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A super-Brownian motion with a locally infinite catalytic mass

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Abstract

A super-Brownian motion X in $\mathbb{R}$ with "hyperbolic" branching rate $\varrho_2(b) = 1/b^2$, $b \in \mathbb{R}$, is constructed, which symbolically could be described by the formal stochastic equation

$$\mathrm{d}X_t = \tfrac{1}{2}\,\Delta X_t\,\mathrm{d}t + \sqrt{2\varrho_2 X_t}\,\mathrm{d}W_t\,, \qquad t \geq 0, \tag{1}$$

(with a space-time white noise $\dot{W}$).

If the finite starting measure X_0 does not have mass at $b = 0$, then this superprocess X will never hit the catalytic center: There is a Brownian stopping time τ strictly smaller than the hitting time of 0 such that Dynkin's stopped measure X_τ vanishes a.s.

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1 Introduction and results

1.1 Motivation and purpose

A continuous *super-Brownian motion (SBM)* $X = \{X_t;\ t \geq 0\}$ in $\mathbb{R}$ with *branching rate* $\varrho(b) \geq 0$, $b \in \mathbb{R}$, can *heuristically* be thought of as follows:

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Many particles with small mass move independently on the line $\mathbb{R}$ according to standard Brownian motions. Additionally each particle at position b may die with a large rate proportional to $\varrho(b)$, or it may split with the same rate into two particles situated again at b which continue to evolve independently and according to the same rules. If we now denote by $X_t(B)$ the mass at time t in the Borel set B, then the measure X_t describes the cloud of mass at time t. Although X_t is not integer-valued (since the mass of particles is asymptotically small), it is useful to interpret $X_t(\mathrm{d}b)$ as the mass of all *particles* situated in b at time t. (For background, we refer to the lecture notes Dawson [Daw93].)

In the simplest case, the branching rate ϱ is a constant. But it may also vary in space and even in time (*varying medium*). For instance, consider the case $\varrho(b) = (2\varepsilon)^{-1} 1\{|b-c| \le \varepsilon\}, \ \ b \in \mathbb{R}$, which means that branching is allowed only if particles are in a small neighborhood of a fixed point $c \in \mathbb{R}$, and then the rate is huge. Even the limiting model as $\varepsilon \to 0$ makes sense non-trivially (in this one-dimensional situation). Then formally one can write $\varrho = \delta_c$ (Dirac δ-function at c), and speak of a single *point catalyst* situated at c; see [DF94, DFLM95, FL95, Dyn95] or the surveys [DFL95, Fle94]. In Dynkin's [Dyn91a] terminology, in this case the branching phenomenon of the approaching particles is governed by the Brownian local time at c.

More generally, ϱ may be a fairly general non-negative Schwartz distribution, that is, the generalized derivative of a measure, which we denote by the same symbol ϱ; see [DF91, DFR91, DF95b, DLM94]. (Or ϱ could additionally be time-dependent, for instance a continuous super-Brownian motion, in which case the catalytic masses itself suffer a branching mechanism; see [DF95a].) But so far we know, a common assumption is that the generalized function ϱ is *locally integrable,* as in the δ-function case $\varrho = \delta_c$; that is, ϱ corresponds to a locally finite measure.

Our *first purpose* in this paper is to demonstrate that a super-Brownian motion X with a *locally infinite* branching rate measure ϱ may make sense (Theorem 3). Then of course the question arises whether such a branching measure-valued process has qualitatively new properties. Intuitively one can expect that X has significantly more extinction features in the area where the branching rate measure is locally unbounded.

Indeed, our *second aim* is to exhibit the following new effect. We consider a particular branching rate ϱ (as ϱ_2 in (1)) which has a sufficiently infinite (accumulated) catalytic mass around a center c. Then, starting X with a finite initial measure without mass in c, the branching population will never hit c. Actually, the infinite catalytic mass around c will *kill* all the hidden particles *before* they reach c (Theorem 5).

The figure shows a simulation of equation (1) with initial condition $X_0(b) \equiv 1$, but with the singular branching rate ϱ_2 replaced by the truncated rate $\varrho_2 \wedge K$ with $K = 10^4$. Large fluctuations around the catalytic center $c = 0$ are clearly exhibited, whereas extinction at $c = 0$ is not apparent, because of the truncation.

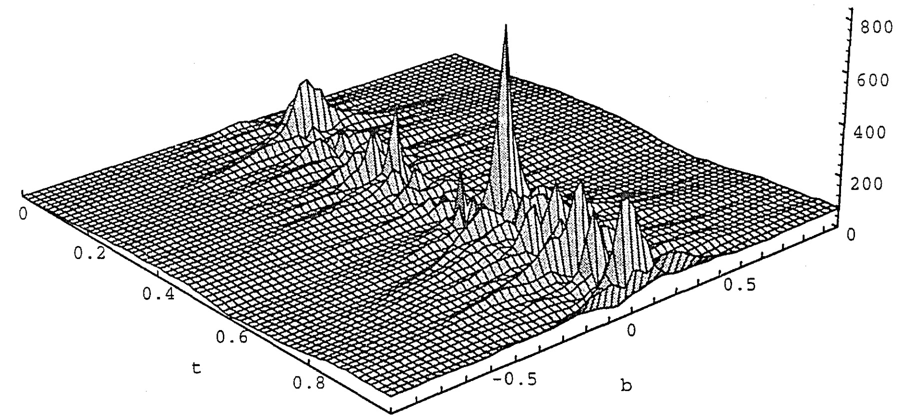

Figure: Simulation of a solution to equation (1)

1.2 Existence in the case of a locally infinite catalytic mass

As started above, we completely concentrate on the *one-dimensional* case, where, by the way, all the super-Brownian motions known so far have absolutely continuous states ([DFR91]). To warm up we will start with the *single point catalytic model* $\varrho = \theta\delta_c$, where $\theta \geq 0$ is an additional weight of the point catalyst, which we let tend to infinity. But it turns out that in this case the *limiting* model is degenerate: The limiting infinite point catalyst will only instantaneously kill the mass, that is, no mass is born. This results into the heat flow with absorption, i.e. the randomness of the model disappears (Proposition 7 at p. 8).

