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A rigidity result for nonlocal semilinear equations

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Abstract

We consider a possibly anisotropic integro-differential semilinear equation, run by a nondecreasing and nontrivial nonlinearity. We prove that if the solution grows at infinity less than the order of the operator, then it must be constant.

1. Introduction

It dates back to Liouville and Cauchy in 1844 that bounded harmonic functions are constant. Several generalizations of this result appeared in the literature, also involving nonlinear equations and more general growth of the solution at infinity (see [F] for a detailed review of this topic).

The purpose of this note is to obtain a rigidity result for integro-differential semilinear equations of fractional order $2s$, with $s \in (0,1)$.

We recall that fractional integro-differential operators are a classical topic in analysis, whose study arises in different fields, including harmonic analysis [St], partial differential equations [C] and probability [B]. Recently, the study of these operators has been further intensified in view of the related real-world applications, such as quantum mechanics [FLl], water waves [CSS], meteorology [CV], crystallography [G], biology [AAVV], finance [Sc] and high technology [ZL], just to name a few.

The type of integro-differential operators that we consider here are of the form

$$\mathcal{I}u(x) := \int_{\mathbb{R}^n} \big(u(x+y) + u(x-y) - 2u(x)\big)\, \mathcal{K}(y)\, dy.$$

We suppose that the kernel $\mathcal{K}$ is elliptic, homogeneous of order $-n-2s$ and possibly anisotropic, that is

$$\mathcal{K}(y) = |y|^{-n-2s} \mathcal{K}_0 \left(\frac{y}{|y|} \right), \tag{1}$$

for some measurable function $\mathcal{K}_0 : \partial B_1 \to [\lambda, \Lambda]$, with $\Lambda \geqslant \lambda > 0$.

We will consider the equation $\mathcal{I}u = f(u)$. This type of equations is often called "semilinear" since the nonlinearity only depends on the values of the solution itself (for these reasons, solutions of semilinear equations may satisfy better geometric properties than solutions of arbitrary equations).

Our main result states that if f is nondecreasing and nontrivial, then solutions of $\mathcal{I}u = f(u)$ whose growth at infinity is bounded by $|x|^\kappa$, with κ less than the order of operator, must be necessarily constant. More precisely, we have:

Theorem 1. *Let $f \in C(\mathbb{R})$ be nondecreasing and not identically zero. Let $u \in C^2(\mathbb{R}^N)$ be a solution of*

$$\mathcal{I}u(x) = f(u(x)) \quad \textit{for any} \quad x \in \mathbb{R}^n. \tag{2}$$

Assume that

$$|u(x)| \leqslant K\,(1+|x|^{\kappa}), \tag{3}$$

for some $K \geqslant 0$ *and* $\kappa \in [0, 2s)$.

Then u *is constant, say* $u(x) = c$ *for any* $x \in \mathbb{R}^n$, *and* $f(c) = 0$.

As far as we know, Theorem 1 is new even in the isotropic case in which $\mathcal{K}_0$ is constant. In this case, the integro-differential operator $\mathcal{I}$ is simply the fractional power of the Laplacian (up to a normalization factor), i.e. $\mathcal{I} = -(-\Delta)^s$.

On the other hand, when $\mathcal{I}$ is replaced by the Laplacian (which is formally the above case with $s = 1$) Theorem 1 is a well known result in the framework of classical Liouville-type theorems: see for instance [F, Se].

We point out that, in general, the assumption that f is not identically zero cannot be removed from Theorem 1: as a counterexample one can consider the linear function $u(x) := x_1$ which satisfies $\mathcal{I}u = 0$ in the whole of $\mathbb{R}^n$, and also (3) when $s \in (1/2, 1)$.

The rest of the paper is organized as follows. First, in Section 2 we present some simple generalizations of Theorem 1, dealing with the case in which (3) is replaced by a one-side inequality and when the notion of solution is taken in the viscosity sense instead in the classical sense. Then, in Section 3 we collect some preliminary integral computations that will be used in Section 4 to construct a useful barrier. Roughly speaking, this barrier replaces the classical paraboloid in our nonlocal framework (of course, checking the properties of the paraboloid in the classical case is much simpler than constructing barriers in nonlocal cases).

The proofs of Theorem 1 and its generalizations occupy Section 5.

