Weierstraß-Institut
für Angewandte Analysis und Stochastik
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An integral equation system approach for electromagnetic scattering by biperiodic structures

Beatrice Bugert

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Weierstrass Institute
Mohrenstr. 39
10117 Berlin
Germany
E-Mail: beatrice.bugert@wias-berlin.de

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Weierstraß-Institut für Angewandte Analysis und Stochastik (WIAS)
Leibniz-Institut im Forschungsverbund Berlin e. V.
Mohrenstraße 39
10117 Berlin
Germany

Fax: +49 30 20372-303
E-Mail: preprint@wias-berlin.de
World Wide Web: http://www.wias-berlin.de/

Abstract

The objective of this paper is the analytical investigation of an integral equation formulation for electromagnetic scattering by 2π-biperiodic multilayered structures with polyhedral Lipschitz regular interfaces. Extending the combined potential ansatz from [6] for the electric fields in the before mentioned electromagnetic scattering problem from single to N profile scattering yields an equivalent system of N integral equations. We present a uniqueness and two existence results for this system depending on the values of the electromagnetic material parameters of the considered biperiodic scatterer. This in particular includes the proof that the system of integral equations is of zero Fredholm index. The general case that the grating interfaces are of polyhedral Lipschitz regularity requires more strict assumptions than the special case of smooth grating interfaces. We exploit the solvability results of this work in a subsequent paper featuring a recursive integral equation algorithm for the 2π-biperiodic multilayered electromagnetic scattering problem.

1 Introduction

In the following, we derive a boundary integral equation method for the treatment of 2π-biperiodic multilayered electromagnetic scattering, which arises from the illumination of a 2π-biperiodic multilayered structure by an electromagnetic plane wave. We model such structures by a finite number of vertically stacked non-self-intersecting grating interfaces of at least polyhedral Lipschitz regularity. The incident, reflected and transmitted waves can be described by the system of time-harmonic Maxwell equations together with transmission conditions across the grating interfaces of the considered multilayered scatterer and suitable outgoing wave conditions. The motivation behind our investigation is that such problems, i.e., particular diffraction problems, offer a variety of considerable and interesting application areas, in particular in micro-optics. Moreover, some results of this article are relevant for the outcome of the consecutive article [7].

In general, periodic structures can be understood in terms of several different geometry settings such as periodic arrays of bounded obstacles, periodically aligned cylinders of infinite extent or surfaces exhibiting a certain periodicity as considered in this paper. There are two main mathematically rigorous methods to treat scattering problems involving periodic structures: integral equation methods and variational approaches. Here, we apply integral equation methods, which lay the foundation for implementations based on boundary element methods. In the periodic framework, the basic idea behind these methods is to assume potential ansatzes in form of integral operators with problem-specific quasiperiodic kernels for the incident and scattered waves occurring in the periodic scattering problems. A clever application of trace operators then makes it possible to obtain boundary integral equations on the boundaries of the considered obstacles. Such techniques were already successfully applied for instance in the articles [9], [12], [17] and [19].

Our precise approach consists in extending the potential ansatz applied in [6] for scattering by a single 2π-biperiodic interface to the multilayered framework. This is done by alternating an α-quasiperiodic Stratton-Chu type integral representation with electric potential ansatzes. For a structure consisting of N interfaces, this approach leads to a system of N singular integral equations that are computationally very expensive to solve, especially for large N. Hence, our focus lies on the analytical investigation of the mentioned system. We depict uniqueness and existence of solutions to the derived integral equation system by applying the results and ideas from [6] for single profile scattering. The solvability of the integral equation system contributes to the proof of an existence result for the recursive integral equation algorithm derived in [7].

The content of this work is also presented in a more extensive form in Section 6.3.1 of the PhD thesis [8] with the title “On Integral Equation Methods for Electromagnetic Scattering by Biperiodic Structures”.

The subsequent section states the 2π-biperiodic N-layered electromagnetic scattering problem for structures composed of $N \geq 2$ non-self-intersecting 2π-biperiodic grating interfaces of polyhedral Lipschitz regularity. Section 3 then provides us with the relevant functional analytic framework necessary to pursue integral equation techniques in the 2π-biperiodic setting, in Section 4. Based on the previously described combined potential ansatz, we derive a system of singular boundary integral equations as well as its adjoint in a certain sense, which are both equivalent to the 2π-biperiodic multilayered electromagnetic scattering problem. This equivalence shall be understood in the sense that any solution of one problem yields a solution of the other and vice versa. The structures of the system and its adjoint are parity dependent. Next, we investigate the solvability of the integral equation system in Section 5. For this, we first determine the Fredholm properties of the integral equation system with the result that it is Fredholm of index zero under certain assumptions on the electromagnetic material parameters. With this result, we can then prove the existence of solutions to the considered integral equation system by extending the techniques in [6] from single to multi-profile scattering. In a similar way, we adapt the ideas in [6] to deduce the uniqueness of solutions to the integral equation system via a variational argumentation. In the final Section 6, we briefly recapitulate the main findings of this article. Moreover, we propose how to continue our work on the treatment of the 2π-biperiodic multilayered electromagnetic scattering problem by integral equation methods.

Notation. For vectors $x \in \mathbb{R}^3$, we denote by $\tilde{x}$ their orthogonal projection to the (x_1, x_2)–plane. We distinguish vector-valued function spaces from scalar-valued ones by writing them in bold font.

2 The multilayered electromagnetic scattering problem

In this section, we want to formulate the 2π-biperiodic multilayered electromagnetic scattering problem treated in this article. For notational reasons, we introduce the index sets

$$K := \{1, \ldots, N-1\}\,, \quad K_0 := K \cup \{0\}, \quad K^N := K \cup \{N\} \quad \text{and} \quad K_0^N := K^N \cup \{0\}.$$

We consider a 2π-biperiodic multilayered structure consisting of $N \geq 2$ non-self-intersecting vertically stacked interfaces $\Sigma_k \subset \mathbb{R}^2$, $k \in K_0$, that can be described by piecewise C^2 parametrizations

$$\sigma_k(t) := \left(t_1, t_2, x_3^{(k)}(t)\right)^{\mathrm{T}} \quad \text{such that} \quad x_3^{(k)}(t + 2\pi\, m) = x_3^{(k)}(t) \tag{2.1}$$

for $t = (t_1, t_2)^{\mathrm{T}}$, $m \in \mathbb{Z}^2$, $k \in K_0$. Speaking visually, each Σ_k is 2π-periodic in both x_1- and x_2-direction and may exhibit edges and corners. From here on, we refer to this kind of regularity as polyhedral Lipschitz regularity. Moreover, the surfaces Σ_k are numbered in descending order from top to bottom, i.e., the top surface is Σ_0 and the bottom one Σ_N. All considerations in this paper focus only on one period of the multilayered scatterer as it is commonly seen in the treatment of periodic problems. This means that we restrict each surface Σ_k, $k \in K_0$, to one period Γ_k:

$$\Gamma_k := \{\sigma_k(t) \;:\; t \in Q\}\,, \quad \text{where } Q := [-\pi, \pi) \times [-\pi, \pi)$$

corresponds to the unit-cell of the periodic lattice. The restricted profiles Γ_k, $k \in K_0$, separate $N+1$ homogeneous material layers $G_k \subset \mathbb{R}^3$, $k \in K_0^N$, of constant electric permittivity ϵ_k and constant magnetic permeability μ_k. The top domain G_0 and the bottom domain G_N are both semi-infinite, whereas all regions G_k, $k \in K$, in between are bounded polyhedral Lipschitz domains. We specify the unit normal vectors $\mathbf{n}_k := \mathbf{n}|_{\Gamma_k}$, $k \in K_0$, of Γ_k in such a way that they point upwards, i.e., into G_k. The electromagnetic material parameters ϵ_k and μ_k, $k \in K_0^N$, are assumed to be 2π-biperiodic in x_1- and in x_2- direction in G_k and to satisfy

$$\mathrm{Im}\,(\epsilon_k) \geq 0 \quad \text{and} \quad \mathrm{Im}\,(\mu_k) \geq 0 \quad \text{in } G_k, k \in K_0^N. \tag{2.2}$$

We exclude the case that $\epsilon_k = 0$ and / or $\mu_k = 0$. This ensures that $\kappa_k \neq 0$. Moreover, we define the piecewise constant wavenumbers

$$\kappa_k := \omega\sqrt{\epsilon_k}\sqrt{\mu_k} \quad \text{in } G_k, k \in K_0^N,$$

where $\omega > 0$ is a fixed frequency. The square root of a complex number $z = re^{\mathrm{i}\varphi}$ is chosen such that $\sqrt{z} = \sqrt{r}e^{\mathrm{i}\frac{\varphi}{2}}$ for $0 \le \varphi < 2\pi$.

In the course of this paper, we will use the auxiliary polyhedral Lipschitz regular domain G^{H} depending on a fixed $\mathrm{H} \in \mathbb{R}_+$, which is chosen such that

$$\Gamma_k \subset G^{\mathrm{H}} := \left\{x = (\tilde{x}, x_3)^{\mathrm{T}} \in Q \times \mathbb{R} \,:\, |x_3| \le \mathrm{H}\right\} \quad \text{for all } k \in K_0. \tag{2.3}$$

Denote by G_0^{H} and G_N^{H} the restrictions of the semi-infinite domains G_0 and G_N to G^{H}, i.e.,

$$G_0^{\mathrm{H}} := G^{\mathrm{H}} \cap G_0 \quad \text{and} \quad G_N^{\mathrm{H}} := G^{\mathrm{H}} \cap G_N.$$

Moreover, we will work with the semi-infinite domains

$$G_k^+ := \{x \in Q \times \mathbb{R} \,:\, x_3 > \sigma_k(\tilde{x})\} \text{ and } G_k^- := \{x \in Q \times \mathbb{R} \,:\, x_3 < \sigma_k(\tilde{x})\}, \quad k \in K_0. \tag{2.4}$$

The interface Γ_0 is now illuminated from G_0 by a time-harmonic electric plane wave $\mathbf{E}^{\mathrm{i}}$ at oblique incidence specified by

$$\mathbf{E}^{\mathrm{i}} := \mathbf{p}e^{\mathrm{i}(\alpha_1 x_1 + \alpha_2 x_2 - \alpha_3 x_3)} \quad \text{with } \alpha_3 > 0. \tag{2.5}$$

It in particular fulfills the relation

$$\mathbf{u}\left(\tilde{x} + 2\pi\, m, x_3\right) = e^{\mathrm{i}2\pi(\alpha_1 m_1 + \alpha_2 m_2)}\mathbf{u}(x) \quad \text{for all } m \in \mathbb{Z}^2.$$

This special type of periodicity up to a phase shift will be called α-quasiperiodicity (abbreviated as α-qp). The wave vector $\alpha = (\alpha_1, \alpha_2, -\alpha_3)^{\mathrm{T}}$ of the incident field exhibits the following properties:

$$|\alpha|^2 = |\kappa_0|^2 \quad \text{and} \quad \alpha \cdot \mathbf{p} = 0. \tag{2.6}$$

The total electric fields are given by $\mathbf{E}^{\mathrm{i}} + \mathbf{E}_0$ in G_0 and by $\mathbf{E}_k$ in G_k, $k \in K^N$. Then the 2π-biperiodic electromagnetic scattering problem written in terms of the electric field is expressed as follows: We look for vector fields $\mathbf{E}_k$, $k \in K_0^N$, of locally finite energy, in the sense that

$$\mathbf{E}_k, \mathbf{curl}\,\mathbf{E}_k \in \mathbf{L}^2_{\mathrm{loc}}(\mathbb{R}^3),$$

solving the time-harmonic Maxwell equations

$$\mathbf{curl}\,\mathbf{curl}\,\mathbf{E}_k - \kappa_k^2\mathbf{E}_k = 0 \quad \text{in } G_k \tag{2.7}$$

with respect to the transmission conditions

$$\gamma^-_{\mathrm{D},0}\mathbf{E}_1 = \gamma^+_{\mathrm{D},0}\mathbf{E}_0 + \gamma^+_{\mathrm{D},0}\mathbf{E}^{\mathrm{i}} \qquad \text{on } \Gamma_0, \tag{2.8}$$

$$\gamma^-_{\mathrm{N}_{\kappa_1},0}\mathbf{E}_1 = \rho_1^{-1}\left(\gamma^+_{\mathrm{N}_{\kappa_0},0}\mathbf{E}_0 + \gamma^+_{\mathrm{N}_{\kappa_0},0}\mathbf{E}^{\mathrm{i}}\right) \qquad \text{on } \Gamma_0, \tag{2.9}$$

$$\gamma^-_{\mathrm{D},k}\mathbf{E}_{k+1} = \gamma^+_{\mathrm{D},k}\mathbf{E}_k \qquad \text{on } \Gamma_k \text{ for } k \in K, \tag{2.10}$$

$$\gamma^-_{\mathrm{N}_{\kappa_{k+1}},k}\mathbf{E}_{k+1} = \rho_{k+1}^{-1}\gamma^+_{\mathrm{N}_{\kappa_k},k}\mathbf{E}_k \qquad \text{on } \Gamma_k \text{ for } k \in K \tag{2.11}$$

and the outgoing wave condition in the sense of Rayleigh series:

$$\mathbf{E}_0(x) = \sum_{n\in\mathbb{Z}^2} \mathbf{E}_n^0 e^{\mathrm{i}\left(\alpha^{(n)}\cdot\tilde{x} + \beta_0^{(n)}x_3\right)}, \qquad x \in G_0 \text{ with } x_3 \ge \mathrm{H}, \tag{2.12}$$

$$\mathbf{E}_N(x) = \sum_{n\in\mathbb{Z}^2} \mathbf{E}_n^N e^{\mathrm{i}\left(\alpha^{(n)}\cdot\tilde{x} - \beta_N^{(n)}x_3\right)}, \qquad x \in G_N \text{ with } x_3 \le -\mathrm{H}. \tag{2.13}$$

Here, $n = (n_1, n_2)^{\mathrm{T}}$, $\alpha^{(n)} := (\alpha_1 + n_1, \alpha_2 + n_2)^{\mathrm{T}}$ and

$$\beta_k^{(n)} := \begin{cases} \sqrt{\kappa_k^2 - |\alpha^{(n)}|^2} & \text{with } 0 \le \arg\left(\beta_k^{(n)}\right) < \pi \quad \text{if } \kappa_k \notin \mathbb{R}_-, \\ -\sqrt{\kappa_k^2 - |\alpha^{(n)}|^2} & \text{if } \kappa_k \in \mathbb{R}_- \text{ and } \kappa_k^2 - |\alpha^{(n)}|^2 > 0, \\ \mathrm{i}\sqrt{\kappa_k^2 - |\alpha^{(n)}|^2} & \text{if } \kappa_k \in \mathbb{R}_- \text{ and } \kappa_k^2 - |\alpha^{(n)}|^2 < 0. \end{cases}$$

Since the electric incident waves are α-quasiperiodic, the sought-after fields are also α-quasiperiodic.

3 Function spaces, traces and electromagnetic potentials

Let Ω be a polyhedral Lipschitz domain in $\mathbb{R}^3$. If Ω is bounded, we denote by $H^s(\Omega)$ the usual scalar-valued Sobolev space of order $s \in \mathbb{R}$ with the common convention $L^2(\Omega) := H^0(\Omega)$. Otherwise, $H^s_{\mathrm{loc}}(\Omega)$ refers to the space of functions contained in $H^s(K)$ for all $K \Subset \Omega$. Their vector-valued counterparts are specified by $\mathbf{H}^s(\Omega)$ and $\mathbf{H}^s_{\mathrm{loc}}(\Omega)$. Let $\mathbf{D}$ be a differential operator. Then

$$\mathbf{H}\left(\mathbf{D}, \Omega\right) := \left\{\mathbf{u} \in \mathbf{L}^2(\Omega) \ : \ \mathbf{D}\mathbf{u} \in \mathbf{L}^2(\Omega) \text{ (or } \mathbf{D}\mathbf{u} \in L^2(\Omega)\right\},$$
$$\mathbf{H}_{\mathrm{loc}}\left(\mathbf{D}, \Omega\right) := \left\{\mathbf{u} \in \mathbf{L}^2_{\mathrm{loc}}(\Omega) \ : \ \mathbf{D}\mathbf{u} \in \mathbf{L}^2_{\mathrm{loc}}(\Omega) \text{ (or } \mathbf{D}\mathbf{u} \in L^2_{\mathrm{loc}}(\Omega)\right\}.$$

Both spaces are endowed with their natural graph norm. We consider the following α-quasiperiodic Sobolev spaces for $s \in \mathbb{R}$:

$$\begin{aligned}
\mathbf{H}^s_\alpha(G_k) &:= \left\{\mathbf{u} \in \mathbf{H}^s(G_k) \ : \ \exists \, \alpha\text{-qp } \mathbf{v} \in \mathbf{H}^s_{\mathrm{loc}}(\mathbb{R}^3) \text{ such that } \mathbf{u} = \mathbf{v}|_{G_k}\right\}, && k \in K,\\
\mathbf{H}^s_\alpha(\mathbf{D}, G_k) &:= \left\{\mathbf{u} \in \mathbf{H}^s(\mathbf{D}, G_k) \ : \ \exists \, \alpha\text{-qp } \mathbf{v} \in \mathbf{H}^s_{\mathrm{loc}}(\mathbf{D}, \mathbb{R}^3) \text{ such that } \mathbf{u} = \mathbf{v}|_{G_k}\right\}, && k \in K,\\
\mathbf{H}^s_{\alpha,\mathrm{loc}}(G_k) &:= \left\{\mathbf{u} \in \mathbf{H}^s_{\mathrm{loc}}(G_k) \ : \ \exists \, \alpha\text{-qp } \mathbf{v} \in \mathbf{H}^s_{\mathrm{loc}}(\mathbb{R}^3) \text{ such that } \mathbf{u} = \mathbf{v}|_{G_k}\right\}, && k \in \{0, N\},\\
\mathbf{H}^s_{\alpha,\mathrm{loc}}(\mathbf{D}, G_k) &:= \left\{\mathbf{u} \in \mathbf{H}^s_{\mathrm{loc}}(\mathbf{D}, G_k) \ : \ \exists \, \alpha\text{-qp } \mathbf{v} \in \mathbf{H}^s_{\mathrm{loc}}(\mathbf{D}, \mathbb{R}^3) \text{ s. t. } \mathbf{u} = \mathbf{v}|_{G_k}\right\}, && k \in \{0, N\}.
\end{aligned}$$

Moreover, we define the space

$$\mathbf{H}^s_\alpha := \left\{\mathbf{u} = \sum_{n \in \mathbb{Z}^2} \mathbf{u}_n e^{\mathrm{i}\alpha^{(n)}\cdot\tilde{x}} \ : \ \|\mathbf{u}\|^2_{\alpha,s} = \sum_{n \in \mathbb{Z}^2} \left(1 + \left|\alpha^{(n)}\right|^2\right)^s |\mathbf{u}_n|^2 < \infty\right\}$$

for $s \geq 0$. Completing $\mathbf{L}^2_\alpha(Q)$ with respect to the norm $\|\mathbf{u}\|_{\alpha,-s} := \sup_{0 \neq \mathbf{v} \in \mathbf{H}^s_\alpha} (|(\mathbf{u}, \mathbf{v})_{\mathbf{L}^2_\alpha(Q)}| / \|\mathbf{v}\|_{\alpha,s})$ provides the dual space $\mathbf{H}^{-s}_\alpha$, $s > 0$, of $\mathbf{H}^s_\alpha$. Moreover, we have

$$\mathbf{H}^s_\alpha(\Gamma_k) := \{\mathbf{u} \ : \ \mathbf{u} \circ \sigma_k \in \mathbf{H}^s_\alpha\} \quad \text{for } s \in [0, 1], k \in K_0.$$

The dual space of $\mathbf{H}^s_\alpha(\Gamma_k)$, $k \in K_0$, denoted by $\mathbf{H}^{-s}_\alpha(\Gamma_k)$ for $s \in (0, 1]$, arises from the completion of $\mathbf{L}^2_\alpha(\Gamma)$ with respect to the norm $\|\mathbf{u}\|_{\mathbf{H}^{-s}_\alpha(\Gamma_k)} := \|(\mathbf{u} \circ \sigma_k)(1 + |\nabla\sigma_k|^2)^{1/2}\|_{\alpha,-s}$. We in particular set

$$\mathbf{V}^k_\alpha := \mathbf{H}^{\frac{1}{2}}_\alpha(\Gamma_k) \quad \text{and} \quad \left(\mathbf{V}^k_\alpha\right)' := \mathbf{H}^{-\frac{1}{2}}_\alpha(\Gamma_k).$$

Finally, we introduce the space $\mathbf{L}^2_{\alpha,t}(\Gamma_k)$, $k \in K_0$, which is defined by

$$\mathbf{L}^2_{\alpha,t}(\Gamma_k) := \left\{\mathbf{u} \in \mathbf{L}^2_\alpha(\Gamma_k) \ : \ \mathbf{u} \cdot \mathbf{n}_k = 0\right\}.$$

This function space is identified with the space of two-dimensional tangential vector fields - sections of the tangent bundle $T\Gamma_k$ of Γ_k for almost every $x \in \Gamma_k$.

Traces of vector fields on each of the scattering surfaces Γ_k, $k \in K_0$, are deduced from the classical traces of vector fields on the boundary of bounded polyhedral Lipschitz domains that contain Γ_k, such as G^{H}, with the help of suitable truncation procedures. For details on the classical traces, we refer the reader to [2]-[5].

Definition 3.1. *Let $\mathbf{u} \in \mathbf{C}^\infty_{\mathrm{c}}(\overline{G_k})$ or $\mathbf{u} \in \mathbf{C}^\infty_{\mathrm{c}}(\overline{G^+_k})$. Then we define the upper Dirichlet, Neumann and Dirichlet tangential components traces of $\mathbf{u}$ on Γ_k as*

$$\gamma^+_{\mathrm{D},k}\mathbf{u} := (\mathbf{n}_k \times \mathbf{u})|_{\Gamma_k}, \qquad \gamma^+_{\mathrm{N}_\kappa,k}\mathbf{u} := \kappa^{-1}\left(\mathbf{n}_k \times \mathbf{curl}\,\mathbf{u}\right)|_{\Gamma_k},$$
$$\pi^+_{\mathrm{D},k}\mathbf{u} := ((\mathbf{n}_k \times \mathbf{u}) \times \mathbf{n}_k)|_{\Gamma_k}.$$

Similarly, we have

$$\gamma^-_{\mathrm{D},k}\mathbf{u} := (\mathbf{n}_k \times \mathbf{u})|_{\Gamma_k}, \qquad \gamma^-_{\mathrm{N}_\kappa,k}\mathbf{u} := \kappa^{-1}\left(\mathbf{n}_k \times \mathbf{curl}\,\mathbf{u}\right)|_{\Gamma_k}$$
$$\pi^-_{\mathrm{D},k}\mathbf{u} := ((\mathbf{n}_k \times \mathbf{u}) \times \mathbf{n}_k)|_{\Gamma_k}$$

for $\mathbf{u} \in \mathbf{C}^\infty_{\mathrm{c}}(\overline{G_{k+1}})$ or $\mathbf{u} \in \mathbf{C}^\infty_{\mathrm{c}}(\overline{G^-_k})$.

Remark 3.2 (Notation)**.** *Let G be a bounded polyhedral Lipschitz domain such that $\Gamma_k \subset \partial G$. Additionally, let $\gamma : \mathbf{H}^1_\alpha(G) \to \mathbf{V}^k_\alpha$ be the standard vector trace operator on Γ_k, $k \in K_0$. We denote by γ^{-1} one of its right inverses. From here on, the Dirichlet trace $\gamma_{\mathrm{D},k}$ and the Dirichlet tangential components trace $\pi_{\mathrm{D},k}$ shall be interpreted as the composite operators $\gamma_{\mathrm{D},k}\gamma^{-1}$ and $\pi_{\mathrm{D},k}\gamma^{-1}$, respectively, if they act on traces - lying, for instance, in the space $\mathbf{V}^k_\alpha$.*

For $k \in K_0$, we define the trace spaces $\mathbf{V}^k_{\alpha,\gamma}$ and $\mathbf{V}^k_{\alpha,\pi}$ by

$$\mathbf{V}^k_{\alpha,\gamma} := \gamma_{\mathrm{D},k}(\mathbf{V}^k_\alpha) \quad \text{and} \quad \mathbf{V}^k_{\alpha,\pi} := \pi_{\mathrm{D},k}(\mathbf{V}^k_\alpha).$$

Endowed with the norms

$$\|\mathbf{u}\|_{\mathbf{V}^k_{\alpha,\gamma}} := \inf_{\mathbf{v}\in\mathbf{V}^k_\alpha} \left\{ \|\mathbf{v}\|_{\mathbf{V}^k_\alpha} \;:\; \gamma_{\mathrm{D},k}\mathbf{v} = \mathbf{u} \right\} \quad \text{and} \quad \|\mathbf{u}\|_{\mathbf{V}^k_{\alpha,\pi}} := \inf_{\mathbf{v}\in\mathbf{V}^k_\alpha} \left\{ \|\mathbf{v}\|_{\mathbf{V}^k_\alpha} \;:\; \pi_{\mathrm{D},k}\mathbf{v} = \mathbf{u} \right\}$$

respectively, the spaces $\mathbf{V}^k_{\alpha,\gamma}$ and $\mathbf{V}^k_{\alpha,\pi}$, $k \in K_0$, are Hilbert spaces. These norms guarantee the continuity of the Dirichlet trace $\gamma_{\mathrm{D},k}$ and the Dirichlet tangential components trace $\pi_{\mathrm{D},k}$. The mappings $\gamma_{\mathrm{D},k} : \mathbf{V}^k_\alpha \to \mathbf{V}^k_{\alpha,\gamma}$ and $\pi_{\mathrm{D},k} : \mathbf{V}^k_\alpha \to \mathbf{V}^k_{\alpha,\pi}$ are isomorphisms by construction (cf. [4, p. 683]). The density of $\mathbf{V}^k_\alpha$ in $\mathbf{L}^2_\alpha(\Gamma_k)$ yields that $\mathbf{V}^k_{\alpha,\gamma}$ and $\mathbf{V}^k_{\alpha,\pi}$ are dense subspaces of $\mathbf{L}^2_{\alpha,t}(\Gamma_k)$. Their dual spaces $(\mathbf{V}^k_{\alpha,\gamma})'$ and $(\mathbf{V}^k_{\alpha,\pi})'$ are given with respect to the pivot space $\mathbf{L}^2_{\alpha,t}(\Gamma_k)$. We emphasize that the spaces $\mathbf{V}^k_{\alpha,\gamma}$, $\mathbf{V}^k_{\alpha,\pi}$, $(\mathbf{V}^k_{\alpha,\gamma})'$ and $(\mathbf{V}^k_{\alpha,\pi})'$ are considered as spaces of tangent fields of regularity $1/2$ and $-1/2$, respectively.

In the following, we denote by $i_{\gamma,k} : \mathbf{L}^2_{\alpha,t}(\Gamma_k) \to \mathbf{L}^2_\alpha(\Gamma_k)$ and $i_{\pi,k} : \mathbf{L}^2_{\alpha,t}(\Gamma_k) \to \mathbf{L}^2_\alpha(\Gamma_k)$ the adjoint operators of $\gamma_{\mathrm{D},k}$ and $\pi_{\mathrm{D},k}$ for $k \in K_0$. They can be extended to the following isomorphisms:

$$i_{\gamma,k} : \left(\mathbf{V}^k_{\alpha,\gamma}\right)' \to \left(\mathcal{N}(\gamma_{\mathrm{D},k}) \cap \mathbf{V}^k_\alpha\right)^\circ \subset \left(\mathbf{V}^k_\alpha\right)', \;\; i_{\pi,k} : \;\; \left(\mathbf{V}^k_{\alpha,\pi}\right)' \to \left(\mathcal{N}(\gamma_{\mathrm{D},k}) \cap \mathbf{V}^k_\alpha\right)^\circ \subset \left(\mathbf{V}^k_\alpha\right)',$$

where $\cdot^\circ$ denotes the polar set (defined, e.g., in [21, pp. 136ff.]).

We define an operator r_k, $k \in K_0$, by

$$r_k : \mathbf{L}^2_{\alpha,t}(\Gamma_k) \to \mathbf{L}^2_{\alpha,t}(\Gamma_k), \quad r_k := i^{-1}_{\pi,k} i_{\gamma,k}.$$

This is the rotation operator corresponding to the geometric operation $\cdot \times \mathbf{n}_k$. The operator r_k can be extended and restricted to mappings $r_k : \mathbf{V}^k_{\alpha,\pi} \to \mathbf{V}^k_{\alpha,\gamma}$ and $r_k : (\mathbf{V}^k_{\alpha,\pi})' \to (\mathbf{V}^k_{\alpha,\gamma})'$. For any choice of spaces r_k, $k \in K_0$, is invertible with $r_k^{-1} = r_k' = -r_k$, where r_k' denotes the adjoint operator of r_k with $\mathbf{L}^2_{\alpha,t}(\Gamma_k)$ as pivot space. These and further insights on the rotation operator r_k, $k \in K_0$, are deduced from its nonperiodic equivalent characterized in [3, p. 851].

From here on, we will frequently come across several surface differential operators on Γ_k, $k \in K_0$: We denote by ∇_Γ the tangential gradient, by div_Γ the surface divergence, by $\mathbf{curl}_\Gamma$ the tangential vector curl and by curl_Γ the surface scalar curl on Γ_k. The definitions of these operators on boundaries of bounded Lipschitz domains can be found in [4]. The corresponding definitions on Γ_k are then easily deduced from the former definitions via suitable truncation procedures. Therefore, we will not give further details in the following but refer to Bugert's PhD thesis [8, Section 2.2].

The spaces defined by

$$\mathbf{H}^{-\frac{1}{2}}_\alpha(\mathrm{div}_\Gamma, \Gamma_k) := \left\{ \mathbf{u} \in \left(\mathbf{V}^k_{\alpha,\pi}\right)', \mathrm{div}_\Gamma\, \mathbf{u} \in H^{-\frac{1}{2}}_\alpha(\Gamma_k) \right\},$$
$$\mathbf{H}^{-\frac{1}{2}}_\alpha(\mathrm{curl}_\Gamma, \Gamma_k) := \left\{ \mathbf{u} \in \left(\mathbf{V}^k_{\alpha,\gamma}\right)', \mathrm{div}_\Gamma\, \mathbf{u} \in H^{-\frac{1}{2}}_\alpha(\Gamma_k) \right\}$$

for $k \in K_0$ are the trace spaces of $\mathbf{H}_\alpha(\mathbf{curl}, G)$ ($\mathbf{H}_{\alpha,\mathrm{loc}}(\mathbf{curl}, G)$) for a bounded (an unbounded) polyhedral Lipschitz domain G such that $\Gamma_k \subset \partial G$. Endowed with the norms

$$\|\mathbf{j}\|_{\mathbf{H}^{-\frac{1}{2}}_\alpha(\mathrm{div}_\Gamma, \Gamma_k)} := \|\mathbf{j}\|_{(\mathbf{V}^k_{\alpha,\pi})'} + \|\mathrm{div}_\Gamma\, \mathbf{j}\|_{H^{-\frac{1}{2}}_\alpha(\Gamma_k)},$$

$$\|\mathbf{j}\|_{\mathbf{H}_\alpha^{-\frac{1}{2}}(\mathrm{curl}_\Gamma,\Gamma_k)} := \|\mathbf{j}\|_{(\mathbf{V}^k_{\alpha,\pi})'} + \|\mathrm{curl}_\Gamma\,\mathbf{j}\|_{H_\alpha^{-\frac{1}{2}}(\Gamma_k)},$$

they are Hilbert spaces.