Going away from this degenerate situation, our *main model* is based on the following *branching rate*

$$\varrho_\sigma(b) := \frac{\theta}{|b-c|^\sigma}, \qquad b \in \mathbb{R}, \tag{2}$$

for $c \in \mathbb{R}$, $\theta > 0$ and $\sigma \geq 0$ fixed. If $0 \leq \sigma < 1$, we get a special case of a model constructed in [DF91, DFR91], since here in particular the catalytic measure $\varrho_\sigma(b)\,\mathrm{d}b$ is locally finite. On the other hand, by a limitation of our methods, as a rule we exclude the "supercritical" case $\sigma > 2$. For convenience, we introduce the following notation.

Definition 1 (hyperbolic branching rate) Under $1 \leq \sigma \leq 2$, the branching rate ϱ_σ of (2) is called *hyperbolic*. Moreover, we distinguish between a *moderate* hyperbolic branching rate if $1 \leq \sigma < 2$, and a *critical* one if $\sigma = 2$. Analogously, the related super-Brownian motions X (to be constructed in § 4.2) are also called *hyperbolic, moderate*, etc., in the respective cases. $\diamond$

Remark 2 The name *critical* hyperbolic branching rate is motivated by the fact that under $\sigma = 2$ (and $c = 0$) the related cumulant equation (26) admits *self-similar* solutions; see Remark 15 at p. 15. $\diamond$

Here is our first theorem (a more precise description will be given with Theorem 21 at p. 21):

Theorem 3 (hyperbolic SBM X)

(a) **(existence)** *There exists a non-degenerate (finite-measure-valued) super-Brownian motion $X = \{X_t;\ t \geq 0\}$ in $\mathbb{R}$ with hyperbolic branching rate ϱ_σ.*

(b) **(total mass process)** *The total mass process $X(\mathbb{R}) = \{X_t(\mathbb{R});\ t \geq 0\}$ is a supermartingale but no longer a martingale. Its variance is finite if and only if $\sigma < 2$.*

As opposed to the previously mentioned model of a point catalyst with a limiting infinite weight, in the present case outside of the catalytic center c now we have a non-degenerate critical branching mechanism allowing a proper stochastic process.

On the other hand, *intuitively speaking*, the infinite catalytic mass around the hyperbolic pole will again kill the Brownian particles eventually arriving at c. Thus, the underlying motion law should "effectively" be the Brownian motion W^c *killed at* c (non-conservative Markov process), and we will indeed finally work with W^c as "underlying" motion process. Note also that this heuristic picture of Brownian particles killed at c says that at c no birth of mass will occur. In particular, the usual criticality of the branching mechanism will "effectively" be violated at c. This also makes transparent why the total mass process $t \mapsto X_t(\mathbb{R})$ is *no longer* a martingale, as opposed to the usual critical super-Brownian motions with a locally finite branching rate measure.

The theorem leaves *open* the case $\sigma > 2$.

Remark 4 In the terminology of Dynkin [Dyn94, § 1.3.1], X is a subcritical superprocess with motion law given by the Brownian motion W^c killed at c. But note that in the case $\sigma = 2$ our existence claim (Theorem 3 (a)) is *not* covered, for instance, by the very general existence Theorem 3.4.1 of [Dyn94], since the branching rate ϱ_σ is *un*bounded and the functional $K(\mathrm{d}s) = \varrho_\sigma(W_s^c)\,\mathrm{d}s$ of Brownian motion W^c killed at c has *infinite* characteristic, that is the expectation of $\int_0^t K(\mathrm{d}s)$ is infinite, [Dyn94, (3.2.2)]. (In fact, the characteristic of K is finite if and only if $\sigma < 2$; to see this, use Lemma 6 below.) $\diamond$

1.3 Strong killing in the critical case $\sigma = 2$

Here is now a more important question: Is it perhaps possible that *all* the hidden Brownian particles *die* already *before* they reach c? The *main result* of the paper will be a *positive* answer to this question. To formulate it, we make use of the "stopped measure" X_τ in the sense of Dynkin [Dyn91a, Dyn91b]: Intuitively, if τ is a (finite) stopping time of Brownian motion, then X_τ describes the cloud of all the branching Brownian particles in their moments τ.

Theorem 5 (strong killing in the case of a critical ϱ_2) *Assume that X is a super-Brownian motion with a critical hyperbolic branching rate ϱ_2 and with starting measure X_0 satisfying $X_0(\{c\}) = 0$. Then there exists a stopping time τ of Brownian motion, which is strictly smaller than the Brownian (first) hitting time τ^c of the catalytic center c, such that the stopped measure X_τ vanishes with probability one.*

Consequently, here all population mass dies already before it reaches the catalytic center c, that is, the superprocess X *does not hit* c. Of course, at this stage the formulation of this theorem is a bit vague. Anyway, a precise description will be given with Theorem 24 at p. 25.

1.4 Tools and outline

An essential tool for our approach is the *historical* superprocess $\widetilde{X}$ related to X, we now roughly want to indicate (for background in the locally finite branching measure case, see [DP91] or [Dyn91b]). To this aim, the measures $X_t(\mathrm{d}a)$ on $\mathbb{R}$ are thought of to be refined by measures $\widetilde{X}_t(\mathrm{d}w)$, where w are continuous functions on $\mathbb{R}_+$ stopped at time t. Heuristically, each particle hidden in the cloud of mass X_t and situated at time t in a is now additionally equipped with its former and all their ancestors motion path w in space (motion path traced back from t to time 0).

A further refinement is used by switching to *stopped historical superprocesses:* Hidden particles are stopped at any stopping time $\tau < \tau^c$ of Brownian motion, instead of t, resulting into a random measure $\widetilde{X}_\tau(\mathrm{d}w)$ defined on paths w stopped at τ. Consequently, $\widetilde{X}_\tau$ describes the "traced back mass distribution of the cloud" from the point of view of the random moment τ.

The *outline of the paper* is as follows. In the next section the case of a single infinite point catalyst is investigated. In Section 3, we deal with the cumulant equation which we tread in a Feynman-Kac approach. The construction of the hyperbolic SBM is provided in Section 4, and indeed in a setting of historical superprocesses. The proof of the strong killing in the case of a critical hyperbolic branching rate then follows in the final section.

2 Single point-catalytic model: degeneration

As announced, in this section we discuss the degenerate case of a single, infinite point catalyst.

2.1 Preliminaries: Some notation

We adopt the following conventions. If E is a topological space then subsets of E will be equipped with the induced topology. Products of topological spaces will be endowed with the product topology. Measures on a topological space E will be defined on the Borel σ-algebra (generated by the open subsets of E). A measure m on E with $m(E \backslash E') = 0$ for some measurable E', that is, m is concentrated on E', will also be regarded as a measure on E' (and conversely).

