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Reverse inequalities for slowly increasing sequences and functions

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Abstract

We consider sharp inequalities involving slowly increasing sequences and functions, i.e., functions $f(t)$ with $f'(t) \leq 1$ and sequences (a_i) with $a_{i+1} - a_i \leq 1$. The inequalities are reverse to mean inequalities, for example. In the continuous case, integrals of powers are estimated by powers of integrals, whereas in the discrete case powers of sums are estimated by sums of powers of sums. The problem is connected with interpolation theory in Banach spaces, one of them $W^{1,\infty}$.

1 Introduction

For general nonnegative functions or sequences we know many classical inequalities estimating lower powers by higher powers (see, e.g., [3]). If the functions or sequences are bounded, it is well known that there can be derived reverse inequalities (see, e.g., [1, 4]).
This is also possible if the functions or sequences are slowly increasing. We consider inequalities, involving continuous differentiable real valued functions $f(t)$ defined on $0 \le t \le a$ with the properties $f(t) \ge 0$, $f(0) = 0$ and $f'(t) \le 1$ (slowly increasing functions); and real valued sequences (a_i), $i = 0, 1, 2, ..., n$ with the properties $a_i \ge 0$, $a_0 = 0$ and $a_{i+1} - a_i \le 1$ (slowly increasing sequences). In the whole paper we do not consider the trivial case $a_i \equiv 0$ or $f(t) \equiv 0$.
For example, such inequalities arise in interpolation theory in Banach spaces, where one of the interpolating spaces is $W^{1,\infty}$ (functions with bounded derivatives). In interpolation theory a common problem is to estimate the norm of an interpolated space by the product of the two norms of the interpolating spaces (see, e.g., [5])

$$\|\psi\|_{[X,Y]_\theta} \le c\|\psi\|_X^{1-\theta}\|\psi\|_Y^{\theta} \ .$$

where $\theta \in [0, 1]$ is the interpolation parameter. An example is the following

Theorem 1 *Let $g(t)$, $0 \le t \le a$ be a continuous differentiable function with $g(0) = 0$ and $g(t) \ge 0$. Then, for $k \ge 1$ the following inequality holds*

$$k\int_0^a g^{2k-1}(t)dt \le \left[2\sup_{t\in[0,a]} g'(t)\right]^{k-1}\left(\int_0^a g(t)dt\right)^k \tag{1}$$

Proof. This Theorem is a special case of Theorem 3, that will be proved in section 3. □
Inequality (1) is similar (after taking the $1/(2k-1)$-th power of the inequality) to

$$\|g\|_{L_{2k-1}} \le C\|g\|_{W^{1,\infty}}^{\frac{k-1}{2k-1}} \cdot \|g\|_{L_1}^{\frac{k}{2k-1}} \tag{2}$$

estimating the norm of a L_q-space ($q = 2k-1$) embedded in the interpolated space $[L_1, W^{1,\infty}]_\theta$ between $L_1[0, a]$ and $W^{1,\infty}[0, a]$ with $\theta = \frac{k}{2k-1}$.
In what follows, we consider functions $f(t)$ with $f'(t) \le 1$. The general case follows from $f(t) = \frac{1}{c}g(t)$ with $c = \sup_{t\in[0,a]} g'(t)$. Inequality (1) for such functions reads

$$k\int_0^a f^{2k-1}(t)dt \le 2^{k-1}\left(\int_0^a f(t)dt\right)^k \ . \tag{3}$$

This inequality is sharp. Taking the linear function $f(t) = t$ we get from (3) the identity

$$k\frac{a^{2k}}{2k} = 2^{k-1}\left(\frac{a^2}{2}\right)^k$$

2 A sharp discrete version

Looking for a discrete version of inequality (3), one could expect

$$2^{k-1}\left(\sum\nolimits_{i=1}^n a_i\right)^k \ \ge \ k\sum\nolimits_{i=1}^n a_i^{2k-1} \tag{4}$$

for non negative sequences $a_0 = 0, a_1, a_2, ..., a_n$ with $a_i - a_{i-1} \leq 1$, $i = 1, ..., n$ This inequality is true, though, it is not sharp unless $k > 2$. For $k = 2$ we get the inequality

$$\left(\sum_{i=1}^{n} a_i\right)^2 \geq \sum_{i=1}^{n} a_i^3 \tag{5}$$

which leads for $a_i = i$ to the identity

$$(1 + 2 + 3 + ... + n)^2 = 1^3 + 2^3 + 3^3 + ... + n^3$$

(well known as Nicomachus's Theorem) showing the sharpness of (4) for $k = 2$.
Inequality (5) is contained in [2] as an exercise.
In general, a sharp inequality is given by the following