2. Generalizations of Theorem 1

In this section we present some more general versions of Theorem 1. A first generalization occurs when assumption (3) is replaced by a one-side bound:

Theorem 2. *Let* $f \in C(\mathbb{R})$ *be nondecreasing and not identically zero. Let* $u \in C^2(\mathbb{R}^N)$ *be a solution of*

$$\mathcal{I}u(x) = f(u(x)) \ \ \textit{for any} \ \ x \in \mathbb{R}^n.$$

Then, if

$$u(x) \leqslant K\,(1+|x|^{\kappa}),$$

for some $K \geqslant 0$ *and* $\kappa \in [0, 2s)$, *we have that*

$$\mathcal{I}u(x) \leqslant 0 \ \ \textit{for any} \ \ x \in \mathbb{R}^n.$$

Similarly, if

$$u(x) \geqslant -K\,(1+|x|^{\kappa}),$$

for some $K \geqslant 0$ and $\kappa \in [0, 2s)$, we have that

$$\mathcal{I}u(x) \geqslant 0 \quad \text{for any} \quad x \in \mathbb{R}^n.$$

Another generalization consists in weakening the regularity assumptions of u. As a matter of fact, one does not need to require u to be smooth to start with, but only to be continuous and satisfy the equation in the viscosity sense (see, e.g., Definition 2.1 in [CS] for the viscosity setting). In this spirit we have:

Theorem 3. *The theses of Theorems 1 and 2 remain valid if the assumption that $u \in C^2(\mathbb{R}^N)$ is replaced by that $u \in C(\mathbb{R}^N)$ and satisfies the equation in the sense of viscosity.*

3. Toolbox

Below are some preliminary integral computations, needed to construct a suitable barrier in Section 4. The calculations will often make use of the scaling properties of the kernel: namely (see (1)) the estimate

$$\mathcal{K}(y) \leqslant \Lambda |y|^{-n-2s}. \tag{4}$$

For convenience, we will also use the notation

$$\mathcal{I}_1 v(x) := \int_{B_1} \big(v(x+y) + v(x-y) - 2v(x) \big) \mathcal{K}(y)\, dy$$

$$\text{and} \qquad \mathcal{I}_2 v(x) := \int_{\mathbb{R}^n \setminus B_1} \big(v(x+y) + v(x-y) - 2v(x) \big) \mathcal{K}(y)\, dy.$$

3.1. Estimates near the origin.

Here we estimate $\mathcal{I}_1 v$ and $\mathcal{I}_2 v$ near the origin according to the following Lemmata 1 and 2:

Lemma 1. *Let $v \in C^2(B_3)$. Then, for any $x \in B_1$,*

$$\mathcal{I}_1 v(x) \leqslant C,$$

for some $C > 0$ possibly depending on n, s and $\|v\|_{C^2(B_2)}$.

Proof. If x, $y \in B_1$ we obtain from a Taylor expansion that

$$\big| v(x+y) + v(x-y) - 2v(x) \big| \leqslant \|D^2 v\|_{L^\infty(B_2)} \, |y|^2.$$

hence the result follows after an integration, recalling (4). □

Lemma 2. *Let*

$$\gamma \in (0, 2s). \tag{5}$$

Let $v : \mathbb{R}^n \to [0, +\infty)$ be a measurable function such that $v(x) \leqslant |x|^\gamma$ for any $x \in \mathbb{R}^n$. Then, for any $x \in B_1$,

$$\mathcal{I}_2 v(x) \leqslant C,$$

for some $C > 0$ possibly depending on n, s and γ.

Proof. Let $x \in B_1$ and $y \in \mathbb{R}^n \setminus B_1$. Then $|x| \leqslant 1 \leqslant |y|$ and so

$$\begin{aligned} & |v(x+y) + v(x-y) - v(x)| \\ & \quad \leqslant |v(x+y)| + |v(x-y)| + |v(x)| \\ & \quad \leqslant |x+y|^\gamma + |x-y|^\gamma + |x|^\gamma \\ & \quad \leqslant 2(|x|+|y|)^\gamma + |x|^\gamma \\ & \quad \leqslant (2^{\gamma+1}+1)\,|y|^\gamma. \end{aligned}$$

So, we multiply the formula above by $\mathcal{K}(y)$, we recall (4) and we integrate over $y \in \mathbb{R}^n \setminus B_1$. Then we use (5) and we obtain the desired result. □

3.2. **Estimates far from the origin.** Now we estimate $\mathcal{I}v = \mathcal{I}_1 v + \mathcal{I}_2 v$ at infinity:

Lemma 3. *Let γ be as in* (5) *and $v : \mathbb{R}^n \to \mathbb{R}$ be a measurable function such that $v(x) \leqslant |x|^\gamma$ for any $x \in \mathbb{R}^n$.*

Assume also that $v(x) = |x|^\gamma$ for any $x \in \mathbb{R}^n \setminus B_1$. Then, for any $x \in \mathbb{R}^n \setminus B_1$,

$$\mathcal{I}v(x) \leqslant C,$$

for some $C > 0$ possibly depending on n, s and γ.