If E_1 and E_2 are topological spaces, let $\mathcal{B}[E_1, E_2]$ denote the space of all *measurable* maps $f : E_1 \mapsto E_2$. Write $b\mathcal{B}[E_1, E_2]$ for the subset of all *bounded* functions. If E_2 is a normed space with norm $\|\cdot\|$, and E_1 compact, as a rule we equip $b\mathcal{B}[E_1, E_2]$ with the supremum norm $\|f\|_\infty := \sup\{\|f(e_1)\|;\ e_1 \in E_1\}$ of *uniform convergence*. Note that $b\mathcal{B}[E_1, E_2]$ is a Banach space if E_2 is. By $\mathcal{C}[E_1, E_2]$ and $b\mathcal{C}[E_1, E_2]$ we denote the spaces of all *continuous* f in $\mathcal{B}[E_1, E_2]$ or $b\mathcal{B}[E_1, E_2]$, respectively.

$\mathcal{M}[E_1]$ refers to the set of all finite (non-negative) measures on a Polish space E_1, endowed with the topology of *weak* convergence. The pairing $\langle \mu, \varphi \rangle$ abbreviates the integral $\int \mu(\mathrm{d}e_1)\, \varphi(e_1)$, $\mu \in \mathcal{M}[E_1]$, $\varphi \in \mathcal{B}[E_1, \mathbb{R}]$ (if it exists).

Write simply $\mathcal{B}$, $\mathcal{C}$, $\mathcal{M}$ etc., if the respective spaces E_i coincide with the real line $\mathbb{R}$. The lower index $+$ on the symbol of a set will always refer to the subset of all of its non-negative members.

2.2 Brownian motion killed at c

Let $W = [W, \Pi_a, a \in \mathbb{R}]$ denote the *standard Brownian motion* in $\mathbb{R}$ starting at time 0. We use the symbol Π_a also to describe the expectation with respect to the law Π_a for the process starting at a, and proceed similarly in related situations. For instance, for $\mu \in \mathcal{M}$, define $\Pi_\mu f(W) := \int \mu(\mathrm{d}a) \int \Pi_a(\mathrm{d}w)\, f(w)$, for reasonable functionals f.

Denote by $S = \{S_t;\ t \geq 0\}$ the Brownian semigroup acting on $\mathcal{B}_+$, and by p the related (continuous) Brownian transition density function,

$$p(t, a, b) = p(t, b-a) := \frac{1}{\sqrt{2\pi t}} \exp\Big[- \frac{(b-a)^2}{2t} \Big], \qquad t > 0, \quad a, b \in \mathbb{R},$$

(fundamental solutions of the heat equation).

Recall that τ^c refers to the *hitting time of* c of the Brownian motion W. Set

$$S_t^c \varphi(a) := \Pi_a \mathbf{1}\{\tau^c > t\} \varphi(W_t), \qquad t > 0, \quad a \in \mathbb{R}, \quad \varphi \in \mathcal{B}_+, \tag{3}$$

for a fixed $c \in \mathbb{R}$, and

$$\langle S_t^c \mu, \varphi \rangle := \langle \mu, S_t^c \varphi \rangle, \qquad t > 0, \quad \mu \in \mathcal{M}, \quad \varphi \in \mathcal{B}_+ .$$

We call $\{S_t^c;\ t > 0\}$ the *semigroup of Brownian motion killed at* c, and the dual $\{S_t^c \mu;\ t > 0\}$ the *heat flow with absorption at* c, starting with μ. This is justified by the fact that restricting to non-negative measurable functions φ defined only on $\{a \neq c\}$ and adding the unity operator, we actually get the semigroup of a non-conservative Markov process on $\{a \neq c\}$, the Brownian motion killed at c. Also, by the reflection principle of Brownian motion,

$$p^c(t, a, b) := 1\Big\{(a-c)(b-c) > 0\Big\}\Big[p(t, a-b) - p(t, a+b-2c)\Big], \qquad (4)$$

$t > 0,\ a, b \in \mathbb{R}$, yields the transition kernels related to S^c. Note that $p^c(t,a,b) > 0$ if and only if $(a-c)(b-c) > 0$, and that the p^c are the fundamental solutions of the heat equation with boundary value 0 at c.

For convenience, we expose at this place the following simple but useful transition density estimate.

Lemma 6 (bounds for p^c) *Let $c = 0$. Then there are constants $0 < c_1 < c_2$ such that*

$$c_1 \exp\big[-\tfrac{2a^2}{t}\big] \le \tfrac{t^{3/2}}{ab}\, p^c(t,a,b) \le c_2 \exp\big[-\tfrac{a^2}{8t}\big], \qquad t > 0, \quad 0 < b < \tfrac{a}{2}.$$

Proof To get the upper bound, use the mean value theorem and $\frac{a}{2} \le a + \vartheta b \le \frac{3}{2}a$ for $|\vartheta| \le 1$ and the assumption on b. On the other hand, to obtain the lower bound, apply the elementary estimate $e^{-x} - e^{-y} \ge (y-x)e^{-y},\ y \ge x \ge 0$, and $a + b \le 2a$. ∎

2.3 Single point-catalytic super-Brownian motion

Consider the *continuous single point catalytic super-Brownian motion* $X^\theta = \{X_t^\theta;\ t > 0\}$ with branching rate $\theta\delta_c$, where $c \in \mathbb{R}$ and, for the moment, also $\theta \ge 0$ is fixed. That is, X^θ is the continuous superprocess related to the formal cumulant equation

$$\frac{\partial}{\partial t} v_\theta = \frac{1}{2}\Delta v_\theta - \theta\,\delta_c\, v_\theta^2. \qquad (5)$$

Consequently, the critical branching phenomenon is restricted to the location c of the point catalyst whereas outside c only the heat flow acts. As usual for superprocesses, the connection to (5) is given by Laplace transition functionals:

$$E_\mu \exp\big\langle X_t^\theta, -\varphi\big\rangle = \exp\big\langle \mu, -v_\theta(t,\cdot)\big\rangle, \qquad \mu \in \mathcal{M}, \quad t > 0, \quad \varphi \in b\mathcal{B}_+,$$

where v_θ solves (5) in a mild sense with initial condition $v_\theta(0+,\cdot) = \varphi$. For a detailed exposition of this point-catalytic SBM X^θ we refer to [DF94] or

[FL95]. Note that μ serves as the initial measure X_0^θ of X^θ although we formally excluded X_0^θ from the notation X^θ for the sake of a simpler formulation of the following proposition.