The trace operators $\gamma^\pm_{\mathrm{D},k}$ and $\gamma^\pm_{\mathrm{N}_\kappa,k}$ can be extended to bounded linear operators

$$\gamma^+_{\mathrm{D},k} : \begin{cases} \mathbf{H}_\alpha(\mathbf{curl}, G_k) & \to \mathbf{H}_\alpha^{-\frac{1}{2}}(\mathrm{div}_\Gamma,\Gamma_k) \quad \textit{if } k \in K, \\ \mathbf{H}_{\alpha,\mathrm{loc}}(\mathbf{curl}, G_k) & \to \mathbf{H}_\alpha^{-\frac{1}{2}}(\mathrm{div}_\Gamma,\Gamma_k) \quad \textit{if } k = 0, \\ \mathbf{H}_{\alpha,\mathrm{loc}}(\mathbf{curl}, G_k^+) & \to \mathbf{H}_\alpha^{-\frac{1}{2}}(\mathrm{div}_\Gamma,\Gamma_k) \quad \textit{if } k \in K_0, \end{cases} \tag{3.1}$$

$$\gamma^-_{\mathrm{D},k} : \begin{cases} \mathbf{H}_\alpha(\mathbf{curl}, G_k) & \to \mathbf{H}_\alpha^{-\frac{1}{2}}(\mathrm{div}_\Gamma,\Gamma_{k-1}) \quad \textit{if } k \in K, \\ \mathbf{H}_\alpha(\mathbf{curl}, G_k) & \to \mathbf{H}_\alpha^{-\frac{1}{2}}(\mathrm{div}_\Gamma,\Gamma_{k-1}) \quad \textit{if } k = N, \\ \mathbf{H}_{\alpha,\mathrm{loc}}(\mathbf{curl}, G_k^-) & \to \mathbf{H}_\alpha^{-\frac{1}{2}}(\mathrm{div}_\Gamma,\Gamma_{k-1}) \quad \textit{if } k \in K^N, \end{cases} \tag{3.2}$$

$$\gamma^+_{\mathrm{N}_\kappa,k} : \begin{cases} \mathbf{H}_\alpha(\mathbf{curlcurl}, G_k) & \to \mathbf{H}_\alpha^{-\frac{1}{2}}(\mathrm{div}_\Gamma,\Gamma_k) \quad \textit{if } k \in K, \\ \mathbf{H}_{\alpha,\mathrm{loc}}(\mathbf{curlcurl}, G_k) & \to \mathbf{H}_\alpha^{-\frac{1}{2}}(\mathrm{div}_\Gamma,\Gamma_k) \quad \textit{if } k = 0, \\ \mathbf{H}_{\alpha,\mathrm{loc}}(\mathbf{curlcurl}, G_k^+) & \to \mathbf{H}_\alpha^{-\frac{1}{2}}(\mathrm{div}_\Gamma,\Gamma_k) \quad \textit{if } k \in K_0, \end{cases} \tag{3.3}$$

$$\gamma^-_{\mathrm{N}_\kappa,k} : \begin{cases} \mathbf{H}_\alpha(\mathbf{curlcurl}, G_k) & \to \mathbf{H}_\alpha^{-\frac{1}{2}}(\mathrm{div}_\Gamma,\Gamma_{k-1}) \quad \textit{if } k \in K, \\ \mathbf{H}_\alpha(\mathbf{curlcurl}, G_k) & \to \mathbf{H}_\alpha^{-\frac{1}{2}}(\mathrm{div}_\Gamma,\Gamma_{k-1}) \quad \textit{if } k = N, \\ \mathbf{H}_{\alpha,\mathrm{loc}}(\mathbf{curlcurl}, G_k^-) & \to \mathbf{H}_\alpha^{-\frac{1}{2}}(\mathrm{div}_\Gamma,\Gamma_{k-1}) \quad \textit{if } k \in K^N. \end{cases} \tag{3.4}$$

The operator r_k can be considered as the mapping $r_k : \mathbf{H}_\alpha^{-1/2}(\mathrm{div}_\Gamma,\Gamma_k) \to \mathbf{H}_\alpha^{-1/2}(\mathrm{curl}_\Gamma,\Gamma_k)$ for $k \in K_0$. This ensures that the bilinear form $\mathcal{B}_k : \mathbf{H}_\alpha^{-1/2}(\mathrm{div}_\Gamma,\Gamma_k) \times \mathbf{H}_{-\alpha}^{-1/2}(\mathrm{div}_\Gamma,\Gamma_k) \to \mathbb{C}$, specified by

$$\mathcal{B}_k(\mathbf{j},\mathbf{m}) := \int_{\Gamma_k} \mathbf{j}\cdot r_k(\mathbf{m})\, d\sigma = -\int_{\Gamma_k} r_k(\mathbf{j})\cdot\mathbf{m}\, d\sigma \quad \text{for } k \in K_0, \tag{3.5}$$

is well-defined. It is non-degenerate in the sense of [18, Definition 1.2.1]. A proof is found in [8, Lemma 2.57].

For technical reasons, we also consider the duality product analogous to $\mathcal{B}_k$, $k \in K_0$, on the boundary $\partial\Omega$ of bounded Lipschitz domains Ω with an unit outer normal vector $\mathbf{n}$:

$$\mathcal{B}_{\partial\Omega} : \mathbf{H}^{-\frac{1}{2}}(\mathrm{div}_\Gamma,\partial\Omega) \times \mathbf{H}^{-\frac{1}{2}}(\mathrm{div}_\Gamma,\partial\Omega) \to \mathbb{C}, \quad \mathcal{B}_{\partial\Omega} := \int_{\partial\Omega} \mathbf{j}\cdot r(\mathbf{m})\, d\sigma = -\int_{\partial\Omega} r(\mathbf{j})\cdot\mathbf{m}\, d\sigma,$$

which is defined in [11, § 3] together with the Hilbert space $\mathbf{H}^{-1/2}(\mathrm{div}_\Gamma,\partial\Omega)$ - the nonperiodic equivalent of the space $\mathbf{H}_\alpha^{-1/2}(\mathrm{div}_\Gamma,\Gamma_k)$, $k \in K_0$. Here, the operator r corresponds to the nonperiodic version of the rotation operator r_k, $k \in K_0$. For all $\mathbf{u},\mathbf{v} \in \mathbf{H}(\mathbf{curl},\Omega)$, we have the Green identity

$$\int_\Omega \mathbf{curl}\,\mathbf{u}\cdot\mathbf{v} - \mathbf{u}\cdot\mathbf{curl}\,\mathbf{v}\, dx = \mathcal{B}_{\partial\Omega}(\gamma_{\mathrm{D}}\mathbf{u}, \gamma_{\mathrm{D}}\mathbf{v}). \tag{3.6}$$

Next, we introduce the α-quasiperiodic potential operators relevant for this work. They are based on G^α_κ, the α-quasiperiodic fundamental solution of the time-harmonic Helmholtz equations, specified by

$$G^\alpha_\kappa(x,y) := \frac{\mathrm{i}}{8\pi^2}\sum_{n\in\mathbb{Z}^2} \frac{e^{\mathrm{i}\left(\alpha^{(n)}\cdot(\tilde{x}-\tilde{y})+\beta^{(n)}|x_3-y_3|\right)}}{\beta^{(n)}}, \tag{3.7}$$

where

$$\beta^{(n)} := \begin{cases} \sqrt{\kappa^2 - \left|\alpha^{(n)}\right|^2} & \text{with } 0 \le \arg\left(\beta^{(n)}\right) < \pi \quad \text{if } \kappa \notin \mathbb{R}_-, \\ -\sqrt{\kappa^2 - \left|\alpha^{(n)}\right|^2} & \text{if } \kappa \in \mathbb{R}_- \text{ and } \kappa^2 - \left|\alpha^{(n)}\right|^2 > 0, \\ \mathrm{i}\sqrt{\kappa^2 - \left|\alpha^{(n)}\right|^2} & \text{if } \kappa \in \mathbb{R}_- \text{ and } \kappa^2 - \left|\alpha^{(n)}\right|^2 > 0. \end{cases}$$

Assuming that $\kappa^2 \neq |\alpha^{(n)}|^2$ for all $n \in \mathbb{Z}^2$, the function G_κ^α converges uniformly on compact sets in $\mathbb{R}^3 \setminus \cup_{n\in\mathbb{Z}^2} (2\pi n_1, 2\pi n_2, 0)^{\mathrm{T}}$. Details on the derivation of G_κ^α and its analytical properties are given in the habilitation thesis [1, §3].

The single layer potential on Γ_k, $k \in K_0$, is given by

$$\left(S_k^{\alpha,\kappa}\mathbf{u}\right)(x) := 2\int_{\Gamma_k} G_\kappa^\alpha(x,y)\mathbf{u}(y)\, d\sigma(y), \quad x \in (Q\times\mathbb{R})\setminus\Gamma_k.$$

The related operator $V_{km}^{\alpha,\kappa}$ is defined by

$$\left(V_{km}^{\alpha,\kappa}\mathbf{u}\right)(x) := 2\int_{\Gamma_k} G_\kappa^\alpha(x,y)\mathbf{u}(y)\, d\sigma(y) \quad \text{for } x \in \Gamma_m, m \in K_0.$$

For $k = m$, the operator $V_{kk}^{\alpha,\kappa}$ corresponds to the classical scalar trace of the potential $S_k^{\alpha,\kappa}$.

Lemma 3.3 ([8, Lemma 6.2])**.** *Let $s \in (0,1)$ and $k \in K_0$. Then the operator $S_k^{\alpha,\kappa}$ gives rise to a continuous linear operator,*

$$\begin{aligned} S_k^{\alpha,\kappa} &: H_\alpha^{s-1}(\Gamma_k) \to H_{\alpha,\mathrm{loc}}^{s+\frac12}(G_k^+) \cup H_{\alpha,\mathrm{loc}}^{s+\frac12}(G_k^-), \quad \textit{or} \\ S_k^{\alpha,\kappa} &: \mathbf{H}_\alpha^{s-1}(\Gamma_k) \to \mathbf{H}_{\alpha,\mathrm{loc}}^{s+\frac12}(G_k^+) \cup \mathbf{H}_{\alpha,\mathrm{loc}}^{s+\frac12}(G_k^-). \end{aligned}$$

The operator $V_{kk}^{\alpha,\kappa}$ exhibits the following mapping properties:

$$V_{kk}^{\alpha,\kappa} :\ H_\alpha^{s-1}(\Gamma_k) \to H_\alpha^s(\Gamma_k) \quad \textit{or} \quad V_{kk}^{\alpha,\kappa} :\ \mathbf{H}_\alpha^{s-1}(\Gamma_k) \to \mathbf{H}_\alpha^s(\Gamma_k).$$

Moreover, for $m \in K_0$ such that $k \neq m$, both the operators $V_{km}^{\alpha,\kappa} : H_\alpha^{s-1}(\Gamma_k) \to H_\alpha^s(\Gamma_m)$ as well as $V_{km}^{\alpha,\kappa} : \mathbf{H}_\alpha^{s-1}(\Gamma_k) \to \mathbf{H}_\alpha^s(\Gamma_m)$ are compact.

These mapping properties hold for all $s \in \mathbb{R}$ if Γ_k and Γ_m are smooth surfaces.

Definition 3.4 (Electric potential)**.** *For a density $\mathbf{j} \in \mathbf{H}_\alpha^{-1/2}(\mathrm{div}_\Gamma, \Gamma_k)$, the electric potential $\Psi_{\mathrm{E}_\kappa,k}^\alpha$ on Γ_k, $k \in K_0$, is defined by*

$$\Psi_{\mathrm{E}_\kappa,k}^\alpha \mathbf{j} := \kappa S_k^{\alpha,\kappa}\mathbf{j} + \kappa^{-1}\nabla S_k^{\alpha,\kappa}\,\mathrm{div}_\Gamma\,\mathbf{j}.$$

By $\mathbf{curl}\,\mathbf{curl} = -\Delta + \nabla\,\mathrm{div}$, it also has a representation as $\Psi_{\mathrm{E}_\kappa,k}^\alpha \mathbf{j} = \kappa^{-1}\,\mathbf{curl}\,\mathbf{curl}\,S_k^{\alpha,\kappa}\mathbf{j}$.

Definition 3.5 (Magnetic potential)**.** *For a density $\mathbf{m} \in \mathbf{H}_\alpha^{-1/2}(\mathrm{div}_\Gamma, \Gamma_k)$, we define the magnetic potential $\Psi_{\mathrm{M}_\kappa,k}^\alpha$ on Γ_k, $k \in K_0$, by*

$$\Psi_{\mathrm{M}_\kappa,k}^\alpha \mathbf{m} := \mathbf{curl}\,S_k^{\alpha,\kappa}\mathbf{m}.$$

We in particular observe that

$$\kappa^{-1}\,\mathbf{curl}\,\Psi_{\mathrm{E}_\kappa,k}^\alpha = \Psi_{\mathrm{M}_\kappa,k}^\alpha \quad \text{and} \quad \kappa^{-1}\,\mathbf{curl}\,\Psi_{\mathrm{M}_\kappa,k}^\alpha = \Psi_{\mathrm{E}_\kappa,k}^\alpha. \tag{3.8}$$

Lemma 3.3 and the identities (3.8) imply the following lemma.

Lemma 3.6. *The electromagnetic potentials $\Psi^{\alpha}_{\mathrm{E}_\kappa,k}$ and $\Psi^{\alpha}_{\mathrm{M}_\kappa,k}$ are continuous operators with the following mapping properties:*

$$\Psi^{\alpha}_{\mathrm{E}_\kappa,k}, \Psi^{\alpha}_{\mathrm{M}_\kappa,k} : \mathbf{H}_\alpha^{-\frac{1}{2}}(\mathrm{div}_\Gamma, \Gamma_k) \to \mathbf{H}_{\alpha,\mathrm{loc}}(\mathbf{curl}, G_k^+) \cup \mathbf{H}_{\alpha,\mathrm{loc}}(\mathbf{curl}, G_k^-) \quad \text{for } k \in K_0,$$

where $G_k^\pm$ are the semi-infinite domains from (2.4). For $\mathbf{j}, \mathbf{m} \in \mathbf{H}_\alpha^{-1/2}(\mathrm{div}_\Gamma, \Gamma_k)$, they satisfy the time-harmonic Maxwell equations

$$\left(\mathbf{curl}\,\mathbf{curl} - \kappa^2\right) \Psi^{\alpha}_{\mathrm{E}_\kappa,k}\mathbf{j} = 0 \quad \text{and} \quad \left(\mathbf{curl}\,\mathbf{curl} - \kappa^2\right) \Psi^{\alpha}_{\mathrm{M}_\kappa,k}\mathbf{m} = 0$$

in $G_k^\pm$ as well as an outgoing wave condition of the form (2.12)-(2.13).

Defining $[\gamma_{*,k}] := \gamma^-_{*,k} - \gamma^+_{*,k}$ for $* \in \{\mathrm{D}, \mathrm{N}_\kappa\}$ and $k \in K_0$, the jump relations

$$[\gamma_{\mathrm{D},k}]\, \Psi^{\alpha}_{\mathrm{E}_\kappa,k} = 0, \qquad [\gamma_{\mathrm{N}_\kappa,k}]\, \Psi^{\alpha}_{\mathrm{E}_\kappa,k} = -2\mathrm{I}, \tag{3.9}$$

$$[\gamma_{\mathrm{D},k}]\, \Psi^{\alpha}_{\mathrm{M}_\kappa,k} = -2\mathrm{I}, \qquad [\gamma_{\mathrm{N}_\kappa,k}]\, \Psi^{\alpha}_{\mathrm{M}_\kappa,k} = 0 \tag{3.10}$$

hold.

The considerations in this article involve the boundary integral operators

$$C^{\alpha,\kappa}_{km} := \{\gamma_{\mathrm{D},k}\}\, \Psi^{\alpha}_{\mathrm{E}_\kappa,m} = \{\gamma_{\mathrm{N}_\kappa,k}\}\, \Psi^{\alpha}_{\mathrm{M}_\kappa,m} \quad \text{and} \quad M^{\alpha,\kappa}_{km} := \{\gamma_{\mathrm{D},k}\}\, \Psi^{\alpha}_{\mathrm{M}_\kappa,m} = \{\gamma_{\mathrm{N}_\kappa,k}\}\, \Psi^{\alpha}_{\mathrm{E}_\kappa,m}$$

for $x \in \Gamma_m$, $m \in K_0$, where $\{\gamma_{*,k}\} := -\frac{1}{2}\left(\gamma^-_{*,k} + \gamma^+_{*,k}\right)$ for $* \in \{\mathrm{D}, \mathrm{N}_\kappa\}$ and $k \in K_0$.

Lemma 3.7. *For $k \in K_0$, the boundary integral operators $C^{\alpha,\kappa}_{kk}$ and $M^{\alpha,\kappa}_{kk}$ give rise to bounded linear operators, $C^{\alpha,\kappa}_{kk}, M^{\alpha,\kappa}_{kk} : \mathbf{H}_\alpha^{-1/2}(\mathrm{div}_\Gamma, \Gamma_k) \to \mathbf{H}_\alpha^{-1/2}(\mathrm{div}_\Gamma, \Gamma_k)$. For $m \in K_0$, $m \neq k$, the operators $C^{\alpha,\kappa}_{km}$, $M^{\alpha,\kappa}_{km} : \mathbf{H}_\alpha^{-1/2}(\mathrm{div}_\Gamma, \Gamma_m) \to \mathbf{H}_\alpha^{-1/2}(\mathrm{div}_\Gamma, \Gamma_k)$ are compact.*

This is easily entailed from Lemma 3.6 and the mapping properties (3.4) of the trace operators.

With the help of the jump relations (3.9) and (3.10), we are able to deduce the technical identities.

$$\gamma^\pm_{\mathrm{D},k} \Psi^{\alpha}_{\mathrm{E}_\kappa,k} = \gamma^\pm_{\mathrm{N}_\kappa,k} \Psi^{\alpha}_{\mathrm{M}_\kappa,k} = -C^{\alpha,\kappa}_{kk}, \tag{3.11}$$

$$\gamma^\pm_{\mathrm{N}_\kappa,k} \Psi^{\alpha}_{\mathrm{E}_\kappa,k} = \gamma^\pm_{\mathrm{D},k} \Psi^{\alpha}_{\mathrm{M}_\kappa,k} = -M^{\alpha,\kappa}_{kk} \pm \mathrm{I}. \tag{3.12}$$

For $k \neq m$, we have

$$\gamma^\pm_{\mathrm{D},k} \Psi^{\alpha}_{\mathrm{E}_\kappa,m} = \gamma^\pm_{\mathrm{N}_\kappa,k} \Psi^{\alpha}_{\mathrm{M}_\kappa,m} = -C^{\alpha,\kappa}_{km} \quad \text{and} \quad \gamma^\pm_{\mathrm{N}_\kappa,k} \Psi^{\alpha}_{\mathrm{E}_\kappa,m} = \gamma^\pm_{\mathrm{D},k} \Psi^{\alpha}_{\mathrm{M}_\kappa,m} = -M^{\alpha,\kappa}_{km}. \tag{3.13}$$

The subsequent lemma provides expressions for the adjoint operators $\left(C^{\alpha,\kappa}_{km}\right)'$, $\left(M^{\alpha,\kappa}_{km}\right)'$ of the boundary integral operators $C^{\alpha,\kappa}_{km}$ and $M^{\alpha,\kappa}_{km}$ with respect to the dual systems (see [18, Definition 1.2.3])

$$\mathcal{B}_m(\mathbf{H}_\alpha^{-\frac{1}{2}}(\mathrm{div}_\Gamma, \Gamma_m), \mathbf{H}_{-\alpha}^{-\frac{1}{2}}(\mathrm{div}_\Gamma, \Gamma_m)) \text{ and } \mathcal{B}_k(\mathbf{H}_\alpha^{-\frac{1}{2}}(\mathrm{div}_\Gamma, \Gamma_k), \mathbf{H}_{-\alpha}^{-\frac{1}{2}}(\mathrm{div}_\Gamma, \Gamma_k)).$$

Lemma 3.8 ([8, Lemma 6.9])**.** *Let $k, m \in K_0$. The adjoint operators $(C^{\alpha,\kappa}_{km})', (M^{\alpha,\kappa}_{km})'$ of the integral operators $C^{\alpha,\kappa}_{km}$ and $M^{\alpha,\kappa}_{km}$ with respect to the dual systems $\mathcal{B}_m(\mathbf{H}_\alpha^{-1/2}(\mathrm{div}_\Gamma, \Gamma_m), \mathbf{H}_{-\alpha}^{-1/2}(\mathrm{div}_\Gamma, \Gamma_m)$ as well as $\mathcal{B}_k(\mathbf{H}_\alpha^{-1/2}(\mathrm{div}_\Gamma, \Gamma_k), \mathbf{H}_{-\alpha}^{-1/2}(\mathrm{div}_\Gamma, \Gamma_k))$ are $(C^{\alpha,\kappa}_{km})' = -C^{-\alpha,\kappa}_{mk}$ and $(M^{\alpha,\kappa}_{km})' = -M^{-\alpha,\kappa}_{mk}$. Thus, we have*

$$\mathcal{B}_k\left(C^{\alpha,\kappa}_{km}\mathbf{m}, \mathbf{j}\right) = -\mathcal{B}_m\left(\mathbf{m}, C^{-\alpha,\kappa}_{mk}\mathbf{j}\right) \quad \text{and} \quad \mathcal{B}_k\left(M^{\alpha,\kappa}_{km}\mathbf{m}, \mathbf{j}\right) = -\mathcal{B}_m\left(\mathbf{m}, M^{-\alpha,\kappa}_{mk}\mathbf{j}\right) \tag{3.14}$$

for all $\mathbf{m} \in \mathbf{H}_\alpha^{-1/2}(\mathrm{div}_\Gamma, \Gamma_m)$ and all $\mathbf{j} \in \mathbf{H}_{-\alpha}^{-1/2}(\mathrm{div}_\Gamma, \Gamma_k)$.

Lemma 3.9 ([6, Lemma 3.13 and Corollary 3.15 for $\Gamma := \Gamma_k$]). *For $k \in K_0$, the boundary integral operators $C_{kk}^{\alpha,\kappa}$ and $\mathrm{I} \pm M_{kk}^{\alpha,\kappa}$ are Fredholm operators of index zero in $\mathbf{H}_\alpha^{-1/2}(\mathrm{div}_\Gamma, \Gamma_k)$.*

The subsequent result is concerned with the invertibility of $C_{kk}^{\alpha,\kappa}$.

Lemma 3.10 ([6, Lemma 3.16 for $\Gamma := \Gamma_k$]). *The boundary integral operator $C_{kk}^{\alpha,\kappa}$ is invertible in the Hilbert space $\mathbf{H}_\alpha^{-1/2}(\mathrm{div}_\Gamma, \Gamma_k)$ if and only if the homogeneous Dirichlet problem,*

$$\begin{cases} \mathbf{curl}\,\mathbf{curl}\,\mathbf{E} - \kappa^2\mathbf{E} = 0,\ \mathrm{div}\,\mathbf{E} = 0,\ \gamma_{\mathrm{D},k}\mathbf{E} = 0 \\ \text{and } \mathbf{E} \text{ satisfies the outgoing wave condition} \end{cases} \tag{3.15}$$

only has the trivial solution in both of the domains G_k^+ and G_k^-.

Remark 3.11. *For several results in this article, we require the invertibility of the boundary integral operator $C_{kk}^{\alpha,\kappa}$ in $\mathbf{H}_\alpha^{-1/2}(\mathrm{div}_\Gamma, \Gamma_k)$, which is equivalent to the uniqueness of (3.15) by Lemma 3.10. Even though there exist several counterexamples to the uniqueness of (3.15) (see, e.g., [15], [16]), we assess the assumption that $C_{kk}^{\alpha,\kappa}$ is invertible not to be very restrictive. For details, we refer to [8, Remark 4.46].*

In the course of this article, the following three integral representations are employed.

Lemma 3.12 (Stratton-Chu integral representation, [8, Theorem 4.24]). *Let $\mathbf{E}$ satisfy time-harmonic Maxwell's equations $\mathbf{curl}\,\mathbf{curl}\,\mathbf{E} - \kappa^2\mathbf{E} = 0$ in $G_k^+ \cup G_k^-$ (see (2.4)) satisfying the outgoing wave condition. Then $\mathbf{E}$ admits the integral representation*

$$\mathbf{E}(x) = -\frac{1}{2}\left(\Psi^\alpha_{\mathrm{E}_\kappa,k}\mathbf{j}(x) + \Psi^\alpha_{\mathrm{M}_\kappa,k}\mathbf{m}(x)\right) \quad \text{for } x \in G_k^+ \cup G_k^-,$$

where $\mathbf{j} := [\gamma_{\mathrm{N}_\kappa,k}]\,\mathbf{E}$ and $\mathbf{m} := [\gamma_{\mathrm{D},k}]\,\mathbf{E}$.

Lemma 3.13 (Stratton-Chu type integral representation, [8, Lemma 6.14]). *Let the electric field $\mathbf{E}$ be an α-quasiperiodic solution of time-harmonic Maxwell's equations $\mathbf{curl}\,\mathbf{curl}\,\mathbf{E} - \kappa^2\mathbf{E} = 0$ in the bounded domain G_k, $k \in K$. Then $\mathbf{E}$ can be represented as*

$$\mathbf{E} = \frac{1}{2}\left(\Psi^\alpha_{\mathrm{E}_\kappa,k}\gamma^+_{\mathrm{N}_\kappa,k}\mathbf{E} + \Psi^\alpha_{\mathrm{M}_\kappa,k}\gamma^+_{\mathrm{D},k}\mathbf{E}\right) - \frac{1}{2}\left(\Psi^\alpha_{\mathrm{E}_\kappa,k-1}\gamma^-_{\mathrm{N}_\kappa,k-1}\mathbf{E} + \Psi^\alpha_{\mathrm{M}_\kappa,k-1}\gamma^-_{\mathrm{D},k-1}\mathbf{E}\right) \tag{3.16}$$

in G_k.

Lemma 3.14 ([8, Lemma 6.16]). *Let the electric field $\mathbf{E}$ be an α-quasiperiodic solution of the time-harmonic Maxwell equations $\mathbf{curl}\,\mathbf{curl}\,\mathbf{E} - \kappa^2\mathbf{E} = 0$ in the bounded domain G_k, $k \in K$. Then $\mathbf{E}$ has a unique representation*

$$\mathbf{E} = \Psi^\alpha_{\mathrm{E}_\kappa,k-1}\mathbf{j} + \Psi^\alpha_{\mathrm{E}_\kappa,k}\mathbf{m} \quad \text{in } G_k \tag{3.17}$$

with the densities $\mathbf{m} \in \mathbf{H}_\alpha^{-1/2}(\mathrm{div}_\Gamma, \Gamma_k)$ and $\mathbf{j} \in \mathbf{H}_\alpha^{-1/2}(\mathrm{div}_\Gamma, \Gamma_{k-1})$ if the boundary integral operators $C_{k-1,k-1}^{\alpha,\kappa}$ and $C_{kk}^{\alpha,\kappa}$ satisfy $\mathcal{N}(C_{k-1,k-1}^{\alpha,\kappa}) = \mathcal{N}(C_{kk}^{\alpha,\kappa}) = \{0\}$.

The last result in this section serves as an auxiliary tool in some of the proofs in this article.

Lemma 3.15. *Let $k \in K$. If we have*

$$\Psi^\alpha_{\mathrm{E}_{\kappa_k},k-1}\mathbf{j}_{k-1} = \Psi^\alpha_{\mathrm{E}_{\kappa_k},k}\mathbf{j}_k \quad \text{in } G_k$$

with $\mathbf{j}_{k-1} \in \mathbf{H}_\alpha^{-1/2}(\mathrm{div}_\Gamma, \Gamma_{k-1})$ and $\mathbf{j}_k \in \mathbf{H}_\alpha^{-1/2}(\mathrm{div}_\Gamma, \Gamma_k)$ for some $k \in K$, then

$$\Psi^\alpha_{\mathrm{E}_{\kappa_k},k-1}\mathbf{j}_{k-1} = \Psi^\alpha_{\mathrm{E}_{\kappa_k},k}\mathbf{j}_k = 0 \quad \text{in } G_k.$$

Remark 3.16 (Notation). *In order to keep the notation as simple and as readable as possible, we introduce the convention to replace the superscript κ_k by (k) for $k \in K_0^N$. If κ_k occurs as a subscript, we abbreviate it by k. Thus, we for example write $C_{kk}^{\alpha,(k)}$ instead of C_{kk}^{α,κ_k} and $\beta_k^{(n)}$ instead of $\beta_{\kappa_k}^{(n)}$.*

Moreover, we separate two-component subscripts as they occur in the operators $V_{km}^{\alpha,\kappa}$, $C_{km}^{\alpha,\kappa}$ and $M_{km}^{\alpha,\kappa}$ by commata if misinterpretations are possible. This notation has already been applied in Lemma 3.14.

4 A system of integral equations

Below, we give an equivalent formulation of the 2π-biperiodic multilayered electromagnetic scattering problem in the sense of boundary integral equations that is deduced by extending the combined potential ansatz used in [6] for the 2π-biperiodic single profile electromagnetic scattering problem to multi-profile scattering. This yields a parity-dependent system of integral equations whose size is directly proportional to the number of scattering interfaces in the multilayered scattering structure.

4.1 Boundary integral equation formulation

We assume an α-quasiperiodic Stratton-Chu representation of the electric field $\mathbf{E}_0$ in the layer G_0 above the scattering structure, which is possible by Lemma 3.12. In the subsequent layers, a two-term α-quasiperiodic electric potential ansatz with unknown densities $\mathbf{j}_k \in \mathbf{H}_\alpha^{-1/2}(\mathrm{div}_\Gamma, \Gamma_k)$ (for $k \in K_0$ if N is even or for $k \in K_0 \setminus \{N-1\}$ if N is odd) alternates with an α-quasiperiodic Stratton-Chu type integral representation in the sense of Lemma 3.13. The field $\mathbf{E}_N$ in G_N below the scatterer is finally either considered to have an α-quasiperiodic Stratton-Chu integral representation if N is even, or to be an α-quasiperiodic electric potential applied to the unknown density $\mathbf{j}_{N-1}$ lying in $\mathbf{H}_\alpha^{-1/2}(\mathrm{div}_\Gamma, \Gamma_{N-1})$ if N is odd. Mathematically speaking, the described potential ansatz reads as

$$\mathbf{E}_0 = \frac{1}{2}\left(\Psi^\alpha_{\mathrm{E}_{\kappa_0},0}\gamma^+_{\mathrm{N}_{\kappa_0},0}\mathbf{E}_0 + \Psi^\alpha_{\mathrm{M}_{\kappa_0},0}\gamma^+_{\mathrm{D},0}\mathbf{E}_0\right) \tag{4.1}$$

in G_0,

$$\mathbf{E}_k = \begin{cases} \frac{1}{2}\Big(\Psi^\alpha_{\mathrm{E}_{\kappa_k},k}\gamma^+_{\mathrm{N}_{\kappa_k},k}\mathbf{E}_k + \Psi^\alpha_{\mathrm{M}_{\kappa_k},k}\gamma^+_{\mathrm{D},k}\mathbf{E}_k \\ \qquad - \Psi^\alpha_{\mathrm{E}_{\kappa_k},k-1}\gamma^-_{\mathrm{N}_{\kappa_k},k-1}\mathbf{E}_k - \Psi^\alpha_{\mathrm{M}_{\kappa_k},k-1}\gamma^-_{\mathrm{D},k-1}\mathbf{E}_k\Big) & \text{for even } k, \\ \Psi^\alpha_{\mathrm{E}_{\kappa_k},k-1}\mathbf{j}_{k-1} + \Psi^\alpha_{\mathrm{E}_{\kappa_k},k}\mathbf{j}_k & \text{for odd } k \end{cases} \tag{4.2}$$

in G_k for $k \in K$,

$$\mathbf{E}_N = \begin{cases} -\frac{1}{2}\left(\Psi^\alpha_{\mathrm{E}_{\kappa_N},N-1}\gamma^-_{\mathrm{N}_{\kappa_N},N-1}\mathbf{E}_N + \Psi^\alpha_{\mathrm{M}_{\kappa_N},N-1}\gamma^-_{\mathrm{D},N-1}\mathbf{E}_N\right) & \text{for even } N, \\ \Psi^\alpha_{\mathrm{E}_{\kappa_N},N-1}\mathbf{j}_{N-1} & \text{for odd } N \end{cases} \tag{4.3}$$

in G_N. By Lemma 3.14, the densities $\mathbf{j}_k$, $k \in K_0$, are uniquely determined if

$$\mathcal{N}\left(C^{\alpha,(k)}_{k-1,k-1}\right) = \mathcal{N}\left(C^{\alpha,(k)}_{kk}\right) = \{0\} \quad \text{for odd } k \in K$$

and additionally $\mathcal{N}(C^{\alpha,(N)}_{N-1,N-1}) = \{0\}$, in case of an odd number of interfaces N, holds.

