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Tracy-Widom at high temperature

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1. INTRODUCTION

One of the most influential random matrix theory (RMT) developments of the last decade was discovery in 2002 of the so-called tridiagonal β-ensembles by Dumitriu and Edelman in [1]. The tridiagonal random matrices of this ensemble have explicit and independent (up to symmetry) entries and their eigenvalues are distributed according to the equilibrium joint probability density function (jpdf) of charged particles in a one dimensional Coulomb gas with electrostatic repulsion, confined in a quadratic potential and subject to a thermal noise at temperature $T = 1/\beta$ for *arbitrary* $\beta > 0$. More precisely, the jpdf of the eigenvalues is given by

$$P_\beta(\lambda_1, \cdots, \lambda_N) = \frac{1}{Z_N^\beta} \prod_{i<j} |\lambda_i - \lambda_j|^\beta \exp(-\frac{1}{4}\sum_{i=1}^N \lambda_i^2)\,. \tag{1}$$

For $\beta = 1$ (respectively $\beta = 2, 4$), this jpdf arises as the joint law of the eigenvalues of the classical Gaussian orthogonal (respectively unitary, symplectic) ensembles, whose linear eigenvalues statistics were extensively studied in the literature (see [2, 3, 4, 5, 6] for a review of RMT and its applications).

The introduction of the tridiagonal random matrices for *arbitrary* β-ensembles has led to considerable progress for the study of linear statistics of the point process with jpdf P_β. They have permitted to prove that the largest eigenvalues converge jointly in distribution to the low-lying eigenvalues of the random Schrodinger operator, also called *stochastic Airy operator*, $-\frac{d^2}{dt^2}+t+\frac{2}{\sqrt{\beta}}b'_t$, restricted to the positive half-line, where b' is a white noise on $\mathbb{R}_+$ [7, 8] (see also [9] for a review on the top eigenvalue statistics of random matrices).

In this paper, we are interested in the limiting marginal distributions of the bottom eigenvalues of the stochastic Airy operator when the parameter β tends to 0. As mentioned above, the stochastic Airy operator appears as the continuous scaling limit of β-ensembles at the edge of the spectrum and we shall see that the question we investigate is in fact related to the largest eigenvalues statistics of β-ensembles when the β index scales with the dimension N such that $\beta_N \to 0$ when $N \to +\infty$.

A somehow related question was investigated in [10] (see also [11]) where the authors consider the empirical eigenvalues density of β-ensembles in the limit of large dimension N and with $\beta = 2c/N$. The limiting family of probability density $\{\rho_c, c \geq 0\}$ is computed explicitly in terms of Parabolic cylinder functions and is proved to interpolate continuously between the Gaussian shape (obtained for $c = 0$) and the Wigner semicircle shape (which is recovered when $c \to +\infty$). The β-Wishart ensemble was handled similarly in [12].

The question of the characterization of an interpolation between the Tracy-Widom β distribution (which governs the typical fluctuations of the top eigenvalue as $N \to \infty$ with $\beta > 0$ fixed) and the Gumbel distribution (which governs the typical fluctuations of the maximum of independent Gaussian variables – corresponding to the $\beta = 0$ case) was raised in [10] and [13].

We answer this question proving that the Tracy-Widom β distribution converges weakly (when properly rescaled and centered) to the Gumbel distribution when β goes to 0. This is the content of Theorem 4.2. We use the characterization of the marginal distributions of the low lying

eigenvalues of the stochastic Airy operator in terms of the explosion times process of the associated Riccati diffusion [8]. We show that the empirical measure of the explosion times converges weakly in the space of Radon measures to an inhomogeneous Poisson point process on $\mathbb{R}_+$ with explicit intensity. The weak convergence of all the marginal distributions of the second, third, etc eigenvalues can be readily deduced. Although we expect the minimal eigenvalues to have Poissonian statistics in the small β limit, the convergences of the joint distribution of the k bottom eigenvalues for any fixed index k seem to be difficult to prove as there is not a simple characterization of this law in terms of a single diffusion. It is still characterized in this setting in terms of a family of coupled diffusions but the interaction between those diffusions is complex and makes the analysis difficult (see Figures 1 and 2 below).

As an application, we investigate (with heuristic and numerical arguments) the weak convergence of the top eigenvalue of β_N-ensembles in the double scaling limit $N \to +\infty$ and $\beta_N \to 0$. We revisit the ideas of [7] which proposes that tridiagonal random matrices of β-ensembles are properly viewed as finite difference schemes of the stochastic Airy operator. From our heuristic discussion in section 5, this relation seems to remain valid also in the regime $\beta_N \to 0$ and permits to establish the weak convergence of the top eigenvalue of β_N ensembles to the Gumbel distribution, for any sequence β_N such that

$$1 \gg \beta_N \gtrapprox \frac{\ln N}{N}\,.$$

We explicit the scaling and centering of this convergence which are in fact the same as in the convergence of the Tracy-Widom β distribution to the Gumbel distribution. Again, the allied results on the second, third, fourth, etc eigenvalues can also be derived from our former results on the Stochastic Airy ensemble (SAE_β for short).

We finally mention that our derivation does not cover the case where $\beta \sim 1/N$, which is highly interesting as the interpolation for the empirical spectral distribution occurs on this range of β_N [10]. For β_N decreasing as slowly as $\ln N/N$ or even more slowly, the typical fluctuations of the top eigenvalues seem to enter the Gumbel regime in the sense that the centerings and scalings are found to be the same as in the classical setting of independent Gaussian variables.

To facilitate the reading, let us draw up a short outline of the paper. In Section 2, we give a brief review on Stochastic Airy Ensemble (SAE_β for short), recalling in particular the correspondence between the law of the eigenvalues and the law of the explosion times process of the associated Riccati diffusion established by [8].

In section 3, we revisit the classical problem of the exit time from a domain of a diffusion which evolves in a *stationary* potential. This problem is the stationary counterpart of our main study and turns to be useful in the next sections to approximate the non stationary Riccati diffusion and in particular its explosion times. We provide a simple characterization of the law of the exit time which permits to prove its weak convergence to an exponential distribution, when the trap gets very deep in comparison to the noise. Then we consider the explosion times process of the stationary Riccati diffusion and we prove that it converges to a (homogeneous) Poisson point process. Finally, we discuss in view of subsection 4.3 the Fokker Planck equation which relates between the transition probability distribution of a diffusion and the flux of probability in the system.

In Section 4, we state our results on the convergence of the distribution of the minimal eigenvalue of the stochastic Airy operator, i.e. of the Tracy-Widom β law, to the Gumbel law. This is

straightforwardly deduced from the convergence of the explosion time process of the diffusion. The convergences of the marginal distributions of the other neighboring minimal eigenvalues can be deduced as well. The proofs of those results appear in subsection 4.2. At the end of the section, we compute with a perturbative *heuristic* method the empirical eigenvalue density of the stochastic Airy operator as $\beta \to 0$ on the macroscopic scale, i.e. without any zooming in the minimal eigenvalues scaling region.

As an application of our results, we discuss in section 5 the marginal statistics of the minimal eigenvalues of β_N-ensembles in the double scaling limit $\beta_N \to 0, N \to +\infty$. We conjecture that they can be readily deduced from our results since the tridiagonal random matrices, when zooming in the edge scaling region, are well approximated, even when $\beta_N \to 0$, by the stochastic Airy operator. Some technical computations and auxiliary proofs are gathered into appendices.

2. Stochastic Airy operator: A short review

In the first subsection, we recall the recent results on the spectral statistics of the stochastic Airy operator obtained in [7, 8, 16, 17, 18]. Then, we give a characterization of the first exit time (also called blow-up time) of the non homogeneous *Riccati* diffusion associated to the stochastic Airy operator.

The stochastic Airy operator $\mathcal{H}_\beta$ is defined formally (see [8, Section 2] for a precise definition) for $\beta > 0$ and $t \geq 0$ as

$$\mathcal{H}_\beta := -\frac{d^2}{dt^2} + t + \frac{2}{\sqrt{\beta}}\, B'_t \tag{2}$$

where B'_t is a white noise on $\mathbb{R}_+$. Following [8], denote by S^* the space of functions f satisfying $f(0) = 0$ and $\int_0^\infty (f')^2 + (1+t)f^2 < \infty$. We will say that $(\phi_\lambda, \lambda) \in S^* \times \mathbb{R}$ is an eigenfunction/ eigenvalue pair for $\mathcal{H}_\beta$ if $||\phi_\lambda||_2 = 1$ and if

$$\phi''_\lambda(t) = (t - \lambda)\phi_\lambda(t) + \frac{2}{\sqrt{\beta}}\, \phi_\lambda(t)\, B'_t \tag{3}$$

holds for all $t \geq 0$ in the following integration by part sense,

$$\phi'_\lambda(t) - \phi'_\lambda(0) = \int_0^t (s - \lambda)\, \phi_\lambda(s)\, ds + \frac{2}{\sqrt{\beta}} \left(B_t\, \phi_\lambda(t) - \int_0^t B_s\, \phi'_\lambda(s)\, ds \right). \tag{4}$$

Ramírez, Rider and Virág proved in [8, Theorem 1.1] that, almost surely, for each $k \geq 0$, the set of eigenvalues of $\mathcal{H}_\beta$ has a well defined $(k+1)$ st lowest element denoted Λ_k^β. Furthermore the law of any eigenvalue Λ_k^β with given index k is characterized in term of the explosion times of a stochastic process $(X_\lambda(t))_{t \geq 0}$ defined through the Riccati change of functions $X_\lambda(t) := \phi'_\lambda(t)/\phi_\lambda(t)$. This stochastic process is a diffusion process whose initial condition and Langevin equation are obtained from the Dirichlet boundary condition $\phi_\lambda(0) = 0$ [1] and (3)

$$X_\lambda(0) = +\infty \quad \text{and} \quad dX_\lambda(t) = \left(t - \lambda - X_\lambda(t)^2\right)\, dt + \frac{2}{\sqrt{\beta}}\, dB_t \quad \text{for} \quad t \geq 0, \tag{5}$$

[1] Necessarily $\phi'_\lambda(0) \neq 0$ (otherwise ϕ_λ is identically 0) and the two signs of $\phi'_\lambda(0), \phi_\lambda(0+)$ are equal.

where B_t is a standard Brownian motion. Solutions of (5) may blow up to $-\infty$ at finite times, as will happen whenever ϕ_λ vanishes. In this case, the diffusion X_λ immediately restarts at $+\infty$ at that time in order to continue the solution corresponding to the Langevin equation (5).

The authors of [8] prove that the operator $\mathcal{H}_\beta$ satisfies a Sturm-Liouville like property in the sense that *the number of eigenvalues of $\mathcal{H}_\beta$ at most λ is equal to the total number of explosions of the diffusion $(X_\lambda(t))$ on $\mathbb{R}_+$* [2].

Before explaining the idea behind this key relation, let us preliminary state the so-called *increasing property* of the coupled family of diffusions $(X_\lambda)_{\lambda\in\mathbb{R}}$, which will be used many times in the paper. The increasing property is rather intuitive and can be enunciated as follows: if $\lambda' \leq \lambda$, then the number of explosions of X_λ is stochastically bounded above by the number of explosions of $X_{\lambda'}$ on any compact interval $[0, T]$. Equivalently, the diffusion $X_{\lambda'}$ remains below X_λ until its first explosion time and can not cross the trajectory of X_λ from below to above. Intuitively this property is rather obvious since the drift of $X_{\lambda'}$ pulls stronger downside than the one of X_λ. Note however that the comparison theorem for sdes (see [38, Proposition 2.18] or [39, Theorem (3.7), Chapter IX]) does not apply directly (the drifts are not Lipschitz) and one needs to use a localization argument before applying it.

Now we explain the key relation. First, we will look at the operator $\mathcal{H}_\beta^L$ defined on the set of functions with support in the truncated interval $[0, L]$ with Dirichlet boundary conditions at both endpoints. This truncated operator is shown to approximate closely the operator $\mathcal{H}_\beta$ when $L \to \infty$ in a precise topology [8]. At fixed L, the definitions of the eigenfunctions and their associated Riccati transform imply that $\lambda \in \mathbb{R}$ is an eigenvalue of the operator $\mathcal{H}_\beta^L$ *if and only if* the Riccati diffusion X_λ explodes precisely at the end point L. We can deduce that the number of eigenvalues of the operator $\mathcal{H}_\beta^L$ smaller than λ equals the number of real numbers $\lambda' \leq \lambda$ such that $(X_{\lambda'})$ blows up exactly in L. Indeed, if λ' slowly decreases from λ to $-\infty$, we can see from the *increasing property* and the continuity of the whole trajectory $(X_{\lambda'}(t))$ with respect to λ' that the explosion waiting times of $X_{\lambda'}$ will simultaneously get longer and longer to finally diverge to $+\infty$ when λ' gets too small for the diffusion to explode in $\mathbb{R}_+$ (see Fig. 1 and 2). In particular, any explosion of X_λ which occurs before time L will slowly translate continuously to occur at some point exactly in L for some $X_{\lambda'}$ with $\lambda' \leq \lambda$. Thus the number of eigenvalues of $\mathcal{H}_\beta^L$ smaller than λ is equal to the number of explosions of X_λ on the interval $[0; L]$. Taking $L \to \infty$ finally establishes this property for the operator $\mathcal{H}_\beta$.

The marginal laws of the eigenvalues are characterized in a rather simple way in term of the distribution of the explosion times process of one single Riccati diffusion X_λ as a function of the parameter λ. We shall use this characterization several times throughout the paper.

In particular, the cumulative distribution of the lowest eigenvalue Λ_0^β satisfies

$$\mathbb{P}\left[\Lambda_0^\beta < \lambda\right] = \mathbb{P}\left[X_\lambda(t) \text{ blows up to } -\infty \text{ in a finite time}\right] . \tag{6}$$

And for $k \geq 1$, we have

$$\mathbb{P}\left[\Lambda_k^\beta < \lambda\right] = \mathbb{P}\left[X_\lambda(t) \text{ blows up to } -\infty \text{ at least } k+1 \text{ times}\right] . \tag{7}$$

[2] It is also proved that the diffusion X_λ has only a finite number of explosions in $\mathbb{R}_+$ and thus that the number of eigenvalues below λ is almost surely finite.

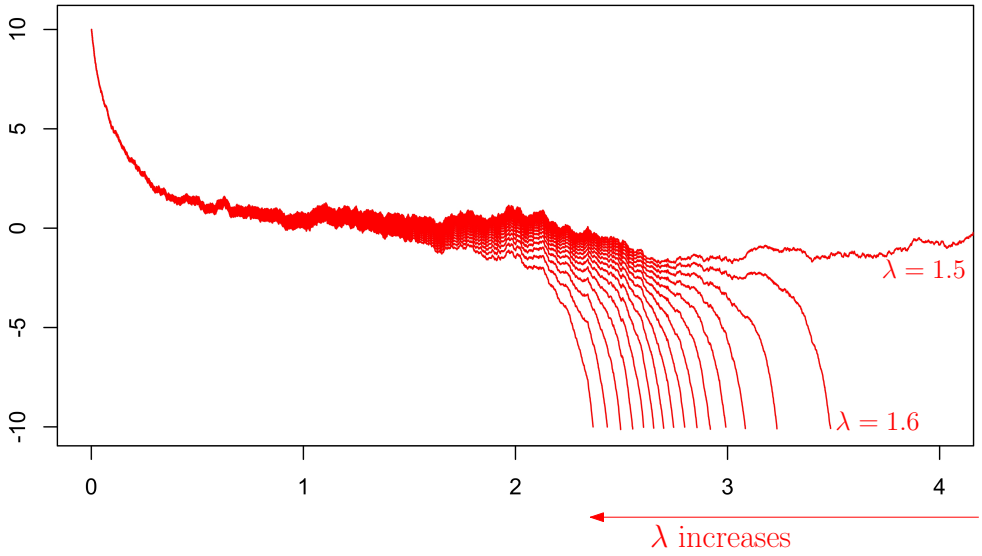

FIGURE 1. Simulated paths of diffusions X_λ driven by the same Brownian motion for several values of λ. The values of λ are between 1.5 and 3 on a grid of mesh 10^{-1}. We took $\beta = 4$. The smallest eigenvalue $-TW(4)$ of the Airy operator is between 1.5 (no explosion) and 1.6 (at least one explosion) on this event.

The joint distribution of the k bottom eigenvalues can also be characterized in this setting but in a more complicated way from the law of the family of coupled diffusions $((X_\lambda(t))_{t\geq 0}, \lambda \in \mathbb{R})$ all satisfying (5) with the *same* driving Brownian motion $(B(t))_{t\geq 0}$. For instance, with $\lambda' < \lambda$,

$$\begin{aligned} &\mathbb{P}[\Lambda_{k-1}^\beta \leq \lambda, \Lambda_k^\beta > \lambda'] \\ &= \mathbb{P}\Big[X_\lambda \text{ blows up to } -\infty \text{ at least } k \text{ times }, X_{\lambda'} \text{ blows up to } -\infty \text{ at most } k \text{ times}\Big]. \end{aligned} \tag{8}$$

Although the two diffusions X_λ and $X_{\lambda'}$ are driven by the same Brownian motion, they have different drifts and (to our knowledge) this makes the right hand side probability of (8) difficult to estimate in practice.