Proposition 7 (degeneration) *As $\theta \to \infty$ and in the sense of weak convergence of all finite-dimensional distributions* (fdd), *the single point-catalytic super-Brownian motion X^θ degenerates to the heat flow with absorption at c :*

$$X^\theta \underset{\theta\to\infty}{\overset{\text{fdd}}{\Longrightarrow}} X^\infty := \bigl\{S_t^c X_0\,;\, t>0\bigr\}.$$

Roughly speaking, if the catalytic mass θ of the point catalyst will be infinite, then all population mass which arrives at the catalyst will immediately be killed, and no branching occurs anymore in the model. In particular, $X_t^\infty \equiv 0$ for $t>0$, provided that $X_0\bigl(\mathbb{R}\backslash\{c\}\bigr)=0$, that is if the initial measure X_0 is concentrated at c.

This proposition will be proved in the next subsection.

Remark 8 (Γ–stable catalysts) Let Γ denote the *stable* random measure on $\mathbb{R}$ with index $0<\gamma<1$ determined by its Laplace functional

$$E\exp\langle\Gamma,-\varphi\rangle \;=\; \exp\Bigl[-\int\mathrm{d}b\;\varphi^\gamma(b)\Bigr], \qquad \varphi\in b\mathcal{B}_+\,.$$

For the moment, consider the super-Brownian motion X^θ with branching rate $\theta\Gamma$, $\theta\geq 0$. That is, X^θ is the superprocess related to the formal equation

$$\frac{\partial}{\partial t}v_\theta \;=\frac{1}{2}\Delta v_\theta-\theta\,\Gamma\,v_\theta^2\,, \qquad v_\theta(0+)=\varphi\geq 0 \tag{6}$$

(see [DF91]). Then Proposition 7 suggests that X_t^θ weakly converges (as $\theta\to\infty$) to the *heat flow with absorption at* Γ, which should degenerate to $X_t^\infty\equiv 0$ for $t>0$, since the atoms of Γ (point catalysts) are *dense* in $\mathbb{R}$. In terms of the related equation (6) this should mean that $v_\theta(t,a)\underset{\theta\to\infty}{\longrightarrow}0,\;\; t>0,\; a\in\mathbb{R}.$ $\diamond$

2.4 Proof of the degeneration proposition

We start the *Proof of Proposition* 7 by recalling first the following approach [FL95] to the continuous single point catalytic super-Brownian motion X^θ with a finite initial state X_0^θ which w.l.o.g. can be assumed to be a deterministic measure $\mu\in\mathcal{M}$. Start with introducing the transition densities q of a standard *stable subordinator* with index $\frac{1}{2}$ on $\mathbb{R}_+$:

$$q(s,a,b)=q(s,b-a):=\frac{s}{\sqrt{2\pi\,(b-a)^3}}\,\exp\Bigl[-\frac{s^2}{2(b-a)}\Bigr], \tag{7}$$

$s > 0,\ \ 0 \le a < b$. Let U^θ denote the related *super-stable subordinator* in $\mathbb{R}_+$ with index $\frac{1}{2}$ and constant branching rate θ. Assume that the initial state $U_0 := U_0^\theta$ of U^θ is given by

$$U_0(\mathrm{d}r) := \mathrm{d}r \int \mu(\mathrm{d}b)\, q\big(|c-b|, r\big), \qquad r \ge 0, \tag{8}$$

with q defined in (7) and using the convention $\mathrm{d}r\, q(0,r) = \delta_0(\mathrm{d}r)$. That is, U_0 is the "law" of the hitting time τ^c of c of a Brownian motion starting at time 0 "distributed" according to the initial measure μ of X^θ (the latter has to be constructed). In particular, $U_0 = \delta_0$ if $\mu = \delta_c$.

Now let $V_\infty^\theta := \int_0^\infty \mathrm{d}s\, U_s^\theta$ denote the *total occupation time* (measure) related to the measure-valued process U^θ. Then the single point catalytic super-Brownian motion X^θ can be defined by

$$X_t^\theta(\mathrm{d}b) := S_t^c\mu(\mathrm{d}b) + \Big(\int_{[0,t)} V_\infty^\theta(\mathrm{d}s)\, q\big(|b-c|, t-s\big)\Big)\mathrm{d}b, \qquad t > 0; \tag{9}$$

see formula (16) in [FL95].

To understand this formula, recall that $q\big(|b|, s\big)$ is also Itô's Brownian excursion from 0 density at time s at $|b|$. Hence, X^θ results from two parts. Namely first from the initial mass described by μ which propagates according to the heat flow $S^c\mu$ with absorption at c. The second contribution comes from some randomly created mass which starts at time s from the catalyst with the amount $V_\infty^\theta(\mathrm{d}s)$ and spreads deterministically away according to the mentioned Itô's excursion density. In particular, V_∞^θ yields the occupation density measure of X^θ at c (super-local time measure at c).

Based on this representation formula (9) and by the Markov property of X^θ, for the proof of Proposition 7 it suffices to show that the total mass $V_\infty^\theta(\mathbb{R})$ of the total occupation measure V_∞^θ converges to 0 in distribution as $\theta \to \infty$. But from the definition of the super-stable subordinator U^θ follows that

$$t \mapsto \int_0^t \mathrm{d}s\ U_s^\theta(\mathbb{R}) =: V_t^\theta(\mathbb{R}), \qquad t \ge 0,$$

is the occupation time process related to *Feller's critical branching diffusion* with branching rate θ, starting with $U_0(\mathbb{R}) = \mu(\mathbb{R})$ (recall (8)). That is,

$$E \exp\big[-\lambda V_t^\theta(\mathbb{R})\big] = \exp\big[-\mu(\mathbb{R})\, u_\theta(t)\big], \qquad \lambda \ge 0,$$

where u_θ is the solution to the ordinary differential equation

$$\frac{\mathrm{d}}{\mathrm{d}t} u_\theta(t) = -\theta\, u_\theta^2(t) + \lambda \quad \text{with} \quad u_\theta(0) = 0.$$

But this equation can explicitly be solved getting

$$u_\theta(t) = \sqrt{\lambda/\theta}\,\tanh\left[t\sqrt{\lambda\theta}\right] \xrightarrow[t\to\infty]{} \sqrt{\lambda/\theta} \xrightarrow[\theta\to\infty]{} 0.$$

Therefore the $\frac{1}{2}$-stable law of $V_\infty^\theta(\mathbb{R})$ has the scaling parameter $\mu(\mathbb{R})/\sqrt{\theta}$ tending to 0, hence it converges weakly to the Dirac measure δ_0 as $\theta \to \infty$. Hence $V_\infty^\theta(\mathbb{R})$ tends to 0 in distribution as $\theta \to \infty$, finishing the proof of Proposition 7. ∎

3 Analytical tool: Feynman-Kac equation

As a preparation for the construction of the hyperbolic SBM X in $\mathbb{R}$ as claimed in the existence Theorem 3, in this section we want to introduce our analytical main tool for this: a Feynman-Kac equation. For convenience, here we restrict our attention to a fixed finite time interval $I := [0, T]$, $T \geq 0$. Since the *historical* superprocess $\widetilde{X}$ related to X we actually need is a time-*in*homogeneous process, it is convenient to work with a *backward* and *historical* setting from the beginning.