The ansatz presented above is inspired by [20] and leads to a system of N integral equations for the unknown densities

$$\mathbf{j}_k \in \mathbf{H}_\alpha^{-\frac{1}{2}}(\mathrm{div}_\Gamma, \Gamma_k), \quad k \in K_0.$$

Written in matrix form, its structure slightly differs depending on whether N is even or odd.

Remark 4.1 (Notation)**.** *In order to simplify the notation in this section, we define the auxiliary index sets K_{even} and K_{odd} connected to the N-index set K as*

$$K_{\mathrm{even}} := \{k \in K \ : \ k \text{ is even}\} \quad \text{and} \quad K_{\mathrm{odd}} := \{k \in K \ : \ k \text{ is odd}\}\,.$$

The rest of this subsection is concerned with the detailed derivation of the already mentioned boundary integral equations, based on the potential ansatz (4.1)-(4.3). Their presentation in terms of a system of linear integral equations is then seen in the subsequent subsection. For convenience, we recall the transmission conditions (2.8)-(2.11). Expressing them in the form

$$\gamma_{\mathrm{D},0}^{-}\mathbf{E}_1 = \gamma_{\mathrm{D},0}^{+}\mathbf{E}_0 + \gamma_{\mathrm{D},0}^{-}\mathbf{E}^{\mathrm{i}} \qquad \text{on } \Gamma_0, \tag{4.4}$$

$$\frac{\kappa_1}{\mu_1}\gamma_{\mathrm{N}_{\kappa_1},0}^{-}\mathbf{E}_1 = \frac{\kappa_0}{\mu_0}\left(\gamma_{\mathrm{N}_{\kappa_0},0}^{+}\mathbf{E}_0 + \gamma_{\mathrm{N}_{\kappa_0},0}^{-}\mathbf{E}^{\mathrm{i}}\right) \qquad \text{on } \Gamma_0, \tag{4.5}$$

$$\gamma_{\mathrm{D},k}^{-}\mathbf{E}_{k+1} = \gamma_{\mathrm{D},k}^{+}\mathbf{E}_k \qquad \text{on } \Gamma_k \text{ for } k \in K, \tag{4.6}$$

$$\frac{\kappa_{k+1}}{\mu_{k+1}}\gamma_{\mathrm{N}_{\kappa_{k+1}},k}^{-}\mathbf{E}_{k+1} = \frac{\kappa_k}{\mu_k}\gamma_{\mathrm{N}_{\kappa_k},k}^{+}\mathbf{E}_k \qquad \text{on } \Gamma_k \text{ for } k \in K \tag{4.7}$$

simplifies the following considerations. Since we require the incident electric field $\mathbf{E}^{\mathrm{i}}$ to solve the time-harmonic Maxwell equations with respect to the wave number κ_0 in absence of the 2π-biperiodic multilayered structure, Lemma 3.12 implies that $\mathbf{E}^{\mathrm{i}}$ can be represented as

$$\mathbf{E}^{\mathrm{i}} = -\frac{1}{2}\left(\Psi_{\mathrm{E}_{\kappa_0},0}^{\alpha}\gamma_{\mathrm{N}_{\kappa_0},0}^{-}\mathbf{E}^{\mathrm{i}} + \Psi_{\mathrm{M}_{\kappa_0},0}^{\alpha}\gamma_{\mathrm{D},0}^{-}\mathbf{E}^{\mathrm{i}}\right) \quad \text{in } G_0^{-}, \tag{4.8}$$

where $G_0^{-} := \{x \in Q \times \mathbb{R} \,:\, x_3 < \sigma_0(\tilde{x})\}$. We then apply the Dirichlet traces $\gamma_{\mathrm{D},0}^{+}$ in (4.1) and $\gamma_{\mathrm{D},0}^{-}$ in (4.8) to arrive at

$$\gamma_{\mathrm{D},0}^{+}\mathbf{E}_0 = -\frac{1}{2}\left(C_{00}^{\alpha,(0)}\gamma_{\mathrm{N}_{\kappa_0},0}^{+}\mathbf{E}_0 + \left(M_{00}^{\alpha,(0)} - \mathrm{I}\right)\gamma_{\mathrm{D},0}^{+}\mathbf{E}_0\right), \tag{4.9}$$

$$\gamma_{\mathrm{D},0}^{-}\mathbf{E}^{\mathrm{i}} = \frac{1}{2}\left(C_{00}^{\alpha,(0)}\gamma_{\mathrm{N}_{\kappa_0},0}^{-}\mathbf{E}^{\mathrm{i}} + \left(M_{00}^{\alpha,(0)} + \mathrm{I}\right)\gamma_{\mathrm{D},0}^{-}\mathbf{E}^{\mathrm{i}}\right) \tag{4.10}$$

with the help of the identities (3.11)-(3.12). We then subtract equation (4.9) from equation (4.10) and multiply the result by the factor $2\kappa_0/\mu_0$:

$$\frac{\kappa_0}{\mu_0}C_{00}^{\alpha,(0)}\left(\gamma_{\mathrm{N}_{\kappa_0},0}^{+}\mathbf{E}_0 + \gamma_{\mathrm{N}_{\kappa_0},0}^{-}\mathbf{E}^{\mathrm{i}}\right) + \frac{\kappa_0}{\mu_0}\left(M_{00}^{\alpha,(0)} + \mathrm{I}\right)\left(\gamma_{\mathrm{D},0}^{+}\mathbf{E}_0 + \gamma_{\mathrm{D},0}^{-}\mathbf{E}^{\mathrm{i}}\right) = 2\frac{\kappa_0}{\mu_0}\gamma_{\mathrm{D},0}^{-}\mathbf{E}^{\mathrm{i}}. \tag{4.11}$$

Exploiting the transmission conditions (4.4)-(4.5) as well as the potential ansatz (4.2) for $k = 1$, we moreover infer that

$$\begin{aligned}\frac{\kappa_1}{\mu_1}C_{00}^{\alpha,(0)}\gamma_{\mathrm{N}_{\kappa_1},0}^{-}&\left(\Psi_{\mathrm{E}_{\kappa_1},0}^{\alpha}\mathbf{j}_0 + \Psi_{\mathrm{E}_{\kappa_1},1}^{\alpha}\mathbf{j}_1\right)\\ &+ \frac{\kappa_0}{\mu_0}\left(M_{00}^{\alpha,(0)} + \mathrm{I}\right)\gamma_{\mathrm{D},0}^{-}\left(\Psi_{\mathrm{E}_{\kappa_1},0}^{\alpha}\mathbf{j}_0 + \Psi_{\mathrm{E}_{\kappa_1},1}^{\alpha}\mathbf{j}_1\right) = 2\frac{\kappa_0}{\mu_0}\gamma_{\mathrm{D},0}^{-}\mathbf{E}^{\mathrm{i}}.\end{aligned}$$

Finally, the identities (3.11)-(3.12) give rise to the boundary integral equation

$$\begin{aligned}&\left[\frac{\kappa_1}{\mu_1}C_{00}^{\alpha,(0)}\left(M_{00}^{\alpha,(1)} + \mathrm{I}\right) + \frac{\kappa_0}{\mu_0}\left(M_{00}^{\alpha,(0)} + \mathrm{I}\right)C_{00}^{\alpha,(1)}\right]\mathbf{j}_0\\ &\qquad + \left[\frac{\kappa_1}{\mu_1}C_{00}^{\alpha,(0)}M_{01}^{\alpha,(1)} + \frac{\kappa_0}{\mu_0}\left(M_{00}^{\alpha,(0)} + \mathrm{I}\right)C_{01}^{\alpha,(1)}\right]\mathbf{j}_1 = -2\frac{\kappa_0}{\mu_0}\gamma_{\mathrm{D},0}^{-}\mathbf{E}^{\mathrm{i}}\end{aligned} \tag{4.12}$$

on Γ_0. Similarly, we obtain boundary integral equations on Γ_{k-1} and Γ_k, $k \in K_{\mathrm{even}} \setminus \{N-1\}$. Indeed, we separately apply the Dirichlet traces $\gamma_{\mathrm{D},k-1}^{-}$ and $\gamma_{\mathrm{D},k}^{+}$ to the electric field $\mathbf{E}_k$, $k \in K_{\mathrm{even}} \setminus \{N-1\}$, represented as in (4.2). Using (3.11)-(3.13), this leads to

$$\begin{aligned}-\frac{\kappa_k}{\mu_k}&\left[C_{k-1,k}^{\alpha,(k)}\gamma_{\mathrm{N}_{\kappa_k},k}^{+}\mathbf{E}_k + M_{k-1,k}^{\alpha,(k)}\gamma_{\mathrm{D},k}^{+}\mathbf{E}_k\right.\\ &\qquad\left. + C_{k-1,k-1}^{\alpha,(k)}\gamma_{\mathrm{N}_{\kappa_k},k-1}^{-}\mathbf{E}_k + \left(M_{k-1,k-1}^{\alpha,(k)} - \mathrm{I}\right)\gamma_{\mathrm{D},k-1}^{-}\mathbf{E}_k\right] = 0\end{aligned}$$

and

$$-\frac{\kappa_k}{\mu_k}\left[C^{\alpha,(k)}_{kk}\gamma^+_{\mathrm{N}_{\kappa_k},k}\mathbf{E}_k+\left(M^{\alpha,(k)}_{kk}+\mathrm{I}\right)\gamma^+_{\mathrm{D},k}\mathbf{E}_k+C^{\alpha,(k)}_{k,k-1}\gamma^-_{\mathrm{N}_{\kappa_k},k-1}\mathbf{E}_k+M^{\alpha,(k)}_{k,k-1}\gamma^-_{\mathrm{D},k-1}\mathbf{E}_k\right]=0,$$

where we additionally multiplied both equations by $2\kappa_k/\mu_k$. Next, we insert the transmission conditions (4.6)-(4.7) as well as the electric potential ansatzes for $\mathbf{E}_{k-1}$ and $\mathbf{E}_{k+1}$ from (4.2) into both equations to arrive at

$$\begin{aligned}
-\frac{\kappa_{k+1}}{\mu_{k+1}}C^{\alpha,(k)}_{k-1,k}\gamma^-_{\mathrm{N}_{\kappa_{k+1}},k}&\left(\Psi^\alpha_{\mathrm{E}_{\kappa_{k+1}},k}\mathbf{j}_k+\Psi^\alpha_{\mathrm{E}_{\kappa_{k+1}},k+1}\mathbf{j}_{k+1}\right)\\
&-\frac{\kappa_k}{\mu_k}M^{\alpha,(k)}_{k-1,k}\gamma^-_{\mathrm{D},k}\left(\Psi^\alpha_{\mathrm{E}_{\kappa_{k+1}},k}\mathbf{j}_k+\Psi^\alpha_{\mathrm{E}_{\kappa_{k+1}},k+1}\mathbf{j}_{k+1}\right)\\
&+\frac{\kappa_{k-1}}{\mu_{k-1}}C^{\alpha,(k)}_{k-1,k-1}\gamma^+_{\mathrm{N}_{\kappa_{k-1}},k-1}\left(\Psi^\alpha_{\mathrm{E}_{\kappa_{k-1}},k-2}\mathbf{j}_{k-2}+\Psi^\alpha_{\mathrm{E}_{\kappa_{k-1}},k-1}\mathbf{j}_{k-1}\right)\\
&+\frac{\kappa_k}{\mu_k}\left(M^{\alpha,(k)}_{k-1,k-1}-\mathrm{I}\right)\gamma^+_{\mathrm{D},k-1}\left(\Psi^\alpha_{\mathrm{E}_{\kappa_{k-1}},k-2}\mathbf{j}_{k-2}+\Psi^\alpha_{\mathrm{E}_{\kappa_{k-1}},k-1}\mathbf{j}_{k-1}\right)=0
\end{aligned}$$

and

$$\begin{aligned}
-\frac{\kappa_{k+1}}{\mu_{k+1}}C^{\alpha,(k)}_{kk}\gamma^+_{\mathrm{N}_{\kappa_k},k}&\left(\Psi^\alpha_{\mathrm{E}_{\kappa_{k+1}},k}\mathbf{j}_k+\Psi^\alpha_{\mathrm{E}_{\kappa_{k+1}},k+1}\mathbf{j}_{k+1}\right)\\
&-\frac{\kappa_k}{\mu_k}\left(M^{\alpha,(k)}_{kk}+\mathrm{I}\right)\gamma^+_{\mathrm{D},k}\left(\Psi^\alpha_{\mathrm{E}_{\kappa_{k+1}},k}\mathbf{j}_k+\Psi^\alpha_{\mathrm{E}_{\kappa_{k+1}},k+1}\mathbf{j}_{k+1}\right)\\
&+\frac{\kappa_{k-1}}{\mu_{k-1}}C^{\alpha,(k)}_{k,k-1}\gamma^-_{\mathrm{N}_{\kappa_k},k-1}\left(\Psi^\alpha_{\mathrm{E}_{\kappa_{k-1}},k-2}\mathbf{j}_{k-2}+\Psi^\alpha_{\mathrm{E}_{\kappa_{k-1}},k-1}\mathbf{j}_{k-1}\right)\\
&+\frac{\kappa_k}{\mu_k}M^{\alpha,(k)}_{k,k-1}\gamma^-_{\mathrm{D},k-1}\left(\Psi^\alpha_{\mathrm{E}_{\kappa_{k-1}},k-2}\mathbf{j}_{k-2}+\Psi^\alpha_{\mathrm{E}_{\kappa_{k-1}},k-1}\mathbf{j}_{k-1}\right)=0.
\end{aligned}$$

Eventually, the technical identities (3.11)-(3.13) yield the boundary integral equation

$$\begin{aligned}
-\Bigg[\frac{\kappa_{k-1}}{\mu_{k-1}}C^{\alpha,(k)}_{k-1,k-1}M^{\alpha,(k-1)}_{k-1,k-2}&+\frac{\kappa_k}{\mu_k}\left(M^{\alpha,(k)}_{k-1,k-1}-\mathrm{I}\right)C^{\alpha,(k-1)}_{k-1,k-2}\Bigg]\mathbf{j}_{k-2}\\
&-\left[\frac{\kappa_{k-1}}{\mu_{k-1}}C^{\alpha,(k)}_{k-1,k-1}\left(M^{\alpha,(k-1)}_{k-1,k-1}-\mathrm{I}\right)+\frac{\kappa_k}{\mu_k}\left(M^{\alpha,(k)}_{k-1,k-1}-\mathrm{I}\right)C^{\alpha,(k-1)}_{k-1,k-1}\right]\mathbf{j}_{k-1}\\
&+\left[\frac{\kappa_{k+1}}{\mu_{k+1}}C^{\alpha,(k)}_{k-1,k}\left(M^{\alpha,(k+1)}_{kk}+\mathrm{I}\right)+\frac{\kappa_k}{\mu_k}M^{\alpha,(k)}_{k-1,k}C^{\alpha,(k+1)}_{kk}\right]\mathbf{j}_k\\
&+\left[\frac{\kappa_{k+1}}{\mu_{k+1}}C^{\alpha,(k)}_{k-1,k}M^{\alpha,(k+1)}_{k,k+1}+\frac{\kappa_k}{\mu_k}M^{\alpha,(k)}_{k-1,k}C^{\alpha,(k+1)}_{k,k+1}\right]\mathbf{j}_{k+1}=0
\end{aligned}$$

on Γ_{k-1} and the boundary integral equation

$$\begin{aligned}
-\Bigg[\frac{\kappa_{k-1}}{\mu_{k-1}}C^{\alpha,(k)}_{k,k-1}M^{\alpha,(k-1)}_{k-1,k-2}&+\frac{\kappa_k}{\mu_k}M^{\alpha,(k)}_{k,k-1}C^{\alpha,(k-1)}_{k-1,k-2}\Bigg]\mathbf{j}_{k-2}\\
&-\left[\frac{\kappa_{k-1}}{\mu_{k-1}}C^{\alpha,(k)}_{k,k-1}\left(M^{\alpha,(k-1)}_{k-1,k-1}-\mathrm{I}\right)+\frac{\kappa_k}{\mu_k}M^{\alpha,(k)}_{k,k-1}C^{\alpha,(k-1)}_{k-1,k-1}\right]\mathbf{j}_{k-1}\\
&+\left[\frac{\kappa_{k+1}}{\mu_{k+1}}C^{\alpha,(k)}_{kk}\left(M^{\alpha,(k+1)}_{kk}+\mathrm{I}\right)+\frac{\kappa_k}{\mu_k}\left(M^{\alpha,(k)}_{kk}+\mathrm{I}\right)C^{\alpha,(k+1)}_{kk}\right]\mathbf{j}_k\\
&+\left[\frac{\kappa_{k+1}}{\mu_{k+1}}C^{\alpha,(k)}_{kk}M^{\alpha,(k+1)}_{k,k+1}+\frac{\kappa_k}{\mu_k}\left(M^{\alpha,(k)}_{kk}+\mathrm{I}\right)C^{\alpha,(k+1)}_{k,k+1}\right]\mathbf{j}_{k+1}=0
\end{aligned}$$

on Γ_k for $k\in K_{\mathrm{even}}\setminus\{N-1\}$. Due to the characteristics of the potential ansatz (4.3) for $\mathbf{E}_N$ in G_N, the boundary integral equations on Γ_{N-1} for even N and those on Γ_{N-2} as well as on Γ_{N-1} for odd N differ slightly from the boundary integral equations derived before. The analogous ideas as already applied above provide

$$-\left[\frac{\kappa_{N-1}}{\mu_{N-1}}C^{\alpha,(N)}_{N-1,N-1}M^{\alpha,(N-1)}_{N-1,N-2}-\frac{\kappa_N}{\mu_N}\left(M^{\alpha,(N)}_{N-1,N-1}-\mathrm{I}\right)C^{\alpha,(N-1)}_{N-1,N-2}\right]\mathbf{j}_{N-2}$$

$$
- \left[\frac{\kappa_{N-1}}{\mu_{N-1}} C^{\alpha,(N)}_{N-1,N-1} \left(M^{\alpha,(N-1)}_{N-1,N-1} - \mathrm{I} \right) - \frac{\kappa_N}{\mu_N} \left(M^{\alpha,(N)}_{N-1,N-1} - \mathrm{I} \right) C^{\alpha,(N-1)}_{N-1,N-1} \right] \mathbf{j}_{N-1} = 0
$$

on Γ_{N-1} for even N,

$$
\begin{aligned}
- \left[\frac{\kappa_{N-2}}{\mu_{N-2}} C^{\alpha,(N-1)}_{N-2,N-2} M^{\alpha,(N-2)}_{N-2,N-3} + \frac{\kappa_{N-1}}{\mu_{N-1}} \left(M^{\alpha,(N-1)}_{N-2,N-2} - \mathrm{I} \right) C^{\alpha,(N-2)}_{N-2,N-3} \right] \mathbf{j}_{N-3} \\
- \left[\frac{\kappa_{N-2}}{\mu_{N-2}} C^{\alpha,(N-1)}_{N-2,N-2} \left(M^{\alpha,(N-2)}_{N-2,N-2} - \mathrm{I} \right) + \frac{\kappa_{N-1}}{\mu_{N-1}} \left(M^{\alpha,(N-1)}_{N-2,N-2} - \mathrm{I} \right) C^{\alpha,(N-2)}_{N-2,N-2} \right] \mathbf{j}_{N-2} \\
+ \left[\frac{\kappa_N}{\mu_N} C^{\alpha,(N-1)}_{N-2,N-1} \left(M^{\alpha,(N-2)}_{N-1,N-1} + \mathrm{I} \right) + \frac{\kappa_{N-1}}{\mu_{N-1}} M^{\alpha,(N-1)}_{N-2,N-1} C^{\alpha,(N)}_{N-1,N-1} \right] \mathbf{j}_{N-1} = 0
\end{aligned}
$$

on Γ_{N-2} for odd N, and

$$
\begin{aligned}
- \left[\frac{\kappa_{N-2}}{\mu_{N-2}} C^{\alpha,(N-1)}_{N-1,N-2} M^{\alpha,(N-2)}_{N-2,N-3} + \frac{\kappa_{N-1}}{\mu_{N-1}} M^{\alpha,(N-1)}_{N-1,N-2} C^{\alpha,(N-2)}_{N-2,N-3} \right] \mathbf{j}_{N-3} \\
- \left[\frac{\kappa_{N-2}}{\mu_{N-2}} C^{\alpha,(N-1)}_{N-1,N-2} \left(M^{\alpha,(N-2)}_{N-2,N-2} - \mathrm{I} \right) + \frac{\kappa_{N-1}}{\mu_{N-1}} M^{\alpha,(N-1)}_{N-1,N-2} C^{\alpha,(N-2)}_{N-2,N-2} \right] \mathbf{j}_{N-2} \\
+ \left[\frac{\kappa_N}{\mu_N} C^{\alpha,(N)}_{N-1,N-1} \left(M^{\alpha,(N)}_{N-1,N-1} + \mathrm{I} \right) + \frac{\kappa_{N-1}}{\mu_{N-1}} \left(M^{\alpha,(N-1)}_{N-1,N-1} + \mathrm{I} \right) C^{\alpha,(N)}_{N-1,N-1} \right] \mathbf{j}_{N-1} = 0
\end{aligned}
$$

on Γ_{N-1} for odd N.

4.2 Structure of the system of linear integral equations

For an even number of interfaces N, the linear system is structured as follows:

$$
\underbrace{\begin{pmatrix}
* & * & & & & & & & & & \\
* & * & * & * & & & & & & & \\
* & * & * & * & & & & & & & \\
& & * & * & * & * & & & & & \\
& & * & * & * & * & & & & & \\
& & & & & & \ddots & & & & \\
& & & & & & & * & * & * & * \\
& & & & & & & * & * & * & * \\
& & & & & & & & & * & *
\end{pmatrix}}_{=:\, M^{\mathrm{even}}_\alpha}
\begin{pmatrix} \mathbf{j}_0 \\ \mathbf{j}_1 \\ \mathbf{j}_2 \\ \mathbf{j}_3 \\ \vdots \\ \mathbf{j}_{N-4} \\ \mathbf{j}_{N-3} \\ \mathbf{j}_{N-2} \\ \mathbf{j}_{N-1} \end{pmatrix}
=
\begin{pmatrix} -\frac{2\kappa_0}{\mu_0} \gamma^-_{\mathrm{D},0} \mathbf{E}^{\mathrm{i}} \\ 0 \\ 0 \\ 0 \\ \vdots \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \tag{4.13}
$$

where the nonvanishing coefficients of M^{even}_α are given by

$$
\begin{aligned}
(M^{\mathrm{even}}_\alpha)_{1,1} &= \frac{\kappa_1}{\mu_1} C^{\alpha,(0)}_{00} \left(M^{\alpha,(1)}_{00} + \mathrm{I} \right) + \frac{\kappa_0}{\mu_0} \left(M^{\alpha,(0)}_{00} + \mathrm{I} \right) C^{\alpha,(1)}_{00}, \\
(M^{\mathrm{even}}_\alpha)_{1,2} &= \frac{\kappa_1}{\mu_1} C^{\alpha,(0)}_{00} M^{\alpha,(1)}_{01} + \frac{\kappa_0}{\mu_0} \left(M^{\alpha,(0)}_{00} + \mathrm{I} \right) C^{\alpha,(1)}_{01}, \\
(M^{\mathrm{even}}_\alpha)_{N,N-1} &= -\frac{\kappa_{N-1}}{\mu_{N-1}} C^{\alpha,(N)}_{N-1,N-1} M^{\alpha,(N-1)}_{N-1,N-2} - \frac{\kappa_N}{\mu_N} \left(M^{\alpha,(N)}_{N-1,N-1} - \mathrm{I} \right) C^{\alpha,(N-1)}_{N-1,N-2}, \\
(M^{\mathrm{even}}_\alpha)_{N,N} &= -\frac{\kappa_{N-1}}{\mu_{N-1}} C^{\alpha,(N)}_{N-1,N-1} \left(M^{\alpha,(N-1)}_{N-1,N-1} - \mathrm{I} \right) - \frac{\kappa_N}{\mu_N} \left(M^{\alpha,(N)}_{N-1,N-1} - \mathrm{I} \right) C^{\alpha,(N-1)}_{N-1,N-1}
\end{aligned}
$$

and

$$
(M^{\mathrm{even}}_\alpha)_{k+1,k} = -\frac{\kappa_k}{\mu_k} C^{\alpha,(k+1)}_{kk} M^{\alpha,(k)}_{k,k-1} - \frac{\kappa_{k+1}}{\mu_{k+1}} \left(M^{\alpha,(k+1)}_{kk} - \mathrm{I} \right) C^{\alpha,(k)}_{k,k-1},
$$

$$
\begin{aligned}
(M_\alpha^{\text{even}})_{k+1,k+1} &= -\frac{\kappa_k}{\mu_k} C_{kk}^{\alpha,(k+1)} \left(M_{kk}^{\alpha,(k)} - \mathrm{I} \right) - \frac{\kappa_{k+1}}{\mu_{k+1}} \left(M_{kk}^{\alpha,(k+1)} - \mathrm{I} \right) C_{kk}^{\alpha,(k)}, \\
(M_\alpha^{\text{even}})_{k+1,k+2} &= \frac{\kappa_{k+2}}{\mu_{k+2}} C_{k,k+1}^{\alpha,(k+1)} \left(M_{k+1,k+1}^{\alpha,(k+2)} + \mathrm{I} \right) + \frac{\kappa_{k+1}}{\mu_{k+1}} M_{k,k+1}^{\alpha,(k+1)} C_{k+1,k+1}^{\alpha,(k+2)}, \\
(M_\alpha^{\text{even}})_{k+1,k+3} &= \frac{\kappa_{k+2}}{\mu_{k+2}} C_{k,k+1}^{\alpha,(k+1)} M_{k+1,k+2}^{\alpha,(k+2)} + \frac{\kappa_{k+1}}{\mu_{k+1}} M_{k,k+1}^{\alpha,(k+1)} C_{k+1,k+2}^{\alpha,(k+2)}, \\
(M_\alpha^{\text{even}})_{k+2,k} &= -\frac{\kappa_k}{\mu_k} C_{k+1,k}^{\alpha,(k+1)} M_{k,k-1}^{\alpha,(k)} - \frac{\kappa_{k+1}}{\mu_{k+1}} M_{k+1,k}^{\alpha,(k+1)} C_{k,k-1}^{\alpha,(k)}, \\
(M_\alpha^{\text{even}})_{k+2,k+1} &= -\frac{\kappa_k}{\mu_k} C_{k+1,k}^{\alpha,(k+1)} \left(M_{kk}^{\alpha,(k)} - \mathrm{I} \right) - \frac{\kappa_{k+1}}{\mu_{k+1}} M_{k+1,k}^{\alpha,(k+1)} C_{kk}^{\alpha,(k)}, \\
(M_\alpha^{\text{even}})_{k+2,k+2} &= \frac{\kappa_{k+2}}{\mu_{k+2}} C_{k+1,k+1}^{\alpha,(k+1)} \left(M_{k+1,k+1}^{\alpha,(k+2)} + \mathrm{I} \right) + \frac{\kappa_{k+1}}{\mu_{k+1}} \left(M_{k+1,k+1}^{\alpha,(k+1)} + \mathrm{I} \right) C_{k+1,k+1}^{\alpha,(k+2)}, \\
(M_\alpha^{\text{even}})_{k+2,k+3} &= \frac{\kappa_{k+2}}{\mu_{k+2}} C_{k+1,k+1}^{\alpha,(k+1)} M_{k+1,k+2}^{\alpha,(k+2)} + \frac{\kappa_{k+1}}{\mu_{k+1}} \left(M_{k+1,k+1}^{\alpha,(k+1)} + \mathrm{I} \right) C_{k+1,k+2}^{\alpha,(k+2)}
\end{aligned}
$$

for $k \in K_{\text{odd}} \setminus \{N-1\}$. Lemma 3.7 implies that

$$
\begin{aligned}
(M_\alpha^{\text{even}})_{1,l} &: \mathbf{H}_\alpha^{-\frac{1}{2}}(\operatorname{div}_\Gamma, \Gamma_{l-1}) \quad \to \mathbf{H}_\alpha^{-\frac{1}{2}}(\operatorname{div}_\Gamma, \Gamma_0) \qquad \text{for } l \in \{1,2\}, \\
(M_\alpha^{\text{even}})_{N,N-l} &: \mathbf{H}_\alpha^{-\frac{1}{2}}(\operatorname{div}_\Gamma, \Gamma_{N-l-1}) \to \mathbf{H}_\alpha^{-\frac{1}{2}}(\operatorname{div}_\Gamma, \Gamma_{N-1}) \quad \text{for } l \in \{0,1\}
\end{aligned}
$$

and, for $k \in K_{\text{odd}} \setminus \{N-1\}$, that

$$
\left.
\begin{aligned}
(M_\alpha^{\text{even}})_{k+1,k+l} &: \mathbf{H}_\alpha^{-\frac{1}{2}}(\operatorname{div}_\Gamma, \Gamma_{k+l-1}) \to \mathbf{H}_\alpha^{-\frac{1}{2}}(\operatorname{div}_\Gamma, \Gamma_k) \\
(M_\alpha^{\text{even}})_{k+2,k+l-1} &: \mathbf{H}_\alpha^{-\frac{1}{2}}(\operatorname{div}_\Gamma, \Gamma_{k+l-2}) \to \mathbf{H}_\alpha^{-\frac{1}{2}}(\operatorname{div}_\Gamma, \Gamma_{k+1})
\end{aligned}
\right\} \qquad \text{for } l \in \{0,\dots,3\}
$$

are bounded linear operators.