Proof. Fix $x \in \mathbb{R}^n \setminus B_1$. Then $v(x) = |x|^\gamma$. Moreover $v(x \pm y) \leqslant |x \pm y|^\gamma$, and so

$$v(x+y) + v(x-y) - 2v(x) \leqslant |x+y|^\gamma + |x-y|^\gamma - 2|x|^\gamma.$$

Therefore, calling $\omega := x/|x|$ and changing variable $y := |x|\eta$, we have that

$$\begin{aligned} \mathcal{I}v(x) &= \int_{\mathbb{R}^n} \big(v(x+y) + v(x-y) - 2v(x)\big)\,\mathcal{K}(y)\,dy \\ &\leqslant \int_{\mathbb{R}^n} \big(|x+y|^\gamma + |x-y|^\gamma - 2|x|^\gamma\big)\,\mathcal{K}(y)\,dy \\ &= |x|^{\gamma+n} \int_{\mathbb{R}^n} \big(|\omega+\eta|^\gamma + |\omega-\eta|^\gamma - 2\big)\mathcal{K}(|x|\eta)\,d\eta \\ &= |x|^{\gamma-2s} \int_{\mathbb{R}^n} \frac{g(\eta)+g(-\eta)-2g(0)}{|\eta|^{n+2s}}\,\mathcal{K}_0\left(\frac{\eta}{|\eta|}\right)\,d\eta, \end{aligned} \tag{6}$$

where $g(\eta) := |\omega + \eta|^\gamma$ and (1) was exploited.

Notice that

$$|g(\eta)| \leqslant (|\omega| + |\eta|)^\gamma = (1+|\eta|)^\gamma. \tag{7}$$

Moreover $g \in C^\infty(B_{1/2})$ and, for any $\eta \in B_{1/2}$ we have that

$$\begin{aligned} & \partial_i g(\eta) = \gamma|\omega+\eta|^{\gamma-2}(\omega_i+\eta_i) \\ \text{and} \quad & \partial^2_{ij} g(\eta) = \gamma(\gamma-2)|\omega+\eta|^{\gamma-4}(\omega_i+\eta_i)(\omega_j+\eta_j) + \gamma|\omega+\eta|^{\gamma-2}\delta_{ij}. \end{aligned}$$

Consequently, for any $\eta \in B_{1/2}$,

$$|D^2 g(\eta)| \leqslant \gamma(\gamma+3)|\omega+\eta|^{\gamma-2},$$

and $|\omega+\eta| \geqslant |\omega| - |\eta| \geqslant 1/2$, therefore

$$\|D^2 g\|_{L^\infty(B_{1/2})} \leqslant 2^{2-\gamma}\gamma(\gamma+3).$$

This, together with a Taylor expansion, implies that, for any $\eta \in B_{1/2}$,

$$|g(\eta)+g(-\eta)-2g(0)| \leqslant \|D^2 g\|_{L^\infty(B_{1/2})}\,|\eta|^2 \leqslant 2^{2-\gamma}\gamma(\gamma+3)\,|\eta|^2.$$

Hence, recalling (7), we obtain that

$$\begin{aligned}
&\int_{\mathbb{R}^n} \frac{g(\eta)+g(-\eta)-2g(0)}{|\eta|^{n+2s}}\,\mathcal{K}_0\left(\frac{\eta}{|\eta|}\right)\,d\eta \\
\leqslant\;& \Lambda \int_{\mathbb{R}^n} \frac{\big|g(\eta)+g(-\eta)-2g(0)\big|}{|\eta|^{n+2s}}\,d\eta \\
\leqslant\;& \Lambda \left[\int_{B_{1/2}} \frac{2^{2-\gamma}\gamma(\gamma+3)\,|\eta|^2}{|\eta|^{n+2s}}\,d\eta + \int_{\mathbb{R}^n\setminus B_{1/2}} \frac{3(1+|\eta|)^\gamma}{|\eta|^{n+2s}}\,d\eta\right] \\
\leqslant\;& C,
\end{aligned}$$

for some $C>0$, thanks to (5). We insert this into (6) and we obtain the desired estimate. □

4. Construction of an auxiliary barrier

Here we use the estimate in Section 3 and we borrow some ideas from [DSV] to construct a useful auxiliary function:

Lemma 4. *Let $\gamma \in (0,2s)$. There exists a function $v \in C^\infty(\mathbb{R}^n)$ such that*

(8) $$v(0)=0,$$

(9) $$0 \leqslant v(x) \leqslant |x|^\gamma \text{ for any } x \in \mathbb{R}^n,$$

(10) $$v(x)=|x|^\gamma \text{ if } |x|\geqslant 1$$

(11) *and* $$\sup_{x\in\mathbb{R}^n} \mathcal{I}v(x) \leqslant C,$$

for some $C>0$.