Theorem 2 *Let $a_0 = 0, a_1, a_2, ..., a_n$ be a slowly increasing sequence of nonnegative real numbers, with $a_i - a_{i-1} \leq 1$, $i = 1, ..., n$ and $k \geq 1$ an integer. Then, the following inequality holds*

$$2^{k-1}\left(\sum_{i=1}^{n} a_i\right)^k \geq \sum_{2\not{|}j\geq 1} \binom{k}{j} \sum_{i=0}^{n} a_i^{2k-j} \tag{6}$$

(the sum on the right hand side is taken over the first odd integers j while $\binom{k}{j}$ is not 0). Equality holds for $a_i = i$ or $k = 1$.

To make the sum more clear, we write down the inequality, for the first k:

$$\begin{aligned}
2^0\left(\sum_{i=0}^{n} a_i\right)^1 &\geq 1\sum_{i=0}^{n} a_i^1 \\
2^1\left(\sum_{i=0}^{n} a_i\right)^2 &\geq 2\sum_{i=0}^{n} a_i^3 \\
2^2\left(\sum_{i=0}^{n} a_i\right)^3 &\geq 3\sum_{i=0}^{n} a_i^5 + \sum_{i=0}^{n} a_i^3 \\
2^3\left(\sum_{i=0}^{n} a_i\right)^4 &\geq 4\sum_{i=0}^{n} a_i^7 + 4\sum_{i=0}^{n} a_i^5 \\
2^4\left(\sum_{i=0}^{n} a_i\right)^5 &\geq 5\sum_{i=0}^{n} a_i^9 + 10\sum_{i=0}^{n} a_i^7 + \sum_{i=0}^{n} a_i^5 \\
2^5\left(\sum_{i=0}^{n} a_i\right)^6 &\geq 6\sum_{i=0}^{n} a_i^{11} + 20\sum_{i=0}^{n} a_i^9 + 6\sum_{i=0}^{n} a_i^7
\end{aligned}$$

Before proving Theorem 2 we define

$$\psi_k(x, y, z) = \left(\frac{z + x + y}{2}\right)^k - \left(\frac{z - x + y}{2}\right)^k - \left(\frac{z + x - y}{2}\right)^k + \left(\frac{z - x - y}{2}\right)^k$$

for real $x, y, z \geq 0$ and an integer $k \geq 0$ and show the following

Lemma 1 *For real $x, y, z \geq 0$ and an integer $k \geq 0$ we have $\psi_k(x, y, z) \geq 0$. Equality holds for $k < 2$, for odd k for $xyz = 0$ and for even k for $xy = 0$.*

Proof. Simple calculations show

$$\begin{aligned}
2^k\psi_k(x,y,z) &= (z+x+y)^k-(z+x-y)^k+(z-x-y)^k-(z-x+y)^k= \\
&= \sum_{i=0}^{k}\binom{k}{i}(z+x)^i\left(y^{k-i}-(-y)^{k-i}\right)+\sum_{i=0}^{k}\binom{k}{i}(z-x)^i\left((-y)^{k-i}-y^{k-i}\right)= \\
&= \sum_{i=0}^{k}\binom{k}{i}\left(y^{k-i}-(-y)^{k-i}\right)\left((z+x)^i-(z-x)^i\right)= \\
&= \sum_{i=0}^{k}\binom{k}{i}\left(y^{k-i}-(-y)^{k-i}\right)\sum_{j=0}^{i}\binom{i}{j}z^j\left(x^{i-j}-(-x)^{i-j}\right)\geq 0
\end{aligned}$$

because $x^i \geq (-x)^i$ for positive x and $i \geq 0$. The cases of equality are obvious. □

Remark 1 *For even k the power function is convex and we have*

$$\frac{z+x+y}{2}+\frac{z-x-y}{2}=\frac{z-x+y}{2}+\frac{z+x-y}{2}$$

thus, the Lemma is a consequence of Karamata's inequality (see, e.g., [3]). For odd k this argument fails, because $z-x-y$ can be negative, and x^k is not convex.