For an odd number of interfaces N, we have a system structure of the form

$$
\underbrace{\begin{pmatrix}
* & * & & & & & & & & \\
* & * & * & * & & & & & & \\
* & * & * & * & & & & & & \\
& & * & * & * & * & & & & \\
& & * & * & * & * & & & & \\
& & & & & \ddots & & & & \\
& & & & & & * & * & * & * \\
& & & & & & * & * & * & * \\
& & & & & & & & * & * & * \\
& & & & & & & & * & * & *
\end{pmatrix}}_{=:\, M_\alpha^{\text{odd}}}
\begin{pmatrix} \mathbf{j}_0 \\ \mathbf{j}_1 \\ \mathbf{j}_2 \\ \mathbf{j}_3 \\ \vdots \\ \mathbf{j}_{N-4} \\ \mathbf{j}_{N-3} \\ \mathbf{j}_{N-2} \\ \mathbf{j}_{N-1} \end{pmatrix}
=
\begin{pmatrix} -\frac{2\kappa_0}{\mu_0}\gamma_{\mathrm{D},0}^{-}\mathbf{E}^{\mathrm{i}} \\ 0 \\ 0 \\ 0 \\ \vdots \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \tag{4.14}
$$

where the nonvanishing elements of the matrix M_α^{odd} are given by

$$
\begin{aligned}
\left(M_\alpha^{\text{odd}}\right)_{1,1} &= (M_\alpha^{\text{even}})_{1,1}\,, \\
\left(M_\alpha^{\text{odd}}\right)_{1,2} &= (M_\alpha^{\text{even}})_{1,2}\,, \\
\left(M_\alpha^{\text{odd}}\right)_{N-1,N-2} &= -\frac{\kappa_{N-2}}{\mu_{N-2}} C_{N-2,N-2}^{\alpha,(N-1)} M_{N-2,N-3}^{\alpha,(N-2)} - \frac{\kappa_{N-1}}{\mu_{N-1}} \left(M_{N-2,N-2}^{\alpha,(N-1)} - \mathrm{I} \right) C_{N-2,N-3}^{\alpha,(N-2)}, \\
\left(M_\alpha^{\text{odd}}\right)_{N-1,N-1} &= -\frac{\kappa_{N-2}}{\mu_{N-2}} C_{N-2,N-2}^{\alpha,(N-1)} \left(M_{N-2,N-2}^{\alpha,(N-2)} - \mathrm{I} \right) - \frac{\kappa_{N-1}}{\mu_{N-1}} \left(M_{N-2,N-2}^{\alpha,(N-1)} - \mathrm{I} \right) C_{N-2,N-2}^{\alpha,(N-2)}, \\
\left(M_\alpha^{\text{odd}}\right)_{N-1,N} &= \frac{\kappa_N}{\mu_N} C_{N-2,N-1}^{\alpha,(N-1)} \left(M_{N-1,N-1}^{\alpha,(N-2)} + \mathrm{I} \right) + \frac{\kappa_{N-1}}{\mu_{N-1}} M_{N-2,N-1}^{\alpha,(N-1)} C_{N-1,N-1}^{\alpha,(N)},
\end{aligned}
$$

$$\left(M_\alpha^{\mathrm{odd}}\right)_{N,N-2} = -\frac{\kappa_{N-2}}{\mu_{N-2}} C_{N-1,N-2}^{\alpha,(N-1)} M_{N-2,N-3}^{\alpha,(N-2)} - \frac{\kappa_{N-1}}{\mu_{N-1}} M_{N-1,N-2}^{\alpha,(N-1)} C_{N-2,N-3}^{\alpha,(N-2)},$$
$$\left(M_\alpha^{\mathrm{odd}}\right)_{N,N-1} = -\frac{\kappa_{N-2}}{\mu_{N-2}} C_{N-1,N-2}^{\alpha,(N-1)} \left(M_{N-2,N-2}^{\alpha,(N-2)} - \mathrm{I}\right) - \frac{\kappa_{N-1}}{\mu_{N-1}} M_{N-1,N-2}^{\alpha,(N-1)} C_{N-2,N-2}^{\alpha,(N-2)},$$
$$\left(M_\alpha^{\mathrm{odd}}\right)_{N,N} = \frac{\kappa_N}{\mu_N} C_{N-1,N-1}^{\alpha,(N-1)} \left(M_{N-1,N-1}^{\alpha,(N)} + \mathrm{I}\right) + \frac{\kappa_{N-1}}{\mu_{N-1}} \left(M_{N-1,N-1}^{\alpha,(N-1)} + \mathrm{I}\right) C_{N-1,N-1}^{\alpha,(N)}$$

and

$$\left(M_\alpha^{\mathrm{odd}}\right)_{k+1,k} = \left(M_\alpha^{\mathrm{even}}\right)_{k+1,k}, \qquad \left(M_\alpha^{\mathrm{odd}}\right)_{k+2,k} = \left(M_\alpha^{\mathrm{even}}\right)_{k+2,k},$$
$$\left(M_\alpha^{\mathrm{odd}}\right)_{k+1,k+1} = \left(M_\alpha^{\mathrm{even}}\right)_{k+1,k+1}, \qquad \left(M_\alpha^{\mathrm{odd}}\right)_{k+2,k+1} = \left(M_\alpha^{\mathrm{even}}\right)_{k+2,k+1},$$
$$\left(M_\alpha^{\mathrm{odd}}\right)_{k+1,k+2} = \left(M_\alpha^{\mathrm{even}}\right)_{k+1,k+2}, \qquad \left(M_\alpha^{\mathrm{odd}}\right)_{k+2,k+2} = \left(M_\alpha^{\mathrm{even}}\right)_{k+2,k+2},$$
$$\left(M_\alpha^{\mathrm{odd}}\right)_{k+1,k+3} = \left(M_\alpha^{\mathrm{even}}\right)_{k+1,k+3}, \qquad \left(M_\alpha^{\mathrm{odd}}\right)_{k+2,k+3} = \left(M_\alpha^{\mathrm{even}}\right)_{k+2,k+3}$$

for $k \in K_{\mathrm{odd}} \setminus \{N-2\}$. Again by Lemma 3.7, they give rise to bounded linear operators

$$\left(M_\alpha^{\mathrm{odd}}\right)_{1,l} : \mathbf{H}_\alpha^{-\frac{1}{2}}(\mathrm{div}_\Gamma, \Gamma_{l-1}) \to \mathbf{H}_\alpha^{-\frac{1}{2}}(\mathrm{div}_\Gamma, \Gamma_0) \qquad \text{for } l \in \{1,2\},$$
$$\left.\begin{aligned}\left(M_\alpha^{\mathrm{odd}}\right)_{N-1,N+l-2} &: \mathbf{H}_\alpha^{-\frac{1}{2}}(\mathrm{div}_\Gamma, \Gamma_{N+l-3}) \to \mathbf{H}_\alpha^{-\frac{1}{2}}(\mathrm{div}_\Gamma, \Gamma_{N-2})\\ \left(M_\alpha^{\mathrm{odd}}\right)_{N,N+l-2} &: \mathbf{H}_\alpha^{-\frac{1}{2}}(\mathrm{div}_\Gamma, \Gamma_{N+l-3}) \to \mathbf{H}_\alpha^{-\frac{1}{2}}(\mathrm{div}_\Gamma, \Gamma_{N-1})\end{aligned}\right\} \text{ for } l \in \{0,1,2\}$$

and, for $k \in K_{\mathrm{odd}} \setminus \{N-2\}$, to bounded linear operators

$$\left.\begin{aligned}\left(M_\alpha^{\mathrm{odd}}\right)_{k+1,k+l} &: \mathbf{H}_\alpha^{-\frac{1}{2}}(\mathrm{div}_\Gamma, \Gamma_{k+l-1}) \to \mathbf{H}_\alpha^{-\frac{1}{2}}(\mathrm{div}_\Gamma, \Gamma_k)\\ \left(M_\alpha^{\mathrm{odd}}\right)_{k+2,k+l-1} &: \mathbf{H}_\alpha^{-\frac{1}{2}}(\mathrm{div}_\Gamma, \Gamma_{k+l-1}) \to \mathbf{H}_\alpha^{-\frac{1}{2}}(\mathrm{div}_\Gamma, \Gamma_{k+1})\end{aligned}\right\} \qquad \text{for } l \in \{0,1,2,3\}.$$

Remark 4.2. *In the special case that the multilayered scatterer just consists of one interface, i.e, $N = 1$, we observe that system (4.14) reduces to the singular integral equation*

$$\rho_1 C_{00}^{\alpha,(0)} \left(M_{00}^{\alpha,(1)} + \mathrm{I}\right) + \left(M_{00}^{\alpha,(0)} + \mathrm{I}\right) C_{00}^{\alpha,(1)} = -2\gamma_{\mathrm{D},0}^{-} \mathbf{E}^{\mathrm{i}} \quad \text{with } \rho_1 = \frac{\mu_0 \kappa_1}{\mu_1 \kappa_0}.$$

This corresponds to the main boundary integral equation in [6], which has already been studied extensively therein. In this paper, we therefore only consider "real" 2π-biperiodic multilayered structures consisting of $N \geq 2$ scattering profiles.

4.3 Structure of the adjoint system of linear integral equations

Next, we reverse our previously considered potential ansatz (4.1)-(4.3): $\mathbf{E}_0$ is now assumed to be an α-quasiperiodic electric potential applied to an unknown density $\mathbf{j}_0 \in \mathbf{H}_\alpha^{-1/2}(\mathrm{div}_\Gamma, \Gamma_0)$. In the layers G_k, $k \in K$, we alternate an α-quasiperiodic Stratton-Chu type integral representation in the sense of Lemma 3.13 and a two-term α-quasiperiodic electric potential ansatz with the unknown densities $\mathbf{j}_k \in \mathbf{H}_\alpha^{-1/2}(\mathrm{div}_\Gamma, \Gamma_k)$ ($k \in K$ if N is even or $k \in K \setminus \{N-1\}$ if N is odd). The field $\mathbf{E}_N$ traveling in the bottom layer G_N is either represented as a simple α-quasiperiodic electric potential applied to the unknown density $\mathbf{j}_{N-1} \in \mathbf{H}_\alpha^{-1/2}(\mathrm{div}_\Gamma, \Gamma_{N-1})$ if N is even, or as a Stratton-Chu integral given by Lemma 3.12 if N is odd. In summary, we assume that

$$\mathbf{E}_0 = \Psi_{\mathrm{E}_{\kappa_0},0}^{\alpha} \mathbf{j}_0 \tag{4.15}$$

in G_0,

$$
\mathbf{E}_k = \begin{cases} \Psi^{\alpha}_{\mathrm{E}_{\kappa_k},k-1}\mathbf{j}_{k-1} + \Psi^{\alpha}_{\mathrm{E}_{\kappa_k},k}\mathbf{j}_k & \text{for even } k, \\[2ex] \dfrac{1}{2}\Big(\Psi^{\alpha}_{\mathrm{E}_{\kappa_k},k}\gamma^{+}_{\mathrm{N}_{\kappa_k},k}\mathbf{E}_k + \Psi^{\alpha}_{\mathrm{M}_{\kappa_k},k}\gamma^{+}_{\mathrm{D},k}\mathbf{E}_k & \\ \qquad - \Psi^{\alpha}_{\mathrm{E}_{\kappa_k},k-1}\gamma^{-}_{\mathrm{N}_{\kappa_k},k-1}\mathbf{E}_k - \Psi^{\alpha}_{\mathrm{M}_{\kappa_k},k-1}\gamma^{-}_{\mathrm{D},k-1}\mathbf{E}_k\Big) & \text{for odd } k \end{cases} \tag{4.16}
$$

in G_k for $k \in K$,

$$
\mathbf{E}_N = \begin{cases} -\dfrac{1}{2}\Big(\Psi^{\alpha}_{\mathrm{E}_{\kappa_N},N-1}\gamma^{-}_{\mathrm{N}_{\kappa_N},N-1}\mathbf{E}_N + \Psi^{\alpha}_{\mathrm{M}_{\kappa_N},N-1}\gamma^{-}_{\mathrm{D},N-1}\mathbf{E}_N\Big) & \text{for even } N, \\[2ex] \Psi^{\alpha}_{\mathrm{E}_{\kappa_N},N-1}\mathbf{j}_{N-1} & \text{for odd } N \end{cases} \tag{4.17}
$$

in G_N. Based on this potential ansatz, we again obtain two systems of integral equations depending on the parity of the number of grating interfaces N. They turn out to be a useful tool in the analysis of the integral equation systems (4.13) and (4.14). The detailed derivation is not carried out here since it is very similar to the one leading to (4.13) and (4.14).

If N is even, the system has the following structure:

$$
\underbrace{\begin{pmatrix} * & * & * & & & & & & & & \\ * & * & * & & & & & & & & \\ & * & * & * & * & & & & & & \\ & * & * & * & * & & & & & & \\ & & & * & * & * & * & & & & \\ & & & & & & \ddots & & & & \\ & & & & & & * & * & * & * & \\ & & & & & & & & * & * & * \\ & & & & & & & & * & * & * \end{pmatrix}}_{=:\,W^{\mathrm{even}}_{\alpha}} \begin{pmatrix} \mathbf{j}_0 \\ \mathbf{j}_1 \\ \mathbf{j}_2 \\ \mathbf{j}_3 \\ \vdots \\ \mathbf{j}_{N-4} \\ \mathbf{j}_{N-3} \\ \mathbf{j}_{N-2} \\ \mathbf{j}_{N-1} \end{pmatrix} = \mathbf{f}_{\alpha} \tag{4.18}
$$

with the nonvanishing elements

$$
\begin{aligned}
(W^{\mathrm{even}}_{\alpha})_{1,1} &= \frac{\kappa_0}{\mu_0}C^{\alpha,(1)}_{00}\left(M^{\alpha,(0)}_{00} - \mathrm{I}\right) + \frac{\kappa_1}{\mu_1}\left(M^{\alpha,(1)}_{00} - \mathrm{I}\right)C^{\alpha,(0)}_{00}, \\
(W^{\mathrm{even}}_{\alpha})_{2,1} &= \frac{\kappa_0}{\mu_0}C^{\alpha,(1)}_{10}\left(M^{\alpha,(0)}_{00} - \mathrm{I}\right) + \frac{\kappa_1}{\mu_1}M^{\alpha,(1)}_{10}C^{\alpha,(0)}_{00}, \\
(W^{\mathrm{even}}_{\alpha})_{N-1,N} &= -\frac{\kappa_N}{\mu_N}C^{\alpha,(N-1)}_{N-2,N-1}\left(M^{\alpha,(N)}_{N-1,N-1} + \mathrm{I}\right) - \frac{\kappa_{N-1}}{\mu_{N-1}}M^{\alpha,(N-1)}_{N-2,N-1}C^{\alpha,(N)}_{N-1,N-1}, \\
(W^{\mathrm{even}}_{\alpha})_{N,N} &= -\frac{\kappa_N}{\mu_N}C^{\alpha,(N-1)}_{N-1,N-1}\left(M^{\alpha,(N)}_{N-1,N-1} + \mathrm{I}\right) - \frac{\kappa_{N-1}}{\mu_{N-1}}\left(M^{\alpha,(N-1)}_{N-1,N-1} + \mathrm{I}\right)C^{\alpha,(N)}_{N-1,N-1}
\end{aligned}
$$

and

$$
\begin{aligned}
(W^{\mathrm{even}}_{\alpha})_{k,k+1} &= -\frac{\kappa_{k+1}}{\mu_{k+1}}C^{\alpha,(k)}_{k-1,k}\left(M^{\alpha,(k+1)}_{kk} + \mathrm{I}\right) - \frac{\kappa_k}{\mu_k}M^{\alpha,(k)}_{k-1,k}C^{\alpha,(k+1)}_{kk}, \\
(W^{\mathrm{even}}_{\alpha})_{k+1,k+1} &= -\frac{\kappa_{k+1}}{\mu_{k+1}}C^{\alpha,(k)}_{kk}\left(M^{\alpha,(k+1)}_{kk} + \mathrm{I}\right) - \frac{\kappa_k}{\mu_k}\left(M^{\alpha,(k)}_{kk} + \mathrm{I}\right)C^{\alpha,(k+1)}_{kk}, \\
(W^{\mathrm{even}}_{\alpha})_{k+2,k+1} &= \frac{\kappa_{k+1}}{\mu_{k+1}}C^{\alpha,(k+2)}_{k+1,k+1}M^{\alpha,(k+1)}_{k+1,k} + \frac{\kappa_{k+2}}{\mu_{k+2}}\left(M^{\alpha,(k+2)}_{k+1,k+1} - \mathrm{I}\right)C^{\alpha,(k+1)}_{k+1,k}, \\
(W^{\mathrm{even}}_{\alpha})_{k+3,k+1} &= \frac{\kappa_{k+1}}{\mu_{k+1}}C^{\alpha,(k+2)}_{k+2,k+1}M^{\alpha,(k+1)}_{k+1,k} + \frac{\kappa_{k+2}}{\mu_{k+2}}M^{\alpha,(k+2)}_{k+2,k+1}C^{\alpha,(k+1)}_{k+1,k},
\end{aligned}
$$

$$
\begin{aligned}
(W_\alpha^{\mathrm{even}})_{k,k+2} &= -\frac{\kappa_{k+1}}{\mu_{k+1}} C_{k-1,k}^{\alpha,(k)} M_{k,k+1}^{\alpha,(k+1)} - \frac{\kappa_k}{\mu_k} M_{k-1,k}^{\alpha,(k)} C_{k,k+1}^{\alpha,(k+1)},\\
(W_\alpha^{\mathrm{even}})_{k+1,k+2} &= -\frac{\kappa_{k+1}}{\mu_{k+1}} C_{kk}^{\alpha,(k)} M_{k,k+1}^{\alpha,(k+1)} - \frac{\kappa_k}{\mu_k} \left(M_{kk}^{\alpha,(k)} + \mathrm{I}\right) C_{k,k+1}^{\alpha,(k+1)},\\
(W_\alpha^{\mathrm{even}})_{k+2,k+2} &= \frac{\kappa_{k+1}}{\mu_{k+1}} C_{k+1,k+1}^{\alpha,(k+2)} \left(M_{k+1,k+1}^{\alpha,(k+1)} - \mathrm{I}\right) + \frac{\kappa_{k+2}}{\mu_{k+2}} \left(M_{k+1,k+1}^{\alpha,(k+2)} - \mathrm{I}\right) C_{k+1,k+1}^{\alpha,(k+1)},\\
(W_\alpha^{\mathrm{even}})_{k+3,k+2} &= \frac{\kappa_{k+1}}{\mu_{k+1}} C_{k+2,k+1}^{\alpha,(k+2)} \left(M_{k+1,k+1}^{\alpha,(k+1)} - \mathrm{I}\right) + \frac{\kappa_{k+2}}{\mu_{k+2}} M_{k+2,k+1}^{\alpha,(k+2)} C_{k+1,k+1}^{\alpha,(k+1)}
\end{aligned}
$$

for $k \in K_{\mathrm{odd}} \setminus \{N-1\}$. The right-hand side $\mathbf{f}_\alpha = (\mathbf{f}_{\alpha,1}, \ldots, \mathbf{f}_{\alpha,N})^{\mathrm{T}} \in \prod_{k=0}^{N-1} \mathbf{H}_\alpha^{-1/2}(\mathrm{div}_\Gamma, \Gamma_k)$ is composed of the following two-dimensional vector fields:

$$
\mathbf{f}_{\alpha,1} = \frac{\kappa_0}{\mu_0} C_{00}^{\alpha,(1)} \gamma_{\mathrm{N}_{\kappa_0},0}^{+} \mathbf{E}^{\mathrm{i}} + \frac{\kappa_1}{\mu_1} \left(M_{00}^{\alpha,(1)} - \mathrm{I}\right) \gamma_{\mathrm{D},0}^{+} \mathbf{E}^{\mathrm{i}}, \tag{4.19}
$$

$$
\mathbf{f}_{\alpha,2} = \frac{\kappa_0}{\mu_0} C_{10}^{\alpha,(1)} \gamma_{\mathrm{N}_{\kappa_0},0}^{+} \mathbf{E}^{\mathrm{i}} + \frac{\kappa_1}{\mu_1} M_{10}^{\alpha,(1)} \gamma_{\mathrm{D},0}^{+} \mathbf{E}^{\mathrm{i}}, \tag{4.20}
$$

$$
\mathbf{f}_{\alpha,k} = 0 \qquad \text{for } k = 3, \ldots, N. \tag{4.21}
$$

With Lemma 3.7, we observe that

$$
\begin{aligned}
(W_\alpha^{\mathrm{even}})_{l,1} &: \mathbf{H}_\alpha^{-\frac{1}{2}}(\mathrm{div}_\Gamma, \Gamma_0) \to \mathbf{H}_\alpha^{-\frac{1}{2}}(\mathrm{div}_\Gamma, \Gamma_{l-1}) && \text{for } l \in \{1,2\},\\
(W_\alpha^{\mathrm{even}})_{N-l,N} &: \mathbf{H}_\alpha^{-\frac{1}{2}}(\mathrm{div}_\Gamma, \Gamma_{N-1}) \to \mathbf{H}_\alpha^{-\frac{1}{2}}(\mathrm{div}_\Gamma, \Gamma_{N-l-1}) && \text{for } l \in \{0,1\}
\end{aligned}
$$

and, for $k \in K_{\mathrm{odd}} \setminus \{N-1\}$,

$$
\left.
\begin{aligned}
(W_\alpha^{\mathrm{even}})_{k+l,k+1} &: \mathbf{H}_\alpha^{-\frac{1}{2}}(\mathrm{div}_\Gamma, \Gamma_k) \to \mathbf{H}_\alpha^{-\frac{1}{2}}(\mathrm{div}_\Gamma, \Gamma_{k+l-1})\\
(W_\alpha^{\mathrm{even}})_{k+l-1,k+2} &: \mathbf{H}_\alpha^{-\frac{1}{2}}(\mathrm{div}_\Gamma, \Gamma_{k+1}) \to \mathbf{H}_\alpha^{-\frac{1}{2}}(\mathrm{div}_\Gamma, \Gamma_{k+l-2})
\end{aligned}
\right\} \qquad \text{for } l \in \{0, \ldots, 3\}
$$

are bounded linear operators.

If the number of grating interfaces N is odd, our potential ansatz leads to a system of the form

$$
\underbrace{\begin{pmatrix}
* & * & * & & & & & & & & \\
* & * & * & & & & & & & & \\
 & * & * & * & * & & & & & & \\
 & * & * & * & * & & & & & & \\
 & & & * & * & * & * & & & & \\
 & & & & & & \ddots & & & & \\
 & & & & & & * & * & * & * & \\
 & & & & & & & & * & * & * & * \\
 & & & & & & & & * & * & * & * \\
 & & & & & & & & & & * & *
\end{pmatrix}}_{=:\, W_\alpha^{\mathrm{odd}}}
\begin{pmatrix}
\mathbf{j}_0\\ \mathbf{j}_1\\ \mathbf{j}_2\\ \mathbf{j}_3\\ \vdots\\ \mathbf{j}_{N-4}\\ \mathbf{j}_{N-3}\\ \mathbf{j}_{N-2}\\ \mathbf{j}_{N-1}
\end{pmatrix}
= \mathbf{f}_\alpha, \tag{4.22}
$$

where the nonvanishing elements of the matrix W_α^{odd} are given by

$$
\begin{aligned}
\left(W_\alpha^{\mathrm{odd}}\right)_{1,1} &= (M_\alpha^{\mathrm{even}})_{1,1},\\
\left(W_\alpha^{\mathrm{odd}}\right)_{2,1} &= (M_\alpha^{\mathrm{even}})_{2,1},\\
\left(W_\alpha^{\mathrm{odd}}\right)_{N-2,N-1} &= -\frac{\kappa_{N-1}}{\mu_{N-1}} C_{N-3,N-2}^{\alpha,(N-2)} \left(M_{N-2,N-2}^{\alpha,(N-1)} + \mathrm{I}\right) - \frac{\kappa_{N-2}}{\mu_{N-2}} M_{N-3,N-2}^{\alpha,(N-2)} C_{N-2,N-2}^{\alpha,(N-1)},
\end{aligned}
$$

$$
\begin{aligned}
\left(W_\alpha^{\text{odd}}\right)_{N-1,N-1} &= -\frac{\kappa_{N-1}}{\mu_{N-1}} C^{\alpha,(N-2)}_{N-2,N-2}\left(M^{\alpha,(N-1)}_{N-2,N-2}+\mathrm{I}\right) - \frac{\kappa_{N-2}}{\mu_{N-2}}\left(M^{\alpha,(N-2)}_{N-2,N-2}+\mathrm{I}\right) C^{\alpha,(N-1)}_{N-2,N-2}, \\
\left(W_\alpha^{\text{odd}}\right)_{N,N-1} &= \frac{\kappa_{N-1}}{\mu_{N-1}} C^{\alpha,(N)}_{N-1,N-1} M^{\alpha,(N-1)}_{N-1,N-2} + \frac{\kappa_N}{\mu_N}\left(M^{\alpha,(N-2)}_{N-1,N-1}-\mathrm{I}\right) C^{\alpha,(N-1)}_{N-1,N-2}, \\
\left(W_\alpha^{\text{odd}}\right)_{N-2,N} &= -\frac{\kappa_{N-2}}{\mu_{N-2}} M^{\alpha,(N-2)}_{N-3,N-2} C^{\alpha,(N-1)}_{N-2,N-1} - \frac{\kappa_{N-1}}{\mu_{N-1}} C^{\alpha,(N-2)}_{N-3,N-2} M^{\alpha,(N-1)}_{N-2,N-1}, \\
\left(W_\alpha^{\text{odd}}\right)_{N-1,N} &= -\frac{\kappa_{N-1}}{\mu_{N-1}} C^{\alpha,(N-2)}_{N-2,N-2} M^{\alpha,(N-1)}_{N-2,N-1} - \frac{\kappa_{N-2}}{\mu_{N-2}}\left(M^{\alpha,(N-2)}_{N-2,N-2}+\mathrm{I}\right) C^{\alpha,(N-1)}_{N-2,N-1}, \\
\left(W_\alpha^{\text{odd}}\right)_{N,N} &= \frac{\kappa_{N-1}}{\mu_{N-1}} C^{\alpha,(N)}_{N-1,N-1}\left(M^{\alpha,(N-1)}_{N-1,N-1}-\mathrm{I}\right) + \frac{\kappa_N}{\mu_N}\left(M^{\alpha,(N)}_{N-1,N-1}-\mathrm{I}\right) C^{\alpha,(N-1)}_{N-1,N-1}
\end{aligned}
$$

and

$$
\begin{aligned}
\left(W_\alpha^{\text{odd}}\right)_{k,k+1} &= \left(W_\alpha^{\text{even}}\right)_{k,k+1}, & \left(W_\alpha^{\text{odd}}\right)_{k,k+2} &= \left(W_\alpha^{\text{even}}\right)_{k,k+2}, \\
\left(W_\alpha^{\text{odd}}\right)_{k+1,k+1} &= \left(W_\alpha^{\text{even}}\right)_{k+1,k+1}, & \left(W_\alpha^{\text{odd}}\right)_{k+1,k+2} &= \left(W_\alpha^{\text{even}}\right)_{k+1,k+2}, \\
\left(W_\alpha^{\text{odd}}\right)_{k+2,k+1} &= \left(W_\alpha^{\text{even}}\right)_{k+2,k+1}, & \left(W_\alpha^{\text{odd}}\right)_{k+2,k+2} &= \left(W_\alpha^{\text{even}}\right)_{k+2,k+2}, \\
\left(W_\alpha^{\text{odd}}\right)_{k+3,k+1} &= \left(W_\alpha^{\text{even}}\right)_{k+3,k+1}, & \left(W_\alpha^{\text{odd}}\right)_{k+3,k+2} &= \left(W_\alpha^{\text{even}}\right)_{k+3,k+2}
\end{aligned}
$$

for $k \in K_{\text{odd}} \setminus \{N-2\}$. The components of the right-hand side $\mathbf{f}_\alpha$, which lies in the product space $\prod_{k=0}^{N-1} \mathbf{H}_\alpha^{-1/2}(\operatorname{div}_\Gamma, \Gamma_k)$, are specified by (4.19)-(4.21). By Lemma 3.7, the nonvanishing elements of W_α^{odd} give rise to bounded linear operators

$$
\begin{aligned}
&\left(W_\alpha^{\text{odd}}\right)_{l,1} : \mathbf{H}_\alpha^{-\frac{1}{2}}(\operatorname{div}_\Gamma, \Gamma_0) \to \mathbf{H}_\alpha^{-\frac{1}{2}}(\operatorname{div}_\Gamma, \Gamma_{l-1}) \qquad \text{for } l \in \{1,2\}, \\
&\left.\begin{aligned}
\left(W_\alpha^{\text{odd}}\right)_{N+l-2,N-1} &: \mathbf{H}_\alpha^{-\frac{1}{2}}(\operatorname{div}_\Gamma, \Gamma_{N-2}) \to \mathbf{H}_\alpha^{-\frac{1}{2}}(\operatorname{div}_\Gamma, \Gamma_{N+l-3}) \\
\left(W_\alpha^{\text{odd}}\right)_{N+l-2,N} &: \mathbf{H}_\alpha^{-\frac{1}{2}}(\operatorname{div}_\Gamma, \Gamma_{N-1}) \to \mathbf{H}_\alpha^{-\frac{1}{2}}(\operatorname{div}_\Gamma, \Gamma_{N+l-3})
\end{aligned}\right\} \text{ for } l \in \{0,1,2\}
\end{aligned}
$$

and, for $k \in K_{\text{odd}} \setminus \{N-2\}$, to bounded linear operators

$$
\left.\begin{aligned}
\left(W_\alpha^{\text{odd}}\right)_{k+l,k+1} &: \mathbf{H}_\alpha^{-\frac{1}{2}}(\operatorname{div}_\Gamma, \Gamma_k) \to \mathbf{H}_\alpha^{-\frac{1}{2}}(\operatorname{div}_\Gamma, \Gamma_{k+l-1}) \\
\left(W_\alpha^{\text{odd}}\right)_{k+l-1,k+2} &: \mathbf{H}_\alpha^{-\frac{1}{2}}(\operatorname{div}_\Gamma, \Gamma_{k+1}) \to \mathbf{H}_\alpha^{-\frac{1}{2}}(\operatorname{div}_\Gamma, \Gamma_{k+l-1})
\end{aligned}\right\} \qquad \text{for } l \in \{0,1,2,3\}.
$$

As the title of this section already indicates, the above described $N \times N$ operators W_α^{even} and W_α^{odd} corresponding to the integral equation systems (4.18) and (4.22) somehow correlate in an adjoint sense with the $N \times N$ operators M_α^{even} and M_α^{odd} from (4.13) and (4.14). In what exact sense this should be understood is explained in the following: Consider an incident electric field $\mathbf{E}^{\mathrm{i}}$ with the wave vector $-\alpha = (-\alpha_1, -\alpha_2, \alpha_3)$. Then the potential ansatz in this subsection involving the densities $\mathbf{j}_k \in \mathbf{H}_{-\alpha}^{-1/2}(\operatorname{div}_\Gamma, \Gamma_k)$, $k \in K_0$, yields the two integral equation systems (4.18) and (4.22) in terms of $(-\alpha)$ with the right-hand sides $\mathbf{f}_{-\alpha}$. The components of the $N \times N$ operators defining these systems are all bounded linear integral operators with kernels based on the $(-\alpha)$-quasiperiodic Green function $G^{-\alpha}_{(k)}$. With the help of Lemma 3.8, we now easily observe that

$$
\begin{aligned}
\mathcal{B}_l\left(\left(M_\alpha^{\text{even}}\right)_{l+1,j+1}\mathbf{j}, \mathbf{l}\right) &= \mathcal{B}_j\left(\mathbf{j}, \left(W_{-\alpha}^{\text{even}}\right)_{j+1,l+1}\mathbf{l}\right), \\
\mathcal{B}_l\left(\left(M_\alpha^{\text{odd}}\right)_{l+1,j+1}\mathbf{j}, \mathbf{l}\right) &= \mathcal{B}_j\left(\mathbf{j}, \left(W_{-\alpha}^{\text{odd}}\right)_{j+1,l+1}\mathbf{l}\right)
\end{aligned}
$$

for all $l, j \in K_0$ as well as all $\mathbf{j} \in \mathbf{H}_\alpha^{-1/2}(\mathrm{div}_\Gamma, \Gamma_j)$ and $\mathbf{l} \in \mathbf{H}_{-\alpha}^{-1/2}(\mathrm{div}_\Gamma, \Gamma_l)$. Define the bilinear form $[\cdot,\cdot] : \prod_{k=0}^{N-1} \mathbf{H}_\alpha^{-1/2}(\mathrm{div}_\Gamma, \Gamma_k) \times \prod_{k=0}^{N-1} \mathbf{H}_{-\alpha}^{-1/2}(\mathrm{div}_\Gamma, \Gamma_k) \to \mathbb{C}$ as

$$[\mathbf{J}, \mathbf{L}] := \sum_{k=0}^{N-1} \mathcal{B}_k(\mathbf{j}_k, \mathbf{l}_k), \tag{4.23}$$

where $\mathcal{B}_k$ is the bilinear form defined in (3.5) and the densities $\mathbf{J}, \mathbf{L}$ are specified as

$$\mathbf{J} = (\mathbf{j}_k)_{k\in K_0} \in \prod_{k=0}^{N-1} \mathbf{H}_\alpha^{-\frac{1}{2}}(\mathrm{div}_\Gamma, \Gamma_k), \quad \mathbf{L} = (\mathbf{l}_k)_{k\in K_0} \in \prod_{k=0}^{N-1} \mathbf{H}_{-\alpha}^{-\frac{1}{2}}(\mathrm{div}_\Gamma, \Gamma_k).$$

Then we can even formulate the adjointness of M_α^{even} and $W_{-\alpha}^{\mathrm{even}}$ as well as of M_α^{odd} and $W_{-\alpha}^{\mathrm{odd}}$ with respect to $[\cdot,\cdot]$.