Proof. Let $\tau \in C^\infty(\mathbb{R}^n)$ be such that $0 \leqslant \tau \leqslant 1$ in the whole of $\mathbb{R}^n$, $\tau=1$ in $B_{1/2}$ and $\tau=0$ in $\mathbb{R}^n\setminus B_1$. We define $v(x) := \big(1-\tau(x)\big)|x|^\gamma$. In this way, conditions (8), (9) and (10) are fulfilled.

Furthermore, v satisfies all the assumptions of Lemmata 1, 2 and 3. Thus, using such results, we obtain condition (11). □

5. Proof of the main results

Proof of Theorem 1. The proof relies on a modification of a classical argument (see for instance [Se, F]). In our setting, the barrier constructed in Lemma 4 will replace (at least from one side) the classical paraboloid. The details of the argument goes as follows. Let f, u, K and κ as in the statement of Theorem 1. Let $\gamma := (2s+\kappa)/2$. By construction,

(12) $$\gamma \in (\kappa, 2s),$$

so we can use the barrier v constructed in Lemma 4. We fix $\epsilon > 0$ and an arbitrary point $x_0 \in \mathbb{R}^n$, and we define

$$
\begin{aligned}
& w_1(x) := u(x) - u(x_0) + 2\epsilon - \epsilon v(x - x_0) \\
\text{and} \quad & w_2(x) := u(x) - u(x_0) - 2\epsilon + \epsilon v(x - x_0)
\end{aligned} \tag{13}
$$

We remark that

$$
\begin{aligned}
& \limsup_{|x|\to+\infty} w_1(x) \leqslant \limsup_{|x|\to+\infty} [\, u(x) + |u(x_0)| + 2\epsilon - \epsilon v(x - x_0)] \\
& \qquad \leqslant \limsup_{|x|\to+\infty} [\, K\,(1 + |x|^\kappa) + |u(x_0)| + 2\epsilon - \epsilon |x - x_0|^\gamma] = -\infty \\
\text{and} \quad & \liminf_{|x|\to+\infty} w_2(x) \geqslant \liminf_{|x|\to+\infty} [\, u(x) - |u(x_0)| - 2\epsilon + \epsilon v(x - x_0)] \\
& \qquad \geqslant \liminf_{|x|\to+\infty} [\, -K\,(1 + |x|^\kappa) - |u(x_0)| - 2\epsilon + \epsilon |x - x_0|^\gamma] = +\infty,
\end{aligned}
$$

where we have used (3), (10) and (12). As a consequence the maximum of w_1 and the minimum of w_2 are attained, i.e. there exists y_1, $y_2 \in \mathbb{R}^n$ such that

$$
w_1(y) \leqslant w_1(y_1) \quad \text{and} \quad w_2(y) \geqslant w_2(y_2) \quad \text{for any } y \in \mathbb{R}^n. \tag{14}
$$

Accordingly, for any $y \in \mathbb{R}^n$,

$$
\begin{aligned}
& w_1(y_1 + y) + w_1(y_1 - y) - 2w_1(y_1) \leqslant 0 \\
\text{and } & w_2(y_1 + y) + w_2(y_1 - y) - 2w_2(y_2) \geqslant 0.
\end{aligned} \tag{15}
$$

On the other hand

$$
\begin{aligned}
& w_1(y_1 + y) + w_1(y_1 - y) - 2w_1(y_1) \\
& \quad = u(y_1 + y) + u(y_1 - y) - 2u(y_1) \\
& \qquad - \epsilon\big(v(y_1 + y - x_0) + v(y_1 - y - x_0) - 2v(y_1 - x_0)\big), \\
\text{and} \quad & w_2(y_2 + y) + w_2(y_2 - y) - 2w_2(y_2) \\
& \quad = u(y_2 + y) + u(y_2 - y) - 2u(y_2) \\
& \qquad + \epsilon\big(v(y_2 + y - x_0) + v(y_2 - y - x_0) - 2v(y_2 - x_0)\big).
\end{aligned} \tag{16}
$$