In view of [8, Theorem 1.1] which states that the top eigenvalue of the tridiagonal matrices properly centered and scaled converges in law to $-\Lambda_0^\beta$, the Tracy-Widom β law ($TW(\beta)$ for short) has been defined for general $\beta > 0$ as the law of the random variable $-\Lambda_0^\beta$.

Using the characterization (6) in terms of the diffusion X_λ, the first two leading terms of the right large deviation tail of the Tracy-Widom β distribution were rigorously obtained by the second author and Virág in [16]. Those two first terms were also computed by Forrester in [18] using a different method. Finally, the right tail of the Tracy-Widom β law was computed to all orders by

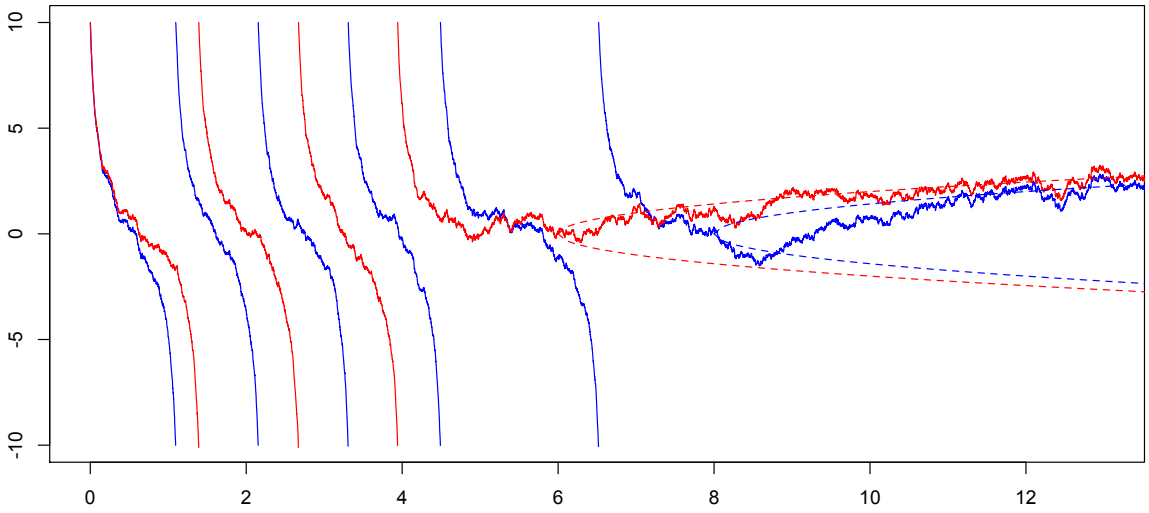

FIGURE 2. (Color online) Simulated paths of two diffusions X_λ and $X_{\lambda'}$ driven by the same Brownian motion with $\lambda = 8$ (blue curve) and $\lambda' = 6$ (red curve), for $\beta = 4$. We add in dashed lines the corresponding parabolas where the drifts cancel. When $t \to +\infty$, the diffusions converge to the upper part of their respective parabolas. On this event, we have $\Lambda_4^\beta < 8 < \Lambda_5^\beta$ and $\Lambda_2^\beta < 6 < \Lambda_3^\beta$.

Borot and Nadal in [17] using heuristic arguments. Their result valid in the $\lambda \to +\infty$ limit reads

$$\mathbb{P}\left[TW(\beta) > \lambda\right] = \frac{\Gamma\left(\frac{\beta}{2}\right)}{(4\beta)^{\frac{\beta}{2}}2\pi}\lambda^{-\frac{3\beta}{4}}\exp\left(-\frac{2}{3}\beta\,\lambda^{3/2}\right)\exp\left(\sum_{m=1}^{+\infty}\frac{\beta}{2}R_m(\frac{2}{\beta})\lambda^{-3m/2}\right), \tag{9}$$

where the R_m are (explicit) polynomials of degree at most $m+1$.

It is in fact possible to obtain an analytical characterization of the law of the minimal eigenvalue Λ_0^β. In [19, Theorem 1.7], the authors prove that the cumulative distribution of Λ_0^β is the unique solution of a boundary value problem. Using similar techniques, we can prove slightly more: The *Laplace transform* of the first explosion time is the solution of a boundary value problem as well. See Remark 3.2 for more details. But it turns out that this boundary value problem is hard to analyse, even in the limit $\beta \to 0$, and this approach in spite of its robustness did not permit us to prove Theorem 4.1.

3. TRAPPING OF A DIFFUSION IN A STATIONARY WELL

We now revisit the classical problem of the exit time from a domain of a diffusion which evolves in a stationary potential. The *small noise limit* was widely studied in the literature (see e.g. [20, 21, 22]) using large deviation theory. The results we derive in this section are not new and hold in a general setting.

The diffusion Y_a considered below in the small noise limit appears in the study [15, 23, 24, 25] of the law of the ground state (minimal eigenvalue) of the *Hill*'s operator, defined as

$$\mathcal{G}_L := -\frac{d^2}{dt^2} + B'(t)$$

where $B'(t)$ is as before a white noise on the segment $[0, L], L > 0$. In this context, due to the stationarity (absence of the linear t term), we need to restrict to a finite perimeter $L > 0$ and we work with Dirichlet boundary conditions $\psi_a(0) = \psi_a(L) = 0$ for the eigenvectors such that $\mathcal{G}_L\psi_a = a\psi_a$. As in the previous section, the law of the minimal eigenvalues A_k^L of the operator $\mathcal{G}_L$ can be characterized in terms of the family of diffusions $(Y_a(t))_{t\geq 0}$ obtained through the Riccati transformation and defined by

$$\begin{cases} dY_a(t) = (a - Y_a(t)^2)\, dt + dB(t) \quad \text{for} \quad t \geq 0\,, \\ Y_a(0) = y\,. \end{cases} \tag{10}$$

where $a \in \mathbb{R}$ is a fixed parameter and $y \in \mathbb{R} \cup \{+\infty\}$ and the diffusion Y_a immediately restarts from $+\infty$ whenever an explosion occurs. The characterization of the marginal distribution of the minimal eigenvalue with index k now reads

$$\mathbb{P}\left[A_k^\beta < a\right] = \mathbb{P}\left[Y_a \text{ blows up to } -\infty \text{ at least } k+1 \text{ times before time } L\right]\,. \tag{11}$$

The forthcoming study of the exit time distribution of the diffusion Y_a in the small noise limit will permit us to analyze the (marginal) distributions of the minimal eigenvalues of the stochastic operator $\mathcal{G}_L$ in the limit $L \to \infty$ (which replaces the limit $\beta \to 0$ in the Airy case).

In view of subsection 4.3, we also provide a discussion in subsection 3.4 on the transition probability density of the diffusion Y_a, and relate this transition pdf with the limiting density of state of the operator $\mathcal{G}_L$. This section corresponds to the stationary counterpart of the main study of this paper. The problem is of course easier to solve for the Hill's operator $\mathcal{G}_L$ than for the Airy operator $\mathcal{H}_\beta$ thanks to the stationarity.

We will actually see later that the law of the minimal eigenvalues of the operators $\mathcal{G}_L$ and $\mathcal{H}_\beta$ are similar in the respective limits $L \to \infty$ and $\beta \to 0$.

3.1. **Definition.** In this section, $\mathbb{P}$ denotes the law of the diffusion Y_a when $y = +\infty$ and $\mathbb{P}_y[\cdot] := \mathbb{P}[\cdot | Y_a(0) = y]$ is the law of the diffusion Y_a conditionally on $Y_a(0) = y$. In particular, $\mathbb{P} = \mathbb{P}_{+\infty}$.

We are interested in the distribution of the exit time (blowup time) $\zeta := \inf\{t \geq 0 : Y_a(t) = -\infty\}$ and in particular in its limit in law when $a \to +\infty$.

The diffusion evolves in a potential $V(y) := -ay + \frac{y^3}{3}$ which presents a local minimum in $y = \sqrt{a}$ and a local maximum in $y = -\sqrt{a}$. The potential barrier $\Delta V = \frac{4}{3}\, a^{3/2}$ gets very large when $a \to +\infty$ while the noise remains constant (see Figure 3).

If the particle starts from above the potential well (from $y = +\infty$, say) at time 0, then it has to cross a very large barrier of size $\Delta V = 4\, a^{3/2}/3$ with a deep well in $\sqrt{a}$. From Kramer's theory [27], we expect the exit time to be distributed according to an exponential law with parameter $\sim \exp(-2\Delta V)$. We give a simple proof of this result by a Laplace transform method (see also [15]).

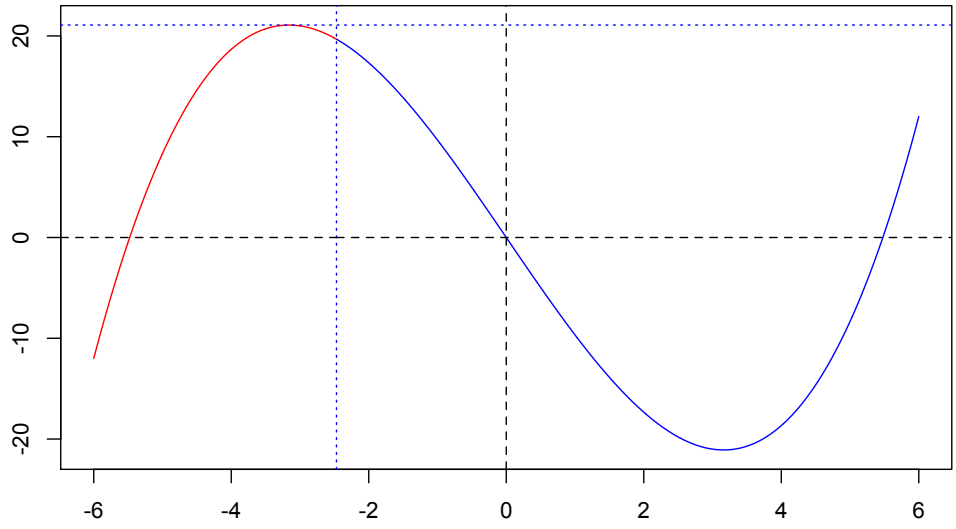

FIGURE 3. The potential $V(y)$ as a function of y.

3.2. **Exit time distribution.** Let us introduce the Laplace transform of the first exit time ζ of the diffusion $Y_a(t)$ with initial position y

$$g_\alpha(y) := \mathbb{E}_y[e^{-\alpha\zeta}]\,. \tag{12}$$

The following proposition characterizes the Laplace transform g_α as the unique solution to a boundary value problem. The proof of this Proposition cand be found in Appendix A.

Proposition 3.1. *Let $\alpha > 0$. Then the function g_α defined in* (12) *is the unique bounded and twice continuously differentiable solution of the boundary value problem*

$$\frac{1}{2}g''_\alpha - (y^2 - a)\,g'_\alpha = \alpha\, g_\alpha \tag{13}$$

satisfying the additional boundary condition

$$g_\alpha(y) \to 1 \quad \text{when} \quad y \to -\infty\,. \tag{14}$$

In addition, it satisfies the fixed point equation

$$g_\alpha(y) = 1 - 2\alpha \int_{-\infty}^{y} dx \int_{x}^{+\infty} ds \exp\left(2\,a(s-x) + \frac{2}{3}(x^3 - s^3)\right)\, g_\alpha(u)\,. \tag{15}$$

This proposition permits to derive a lot of information about the behavior of the diffusion. A first natural question one could ask is: what is the probability that the diffusion (Y_a) explodes in a finite time? The answer is straightforward from Proposition 3.1 by taking $\alpha = 0$ and noting that

$$g_0(y) := \lim_{\alpha\downarrow 0} \mathbb{E}_y[e^{-\alpha\zeta}] = \mathbb{P}_y[\zeta < +\infty] = 1\,.$$

The mean exit time of the diffusion starting at time $t = 0$ from position y can also be computed explicitly from Proposition 3.1. It suffices to differentiate $g_\alpha(y)$ with respect to α and then take $\alpha = 0$. Eq. (13) then transforms into a second order differential equation which can be solved explicitly. We find the mean exit time $m(a, y)$ starting from position y

$$m(a,y) = 2\int_{-\infty}^{y} dx \int_{x}^{+\infty} du \exp\left(2a(u-x) + \frac{2}{3}(x^3 - u^3)\right) . \tag{16}$$

For $y = +\infty$, this expression $m(a, +\infty)$ (simply denoted as $m(a)$ in the sequel) simplifies, after two changes of variables and a further Gaussian integration, in a single integral expression

$$m(a) = \sqrt{2\pi} \int_0^{+\infty} \frac{dv}{\sqrt{v}} \exp\left(2av - \frac{1}{6}\, v^3\right) . \tag{17}$$

This explicit integral form for $m(a)$ is convenient to determine its asymptotic when $a \to +\infty$ using the saddle point method. This is done in Appendix C and the estimate (85) will be useful throughout the paper.

All the moments $\mathbb{E}_y[\zeta^n]$ for $n \in \mathbb{N}$ of the exit time ζ (when the diffusion starts from y at time 0) can also be derived by iterating this argument (see Appendix A). In particular all the moments are finite for all starting point y and fixed a. They actually satisfy $\mathbb{E}_y[\zeta^n] \leq n!\, m(a)^n$.

Remark 3.2. *As mentioned in the previous paragraph, it is interesting to note that we can derive a similar theorem for the Laplace transform of the first exit time of the non-stationary diffusion under consideration in the latter section* X_λ (5)*. Indeed, let us denote by* $\zeta := \zeta(x,t)$ *the first explosion time of the diffusion process* $(X_\lambda(s))_{s\geq t}$ *conditioned to start at time* t *in* x *and let* $f_\alpha(x,t)$ *for* $\alpha > 0$ *be its Laplace transform. For* $\alpha = 0$*, the Laplace transform is extended by continuity*

$$f_0(x,t) := \lim_{\alpha \downarrow 0} f_\alpha(x,t) = \mathbb{P}[\zeta(x,t) < +\infty] . \tag{18}$$

Similarly to Proposition 3.1, we can show that the function $f_\alpha(x,t)$ *is the unique solution of the following boundary value problem:*

$$\frac{\partial f_\alpha}{\partial t} + \left(t - \lambda - x^2\right) \frac{\partial f_\alpha}{\partial x} + \frac{2}{\beta} \frac{\partial^2 f_\alpha}{\partial x^2} = \alpha\, f_\alpha , \tag{19}$$

satisfying the additional boundary conditions

$$f_\alpha(x,t) \to 1 \quad \text{when} \quad x \to -\infty \quad \text{with} \quad t \quad \text{fixed} , \tag{20}$$

$$f_\alpha(x,t) \to 0 \quad \text{when} \quad x, t \to +\infty \quad \text{together.} \tag{21}$$

Note that this characterization is also true for $\alpha = 0$*: We recover the result of* [19, Theorem 1.7 (ii)] *which permits to find the cumulative distribution of the minimal eigenvalue thanks to Eq.* (6) *which rewrites as*

$$\mathbb{P}[\Lambda_0^\beta > \lambda] = \lim_{x \to +\infty} \mathbb{P}[\zeta(x,0) = +\infty] .$$

The generalization given here has the advantage to contain the full information about the law of the first exit time. Nevertheless, it did not permit us to derive the convergence of the point process of the explosion times.

We can now establish the weak convergence when $a \to +\infty$ of the explosion waiting time ζ when rescaled by its mean $m(a)$ to the exponential distribution with parameter 1. The result is

valid for a large range of starting points, mainly all the points which are *in* the potential well (*i.e. above* the local maximum of the potential).

Theorem 3.3. *Let $f : \mathbb{R} \to \mathbb{R}$ such that $a^{1/4}(f(a) + a^{1/2}) \to_{a\to+\infty} +\infty$. Then,*

$$\sup_{y\geq f(a)} \left| g_{\alpha/m(a)}(y) - \frac{1}{1+\alpha} \right| \longrightarrow 0\,.$$

In particular, for any $y \geq f(a)$ the first blowup time to $-\infty$ of the diffusion $(Y_a(t))$ starting from position y at time 0, rescaled as $\zeta/m(a)$, converges weakly when $a \to +\infty$ to an exponential law with parameter 1.

Remark 3.4. *We recover here the prediction of Kramer's theory. Indeed the exit time (starting from a point inside the well) is distributed in the limit $a \to +\infty$ according to an exponential law with parameter $m(a)^{-1}$, for which we have found a logarithmic equivalent $m(a)^{-1} \asymp \exp(-8/3\, a^{3/2})$ in Appendix C.*

From Theorem 3.3 and the characterization (11) of the law minimal eigenvalues of the operator $\mathcal{G}_L$, we get back the result due to McKean in [25] about the fluctuations of the ground state of the Hill's operator.