3.1 Preliminaries: Terminology and spaces

We start by introducing some terminology. If A, B are sets and $a \mapsto B^a$ is a map of A into the set of all subsets of B, then we write

$$A\widehat{\times}B^\bullet := \left\{[a,b];\ a \in A,\ b \in B^a\right\} = \bigcup\nolimits_{a\in A}\{a\} \times B^a \qquad (10)$$

for the *graph* of this map. Note that $A\widehat{\times}B^\bullet \subseteq A\times B$.

To each path w in the Banach space $\mathbf{C} := \mathcal{C}[I, \mathbb{R}]$, and $t \in I = [0, T]$, we associate the corresponding *stopped path* w^t by setting $w_s^t := w_{t\wedge s}$, $s \in I$. That is, the path is held constant after time t. The set of all *stopped paths* $w^t = \{w_s^t;\ s \in I\}$ is denoted by $\mathbf{C}^t$, getting (for t fixed) a closed subspace of $\mathbf{C}$. Note that $\mathbf{C}^s \subseteq \mathbf{C}^t$ if $s \leq t$, that $\mathbf{C}^T = \mathbf{C}$, and that $\mathbf{C}^0$ can be identified with $\mathbb{R}$, whereas $\mathbf{C}^t$ could also be considered as $\mathcal{C}\big[[0,t], \mathbb{R}\big]$.

To each path w in $\mathbf{C}$, we can also associate the corresponding *stopped path trajectory* $\widetilde{w}$ by setting: $\widetilde{w}_t := w^t$, $t \in I$. Note that $\widetilde{w}$ is a mapping of I into $\mathbf{C}$. Since, for $0 \leq s \leq t \leq T$,

$$\|\widetilde{w}_t - \widetilde{w}_s\|_\infty = \|w^t - w^s\|_\infty = \sup\nolimits_{s\leq r\leq t}\left|w_r - w_s\right| \xrightarrow[t-s\to 0]{} 0,$$

$\widetilde{w}$ actually belongs to the closed subspace

$$\widetilde{\mathbf{C}}(I) := \Big\{\omega \in \mathcal{C}[I, \mathbf{C}];\ \omega_t \in \mathbf{C}^t,\ t \in I\Big\} \qquad (11)$$

of the Banach space $\mathcal{C}[I, \mathbf{C}]$. Moreover,

$$\|\widetilde{v} - \widetilde{w}\|_\infty = \sup\nolimits_{t\in I}\|\widetilde{v}_t - \widetilde{w}_t\|_\infty = \|v^T - w^T\|_\infty = \|v - w\|_\infty, \qquad v, w \in \mathbf{C},$$

hence $w \mapsto \widetilde{w}$ maps $\mathbf{C}$ continuously into the space $\widetilde{\mathbf{C}}(I)$ of stopped path trajectories. Note also that $\widetilde{\mathbf{C}}(I) \subseteq I \widehat{\times} \mathbf{C}^{\bullet}$ where the latter is a closed subset of $I \times \mathbf{C}$.

3.2 Brownian path processes $\widetilde{W}$ and $\widetilde{W}^c$

Recall that Π_a denotes the distribution of a standard Brownian path W starting at $W_0 = a$. Now we regard it as a probability law at $\mathbf{C} = \mathcal{C}[I, \mathbb{R}]$. Applying then the map $w \mapsto \widetilde{w}$ from the previous subsection to W, we get the so-called *Brownian path process* $\widetilde{W} = \left[\widetilde{W}, \widetilde{\Pi}_{s,w}, s \in I, w \in \mathbf{C}^s\right]$ which is a time-inhomogeneous strong Markov process. In other words, at time s we start with a path $w = \widetilde{W}_s$ stopped at time s, and let evolve a path trajectory $\{\widetilde{W}_t;\ t \in [s,T]\}$ with law $\widetilde{\Pi}_{s,w}$ determined by a Brownian path $\{W_t;\ s \le t \le T\}$ starting at time s at w_s. We may and will regard $\widetilde{\Pi}_{s,w}$ as a probability law on $\widetilde{\mathbf{C}}([s,T])$ (recall (11)).

The *semigroup* of $\widetilde{W}$ will be denoted by

$$\widetilde{S}_{s,t}\,\varphi(w) := \widetilde{\Pi}_{s,w}\,\varphi(\widetilde{W}_t), \quad 0 \le s \le t \le T, \ \ w \in \mathbf{C}^s, \ \ \varphi \in b\mathcal{B}[\mathbf{C}^t, \mathbb{R}], \tag{12}$$

and the related *generator* by $\{\widetilde{A}_s;\ s \in I\}$:

$$\widetilde{A}_s \psi(w) = \lim_{h \downarrow 0} h^{-1}\left[\widetilde{S}_{s-h,s}\,\psi(w^{s-h}) - \psi(w)\right], \qquad w \in \mathbf{C}^s,$$

$\psi \in \mathcal{D}(\widetilde{A}_s)$ (that is $\psi \in b\mathcal{B}[\mathbf{C}^s, \mathbb{R}]$ such that the limit exists).