Proof. of Theorem 2: It is easy to verify

$$\frac{1}{2}\sum_{i=1}^{n}\left[(a_i^2+a_i)^k-(a_i^2-a_i)^k\right]=\sum\nolimits_{2\not| j\geq 1}\binom{k}{j}\sum\nolimits_{i=0}^{n}a_i^{2k-j}$$

Thus, we define

$$\varphi_n=2^{k-1}\left(\sum\nolimits_{i=1}^{n}a_i\right)^k-\frac{1}{2}\sum_{i=1}^{n}\left[(a_i^2+a_i)^k-(a_i^2-a_i)^k\right]$$

and show $\varphi_n \geq 0$. We set $A_1=0$ and

$$A_n=\sum\nolimits_{i=1}^{n-1}a_i\ .$$

Then, we get

$$\begin{aligned}
\varphi_1 &= 2^{k-1}a_1^k-\frac{1}{2}\left[(a_1^2+a_1)^k-(a_1^2-a_1)^k\right]= \\
&= \frac{1}{2}\left[(2a_1)^k-(0a_1)^k-(a_1^2+a_1)^k+(a_1^2-a_1)^k\right]= \\
&= \frac{1}{2}\psi_k(2a_1,a_1-a_1^2,a_1^2+a_1)
\end{aligned}$$

$$\begin{aligned}
\varphi_n &= 2^{k-1}\left(\sum\nolimits_{i=1}^{n} a_i\right)^k - \frac{1}{2}\sum_{i=1}^{n}\left[(a_i^2+a_i)^k-(a_i^2-a_i)^k\right] = \\
&= 2^{k-1}(A_n+a_n)^k - \frac{1}{2}\sum_{i=1}^{n-1}\left[(a_i^2+a_i)^k-(a_i^2-a_i)^k\right] - \frac{1}{2}\left[(a_n^2+a_n)^k-(a_n^2-a_n)^k\right] = \\
&= 2^{k-1}A_n^k - \frac{1}{2}\sum_{i=1}^{n-1}\left[(a_i^2+a_i)^k-(a_i^2-a_i)^k\right] + \\
&= \frac{1}{2}\left[(2A_n+2a_n)^k-(2A_n)^k-(a_n^2+a_n)^k+(a_n^2-a_n)^k\right] = \\
&= \varphi_{n-1}+\frac{1}{2}\psi_k(2a_n, 2A_n+a_n-a_n^2, 2A_n+a_n+a_n^2)
\end{aligned}$$

It follows

$$\varphi_n = \frac{1}{2}\sum_{i=1}^{n}\psi_k(2a_i, 2A_i+a_i-a_i^2, 2A_i+a_i+a_i^2)$$

To complete the proof, we use Lemma 1. For this purpose, we have to show that the arguments of ψ_k are nonnegative. The only nontrivial case is $2A_i+a_i-a_i^2 \geq 0$. We obtain

$$\begin{aligned}
2A_i+a_i-a_i^2 &= 2\sum\nolimits_{j=1}^{i-1} a_j + a_i - a_i^2 = \\
&= \sum\nolimits_{j=0}^{i-1} a_j + \sum\nolimits_{j=1}^{i} a_{j-1} + a_i - a_i^2 = \\
&= \sum\nolimits_{j=1}^{i}\left(a_j+a_{j-1}\right) - \sum\nolimits_{j=1}^{i}\left(a_j^2-a_{j-1}^2\right) = \\
&= \sum\nolimits_{j=1}^{i}\left(a_j+a_{j-1}\right)\left(1-a_j+a_{j-1}\right) \geq 0
\end{aligned}$$