Corollary 3.5. *In the limit $L \to \infty$, the fluctuations of the minimal eigenvalue A_0^L of the stochastic Hill operator $\mathcal{G}_L$ are governed by the Gumbel distribution. More precisely, we have the convergence in law as $L \to \infty$,*

$$-2 \cdot 3^{1/3} (\ln L)^{1/3} \left[A_0^L + \left(\frac{3}{8} \ln \frac{L}{\pi} \right)^{2/3} \right] \Rightarrow e^{-x} \exp(-e^{-x})\, dx\,.$$

Proof.

Using the asymptotic estimate (85) for $m(a)$ as $a \to +\infty$, the convergence in law of the exit time towards the exponential distribution provided by Theorem 3.3 and the characterization (11), it is easy to see that

$$\mathbb{P}\left[A_0^L \leq -\left(\frac{3}{8} \ln \frac{L}{\pi} \right)^{2/3} + \frac{1}{2 \cdot 3^{1/3}} \frac{x}{(\ln L)^{1/3}} \right] \to_{L\to+\infty} 1 - e^{-e^x}\,,$$

which yields the result. □

3.3. **Exit times point process.** Although the diffusion $(Y_a(t))$ blows up to $-\infty$ in a finite time almost surely, we can again define the trajectory for all time $t \geq 0$ by restarting the diffusion in $+\infty$ immediately after any blow up to $-\infty$.

Endowed with this new definition, we introduce the empirical measure μ_a of the explosion times $\zeta_1 < \zeta_2 < \zeta_3 \ldots$ (with a further rescaling) defined for any Borel set B of $\mathbb{R}_+$ as

$$\mu_a(B) = \sum_{i=1}^{+\infty} \delta_{\zeta_i/m(a)}(B)\,. \tag{22}$$

In particular, the random variable $\mu_a([0,t])$ is the number of explosions of the diffusion in the interval $[0, m(a)t]$.

We have seen (see Theorem 3.3) that the sequence of the waiting time $(\zeta_i - \zeta_{i-1})/m(a)$ until the next increment for μ_a converges in law when $a \to +\infty$ to an exponential distribution with parameter 1. It is therefore easy to deduce the convergence of the point process μ_a to a Poisson point process on $\mathbb{R}_+$. More precisely, we have the following result.

Theorem 3.6. *Let Y_a the diffusion* (10) *with initial position $Y_a(0) = +\infty$. The associated explosion times point process μ_a* (22) *converges weakly (in the space of Radon measures on $\mathbb{R}_+$ equipped with the topology of vague convergence* [28]*) when $a \to +\infty$ to a Poisson point process with intensity 1 on $\mathbb{R}_+$.*

The end of this section is devoted the proof of Theorem 3.6.

Proof of Theorem 3.6.

We will use the useful criterion from Kallenberg [28], which states that it is sufficient to prove that, for any finite union I of disjoint and bounded intervals, we have the following convergences when $a \to +\infty$,

$$\mathbb{E}[\mu_a(I)] \longrightarrow |I|\,, \tag{23}$$

$$\mathbb{P}[\mu_a(I) = 0] \longrightarrow \exp(-|I|)\,, \tag{24}$$

where $|I|$ denotes the length of the set I.

Towards (23), by linearity we just need to prove that $\mathbb{E}[\mu_a[0,t]] \to t$. The advantage here is that the starting point of the diffusion Y_a at time 0 is $+\infty$ so that the waiting time $\zeta/m(a)$ until the first explosion converges weakly to an exponential distribution according to Theorem 3.3. We have

$$\mathbb{E}[\mu_a[0,t]] = \sum_{k=0}^{+\infty} \mathbb{P}\left[\mu_a[0,t] \geq k\right] = 1 + \sum_{k=1}^{+\infty} \mathbb{P}\left[\frac{\zeta_k}{m(a)} \leq t\right]\,,$$

where ζ_k is the k th exit time of the diffusion Y_a started at $+\infty$. The strong Markov property implies that, for each k, the random variable $\zeta_{k+1} - \zeta_k$ is independent of $(\zeta_1, \zeta_2 - \zeta_1, \ldots, \zeta_k - \zeta_{k-1})$ and has the same distribution as ζ_1. It is then easy to deduce that for any fixed k, $\zeta_k/m(a)$ converges in law when $a \to +\infty$ to the Gamma distribution $\Gamma(k,1)$ with shape and scale parameter k and 1 [Recall $\Gamma(k,1)$ is simply the law of a sum of k independent exponential random variables with parameter 1]. One can easily prove existence of a constant $C > 0$ independent of k such that $\mathbb{P}[\zeta_k/m(a) \leq t] \leq C/k^2$ using for instance Chebyshev's inequality since we have $\mathbb{E}[(\zeta_i - \zeta_{i-1})^2/m(a)^2] \leq 1$ for all $i \leq k$ or the Cramer's large deviation principle which would give a much stronger bound. The bounded convergence theorem finally applies and gives, when $a \to +\infty$,

$$\mathbb{E}[\mu_a[0,t]] \longrightarrow 1 + \sum_{k=1}^{+\infty} \mathbb{P}\left[\Gamma(k,1) \leq t\right] = t. \tag{25}$$

The second equality (24) is proved with the same idea. For any $k \geq 1$, using the strong Markov property for the diffusion Y_a, we easily prove that the first k explosions times converge jointly in law to the first k occurrence times $\xi_1 < \xi_2 < \cdots < \xi_k$ of a Poisson point process with intensity 1 (the increments are independent and each of them converges to an exponential distribution

thanks to Theorem 3.3) *i.e.*

$$\frac{1}{m(a)}(\zeta_1, \zeta_2, \dots, \zeta_k) \Rightarrow_{a\to+\infty} (\xi_1, \xi_2, \dots, \xi_k)\,. \tag{26}$$

The convergence (24) follows by proving first that we can consider only the first k explosion times for k large enough using a large deviation argument as above (in order to bound the probability of having more than k explosion times before time $m(a)t$) and then by using the convergence (26) on the overwhelming event $\{\zeta_k > m(a)t\}$. □

3.4. **Transition probability density.** We aim at describing also the probability density $p(y,t)$ of the diffusion $Y_a(t)$ at time t and other related quantities. The main tool is the Fokker Planck equation which gives the evolution of $p(y,t)$ and writes as

$$\frac{\partial p}{\partial t} = \frac{\partial}{\partial y}\left[(y^2 - a)\, p(y,t) + \frac{1}{2}\frac{\partial^2}{\partial y^2} p(y,t)\right]\,. \tag{27}$$

This equation takes the form of a *continuity equation*. Indeed, introducing the flux

$$j(y,t) := (y^2 - a)\, p(y,t) + \frac{1}{2}\frac{\partial^2}{\partial y^2} p(y,t)\,,$$

Eq. (27) rewrites as

$$\frac{\partial p}{\partial t} = \frac{\partial j}{\partial y}\,.$$

The equilibrium of the system can be characterized by finding the stationary solution $p_0(y)$ to the Fokker Planck equation (27) which satisfies

$$\frac{1}{2}\, p_0'(y) + (y^2 - a)\, p_0(y) = J_0\,, \tag{28}$$

where J_0 is a constant, which does not depend on y or t. We can solve the ode Eq. (28) explicitly and find the constant J_0 using the additional normalization constraint $\int_{\mathbb{R}} p_0 = 1$. We obtain

$$p_0(y) = 2\, J_0(a) \int_{-\infty}^{y} du\, \exp\left[2a(y-u) + \frac{2}{3}(u^3 - y^3)\right]\,, \tag{29}$$

where

$$J_0(a) = \frac{1}{m(a)} = \frac{1}{\sqrt{2\pi}}\left[\int_0^{+\infty} \frac{dv}{\sqrt{v}}\, \exp\left(2av - \frac{1}{6}\, v^3\right)\right]^{-1}\,. \tag{30}$$

Here we stress that $J_0(a)$ is precisely equal to the inverse of the expected exit time of the diffusion starting from $+\infty$ [23, 24]. In the limit of large L, the number n_L of blow-ups to $-\infty$ of the diffusion $(Y_a(t))_{t\geq 0}$ during the interval $[0; L]$ (or equivalently – using the characterization (11) – the number of $\mathcal{G}_L$-eigenvalues lying below the level a) is proportional (at leading order) to the inverse of the expected exit time of the diffusion,

$$n_L = L\, J_0(a) + O(\sqrt{L})\,. \tag{31}$$

We recover the formula of McKean [25] for the limiting integrated density of states of the operator $\mathcal{G}_L$ introduced above, as $L \to \infty$. The scaling order $\sqrt{L}$ for the fluctuations comes from the central limit theorem.

Let us finally mention that this study corresponds to the $N = 1$ case of a more general model of N interacting particles. In this context, we expect the number of explosions per unit of time to

display a different fluctuation order when N goes to infinity because of the non-trivial interaction. See [29] for more details about the model and this conjecture.

4. Statistics of SAE_β when $\beta \to 0$

We first state our results on the limiting marginal distributions of the minimal eigenvalues of the stochastic Airy operator $\mathcal{H}_\beta$ when $\beta \to 0$. Then, we provide a more global information on the spectrum by computing the empirical eigenvalue density on the macroscopic scale.

Let us introduce a more convenient stochastic linear operator defined for $t > 0$ as

$$\mathcal{L}_\beta := -\frac{d^2}{dt^2} + \frac{\beta}{4}\, t + B'(t)\,.$$

Denoting by $L_0^\beta < L_1^\beta < \cdots < L_k^\beta \cdots$ its eigenvalues, it is easy to show that in law

$$\{L_k^\beta,\, k \in \mathbb{N}\} \overset{(d)}{=} \left(\frac{\beta}{4}\right)^{2/3} \{\Lambda_k^\beta,\, k \in \mathbb{N}\}\,, \tag{32}$$

where $\Lambda_0^\beta < \Lambda_1^\beta < \cdots < \Lambda_k^\beta \cdots$ are the eigenvalues of the operator $\mathcal{H}_\beta$. Indeed, (32) follows after the change of function $\psi(t) = c\phi(t/c)$ with $c = (\frac{4}{\beta})^{1/3}$ in the differential equation (3) satisfied by the eigenfunction ϕ of $\mathcal{H}_\beta$.

The Riccati diffusion Z_ℓ associated to the stochastic linear operator $\mathcal{L}_\beta$ satisfies

$$Z_\ell(0) = +\infty \quad \text{and} \quad dZ_\ell(t) = \left(\frac{\beta}{4}\, t - \ell - Z_\ell(t)^2\right) dt + dB(t) \quad \text{for} \quad t \geq 0 \tag{33}$$

where B is a Brownian motion. The law of the diffusion Z_ℓ is denoted $\mathbb{P}$ and $\mathbb{P}_{z,t}[\cdot] := \mathbb{P}[\cdot | Z_\ell(t) = z]$ is the law of the diffusion $(Z_\ell(s))_{s \geq t}$ conditionally on $Z_\ell(t) = z$.

Recall that when blowing up to $-\infty$ at some time t, the diffusion Z_ℓ immediately restarts in $+\infty$ at this time t and that we have the key relation

$$\mathbb{P}[L_k^\beta \leq \ell] = \mathbb{P}\left[Z_\ell \text{ blows up to } -\infty \text{ at least } k \text{ times in } \mathbb{R}_+\right]\,. \tag{34}$$

4.1. Minimal eigenvalues of $\mathcal{L}_\beta$.

We investigate the minimal eigenvalues statistics of the linear stochastic operator $\mathcal{L}_\beta$ when $\beta \to 0$ and in particular the convergence of the marginal distributions of the bottom eigenvalues.

When $\beta \to 0$, we can check that for any *fixed* ℓ and k, $\mathbb{P}[L_k^\beta \leq \ell] \to 1$. Indeed, if β tends to 0 while ℓ is fixed, the diffusion Z_ℓ defined in (33) converges in law to the diffusion Y_a of the previous section for $a = -\ell$. The probability for the diffusion Z_ℓ to explode k times in $\mathbb{R}_+$ will therefore tend to the corresponding probability for the diffusion Y_a, which is exactly equal to 1. Recalling (34), we have the claim.

Therefore if we look for a non trivial limit in law for the eigenvalue L_k^β when $\beta \to 0$, the parameter ℓ should decrease to $-\infty$ as $\beta \to 0$ and we need to determine the rescaling of ℓ as a function of β. We can actually make the guess $\ell_\beta \sim -\ln(1/\beta)^{2/3}$ using the right tail asymptotic of the Tracy-Widom β distribution (9) and the relation (32) between the laws of L_0^β and $\Lambda_0^\beta = -TW(\beta)$.

In agreement with this heuristic derivation, we fix $x \in \mathbb{R}$ and set

$$\ell_\beta = \ell_\beta(x) := -\left(\frac{3}{8}\ln\frac{1}{\beta\pi}\right)^{2/3} + \frac{1}{2}\frac{1}{3^{1/3}}\left(\ln\frac{1}{\beta}\right)^{-1/3} x\,. \tag{35}$$

We shall prove that the function $x \to \mathbb{P}[L_k^\beta \le \ell_\beta(x)]$ converges to a non trivial cumulative distribution function on $\mathbb{R}$ when $\beta \to 0$.

Similarly to the previous section, we first consider the empirical measure ν_β of the explosions times $(\zeta_k)_{k\in\mathbb{N}}$ of the diffusion $Z_{\ell_\beta(x)}$ after a further (well chosen) rescaling of time. For a Borel set $A \subset \mathbb{R}_+$,

$$\nu_\beta(A) = \sum_{k=1}^{+\infty} \delta_{\beta(\frac{3}{8}\ln(1/\beta))^{1/3}\,\zeta_k}(A)\,. \tag{36}$$

Finally, recall the key point: for almost all x, the number $\nu_\beta(\mathbb{R}_+)$ of explosions in $\mathbb{R}_+$ is equal to the number of eigenvalues smaller than $\ell_\beta(x)$.

Theorem 4.1. *The explosion times point process ν_β associated to the diffusion $Z_{\ell_\beta(x)}$ converges weakly (in the space of Radon measures equipped with the topology of vague convergence* [28]*) when $\beta \to 0$ to a Poisson point process with inhomogeneous intensity $e^x \times e^{-t}\,dt$.*

It readily implies the following convergence: for any $t < t'$, $k \in \mathbb{N}$,

$$\mathbb{P}\left[\nu_\beta\left[t;t'\right] = k\right] \longrightarrow_{\beta\to 0} \exp\left(-e^x\int_t^{t'} e^{-s}\,ds\right)\frac{\left(e^x\int_t^{t'} e^{-s}\,ds\right)^k}{k!}\,.$$

It might be useful to compare the time scale of the measure ν_β with the one used in the definition of μ_a (22). Using the asymptotic formula Eq. 85 given in Appendix C, we can check that

$$m(-\ell_\beta(0))^{-1} = m\Big(\big(\frac{3}{8}\ln(1/(\beta\pi))\big)^{2/3}\Big)^{-1} \sim_{\beta\to 0} \beta\left(\frac{3}{8}\ln\frac{1}{\beta}\right)^{1/3}.$$

In such a way, the two time scales are actually the same up to exchanging the parameters a and $-\ell_\beta(0) = (\frac{3}{8}\ln\frac{1}{\beta\pi})^{2/3}$. More generally, note that for any $x \in \mathbb{R}, t \in \mathbb{R}_+$,

$$m\left(-\ell_\beta(x-t)\right))^{-1} \sim_{\beta\to 0} \beta\left(\frac{3}{8}\ln\frac{1}{\beta}\right)^{1/3} e^{x-t}\,.$$

The scenario described in Theorem 4.1 is therefore fairly simple. When $\beta \to 0$, the diffusion $Z_{\ell_\beta(x)}$ feels the evolution in time due to the linear term $\frac{\beta}{4}t$ in the drift but in a rather trivial way: setting

$$s := \frac{t}{\beta(\frac{3}{8}\ln(1/\beta))^{1/3}}$$

to work in the appropriate time scale, the explosion times process of $Z_{\ell_\beta(x)}$ is somehow the same as the one of a stationary "frozenßystem which evolves with time t *adiabatically* such that the parameter a in the drift of the diffusion Y_a evolves slowly with time t as

$$a := -\ell_\beta(x) + \frac{\beta}{4}s = -\ell_\beta(x-t)\,. \tag{37}$$

Finally let us come to the initial motivation of this paper which comes from interesting questions, recently asked in the literature in [10, 12, 13], about the existences and characterizations of possible distributions in extreme value theory which would interpolate between the Tracy-Widom $\beta = 1, 2, 4$ (maximum of *highly* correlated random variables) and the Gumbel law (maximum of *weakly* correlated random variables).

The following Theorem, obtained as a straightforward application of Theorem 4.1, establishes and describes precisely the progressive deformation of the Tracy-Widom β laws into a Gumbel law when $\beta \to 0$.