By comparing (15) and (16), we obtain that

$$
\begin{aligned}
& 0 \geqslant \int_{\mathbb{R}^n} \big(w_1(y_1 + y) + w_1(y_1 - y) - 2w_1(y_1)\big)\, \mathcal{K}(y)\, dy \\
& \quad = \mathcal{I}u(y_1) - \epsilon \mathcal{I}v(y_1 - x_0) \\
\text{and} \quad & 0 \leqslant \int_{\mathbb{R}^n} \big(w_2(y_2 + y) + w_2(y_2 - y) - 2w_2(y_2)\big)\, \mathcal{K}(y)\, dy \\
& \quad = \mathcal{I}u(y_2) + \epsilon \mathcal{I}v(y_2 - x_0).
\end{aligned} \tag{17}
$$

Therefore, using and (2) and (11), we obtain that

$$
0 \geqslant f\big(u(y_1)\big) - C\epsilon \qquad \text{and} \qquad 0 \leqslant f\big(u(y_2)\big) + C\epsilon. \tag{18}
$$

Now we observe that $w_1(x_0) = 2\epsilon \geqslant 0$ and $w_2(x_0) = -2\epsilon \leqslant 0$, thanks to (13) and (8) So, if we evaluate (14) at the point $y := x_0$, we obtain that

$$0 \leqslant w_1(x_0) \leqslant w_1(y_1) \quad \text{and} \quad 0 \geqslant w_2(x_0) \geqslant w_2(y_2). \tag{19}$$

Furthermore, using that $v \geqslant 0$ (recall (9)), we see from (13) that

$$w_1(y_1) \leqslant u(y_1) - u(x_0) + 2\epsilon \quad \text{and} \quad w_2(y_2) \geqslant u(y_2) - u(x_0) - 2\epsilon.$$

By comparing this with (19), we conclude that

$$u(y_1) \geqslant u(x_0) - 2\epsilon \quad \text{and} \quad u(y_2) \leqslant u(x_0) + 2\epsilon.$$

Therefore, since f is nondecreasing, we deduce that

$$f\big(u(y_1)\big) \geqslant f\big(u(x_0) - 2\epsilon\big) \quad \text{and} \quad f\big(u(y_2)\big) \leqslant f\big(u(x_0) + 2\epsilon\big).$$

We plug this information into (18), and we obtain that

$$0 \geqslant f\big(u(x_0) - 2\epsilon\big) - C\epsilon \qquad \text{and} \qquad 0 \leqslant f\big(u(x_0) + 2\epsilon\big) + C\epsilon. \tag{20}$$

We remark that x_0 was fixed at the beginning and so it is independent of ϵ (conversely, the points y_1 and y_2 in general may depend on ϵ). This says that we can pass to the limit as $\epsilon \to 0^+$ in (20) and use the continuity of f to obtain that

$$0 \geqslant f\big(u(x_0)\big) \qquad \text{and} \qquad 0 \leqslant f\big(u(x_0)\big),$$

that is $f\big(u(x_0)\big) = 0$. Since x_0 is an arbitrary point of $\mathbb{R}^n$, we have proved that

$$f\big(u(x)\big) = 0 \text{ for any } x \in \mathbb{R}^n. \tag{21}$$

Thus, using again (2), we obtain that

$$\mathcal{I}u = 0 \text{ in } \mathbb{R}^n. \tag{22}$$

We claim that

(23) either u is bounded from above, or it is bounded from below.

Indeed, suppose not: then the image of $u(\mathbb{R}^n)$ would cover the whole of $(-\infty, +\infty)$. In particular, for any $r \in \mathbb{R}$ there would exist $x_r \in \mathbb{R}^n$ for which $u(x_r) = r$. Hence, by (21), we would have that $f(r) = f\big(u(x_r)\big) = 0$, and so f would vanish identically, in contradiction with the assumptions of Theorem 1.

This proves (23). From it, (22) and the integro-differential Liouville Theorem (see e.g. Theorem 10.1 in [CS], applied here with $M^+ := M^- := \mathcal{I}$ and $C_0 := 0$), we deduce that u is constant, say $u(x) = c$ for any $x \in \mathbb{R}^n$.

Finally, we use (21) once more and we obtain that $f(c) = f\big(u(0)\big) = 0$, thus completing the proof of Theorem 1. □

Proof of Theorem 2. The proof of Theorem 1 goes through in this case, just considering only the function w_1 (to obtain the first statement of Theorem 2), or only the function w_2 (to obtain the second statement). □

Proof of Theorem 3. The proof of Theorem 1 goes through in this case, simply by using the viscosity definition in (17). □

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