Analogously, we introduce the *standard Brownian motion killed at* c:

$$W^c = \left[W^c, \Pi_a^c, a \in \mathbb{R}\right],$$

and the related *Brownian path process killed at* c:

$$\widetilde{W}^c = \left[\widetilde{W}^c, \widetilde{\Pi}^c_{s,w}, s \in I, w \in \mathbf{C}^s\right].$$

Here the Π_a^c are *sub*probability laws satisfying, in particular,

$$\Pi_a^c(W_t^c \in \cdot\,) = \Pi_a\big(\tau^c > t,\ W_t \in \cdot\,\big), \qquad t \in I, \ \ a \in \mathbb{R}, \tag{13}$$

(where τ^c is the hitting time of the catalytic center c). Analogously,

$$\widetilde{\Pi}^c_{s,w}\big(\widetilde{W}_t^c \in \cdot\,\big) = \widetilde{\Pi}_{s,w}\big(\tau^c > t,\ \widetilde{W}_t \in \cdot\,\big), \qquad 0 \le s \le t \le T, \ \ w \in \mathbf{C}^s. \tag{14}$$

Recall that S^c is the semigroup related to the Brownian motion W^c killed at c, introduced in (3). Denote by $\widetilde{S}^c = \{\widetilde{S}^c_{s,t};\ 0 \le s \le t \le T\}$ the semigroup of $\widetilde{W}^c$.

As in the case of Brownian motion, we use notations as

$$\widetilde{\Pi}_{s,\mu}\,\varphi\big(\widetilde{W}_t\big) = \int \mu(\mathrm{d}w)\,\widetilde{\Pi}_{s,w}\,\varphi\big(\widetilde{W}_t\big), \tag{15}$$

$0 \le s \le t \le T,\ \mu \in \mathcal{M}(\mathbf{C}^s),\ \varphi \in b\mathcal{B}[\mathbf{C}^t, \mathbb{R}]$.

Of course, from $\widetilde{W}$ and $\widetilde{W}^c$ we can gain back W and W^c by projection. For instance, $W_t := (\widetilde{W}_t)_t$, which will repeatedly be used.

3.3 Truncated equation

Fix a constant $K > 1$ and consider the *truncated rate function* $\varrho_\sigma \wedge K$, where ϱ_σ is the hyperbolic branching rate from (2) (recall that $1 \le \sigma \le 2$).

For $\varphi \in b\mathcal{B}_+[\mathbf{C}, \mathbb{R}]$ fixed, let $V^K\varphi := v_K \ge 0$ denote the unique element in $b\mathcal{B}\big[I\widehat{\times}\mathbf{C}^\bullet, \mathbb{R}\big]$ (recall (10)) which solves the following non-linear equation (*"truncated" cumulant equation*)

$$v_K(s,\omega_s) \;=\; \widetilde{\Pi}_{s,\omega_s}\Big[\varphi(\widetilde{W}_T) - \int_s^T \mathrm{d}r\ (\varrho_\sigma \wedge K)(W_r)\ v_K^2\big(r, \widetilde{W}_r\big)\Big], \qquad (16)$$

$[s,\omega_s] \in I\widehat{\times}\mathbf{C}^\bullet$ (cf. e.g. Dynkin [Dyn91b, Theorem 1.1]). Note that by a formal differentiation using the semigroup $\widetilde{S} = \big\{\widetilde{S}_{s,t}\,;\ 0 \le s \le t \le T\big\}$ of $\widetilde{W}$, from this integral equation we get the partial differential equation

$$\begin{aligned} -\tfrac{\partial}{\partial s}\, v_K(s,\omega_s) \;&=\; \widetilde{A}_s v_K(s,\omega_s) - (\varrho_\sigma \wedge K)\big((\omega_s)_s\big)\, v_K^2\,(s,\omega_s), \\ &\text{with terminal condition } v_K(T,\omega_T) = \varphi(\omega_T). \end{aligned} \qquad (17)$$

(Here $-\frac{\partial}{\partial s}\, v_K(s,\omega_s) = \lim_{h\downarrow 0} h^{-1}\big[v_K\big(s-h, (\omega_s)^{s-h}\big) - v_K(s,\omega_s)\big]$.) Moreover, if φ is in the domain $\mathcal{D}(\widetilde{A}_T)$ of $\widetilde{A}_T$, then v_K actually solves (17). But then it also uniquely solves the following *"truncated Feynman-Kac equation"* (that is, Feynman-Kac version of (16)):

$$v_K(s,\omega_s) \;=\; \widetilde{\Pi}_{s,\omega_s}\, \varphi(\widetilde{W}_T) \exp\Big[-\int_s^T \mathrm{d}r\ (\varrho_\sigma \wedge K)(W_r)\, v_K\big(r, \widetilde{W}_r\big)\Big], \qquad (18)$$

$[s,\omega_s] \in I\widehat{\times}\mathbf{C}^\bullet$ (cf. Dynkin [Dyn94, Theorem 4.2.1]). By dominated convergence, also in this equation we can go back to any $\varphi \in b\mathcal{B}_+[\mathbf{C}, \mathbb{R}]$.

3.4 Simplified terminal functions

If the terminal function φ (defined on paths $w \in \mathbf{C}^T = \mathbf{C}$) in the truncated equation only depends on $|w_T - c|$ and even in a non-decreasing way, then also the solution $v_K = V^K\varphi$ has a similar property, which we now want to expose in a lemma. Recall that $K > 1$ is fixed.

Lemma 9 (simplified terminal condition) *Assume that $\varphi \in b\mathcal{B}_+[\mathbf{C}, \mathbb{R}]$ can be represented as*

$$\varphi(w) = f\big(|w_T - c|\big), \qquad w \in \mathbf{C} = \mathbf{C}^T,$$

with $f \in b\mathcal{B}[\mathbb{R}_+, \mathbb{R}_+]$ *being non-decreasing. Then the (unique) solution* $v_K = V^K\varphi$ *to the truncated equation* (16) *or* (18) *has a representation*

$$v_K(s, \omega_s) = g\big(s, \big|(\omega_s)_s - c\big|\big), \qquad [s, \omega_s] \in I \widehat{\times} \mathbf{C}^\bullet, \tag{19}$$

with $g \in b\mathcal{B}[I \times \mathbb{R}_+, \mathbb{R}_+]$ *being non-decreasing in the second coordinate.*

Remark 10 By an abuse of notation, in cases as in the lemma (and in similar situations), we simply write

$$v_K(s, \omega_s) = v_K\big(s, (\omega_s)_s - c\big) = v_K\big(s, \big|(\omega_s)_s - c\big|\big). \qquad \diamond$$

Proof of Lemma 9 Without loss of generality, we may assume that $c = 0$.

1° (*Trotter's product formula*) v_K can be thought of as arising in the following way. Fix $n >> 1$ and decompose the interval $[0, T]$ into small pieces of length T/n. Apply now alternatively both terms at the r.h.s. of (17), that is consider separately the pure Brownian path process and the pure quadratic absorption. (That this Trotter's product formula like procedure converges as $n \to \infty$ to v_K also in the present *non*-linear situation can be seen as follows: Via Laplace transition functionals, as in (41) below, one can switch to the corresponding Markov processes, and for their *linear* semigroups one can apply Trotter's product formula, as e.g. in [EK86, Corollary 1.6.7], to get the desired convergence result.)