Theorem 4.2. *When properly rescaled and centered, the Tracy-Widom β law converges weakly to the Gumbel law. More precisely, when $\beta \to 0$, the following convergence in law holds*

$$2 \cdot 3^{1/3} \cdot \left(\ln \frac{1}{\beta}\right)^{1/3} \left[\left(\frac{\beta}{4}\right)^{2/3} TW(\beta) - \left(\frac{3}{8}\right)^{2/3} \left(\ln \frac{1}{\beta\pi}\right)^{2/3} \right] \Longrightarrow e^{-x} \exp(-e^{-x})\, dx\,.$$

Comparing Theorem 4.2 to Corollary 3.5, we see that the fluctuations of the ground state of the stochastic Hill and Airy operators are very similar, governed by the Gumbel distribution, with the same scalings under the relation $L = 1/\beta$.

The convergences of the marginal distributions of the other minimal eigenvalues with general index k can also be deduced from Theorem 4.1 which implies that

$$\mathbb{P}\left[L_k^\beta \leq \ell_\beta(x)\right] = \mathbb{P}\left[\nu_\beta[0;+\infty] \geq k+1\right] \to_{\beta\to 0} 1 - e^{-e^x} \sum_{i=0}^{k} \frac{(e^x)^i}{i!}\,. \tag{38}$$

We can check that the function of x in the right hand side is indeed a cumulative distribution function and the weak convergence of L_k^β follows. Note also that the right hand side of (38) corresponds exactly to the cumulative distribution of the point with index k of a Poisson point process on $\mathbb{R}$ with intensity $e^x dx$.

It is therefore natural to conjecture that the empirical measure ρ_β of the eigenvalues of $\mathcal{L}_\beta$ defined, for any A Borel set of $\mathbb{R}$, as

$$\rho_\beta(A) = \sum_{k=0}^{+\infty} \delta_{L_k^\beta}(A)\,, \tag{39}$$

converges weakly when considered on the *microscopic scaling region of the minimal energies* as in this paragraph to a Poisson point process on $\mathbb{R}$ with intensity $e^x dx$, when $\beta \to 0$.

Let us finally consider the matching between the microscopic (where one zooms in the measure ρ_β on the microscopic region of the bottom eigenvalues) and the macroscopic regime where the sets $A \subset \mathbb{R}$ remains fixed as $\beta \to 0$.

In section 4.3, we compute asymptotically when $\beta \to 0$ the empirical spectral measure on the macroscopic scale. Our result reads

$$\rho_\beta(A) = \frac{4}{\beta} \int_A J_0(\ell) d\ell + O(1) \tag{40}$$

where $O(1)$ is a constant of order 1 as $\beta \to 0$. In particular, $\rho_\beta(]-\infty;\ell])$ is the number of eigenvalues below the level ℓ. Using the result obtained in Appendix D, — where we check

that the second order term $O(1)$ in (40) remains negligible compared to the leading order when $\beta \to 0$ and $A =]-\infty; \ell_\beta(x)]$— we see that for any fixed x,

$$\rho_\beta\left(]-\infty; \ell_\beta(x)]\right) \sim_{\beta\to 0} e^x\,,$$

where we have also used the asymptotic (87).

Therefore, if one considers a Poisson point process $\mathcal{P}$ on $\mathbb{R}$ with intensity $4J_0(\cdot)/\beta$, then the probability that the $k+1$ bottom points of $\mathcal{P}$ are below the level $\ell_\beta(x)$ converges when $\beta \to 0$ as

$$\mathbb{P}\left[\mathcal{P}\left(]-\infty; \ell_\beta(x)]\right) \geq k+1\right] \longrightarrow 1 - e^{-e^x} \sum_{i=0}^{k} \frac{(e^x)^i}{i!}\,.$$

The matching between the two microscopic and macroscopic regimes, if the Poisson point process convergence be true, would be smooth.

Let us mention two articles [30, 31] from the literature on random Schrodinger operators (with *stationary* potential) which establish that the bottom eigenvalues have Poissonian statistics in the limit of infinite support for different kind of potential, including sums of Dirac masses and smooth functions of diffusions.

4.2. **Proof of Theorem 4.1.** Let us fix $x \in \mathbb{R}$ and denote simply by Z the diffusion $Z_{\ell_\beta(x)}$ in this proof. From Kallenberg's theorem [28], we just need to see that, for any finite union I of disjoint and bounded intervals, we have when $\beta \to 0$,

$$\mathbb{E}[\nu_\beta(I)] \longrightarrow e^x \int_I e^{-t}\,dt\,, \tag{41}$$

$$\mathbb{P}[\nu_\beta(I) = 0] \longrightarrow \exp(-e^x \int_I e^{-t}\,dt)\,. \tag{42}$$

Denote by $[t_1; t_2]$ the right most interval of I, by J the union of disjoint and bounded intervals such that $I = J \cup [t_1; t_2]$ and by t_0 the supremum of J. Note that $t_0 < t_1$.

To simplify notations, set $s_i := t_i/(\beta(\frac{3}{8}\ln\frac{1}{\beta})^{1/3})$ for $i = 0, 1, 2$ (the times s_i will correspond to the real time scale for the diffusion Z).

Thanks to the linearity of the expectation, it is enough to prove (41) for intervals I of the form $I = [0, t]$.

On the other hand for (42), the simple Markov property yields

$$\mathbb{P}[\nu_\beta(I) = 0] = \mathbb{P}\left[1_{\nu_\beta(J)=0}\mathbb{P}_{Z(s_0),s_0}\left[\nu_\beta[t_1; t_2] = 0\right]\right]\,.$$

Showing (42) therefore reduces to see that with probability going to 1 as $\beta \to 0$,

$$\mathbb{P}_{Z(s_0),s_0}\left[\nu_\beta[t_1; t_2] = 0\right] \longrightarrow_{\beta\to 0} \exp(-e^x \int_{t_1}^{t_2} e^{-t}\,dt)\,.$$

For both (41) and (42), the idea is to decompose the interval $[s_1; s_2]$ (with $s_1 = 0$ for (41)) into a finite number of small intervals of length $\delta := \varepsilon/(\beta(\frac{3}{8}\ln(1/\beta))^{1/3})$ and to approximate the number of explosions of Z on each small interval of the subdivision by those of stationary diffusions, thanks to the *increasing property*.

We will use in fact a random subdivision instead of a deterministic one to avoid technical issues due to special points. To this end, let us define a sequence $(\tau_k)_{k\in\mathbb{N}}$ of i.i.d. random variables with uniform law in $[0;1]$, independent of the diffusion Z. Let δ small enough such that $0 < \delta < s_1 - s_0$. Then, we construct iteratively the sequence of random times $(S_k)_{k\geq 0}$ such that $S_0 := s_1 - \delta\tau_0$ [3], $S_k := S_{k-1} + \delta\tau_k$ for $k \geq 1$ and the stopping times defined as:

$$K_1 := \inf\{k \geq 1 \ : \ S_k \geq s_1\}\,,$$
$$K_2 := \inf\{k \geq 1 \ : \ S_k \geq s_2\}\,.$$

In the following $\mathbb{P}$ refers to expectation with respect to both Z and the τ_k.

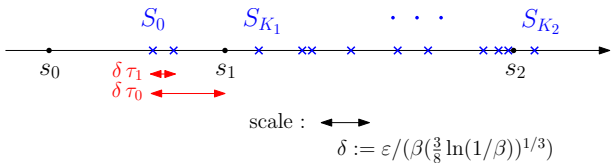

FIGURE 4. Definition of the times $(S_k, k \geq 0)$.

Set $a := -\ell_\beta(x)$ such that the drift of Z at time s simply equals $a + \beta s/4 - Z(s)^2$. On each interval $[S_k, S_{k+1}]$ of the random subdivision, we define the following two diffusions m_k and M_k (independent of the times τ_k) driven by the same Brownian motion as Z.

$$m_k(S_k) = Z(S_k)\,, \quad \text{and} \quad dm_k(s) = \left(a + \frac{\beta}{4}S_k \ - m_k^2(s)\right)ds + dB_s \text{ for } s \in [S_k; S_{k+1}]\,,$$

$$M_k(S_k) = Z(S_k)\,, \quad \text{and} \quad dM_k(s) = \left(a + \frac{\beta}{4}S_{k+1} - M_k^2(s)\right)ds + dB_s \text{ for } s \in [S_k; S_{k+1}]\,.$$

The *increasing property* implies that the number of explosions $\nu_\beta[S_k; S_{k+1}]$ of the diffusion Z is stochastically dominated from above (respectively from below) by the number of explosions of the diffusion m_k (respectively M_k). Indeed, the drifts of the diffusion $m_k(s), Z(s)$ and $M_k(s)$ are in increasing order for $s \in [S_k; S_{k+1}]$

$$a + \frac{\beta}{4}S_k \leq a + \frac{\beta}{4}s \leq a + \frac{\beta}{4}S_{k+1}\,.$$

The drifts of the diffusions m_k and M_k do not depend on time s so that the previous section applies on each $[S_k; S_{k+1}]$.

Let us introduce the so-called downside region D and its complementary upside one U we define as

$$D := \{y \in \mathbb{R} : y \leq -a^{1/2} + \frac{(\ln a)^{1/4}}{a^{1/4}}\}\,, \quad U := \mathbb{R} \setminus D\,. \tag{43}$$

The choice $f(a) = (\ln a)^{1/4}$ is large enough for our purposes (it tends to infinity) and small enough such that the process will spend little time in the region D (see Lemma 4.3).

From the previous section, we know that *as long as the starting point $Z(S_k)$ belongs to the region U* (which indeed is contained in the interval where the convergence applies), the two respective explosions point processes of the diffusions m_k and M_k converge weakly in the space of Radon measure, when the time scale is renormalized respectively by $m(a + \beta S_k/4)$

[3] In the case where $s_0 = 0$, we simply set $S_0 := 0$.

and $m(a+\beta S_{k+1}/4)$, to Poisson point processes with intensity 1, independently of the exact location of $Z(S_k)$.

Therefore we need to prove that

$$\mathcal{C} := \bigcap_{k=0}^{K_2-1} \{Z(S_k) \in U\} \tag{44}$$

has a probability going to 1 when $\beta \to 1$.

We first prove that the occupation time of the region D by the particle Z on the region of times considered tends to 0 in expectation. This is the content of the following Lemma whose proof is deferred at the end of this section.

Lemma 4.3. *Fix $t'_0 < t'_1 < t'_2$ and as before $s'_i := t'_i/[\beta\,(\frac{3}{8}\ln\frac{1}{\beta})^{1/3}], i = 1, 2$. Define*

$$T_\beta(s'_1, s'_2) := \int_{s'_1}^{s'_2} 1_{\{Z(u)\in D\}}\, du\,.$$

Then, for any $z \in \mathbb{R}$ there exists $\eta > 0$ and a positive constant C independent of β such that for all $\beta > 0$,

$$\mathbb{E}_{z,s'_0}[T_\beta(t'_1, t'_2)] \le C\left(\ln\frac{1}{\beta}\right)^{-\eta}.$$

We are now ready to prove that $\mathbb{P}(\mathcal{C}) \to 1$. Indeed, using the notations introduced above, the probability of $\mathcal{C}^c$ is easily bounded from above by

$$\mathbb{P}_{Z(s_0),s_0}\left[\bigcup_{k=0}^{K_2-1}\{Z(S_k)\in D\}\right] \le \sum_{k=0}^{4(t_2-t_1)/\varepsilon} \mathbb{P}_{Z(s_0),s_0}\left[Z(S_k)\in D\right] + \mathbb{P}\Big[\sum_{k=1}^{4(t_2-t_1)/\varepsilon}\tau_k \le \frac{t_2-t_1}{\varepsilon}\Big]\,. \tag{45}$$

The second probability on the RHS of (45) is easily bounded by noting simply that the τ_k have mean $1/2$ so that the empirical sum of the τ_k should be of order $2(t_2-t_1)/\varepsilon$. From large deviation theory, we know that this probability is of order $\exp(-c/\varepsilon)$ where c is a positive constant independent of ε.

Thanks to the independence between the diffusion Z and the sequence (S_k), the sum in (45) can be bounded as

$$\begin{aligned}\sum_{k=1}^{4(t_2-t_1)/\varepsilon} \mathbb{P}_{Z(s_0),s_0}\left[Z(S_k)\in D\right] &= \sum_{k=1}^{4(t_2-t_1)/\varepsilon} \mathbb{E}_{Z(s_0),s_0}\Big[\int_{S_{k-1}}^{S_{k-1}+\delta} 1_{\{Z(\tau)\in D\}}\,\frac{d\tau}{\delta}\Big]\\ &\le 2\beta(\frac{3}{8}\ln\frac{1}{\beta})^{1/3}\varepsilon^{-1}\,\mathbb{E}_{Z(s_0),s_0}\left[\int_{S_0}^{S_{4(t_2-t_1)/\varepsilon}} 1_{\{Z(\tau)\in D\}}\,d\tau\right]\,.\end{aligned}$$

Using again the Cramer's theorem of large deviation theory, we can show that, with probability going to 1 when $\varepsilon \to 0$, the empirical sum $S_{4(t_2-t_1)/\varepsilon} - S_0 = \delta\sum_{k=1}^{4(t_2-t_1)/\varepsilon}\tau_k$ is smaller than $\delta + 3(s_2-s_1)$ [4]. Lemma 4.3, applied for fixed times $s'_0 := s_0 < s'_1 < s'_2$ such that

[4] the choice of 3 is arbitrary.

$[s_1-\delta; s_1+3(s_2-s_1)] \subset [s_1'; s_2']$ and for $z = Z(s_0)$, permits to conclude that there exist $\eta > 0$ and two positive constants c and $C > 0$ both independent of β, ε, η such that

$$\mathbb{P}_{Z(s_0),s_0}\left[\bigcup_{k=1}^{K_2-1}\{Z(S_k)\in D\}\right] \leq C\beta\,(\ln\frac{1}{\beta})^{1/3-\eta} + e^{-c/\varepsilon}\,.$$

We are now ready to prove (41) and (42). To simplify notations, set N_k, N_k^- and N_k^+ for the number of explosions of the diffusions Z, M_k and m_k in the interval $[S_k; S_{k+1}]$. Recall that a.s. for all k, $N_k^- \leq N_k \leq N_k^+$

We begin with (41). Similarly to the stationary case, we could take $s_1 = 0$ in this case for which the situation is slightly easier. Indeed, at the starting point $s_1 = 0$, $Z(s_1) = +\infty \in U$ and we do not need to introduce an independent random time just before s_1 to start from a nice position. Nevertheless, we treat the case $s_1 > 0$ to have consistent notations with the ones required in the proof of (42).

Thanks to the increasing property, we can bound from above the mean number of explosions of Z as

$$\mathbb{E}[\nu_\beta[t_1,t_2]] \leq \mathbb{E}\left[\sum_{k=0}^{K_2-1}\mathbb{E}\left[N_k^+\,|Z(S_k)\in U\right]\right] + \mathbb{E}[\nu_\beta[t_1,t_2]1_{\mathcal{C}^c}]\,. \tag{46}$$

The second expectation in (46) is negligible. To show this, we use the Cauchy Schwartz inequality and the increasing property which permits to bound the second moment of $\nu_\beta[0,t_2]$ by the second moment of the number of explosions in the interval $[s_1, s_2]$ of the diffusion $Y_{a+\beta s_1/4}$ studied in section 3. It is easily seen that this second moment is bounded independently of a thanks to Cramer's theorem for example as was done to bound the expectation of the number of explosions in the proof of Theorem 3.6.