Lemma 4.3. *For any wave vector α, the operators $W_{-\alpha}^{\mathrm{even}}$ and $W_{-\alpha}^{\mathrm{odd}}$ are the adjoint operators of M_α^{even} and M_α^{odd} with respect to the bilinear form $[\cdot,\cdot]$ from (4.23). Thus, we have*

$$[M_\alpha^{\mathrm{even}}\mathbf{J}, \mathbf{L}] = [\mathbf{J}, W_{-\alpha}^{\mathrm{even}}\mathbf{L}] \quad \text{and} \quad \left[M_\alpha^{\mathrm{odd}}\mathbf{J}, \mathbf{L}\right] = \left[\mathbf{J}, W_{-\alpha}^{\mathrm{odd}}\mathbf{L}\right]$$

for all $\mathbf{J} \in \prod_{k=0}^{N-1} \mathbf{H}_\alpha^{-1/2}(\mathrm{div}_\Gamma, \Gamma_k)$ and $\mathbf{L} \in \prod_{k=0}^{N-1} \mathbf{H}_{-\alpha}^{-1/2}(\mathrm{div}_\Gamma, \Gamma_k)$.

4.4 Equivalence

The following lemma ensures the equivalence of the integral systems (4.13) and (4.14) to the electromagnetic scattering problem (2.7)-(2.13). We recall that the constant ρ_{k+1} for $k \in K_0$ is specified by

$$\rho_{k+1} = \frac{\mu_k \kappa_{k+1}}{\mu_{k+1}\kappa_k}.$$

Lemma 4.4 (Equivalence for the systems (4.13) and (4.14))**.** *Let the vector-valued density*

$$\mathbf{j} = (\mathbf{j}_0, \mathbf{j}_1, \ldots, \mathbf{j}_{N-1})^{\mathrm{T}} \in \prod_{k=0}^{N-1} \mathbf{H}_\alpha^{-\frac{1}{2}}(\mathrm{div}_\Gamma, \Gamma_k)$$

be a solution of the linear system (4.13) if N is even, or of (4.14) if N is odd. Moreover, let $N \geq 2$ and assume that

$$\mathcal{N}\left(C_{00}^{\alpha,(0)}\right) = \{0\}, \quad \mathcal{N}\left(C_{kk}^{\alpha,(k+1)}\right) = \mathcal{N}\left(C_{k+1,k+1}^{\alpha,(k+1)}\right) = \{0\} \quad \text{for } k \in K_{\mathrm{odd}} \setminus \{N-1\}$$

and additionally $\mathcal{N}(C_{N-1,N-1}^{\alpha,(N)}) = \{0\}$ if N is even. Then the functions

$$\left.\begin{aligned} \mathbf{E}_0 = &-\frac{1}{2}\left[\rho_1 \Psi_{\mathrm{E}_{\kappa_0},0}^\alpha \left(M_{00}^{\alpha,(1)} + \mathrm{I}\right) + \Psi_{\mathrm{M}_{\kappa_0},0}^\alpha C_{00}^{\alpha,(1)}\right]\mathbf{j}_0 \\ &-\frac{1}{2}\left[\rho_1 \Psi_{\mathrm{E}_{\kappa_0},0}^\alpha M_{01}^{\alpha,(1)} + \Psi_{\mathrm{M}_{\kappa_0},0}^\alpha C_{01}^{\alpha,(1)}\right]\mathbf{j}_1 \end{aligned}\right\} \text{ in } G_0,$$

$$\mathbf{E}_k = \Psi_{\mathrm{E}_{\kappa_k},k-1}^\alpha \mathbf{j}_{k-1} + \Psi_{\mathrm{E}_{\kappa_k},k}^\alpha \mathbf{j}_k \qquad \text{in } G_k$$

for $k \in K_{\mathrm{odd}}$,

$$\left.\begin{aligned} \mathbf{E}_k = &\frac{1}{2}\left[\rho_k^{-1} \Psi_{\mathrm{E}_{\kappa_k},k-1}^\alpha M_{k-1,k-2}^{\alpha,(k-1)} + \Psi_{\mathrm{M}_{\kappa_k},k-1}^\alpha C_{k-1,k-2}^{\alpha,(k-1)}\right]\mathbf{j}_{k-2} \\ &+\frac{1}{2}\left[\rho_k^{-1} \Psi_{\mathrm{E}_{\kappa_k},k-1}^\alpha \left(M_{k-1,k-1}^{\alpha,(k-1)} - \mathrm{I}\right) + \Psi_{\mathrm{M}_{\kappa_k},k-1}^\alpha C_{k-1,k-1}^{\alpha,(k-1)}\right]\mathbf{j}_{k-1} \\ &-\frac{1}{2}\left[\rho_{k+1} \Psi_{\mathrm{E}_{\kappa_k},k}^\alpha \left(M_{kk}^{\alpha,(k+1)} + \mathrm{I}\right) + \Psi_{\mathrm{M}_{\kappa_k},k}^\alpha C_{kk}^{\alpha,(k+1)}\right]\mathbf{j}_k \\ &-\frac{1}{2}\left[\rho_{k+1} \Psi_{\mathrm{E}_{\kappa_k},k}^\alpha M_{k,k+1}^{\alpha,(k+1)} + \Psi_{\mathrm{M}_{\kappa_k},k}^\alpha C_{k,k+1}^{\alpha,(k+1)}\right]\mathbf{j}_{k+1} \end{aligned}\right\} \text{ in } G_k$$

for $k \in K_{\mathrm{even}} \setminus \{N-1\}$,

$$\left.\begin{aligned}\mathbf{E}_{N-1} = {} & \frac{1}{2}\left[\rho_{N-1}^{-1}\Psi^{\alpha}_{\mathrm{E}_{\kappa_{N-1}},N-2}M^{\alpha,(N-2)}_{N-2,N-3} + \Psi^{\alpha}_{\mathrm{M}_{\kappa_{N-1}},N-2}C^{\alpha,(N-2)}_{N-2,N-3}\right]\mathbf{j}_{N-3}\\ & + \frac{1}{2}\left[\rho_{N-1}^{-1}\Psi^{\alpha}_{\mathrm{E}_{\kappa_{N-1}},N-2}\left(M^{\alpha,(N-2)}_{N-2,N-2} - \mathrm{I}\right) + \Psi^{\alpha}_{\mathrm{M}_{\kappa_{N-1}},N-2}C^{\alpha,(N-2)}_{N-2,N-2}\right]\mathbf{j}_{N-2}\\ & - \frac{1}{2}\left[\rho_{N}\Psi^{\alpha}_{\mathrm{E}_{\kappa_{N-1}},N-1}\left(M^{\alpha,(N-2)}_{N-1,N-1} + \mathrm{I}\right) + \Psi^{\alpha}_{\mathrm{M}_{\kappa_{N-1}},N-1}C^{\alpha,(N)}_{N-1,N-1}\right]\mathbf{j}_{N-1}\end{aligned}\right\} \text{ in } G_{N-1},$$

$$\mathbf{E}_N = \Psi^{\alpha}_{\mathrm{E}_{\kappa_N},N-1}\mathbf{j}_{N-1} \qquad \text{in } G_N$$

for odd N *and*

$$\left.\begin{aligned}\mathbf{E}_{N} = {} & -\frac{1}{2}\left[\rho_{N}^{-1}\Psi^{\alpha}_{\mathrm{E}_{\kappa_{N}},N-1}M^{\alpha,(N-1)}_{N-1,N-2} + \Psi^{\alpha}_{\mathrm{M}_{\kappa_{N}},N-1}C^{\alpha,(N-1)}_{N-1,N-2}\right]\mathbf{j}_{N-2}\\ & - \frac{1}{2}\left[\rho_{N}^{-1}\Psi^{\alpha}_{\mathrm{E}_{\kappa_{N}},N-1}\left(M^{\alpha,(N-1)}_{N-1,N-1} - \mathrm{I}\right) + \Psi^{\alpha}_{\mathrm{M}_{\kappa_{N}},N-1}C^{\alpha,(N-1)}_{N-1,N-1}\right]\mathbf{j}_{N-1}\end{aligned}\right\} \text{ in } G_{N}$$

for even N *solve the electromagnetic scattering problem (2.7)-(2.13).*

On the other hand, if

$$\mathcal{N}\left(C^{\alpha,(k)}_{k-1,k-1}\right) = \mathcal{N}\left(C^{\alpha,(k)}_{kk}\right) = \{0\} \quad \text{for } k \in K_{\mathrm{odd}}$$

and additionally $\mathcal{N}(C^{\alpha,(N)}_{N-1,N-1}) = \{0\}$ *if* N *is odd, then any solution* $\mathbf{E}$ *of the electromagnetic scattering problem (2.7)-(2.13) provides a solution of the integral equation system (4.13) in case of an even number of interfaces* N *and of the integral equation system (4.14) in case of an odd number of interfaces* N*, respectively.*

Proof. We first consider the situation that a density $\mathbf{j} = (\mathbf{j}_0, \mathbf{j}_1, \ldots, \mathbf{j}_{N-1})^{\mathrm{T}} \in \prod_{k=0}^{N-1}\mathbf{H}_{\alpha}^{-1/2}(\mathrm{div}_{\Gamma}, \Gamma_k)$ solves the integral equation systems (4.13) if N is even, or (4.14) if N is odd. Then the functions

$$\mathbf{E}_k = \Psi^{\alpha}_{\mathrm{E}_{\kappa_k},k-1}\mathbf{j}_{k-1} + \Psi^{\alpha}_{\mathrm{E}_{\kappa_k},k}\mathbf{j}_k \qquad \text{in } G_k, k \in K_{\mathrm{odd}}, \tag{4.24}$$

and additionally

$$\mathbf{E}_N = \Psi^{\alpha}_{\mathrm{E}_{\kappa_N},N-1}\mathbf{j}_{N-1} \qquad \text{in } G_N \tag{4.25}$$

if N is odd are solutions of the time-harmonic Maxwell equations $\mathbf{curl}\,\mathbf{curl}\,\mathbf{E} - \kappa_k^2\mathbf{E} = 0$ in G_k and of $\mathbf{curl}\,\mathbf{curl}\,\mathbf{E} - \kappa_N^2\mathbf{E} = 0$ in G_N, respectively. This is easily justified by Lemma 3.6. We recall that these representations are unique according to Lemma 3.14 and the assumptions of this lemma. The mapping properties of the Dirichlet trace and the Neumann trace by (3.4) imply that

$$\gamma^{-}_{\mathrm{D},k-1}\mathbf{E}_k, \gamma^{-}_{\mathrm{N}_{\kappa_k},k-1}\mathbf{E}_k \in \mathbf{H}_{\alpha}^{-\frac{1}{2}}(\mathrm{div}_{\Gamma}, \Gamma_{k-1}) \quad \text{and} \quad \gamma^{+}_{\mathrm{D},k}\mathbf{E}_k, \gamma^{+}_{\mathrm{N}_{\kappa_k},k}\mathbf{E}_k \in \mathbf{H}_{\alpha}^{-\frac{1}{2}}(\mathrm{div}_{\Gamma}, \Gamma_k)$$

for $k \in K_{\mathrm{odd}}$. Moreover, we observe by (3.4) that $\gamma^{-}_{\mathrm{D},N-1}\mathbf{E}_N, \gamma^{-}_{\mathrm{N}_{\kappa_N},N-1}\mathbf{E}_N \in \mathbf{H}_{\alpha}^{-1/2}(\mathrm{div}_{\Gamma}, \Gamma_{N-1})$ if N is odd. Hence, the functions

$$\mathbf{E}_0 = \frac{1}{2}\left(\rho_1\Psi^{\alpha}_{\mathrm{E}_{\kappa_0},0}\gamma^{-}_{\mathrm{N}_{\kappa_1},0}\mathbf{E}_1 + \Psi^{\alpha}_{\mathrm{M}_{\kappa_0},0}\gamma^{-}_{\mathrm{D},0}\mathbf{E}_1\right) \qquad \text{in } G_0, \tag{4.26}$$

$$\begin{aligned}\mathbf{E}_k = \frac{1}{2}\Big(&\rho_{k+1}\Psi^{\alpha}_{\mathrm{E}_{\kappa_k},k}\gamma^{+}_{\mathrm{N}_{\kappa_{k+1}},k}\mathbf{E}_{k+1} + \Psi^{\alpha}_{\mathrm{M}_{\kappa_k},k}\gamma^{+}_{\mathrm{D},k}\mathbf{E}_{k+1}\\ &-\rho_k^{-1}\Psi^{\alpha}_{\mathrm{E}_{\kappa_k},k-1}\gamma^{-}_{\mathrm{N}_{\kappa_{k-1}},k-1}\mathbf{E}_{k-1} - \Psi^{\alpha}_{\mathrm{M}_{\kappa_k},k-1}\gamma^{-}_{\mathrm{D},k-1}\mathbf{E}_{k-1}\Big)\end{aligned} \qquad \text{in } G_{k\in K_{\mathrm{even}}}, \tag{4.27}$$

and, if N is even, also

$$\mathbf{E}_N = \frac{1}{2}\left(\rho_N^{-1}\Psi^{\alpha}_{\mathrm{E}_{\kappa_N},N-1}\gamma^{-}_{\mathrm{N}_{\kappa_{N-1}},N-1}\mathbf{E}_{N-1} + \Psi^{\alpha}_{\mathrm{M}_{\kappa_{N-1}},N-1}\gamma^{-}_{\mathrm{D},N-1}\mathbf{E}_{N-1}\right) \qquad \text{in } G_N \tag{4.28}$$

solve the time-harmonic Maxwell equations $\mathbf{curl}\,\mathbf{curl}\,\mathbf{E} - \kappa_0^2\mathbf{E} = 0$ in G_0, $\mathbf{curl}\,\mathbf{curl}\,\mathbf{E} - \kappa_k^2\mathbf{E} = 0$ in G_k, $k \in K_{\text{even}}$, and moreover $\mathbf{curl}\,\mathbf{curl}\,\mathbf{E} - \kappa_N^2\mathbf{E} = 0$ in G_N in case of an even N, respectively. This goes back to Lemma 3.6. Furthermore, the latter lemma yields that the fields $\mathbf{E}_0$ and $\mathbf{E}_N$ for even N fulfill the outgoing wave condition (2.12)-(2.13). Therefore, it remains to prove the validity of the transmission conditions (2.8)-(2.11).

We first address the verification of the transmission conditions (2.8)-(2.9) across the grating interface Γ. For this, we apply the Dirichlet trace $\gamma_{\mathrm{D},0}^{+}$ to the electric field $\mathbf{E}_0$ represented as in (4.26):

$$\gamma_{\mathrm{D},0}^{+}\mathbf{E}_0 \overset{(3.11),(3.12)}{=} -\frac{1}{2}\left(\rho_1 C_{00}^{\alpha,(0)}\gamma_{\mathrm{N}_{\kappa_1},0}^{-}\mathbf{E}_1 + \left(M_{00}^{\alpha,(0)} - \mathrm{I}\right)\gamma_{\mathrm{D},0}^{-}\mathbf{E}_1\right). \tag{4.29}$$

The trace expressions in (4.29) can be reformulated with the help of the potential ansatz (4.24) for $k = 1$ as

$$\gamma_{\mathrm{D},0}^{-}\mathbf{E}_1 \overset{(3.11),(3.12)}{=} -C_{00}^{\alpha,(1)}\mathbf{j}_0 - C_{01}^{\alpha,(1)}\mathbf{j}_1, \tag{4.30}$$

$$\gamma_{\mathrm{N}_{\kappa_0},0}^{-}\mathbf{E}_1 \overset{(3.11),(3.12)}{=} -\left(M_{00}^{\alpha,(1)} + \mathrm{I}\right)\mathbf{j}_0 - M_{01}^{\alpha,(1)}\mathbf{j}_1. \tag{4.31}$$

Inserting these into (4.29) yields

$$\begin{aligned}\gamma_{\mathrm{D},0}^{+}\mathbf{E}_0 = &\frac{1}{2}\left[\rho_1 C_{00}^{\alpha,(0)}\left(M_{00}^{\alpha,(1)} + \mathrm{I}\right) + \left(M_{00}^{\alpha,(0)} - \mathrm{I}\right)C_{00}^{\alpha,(1)}\right]\mathbf{j}_0 \\ &+ \frac{1}{2}\left[\rho_1 C_{00}^{\alpha,(0)}M_{01}^{\alpha,(1)} + \left(M_{00}^{\alpha,(0)} - \mathrm{I}\right)C_{01}^{\alpha,(1)}\right]\mathbf{j}_1.\end{aligned}$$

Since the densities $\mathbf{j}_0$ and $\mathbf{j}_1$ satisfy the first equation of both the linear systems (4.13) and (4.14), i.e.,

$$\begin{aligned}&\left[\frac{\kappa_1}{\mu_1}C_{00}^{\alpha,(0)}\left(M_{00}^{\alpha,(1)} + \mathrm{I}\right) + \frac{\kappa_0}{\mu_0}\left(M_{00}^{\alpha,(0)} + \mathrm{I}\right)C_{00}^{\alpha,(1)}\right]\mathbf{j}_0 \\ &\quad + \left[\frac{\kappa_1}{\mu_1}C_{00}^{\alpha,(0)}M_{01}^{\alpha,(1)} + \frac{\kappa_0}{\mu_0}\left(M_{00}^{\alpha,(0)} + \mathrm{I}\right)C_{01}^{\alpha,(1)}\right]\mathbf{j}_1 = -\frac{2\kappa_0}{\mu_0}\gamma_{\mathrm{D},0}^{-}\mathbf{E}^{\mathrm{i}},\end{aligned}$$

we conclude that

$$\gamma_{\mathrm{D},0}^{+}\mathbf{E}_0 = -\gamma_{\mathrm{D},0}^{-}\mathbf{E}^{\mathrm{i}} - \left(C_{00}^{\alpha,(1)}\mathbf{j}_0 + C_{01}^{\alpha,(1)}\mathbf{j}_1\right) \overset{(4.30)}{=} -\gamma_{\mathrm{D},0}^{-}\mathbf{E}^{\mathrm{i}} + \gamma_{\mathrm{D},0}^{-}\mathbf{E}_1.$$

This corresponds to the first transmission condition (2.8) in the electromagnetic scattering problem (2.7)-(2.13) rewritten in the form $\gamma_{\mathrm{D},0}^{-}\mathbf{E}_1 = \gamma_{\mathrm{D},0}^{+}\mathbf{E}_0 + \gamma_{\mathrm{D},0}^{-}\mathbf{E}^{\mathrm{i}}$.

For the proof of the second transmission condition (2.9), we recall the representation (4.26) of the electric field $\mathbf{E}_1$. Exploiting the previously verified first transmission condition (2.8), we arrive at

$$\mathbf{E}_0 = \frac{1}{2}\left[\rho_1\Psi_{\mathrm{E}_{\kappa_0},0}^{\alpha}\gamma_{\mathrm{N}_{\kappa_1},0}^{-}\mathbf{E}_1 + \Psi_{\mathrm{M}_{\kappa_0},0}^{\alpha}\left(\gamma_{\mathrm{D},0}^{+}\mathbf{E}_0 + \gamma_{\mathrm{D},0}^{-}\mathbf{E}^{\mathrm{i}}\right)\right]. \tag{4.32}$$

With the identities

$$\mathbf{E}_0 = \frac{1}{2}\left(\Psi_{\mathrm{E}_{\kappa_0},0}^{\alpha}\gamma_{\mathrm{D},0}^{+}\mathbf{E}_0 + \Psi_{\mathrm{M}_{\kappa_0},0}^{\alpha}\gamma_{\mathrm{N}_{\kappa_0},0}^{+}\mathbf{E}_0\right) \quad \text{and} \quad \Psi_{\mathrm{M}_{\kappa_0},0}^{\alpha}\gamma_{\mathrm{D},0}^{-}\mathbf{E}^{\mathrm{i}} = -\Psi_{\mathrm{E}_{\kappa_0},0}^{\alpha}\gamma_{\mathrm{N}_{\kappa_0},0}^{-}\mathbf{E}^{\mathrm{i}},$$

which arise as special cases of the α-quasiperiodic Stratton-Chu integral representation from Lemma 3.12, equation (4.32) can be reformulated to

$$\begin{aligned}&\frac{1}{2}\Psi_{\mathrm{E}_{\kappa_0},0}^{\alpha}\left(\gamma_{\mathrm{N}_{\kappa_0},0}^{+}\mathbf{E}_0 + \gamma_{\mathrm{N}_{\kappa_0},0}^{-}\mathbf{E}^{\mathrm{i}}\right) = \frac{1}{2}\Psi_{\mathrm{E}_{\kappa_0},0}^{\alpha}\rho_1\gamma_{\mathrm{N}_{\kappa_1},0}^{-}\mathbf{E}_1 \\ \xrightarrow{\gamma_{\mathrm{D},0}^{+},(3.11)}\quad &C_{00}^{\alpha,(0)}\left(\gamma_{\mathrm{N}_{\kappa_0},0}^{+}\mathbf{E}_0 + \gamma_{\mathrm{N}_{\kappa_0},0}^{-}\mathbf{E}^{\mathrm{i}}\right) = C_{00}^{\alpha,(0)}\rho_1\gamma_{\mathrm{N}_{\kappa_1},0}^{-}\mathbf{E}_1.\end{aligned}$$

The assumption that $\mathcal{N}(C_{00}^{\alpha,(0)}) = \{0\}$ already ensures that $C_{00}^{\alpha,(0)}$ is invertible in $\mathbf{H}_\alpha^{-1/2}(\mathrm{div}_\Gamma, \Gamma_0)$ due to its Fredholm properties given by Lemma 3.9. Therefore, we deduce that

$$\rho_1 \gamma^-_{\mathrm{N}_{\kappa_1},0}\mathbf{E}_1 = \gamma^+_{\mathrm{N}_{\kappa_0},0}\mathbf{E}_0 + \gamma^-_{\mathrm{N}_{\kappa_0},0}\mathbf{E}^{\mathrm{i}},$$

i.e., the transmission condition (2.9) holds.

Next, we simultaneously derive the transmission conditions (2.10) and (2.11) across the surfaces Γ and Γ_k for $k \in K_{\mathrm{even}}$. The argumentation resembles the one seen above. We apply the Dirichlet traces $\gamma^-_{\mathrm{D},k-1}$ and $\gamma^+_{\mathrm{D},k}$ to the vector field $\mathbf{E}_k$ represented as in (4.27), which gives rise to

$$\begin{aligned}\gamma^-_{\mathrm{D},k-1}\mathbf{E}_k = \frac{1}{2}\Big(&\rho_k^{-1} C^{\alpha,(k)}_{k-1,k-1}\gamma^-_{\mathrm{N}_{\kappa_{k-1}},k-1}\mathbf{E}_{k-1} + \left(M^{\alpha,(k)}_{k-1,k-1} + \mathrm{I}\right)\gamma^-_{\mathrm{D},k-1}\mathbf{E}_{k-1} \\ &-\rho_{k+1} C^{\alpha,(k)}_{k-1,k}\gamma^+_{\mathrm{N}_{\kappa_{k+1}},k}\mathbf{E}_{k+1} - M^{\alpha,(k)}_{k-1,k}\gamma^+_{\mathrm{D},k}\mathbf{E}_{k+1}\Big)\end{aligned} \tag{4.33}$$

and

$$\begin{aligned}\gamma^+_{\mathrm{D},k}\mathbf{E}_k = \frac{1}{2}\Big(&\rho_k^{-1} C^{\alpha,(k)}_{k,k-1}\gamma^-_{\mathrm{N}_{\kappa_{k-1}},k-1}\mathbf{E}_{k-1} + M^{\alpha,(k)}_{k,k-1}\gamma^-_{\mathrm{D},k-1}\mathbf{E}_{k-1} \\ &-\rho_{k+1} C^{\alpha,(k)}_{kk}\gamma^+_{\mathrm{N}_{\kappa_{k+1}},k}\mathbf{E}_{k+1} - \left(M^{\alpha,(k)}_{kk} - \mathrm{I}\right)\gamma^+_{\mathrm{D},k}\mathbf{E}_{k+1}\Big).\end{aligned} \tag{4.34}$$

For this, we in particular used the identities (3.11)-(3.12) and (3.13). With the help of the representation (4.24) in terms of the indices $k-1$ and $k+1$, it is now possible to rewrite the Dirichlet and Neumann traces of $\mathbf{E}_{k-1}$ and $\mathbf{E}_{k+1}$ occurring in the expressions (4.33) and (4.34) as follows:

$$\begin{aligned}\gamma^+_{\mathrm{D},k-1}\mathbf{E}_{k-1} &\overset{(3.11),(3.13)_1}{=} -C^{\alpha,(k-1)}_{k-1,k-2}\mathbf{j}_{k-2} - C^{\alpha,(k-1)}_{k-1,k-1}\mathbf{j}_{k-1}, \\ \gamma^+_{\mathrm{N}_{\kappa_{k-1}},k-1}\mathbf{E}_{k-1} &\overset{(3.12),(3.13)_1}{=} -M^{\alpha,(k-1)}_{k-1,k-2}\mathbf{j}_{k-2} - \left(M^{\alpha,(k-1)}_{k-1,k-1} - \mathrm{I}\right)\mathbf{j}_{k-1}, \\ \gamma^-_{\mathrm{D},k}\mathbf{E}_{k+1} &= \begin{cases} -C^{\alpha,(k+1)}_{kk}\mathbf{j}_k - C^{\alpha,(k+1)}_{k,k+1}\mathbf{j}_{k+1} & \text{if } k \neq N-1, \\ -C^{\alpha,(N)}_{N-1,N-1}\mathbf{j}_{N-1} & \text{if } k = N-1, \end{cases} \\ \gamma^-_{\mathrm{N}_{\kappa_{k+1}},k}\mathbf{E}_{k+1} &= \begin{cases} -\left(M^{\alpha,(k+1)}_{kk} + \mathrm{I}\right)\mathbf{j}_k - M^{\alpha,(k+1)}_{k,k+1}\mathbf{j}_{k+1} & \text{if } k \neq N-1, \\ -\left(M^{\alpha,(N)}_{N-1,N-1} + \mathrm{I}\right)\mathbf{j}_{N-1} & \text{if } k = N-1. \end{cases}\end{aligned} \tag{4.35}$$

We recall that the densities $\mathbf{j}_l$, $l \in \{k-2, k-1, k, k+1\}$, solve the kth and the $(k+1)$st integral equation in the systems (4.13) and (4.14). Exploiting this after inserting the expressions (4.35) into (4.33)-(4.34), we can conclude that

$$\begin{aligned}\gamma^-_{\mathrm{D},k-1}\mathbf{E}_k &= -C^{\alpha,(k-1)}_{k-1,k-2}\mathbf{j}_{k-2} - C^{\alpha,(k-1)}_{k-1,k-1}\mathbf{j}_{k-1} \overset{(4.35)}{=} \gamma^+_{\mathrm{D},k-1}\mathbf{E}_{k-1} \qquad \text{for } k \neq N-1, \\ \gamma^-_{\mathrm{D},N-2}\mathbf{E}_{N-1} &= -C^{\alpha,(N-2)}_{N-2,N-3}\mathbf{j}_{N-3} - C^{\alpha,(N-2)}_{N-2,N-2}\mathbf{j}_{N-2} \overset{(4.35)}{=} \gamma^+_{\mathrm{D},N-2}\mathbf{E}_{N-2}\end{aligned}$$

and that

$$\begin{aligned}\gamma^+_{\mathrm{D},k}\mathbf{E}_k &= -C^{\alpha,(k+1)}_{kk}\mathbf{j}_k - C^{\alpha,(k+1)}_{k,k+1}\mathbf{j}_{k+1} \overset{(4.35)}{=} \gamma^-_{\mathrm{D},k}\mathbf{E}_{k+1}, \qquad \text{for } k \neq N-1, \\ \gamma^+_{\mathrm{D},N-1}\mathbf{E}_{N-1} &= -C^{\alpha,(N)}_{N-1,N-1}\mathbf{j}_{N-1} \overset{(4.35)}{=} \gamma^-_{\mathrm{D},N-1}\mathbf{E}_N.\end{aligned}$$

This proves the transmission condition (2.10) for $k \in K$ if N is odd and for $k \in K \setminus \{N-1\}$ if N is even. Thus, we are left to verify the transmission condition (2.10) for the index $N-1$. Applying the Dirichlet trace $\gamma^-_{\mathrm{D},N-1}$ to $\mathbf{E}_N$, given by (4.28), yields

$$\gamma^-_{\mathrm{D},N-1}\mathbf{E}_N = -\frac{1}{2}\left(\rho_N^{-1} C^{\alpha,(N)}_{N-1N-1}\gamma^+_{\mathrm{N}_{\kappa_{N-1}},N-1}\mathbf{E}_{N-1}\left(M^{\alpha,(N)}_{N-1N-1} + \mathrm{I}\right)\gamma^+_{\mathrm{D},N-1}\mathbf{E}_{N-1}\right) \tag{4.36}$$

with the help of $(3.11) - (3.12)$. Inserting the expressions (4.35) for the traces $\gamma^+_{\mathrm{N}_{\kappa_{N-1}},N-1}\mathbf{E}_{N-1}$ and $\gamma^+_{\mathrm{D},N-1}\mathbf{E}_{N-1}$ in (4.36) and exploiting the validity of the Nth equation of the integral equation system (4.13) leads to

$$\gamma^-_{\mathrm{D},N-1}\mathbf{E}_N = -C^{\alpha,(N-1)}_{N-1,N-2}\mathbf{j}_{N-2} - C^{\alpha,(N-1)}_{N-1,N-1}\mathbf{j}_{N-1} \overset{(4.35)}{=} \gamma^+_{\mathrm{D},N-1}\mathbf{E}_{N-1}.$$

This clearly corresponds to the desired transmission condition.