Equality holds for $\psi_k(2a_i, 2A_i+a_i-a_i^2, 2A_i+a_i+a_i^2)=0$ for all $i=1,...,n$. From the Lemma follows that this is the case for $2A_i+a_i-a_i^2=0$. The last calculation shows that this happens for $a_j-a_{j-1}=1$ for all $j=1,...,n$, i.e., $a_j=j$.
Finally, we show equality for $a_i=i$. Using $\sum_{i=1}^{n} i = \frac{1}{2}n(n+1)$ we have to show

$$n^k(n+1)^k = 2\sum\nolimits_{2\not| j\geq 1}\binom{k}{j}\sum\nolimits_{i=0}^{n} i^{2k-j}$$

From $\sum\limits_{j=0}^{k}\binom{k}{j}i^j=(i+1)^k$ we can conclude

$$2\sum\nolimits_{2\not| j\geq 1}\binom{k}{j} i^{2k-j} = \left[i(i+1)\right]^k - \left[i(i-1)\right]^k$$

Now, the claim follows from induction over n, since

$$n^k(n+1)^k + \left(\left[(n+1)(n+2)\right]^k - \left[(n+1)n\right]^k\right) = (n+2)^k(n+1)^k$$

The induction basis is the well known formulae $\sum_{2\not| j\geq 1}\binom{k}{j} = 2^{k-1}$.
The case $k=1$ is $2^0\left(\sum_{i=0}^{n} a_i\right)^1 = 1\sum_{i=0}^{n} a_i^1$. $\square$

3 An estimate for integrals of powers of a function

Theorem 3 *Let $f(t)$, $0 \le t \le a$ be a continuous differentiable function with $f(0) = 0$, $f(t) \ge 0$ and $f'(t) \le 1$. Then, for real $k \ge 1$ and real $p > 0$ such that f^{kp-1} and f^{p-1} are integrable, the following inequality holds*

$$k \int_0^a f^{kp-1}(t)dt \le p^{k-1} \left(\int_0^a f^{p-1}(t)dt \right)^k \tag{7}$$

For $0 < k \le 1$ we get the opposite inequality. Equality holds for the linear function $f(t) = t$ or $k = 1$.

Proof. Let $0 \le t' \le a$ and $k \ge 1$. From $f'(t') \le 1$ and $f \ge 0$ follows

$$f'(t')f^{p-1}(t') \le f^{p-1}(t').$$

After integration we get

$$\int_0^t f'(t')f^{p-1}(t')dt' = \frac{1}{p} \int_0^t \frac{d}{dt'} f^p(t')dt' = \frac{1}{p} f^p(t) \le \int_0^t f^{p-1}(t')dt'$$

multiplying by $p > 0$ and taking the power to $k - 1$ it follows

$$f^{p(k-1)}(t) \le \left(p \int_0^t f^{p-1}(t')dt' \right)^{k-1} \tag{8}$$

multiplying by $kf^{p-1}(t)$ we obtain

$$kf^{kp-1}(t) \le kp^{k-1} f^{p-1}(t) \left(\int_0^t f^{p-1}(t')dt' \right)^{k-1}$$

and after integration

$$\begin{aligned} k \int_0^a f^{kp-1}(t)dt &\le p^{k-1} \int_0^a kf^{p-1}(t) \left(\int_0^t f^{p-1}(t')dt' \right)^{k-1} dt = \\ &= p^{k-1} \int_0^a \left(\frac{d}{dt} \left(\int_0^t f^{p-1}(t')dt' \right)^k \right) dt = p^{k-1} \left(\int_0^a f^{p-1}(t)dt \right)^k \end{aligned}$$

For $0 < k \le 1$ we get – beginning with (8) – the opposite inequalities.
Equality for $f(t) = t$ follows from the identity

$$k \int_0^a t^{kp-1}dt = \frac{k}{kp} a^{kp} = p^{k-1} \left(\frac{1}{p} a^p \right)^k = p^{k-1} \left(\int_0^a t^{p-1}dt \right)^k$$

The case $k = 1$ is obvious. □

Remark 2 *Inequality (3) is the special case $p = 2$ of Theorem 3.*

4 An estimate by products of integrals

Theorem 4 *Let $f(t)$, $0 \le t \le a$ be a continuous differentiable function with $f(0) = 0$, $f(t) \ge 0$ and $f'(t) \le 1$. Define*

$$F(p) = p \int_0^a f^{p-1}(t)dt \ . \tag{9}$$

Then, for real $k_1, k_2 \ge 1$ and real $p_1, p_2 > 0$ such that f^{p_1-1} and f^{p_2-1} are integrable the following inequality holds

$$F\left(k_1p_1 + k_2p_2\right) \le F^{k_1}(p_1)F^{k_2}(p_2) \tag{10}$$

Equality holds for the linear function $f(t) = t$ or $k_1 = k_2 = 1$.