Gathering the above arguments, we can deduce the convergence of the first term. Indeed, conditionally on $\{Z(S_k) \in U\}$, the explosion times process of m_k converges when time is renormalized by $m(a+\beta S_k/4)$, to a Poisson point process with intensity 1. Here we must pay attention to the renormalization of time: Noting that

$$a+\beta S_k/4 = \left(\frac{3}{8}\ln\frac{1}{\beta\pi}\right)^{2/3} + \frac{1}{2}\frac{1}{3^{1/3}}\left(\ln\frac{1}{\beta}\right)^{-1/3}\left(t_1 - x + \varepsilon(-\tau_0 + \sum_{i=1}^{k}\tau_i)\right)\,,$$

we can find the limit of the time scales ratio

$$\beta\left(\frac{3}{8}\ln\frac{1}{\beta\pi}\right)^{1/3} m(a+\beta S_k/4) \longrightarrow_{\beta\to 0} \exp(t_1 - x + \varepsilon(-\tau_0 + \sum_{i=1}^{k}\tau_i))\,.$$

Therefore, when $\beta \to 0$, we have

$$\mathbb{E}\left[N_k^+\,|Z(S_k)\in U\right] \longrightarrow \mathbb{E}[\tau_{k+1}\exp(x - t_1 - \varepsilon(-\tau_0 + \sum_{i=1}^{k}\tau_i))]\,,$$

which finally gives taking $\varepsilon \to 0$ and thanks to the convergence of the Riemann sum associated to the subdivision $\tau_1 < \tau_1+\tau_2 < \cdots < \tau_1+\cdots+\tau_k < \cdots$ that

$$\limsup_{\beta\to 0}\mathbb{E}[\nu_\beta[t_1,t_2]] \leq e^x\int_{t_1}^{t_2} e^{-u}\,du\,. \tag{47}$$

For the lower bound,

$$\mathbb{E}[\nu_\beta[t_1,t_2]] = \mathbb{E}\left[\sum_{k=0}^{\infty}\mathbb{E}\left[N_k^-\ 1_{\{s_1\le S_k\le S_{k+1}\le s_2\}}1_{\{Z(S_k)\in U\}}\right]\right]$$
$$\ge \mathbb{E}\left[\sum_{k=0}^{4(t_1-t_0)/\varepsilon}\mathbb{E}\left[N_k^-\ 1_{\{s_1\le S_k\le S_{k+1}\le s_2\}}\,|Z(S_k)\in U\right]\right]$$
$$-\mathbb{E}\left[\sum_{k=0}^{4(t_1-t_0)/\varepsilon}\mathbb{E}\left[N_k^-\ 1_{\{s_1\le S_k\le S_{k+1}\le s_2\}}\,|Z(S_k)\in U\right]\mathbb{P}\left[Z(S_k)\in D\right]\right].$$

It suffices to show that the second term is negligible and apply the same method as was done above to obtain (47). The second term is bounded from below by

$$-\mathbb{E}\left[\text{Number of explosion of } Y_{a+\beta s_1/4} \text{ in } [s_1,s_2]\right]\times \sup_{0\le k\le \frac{4(t_1-t_0)}{\varepsilon}}\frac{\mathbb{P}[Z(S_k)\in D]}{1-\mathbb{P}[Z(S_k)\in D]}\,. \tag{48}$$

Lemma 4.3 applied for $t_0' := 0 < t_1' < t_2'$ such that $t_1' < t_1 - \varepsilon$ and $t_2' > t_1 + 4(t_2-t_1)$ gives a uniform (independent of $k \le 4(t_2-t_1)/\varepsilon$) bound of $\mathbb{P}[Z(S_k)\in D]$ which goes to 0 as $\beta\to 0$. It easily follows that the quantity (48) tends to 0 (we have already seen that the first term is bounded independently of a).

We now turn to the convergence (42). By the increasing property, we have

$$\mathbb{P}_{Z(s_0),s_0}\left[\nu_\beta[t_1;t_2]=0\right]$$
$$\le \mathbb{P}_{Z(s_0),s_0}\left[\prod_{k=K_1}^{K_2-2}\mathbb{P}_{Z(S_k),S_k}\left[N_k^-=0\,|Z(S_k)\in U,(\tau_k)_{k\in\mathbb{N}}\right]\right]+\mathbb{P}_{Z(s_0),s_0}[\mathcal{C}^c]\,.$$

Using the previous arguments, we get

$$\limsup_{\beta\to 0}\ \mathbb{P}_{Z(s_0),s_0}\left[\nu_\beta[t_1;t_2]=0\right]\le \mathbb{E}\left[\prod_{k=K_1}^{K_2-2}\exp\left(-\varepsilon\tau_{k+1}\exp(x-t_1-\varepsilon(-\tau_0+\sum_{i=1}^{k+1}\tau_i))\right)\right]+e^{-c/\varepsilon}\,.$$

Taking the limit $\varepsilon\to 0$, we obtain as before

$$\limsup_{\beta\to 0}\mathbb{P}_{Z(s_0),s_0}\left[\nu_\beta[t_1;t_2]=0\right]\le \exp(-e^x\int_{t_1}^{t_2}e^{-u}\,du)\,.$$

For the lower bound, we use again the increasing property which allows to write

$$\mathbb{P}_{Z(s_0),s_0}\left[\nu_\beta[t_1;t_2]=0\right]$$
$$\ge \mathbb{P}_{Z(s_0),s_0}\left[\prod_{k=0}^{K_2-1}\mathbb{P}_{Z(S_k),S_k}\left[N_k^+=0\,|Z(S_k)\in U,(\tau_k)_{k\in\mathbb{N}}\right]\right]-\mathbb{P}_{Z(s_0),s_0}[\mathcal{C}^c]\,.$$

Using the same arguments as above, we can deduce that

$$\liminf_{\beta\to 0}\mathbb{P}_{Z(s_0),s_0}\left[\nu_\beta[t_1;t_2]=0\right]\ge \exp(-e^x\int_{t_1}^{t_2}e^{-u}\,du)\,,$$

and (42) is proved.

Proof of Lemma 4.3.

The key estimate to prove this Lemma is given by [16, Proposition 10], which is recalled in Appendix B. Let us recall it here. We denote by $\zeta^{(u)} := \inf\{v \geq 0 : Z(u+v) = -\infty\}$ the waiting time of the first explosion after u. There exists a constant $c > 0$ independent of β such that for all $u \in [s_1'; s_2']$ [5]

$$\mathbb{P}_{-a^{1/2}+\frac{(\ln a))^{1/4}}{a^{1/4}},u}\left[\zeta^{(u)} < \frac{\ln a}{\sqrt{a}}\right] \geq \exp(-c\sqrt{\ln a})\,. \tag{49}$$

We can rewrite the inequality (49) in the following way, using the asymptotic expansion of a as a function of β:

$$\mathbb{P}_{-a^{1/2}+\frac{(\ln a)^{1/4}}{a^{1/4}},u}\left[\zeta^{(u)} < C\,(\ln\frac{1}{\beta})^{-\eta}\right] \geq \exp(-c\,(\ln\frac{1}{\beta})^{1/3})\,. \tag{50}$$

Eq. (50) gives a lower bound of the probability for the diffusion Z starting from the right most point of the interval D at time u to explode to $-\infty$ in a short time. For the sake of completeness, we rewrite the proof of the estimate Eq. (49) in Appendix B with more details than originally given in [16].

The idea to obtain the Lemma is to use (50) to relate between the time spent in the region D by the process Z and the number of its explosions which is bounded on the interval $[s_1'; s_2']$. In the following, the probability $\mathbb{P}$ is with respect to the diffusion starting from z at time s_0'.

From (49), we can control for any given $u > 0$, the probability that the diffusion Z is in the region D at time u: Indeed, we have

$$\begin{aligned}\mathbb{P}[Z(u)\in D] &= \mathbb{P}\left[Z(u)\in D, \zeta^{(u)} \leq C\,(\ln\frac{1}{\beta})^{-\eta}\right] + \mathbb{P}\left[Z(u)\in D, \zeta^{(u)} > C\,(\ln\frac{1}{\beta})^{-\eta}\right] \\ &\leq \mathbb{P}\left[Z(u)\in D, \zeta^{(u)} \leq C\,(\ln\frac{1}{\beta})^{-\eta}\right] + \left(1-\exp(-c\,(\ln\frac{1}{\beta})^{1/3})\right)\,\mathbb{P}\left[Z(u)\in D\right]\end{aligned} \tag{51}$$

where we have used the simple Markov property and the increasing property in the second line. Eq. (51) readily rewrites as

$$\begin{aligned}\mathbb{P}\left[Z(u)\in D\right] &\leq \exp(c\,(\ln\frac{1}{\beta})^{1/3})\,\mathbb{P}\left[Z(u)\in D, \zeta^{(u)} \leq C\,(\ln\frac{1}{\beta})^{-\eta}\right] \\ &\leq \exp(c\,(\ln\frac{1}{\beta})^{1/3})\,\mathbb{P}\left[\text{The interval } \left[u, u + C\,(\ln\frac{1}{\beta})^{-\eta}\right] \text{ contains at least one explosion}\right]\,.\end{aligned} \tag{52}$$

Note that, denoting by $k = \nu_\beta[t_1', t_2']$ the (random) number of explosions in the interval $I_{t,t'}$ and as before by $0 < \zeta_1 < \zeta_2 < \cdots < \zeta_k$ the explosion times, we have almost surely

$$\int_{s_1'}^{s_2'} 1_{\{\exists i:\zeta_i\in[u,u+C\,(\ln\frac{1}{\beta})^{-\eta}]\}} du \leq C\,(\ln\frac{1}{\beta})^{-\eta}\,\nu_\beta[t_1', t_2']\,.$$

Therefore, integrating Eq. (52) with respect to u in the interval $I_{t,t'}$, we finally obtain the inequality

$$\mathbb{E}[T_\beta(s_1', s_2')] \leq C\,(\ln\frac{1}{\beta})^{-\eta}\,\exp(c\,(\ln\frac{1}{\beta})^{1/3})\;\mathbb{E}\left[\nu_\beta[t_1', t_2']\right]\,.$$

[5] $[s_1'; s_2']$ indeed is contained in the region where Lemma B.1 applies.

In addition we easily check, using the increasing property and the convergence (25) proved in the previous section, that the mean number of explosions $\mathbb{E}[\nu_\beta[t'_1, t'_2]]$ is bounded independently of β. The Lemma follows.

□

4.3. **Empirical spectral measure on the macroscopic scale.** In this subsection, we derive the empirical spectral measure ρ_β defined in (39) of the stochastic linear operator $\mathcal{L}_\beta$ on the macroscopic scale, i.e. *without zooming* in the minimal eigenvalues scaling region.

As mentioned above, we expect the number $\rho_\beta(]-\infty;\ell])$, which is almost surely finite for any $\beta > 0$ (see [8]), to tend to $+\infty$ when $\beta \to 0$. The minimal eigenvalues are indeed going to $-\infty$ as $\beta \to 0$ and if one sets $\beta = 0$ abruptly so that the linear confining term disappears, then the operator $\mathcal{L}_0 := -\frac{d^2}{dt^2} + B'_t$ has an infinite number of eigenvalues below any given level $\ell \in \mathbb{R}$ (see subsection 3). The spectral statistics of this operator have in fact been extensively studied in the literature (see [23, 24, 25, 15]). For this study, the eigenfunctions are restricted to a finite interval $[0; L]$ with Dirichlet boundary conditions in $t = 0$ and $t = L$ so that the number of eigenvalues below a certain level remains finite. The authors investigate the spectrum of $\mathcal{L}_0$ in the large L limit. The minimal eigenvalues statistics of the operator $\mathcal{L}_\beta$ when $\beta \to 0$ as described above are different from the statistics found in [25, 15]. We shall see that the limiting empirical eigenvalue density differs as well (to be compared with the density found in [23, 24]).

To compute $\rho_\beta(]-\infty;\ell])$, we use again the fact that the number of $\mathcal{L}_\beta$-eigenvalues strictly less than ℓ equals the number of explosions of the diffusion $(Z_t)_{t\geq 0}$ (defined in Eq. (33)) on $\mathbb{R}_+$. Counting the blow-ups along the trajectory $(Z_t)_{t\geq 0}$ can be done computing the flux of particles at $z = -\infty$ in the system, i.e. the number of particles going through $-\infty$ per unit of time. In this non stationary system, the flux depends on time t and on position z and will be denoted $J_\beta(z,t)$ in the following.

As the particle immediately restarts in $z = +\infty$ when blowing up to $-\infty$, the flux in $z = -\infty$ is equal to the flux in $z = +\infty$. We can again write the Fokker Planck equation which gives the evolution of the transition probability density $q_\beta(z,t)$ of the diffusion Z_t as

$$\frac{\partial}{\partial t} q_\beta(z,t) = \frac{\partial}{\partial z}\left[(z^2 + \ell - \frac{\beta}{4}t) q_\beta(z,t) + \frac{1}{2}\frac{\partial}{\partial z} q_\beta(z,t)\right] . \tag{53}$$

This equality can in turn be rewritten as a continuity equation

$$\frac{\partial q_\beta}{\partial t} = \frac{\partial J_\beta}{\partial z}$$

where $J_\beta(z,t) = (z^2 + \ell - \frac{\beta}{4}t) q_\beta(z,t) + \frac{1}{2}\frac{\partial}{\partial z} q_\beta(z,t)$ is the flux of probability in z at time t. By definition of the flux, the number of explosions to $-\infty$ of the diffusion $(Z_t)_{t\geq 0}$ on $\mathbb{R}_+$ is

$$\rho_\beta(]-\infty;\ell]) = \int_0^{+\infty} J_\beta(-\infty, t)\, dt\, .$$

Re scaling time as before by setting $p_\beta(z,t) := q_\beta(z, \frac{4}{\beta}t)$, Eq. (53) becomes

$$\frac{\beta}{4}\frac{\partial}{\partial t} p_\beta(z,t) = \frac{\partial}{\partial z}\left[(z^2 + \ell - t)\, p_\beta(z,t) + \frac{1}{2}\frac{\partial}{\partial z} p_\beta(z,t)\right] . \tag{54}$$

We now use perturbation theory in the limit $\beta \to 0$ in Eq. (53) in order to obtain an approximation for p_β for small β. The method consists in searching a solution of Eq. (53) valid at small β

under the form

$$p_\beta(z,t) = p_0(z,t) + \beta\, p_1(z,t) + o(\beta) \tag{55}$$

At leading order, we find the following ordinary differential equation for p_0

$$\frac{d}{dz}\left((z^2 + \ell - t)\, p_0(z,t) + \frac{1}{2}\frac{d}{dz} p_0(z,t) \right) = 0\,. \tag{56}$$

Equation (56) is the same as Eq. (27) of the previous section with the parameter a replaced by $-\ell + t$. The solution is

$$p_0(z,t) = 2\, J_0\,(\ell - t) \int_{-\infty}^{z} du\, e^{2(\ell-t)(u-z)+\frac{2}{3}(u^3-z^3)}$$

where J_0 is the stationary flux given by Eq. (30) such that $\int_{\mathbb{R}} p_0(z,t)dz = 1$. Hence, recalling that $p_\beta(z,t)$ is the law of $Z_{4t/\beta}$, we conclude that the flux in $-\infty$ at time t is $J_0\,(\ell - \beta t/4)$ and thus,

$$\rho_\beta(]-\infty;\ell]) = \int_0^{+\infty} J_\beta\,(-\infty,t)\; dt = \frac{4}{\beta}\int_{-\infty}^{\ell} J_0\,(u)\; du + O(1)\,, \tag{57}$$

where $O(1)$ is a correction of order 1 when $\beta \to 0$. It is now straightforward to deduce the empirical eigenvalue density

$$\rho_\beta(\ell) \sim_{\beta\to 0} \frac{4}{\beta} J_0\,(\ell)\;. \tag{58}$$

This formula contrasts with the result found in [23, 24, 25] where the empirical eigenvalue density of the operator $\mathcal{L}_0$ restricted on a finite interval $[0;L]$ is proportional to the length L of the time interval

$$n'_L(\ell) \sim_{L\to-\infty} L\, J'_0(\ell)\,.$$

The computation leading to this result was in fact recalled above in (31).

It is interesting to study the behaviors at $\ell \to \pm\infty$ of the empirical eigenvalue density $\rho_\beta(\ell)$ and of the integrated density $\rho_\beta(]-\infty;\ell])$. This can be done from the integral forms with the saddle point route. When $\ell \to -\infty$, we have (see again end of Appendix C)

$$\rho_\beta(\ell) \sim \frac{4}{\pi\beta}\,|\ell|^{1/2}\,\exp\left(-\frac{8}{3}|\ell|^{3/2}\right) \quad \text{and} \quad \rho_\beta(]-\infty;\ell]) \sim \frac{1}{\pi\beta}\exp\left(-\frac{8}{3}|\ell|^{3/2}\right)\,. \tag{59}$$

On the other side, when $\ell \to +\infty$, we obtain

$$\rho_\beta(\ell) \sim \frac{1}{\pi\beta}\,\ell^{1/2} \quad \text{and} \quad \rho_\beta(]-\infty;\ell]) \sim \frac{8}{3\pi\beta}\,\ell^{3/2}\,.$$

We note the correct matching of the tails of the empirical eigenvalue density $\rho_\beta(\ell)$ as $\ell \to \pm\infty$ with those of the crossover density of eigenvalues (in random matrix models) near the edge scaling region of width $N^{-2/3}$ found by Bowick and Brézin in [14, 36] (see Figure 5).

We discuss the validity of perturbation theory as applied here in Appendix D.