Next, we turn to the proof of the transmission condition (2.11). First, consider an index $k \in K_{\mathrm{even}}$. We insert the transmission condition (2.10) into the representation of the field $\mathbf{E}_k$ as in (4.27) to obtain

$$\begin{aligned}\mathbf{E}_k = &\frac{1}{2}\left(\rho_{k+1}\Psi^\alpha_{\mathrm{E}_{\kappa_k},k}\gamma^-_{\mathrm{N}_{\kappa_{k+1}},k}\mathbf{E}_{k+1} + \Psi^\alpha_{\mathrm{M}_{\kappa_k},k}\gamma^+_{\mathrm{D},k}\mathbf{E}_k\right)\\ &- \frac{1}{2}\left(\rho_k^{-1}\Psi^\alpha_{\mathrm{E}_{\kappa_k},k-1}\gamma^+_{\mathrm{N}_{\kappa_{k-1}},k-1}\mathbf{E}_{k-1} + \Psi^\alpha_{\mathrm{M}_{\kappa_k},k-1}\gamma^-_{\mathrm{D},k-1}\mathbf{E}_k\right).\end{aligned}$$

Identifying this equation with the Stratton-Chu type representation (3.16) of $\mathbf{E}_k$ guaranteed by Lemma 3.13 yields

$$\Psi^\alpha_{\mathrm{E}_{\kappa_k},k}\left(\rho_{k+1}\gamma^-_{\mathrm{N}_{\kappa_{k+1}},k}\mathbf{E}_{k+1} - \gamma^+_{\mathrm{N}_{\kappa_k},k}\mathbf{E}_k\right) = \Psi^\alpha_{\mathrm{E}_{\kappa_k},k-1}\left(\rho_k^{-1}\gamma^+_{\mathrm{N}_{\kappa_{k-1}},k-1} - \gamma^-_{\mathrm{N}_{\kappa_k},k-1}\mathbf{E}_k\right).$$

Since

$$\rho_{k+1}\gamma^-_{\mathrm{N}_{\kappa_{k+1}},k}\mathbf{E}_{k+1} - \gamma^+_{\mathrm{N}_{\kappa_k},k}\mathbf{E}_k \in \mathbf{H}_\alpha^{-\frac{1}{2}}(\mathrm{div}_\Gamma,\Gamma_k) \qquad \text{and}$$
$$\rho_k^{-1}\gamma^+_{\mathrm{N}_{\kappa_{k-1}},k-1} - \gamma^-_{\mathrm{N}_{\kappa_k},k-1}\mathbf{E}_k \in \mathbf{H}_\alpha^{-\frac{1}{2}}(\mathrm{div}_\Gamma,\Gamma_{k-1}),$$

we deduce from Lemma 3.15 that

$$\Psi^\alpha_{\mathrm{E}_{\kappa_k},k}\left(\rho_{k+1}\gamma^-_{\mathrm{N}_{\kappa_{k+1}},k}\mathbf{E}_{k+1} - \gamma^+_{\mathrm{N}_{\kappa_k},k}\mathbf{E}_k\right) = 0 \quad \text{and} \tag{4.37}$$

$$\Psi^\alpha_{\mathrm{E}_{\kappa_k},k-1}\left(\rho_k^{-1}\gamma^+_{\mathrm{N}_{\kappa_{k-1}},k-1} - \gamma^-_{\mathrm{N}_{\kappa_k},k-1}\mathbf{E}_k\right) = 0. \tag{4.38}$$

At this point, we call attention to the fact that the operators $C^{\alpha,(k)}_{kk}$ and $C^{\alpha,(k)}_{k-1,k-1}$ are invertible because they are Fredholm operators of index zero by Lemma 3.9 and $\mathcal{N}(C^{\alpha,(k)}_{kk}) = \mathcal{N}(C^{\alpha,(k)}_{k-1,k-1}) = \{0\}$ by assumption. We now apply the Dirichlet trace $\gamma^+_{\mathrm{D},k}$ to (4.37) and the Dirichlet trace $\gamma^-_{\mathrm{D},k-1}$ to (4.38), respectively, to obtain

$$0 \overset{(4.37)}{=} \gamma^+_{\mathrm{D},k}\Psi^\alpha_{\mathrm{E}_{\kappa_k},k}\left(\rho_{k+1}\gamma^-_{\mathrm{N}_{\kappa_{k+1}},k}\mathbf{E}_{k+1} - \gamma^+_{\mathrm{N}_{\kappa_k},k}\mathbf{E}_k\right) = -C^{\alpha,(k)}_{kk}\left(\rho_{k+1}\gamma^-_{\mathrm{N}_{\kappa_{k+1}},k}\mathbf{E}_{k+1} - \gamma^+_{\mathrm{N}_{\kappa_k},k}\mathbf{E}_k\right)$$

and

$$\begin{aligned}0 &\overset{(4.38)}{=} \gamma^-_{\mathrm{D},k-1}\Psi^\alpha_{\mathrm{E}_{\kappa_k},k-1}\left(\rho_k^{-1}\gamma^+_{\mathrm{N}_{\kappa_{k-1}},k-1} - \gamma^-_{\mathrm{N}_{\kappa_k},k-1}\mathbf{E}_k\right)\\ &= -C^{\alpha,(k)}_{k-1,k-1}\Psi^\alpha_{\mathrm{E}_{\kappa_k},k-1}\left(\rho_k^{-1}\gamma^+_{\mathrm{N}_{\kappa_{k-1}},k-1} - \gamma^-_{\mathrm{N}_{\kappa_k},k-1}\mathbf{E}_{k-1}\right).\end{aligned}$$

Exploiting the invertibility of the operators $C^{\alpha,(k)}_{kk}$ and $C^{\alpha,(k)}_{k-1,k-1}$ eventually gives

$$\rho_{k+1}\gamma^-_{\mathrm{N}_{\kappa_{k+1}},k}\mathbf{E}_{k+1} = \gamma^+_{\mathrm{N}_{\kappa_k},k}\mathbf{E}_k \quad \text{and} \quad \rho_k^{-1}\gamma^+_{\mathrm{N}_{\kappa_{k-1}},k-1} = \gamma^-_{\mathrm{N}_{\kappa_k},k-1}\mathbf{E}_k.$$

This validates the transmission condition (2.11) for $k \in K$ if N is odd as well as for $k \in K \setminus \{N-1\}$ if N is even. Thus, only the proof of (2.11) for the index $N-1$ for an even number of interfaces N remains open. We treat this case analogously. We start by inserting the transmission condition (2.10) into the representation (4.28) of the electric field $\mathbf{E}_N$:

$$\mathbf{E}_N = \frac{1}{2}\left(\rho_N^{-1}\Psi^\alpha_{\mathrm{E}_{\kappa_N},N-1}\gamma^+_{\mathrm{N}_{\kappa_{N-1}},N-1}\mathbf{E}_{N-1} + \Psi^\alpha_{\mathrm{M}_{\kappa_N},N-1}\gamma^-_{\mathrm{D},N-1}\mathbf{E}_N\right).$$

By Lemma 3.12, we have an alternative representation of $\mathbf{E}_N$ via the α-quasiperiodic Stratton-Chu integral representation that we can insert above. We arrive at

$$\begin{aligned} &\Psi^{\alpha}_{\mathrm{E}_{\kappa_N},N-1}\left(\rho_N^{-1}\gamma^{+}_{\mathrm{N}_{\kappa_{N-1}},N-1}\mathbf{E}_{N-1} - \gamma^{-}_{\mathrm{N}_{\kappa_N},N-1}\mathbf{E}_N\right) = 0 \\ \xrightarrow{\gamma^{-}_{\mathrm{D},N-1}} \quad & - C^{\alpha,(N)}_{N-1,N-1}\left(\rho_N^{-1}\gamma^{+}_{\mathrm{N}_{\kappa_{N-1}},N-1}\mathbf{E}_{N-1} - \gamma^{-}_{\mathrm{N}_{\kappa_N},N-1}\mathbf{E}_N\right) = 0. \end{aligned}$$

The invertibility of the Fredholm operator of index zero $C^{\alpha,(N)}_{N-1,N-1}$, justified by $\mathcal{N}(C^{\alpha,(N)}_{N-1,N-1}) = \{0\}$, then yields

$$\rho_N^{-1}\gamma^{+}_{\mathrm{N}_{\kappa_{N-1}},N-1}\mathbf{E}_{N-1} = \gamma^{-}_{\mathrm{N}_{\kappa_N},N-1}\mathbf{E}_N.$$

This corresponds to the transmission condition (2.11) for the index $N-1$ if N is even and therefore completes our consideration.

Next, we assume that a solution $\mathbf{E}$ of the 2π-biperiodic multilayered electromagnetic scattering problem (2.7)-(2.13) is given. We denote by $\mathbf{E}_k$ the restriction of the electric field $\mathbf{E}$ to G_k, $k \in K_0^N$. Lemma 3.15 and its proof imply that for every $k \in K_{\mathrm{even}}$, there exist two unique densities $\mathbf{j}_{k-2}$ and $\mathbf{j}_{k-1}$ lying in $\mathbf{H}_\alpha^{-1/2}(\mathrm{div}_\Gamma, \Gamma_{k-2})$ and $\mathbf{H}_\alpha^{-1/2}(\mathrm{div}_\Gamma, \Gamma_{k-1})$, respectively, such that

$$\Psi^{\alpha}_{\mathrm{E}_{\kappa_{k-1}},k-2}\mathbf{j}_{k-2} = \frac{1}{2}\left(\Psi^{\alpha}_{\mathrm{E}_{\kappa_{k-1}},k-2}\gamma^{-}_{\mathrm{N}_{\kappa_{k-1}},k-2}\mathbf{E}_{k-1} + \Psi^{\alpha}_{\mathrm{M}_{\kappa_{k-1}},k-2}\gamma^{-}_{\mathrm{D},k-2}\mathbf{E}_{k-1}\right), \tag{4.39}$$

$$\Psi^{\alpha}_{\mathrm{E}_{\kappa_{k-1}},k-1}\mathbf{j}_{k-1} = -\frac{1}{2}\left(\Psi^{\alpha}_{\mathrm{E}_{\kappa_{k-1}},k-1}\gamma^{+}_{\mathrm{N}_{\kappa_{k-1}},k-1}\mathbf{E}_{k-1} + \Psi^{\alpha}_{\mathrm{M}_{\kappa_{k-1}},k-1}\gamma^{+}_{\mathrm{D},k-1}\mathbf{E}_{k-1}\right) \tag{4.40}$$

holds. From the assumption that $\mathcal{N}(C^{\alpha,(k-1)}_{k-2,k-2}) = \mathcal{N}(C^{\alpha,(k-1)}_{k-1,k-1}) = \{0\}$ and the fact that the boundary integral operators $C^{\alpha,(k-1)}_{k-2,k-2}$ and $C^{\alpha,(k-1)}_{k-1,k-1}$ are Fredholm operators of index zero according to Lemma 3.9, we deduce that the latter are also invertible in $\mathbf{H}_\alpha^{-1/2}(\mathrm{div}_\Gamma, \Gamma_{k-2})$ and $\mathbf{H}_\alpha^{-1/2}(\mathrm{div}_\Gamma, \Gamma_{k-1})$, respectively. With this, we easily derive, that

$$\begin{aligned} \mathbf{j}_{k-2} &= -\frac{1}{2}\left[\gamma^{-}_{\mathrm{N}_{\kappa_{k-1}},k-2}\mathbf{E}_{k-1} + \left(C^{\alpha,(k-1)}_{k-2,k-2}\right)^{-1}\left(M^{\alpha,(k-1)}_{k-2,k-2} + \mathrm{I}\right)\gamma^{-}_{\mathrm{D},k-2}\mathbf{E}_{k-1}\right], \\ \mathbf{j}_{k-1} &= \frac{1}{2}\left[\gamma^{+}_{\mathrm{N}_{\kappa_{k-1}},k-1}\mathbf{E}_{k-1} + \left(C^{\alpha,(k-1)}_{k-1,k-1}\right)^{-1}\left(M^{\alpha,(k-1)}_{k-1,k-1} - \mathrm{I}\right)\gamma^{+}_{\mathrm{D},k-1}\mathbf{E}_{k-1}\right] \end{aligned}$$

after applying the Dirichlet trace $\gamma^{-}_{\mathrm{D},k-2}$ to (4.39) and $\gamma^{+}_{\mathrm{D},k-1}$ to (4.40), respectively. It remains to show the existence of the density $\mathbf{j}_{N-1} \in \mathbf{H}_\alpha^{-1/2}(\mathrm{div}_\Gamma, \Gamma_{N-1})$ in the case that N is odd. Then we can represent the electric field $\mathbf{E}_N$ as

$$\mathbf{E}_N = \Psi^{\alpha}_{\mathrm{E}_{\kappa_N},N-1}\mathbf{j}_{N-1}$$

and conclude that

$$\mathbf{j}_{N-1} = -\left(C^{\alpha,(N)}_{N-1,N-1}\right)^{-1}\mathbf{E}_N$$

due to the invertibility of $C^{\alpha,(N)}_{N-1,N-1}$ by $\mathcal{N}(C^{\alpha,(N)}_{N-1,N-1}) = \{0\}$ and Lemma 3.9. Going back to the derivation of the integral equation systems (4.13) and (4.14) presented in Section 4.1 clearly reveals that the densities $\mathbf{j}_k \in \mathbf{H}_\alpha^{-1/2}(\mathrm{div}_\Gamma, \Gamma_k)$, $k \in K_0$, solve the before mentioned systems of integral equations. □

We arrive at a similar equivalence result for the adjoint systems (4.18) and (4.22). In order to enable a readable formulation of this equivalence, we distinguish between the cases $N = 2$ and $N > 2$.

Lemma 4.5 (Equivalence for the systems (4.18) and (4.22), case $N = 2$). *Let $N = 2$ and let the vector-valued density $\mathbf{j} = (\mathbf{j}_0, \mathbf{j}_1)^{\mathrm{T}} \in \prod_{k=0}^{1} \mathbf{H}_\alpha^{-1/2}(\mathrm{div}_\Gamma, \Gamma_k)$ be a solution of the linear system (4.18). Moreover, assume that*

$$\mathcal{N}\left(C_{00}^{\alpha,(1)}\right) = \mathcal{N}\left(C_{11}^{\alpha,(1)}\right) = \{0\}.$$

Then the functions

$$\begin{aligned}
&\mathbf{E}_0 = \Psi^\alpha_{\mathrm{E}_{\kappa_0},0}\mathbf{j}_0 && \text{in } G_0,\\
&\left.\begin{aligned}
\mathbf{E}_1 = &\frac{1}{2}\left[\rho_1^{-1}\Psi^\alpha_{\mathrm{E}_{\kappa_1},0}\left(M_{00}^{\alpha,(0)} - \mathrm{I}\right) + \Psi^\alpha_{\mathrm{M}_{\kappa_1},0}C_{00}^{\alpha,(0)}\right]\mathbf{j}_0\\
&- \frac{1}{2}\left[\rho_2\Psi^\alpha_{\mathrm{E}_{\kappa_1},1}\left(M_{11}^{\alpha,(2)} + \mathrm{I}\right) + \Psi^\alpha_{\mathrm{M}_{\kappa_1},1}C_{11}^{\alpha,(2)}\right]\mathbf{j}_1
\end{aligned}\right\} && \text{in } G_1,\\
&\left.\begin{aligned}
\mathbf{E}_2 = &-\frac{1}{2}\left[\rho_2^{-1}\Psi^\alpha_{\mathrm{E}_{\kappa_2},1}M_{10}^{\alpha,(1)} + \Psi^\alpha_{\mathrm{M}_{\kappa_2},1}C_{10}^{\alpha,(1)}\right]\mathbf{j}_0\\
&- \frac{1}{2}\left[\rho_2^{-1}\Psi^\alpha_{\mathrm{E}_{\kappa_2},1}\left(M_{11}^{\alpha,(1)} - \mathrm{I}\right) + \Psi^\alpha_{\mathrm{M}_{\kappa_2},1}C_{11}^{\alpha,(1)}\right]\mathbf{j}_1
\end{aligned}\right\} && \text{in } G_2
\end{aligned}$$

solve the electromagnetic scattering problem (2.7)-(2.13).

On the other hand, if

$$\mathcal{N}\left(C_{00}^{\alpha,(0)}\right) = \{0\}, \quad \mathcal{N}\left(C_{11}^{\alpha,(2)}\right) = \{0\},$$

then any solution $\mathbf{E}$ of the electromagnetic scattering problem (2.7)-(2.13) provides a solution of the integral equation system (4.13) for $N = 2$.

Lemma 4.6 (Equivalence for the systems (4.18) and (4.22), case $N > 2$). *Let $N > 2$ and let the vector-valued density $\mathbf{j} = (\mathbf{j}_0, \mathbf{j}_1, \ldots, \mathbf{j}_{N-1})^{\mathrm{T}} \in \prod_{k=0}^{N-1} \mathbf{H}_\alpha^{-1/2}(\mathrm{div}_\Gamma, \Gamma_k)$ be a solution of the linear system (4.18) if N is even, or of (4.22) if N is odd. Moreover, assume that*

$$\mathcal{N}\left(C_{k-1,k-1}^{\alpha,(k)}\right) = \mathcal{N}\left(C_{kk}^{\alpha,(k)}\right) = \{0\} \quad \text{for } k \in K_{\mathrm{odd}}$$

and additionally $\mathcal{N}(C_{N-1,N-1}^{\alpha,(N)})$ if N is odd. Then the functions

$$\begin{aligned}
&\mathbf{E}_0 = \Psi^\alpha_{\mathrm{E}_{\kappa_0},0}\mathbf{j}_0 && \text{in } G_0,\\
&\left.\begin{aligned}
\mathbf{E}_1 = &\frac{1}{2}\left[\rho_1^{-1}\Psi^\alpha_{\mathrm{E}_{\kappa_1},0}\left(M_{00}^{\alpha,(0)} - \mathrm{I}\right) + \Psi^\alpha_{\mathrm{M}_{\kappa_1},0}C_{00}^{\alpha,(0)}\right]\mathbf{j}_0\\
&- \frac{1}{2}\left[\rho_2\Psi^\alpha_{\mathrm{E}_{\kappa_1},1}\left(M_{11}^{\alpha,(2)} + \mathrm{I}\right) + \Psi^\alpha_{\mathrm{M}_{\kappa_1},1}C_{11}^{\alpha,(2)}\right]\mathbf{j}_1\\
&- \frac{1}{2}\left[\rho_2\Psi^\alpha_{\mathrm{E}_{\kappa_1},1}M_{12}^{\alpha,(2)} + \Psi^\alpha_{\mathrm{M}_{\kappa_1},1}C_{12}^{\alpha,(2)}\right]\mathbf{j}_2
\end{aligned}\right\} && \text{in } G_1,\\
&\left.\begin{aligned}
\mathbf{E}_k = &\frac{1}{2}\left[\rho_k^{-1}\Psi^\alpha_{\mathrm{E}_{\kappa_k},k-1}M_{k-1,k-2}^{\alpha,(k-1)} + \Psi^\alpha_{\mathrm{M}_{\kappa_k},k-1}C_{k-1,k-2}^{\alpha,(k-1)}\right]\mathbf{j}_{k-2}\\
&+ \frac{1}{2}\left[\rho_k^{-1}\Psi^\alpha_{\mathrm{E}_{\kappa_k},k-1}\left(M_{k-1,k-1}^{\alpha,(k-1)} - \mathrm{I}\right) + \Psi^\alpha_{\mathrm{M}_{\kappa_k},k-1}C_{k-1,k-1}^{\alpha,(k-1)}\right]\mathbf{j}_{k-1}\\
&- \frac{1}{2}\left[\rho_{k+1}\Psi^\alpha_{\mathrm{E}_{\kappa_k},k}\left(M_{kk}^{\alpha,(k+1)} + \mathrm{I}\right) + \Psi^\alpha_{\mathrm{M}_{\kappa_k},k}C_{kk}^{\alpha,(k+1)}\right]\mathbf{j}_k\\
&- \frac{1}{2}\left[\rho_{k+1}\Psi^\alpha_{\mathrm{E}_{\kappa_k},k}M_{k,k+1}^{\alpha,(k+1)} + \Psi^\alpha_{\mathrm{M}_{\kappa_k},k}C_{k,k+1}^{\alpha,(k+1)}\right]\mathbf{j}_{k+1}
\end{aligned}\right\} && \text{in } G_k
\end{aligned}$$

for $k \in K_{\mathrm{odd}} \setminus \{1, N-1\}$,

$$\mathbf{E}_k = \Psi^\alpha_{\mathrm{E}_{\kappa_k},k-1}\mathbf{j}_{k-1} + \Psi^\alpha_{\mathrm{E}_{\kappa_k},k}\mathbf{j}_k \qquad \text{in } G_k$$

for $k \in K_{\mathrm{even}}$,

$$\left.\begin{aligned}\mathbf{E}_{N-1} = \frac{1}{2}&\left[\rho_{N-1}^{-1}\Psi^{\alpha}_{\mathrm{E}_{\kappa_{N-1}},N-2}M^{\alpha,(N-2)}_{N-2,N-3} + \Psi^{\alpha}_{\mathrm{M}_{\kappa_{N-1}},N-2}C^{\alpha,(N-2)}_{N-2,N-3}\right]\mathbf{j}_{N-3}\\ &+\frac{1}{2}\left[\rho_{N-1}^{-1}\Psi^{\alpha}_{\mathrm{E}_{\kappa_{N-1}},N-2}\left(M^{\alpha,(N-2)}_{N-2,N-2}-\mathrm{I}\right) + \Psi^{\alpha}_{\mathrm{M}_{\kappa_{N-1}},N-2}C^{\alpha,(N-2)}_{N-2,N-2}\right]\mathbf{j}_{N-2}\\ &-\frac{1}{2}\left[\rho_{N}\Psi^{\alpha}_{\mathrm{E}_{\kappa_{N-1}},N-1}\left(M^{\alpha,(N-2)}_{N-1,N-1}+\mathrm{I}\right) + \Psi^{\alpha}_{\mathrm{M}_{\kappa_{N-1}},N-1}C^{\alpha,(N)}_{N-1,N-1}\right]\mathbf{j}_{N-1}\end{aligned}\right\}\text{ in } G_{N-1},$$

$$\mathbf{E}_N = \Psi^{\alpha}_{\mathrm{E}_{\kappa_N},N-1}\mathbf{j}_{N-1} \qquad \text{in } G_N$$

for odd N and

$$\left.\begin{aligned}\mathbf{E}_{N} = -\frac{1}{2}&\left[\rho_{N}^{-1}\Psi^{\alpha}_{\mathrm{E}_{\kappa_{N}},N-1}M^{\alpha,(N-1)}_{N-1,N-2} + \Psi^{\alpha}_{\mathrm{M}_{\kappa_{N}},N-1}C^{\alpha,(N-1)}_{N-1,N-2}\right]\mathbf{j}_{N-2}\\ &-\frac{1}{2}\left[\rho_{N}^{-1}\Psi^{\alpha}_{\mathrm{E}_{\kappa_{N}},N-1}\left(M^{\alpha,(N-1)}_{N-1,N-1}-\mathrm{I}\right) + \Psi^{\alpha}_{\mathrm{M}_{\kappa_{N}},N-1}C^{\alpha,(N-1)}_{N-1,N-1}\right]\mathbf{j}_{N-1}\end{aligned}\right\}\text{ in } G_N$$

for even N solve the electromagnetic scattering problem (2.7)-(2.13).

On the other hand, if

$$\mathcal{N}\left(C^{\alpha,(0)}_{00}\right) = \{0\}, \quad \mathcal{N}\left(C^{\alpha,(k+1)}_{kk}\right) = \mathcal{N}\left(C^{\alpha,(k+1)}_{k+1,k+1}\right) = \{0\} \quad \text{for } k \in K_{\mathrm{odd}} \setminus \{N-1\}$$

and additionally $\mathcal{N}(C^{\alpha,(N)}_{N-1,N-1}) = \{0\}$ if N is even, then any solution $\mathbf{E}$ of the electromagnetic scattering problem (2.7)-(2.13) provides a solution of the integral equation system (4.13), in case of an even number of interfaces N, and of the integral equation system (4.14), in case of an odd number of interfaces N, respectively.

The proofs of Lemmata 4.6 and 4.5 are based on the same ideas as the proof of Lemma 4.4 and are therefore left to the reader.

5 Solvability of the system of integral equations

In the rest of this paper, we want to discuss the solvability of the linear integral equation systems (4.13) and (4.14). Since the potential approach applied here arises from the extension of the combined potential ansatz in [6] for electromagnetic scattering by a single 2π-biperiodic grating profile, we can also adapt the techniques of proof employed in [6]. We first verify that our integral equation systems are Fredholm of index zero under quite general assumptions on the electromagnetic material parameters if the grating interfaces of the considered multilayered structure are smooth, and under more restrictive assumptions if they are only polyhedral Lipschitz regular. Then it is possible to entail the existence of (possibly unique) solutions to (4.13) and (4.14) depending on the values of the electric permittivity and the magnetic permeability in each of the material layers. The uniqueness of solutions to the integral equation systems is separately studied with the help of a variational argumentation. The solvability of the integral equation systems (4.13) and (4.14) contributes to the proof of an existence result for the recursive integral equation algorithm derived in [7].

5.1 Fredholmness

Below, we study the Fredholm properties of the linear integral equation systems (4.13), (4.14), (4.18) and (4.22). Our main result states that the left-hand sides of these equations, i.e., $M^{\mathrm{even}}_{\alpha}$, $M^{\mathrm{odd}}_{\alpha}$, $W^{\mathrm{even}}_{\alpha}$ and $W^{\mathrm{odd}}_{\alpha}$, are Fredholm operators of index zero in the Hilbert space $\prod_{k=0}^{N-1}\mathbf{H}_{\alpha}^{-1/2}(\mathrm{div}_{\Gamma},\Gamma_k)$ under certain assumptions on the electromagnetic material parameters.

Theorem 5.1 (Fredholmness)**.** *Assume the electromagnetic material parameters ϵ_k, μ_k, $k \in K_0^N$, of the considered 2π-biperiodic N-layered structure to satisfy (2.2) such that*

$$\epsilon_{k+1} \neq -\epsilon_k \quad \textit{and} \quad \mu_{k+1} \neq -\mu_k \quad \textit{for } k \in K_0$$

holds if Γ_k is smooth, or

$$\mathrm{Re}(\epsilon_k)\,\mathrm{Re}(\epsilon_{k+1}) + \mathrm{Im}(\epsilon_k)\,\mathrm{Im}(\epsilon_{k+1}) \geq 0 \quad \textit{and} \quad \mathrm{Re}(\mu_k)\,\mathrm{Re}(\mu_{k+1}) + \mathrm{Im}(\mu_k)\,\mathrm{Im}(\mu_{k+1}) \geq 0$$

holds if Γ_k is only polyhedral Lipschitz regular. Then the $N \times N$ sized operators

$$M_\alpha^{\mathrm{even}}, M_\alpha^{\mathrm{odd}} : \prod_{k=0}^{N-1} \mathbf{H}_\alpha^{-\frac{1}{2}}(\mathrm{div}_\Gamma, \Gamma_k) \to \prod_{k=0}^{N-1} \mathbf{H}_\alpha^{-\frac{1}{2}}(\mathrm{div}_\Gamma, \Gamma_k),$$
$$W_{-\alpha}^{\mathrm{even}}, W_{-\alpha}^{\mathrm{odd}} : \prod_{k=0}^{N-1} \mathbf{H}_{-\alpha}^{-\frac{1}{2}}(\mathrm{div}_\Gamma, \Gamma_k) \to \prod_{k=0}^{N-1} \mathbf{H}_{-\alpha}^{-\frac{1}{2}}(\mathrm{div}_\Gamma, \Gamma_k)$$

corresponding to the linear integral equation systems (4.13), (4.14), (4.18) and (4.22) are Fredholm operators of index zero for all wave vectors α fulfilling $\alpha_3 > 0$.

In order to give the proof of Theorem 5.1 a nice structure, we formulate two auxiliary lemmata in advance.

Lemma 5.2. *The $N \times N$ sized linear operators*

$$M_\alpha^{\mathrm{even}}, M_\alpha^{\mathrm{odd}} : \prod_{k=0}^{N-1} \mathbf{H}_\alpha^{-\frac{1}{2}}(\mathrm{div}_\Gamma, \Gamma_k) \to \prod_{k=0}^{N-1} \mathbf{H}_\alpha^{-\frac{1}{2}}(\mathrm{div}_\Gamma, \Gamma_k)$$

from (4.13) and (4.14) are Fredholm operators of index zero if and only if their diagonal elements

$$(M_\alpha^{\mathrm{even}})_{k+1,k+1} : \mathbf{H}_\alpha^{-\frac{1}{2}}(\mathrm{div}_\Gamma, \Gamma_k) \to \mathbf{H}_\alpha^{-\frac{1}{2}}(\mathrm{div}_\Gamma, \Gamma_k),$$
$$\left(M_\alpha^{\mathrm{odd}}\right)_{k+1,k+1} : \mathbf{H}_\alpha^{-\frac{1}{2}}(\mathrm{div}_\Gamma, \Gamma_k) \to \mathbf{H}_\alpha^{-\frac{1}{2}}(\mathrm{div}_\Gamma, \Gamma_k)$$

are Fredholm operators of index zero for all $k \in K_0$.

Proof. We recall that all integral operators occurring in the elements of M_α^{even} and M_α^{odd} are linear and bounded. For $k \neq j$, the kernels of the operators

$$C_{kj}^{\alpha,\kappa}, M_{kj}^{\alpha,\kappa} : \mathbf{H}_\alpha^{-\frac{1}{2}}(\mathrm{div}_\Gamma, \Gamma_j) \to \mathbf{H}_\alpha^{-\frac{1}{2}}(\mathrm{div}_\Gamma, \Gamma_k)$$

are smooth on $\Gamma_k \times \Gamma_j$. Therefore, the operators $C_{kj}^{\alpha,\kappa}$ and $M_{kj}^{\alpha,\kappa}$, $k \neq j$, are compact. From this, we easily deduce the compactness of all off-diagonal elements ($k \neq j$)

$$(M_\alpha^{\mathrm{even}})_{k+1,j+1} : \mathbf{H}_\alpha^{-\frac{1}{2}}(\mathrm{div}_\Gamma, \Gamma_j) \to \mathbf{H}_\alpha^{-\frac{1}{2}}(\mathrm{div}_\Gamma, \Gamma_k),$$
$$\left(M_\alpha^{\mathrm{odd}}\right)_{k+1,j+1} : \mathbf{H}_\alpha^{-\frac{1}{2}}(\mathrm{div}_\Gamma, \Gamma_j) \to \mathbf{H}_\alpha^{-\frac{1}{2}}(\mathrm{div}_\Gamma, \Gamma_k),$$

i.e., M_α^{even} and M_α^{odd} are compact perturbations of the diagonal operators

$$\mathrm{diag}\left((M_\alpha^{\mathrm{even}})_{11}, (M_\alpha^{\mathrm{even}})_{22}, \ldots, (M_\alpha^{\mathrm{even}})_{NN}\right),$$
$$\mathrm{diag}\left(\left(M_\alpha^{\mathrm{odd}}\right)_{11}, \left(M_\alpha^{\mathrm{odd}}\right)_{22}, \ldots, \left(M_\alpha^{\mathrm{odd}}\right)_{NN}\right).$$

Thus, M_α^{even} and M_α^{odd} are Fredholm operators of index zero in the product space $\prod_{k=0}^{N-1} \mathbf{H}_\alpha^{-1/2}(\mathrm{div}_\Gamma, \Gamma_k)$ if and only if $(M_\alpha^{\mathrm{even}})_{k+1,k+1}$ and $\left(M_\alpha^{\mathrm{odd}}\right)_{k+1,k+1}$ are Fredholm operators of index zero in $\mathbf{H}_\alpha^{-1/2}(\mathrm{div}_\Gamma, \Gamma_k)$. □

Lemma 5.3. *Under the assumptions of Theorem 5.1, the operators*

$$\begin{aligned}(M_\alpha^{\mathrm{even}})_{k+1,k+1} &: \mathbf{H}_\alpha^{-\frac{1}{2}}(\mathrm{div}_\Gamma,\Gamma_k) \to \mathbf{H}_\alpha^{-\frac{1}{2}}(\mathrm{div}_\Gamma,\Gamma_k),\\ \left(M_\alpha^{\mathrm{odd}}\right)_{k+1,k+1} &: \mathbf{H}_\alpha^{-\frac{1}{2}}(\mathrm{div}_\Gamma,\Gamma_k) \to \mathbf{H}_\alpha^{-\frac{1}{2}}(\mathrm{div}_\Gamma,\Gamma_k)\end{aligned}$$

are Fredholm of index zero.