Proof. Analogously to the proof of Theorem 3, inequality (8) we obtain for $k_1, k_2 \ge 1$ the inequalities

$$\begin{aligned}
f^{p_1(k_1-1)}(t) &\le \left(p_1 \int_0^t f^{p_1-1}(t')dt'\right)^{k_1-1} \\
f^{p_2k_2}(t) &\le \left(p_2 \int_0^t f^{p_2-1}(t')dt'\right)^{k_2} \\
f^{p_2(k_2-1)}(t) &\le \left(p_2 \int_0^t f^{p_2-1}(t')dt'\right)^{k_2-1} \\
f^{p_1k_1}(t) &\le \left(p_1 \int_0^t f^{p_1-1}(t')dt'\right)^{k_1}
\end{aligned}$$

Multiplying the first two and the last two inequalities, we get

$$\begin{aligned}
f^{p_1(k_1-1)+p_2k_2}(t) &\le p_1^{k_1-1}p_2^{k_2}\left(\int_0^t f^{p_1-1}(t')dt'\right)^{k_1-1}\left(\int_0^t f^{p_2-1}(t')dt'\right)^{k_2} \\
f^{p_1k_1+p_2(k_2-1)}(t) &\le p_2^{k_2-1}p_1^{k_1}\left(\int_0^t f^{p_1-1}(t')dt'\right)^{k_1}\left(\int_0^t f^{p_2-1}(t')dt'\right)^{k_2-1}
\end{aligned}$$

Multiplying the first inequality by $k_1p_1f^{p_1-1}(t)$, the second by $k_2p_2f^{p_2-1}(t)$ and adding them, yields for the left hand side

$$\text{l.h.s.} = \left(k_1p_1 + k_2p_2\right)f^{p_1k_1+p_2k_2-1}(t)$$

because of $p_1(k_1 - 1) + p_2k_2 + p_1 = p_1k_1 + p_2(k_2 - 1) + p_2 = p_1k_1 + p_2k_2 - 1$ and for the right hand side

$$\begin{aligned}
\text{r.h.s.} &= p_1^{k_1}p_2^{k_2}\Bigg(k_1f^{p_1-1}(t)\left(\int_0^t f^{p_1-1}(t')dt'\right)^{k_1-1}\left(\int_0^t f^{p_2-1}(t')dt'\right)^{k_2} + \\
&= k_2f^{p_2-1}(t)\left(\int_0^t f^{p_1-1}(t')dt'\right)^{k_1}\left(\int_0^t f^{p_2-1}(t')dt'\right)^{k_2-1}\Bigg) = \\
&= p_1^{k_1}p_2^{k_2}\frac{d}{dt}\left[\left(\int_0^t f^{p_1-1}(t')dt'\right)^{k_1}\left(\int_0^t f^{p_2-1}(t')dt'\right)^{k_2}\right]
\end{aligned}$$

For integrable f^{p_i-1} and $k_i \geq 1$, $f^{k_1p_1+k_2p_2-1}$ is integrable, too. Now, the desired inequality follows from integration.
The cases of equality are analogously to the proof of Theorem 3. □

Remark 3 *For $p_1 = p_2 =: p$ we get Theorem 3 with $k := k_1 + k_2$.*

By induction it is easy to obtain the following

Theorem 5 *Let $f(t)$, $0 \leq t \leq a$ be a continuous differentiable function with $f(0) = 0$, $f(t) \geq 0$ and $f'(t) \leq 1$. With definition (9) we have for real $k_i \geq 1$ and real $p_i > 0$ such that all f^{p_i-1} are integrable with $i = 1, ..., m$ the following inequality*

$$F\left(\sum_{i=1}^{m} k_i p_i\right) \leq \prod_{i=1}^{m} F^{k_i}(p_i) \tag{11}$$

Equality holds for the linear function $f(t) = t$ or $k_i = 1$.