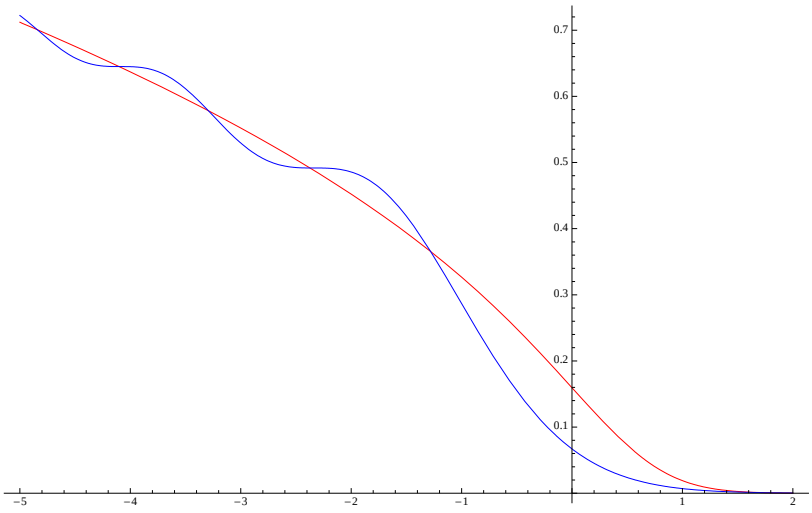

FIGURE 5. (Color online). Red non-oscillatory curve: $4J_0(\lambda)$ as a function of λ. Blue oscillatory curve: density $K_{\mathrm{Airy}}(\lambda,\lambda) = \mathrm{Ai}'(\lambda)^2 - \lambda \mathrm{Ai}^2(\lambda)$ at the edge of the spectrum for $\beta = 2$ found by Bowick and Brézin. The bumps of the $\beta = 2$ curve are reminiscent to the electrostatic repulsion between the particles.

5. APPLICATIONS TO RANDOM MATRIX THEORY

5.1. **Top eigenvalue of β-ensembles.** In this subsection, we use the connection between random matrices and stochastic linear operators to study the top eigenvalue statistics of β-ensembles with an index β_N depending of N such that

$$\beta_N \to_{N\to+\infty} 0 \quad \text{and} \quad N\beta_N \to_{N\to+\infty} +\infty\,. \tag{60}$$

Example of such a sequence is $\beta_N = 1/N^\alpha$ where $0 < \alpha < 1$. The tridiagonal random matrices introduced by Dumitriu and Edelman in [1], whose eigenvalues are distributed according to the jpdf P_β defined in (1), can be written as

$$X_N = \begin{pmatrix} \sqrt{2}\,g_1 & \chi_{(N-1)\beta_N} & & & \\ \chi_{(N-1)\beta_N} & \sqrt{2}\,g_2 & \chi_{(N-2)\beta_N} & & \\ & \ddots & \ddots & \ddots & \\ & & \chi_{2\beta} & \sqrt{2}\,g_{N-1} & \chi_{\beta_N} \\ & & & \chi_{\beta_N} & \sqrt{2}\,g_N \end{pmatrix} \tag{61}$$

where the g_k are independent Gaussian random variables with variance 1 and where the $\chi_{k\beta_N}$ are independent χ distributed random variables with $k\beta_N$ degrees of freedom and scale parameter 2.

In [8], the authors prove that the largest eigenvalues of those tridiagonal random matrices converge in distribution to the low lying eigenvalues of the stochastic Airy operator $\mathcal{H}_\beta$ introduced above. Their proof is rather technical, but a simple heuristic of this convergence can be found in [7].

In the following, we apply this heuristic to analyse the present case with a vanishing repulsion coefficient β_N satisfying (60). The main idea is that, under the assumption (60), the information on the top eigenvalues of the matrix X_N is contained in the upper left sub matrices of X_N. In

order to analyze the law of those eigenvalues, it is sufficient to consider only the entries with column or line numbers very small compared to the dimension N of the matrix. Developing for $k \ll N$ the χ variables as $\chi_{\beta_N(N-k)} \approx \sqrt{\beta_N(N-k)} + \frac{h_k}{\sqrt{2}}$ where the h_k are independent Gaussian variables, we obtain the following decomposition for the translated and rescaled matrix

(62)

$$A_N := (N\beta_N)^{2/3}\left(2I_N - \frac{X_N}{(N\beta_N)^{1/2}}\right) = -(N\beta_N)^{2/3}\begin{pmatrix} -2 & 1 & & \\ 1 & -2 & 1 & \\ & \ddots & \ddots & \ddots \end{pmatrix}$$
$$+ \beta_N \frac{1}{2}\frac{1}{(N\beta_N)^{1/3}}\begin{pmatrix} 0 & 1 & & \\ 1 & 0 & 2 & \\ & 2 & \ddots & \ddots \\ & & \ddots & \end{pmatrix} - \frac{1}{\sqrt{2}}(N\beta_N)^{1/6}\begin{pmatrix} 2g_1 & h_1 & & \\ h_1 & 2g_2 & h_2 & \\ & \ddots & \ddots & \ddots \end{pmatrix} + o\left(\frac{1}{N}\right).$$

Using the same argument as in [7], (62) may be rewritten in short as

(63) $$A_N \approx -\frac{d^2}{dt^2} + \beta_N\, t + 2\, b'_t\,.$$

In view of the previous section, it is important to keep the linear term $\beta_N t$ (even for $N \to +\infty$) so that the least eigenvalue of the operator on the right hand side of (63) remains finite. This term is small for small values of t but gets large for values of $t \gg \beta_N^{-1}$ and should not neglected.

The top eigenvalue λ_0^N of the matrix X_N should therefore approximately satisfy, for large values of N, the equality in law

(64) $$(N\beta_N)^{2/3}\left(2 - \frac{\lambda_0^N}{(N\beta_N)^{1/2}}\right) \overset{(law)}{\approx} 4^{2/3}\, L_0^N$$

where L_0^N is the least eigenvalue of the operator $-\frac{d^2}{dt^2} + \frac{\beta_N}{4}t + b'_t$. Using the previous results, Eq. (64) can be rewritten

(65)

$$(N\beta_N)^{2/3}\left(\frac{\lambda_0^N}{(N\beta_N)^{1/2}} - 2\right) \overset{(law)}{\approx} \left(\frac{3}{2}\right)^{2/3}\left(\ln\frac{1}{\pi\beta_N}\right)^{2/3} + \left(\frac{2}{3}\right)^{1/3}\left(\ln\frac{1}{\pi\beta_N}\right)^{-1/3} G\,.$$

where G is a random variable distributed according to the Gumbel distribution. This result (65) was checked numerically with very good agreement.

From this proposal that the tridiagonal random matrices in the β ensembles may be regarded at the edge of the spectrum as finite different schemes of the stochastic Airy operator, we argue that the results proved in the continuous setting for the stochastic Airy operator can be extended to the discrete setting of random matrices. The convergence we obtained for the marginal distribution of any eigenvalue with fixed index k in (38) can be rewritten in the context of β_N ensembles as was done in (65) for the maximal eigenvalue.

At this point, it is tempting to conjecture, as in the continuous setting of the stochastic Airy operator, that the maximal eigenvalues of β_N-ensembles have Poissonian statistics in the double scaling limit $N \to +\infty, \beta_N \to 0$. Nevertheless the transition between Wigner/ Airy and Poisson statistics is not very well known up to now and many questions have been asked in

the literature of RMT (see e.g. [32, 33, 34, 35] for related questions on eigenvectors localization/delocalization).

5.2. **Break down at $\beta = 2c/N$.** The discussion of the previous subsection breaks down if β scales with N as $\beta_N = 2c/N$ where c is a positive constant. In such cases (and for even more rapidly decreasing β_N), the lower part of the matrix X_N is no longer negligible compared to the upper part. One has to keep all the entries in the matrix X_N and the approach through the stochastic linear operator is no longer valid.

As mentioned in the introduction, the case $\beta = 2c/N$ was studied before. In [10], the authors prove that, if β scales with N as $\beta = 2c/N$ where c is a positive constant, then the empirical eigenvalue density $1/N \sum_{i=1}^{N} \delta(\lambda - \lambda_i)$ of the matrix X_N converges in the large N limit to a continuous probability density ρ_c with non compact support. The density ρ_c is computed explicitly in [10] and one can easily recover the Gaussian tails of ρ_c for $\lambda \to \pm\infty$

$$\rho_c(\lambda) \sim C\lambda^{2c} e^{-\lambda^2/4} \tag{66}$$

where C is an explicit multiplicative constant.

The reader may wonder whether Eq. (65) is coherent with the results found in [10] where the limiting empirical eigenvalue density of the matrix X_N was computed in the double scaling limit $\beta_N = 2c/N$ with $N \to \infty$. The developments of the previous subsection should a priori apply for any sequence β_N such that $\beta_N \to 0$ and $N\beta_N \to +\infty$ as $N \to +\infty$. In particular, it should hold for $\beta_N = \ln N/N$. For such a β_N, Eq. (65) rewrites under the form

$$\lambda_0^N = c_1\sqrt{\ln N} + c_2 \frac{\ln\ln N}{\sqrt{\ln N}} + c_4 \frac{1}{\sqrt{\ln N}} + c_3 \frac{1}{\sqrt{\ln N}} G\,, \tag{67}$$

where c_1, c_2, c_3, c_4 are (explicit) constant (their values are irrelevant in the present discussion). This scaling form matches the one would find for the maximum of N particles allocated according to the density ρ_c derived in [10] when $\beta_N = 2c/N$.

We therefore conjecture that, even though λ_0^N has Gumbel fluctuations for any sequences β_N going to 0 in the large N limit, it has different scalings depending on whether

- β_N decreases faster than $\ln N/N$, then the centering and scaling in the convergence of λ_0^N are as in (67).
- β_N decreases slower than $\ln N/N$, then the centering and scaling are as in Eq. (65).

APPENDIX A. PROOFS OF AUXILIARY RESULTS

Proof of Proposition 3.1.

It is classical to show that the function g_α defined in Eq. (12) is a weak solution to the boundary value problem Eq. (13) and (14) [37]. Note simply that the differential equation is $\mathcal{G}g_\alpha = \alpha g_\alpha$ where $\mathcal{G}$ is the infinitesimal generator associated to the diffusion process $(Y_a(t))$. The boundary condition (14) is satisfied by g_α as defined in Eq. (12) since the diffusion, when starting from y, will explode in a time going to 0 when $y \to -\infty$.

At this point, we do not know yet that the function g_α is a C^2 function. Nevertheless, we can (at least formally) perform the change of function $u(y) = \exp[-2(y^3/3 - ay)]g'_\alpha(y)$ and show that the function g_α satisfies the fixed point equation (15). This computation can be made rigorous

by considering regularizations of g_α and taking the limit in the end. It immediately follows from Eq. (15) that the function g_α is C^∞.

We can now prove that there exists a unique bounded C^2 function g_α satisfying the boundary value problem specified by Eq. (13) and (14). If h is another solution, it is straightforward to check from Itô's formula that the process $e^{-\alpha t}h(Y_a(t))$ is a martingale. Besides, it is bounded and hence we can apply the stopping time Theorem which yields

$$h(y) = \mathbb{E}_y[e^{-\alpha\zeta}\,\mathbf{1}_{\{\zeta<\infty\}}\,h(Y_\zeta)] = \mathbb{E}_y[e^{-\alpha\zeta}] = g_\alpha(y)\,,$$

where we have used the boundary condition at $y \to -\infty$ satisfied by h as well as the fact that $\zeta < \infty$ almost surely. The unicity is proved. □

Proof of Theorem 3.3.

First note that $g_{\alpha/m(a)}(y)$ is non increasing with respect to y. In addition we have $0 \le g_{\alpha/m(a)}(y) \le 1$ and thus $g_{\alpha/m(a)}(y)$ admits a limit when $y \to +\infty$. We will denote this limit $g_{\alpha/m(a)}(+\infty)$.

From Theorem 3.1, $g_{\alpha/m(a)}(y)$ satisfies for any $\alpha \in (0;1)$ the fixed point equation
(68)

$$g_{\alpha/m(a)}(y) = 1 - 2\frac{\alpha}{m(a)}\int_{-\infty}^{y} dx \int_x^{+\infty} du \exp\left(2a(x-u) + \frac{2}{3}(x^3-u^3)\right) g_{\alpha/m(a)}(u)\,.$$

Let us define recursively a sequence $R_n(y,a)$ such that $R_0(y,a) = 1$ and for $n \ge 1$

$$R_n(y,a) := \frac{2}{m(a)}\int_{-\infty}^{y} dx \int_x^{+\infty} du \exp\left(2a(x-u) + \frac{2}{3}(x^3-u^3)\right) R_{n-1}(u,a)\,. \tag{69}$$

To begin note that, for $\alpha \ge 0$, and for all $u \in \mathbb{R}$,

$$0 \le g_{\alpha/m(a)}(u) = \mathbb{E}_u[e^{-\alpha\zeta/m(a)}] \le 1 \tag{70}$$

Using the upper bound of Eq. (70), we obtain from Eq. (68) a new lower bound

$$1 - \alpha\, R_1(y,a) \le g_{\alpha/m(a)}(y) \le 1\,. \tag{71}$$

Using now the lower bound of Eq. (71), we obtain from Eq. (68) a new upper bound

$$1 - \alpha\, R_1(y,a) \le g_{\alpha/m(a)}(y) \le 1 - \alpha\, R(y,a) + \alpha^2\, R_2(y,a)\,.$$

We can check iteratively that for all N,

$$\sum_{n=0}^{2N-1}(-1)^n\alpha^n R_n(y,a) \le g_{\alpha/m(a)}(y) \le \sum_{n=0}^{2N}(-1)^n\alpha^n R_n(y,a)\,.$$

In particular, we have by letting $N \to +\infty$ that, for any $\alpha \in (0;1)$,

$$g_{\alpha/m(a)}(y) = \sum_{n=0}^{+\infty}(-1)^n\alpha^n R_n(y,a)\,. \tag{72}$$

Note that (72) permits to deduce the values of all the moments $\mathbb{E}_y[\zeta^n] = n!\,m(a)^n\,R_n(y,a)$.

Now it suffices to prove that for all fixed $n \in \mathbb{N}$ and for any $y \in \mathbb{R}$, $R_n(y,a)$ converges to 1 when $a \to +\infty$. This is the content of the following Lemma A.1 whose proof can be found at the end of this section.

Lemma A.1. *Let $f(a)$ such that $a^{1/4}(f(a)+a^{1/2}) \to +\infty$. Then, for each $n \in \mathbb{N}$ and for any $y \ge f(a)$, $R_n(y,a)$ converges to 1 when $a \to +\infty$.*

It is then straightforward to see that

$$g_{\alpha/m(a)}(y,a) \to \frac{1}{1+\alpha}\,.$$

This convergence holds for any $\alpha \in (0;1)$. The theorem follows since $1/(1+\alpha)$ is the Laplace transform of an exponential law with parameter 1. □

Proof of Lemma A.1.

First note that for each n, $R_n(y,a)$ is increasing with respect to y and uniformly bounded by 1. This can be seen easily by induction over n recalling the expression of $m(a)(= m(a,+\infty))$ given in (16). It follows that the $\lim_{y\to+\infty} R_n(y,a)$ exists and is denoted as usual $R_n(+\infty,a)$. We proceed by induction over n to show that $R_n(y,a)$ converges to 1 for any n and for any $y \geq f(a)$. By monotonicity, it is enough to prove that both $R_n(f(a),a)$ and $R_n(+\infty,a)$ converges to 1 when $a \to +\infty$.

By the definitions of $R_1(y,a)$ (Eq. (69) for $n=1$) and of $m(a)$ (see Eq. (16)), it is clear that, for $y \in \mathbb{R}$ fixed and $a \to +\infty, R_1(y,a) \to 1$. Moreover, by monotonicity arguments, we easily check that this convergence holds uniformly on the interval $[f(a);+\infty[$.

At step n, performing two changes of variables in the recursive integral expression Eq. (69) of $R_n(y,a)$, we obtain for all $a>0$,

$$\begin{aligned} R_n(y,a) &= \frac{2}{m(a)}\, a \int_{-\infty}^{y/a^{1/2}} dx \int_x^{+\infty} du \exp\left(2a^{3/2}\left(\frac{1}{3}(x^3-u^3)-x+u\right)\right) R_{n-1}(a^{1/2}u,a) \\ &= \frac{2}{m(a)}\frac{1}{a^{1/2}}\,\exp\left(\frac{8}{3}a^{3/2}\right) \int_{-\infty}^{a^{1/4}(y+a^{1/2})} dx \int_{-2a^{3/4}+x}^{+\infty} du \\ &\quad \exp\left(-2x^2-2u^2+\frac{2}{3}(\frac{x^3}{a^{3/4}}-\frac{u^3}{a^{3/4}})\right) R_{n-1}(a^{1/2}(1+\frac{u}{a^{3/4}}),a)\,. \end{aligned}$$

By monotonicity again, we simply need to consider $y \in \{f(a);+\infty\}$. From the second line and provided that $a^{1/4}(y+a^{1/2}) \to +\infty$ (which is insured by assumption), we see that this integral is concentrated in the regions $x = O(1), u = O(1)$. One can in fact show that the other regions bring negligible contributions. In this regime $u = O(1)$ and we have, by the induction hypothesis, $R_{n-1}(a^{1/2}(1+\frac{u}{a^{3/4}}),a) \to 1$ uniformly for $u \in [-a^\epsilon;+\infty], \epsilon > 0$ small. The convergence of $R_n(f(a),a)$ and $R_n(+\infty,a)$ follow from the previous arguments and the asymptotic expression for $m(a)$ given in (85). □

APPENDIX B. PROOF OF (49)

Setting $a = -\ell$ to work with a positive parameter and writing Z_a instead of Z_ℓ, recall the definition of the diffusion Z_a,

$$dZ_a(t) = (\frac{\beta}{4}\,t + a - Z_a(t)^2)\,dt + dB(t)\,.$$

Recall also that $\mathbb{P}_{z,t}$ refers to the law of the diffusion Z_a starting from z at time t. We denote in this paragraph by $\zeta^{(u)}$ the first blowup time after time u of Z_a.