Proof. For $k \in K_0$, we define the operator $\mathbf{A}_\alpha^k : \mathbf{H}_\alpha^{-1/2}(\mathrm{div}_\Gamma,\Gamma_k) \to \mathbf{H}_\alpha^{-1/2}(\mathrm{div}_\Gamma,\Gamma_k)$ by

$$\mathbf{A}_\alpha^k = \rho_{k+1} C_{kk}^{\alpha,(k)} \left(M_{kk}^{\alpha,(k+1)} + \mathrm{I}\right) + \left(M_{kk}^{\alpha,(k)} + \mathrm{I}\right) C_{kk}^{\alpha,(k+1)}$$

with $\rho_{k+1} = \frac{\mu_k \kappa_{k+1}}{\mu_{k+1}\kappa_k}$. This corresponds to the boundary integral operator $\mathbf{A}_\alpha$ for $\Gamma := \Gamma_k$ from [6], which is a Fredholm operator of index zero under the assumptions of [6, Corollary 5.2] if Γ is smooth and under the assumptions of [6, Corollary 5.7] if Γ is polyhedral Lipschitz regular. By Lemma 3.8, the adjoint operator of $\mathbf{A}_{-\alpha}^k$ with respect to the bilinear form $\mathcal{B}_k$ is

$$\left(\mathbf{A}_{-\alpha}^k\right)' := \rho_{k+1} \left(M_{kk}^{\alpha,(k+1)} - \mathrm{I}\right) C_{kk}^{\alpha,(k)} + C_{kk}^{\alpha,(k+1)} \left(M_{kk}^{\alpha,(k)} - \mathrm{I}\right).$$

This operator inherits the Fredholm properties of $\mathbf{A}_{-\alpha}^k$. Taking a closer look at the boundary integral operators $(M_\alpha^{\mathrm{even}})_{k+1,k+1}$ and $(M_\alpha^{\mathrm{odd}})_{k+1,k+1}$, we realize that

$$\begin{aligned}(M_\alpha^{\mathrm{even}})_{k+1,k+1} &= \left(M_\alpha^{\mathrm{odd}}\right)_{k+1,k+1} = \frac{\kappa_k}{\mu_k}\mathbf{A}_\alpha^k && \text{for } k \in K_{\mathrm{even}} \cup \{0\},\\ (M_\alpha^{\mathrm{even}})_{k+1,k+1} &= \left(M_\alpha^{\mathrm{odd}}\right)_{k+1,k+1} = -\frac{\kappa_k}{\mu_k}\left(\mathbf{A}_{-\alpha}^k\right)' && \text{for } k \in K_{\mathrm{odd}}.\end{aligned}$$

Since the assumptions of this theorem are in accordance with the assumptions of [6, Corollaries 5.2 and 5.7], we apply them to conclude that $(M_\alpha^{\mathrm{even}})_{k+1,k+1}$ and $(M_\alpha^{\mathrm{odd}})_{k+1,k+1}$ are Fredholm operators of index zero in $\mathbf{H}_\alpha^{-1/2}(\mathrm{div}_\Gamma,\Gamma_k)$ for all $k \in K_0$. ☐

Proof of Theorem 5.1. The auxiliary Lemmata 5.2 and 5.3 immediately yield that M_α^{even} and M_α^{odd} are Fredholm operators of index zero in $\prod_{k=0}^{N-1} \mathbf{H}_\alpha^{-1/2}(\mathrm{div}_\Gamma,\Gamma_k)$. Together with Lemma 4.3, we moreover infer that $W_{-\alpha}^{\mathrm{even}}$ and $W_{-\alpha}^{\mathrm{odd}}$ are Fredholm operators of index zero in $\prod_{k=0}^{N-1} \mathbf{H}_\alpha^{-1/2}(\mathrm{div}_\Gamma,\Gamma_k)$. ☐

5.2 Uniqueness

This subsection is concerned with the uniqueness of solutions to the systems of linear integral equations (4.13) and (4.14). Our main result reads as follows.

Theorem 5.4 (Uniqueness)**.** *Let the electromagnetic material parameters ϵ_k, μ_k, $k \in K_0^N$, of the considered 2π-biperiodic N-layered structure satisfy (2.2) such that $\epsilon_0, \mu_0 \notin \mathbb{R}_-$ and $\epsilon_N, \mu_N \notin \mathbb{R}_-$. Moreover, assume that one of the following situations holds for ϵ_j, ϵ_{j+1}, μ_j and μ_{j+1} for some $j \in K_0$:*

(i) $\epsilon_j, \mu_j \in \mathbb{R}$ *such that at least one of them is positive and*

$$\mathrm{Im}\,(\epsilon_{j+1}) \geq 0 \quad \textit{and} \quad \mathrm{Im}\,(\mu_{j+1}) \geq 0 \quad \textit{with} \quad \mathrm{Im}\,(\epsilon_{j+1} + \mu_{j+1}) > 0;$$

(ii) $\epsilon_{j+1}, \mu_{j+1} \in \mathbb{R}$ *such that at least one of them is positive and*

$$\mathrm{Im}\,(\epsilon_j) \geq 0 \quad \textit{and} \quad \mathrm{Im}\,(\mu_j) \geq 0 \quad \textit{with} \quad \mathrm{Im}\,(\epsilon_j + \mu_j) > 0;$$

(iii) $\mathrm{Im}\left(\epsilon_j\right), \mathrm{Im}\left(\epsilon_{j+1}\right), \mathrm{Im}\left(\mu_j\right), \mathrm{Im}\left(\mu_{j+1}\right) \geq 0$ *with*

$$\mathrm{Im}\left(\epsilon_j + \mu_j\right) > 0 \quad \text{and} \quad \mathrm{Im}\left(\epsilon_{j+1} + \mu_{j+1}\right) > 0.$$

Then, depending on the parity of N, the linear integral equation systems (4.13) or (4.14) have at most one solution $\mathbf{J} \in \prod_{k=0}^{N-1} \mathbf{H}_\alpha^{-1/2}(\mathrm{div}_\Gamma, \Gamma_k)$ if

$$\mathcal{N}\left(C_{k-1,k-1}^{\alpha,(k)}\right) = \mathcal{N}\left(C_{kk}^{\alpha,(k)}\right) = \{0\} \quad \text{for } k \in K_{\mathrm{odd}}$$

and additionally $\mathcal{N}(C_{N-1,N-1}^{\alpha,(N)}) = \{0\}$ in case of an odd number of interfaces N.

The proof of Theorem 5.4 requires several auxiliary lemmata, which are presented hereafter.

Lemma 5.5. *Let the electric permittivities ϵ_k and the magnetic permeabilities μ_k, $k \in K_0^N$, satisfy (2.2). Then, if $\kappa_k \in \mathbb{R}$ for some $k \in K_0^N$, we have*

$$\mathrm{Im}\left(\beta_k^{(n)}\right) > 0 \quad \text{for all except of a finite number } N_k \text{ of } n \in \mathbb{Z}^2.$$

The excluded $n \in N_k$ satisfy $\mathrm{Im}(\beta_k^{(n)}) = 0$. For all other values of κ_k, the imaginary part of $\beta_k^{(n)}$ is non-negative for all $n \in \mathbb{Z}^2$, i.e., $\mathrm{Im}(\beta_k^{(n)}) > 0$ for all $n \in \mathbb{Z}^2$.

Lemma 5.6. *Let the electromagnetic material parameters ϵ_k and μ_k, $k \in K_0^N$, satisfy (2.2). Then we have*

$$\mathrm{Im}\left(\frac{\epsilon_k}{\kappa_k^2}\right) \leq 0 \quad \text{for all } k \in K_0^N. \tag{5.1}$$

Both Lemma 5.5 and Lemma 5.6 are shown by simple computations.

The next auxiliary result is a particular type of Holmgren's uniqueness theorem (HUT) for the time-harmonic Maxwell equations. The original version of Holmgren's theorem is found in [14].

Theorem 5.7 (HUT for time-harmonic Maxwell's equations)**.** *Let G be a connected and bounded polyhedral Lipschitz domain and assume that $\mathbf{E} \in \mathbf{H}(\mathbf{curl}, G)$ is a solution of the time-harmonic Maxwell equations $\mathbf{curl}\,\mathbf{curl}\,\mathbf{E} - \kappa^2\mathbf{E} = 0$ in G. If there exists an open set U such that $U \cap \partial G \neq \emptyset$ and*

$$\gamma_{\mathrm{D}}\mathbf{E} = \gamma_{\mathrm{N}_\kappa}\mathbf{E} = 0 \quad \text{on } U \cap \partial G \tag{5.2}$$

holds, then $\mathbf{E}$ already vanishes in all of G.

Theorem 5.7 can be verified by adapting the proof of Theorem 3.5 in [13], which presents the corresponding result for acoustics, to electromagnetics (see also [10, Theorem 6.5]).

Proof of Theorem 5.4. For the verification of Theorem 5.4, we reuse the ideas of the proof of Theorem 5.9 from [6] and make a proof by contradiction. Depending on the parity of N, let

$$\mathbf{J} \in \prod_{k=0}^{N-1} \mathbf{H}_\alpha^{-\frac{1}{2}}(\mathrm{div}_\Gamma, \Gamma_k)$$

be a nontrivial solution of $M_\alpha^{\mathrm{even}} = 0$ or $M_\alpha^{\mathrm{odd}} = 0$. With the help of Lemma 4.4, it is then possible to compose an α-quasiperiodic electric field $\mathbf{E}$ in $G_k, k \in K_0^N$, from $\mathbf{J}$, which solves the homogeneous 2π-biperiodic multilayered electromagnetic scattering problem with respect to the transmission conditions

$$\gamma_{\mathrm{D},k}^{+}\mathbf{E}_k = \gamma_{\mathrm{D},k}^{-}\mathbf{E}_{k+1} \quad \text{and} \quad \mu_{k+1}\gamma_{\mathrm{D},k}^{+}\left(\mathbf{curl}\,\mathbf{E}_k\right) = \mu_k\gamma_{\mathrm{D},k}^{-}\left(\mathbf{curl}\,\mathbf{E}_{k+1}\right) \tag{5.3}$$

for $k \in K_0$. Next, we want to derive a variational formulation in terms of $\mathbf{E}$ in the domain G^{H} introduced in (2.3) for a fixed $\mathrm{H} \in \mathbb{R}_+$. Speaking visually, G^{H} is a periodically extendable cell of width 2π in both x_1- and x_2-direction that contains all considered grating interfaces Γ_k, $k \in K_0$, of the considered multilayered structure and is bounded by the plane surfaces

$$\Gamma_{\pm}^{\mathrm{H}} \coloneqq \{x \in Q \times \mathbb{R} \,:\, x_3 = \pm \mathrm{H}\}$$

with the outer normals $\mathbf{n}_{\pm}^{\mathrm{H}} = (0,0,\pm 1)^{\mathrm{T}}$. Furthermore, we recall the definition of the bounded domains $G_0^{\mathrm{H}} = G^{\mathrm{H}} \cap G_0$ and $G_N^{\mathrm{H}} = G^{\mathrm{H}} \cap G_N$. Our first step now consists in multiplying the time-harmonic Maxwell equations (2.7) by $\frac{\epsilon}{\kappa^2}\overline{\mathbf{E}}$. Afterwards, we integrate the resulting expression over the polyhedral Lipschitz domain $\Omega \coloneqq G_0^{\mathrm{H}} \cup_{k=1}^{N-1} G_k \cup G_N^{\mathrm{H}}$ and apply Green's identity (3.6) for the curl operator in the polyhedral Lipschitz domains $G_0^{\mathrm{H}}, G_N^{\mathrm{H}}$ and G_k, $k \in K$, in terms of their outer unit normals:

$$\begin{aligned} 0 &= \int_{\Omega} \left(\mathbf{curl}\,\mathbf{curl}\,\mathbf{E} - \kappa^2 \mathbf{E}\right) \cdot \frac{\epsilon}{\kappa^2} \overline{\mathbf{E}}\, dx \\ &\overset{(3.6)}{=} \int_{G^{\mathrm{H}}} \frac{\epsilon}{\kappa^2} \left|\mathbf{curl}\,\mathbf{E}\right|^2 - \epsilon \left|\mathbf{E}\right|^2 \, dx + \mathcal{B}_{\partial G_0^{\mathrm{H}}}\left(\gamma_{\mathrm{D}}\left(\mathbf{curl}\,\mathbf{E}\right), \frac{\epsilon_0}{\kappa_0^2}\gamma_{\mathrm{D}}\overline{\mathbf{E}}\right) \\ &\quad + \sum_{k=1}^{N-1} \mathcal{B}_{\partial G_k}\left(\gamma_{\mathrm{D}}\left(\mathbf{curl}\,\mathbf{E}\right), \frac{\epsilon_k}{\kappa_k^2}\gamma_{\mathrm{D}}\overline{\mathbf{E}}\right) + \mathcal{B}_{\partial G_N^{\mathrm{H}}}\left(\gamma_{\mathrm{D}}\left(\mathbf{curl}\,\mathbf{E}\right), \frac{\epsilon_N}{\kappa_N^2}\gamma_{\mathrm{D}}\overline{\mathbf{E}}\right). \end{aligned}$$

The α-quasiperiodicity of the integrands implies that

$$\begin{aligned} &\mathcal{B}_{\{x \in \partial G_*^{\mathrm{H}} \,:\, x_1 = -\pi\}}\left(\gamma_{\mathrm{D}}\left(\mathbf{curl}\,\mathbf{E}\right), \frac{\epsilon_*}{\kappa_*^2}\gamma_{\mathrm{D}}\overline{\mathbf{E}}\right) \\ &\qquad + \mathcal{B}_{\{x \in \partial G_*^{\mathrm{H}} \,:\, x_1 = \pi\}}\left(\gamma_{\mathrm{D}}\left(\mathbf{curl}\,\mathbf{E}\right), \frac{\epsilon_*}{\kappa_*^2}\gamma_{\mathrm{D}}\overline{\mathbf{E}}\right) = 0, \\ &\mathcal{B}_{\{x \in \partial G_*^{\mathrm{H}} \,:\, x_2 = -\pi\}}\left(\left(\mathbf{curl}\,\mathbf{E}\right), \frac{\epsilon_*}{\kappa_*^2}\gamma_{\mathrm{D}}\overline{\mathbf{E}}\right) \\ &\qquad + \mathcal{B}_{\{x \in \partial G_*^{\mathrm{H}} \,:\, x_2 = \pi\}}\left(\gamma_{\mathrm{D}}\left(\mathbf{curl}\,\mathbf{E}\right), \frac{\epsilon_*}{\kappa_*^2}\gamma_{\mathrm{D}}\overline{\mathbf{E}}\right) = 0 \end{aligned}$$

for $* \in \{0, N\}$ and

$$\begin{aligned} &\mathcal{B}_{\{x \in \partial G_k \,:\, x_1 = -\pi\}}\left(\gamma_{\mathrm{D}}\left(\mathbf{curl}\,\mathbf{E}\right), \frac{\epsilon_k}{\kappa_k^2}\gamma_{\mathrm{D}}\overline{\mathbf{E}}\right) \\ &\qquad + \mathcal{B}_{\{x \in \partial G_k \,:\, x_1 = \pi\}}\left(\gamma_{\mathrm{D}}\left(\mathbf{curl}\,\mathbf{E}\right), \frac{\epsilon_k}{\kappa_k^2}\gamma_{\mathrm{D}}\overline{\mathbf{E}}\right) = 0, \\ &\mathcal{B}_{\{x \in \partial G_k \,:\, x_2 = -\pi\}}\left(\gamma_{\mathrm{D}}\left(\mathbf{curl}\,\mathbf{E}\right), \frac{\epsilon_k}{\kappa_k^2}\gamma_{\mathrm{D}}\overline{\mathbf{E}}\right) \\ &\qquad + \mathcal{B}_{\{x \in \partial G_k \,:\, x_2 = \pi\}}\left(\gamma_{\mathrm{D}}\left(\mathbf{curl}\,\mathbf{E}\right), \frac{\epsilon_k}{\kappa_k^2}\gamma_{\mathrm{D}}\overline{\mathbf{E}}\right) = 0 \end{aligned}$$

for $k \in K$. Our equation so far can thus be reformulated to

$$\begin{aligned} 0 &= \int_{G^{\mathrm{H}}} \frac{\epsilon}{\kappa^2} \left|\mathbf{curl}\,\mathbf{E}\right|^2 - \epsilon \left|\mathbf{E}\right|^2 \, dx + \mathcal{B}_{\partial G_0^{\mathrm{H}}}\left(\gamma_{\mathrm{D}}\left(\mathbf{curl}\,\mathbf{E}\right), \frac{\epsilon_0}{\kappa_0^2}\gamma_{\mathrm{D}}\overline{\mathbf{E}}\right)\Big|_{\Gamma_0} \\ &\quad + \sum_{k=1}^{N-1} \mathcal{B}_{\partial G_k}\left(\gamma_{\mathrm{D}}\left(\mathbf{curl}\,\mathbf{E}\right), \frac{\epsilon_k}{\kappa_k^2}\gamma_{\mathrm{D}}\overline{\mathbf{E}}\right)\Big|_{\Gamma_{k-1}} + \mathcal{B}_{\partial G_k}\left(\gamma_{\mathrm{D}}\left(\mathbf{curl}\,\mathbf{E}\right), \frac{\epsilon_k}{\kappa_k^2}\gamma_{\mathrm{D}}\overline{\mathbf{E}}\right)\Big|_{\Gamma_k} \\ &\quad + \mathcal{B}_{\partial G_N^{\mathrm{H}}}\left(\gamma_{\mathrm{D}}\left(\mathbf{curl}\,\mathbf{E}\right), \frac{\epsilon_k}{\kappa_k^2}\gamma_{\mathrm{D}}\overline{\mathbf{E}}\right)\Big|_{\Gamma_{N-1}} \end{aligned}$$

$$
\begin{aligned}
&- \int_{\Gamma^{\mathrm{H}}_{+}} \frac{\epsilon_0}{\kappa_0^2} r\left(\gamma_{\mathrm{D}}|_{\Gamma^{\mathrm{H}}_{+}} (\mathbf{curl}\, \mathbf{E}_0)\right) \gamma_{\mathrm{D}}|_{\Gamma^{\mathrm{H}}_{+}} \overline{\mathbf{E}}_0 \, d\sigma - \int_{\Gamma^{\mathrm{H}}_{-}} \frac{\epsilon_N}{\kappa_N^2} r\left(\gamma_{\mathrm{D}}|_{\Gamma^{\mathrm{H}}_{-}} (\mathbf{curl}\, \mathbf{E}_N)\right) \gamma_{\mathrm{D}}|_{\Gamma^{\mathrm{H}}_{-}} \overline{\mathbf{E}}_N \, d\sigma \\
\overset{(5.3)_1}{=} & \int_{G^{\mathrm{H}}} \frac{\epsilon}{\kappa^2} |\mathbf{curl}\, \mathbf{E}|^2 - \epsilon |\mathbf{E}|^2 \; dx \\
&+ \sum_{k=0}^{N-1} \left[\mathcal{B}_k \left(\frac{\epsilon_{k+1}}{\kappa_{k+1}^2} \gamma^{-}_{\mathrm{D},k} (\mathbf{curl}\, \mathbf{E}_{k+1}), \gamma^{+}_{\mathrm{D},k} \overline{\mathbf{E}}_k \right) - \mathcal{B}_k \left(\frac{\epsilon_k}{\kappa_k^2} \gamma^{+}_{\mathrm{D},k} (\mathbf{curl}\, \mathbf{E}_k), \gamma^{+}_{\mathrm{D},k} \overline{\mathbf{E}}_k \right) \right] \\
&+ \int_{\Gamma^{\mathrm{H}}_{+}} \frac{\epsilon_0}{\kappa_0^2} \left[(\mathbf{curl}\, \mathbf{E}_0)_1 \left(\overline{\mathbf{E}}_0\right)_2 - (\mathbf{curl}\, \mathbf{E}_0)_2 \left(\overline{\mathbf{E}}_0\right)_1 \right] \, d\sigma \\
&+ \int_{\Gamma^{\mathrm{H}}_{-}} \frac{\epsilon_N}{\kappa_N^2} \left[(\mathbf{curl}\, \mathbf{E}_N)_2 \left(\overline{\mathbf{E}}_N\right)_1 - (\mathbf{curl}\, \mathbf{E}_N)_1 \left(\overline{\mathbf{E}}_N\right)_2 \right] \, d\sigma \\
\overset{(5.3)_2}{=} & \int_{G^{\mathrm{H}}} \frac{\epsilon}{\kappa^2} |\mathbf{curl}\, \mathbf{E}|^2 - \epsilon |\mathbf{E}|^2 \; dx \\
&+ \sum_{k=0}^{N-1} \mathcal{B}_k \left(\underbrace{\left(\frac{\mu_{k+1}\epsilon_{k+1}}{\mu_k \kappa_{k+1}^2} - \frac{\epsilon_k}{\kappa_k^2} \right)}_{=0} \gamma^{+}_{\mathrm{D},k} (\mathbf{curl}\, \mathbf{E}_k), \gamma^{+}_{\mathrm{D},k} \overline{\mathbf{E}}_k \right) \\
&+ \int_{\Gamma^{\mathrm{H}}_{+}} \frac{\epsilon_0}{\kappa_0^2} \left[(\mathbf{curl}\, \mathbf{E}_0)_1 \left(\overline{\mathbf{E}}_0\right)_2 - (\mathbf{curl}\, \mathbf{E}_0)_2 \left(\overline{\mathbf{E}}_0\right)_1 \right] \, d\sigma \\
&+ \int_{\Gamma^{\mathrm{H}}_{-}} \frac{\epsilon_N}{\kappa_N^2} \left[(\mathbf{curl}\, \mathbf{E}_N)_2 \left(\overline{\mathbf{E}}_N\right)_1 - (\mathbf{curl}\, \mathbf{E}_N)_1 \left(\overline{\mathbf{E}}_N\right)_2 \right] \, d\sigma.
\end{aligned}
$$

In the above calculation, the expressions $r(\gamma_{\mathrm{D}}|_{\Gamma^{\mathrm{H}}_{\pm}} \cdot)$ on the plane surfaces $\Gamma^{\mathrm{H}}_{\pm}$ are computed via the classical cross product as $(\gamma_{\mathrm{D}}|_{\Gamma^{\mathrm{H}}_{\pm}} \cdot \times \mathbf{n}^{\mathrm{H}}_{\pm})$. The electric fields $\mathbf{E}_0$ and $\mathbf{E}_N$ solve the electromagnetic scattering problem in the semi-infinite domains G_0 and G_N and thus in particular fulfill the outgoing wave condition (2.12)-(2.13). Combined with $\operatorname{div} \mathbf{E} = 0$, this yields the identities

$$
\alpha_1^{(n)} \left(\overline{\mathbf{E}}_n^0\right)_1 + \alpha_2^{(n)} \left(\overline{\mathbf{E}}_n^0\right)_2 + \overline{\beta_0^{(n)}} \left(\overline{\mathbf{E}}_n^0\right)_3 = 0 \qquad \text{on } \Gamma^{\mathrm{H}}_{+}, \tag{5.4}
$$

$$
\alpha_1^{(n)} \left(\overline{\mathbf{E}}_n^N\right)_1 + \alpha_2^{(n)} \left(\overline{\mathbf{E}}_n^N\right)_2 - \overline{\beta_N^{(n)}} \left(\overline{\mathbf{E}}_n^N\right)_3 = 0 \qquad \text{on } \Gamma^{\mathrm{H}}_{-} \tag{5.5}
$$

for the complex-valued Rayleigh coefficients $\mathbf{E}_n^0$ and $\mathbf{E}_n^N$, $n \in \mathbb{Z}^2$, on Γ^{H}_{+} and Γ^{H}_{-}, respectively. Together with the outgoing wave condition (2.12)-(2.13) these relations give rise to

$$
\int_{\Gamma^{\mathrm{H}}_{+}} \frac{\epsilon_0}{\kappa_0^2} \left[(\mathbf{curl}\, \mathbf{E}_0)_1 \left(\overline{\mathbf{E}}_0\right)_2 - (\mathbf{curl}\, \mathbf{E}_0)_2 \left(\overline{\mathbf{E}}_0\right)_1 \right] \, d\sigma \overset{(5.4)}{=} - \sum_{n \in \mathbb{Z}^2} M_n^{\alpha,0} \mathbf{E}_n^0 \cdot \overline{\mathbf{E}}_n^0 e^{-2\,\mathrm{Im}\left(\beta_0^{(n)}\right)\mathrm{H}}
$$

and

$$
\int_{\Gamma^{\mathrm{H}}_{-}} \frac{\epsilon_N}{\kappa_N^2} \left[(\mathbf{curl}\, \mathbf{E}_N)_2 \left(\overline{\mathbf{E}}_N\right)_1 - (\mathbf{curl}\, \mathbf{E}_N)_1 \left(\overline{\mathbf{E}}_N\right)_2 \right] \, d\sigma \overset{(5.5)}{=} - \sum_{n \in \mathbb{Z}^2} M_n^{\alpha,N} \mathbf{E}_n^N \cdot \overline{\mathbf{E}}_n^N e^{-2\,\mathrm{Im}\left(\beta_N^{(n)}\right)\mathrm{H}}
$$

with

$$
M_n^{\alpha,0} := \frac{\mathrm{i} 4\pi^2 \epsilon_0}{\kappa_0^2} \begin{pmatrix} \beta_0^{(n)} & 0 & 0 \\ 0 & \beta_0^{(n)} & 0 \\ 0 & 0 & \overline{\beta_0^{(n)}} \end{pmatrix}, \quad M_n^{\alpha,N} := \frac{\mathrm{i} 4\pi^2 \epsilon_N}{\kappa_N^2} \begin{pmatrix} \beta_N^{(n)} & 0 & 0 \\ 0 & \beta_N^{(n)} & 0 \\ 0 & 0 & \overline{\beta_N^{(n)}} \end{pmatrix}.
$$

Inserting this into the variational equation for $\mathbf{E}$ from above yields

$$
\begin{aligned}
&\int_{G^{\mathrm{H}}} \frac{\epsilon}{\kappa^2} |\mathbf{curl}\, \mathbf{E}|^2 - \epsilon |\mathbf{E}|^2 \; dx \\
&\qquad = \sum_{n \in \mathbb{Z}^2} \left(M_n^{\alpha,0} \mathbf{E}_n^0 \cdot \overline{\mathbf{E}}_n^0 e^{-2\,\mathrm{Im}\left(\beta_0^{(n)}\right)} + M_n^{\alpha,N} \mathbf{E}_n^N \cdot \overline{\mathbf{E}}_n^N e^{-2\,\mathrm{Im}\left(\beta_N^{(n)}\right)} \right).
\end{aligned} \tag{5.6}
$$

We now take the imaginary part of (5.6) and let $\mathrm{H} \to \infty$: Exploiting that, by Lemma 5.5, we have $\operatorname{Im}(\beta_0^{(n)}) \geq 0$ and $\operatorname{Im}(\beta_N^{(n)}) \geq 0$ for all $n \in \mathbb{Z}^2$ with $\operatorname{Im}\left(\beta_0^{(n)}\right) = 0$ and $\operatorname{Im}\left(\beta_N^{(n)}\right) = 0$ only for a finite number of $n \in \mathbb{Z}^2$ if $\kappa_0^2 \in \mathbb{R}$ and $\kappa_N^2 \in \mathbb{R}$, we then obtain that

$$\begin{aligned}
&\lim_{\mathrm{H}\to\infty} \int_{G_0^{\mathrm{H}}} \operatorname{Im}\left(\frac{\epsilon_0}{\kappa_0^2}\right) |\mathbf{curl}\, \mathbf{E}_0|^2 - \operatorname{Im}(\epsilon_0) |\mathbf{E}_0|^2 \, dx \\
&\quad + \sum_{k=1}^{N-1} \int_{G_k} \operatorname{Im}\left(\frac{\epsilon_k}{\kappa_k^2}\right) |\mathbf{curl}\, \mathbf{E}_k|^2 - \operatorname{Im}(\epsilon_k) |\mathbf{E}_k|^2 \, dx \\
&\quad + \lim_{\mathrm{H}\to\infty} \int_{G_N^{\mathrm{H}}} \operatorname{Im}\left(\frac{\epsilon_N}{\kappa_N^2}\right) |\mathbf{curl}\, \mathbf{E}_N|^2 - \operatorname{Im}(\epsilon_N) |\mathbf{E}_N|^2 \, dx \\
&= 4\pi^2 \left[\operatorname{Im}\left(\mathrm{i}\frac{\epsilon_0}{\kappa_0^2}\right) \sum_{B_0} \beta_0^{(n)} \left|\mathbf{E}_n^0\right|^2 + \operatorname{Im}\left(\mathrm{i}\frac{\epsilon_N}{\kappa_N^2}\right) \sum_{B_N} \beta_N^{(n)} \left|\mathbf{E}_n^N\right|^2\right]
\end{aligned} \tag{5.7}$$

with $B_0 := \left\{n \in \mathbb{Z}^2 \,:\, \beta_0^{(n)} > 0\right\}$ and $B_N := \left\{n \in \mathbb{Z}^2 \,:\, \beta_N^{(n)} > 0\right\}$ as $\kappa_0, \kappa_N \notin \mathbb{R}_-$. This means that in particular the limit expression on the left-hand side exists. The assumptions of this theorem on the electromagnetic material parameters make an application of Lemma 5.6 possible. In fact, with $\operatorname{Im}(\epsilon_k/\kappa_k^2) \leq 0$, given by (5.1), and $-\operatorname{Im}(\epsilon_k) \leq 0$ for all $k \in K_0^N$, we arrive at

$$\begin{aligned}
&\lim_{\mathrm{H}\to\infty} \int_{G_0^{\mathrm{H}}} \operatorname{Im}\left(\frac{\epsilon_0}{\kappa_0^2}\right) |\mathbf{curl}\, \mathbf{E}_0|^2 - \operatorname{Im}(\epsilon_0) |\mathbf{E}_0|^2 \, dx \\
&\quad + \sum_{k=1}^{N-1} \int_{G_k} \operatorname{Im}\left(\frac{\epsilon_k}{\kappa_k^2}\right) |\mathbf{curl}\, \mathbf{E}_k|^2 - \operatorname{Im}(\epsilon_k) |\mathbf{E}_k|^2 \, dx \\
&\quad + \lim_{\mathrm{H}\to\infty} \int_{G_N^{\mathrm{H}}} \operatorname{Im}\left(\frac{\epsilon_N}{\kappa_N^2}\right) |\mathbf{curl}\, \mathbf{E}_N|^2 - \operatorname{Im}(\epsilon_N) |\mathbf{E}_N|^2 \, dx \leq 0.
\end{aligned} \tag{5.8}$$

Next, we take a look at the right-hand side of equation (5.7). Lemma 5.5 implies that $\beta_0^{(n)} \in \mathbb{R} \setminus \{0\}$ and $\beta_N^{(n)} \in \mathbb{R} \setminus \{0\}$ if and only if $\kappa_0 \in \mathbb{R}$ and $\kappa_N \in \mathbb{R}$. Since, by assumption, we excluded the case that $\epsilon_0, \mu_0 \in \mathbb{R}_-$ and $\epsilon_N, \mu_N \in \mathbb{R}_-$, the latter requirement is only satisfied if $\epsilon_0, \mu_0 \in \mathbb{R}_+$ and $\epsilon_N, \mu_N \in \mathbb{R}_+$. Then the right-hand side of equation (5.7) is non-negative and we altogether obtain

$$\begin{aligned}
0 &\overset{(5.8)}{\geq} \lim_{\mathrm{H}\to\infty} \int_{G_0^{\mathrm{H}}} \operatorname{Im}\left(\frac{\epsilon_0}{\kappa_0^2}\right) |\mathbf{curl}\, \mathbf{E}_0|^2 - \operatorname{Im}(\epsilon_0) |\mathbf{E}_0|^2 \, dx \\
&\quad + \sum_{k=1}^{N-1} \int_{G_k} \operatorname{Im}\left(\frac{\epsilon_k}{\kappa_k^2}\right) |\mathbf{curl}\, \mathbf{E}_k|^2 - \operatorname{Im}(\epsilon_k) |\mathbf{E}_k|^2 \, dx \\
&\quad + \lim_{\mathrm{H}\to\infty} \int_{G_N^{\mathrm{H}}} \operatorname{Im}\left(\frac{\epsilon_N}{\kappa_N^2}\right) |\mathbf{curl}\, \mathbf{E}_N|^2 - \operatorname{Im}(\epsilon_N) |\mathbf{E}_N|^2 \, dx \\
&= 4\pi^2 \left[\operatorname{Im}\left(\mathrm{i}\frac{\epsilon_0}{\kappa_0^2}\right) \sum_{B_0} \beta_0^{(n)} \left|\mathbf{E}_n^0\right|^2 + \operatorname{Im}\left(\mathrm{i}\frac{\epsilon_N}{\kappa_N^2}\right) \sum_{B_N} \beta_N^{(n)} \left|\mathbf{E}_n^N\right|^2\right] \geq 0.
\end{aligned} \tag{5.9}$$

In fact, this in particular gives

$$\lim_{\mathrm{H}\to\infty} \int_{G_0^{\mathrm{H}}} \operatorname{Im}\left(\frac{\epsilon_0}{\kappa_0^2}\right) |\mathbf{curl}\, \mathbf{E}_0|^2 - \operatorname{Im}(\epsilon_0) |\mathbf{E}_0|^2 \, dx = 0, \tag{5.10}$$

$$\lim_{\mathrm{H}\to\infty} \int_{G_N^{\mathrm{H}}} \operatorname{Im}\left(\frac{\epsilon_N}{\kappa_N^2}\right) |\mathbf{curl}\, \mathbf{E}_N|^2 - \operatorname{Im}(\epsilon_N) |\mathbf{E}_N|^2 \, dx = 0, \tag{5.11}$$

$$\int_{G_k} \mathrm{Im}\left(\frac{\epsilon_k}{\kappa_k^2}\right) |\mathbf{curl}\,\mathbf{E}_k|^2 - \mathrm{Im}\,(\epsilon_k)\,|\mathbf{E}_k|^2 \;dx = 0 \quad \text{for } k \in K. \tag{5.12}$$

Denote by j the index in K for which the selected electromagnetic material parameters ϵ_j, ϵ_{j+1}, μ_j and μ_{j+1} satisfy the assumptions of one of the cases (i)-(iii). If $j = 0$ or $j = N$, we deduce from (5.10) and (5.11) that $\mathbf{E}_j = 0$ a.e. in G_j. If $j \in K$, we immediately observe that either

$$\mathbf{E}_j = 0 \quad \text{or} \quad \mathbf{curl}\,\mathbf{E}_j = 0 \qquad\qquad \text{in cases (ii), (iii)} \tag{5.13}$$

or

$$\mathbf{E}_{j+1} = 0 \quad \text{or} \quad \mathbf{curl}\,\mathbf{E}_{j+1} = 0 \qquad\qquad \text{in cases (i), (iii)} \tag{5.14}$$

can be inferred from (5.12) for $k = j$ and $k = j+1$. If $\mathbf{curl}\,\mathbf{E}_j = 0$ holds in (5.13), the time-harmonic Maxwell equations $\mathbf{curl}\,\mathbf{curl}\,\mathbf{E}_j - \kappa_j^2\mathbf{E}_j = 0$ imply that then also $\mathbf{E}_j = 0$. In (5.14), we similarly conclude that also $\mathbf{E}_{j+1} = 0$ is true if $\mathbf{curl}\,\mathbf{E}_{j+1} = 0$. Furthermore, if one of the identities in (5.14) is satisfied, we have

$$\gamma^-_{\mathrm{D},j}\mathbf{E}_j \overset{(5.3)}{=} \gamma^+_{\mathrm{D},j}\mathbf{E}_{j+1} = 0 \quad \text{and} \quad \gamma^-_{\mathrm{N}_j,j}\mathbf{E}_j \overset{(5.3)}{=} \gamma^+_{\mathrm{N}_{j+1},j}\mathbf{E}_{j+1} = 0$$

and an application of Holmgren's uniqueness theorem in the version of Theorem 5.7 to the bounded domain G_j implies that $\mathbf{E}_j = 0$ in G_j. Thus, all in all, we conclude that

$$\mathbf{E}_j = 0 \quad \text{a.e. in } G_j \text{ in situations (i)-(iii)} \tag{5.15}$$

for the characteristic index $j \in K_0^N$.