Remark 4 *Remark 3 shows that an opposite inequality for $0 < k_1, k_2 \leq 1$ is not true in general, because it may happens that $0 < k_1, k_2 \leq 1$ but $k_1 + k_2 > 1$.*

Remark 5 *The case $k_i \geq 0$, $\sum_{i=1}^{m} k_i = 1$ would be the exponential version of Jensen's inequality. Indeed, $F(p)$ is log-concave, since*

$$\begin{aligned}
F^2(p)\big(\log F(p)\big)'' &= F''(p)F(p) - F'^2(p) = \\
&= \left(\int_0^a \big(2 + p\log f(t)\big)\log f(t) f^{p-1}(t)dt\right)\left(\int_0^a p f^{p-1}(t')dt'\right) - \\
&- \left(\int_0^a \big(1 + p\log f(t)\big) f^{p-1}(t)dt\right)\left(\int_0^a \big(1 + p\log f(t')\big) f^{p-1}(t')dt'\right)
\end{aligned}$$

Hence, for $k_i \geq 0$, $\sum_{i=1}^{m} k_i = 1$ we have

$$F\left(\sum_{i=1}^{m} k_i p_i\right) \geq \prod_{i=1}^{m} F^{k_i}(p_i)$$

5 A special case: power functions

For $\alpha \geq 1$, $0 \leq t \leq a$ the function

$$f(t) = \frac{1}{\alpha a^{\alpha-1}} t^{\alpha}$$

satisfy $f(0) = 0$, $f(t) \geq 0$ and

$$f'(t) = \left(\frac{t}{a}\right)^{\alpha-1} \leq 1\ .$$

f^{p_i-1} and $f^{\sum_{i=1}^{m} k_i p_i - 1}$ are integrable, if the conditions $\alpha(p_i - 1) > -1$ or, equivalently,

$$\alpha p_i - \alpha + 1 > 0 \quad \Longleftrightarrow \quad p_i > \frac{\alpha - 1}{\alpha} \tag{12}$$

hold. $f^{\sum_{i=1}^m k_i p_i - 1}$ is automatically integrable since

$$\alpha\left(\sum_{i=1}^m k_i p_i - 1\right) + 1 = \sum_{i=1}^m \big(\alpha p_i - \alpha + 1\big) + \alpha \sum_{i=1}^m p_i(k_i - 1) + (m-1)(\alpha - 1) > 0$$

It follows

$$F(p) = p\int_0^a f^{p-1}(t)dt = p\left(\frac{1}{\alpha a^{\alpha-1}}\right)^{p-1} \int_0^a t^{\alpha p - \alpha} dt = \frac{p a^p}{\alpha^{p-1}(\alpha p - \alpha + 1)}$$

and therefore

$$\frac{\sum_{i=1}^m k_i p_i a^{\sum_{i=1}^m k_i p_i}}{\alpha^{\sum_{i=1}^m k_i p_i - 1}\left(\alpha \sum_{i=1}^m k_i p_i - \alpha + 1\right)} \le \prod_{i=1}^m \left(\frac{p_i a^{p_i}}{\alpha^{p_i - 1}(\alpha p_i - \alpha + 1)}\right)^{k_i} .$$

From this follows (the powers of a cancel)

$$\prod_{i=1}^m \left(\frac{\alpha p_i - \alpha + 1}{\alpha p_i}\right)^{k_i} \le \frac{\alpha \sum_{i=1}^m k_i p_i - \alpha + 1}{\alpha \sum_{i=1}^m k_i p_i} \tag{13}$$

with equality for $\alpha = 1$ or $k_i = 1$.
This inequality is true even for $0 < k_i \le 1$ with $\sum k_i = 1$. This follows from Jensen's inequality: The function

$$h(p) = \log\left(\frac{\alpha p - \alpha + 1}{\alpha p}\right) = \log\left(1 - \frac{\alpha - 1}{\alpha p}\right)$$

is concave because of

$$h''(p) = -\frac{(\alpha - 1)(2\alpha p - \alpha + 1)}{p^2(\alpha p - \alpha + 1)^2} \le 0 .$$