Proposition 10 of [16] yields that

Lemma B.1. *Fix $c' > 0$. There exists a positive constant c independent of a and $\beta \in [0;1]$ such that for all a large enough and for all $u \leq c' \ln(a)/(\beta\sqrt{a})$*

$$\mathbb{P}_{-\sqrt{a}+\frac{(\ln a)^{1/4}}{a^{1/4}},u}\left[\zeta^{(u)} < \frac{\ln a}{\sqrt{a}}\right] \geq \exp(-c\sqrt{\ln a})\,. \tag{73}$$

Remark B.2. *The constant in front of $\ln a/\sqrt{a}$ on the LHS is not optimal and can be replaced by any value strictly greater than $3/8$.*

Proof. For $x \in \mathbb{R} \cup \{-\infty\}$, set $T_x := \inf\{t \geq 0 : Z_a(t) = x\}$ and let $\delta := (\ln a)^{1/4}/a^{1/4}$ and $\varepsilon := \sqrt{\ln a}/a^{1/4}$. Without loss of generality we can suppose that $B_0 = 0$.

Moreover, to simplify the proof, we will restrict to the case $u = 0$. The other cases follow the same lines (the only change to make is the value of A introduced below (81) which should be replaced by $A' := a + (1 + c')\frac{\ln a}{\sqrt{a}}$).

Using the strong Markov property and the increasing property [16, 8], we can lower bound the left hand side of (73) by

$$\mathbb{P}_{-\sqrt{a}+\delta,0}\left[T_{-\sqrt{a}-\delta} < \frac{1}{\sqrt{a}} \wedge T_{-\sqrt{a}+2\delta}\right] \times \mathbb{P}_{-\sqrt{a}-\delta,\frac{1}{\sqrt{a}}}\left[T_{-\sqrt{a}-\varepsilon} < \frac{\ln a}{8\sqrt{a}} \wedge T_{-\sqrt{a}}\right] \tag{74}$$

$$\times \mathbb{P}_{-\sqrt{a}-\varepsilon,\frac{\ln a}{8\sqrt{a}}}\left[T_{-\infty} < \frac{3}{8}\frac{\ln a}{\sqrt{a}}\right]\,. \tag{75}$$

• The first probability gives the main cost. Using the comparison Theorem for sde (see [38, Proposition 2.18] or [39, Theorem (3.7), Chapter IX]), we see that the process Z_a starting from $Z_a(0) = \sqrt{a} + \delta$ and until the stopping time $T_{-\sqrt{a}-2\delta}$ is stochastically dominated for a large enough by the drifted Brownian motion $-\sqrt{a}+\delta+4\sqrt{a}\,\delta\,t+B(t)$. More precisely, this means that for all $t \leq T_{-\sqrt{a}+2\delta}$, we have a.s.

$$Z_a(t) \leq -\sqrt{a} + \delta + 4\sqrt{a}\,\delta\,t + B(t)\,.$$

Therefore, using the fact that $B(t) + 4\sqrt{a}\,\delta\,t \geq B(t)$ for all $t \geq 0$, we can lower bound the first probability of (74) by

$$\mathbb{P}\left[B(\frac{1}{\sqrt{a}}) < -5\frac{(\ln a)^{1/4}}{a^{1/4}}, \sup_{0\leq t\leq 1/\sqrt{a}} B(t) < \frac{(\ln a)^{1/4}}{a^{1/4}}\right]\,. \tag{76}$$

By the reflection principle [6] and Brownian scaling, this later probability is

$$\mathbb{P}[B(1) \leq -5(\ln a)^{1/4}] - \mathbb{P}[B(1) \geq 7(\ln a)^{1/4}] = \mathbb{P}[-7(\ln a)^{1/4} \leq B(1) \leq -5(\ln a)^{1/4}]$$
$$\geq c'\,\exp(-\frac{25}{2}\sqrt{\ln a})\,,$$

where the last inequality holds for a large enough and for a positive constant c'.

• For the second part, using again the comparison theorem for sde [38, Proposition 2.18], we can see that the process Z_a starting from $Z_a(\frac{1}{\sqrt{a}}) = -\sqrt{a} - \delta$ is almost surely below the Brownian motion for $t \in [\frac{1}{\sqrt{a}}; T_{-\sqrt{a}}]$. More precisely,

$$Z_a(t) \leq -\sqrt{a} - \delta + B(t) - B(\frac{1}{\sqrt{a}})$$

[6] For $b \leq a$ and $t \geq 0$, $\mathbb{P}[\sup_{0\leq s\leq t} B(s) \geq a, B(t) \leq b,] = \mathbb{P}[B_t \geq 2a - b]$.

for all $t \in [\frac{1}{\sqrt{a}}; T_{-\sqrt{a}}]$.

Therefore the second probability is bounded from below by

$$\mathbb{P}\left[B(\frac{\ln a}{8\sqrt{a}}) \leq \delta - \varepsilon, \sup_{0\leq t\leq \frac{\ln a}{8\sqrt{a}}} B(t) \leq \delta\right] .$$

Again from reflexion principle and Brownian scaling property, this probability is equal to

$$\mathbb{P}\left[-\sqrt{8} - \frac{\sqrt{8}}{(\ln a)^{1/4}} \leq B_1 \leq -\sqrt{8} + \frac{\sqrt{8}}{(\ln a)^{1/4}}\right] =_{a\to+\infty} O(\frac{1}{(\ln a)^{1/4}}) .$$

This handles the second probability of (74) which does not contribute to the main cost as it is much larger than the first probability.

• For the last probability, the idea of [16] is to compare the diffusion Z_a with the solution of an (random) autonomous differential equation which can be computed explicitly. Set

$$G_a(t) := Z_a(t) - B(t) . \tag{77}$$

Denote

$$M := \sup_{0\leq t\leq \frac{3}{8}\frac{\ln a}{\sqrt{a}}} |B(t)| .$$

For $t \leq T_{-\sqrt{a}} \wedge \frac{3}{8}\frac{\ln a}{\sqrt{a}}$, we have

$$G_a(t) \leq -\sqrt{a} + M . \tag{78}$$

The function $G_a(t)$ satisfies the following (random) first order differential equation for $t > \frac{\ln a}{\sqrt{a}}$

$$G_a'(t) = \frac{\beta}{4}\, t + a - G_a(t)^2 \left(1 + \frac{B(t)}{G_a(t)}\right)^2 , \quad G_a(\frac{\ln a}{\sqrt{a}}) = -\sqrt{a} - \frac{\sqrt{\ln a}}{a^{1/4}} . \tag{79}$$

Now consider the event

$$\{M \leq \frac{1}{2}\frac{\sqrt{\ln a}}{a^{1/4}}\} ,$$

under which we are able control the solution G_a and its explosion time. Note that this event happens with a positive probability p independent of a. Moreover, under this event, we have $T_{-\sqrt{a}} \geq \frac{3}{8}\frac{\ln a}{\sqrt{a}}$ so that the inequality (78) is valid for any $t \leq \frac{3}{8}\frac{\ln a}{\sqrt{a}}$.

Thus, under the event $\{M \leq \frac{1}{2}\frac{\sqrt{\ln a}}{a^{1/4}}\}$ and for $t \leq \frac{3}{8}\frac{\ln a}{\sqrt{a}}$, we easily check that

$$\left(1 + \frac{B(t)}{G_a(t)}\right)^2 \geq 1 - 2\frac{M}{\sqrt{a} - M} \geq 1 - \frac{\sqrt{\ln a}}{a^{3/4} - \frac{\sqrt{\ln a}}{2}} .$$

Introduce now the function $H_a(t)$ solution of the (random) autonomous first order differential equation

$$H_a'(t) = A - B\, H_a(t)^2 , \quad H_a(0) = -\sqrt{a} - \frac{\sqrt{\ln a}}{a^{1/4}} \tag{80}$$

with

$$A := a + \frac{\ln a}{\sqrt{a}} \quad \text{and} \quad B := 1 - \frac{\sqrt{\ln a}}{a^{3/4} - \frac{\sqrt{\ln a}}{2}} \,. \tag{81}$$

It is easy to check (with the argument used to prove the increasing property) that, under the event $\{M \leq \frac{1}{2}\frac{\sqrt{\ln a}}{a^{1/4}}\}$, the function H_a dominates the function G_a a.s. for all $t \leq \frac{3}{8}\frac{\ln a}{\sqrt{a}}$,

$$G_a(\frac{\ln a}{\sqrt{a}} + t) \leq H_a(t) \,. \tag{82}$$

But Eq. (80) can be integrated with respect to t leading to

$$\frac{H_a(t) + \sqrt{\frac{A}{B}}}{H_a(t) - \sqrt{\frac{A}{B}}} = C\, e^{2\sqrt{AB}t}$$

where C can be computed explicitly from the initial condition in $t = 0$. From this expression, we see that the function H_a explodes after a finite time τ which can be computed explicitly and bounded as

$$\tau = \frac{1}{2\sqrt{AB}} \, \ln \frac{1}{C} \leq \frac{3}{8}\frac{\ln a}{\sqrt{a}} \,. \tag{83}$$

Using (77) and (82), we deduce that under the event $\{M \leq \frac{1}{2}\frac{\sqrt{\ln a}}{a^{1/4}}\}$, the diffusion Z_a explodes in a time smaller than $(1 + \frac{3}{8})\frac{\ln a}{\sqrt{a}}$. The Lemma is proved. □

APPENDIX C. ASYMPTOTIC OF MAIN INTEGRALS

With two consecutive change of variables, we obtain for $a > 0$,

$$\begin{aligned} m(a) &= \sqrt{2\pi}\, a^{1/4} \int_0^{+\infty} \frac{dx}{\sqrt{x}} \exp\left(2a^{3/2}(x - \frac{1}{12}x^3)\right) \\ &= \frac{\sqrt{2\pi}}{\sqrt{a}} \exp\left(\frac{8}{3}a^{3/2}\right) \int_{-2a^{3/4}}^{+\infty} \frac{dy}{\sqrt{2 + \frac{y}{a^{3/4}}}} \exp\left(-y^2 - \frac{1}{6}\frac{y^3}{a^{3/4}}\right) \\ &\sim_{a\to+\infty} \frac{\pi}{a^{1/2}} \exp\left(\frac{8}{3}a^{3/2}\right) \,. \end{aligned} \tag{84}$$

With a more precise analysis of the integral (84), we can in fact check that

$$m(a) = \frac{\pi}{\sqrt{a}} \exp\left(\frac{8}{3}a^{3/2}\right) \left(1 + \frac{5}{48}\frac{1}{a^{3/2}} + o(\frac{1}{a^3})\right) \,, \tag{85}$$

Then it is plain to deduce that, with $J_0(a) = 1/m(a)$,

$$J_0(a) = \frac{a^{1/2}}{\pi} \exp\left(-\frac{8}{3}a^{3/2}\right) \left(1 - \frac{5}{48}\frac{1}{a^{3/2}} + o(\frac{1}{a^3})\right) \,, \tag{86}$$

and

$$\int_{-\infty}^{\ell} J_0(t)\, dt \sim_{\ell\to-\infty} \frac{1}{4\pi} \exp\left(-\frac{8}{3}|\ell|^{3/2}\right) \left(1 - \frac{5}{48}\frac{1}{|\ell|^{3/2}} + o(\frac{1}{|\ell|^3})\right) \,. \tag{87}$$

Differentiating $J_0(a)$ with respect to a, we obtain for $a > 0$

$$\begin{aligned} J_0'(a) &= -2\sqrt{2\pi}\, J_0(a)^2 \int_0^{+\infty} dv\, \sqrt{v}\, \exp\left(2a\, v - \frac{1}{6}v^3\right) \\ &= -2\sqrt{2\pi}\, J_0(a)^2\, a^{3/4} \int_0^{+\infty} dx\, \sqrt{x}\, \exp\left(2a^{3/2}(x - \frac{1}{12}x^3)\right) \\ &= -2\sqrt{2\pi}\, J_0(a)^2 \int_{-2a^{3/4}}^{+\infty} dy\, \sqrt{2 + \frac{y}{a^{3/4}}}\, \exp\left(-y^2 - \frac{1}{6}\frac{y^3}{a^{3/4}}\right) \\ &\sim_{a\to+\infty} -\frac{4}{\pi}\, a\, \exp\left(-\frac{8}{3}a^{3/2}\right). \end{aligned} \tag{88}$$

APPENDIX D. CONSISTENCY OF PERTURBATION THEORY

Perturbation theory was used to obtain approximation at leading order of the solution of the partial differential equation Eq. (53).

The number $\rho_\beta(]-\infty;\ell])$ of eigenvalues below the level ℓ was then computed from the perturbative solution of the pde Eq. (53).

In this section, we check the validity of perturbation theory by verifying that the correction terms remain negligible compared to the leading solution obtained.

D.1. **Correction to the leading order.** We compute the $O(1)$ correction to the leading order in the expansion Eq. (57) of $\rho_\beta(]-\infty;\ell])$ and show that, for all values of ℓ, perturbation theory leads to a consistent expansion at small β of $\rho_\beta(]-\infty;\ell])$ such that the $O(1)$ correction is much smaller than the leading term of order $1/\beta$. In particular this holds for $\ell := \ell_\beta$ scaling with β as $\ell_\beta \sim -\ln(1/\beta)^{2/3}$ when $\beta \to 0$. This was not obvious in the first place and needed to be checked in particular for ℓ_β in the scaling region of the minimal eigenvalues at small β.

The $O(1)$ correction denoted $\Gamma^1_\beta(\ell)$ such that

$$\rho_\beta(]-\infty;\ell]) = \frac{4}{\beta}\int_{-\infty}^{\ell} J_0(u)\, du + \Gamma^1_\beta(\ell) + o(\beta)\,. \tag{89}$$

can be obtained by computing first the linear correction $J_1(z,t)$ to the flux $J^q_\beta(z, 4t/\beta) = J^p_\beta(z,t) = J_0(\ell - t) + \beta J_1(z,t) + o(\beta)$. The flux J_1 is related to the function p_1 introduced in Eq. (55) and satisfies $J_1(z,t) = (z^2 + \ell - t)p_1(z,t) + \frac{1}{2}\frac{d}{dz}p_1(z,t)$. We thus need to compute p_1. This can be done by identifying the linear terms in β on both sides of equation (54), we obtain the following ordinary differential equation for $p_1(z,t)$

$$\frac{d}{dz}\left[(z^2 + \ell - t)\, p_1(z,t) + \frac{1}{2}\frac{d}{dz}p_1(z,t)\right] = \frac{d}{dt}p_0^{\ell-t}(z)\,. \tag{90}$$

By a further integration with respect to z, Eq. (90) becomes

$$J_1(z,t) = \frac{d}{dt}\int_{-\infty}^{z} p_0^{\ell-t}(u)\, du + j_1(t)$$

where $j_1(t)$ is a constant which does not depend on z. As mentioned above, we are interested only in $\lim_{z\to-\infty} J_1(z,t) = j_1(t)$, which can be computed easily using the normalization

constraint $\int_{\mathbb{R}} p_1(z,t)dz = 0$ [7]. We find

$$j_1(t) = 4J_0'(\ell-t)J_0(\ell-t)\int_{-\infty}^{+\infty} dz \int_{-\infty}^{z} du \int_{-\infty}^{u} dv \int_{-\infty}^{v} dw\, e^{2(\ell-t)(u-z+w-v)+\frac{2}{3}(u^3-z^3+w^3-v^3)}$$
$$- 8J_0(\ell-t)^2 \int_{-\infty}^{+\infty} dz \int_{-\infty}^{z} du \int_{-\infty}^{u} dv \int_{-\infty}^{v} dw\,(v-w)\, e^{2(\ell-t)(u-z+w-v)+\frac{2}{3}(u^3-z^3+w^3-v^3)}\,.$$

We can finally deduce the correction $\Gamma^1_\beta(\ell)$ as a function of the flux $j_1(t)$

$$\Gamma^1_\beta(\ell) = \int_0^{+\infty} j_1(t)\, dt\,,$$

leading us to

(91)

$$\Gamma^1_\beta(\ell) = 4\int_{-\infty}^{\ell} d\lambda J_0'(\lambda)J_0(\lambda)\int_{-\infty}^{+\infty} dz \int_{-\infty}^{z} du \int_{-\infty}^{u} dv \int_{-\infty}^{v} dw\, e^{2\lambda(u-z+w-v)+\frac{2}{3}(u^3-z^3+w^3-v^3)}$$
$$- 8\int_{-\infty}^{\ell} d\lambda J_0(\lambda)^2 \int_{-\infty}^{+\infty} dz \int_{-\infty}^{z} du \int_{-\infty}^{u} dv \int_{-\infty}^{v} dw(v-w)e^{2\lambda(u-z+w-v)+\frac{2}{3}(u^3-z^3+w^3-v^3)}\,.$$

A careful analysis of this integral (see Appendix D.2) permits to extract the asymptotic behavior of $\Gamma^1_\beta(\ell)$ when $\ell \to -\infty$

$$\Gamma^1_\beta(\ell) \sim_{\ell\to-\infty} -\frac{3}{\pi}\,\ln|\ell|\,\exp\left(-4|\ell|^{3/2}\right)\,. \tag{92}$$

Using this estimate, we can check that the asymptotic range $\ell_\beta(x)$ for $x \sim O(1)$ (going to $-\infty$ when $\beta \to 0$) of the minimal eigenvalue L_0^β satisfies

$$\Gamma^1_\beta(\ell_\beta(x)) \propto -\ln\left(\ln\frac{1}{\beta}\right)\,\beta^{3/2} \ll \Gamma_\beta(\ell_\beta(x)) = O(1)\,.$$

Thus we see that the correction $\Gamma^1_\beta(\ell)$ remains negligible compared to $\rho_\beta(]-\infty;\ell])$ in the region $\ell \sim \ell_\beta(x)$.