For $j \in K_0^N$, the conclusion (5.15) in particular gives rise to

$$\gamma^-_{\mathrm{D},j-1}\mathbf{E}_j = \gamma^-_{\mathrm{N}_{\kappa_j},j-1}\mathbf{E}_j = \gamma^+_{\mathrm{D},j}\mathbf{E}_j = \gamma^+_{\mathrm{N}_{\kappa_j},j}\mathbf{E}_j = 0 \quad \text{a.e. in } G_j$$

for the characteristic index $j \in K_0^N$ in all situations (i)-(iii). With the transmission conditions (5.3), we derive that

$$\gamma^+_{\mathrm{D},j-1}\mathbf{E}_{j-1} = \gamma^+_{\mathrm{N}_{\kappa_{j-1}},j-1}\mathbf{E}_{j-1} = 0 \qquad\qquad \text{for all } j \in K^N \text{ and} \tag{5.16}$$

$$\gamma^-_{\mathrm{D},j}\mathbf{E}_{j+1} = \gamma^-_{\mathrm{N}_{\kappa_{j+1}},j}\mathbf{E}_{j+1} = 0 \qquad\qquad \text{for all } j \in K_0. \tag{5.17}$$

We recapitulate that the electric fields $\mathbf{E}_{j-1}$ and $\mathbf{E}_{j+1}$ are solutions of the time-harmonic Maxwell equations $\mathbf{curl}\,\mathbf{curl}\,\mathbf{E} - \kappa_{j-1}^2\mathbf{E} = 0$ and $\mathbf{curl}\,\mathbf{curl}\,\mathbf{E} - \kappa_{j+1}^2\mathbf{E} = 0$, respectively, in addition to (5.16) and (5.17), respectively. Thus, we are able to apply Holmgren's uniqueness theorem (see Theorem 5.7) if $j-1 \neq 0$ and $j+1 \neq N$. This results in $\mathbf{E}_{j-1} = 0$ in G_{j-1} as well as $\mathbf{E}_{j+1} = 0$ in G_{j+1}. If $j-1 = 0$ or $j+1 = N$, the Stratton-Chu integral representation from Lemma 3.12 also shows that

$$\mathbf{E}_0 = \frac{1}{2}\left(\Psi^\alpha_{\mathrm{E}_{\kappa_0},0}\gamma^+_{\mathrm{N}_{\kappa_0},0}\mathbf{E}_0 + \Psi^\alpha_{\mathrm{M}_{\kappa_0},0}\gamma^+_{\mathrm{D},0}\mathbf{E}_0\right) \overset{(5.17)}{=} 0 \qquad \text{in } G_0 \quad \text{or}$$

$$\mathbf{E}_N = -\frac{1}{2}\left(\Psi^\alpha_{\mathrm{E}_{\kappa_N},N-1}\gamma^-_{\mathrm{N}_{\kappa_N},N-1}\mathbf{E}_N + \Psi^\alpha_{\mathrm{M}_{\kappa_N},N-1}\gamma^-_{\mathrm{D},N-1}\mathbf{E}_N\right) \overset{(5.17)}{=} 0 \qquad \text{in } G_N.$$

This type of argumentation can easily be applied iteratively. We altogether obtain that

$$\mathbf{E}_k = 0 \quad \text{in } G_k \text{ for } k \in K_0^N.$$

Due to the considered potential ansatz, this means that for $k \in K_{\mathrm{odd}}$

$$\mathbf{E}_k = \Psi^\alpha_{\mathrm{E}_{\kappa_k},k-1}\mathbf{j}_{k-1} + \Psi^\alpha_{\mathrm{E}_{\kappa_k},k}\mathbf{j}_k = 0 \quad \xrightarrow{\text{Lemma 3.15}} \quad \Psi^\alpha_{\mathrm{E}_{\kappa_k},k-1}\mathbf{j}_{k-1} = \Psi^\alpha_{\mathrm{E}_{\kappa_k},k}\mathbf{j}_k = 0 \tag{5.18}$$

and moreover

$$\Psi^{\alpha}_{\mathrm{E}_{\kappa_N},N-1}\mathbf{j}_{N-1} = 0 \tag{5.19}$$

if N is odd. At this point, we recall the assumption that

$$\mathcal{N}\left(C^{\alpha,(k)}_{k-1,k-1}\right) = \mathcal{N}\left(C^{\alpha,(k)}_{kk}\right) = \{0\} \quad \text{for } k \in K_{\mathrm{odd}}$$

and additionally $\mathcal{N}(C^{\alpha,(N)}_{N-1,N-1}) = \{0\}$ in case of an odd number of interfaces N. All involved boundary integral operators

$$C^{\alpha,(k)}_{k-1,k-1} : \mathbf{H}^{-\frac{1}{2}}_{\alpha}(\mathrm{div}_\Gamma, \Gamma_{k-1}) \to \mathbf{H}^{-\frac{1}{2}}_{\alpha}(\mathrm{div}_\Gamma, \Gamma_{k-1}),$$
$$C^{\alpha,(k)}_{kk} : \mathbf{H}^{-\frac{1}{2}}_{\alpha}(\mathrm{div}_\Gamma, \Gamma_{k}) \quad \to \mathbf{H}^{-\frac{1}{2}}_{\alpha}(\mathrm{div}_\Gamma, \Gamma_{k})$$

for $k \in K_{\mathrm{odd}}$ as well as additionally $C^{\alpha,(N)}_{N-1,N-1} : \mathbf{H}^{-1/2}_{\alpha}(\mathrm{div}_\Gamma, \Gamma_{N-1})$ if N is odd are all Fredholm operators of index zero by Lemma 3.9 and are therefore already invertible. Then

$$-C^{\alpha,(k)}_{k-1,k-1}\mathbf{j}_{k-1} \overset{(3.11)}{=} \gamma^{+}_{\mathrm{D},k-1}\Psi^{\alpha}_{\mathrm{E}_{\kappa_k},k-1}\mathbf{j}_{k-1} \overset{(5.18)}{=} 0 \quad \Longrightarrow \mathbf{j}_{k-1} = 0,$$
$$-C^{\alpha,(k)}_{k,k}\mathbf{j}_{k} \overset{(3.11)}{=} \gamma^{-}_{\mathrm{D},k}\Psi^{\alpha}_{\mathrm{E}_{\kappa_k},k}\mathbf{j}_{k} \overset{(5.18)}{=} 0 \quad \Longrightarrow \mathbf{j}_{k} = 0$$

for $k \in K_{\mathrm{odd}}$ and in addition

$$-C^{\alpha,(N)}_{N-1,N-1}\mathbf{j}_{N-1} \overset{(3.11)}{=} \gamma^{+}_{\mathrm{D},N-1}\Psi^{\alpha}_{\mathrm{E}_{\kappa_N},N-1}\mathbf{j}_{N-1} \overset{(5.19)}{=} 0 \quad \Longrightarrow \mathbf{j}_{N-1} = 0$$

if the number of interfaces N is odd. In summary, we derived that $\mathbf{J} = 0$ in all of the situations (i)-(iii), which contradicts the assumed nontriviality of $\mathbf{J}$. Thus, under the assumptions of this theorem, solutions $\mathbf{J}$ lying in $\prod_{k=0}^{N-1}\mathbf{H}^{-1/2}_{\alpha}(\mathrm{div}_\Gamma, \Gamma_k)$ to the linear integral equation systems (4.13) for even N and (4.14) for odd N are unique. ☐

5.3 Existence

Finally, the existence of solutions to the linear integral equation systems (4.13) and (4.14) is studied. We assume that the requirements of Theorem 5.1 are satisfied, which entails that $M^{\mathrm{even}}_{\alpha}$ and $M^{\mathrm{odd}}_{\alpha}$ are Fredholm operators of index zero in the product space $\prod_{k=0}^{N-1}\mathbf{H}^{-1/2}_{\alpha}(\mathrm{div}_\Gamma, \Gamma_k)$. Then we separately consider their left-hand sides $M^{\mathrm{even}}_{\alpha}$ and $M^{\mathrm{odd}}_{\alpha}$ to either have a trivial nullspace, i.e., to be invertible, or to have a nontrivial nullspace. In the latter case,

$$\mathcal{R}\left(M^{\mathrm{even}}_{\alpha}\right) \neq \prod_{k=0}^{N-1}\mathbf{H}^{-\frac{1}{2}}_{\alpha}(\mathrm{div}_\Gamma, \Gamma_k) \quad \text{and} \quad \mathcal{R}\left(M^{\mathrm{odd}}_{\alpha}\right) \neq \prod_{k=0}^{N-1}\mathbf{H}^{-\frac{1}{2}}_{\alpha}(\mathrm{div}_\Gamma, \Gamma_k)$$

holds and the existence of (possibly nonunique) solutions to (4.13) and (4.14) is no longer guaranteed.

Theorem 5.8 (Solvability of $M^{\mathrm{even}}_{\alpha}$ and $M^{\mathrm{odd}}_{\alpha}$)**.** *Let the assumptions of Theorem 5.4 hold. Moreover, assume that the electromagnetic material parameters ϵ_k and μ_k, $k \in K_0^N$, satisfy (2.2) such that*

$$\epsilon_{k+1} \neq -\epsilon_k \quad \textit{and} \quad \mu_{k+1} \neq -\mu_k \quad \textit{for } k \in K_0$$

if Γ_k is smooth, or

$$\mathrm{Re}(\epsilon_k)\,\mathrm{Re}(\epsilon_{k+1}) + \mathrm{Im}(\epsilon_k)\,\mathrm{Im}(\epsilon_{k+1}) \geq 0 \quad \textit{and} \quad \mathrm{Re}(\mu_k)\,\mathrm{Re}(\mu_{k+1}) + \mathrm{Im}(\mu_k)\,\mathrm{Im}(\mu_{k+1}) \geq 0$$

if Γ_k is only polyhedral Lipschitz regular. Then there exists a density $\mathbf{J} \in \prod_{k=0}^{N-1}\mathbf{H}^{-1/2}_{\alpha}(\mathrm{div}_\Gamma, \Gamma_k)$ that uniquely solves either the system (4.13) if N is even or the system (4.14) if N is odd, i.e., either

$$M^{\mathrm{even}}_{\alpha}\mathbf{J} = \mathbf{f}_\alpha \quad \textit{for even } N \quad \textit{or} \quad M^{\mathrm{odd}}_{\alpha}\mathbf{J} = \mathbf{f}_\alpha \quad \textit{for odd } N,$$

where $\mathbf{f}_\alpha$ is given by (4.19)-(4.21).

Since, under the assumptions of Theorem 5.8, M_α^{even} and M_α^{odd} are both Fredholm operators of index zero in $\prod_{k=0}^{N-1}\mathbf{H}_\alpha^{-1/2}(\operatorname{div}_\Gamma,\Gamma_k)$ and a uniqueness result in form of Theorem 5.4 holds, the operators M_α^{even} and M_α^{odd} are already invertible. This easily proves Theorem 5.8.

Finally, we investigate the existence of solutions to (4.13) and (4.14) for material parameter choices such that Theorem 5.4 can not be applied, i.e., in situations in which M_α^{even} and M_α^{odd} are no longer invertible in $\prod_{k=0}^{N-1}\mathbf{H}_\alpha^{-1/2}(\operatorname{div}_\Gamma,\Gamma_k)$. Indeed, we consider electromagnetic material parameters $\epsilon_k,\mu_k,\ k\in K_0^N$, satisfying (2.2) such that $\epsilon_k,\mu_k\in\mathbb{R}$. Unfortunately, we fail to verify a general existence result in the mentioned situations. However, the next theorem still provides rather general conditions on the electromagnetic material parameters that ensure the existence of solutions to the systems (4.13) and (4.14) for real-valued ϵ_k and μ_k, $k\in K_0^N$.

Theorem 5.9 (Existence of solutions to (4.13) and (4.14))**.** *Let the electromagnetic material parameters ϵ_k, $\mu_k\in\mathbb{R}$, $k\in K_0^N$, satisfy (2.2) such that $\operatorname{sgn}(\epsilon_0\mu_0)>0$ and $\operatorname{sgn}(\mu_0\mu_N)>0$ if $\operatorname{sgn}(\epsilon_N\mu_N)>0$. Moreover, assume, for $k\in K_0$, that*

$$\epsilon_{k+1}\neq-\epsilon_k \quad \textit{and} \quad \mu_{k+1}\neq-\mu_k$$

if Γ_k is smooth, or

$$\operatorname{Re}(\epsilon_k)\operatorname{Re}(\epsilon_{k+1})+\operatorname{Im}(\epsilon_k)\operatorname{Im}(\epsilon_{k+1})\geq 0 \quad \textit{and} \quad \operatorname{Re}(\mu_k)\operatorname{Re}(\mu_{k+1})+\operatorname{Im}(\mu_k)\operatorname{Im}(\mu_{k+1})\geq 0$$

if Γ_k is only polyhedral Lipschitz regular. Then, if

$$\mathcal{N}\left(C_{k-1,k-1}^{\alpha,(k)}\right)=\mathcal{N}\left(C_{kk}^{\alpha,(k)}\right)=\{0\} \quad \textit{for } k\in K_{\mathrm{odd}}$$

and additionally $\mathcal{N}(C_{N-1,N-1}^{\alpha,(N)})=\{0\}$ if N is odd, there exists at least one solution $\mathbf{J}$, lying in the product space $\prod_{k=0}^{N-1}\mathbf{H}_\alpha^{-1/2}(\operatorname{div}_\Gamma,\Gamma_k)$, of either the integral equation system (4.13) in the case that N is even or the integral equation system (4.14) in the case that N is odd.

The proof strategy for Theorem 5.9 is to extend the proof of Theorem 5.13 from [6] from single to multi-profile scattering. We recall the adjoint relation of the systems (4.13) and (4.18) as well as of the systems (4.14) and (4.22) with respect to the bilinear form $[\cdot,\cdot]$ from (4.23) in the sense of Lemma 4.3, i.e.,

$$\left[M_\alpha^{\mathrm{even}}\mathbf{J},\mathbf{L}\right]=\left[\mathbf{J},W_{-\alpha}^{\mathrm{even}}\mathbf{L}\right] \quad \text{and} \quad \left[M_\alpha^{\mathrm{odd}}\mathbf{J},\mathbf{L}\right]=\left[\mathbf{J},W_{-\alpha}^{\mathrm{odd}}\mathbf{L}\right]$$

for all $\mathbf{J}\in\prod_{k=0}^{N-1}\mathbf{H}_\alpha^{-1/2}(\operatorname{div}_\Gamma,\Gamma_k)$ and $\mathbf{L}\in\prod_{k=0}^{N-1}\mathbf{H}_{-\alpha}^{-1/2}(\operatorname{div}_\Gamma,\Gamma_k)$. The vector $\mathbf{f}$ consisting of N components - each corresponding to a two-dimensional tangent vector - is defined by

$$\mathbf{f}:=\left(-\frac{2\kappa_0}{\mu_0}\gamma_{\mathrm{D},0}^{-}\mathbf{E}^{\mathrm{i}},0,\ldots,0\right)^{\mathrm{T}}.$$

It describes both the right-hand sides of (4.13) and (4.14). We are then able to reduce the proof of Theorem 5.9 to showing that either

$$[\mathbf{f},\mathbf{L}]=0 \quad \text{for all } \mathbf{L}\in\prod_{k=0}^{N-1}\mathbf{H}_{-\alpha}^{-\frac{1}{2}}(\operatorname{div}_\Gamma,\Gamma_k) \text{ with } W_{-\alpha}^{\mathrm{even}}\mathbf{L}=0 \tag{5.20}$$

if the number of interfaces N in the considered 2π-biperiodic multilayered structure is even, or

$$[\mathbf{f},\mathbf{L}]=0 \quad \text{for all } \mathbf{L}\in\prod_{k=0}^{N-1}\mathbf{H}_{-\alpha}^{-\frac{1}{2}}(\operatorname{div}_\Gamma,\Gamma_k) \text{ with } W_{-\alpha}^{\mathrm{odd}}\mathbf{L}=0 \tag{5.21}$$

if N is odd. This essentially goes back to the fact that, under the assumptions of Theorem 5.9, the $N\times N$ integral operators M_α^{even} and M_α^{odd} are Fredholm operators of index zero by Theorem 5.1. In fact, then the ranges of M_α^{even} and M_α^{odd} are closed.

Proof of Theorem 5.9. Let $\mathbf{L} := (\mathbf{l}_0, \ldots, \mathbf{l}_{N-1})^{\mathrm{T}}$ be an arbitrary density lying in $\prod_{k=0}^{N-1} \mathbf{H}_{-\alpha}^{-1/2}(\mathrm{div}_\Gamma, \Gamma_k)$ such that either $\mathbf{L} \in \mathcal{N}(W_{-\alpha}^{\mathrm{even}})$ if the number of grating surfaces N is even or $\mathbf{L} \in \mathcal{N}(W_{-\alpha}^{\mathrm{odd}})$ if N is odd. Then Lemma 4.6 provides us with a $(-\alpha)$-quasiperiodic solution $\mathbf{E} := \mathbf{E}_k$ in G_k, $k \in K_0^N$, of the homogeneous version of the 2π-biperiodic electromagnetic scattering problem (2.7)-(2.13) since

$$\mathcal{N}\left(C_{k-1,k-1}^{\alpha,(k)}\right) = \mathcal{N}\left(C_{kk}^{\alpha,(k)}\right) = \{0\} \quad \text{for } k \in K_{\mathrm{odd}}$$

and additionally $\mathcal{N}(C_{N-1,N-1}^{\alpha,(N)}) = \{0\}$ if N is odd holds. Such a solution in particular satisfies a variational equation similar to (5.7) in terms of the complex-valued Rayleigh coefficients $\mathbf{E}_n^0$ and $\mathbf{E}_n^N$ - defined in $Q \times \mathbb{R}$ above Γ_+^{H} and below Γ_-^{H}, respectively - after replacing the wave vector α by $(-\alpha)$:

$$\begin{aligned}
&\lim_{\mathrm{H}\to\infty} \int_{G_0^{\mathrm{H}}} \mathrm{Im}\left(\frac{\epsilon_0}{\kappa_0^2}\right) |\mathbf{curl}\,\mathbf{E}_0|^2 - \mathrm{Im}\left(\epsilon_0\right) |\mathbf{E}_0|^2 \; dx \\
&\quad + \sum_{k=1}^{N-1} \int_{G_k} \mathrm{Im}\left(\frac{\epsilon_k}{\kappa_k^2}\right) |\mathbf{curl}\,\mathbf{E}_k|^2 - \mathrm{Im}\left(\epsilon_k\right) |\mathbf{E}_k|^2 \; dx \\
&\quad + \lim_{\mathrm{H}\to\infty} \int_{G_N^{\mathrm{H}}} \mathrm{Im}\left(\frac{\epsilon_N}{\kappa_N^2}\right) |\mathbf{curl}\,\mathbf{E}_N|^2 - \mathrm{Im}\left(\epsilon_N\right) |\mathbf{E}_N|^2 \; dx \\
&= 4\pi^2 \left[\mathrm{Im}\left(\mathrm{i}\frac{\epsilon_0}{\kappa_0^2}\right) \sum_{B_0} \beta_0^{(n)} \left|\mathbf{E}_n^0\right|^2 + \mathrm{Im}\left(\mathrm{i}\frac{\epsilon_N}{\kappa_N^2}\right) \sum_{B_N} \beta_N^{(n)} \left|\mathbf{E}_n^N\right|^2 \right],
\end{aligned}$$

where $B_0 := \left\{ n \in \mathbb{Z}^2 \ : \ \beta_0^{(n)} \in \mathbb{R} \setminus \{0\} \right\}$ and $B_N := \left\{ n \in \mathbb{Z}^2 \ : \ \beta_N^{(n)} \in \mathbb{R} \setminus \{0\} \right\}$. Since all considered electromagnetic material parameters are real-valued, we remain with

$$\mathrm{Im}\left(\mathrm{i}\frac{\epsilon_0}{\kappa_0^2}\right) \sum_{B_0} \beta_0^{(n)} \left|\mathbf{E}_n^0\right|^2 + \mathrm{Im}\left(\mathrm{i}\frac{\epsilon_N}{\kappa_N^2}\right) \sum_{B_N} \beta_N^{(n)} \left|\mathbf{E}_n^N\right|^2 = 0. \tag{5.22}$$

The specific assumptions on ϵ_0, μ_0, ϵ_N and μ_N, i.e., $\mathrm{sgn}(\epsilon_0\mu_0) > 0$ and $\mathrm{sgn}(\mu_0\mu_N) > 0$ if $\mathrm{sgn}(\epsilon_N\mu_N) > 0$, guarantee that

$$\mathrm{sgn}\left(\mathrm{Im}\left(\mathrm{i}\frac{\epsilon_0}{\kappa_0^2}\right) \mathrm{Im}\left(\mathrm{i}\frac{\epsilon_N}{\kappa_N^2}\right)\right) = \mathrm{sgn}\left(\frac{1}{\omega^4\mu_0\mu_N}\right) > 0$$

if $\mathrm{sgn}(\epsilon_N\mu_N) > 0$. In the remaining case that $\mathrm{sgn}(\epsilon_N\mu_N) < 0$, the electric permittivity ϵ_N and the magnetic permeability μ_N are of different sign. Therefore, $\mathrm{Re}(\kappa_N^2) = 0$ and thus $\mathrm{Re}(\beta_N^{(n)}) = 0$, from which we infer that the second term on the right-hand side of (5.22) is equal to zero. All in all, we can then conclude that

$$\mathbf{E}_n^0 = 0 \quad \text{in } Q \times \mathbb{R} \text{ above } \Gamma_+^{\mathrm{H}} \text{ for those } n \in \mathbb{Z}^2 \text{ such that } \beta_0^{(n)} \in \mathbb{R} \setminus \{0\}.$$

From the properties of the wave vector $(-\alpha)$ of the incident plane wave $\mathbf{E}^{\mathrm{i}}$ occurring in the linear integral equation systems (4.13) and (4.14), it is clear that

$$\beta_0^{(0)} = \sqrt{\kappa_0^2 - \left|-\alpha^{(0)}\right|^2} = \sqrt{\kappa_+^2 - |-\tilde{\alpha}|^2} = \alpha_3 > 0.$$

This insight leads to

$$\mathbf{E}_0^0 = 0 \quad \text{in } Q \times \mathbb{R} \text{ above } \Gamma_+^{\mathrm{H}}. \tag{5.23}$$

The Rayleigh coefficient $\mathbf{E}_0^0$ can be computed explicitely by using the potential ansatz $\mathbf{E}_0 = \Psi_{\mathrm{E}_{\kappa_0},0}^{-\alpha}\mathbf{l}_0$ in G_0. Executing this, leads together with the identity (5.23) - in the same manner as in the proof of Theorem 5.13 from [6] - to the conclusion that

$$\mathbf{g} = 0 \quad \text{or} \quad \mathbf{g} \parallel \alpha \tag{5.24}$$

for all vector-valued densities $\mathbf{L} \in \prod_{k=0}^{N-1} \mathbf{H}_{-\alpha}^{-1/2}(\mathrm{div}_\Gamma, \Gamma_k)$ such that either $\mathbf{L} \in \mathcal{N}(W_{-\alpha}^{\mathrm{even}})$ if N is even or $\mathbf{L} \in \mathcal{N}(W_{-\alpha}^{\mathrm{odd}})$ if N is odd, where

$$\mathbf{g}_j := \int_{\Gamma_0} (i_{\pi,0}\mathbf{l}_0)_j \, e^{\mathrm{i}(\tilde{\alpha}\cdot\tilde{y}-\alpha_3 y_3)} \, d\sigma(y) \quad \text{for } j \in \{1,2,3\}.$$

Recalling that

$$\mathbf{f} = \left(-\frac{2\kappa_0}{\mu_0}\gamma_{\mathrm{D},0}^{-}\mathbf{E}^{\mathrm{i}}, 0, \ldots, 0\right)^{\mathrm{T}}$$

is the left-hand side of each of the integral equation systems (4.13) and (4.14) and $\mathbf{n}_0$ denotes the upwards pointing normal on Γ_0, we obtain by simple manipulations that

$$[\mathbf{f}, \mathbf{L}] = \sum_{k=0}^{N-1} \mathcal{B}_k\left(\mathbf{f}_{k+1}, \mathbf{l}_k\right) = \mathcal{B}_0\left(-\frac{2\kappa_0}{\mu_0}\gamma_{\mathrm{D},0}^{-}\mathbf{E}^{\mathrm{i}}, \mathbf{l}_0\right) = \frac{2\kappa_0}{\mu_0}\int_{\Gamma_0} \mathbf{E}^{\mathrm{i}}(y) \cdot (i_{\pi,0}\mathbf{l}_0)(y) \, d\sigma(y).$$

Inserting the respresentation of the incident plane wave $\mathbf{E}^{\mathrm{i}}$ as $\mathbf{E}^{\mathrm{i}} = \mathbf{p}e^{\mathrm{i}(\alpha\cdot\tilde{y}-\alpha_3 y_3)}$ in the equation above and exploiting that the property $\alpha \parallel \mathbf{g}$ from (5.24) is equivalent to

$$\mathbf{p}\cdot\mathbf{g} = 0 \quad \text{due to } \alpha\cdot\mathbf{p} \overset{(2.6)}{=} 0,$$

we finally conclude that

$$[\mathbf{f}, \mathbf{L}] = \frac{2\kappa_0}{\mu_0}\mathbf{p}\cdot\mathbf{g} \overset{(5.24)}{=} 0 \quad \text{for all } \mathbf{L} \in \mathbf{H}_{-\alpha}^{-\frac{1}{2}}(\mathrm{div}_\Gamma, \Gamma_k) \text{ with } W_{-\alpha}^{\mathrm{even}}\mathbf{L} = 0$$

if the number of grating interfaces N is even, and

$$[\mathbf{f}, \mathbf{L}] = \frac{2\kappa_0}{\mu_0}\mathbf{p}\cdot\mathbf{g} \overset{(5.24)}{=} 0 \quad \text{for all } \mathbf{L} \in \mathbf{H}_{-\alpha}^{-\frac{1}{2}}(\mathrm{div}_\Gamma, \Gamma_k) \text{ with } W_{-\alpha}^{\mathrm{odd}}\mathbf{L} = 0$$

if N is odd. This proves our claim. □

6 Conclusion

In this article, we presented an integral equation method for the treatment of electromagnetic scattering by 2π-biperiodic multilayered structures composed of $N \geq 2$ vertically stacked non-self-intersecting grating interfaces of polyhedral Lipschitz regularity. It led to a parity-dependent system of integral equations equivalent to the 2π-biperiodic N-layered electromagnetic scattering problem. In order to achieve this, we applied a particular combined potential ansatz, which is the natural extension of the combined potential ansatz used in [6] for the corresponding problem of single profile scattering: Above the structure, we assumed an α-quasiperiodic Stratton-Chu integral representation and then alternated a two-term electric potential ansatz with two unknown densities with an α-quasiperiodic Stratton-Chu type integral representation. Below the scatterer we either assumed a Stratton-Chu integral representation or a simple electric potential ansatz. Due to this approach, we encounter boundary integral equations that are structurally similar to the ones occurring in the study of single profile scattering as in [6]. With the help of the same techniques as those employed in the presence of only one grating interface, we were therefore able to prove analogous results on the Fredholmness of the system of integral equations as well as on existence and uniqueness of its solution.

It is clear that the numerical solution of a system of N integral equations is computationally very expensive to obtain, in particular for a large N. Therefore, we are interested in the development of a more sophisticated method. That this is possible is shown in the consecutive article [7], in which we introduce a recursive integral equation algorithm. In the course of its study, we exploit the analytical findings of the present paper.

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