E).$$

As in (5.15), by means of (2.3) and again the fractional isoperimetric inequality we then get

$$\begin{aligned}
P_t(E, B(x,R)) &\geqslant P_t(F, B(x,R)) - C(N,\delta_0)|F|^{-\frac{N-s}{N}} |F\Delta E|^{\frac{t-s}{N}} P_t(F\Delta E) \\
&= \left(1 - C(N,\delta_0)|F|^{-\frac{N-s}{N}} R^{t-s}\right) P_t(F, B(x,R))\,,
\end{aligned}$$

which gives

$$P_t(F, B(x,R)) \leqslant \frac{|F|^{-\frac{N-s}{N}}}{1 - C(N,\delta_0)R^{t-s}}\, P_t(E, B(x,R))\,.$$

□

Reasoning as in Section 5, from Proposition (7.6) we obtain the following regularity result.

Corollary 7.7. *There exists $\beta = \beta(N, \delta_0) < 1$ such that any minimizer F of (7.1) is bounded and has boundary of class $C^{1,\beta}$, outside of a closed singular set of Hausdorff dimension at most $N - 2$ (respectively $N - 8$ if $t = 1$).*

Proof of Theorem 1.3. The existence claim is a consequence of Theorem 7.2 and the regularity follows from Corollary 7.7. □

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