D.2. **Other integrals.** If $\lambda < 0$, then

$$I_\lambda := \int_{-\infty}^{+\infty} dz \int_{-\infty}^{z} du \int_{-\infty}^{u} dv \int_{-\infty}^{v} dw\, \exp\left(2\lambda(u-z+w-v)+\frac{2}{3}(u^3-z^3+w^3-v^3)\right)$$
$$= \int_{-\infty}^{+\infty} dz \int_{-\infty}^{z} du \int_{-\infty}^{u} dv \int_{-\infty}^{v} dw\, \exp\left(2|\lambda|(z-u+v-w)+\frac{2}{3}(u^3-z^3+w^3-v^3)\right)$$
$$= |\lambda|^2 \int_{-\infty}^{+\infty} dz \int_{-\infty}^{z} du \int_{-\infty}^{u} dv \int_{-\infty}^{v} dw \exp\left(2|\lambda|^{3/2}\left((z-u+v-w)+\frac{1}{3}(u^3-z^3+w^3-v^3)\right)\right)\,.$$

In the limit $\lambda \to -\infty$, we can determine the leading order of this integral with the saddle point method. We have to compute the maximum of the function

$$f(z,u,v,w) = (z-u+v-w)+\frac{1}{3}(u^3-z^3+w^3-v^3)$$

[7] p_1 is the coefficient of the linear correction in β to the probability density p_β: It should not bring mass in the integral.

on the domain of integration $D := \{(z,u,v,w) \in \mathbb{R}^4, u \leq z, v \leq u, w \leq v\}$. It is easily seen that it is reached for $z = 1$, $u = v$, $w = -1$ and is equal to $4/3$. Let us perform the following change of variables

$$z = 1 + \frac{x_1}{|\lambda|^{3/4}}, \quad u = x_2, \quad v = x_2 - \frac{x_3}{|\lambda|^{3/2}}, \quad w = -1 + \frac{x_4}{|\lambda|^{3/4}}\,.$$

Up to small corrections, when $\lambda \to -\infty$, the integral is equivalent to

$$|\lambda|^{-1} \exp\left(\frac{4}{3}|\lambda|^{3/2}\right) \int_{-\infty}^{+\infty} dx_1 e^{-x_1^2} \int_{-\infty}^{+\infty} dx_4 e^{-x_4^2} \int_0^{+\infty} dx_3 \int_{-1+\frac{x_3}{|\lambda|^{3/2}}+\frac{x_4}{|\lambda|^{3/4}}}^{1+\frac{x_1}{|\lambda|^{3/4}}} dx_2 \exp\left((x_2^2-1)x_3 - x_2\frac{x_3^2}{|\lambda|^{3/2}}\right)$$

Two contributions have to be accounted for. The variables x_1 and x_4 being fixed of order 1, we look separately at the integration over x_3 with $x_2 \leq 0$ and then with $x_2 \geq 0$.

$$\int_0^{+\infty} dx_3 \int_{-1+\frac{x_3}{|\lambda|^{3/2}}+\frac{x_4}{|\lambda|^{3/4}}}^{0} dx_2 \exp\left((x_2^2-1)x_3 - x_2\frac{x_3^2}{|\lambda|^{3/2}}\right)$$

$$\sim_{\lambda\to-\infty} \int_0^{+\infty} dx_3 \int_{-1+\frac{x_3}{|\lambda|^{3/2}}}^{0} dx_2 \exp\left(-(1-x_2^2)x_3\right) = \int_0^1 dx_2 \int_0^{(1-x_2)|\lambda|^{3/2}} dx_3 \exp\left(-(1-x_2^2)x_3\right)$$

$$= \int_0^1 \frac{dx_2}{1-x_2^2}\left(1 - \exp\left(-(1-x_2^2)(1-x_2)|\lambda|^{3/2}\right)\right) = \int_0^1 \frac{dt}{t(2-t)}\left(1 - \exp(-t^2(2+t)|\lambda|^{3/2})\right)$$

$$= \int_0^{|\lambda|^{3/4}} \frac{dx}{x\left(2-\frac{x}{|\lambda|^{3/4}}\right)}\left(1 - \exp\left(-x^2(2+\frac{x}{|\lambda|^{3/4}})\right)\right) \sim_{\lambda\to-\infty} \frac{3}{8}\ln|\lambda|\,.$$

Integrating now with respect to x_1 and x_4 leads to a contribution to the total integral I_λ of order

$$\pi\,|\lambda|^{-1}\,\exp\left(\frac{4}{3}|\lambda|^{3/2}\right)\,\frac{3}{8}\ln|\lambda|\,.$$

The region $x_2 \geq 0$ has to be treated differently: Here we have to keep the correction term $-x_2 x_3^2/|\lambda|^{3/2}$ in the exponential which prevents the integral to be infinite.

$$\begin{aligned}
&\int_0^{+\infty} dx_3 \int_0^{1+\frac{x_1}{|\lambda|^{3/4}}} dx_2 \exp\left((x_2^2-1)x_3 - x_2\frac{x_3^2}{|\lambda|^{3/2}}\right) \sim_{\lambda\to-\infty} \int_0^{+\infty} dx_3 \int_0^{1} dx_2 \exp\left((x_2^2-1)x_3 - x_2\frac{x_3^2}{|\lambda|^{3/2}}\right)\\
&= \frac{1}{|\lambda|^{3/2}}\int_0^{+\infty} dx_3 \int_0^{|\lambda|^{3/2}} dt \exp\left(-\frac{t}{|\lambda|^{3/2}}(2-\frac{t}{|\lambda|^{3/2}})x_3 - (1-\frac{t}{|\lambda|^{3/2}})\frac{x_3^2}{|\lambda|^{3/2}}\right)\\
&\sim_{\lambda\to-\infty} \frac{1}{|\lambda|^{3/2}}\int_0^{+\infty} dx \int_0^{|\lambda|^{3/2}} dt \exp\left(-\frac{2tx}{|\lambda|^{3/2}} - \frac{x^2}{|\lambda|^{3/2}}\right)\\
&= \frac{1}{|\lambda|^{3/4}}\int_0^{+\infty} dy \int_0^{|\lambda|^{3/2}} dt \exp\left(-\frac{2t}{|\lambda|^{3/4}}y - y^2\right)\\
&= \frac{1}{2}\int_0^{+\infty}\frac{dy}{y}e^{-y^2}\left(1-e^{-2y|\lambda|^{3/4}}\right)\\
&\sim_{\lambda\to-\infty} \frac{1}{2}\int_0^{1}\frac{dy}{y}e^{-y^2}\left(1-e^{-2y|\lambda|^{3/4}}\right) = \frac{1}{2}\int_0^{|\lambda|^{3/4}}\frac{dt}{t}e^{-t^2/|\lambda|^{3/2}}\left(1-e^{-2t}\right)\\
&\sim_{\lambda\to-\infty} \frac{3}{8}\ln|\lambda|\,.
\end{aligned}$$

Hence the region $x_2 \geq 0$ leads to the same contribution as the region $x_2 \leq 0$. We finally have

$$I_\lambda \sim_{\lambda\to-\infty} \frac{3\pi}{4}\frac{\ln|\lambda|}{|\lambda|}\exp\left(\frac{4}{3}|\lambda|^{3/2}\right)\,. \tag{93}$$

Next we need to analyse the integral

$$\begin{aligned}
K_\lambda &:= \int_{-\infty}^{+\infty} dz \int_{-\infty}^{z} du \int_{-\infty}^{u} dv \int_{-\infty}^{v} dw (v-w) e^{2\lambda(u-z+w-v)+\frac{2}{3}(u^3-z^3+w^3-v^3)}\\
&= |\lambda|^{5/2}\int_{-\infty}^{z} du \int_{-\infty}^{u} dv \int_{-\infty}^{v} dw (v-w)\exp\left(2|\lambda|^{3/2}\left((z-u+v-w)+\frac{1}{3}(u^3-z^3+w^3-v^3)\right)\right)
\end{aligned}$$

Up to small corrections, the integral K_λ takes the form

$$|\lambda|^{-1/2}\exp\left(\frac{4}{3}|\lambda|^{3/2}\right)$$

$$\int_{-\infty}^{+\infty} dx_1 e^{-x_1^2}\int_{-\infty}^{+\infty} dx_4 e^{-x_4^2}\int_0^{+\infty} dx_3 \int_{-1+\frac{x_3}{|\lambda|^{3/2}}+\frac{x_4}{|\lambda|^{3/4}}}^{1+\frac{x_1}{|\lambda|^{3/4}}} dx_2 (x_2+1)\exp\left((x_2^2-1)x_3 - x_2\frac{x_3^2}{|\lambda|^{3/2}}\right)\,.$$

The integral in the second line can be analysed following the previous lines. We check that the mass in the integral is carried this time exclusively by the region $x_2 \geq 0$. We obtain finally

$$K_\lambda \sim_{\lambda\to-\infty} \frac{3\pi}{4}\frac{\ln|\lambda|}{\sqrt{|\lambda|}}\exp\left(\frac{4}{3}|\lambda|^{3/2}\right)\,. \tag{94}$$

Gathering (86), (88), (93) and (94), we can deduce from (91) that

$$\Gamma_\beta^1(\ell) \sim_{\ell\to-\infty} -\frac{18}{\pi}\int_{-\infty}^{\ell} d\lambda\,\sqrt{|\lambda|}\,\ln|\lambda|\,\exp\left(-4|\lambda|^{3/2}\right)$$
$$\sim_{\ell\to-\infty} -\frac{3}{\pi}\,\ln|\ell|\,\exp\left(-4|\ell|^{3/2}\right)\,.$$

REFERENCES

[1] I. Dumitriu and A. Edelman. *Matrix Models for Beta Ensembles* J. Math. Phys. **43**, 5830-5847 (2002).

[2] G.W. Anderson, A. Guionnet, and O. Zeitouni.*An Introduction to Random Matrices*, Cambridge Studies in Advanced Mathematics (Cambridge University Press, Cambridge, 2009).

[3] Z. Bai and J. Silverstein. *Spectral Analysis of Large Dimensional Random Matrices* (Springer, New York, 2010), 2nd ed., see Theorem 9.2.

[4] M.L. Mehta. *Random Matrices* (Elsevier, New York, 2004).

[5] P.J. Forrester. *Log Gases and Random Matrices* (Princeton University Press, Princeton, 2010).

[6] G. Akemann, J. Baik, and Ph. Di Francesco. *The Oxford Handbook of Random Matrix Theory* (Oxford University Press, New York, 2011).

[7] A. Edelman and B. D. Sutton. *From Random matrices to stochastic operators*. J. Stat. Phys. **127** 6, 1121-1165 (2007).

[8] J. A. Ramírez, B. Rider and B. Virág. *Beta ensembles, stochastic Airy spectrum, and a diffusion*. J. Amer. Math. Soc. **24** 919-944 (2011).

[9] S. N. Majumdar and G. Schehr. *Top eigenvalue of a random matrix: large deviations and third order phase transition*. J. Stat. Mech. **1** 12 (2014).

[10] R. Allez, J.-P. Bouchaud and A. Guionnet. *Invariant β-ensembles and the Gauss-Wigner Crossover*. Phys. Rev. Lett. **109**, 094102 (2012).

[11] R. Allez and A. Guionnet. *A diffusive matrix model for invariant ?-ensembles*. Electron. J. Probab. **18** 62, 1-30 (2013).

[12] R. Allez, J.-P. Bouchaud, S. N. Majumdar, P. Vivo. *Invariant ?-Wishart ensembles, crossover densities and asymptotic corrections to the Marchenko-Pastur law*. J. Phys. A: Math. Theor. **46** 015001 (2013).

[13] K. Johansson. *From Gumbel to Tracy-Widom*. Prob. Th. Rel. Fields **138** 75-112 (2007).

[14] M. J. Bowick and E. Brézin. *Universal scaling of the tail of the density of eigenvalues in random matrix models*. Phys. Lett. B **268** 21-28 (1991).

[15] C. Texier. *Individual energy level distributions for one-dimensional diagonal and off-diagonal disorder*. J. Phys. A: Math. Gen. **33** 6095 (2000).

[16] L. Dumaz and B. Virág. *The right tail exponent of the Tracy-Widom-beta distribution*. Ann. Inst. H. Poincaré Probab. Statist. **49**, 4, 915-933, (2013).

[17] G. Borot and C. Nadal. *Right tail expansion of Tracy-Widom beta laws*. Random Matrices: Theory Appl. **1** 1250006 (2012).

[18] P. J. Forrester. *Spectral density asymptotics for Gaussian and Laguerre ?- ensembles in the exponentially small region*. J. Phys. A Math. Gen. **45** 075206 (2012).

[19] A. Bloemendal and B. Virag. *Limits of spiked random matrices I*. Probab. Theory Relat. Fields (online first) (2012).

[20] M. Freidlin and A. Wentzell. *Random perturbations of dynamical systems, 3rd edition*. Grundlehren der Mathematischen Wissenschaften, Springer (2012).

[21] A. Dembo and O. Zeitouni. *Large deviations techniques and applications (second edition)*, Springer (1998).

[22] S. Varadhan. *Large Deviations and Applications*. École d'Été de Probabilités de Saint-Flour, Springer, Academic Press, New York, London; Akademie-Verlag, Berlin (1986).

[23] H. L. Frisch and S. P. Lloyd. *Electron Levels in a One-Dimensional Random Lattice*. Phys. Rev. **120** 1175-1189 (1960).

[24] B. I. Halperin. *Green's Functions for a Particle in a One-Dimensional Random Potential*. Phys. Rev. **139** A 104-A117 (1965).

[25] H. P. McKean. *A Limit Law for the Ground State of Hill's Equation*. J. Stat. Phys. **74**, 1227 (1994).

[26] M. H. Holmes. *Introduction to Perturbation Methods*. Texts Appl. Math., **20**, Springer-Verlag, New-York, (1998).

[27] P. Hanggi, P. Talkner and M. Borkonec. *Reaction-rate theory: fifty years after Kramers*. Rev. Mod. Phys. **62** 251-341 (1990).

[28] O. Kallenberg. *Random Measures, 4th edition*. Academic Press, New York, London; Akademie-Verlag, Berlin (1986).

[29] R. Allez and L. Dumaz. *Random matrices in non-confining potentials*. arXiv:1404.5265 (2014).

[30] S. A. Molcanov. *The local structure of the spectrum of the one-dimensional Schrödinger operator*. Commun. Math. Phys. **78**,429-446 (1981).

[31] L. N. Grenkova, S. A. Molcanov, and Ju. N. Sudarev. *On the basic states of one-dimensional disordered structures*. Commun. Math. Phys. **90**,101-123 (1983).

[32] J.-P. Bouchaud and P. Cizeau. *Theory of Lévy matrices*. Phys. Rev. E **3**, 1810-1822 (1994).

[33] C. Bordenave and A. Guionnet. *Localization and delocalization of eigenvectors for heavy-tailed random matrices*. Probab. Theory Relat. Fields (online first) (2013)

[34] V. Bapst. *The Large Connectivity Limit of the Anderson Model on Tree Graphs*. arXiv:1303.4908 (2013).

[35] G. Biroli, A. C. Ribeiro-Teixeira and M. Tarzia. *Difference between level statistics, ergodicity and localization transitions on the Bethe lattice* arXiv:1211.7334 (2013).

[36] P. J. Forrester. *The spectrum edge of random matrix ensembles*. Nucl. Phys. B **402**, 709-728 (1993).

[37] K. Itô and H. P. McKean. *Diffusion processes and their Sample Paths*. Springer.

[38] I. Karatzas and S. E. Shreve. *Brownian Motion and Stochastic Calculus, Second Edition*. Graduate Texts in Mathematics, Springer.

[39] D. Revuz and M. Yor. *Continuous Martingales and Brownian Motion, Third edition*, Springer.