This is a consequence of the assumption $\alpha p - \alpha + 1 > 0$ since $2\alpha p - \alpha + 1 > \alpha p - \alpha + 1 > 0$. Thus, Jensen's inequality reads

$$\sum_{i=1}^m k_i h(p_i) \le h\left(\sum_{i=1}^m k_i p_i\right) \tag{14}$$

Taking the exponential function of this inequality shows (13).
If the sum of the k_i is smaller then 1, (13) fails, in general. This can be seen, looking at (13) for small k_i. Then, the left hand side of (13) is around 1, whereas the right hand side becomes negative.
If $\sum_{i=1}^m k_i = K > 1$, (14) holds as well, even if some of the k_i are less then 1. We have

$$K h(p/K) \le h(p), \quad \frac{p}{K} > \frac{\alpha - 1}{\alpha}$$

as a consequence of the inequality $\log(1 - 1/x) \ge K \log(1 - K/x)$ for $K > 1$ and $x \in (K, \infty)$. Then, from (14) follows (we can use it with $k_i := k_i/K$)

$$\frac{1}{K}\sum_{i=1}^m k_i h(p_i) \le h\left(\frac{1}{K}\sum_{i=1}^m k_i p_i\right) \le \frac{1}{K} h\left(\sum_{i=1}^m k_i p_i\right)$$

Thus, for such special functions, inequality (11) is a consequence of Jensen's inequality, too.

6 Some remarks on the general discrete case

Comparing inequality (3) with its discrete version (4), having equality for $a_i = i$, one can expect that the discrete version of

$$p^{k-1}\left(\int_0^a f^{p-1}(t)dt\right)^k \geq k\int_0^a f^{kp-1}(t)dt$$

is

$$p^{k-1}\left(\sum\nolimits_{i=1}^n a_i^{p-1}\right)^k \ \geq\ \sum\nolimits_j \alpha_j \sum\nolimits_{i=0}^n a_i^{pk-j}$$

with some suitable coefficients α_j and equality for $a_i = i$. For $p = 3$ this is

$$3^{k-1}\left(\sum\nolimits_{i=1}^n a_i^2\right)^k \ \geq\ \sum\nolimits_j \alpha_j \sum\nolimits_{i=0}^n a_i^{3k-j}$$

Setting $s_m = \sum_{i=0}^n i^m$, simple calculations show for the first $k = 1, 2, ..., 6$

$$\begin{aligned}
s_2 &= s_2 \\
2^1 \cdot 3^1 \cdot s_2^2 &= 2s_3 + 2 \cdot 2^1 s_5 \\
2^2 \cdot 3^2 \cdot s_2^3 &= 3s_4 + 21s_6 + 3 \cdot 2^2 s_8 \\
2^3 \cdot 3^3 \cdot s_2^4 &= 4s_5 + 60s_7 + 120s_9 + 4 \cdot 2^3 s_{11} \\
2^4 \cdot 3^4 \cdot s_2^5 &= 5s_6 + 130s_8 + 11 \cdot 51s_{10} + 13 \cdot 40s_{12} + 5 \cdot 2^4 s_{14} \\
2^5 \cdot 3^5 \cdot s_2^6 &= 6s_7 + 240s_9 + 42 \cdot 43s_{11} + 6 \cdot 14 \cdot 43s_{13} + 16 \cdot 120s_{15} + 6 \cdot 2^5 s_{17}
\end{aligned}$$

For $k = 2$, this leads to the conjecture

$$3\left(\sum\nolimits_{i=1}^n a_i^2\right)^2 \ \overset{?}{\geq}\ \sum\nolimits_{i=0}^n a_i^3 + 2\sum\nolimits_{i=0}^n a_i^5$$

Setting $a_i = i/m$ for some $m \geq 1$, we obtain

$$3\left(\sum\nolimits_{i=1}^n a_i^2\right)^2 - \sum\nolimits_{i=0}^n a_i^3 - 2\sum\nolimits_{i=0}^n a_i^5 \ =\ \frac{(m-1)n^2(1+n)^2\big(4n(1+n)-(2+3m)\big)}{12m^5}$$

For a given $m > 1$ this is positive only for sufficiently large n. Or, reversely, The inequality fails for to slowly increasing sequences.
Thus, it seems, there is no general discrete analogon of inequality (7) with equality for $a_i = i$.

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