For a proof by induction, assume that for some k, $0 \le k < n$, at time $s_k := (n-k)T/n$ a representation as in (19) is given (which is certainly true for $k = 0$). Now it suffices to show that such representation is reproduced at time s_{k+1}, if either only the Brownian path process acts, or only the quadratic absorption is effective. In fact, then the claim follows by taking the limit as $n \to \infty$.

2° (*pure Brownian path process semigroup*) By the semigroup property, we have

$$v_K(s_{k+1}, \omega_{s_{k+1}}) = \widetilde{\Pi}_{s_{k+1}, \omega_{s_{k+1}}} v_K\big(s_k, \widetilde{W}_{s_k}\big).$$

By the induction hypothesis and (19), we may continue with

$$= \Pi_{s_{k+1}, (\omega_{s_{k+1}})_{s_{k+1}}} g\big(s_k, |W_{s_k}|\big).$$

But this expression depends only on $a := \big|(\omega_{s_{k+1}})_{s_{k+1}}\big|$, and, moreover, in a non-decreasing way. To see this monotonicity, use a simple coupling argument. In fact, for fixed $0 \le a_1 < a_2$, consider a pair of coupled reflected standard Brownian motions denoted by $[Z^1, Z^2]$, starting at $[a_1, a_2]$, which evolve independently until they hit each other, and are identical afterwards. Then $Z^1 \le Z^2$, hence $g(Z^1) \le g(Z^2)$ from the assumed monotonicity of g, and the claim follows by taking expectations.

3° (*pure quadratic absorption*) We have to solve (in a mild sense) the equation

$$-\frac{\partial}{\partial s} v_K(s,\omega_s) = -(\varrho_\sigma \wedge K)\big((\omega_s)_s\big)\, v_K^2(s,\omega_s), \qquad [s,\omega_s] \in I \widehat{\times} \mathbf{C}^\bullet,$$

at time $s = s_{k+1}$. By the semigroup property of solutions, we may fix here our attention to the terminal condition $v_K(s_k, \omega_{s_k}) = g\big(s_k, |(\omega_{s_k})_{s_k}|\big)$, according to the induction hypothesis. As solution we get

$$v_K\big(s_{k+1}, \omega_{s_{k+1}}\big) = g(s_k, a)\Big[1 + g(s_k, a)(s_k - s_{k+1})\,(\varrho_\sigma \wedge K)(a)\Big]^{-1},$$

with $a := \big|(\omega_{s_{k+1}})_{s_{k+1}}\big|$, which is obviously non-decreasing in a. This finishes the proof. ■

3.5 Constant terminal functions

Lemma 9 is applicable in particular if the terminal function φ is a constant. Then we can complement that lemma by the following result. Recall that $K > 1$ is fixed.

Lemma 11 (temporal monotonicity at the catalytic center) *Assume that $\varphi \in b\mathcal{B}_+[\mathbf{C}, \mathbb{R}]$ equals the constant $m \geq 0$. Then the solution $v_K = V^K m$ to the truncated equation* (16)–(18) *has the following property. Fix $s \in [0,T)$. Consider only $\omega_t \in \mathbf{C}^t$ with $(\omega_t)_t = c$ for all $t \in [s,T)$. Then $v_K(t,\omega_t)$ is non-decreasing in $t \in [s,T)$.*

Proof Again we may set $c = 0$.

1° (*reformulation*) Fix $s < T$, and consider only those $[t,\omega_t] \in [s,T)\widehat{\times}\mathbf{C}^\bullet$ such that $(\omega_t)_t \equiv a$ on $[s,T)$ for some $a \in \mathbb{R}$. By Lemma 9 we may write $v_K(t,\omega_t) = v_K(t,a)$, and from (17) we get

$$-\frac{\partial}{\partial t} v_K(t,a) = \frac{1}{2}\Delta v_K(t,a) - (\varrho_\sigma \wedge K)(a)\, v_K^2(t,a), \qquad s \leq t < T, \;\; a \in \mathbb{R},$$

with constant terminal condition $v_K(T,a) \equiv m$. But for $a \in \mathbb{R}$ fixed, $v_K(t,a)$ only depends on $T - t$, and since we intent to use a scaling argument it is convenient to turn to a forward setting. Then it suffices to show that the solution v_K to

$$v_K(t,a) = m - \Pi_a \int_0^t \mathrm{d}r\; (\varrho_\sigma \wedge K)(W_r)\, v_K^2(t-r, W_r) \tag{20}$$

is non-increasing in $t > 0$ if $a = c = 0$.

2° (*scaling*) For $t > 0$ fixed, introduce

$$u_t(s,a) := v_K\big(ts, \sqrt{t}\,a\big), \qquad s > 0, \;\; a \in \mathbb{R}.$$

Then by Brownian scaling, from (20) we conclude

$$u_t(s,a) = m - \Pi_a \int_0^s \mathrm{d}r \; \varrho^t(W_r) \; u_t^2(s-r, W_r) \tag{21}$$

with

$$\varrho^t(b) := t(\varrho_\sigma \wedge K)\big(\sqrt{t}\, b\big) = \frac{\theta\, t^{1-\sigma/2}}{|b|^\sigma} \wedge (tK), \qquad b \in \mathbb{R}. \tag{22}$$

These new bounded coefficients ϱ^t in the absorption term of (21) are non-decreasing in $t > 0$ since $\sigma \le 2$. Hence the (unique) solutions u_t of equation (21) are non-increasing in $t > 0$. In particular, $u_t(1,0) = v_K(t,0)$ is non-increasing in $t > 0$. This finishes the proof. ∎

Remark 12 (limitation to $\sigma \le 2$) This is the first time in the present development we needed to restrict to $\sigma \le 2$. ◇

3.6 Limiting functions

Turning back to the truncated equation (16) or (18), we now replace the terminal function φ by $\varphi_K \in b\mathcal{B}_+[\mathbf{C}, \mathbb{R}]$, and assume that $\varphi_K \downarrow \varphi \in b\mathcal{B}_+[\mathbf{C}, \mathbb{R}]$ as $K \to \infty$. Then from monotonicity in both φ_K and $\varrho_\sigma \wedge K$, we obtain, for the corresponding solutions $v_K := V^K \varphi_K$, the following *pointwise limit assertion:*

$$v_K = V^K \varphi_K \downarrow \text{some } v =: V\varphi \ge 0 \text{ as } K \to \infty. \tag{23}$$

Note that at this stage the limiting $v = V\varphi$ could depend on the choice of the approaching sequence φ_K.

Lemma 13 (independence of the choice of the φ_K) *For φ in $b\mathcal{B}_+[\mathbf{C}, \mathbb{R}]$ fixed, the limiting function $v = V\varphi$ of (23) is independent of the choice of the approximating functions $\varphi_K \downarrow \varphi$.*

Proof Consider two sequences $\varphi_K \downarrow \varphi$ and $\psi_K \downarrow \varphi$. We may assume that $\varphi_K \le \psi_K$ (otherwise bound φ_K and ψ_K in-between $\varphi_K \wedge \psi_K$ and $\varphi_K \vee \psi_K$). Then $V^K \varphi_K \le V^K \psi_K$, and using the equation (16) or (18), and monotonicity, we may continue with

$$\le V^K \varphi_K + \widetilde{\Pi}_{s,\omega_s}\Big[\psi_K(\widetilde{W}_T) - \varphi_K(\widetilde{W}_T)\Big].$$

Then the claim immediately follows by bounded convergence as $K \to \infty$. ∎

Remark 14 (monotonicities) It is clear that the statements of the Lemmas 9 and 11 remain valid also for the limiting function $v = V\varphi$. ◇

Remark 15 (self-similarity) In the case $\sigma = 2$, $c = 0$, and for constant terminal functions m, the limiting functions $v = Vm$ are *self-similar* with respect to the Brownian scaling. In fact, for approximating v_K in a forward setting we have $v_K(L^2 t, La) = v_{L^2 K}(t,a)$, for each $L > 0$, since (20) is uniquely solvable. Letting $K \to \infty$ gives the claim. ◇

3.7 Disappearance at the catalytic center

Without any additional assumptions on φ, the limiting functions vanish at the catalytic center c in the following sense:

Lemma 16 (disappearance at the catalytic center) *For φ in $b\mathcal{B}_+[\mathbf{C}, \mathbb{R}]$, the limiting function $v = V\varphi$ of (23) has the following property:*

$$v(s, \omega_s) = 0 \quad if \quad (\omega_s)_s = c, \ \omega_s \in \mathbf{C}^s, \ s \in [0, T).$$

Proof Set again $c = 0$. Since $V\varphi \geq 0$ is non-decreasing in φ, without loss of generality we may assume that φ equals a constant $m > 0$. Start with considering an approximating solution v_K of (16) with terminal condition m, for a fixed $K > 1$.

For any $[s, \omega_s] \in I\widehat{\times}\mathbf{C}^\bullet$, by Lemma 9 (and recalling Remark 10) we may write $v_K(s, \omega_s) = v_K(s, |(\omega_s)_s|)$. Additionally, by the monotonicity statement in that lemma, we may continue with $\geq v_K(s, 0) \geq v(s, 0)$. Applying this to the integral term in (16) we get in particular

$$0 \leq \widetilde{\Pi}_{0,0} \int_0^T \mathrm{d}r \ (\varrho_\sigma \wedge K)(W_r) \ v^2(r, 0) \leq m.$$

Additionally, by monotone convergence as $K \to \infty$ we therefore conclude that

$$\Pi_0 \int_0^T \mathrm{d}r \ \varrho_\sigma(W_r) \ v^2(r, 0) \leq m.$$

However,

$$\Pi_0 \, \varrho_\sigma(W_r) = S_r \, \varrho_\sigma(0) \equiv +\infty, \qquad r > 0,$$

since $\sigma \geq 1$. Thus

$$v(r, 0) = 0 \quad \text{for almost all} \quad r \in [0, T). \tag{24}$$

Finally, combined with the monotonicity statement in Lemma 11 (recall Remark 14)), we get $v(r, 0) \equiv 0$ on $[0, T)$. Hence, the claim $v(s, \omega_s) = 0$ follows, finishing the proof. ■

Remark 17 (supercritical σ) Under $\sigma > 2$, letting in (22) first $K \to \infty$, we get a non-*in*creasing function in t which suggests that (instead of Lemma 11) $v(t, \omega_t)$ should be non-*in*creasing in t, provided that $(\omega_t)_t \equiv c$ on $[s, T)$. Then (24), which was derived by using only $\sigma \geq 1$, would imply that again v disappeared at the catalytic center in the sense of Lemma 16. ◇

3.8 Limiting equation: Existence of solutions

Here we want to show that, for φ and $T > 0$ fixed, the limiting function v introduced in (23) solves the formal limiting equation (as $K \to \infty$) arising from the Feynman-Kac equation (18), if we additionally switch to the Brownian path process $\widetilde{W}^c$ *killed* at c :

Proposition 18 (limiting equation: existence of solutions) *Given φ in $b\mathcal{B}_+[\mathbf{C}, \mathbb{R}]$, the limiting function $v = V\varphi$ of (23) satisfies the Feynman-Kac equation*

$$v(s,\omega_s) = \widetilde{\Pi}^c_{s,\omega_s} \varphi(\widetilde{W}^c_T) \exp\Big[-\int_s^T \mathrm{d}r \ \varrho_\sigma(W^c_r)\, v\big(r, \widetilde{W}^c_r\big)\Big], \tag{25}$$

$[s,\omega_s] \in [0,T)\widehat{\times}\mathbf{C}^\bullet$.

Note that (25) can symbolically be written as

$$-\frac{\partial}{\partial s} v(s,\omega_s) = \widetilde{A}_s v(s,\omega_s) - \varrho_\sigma\big((\omega_s)_s\big)\, v^2(s,\omega_s), \tag{26}$$

with terminal condition $v(T,\cdot) = \varphi$, and boundary condition $v(s,\omega_s)\big|_{(\omega_s)_s=c} = 0, \ \ s < T$.

Proof of Proposition 18 Fix $[s,\omega_s] \in [0,T)\widehat{\times}\mathbf{C}^\bullet$. If $(\omega_s)_s = c$ holds, then the r.h.s. of (25) disappears by the property (14) of the subprobabilities $\widetilde{\Pi}^c$. But by Lemma 16 also the l.h.s. vanishes. Therefore we may restrict our attention to the case $(\omega_s)_s \neq c$.

Going back to the approximating functions $v_K = V_K\varphi$, look at the truncated Feynman-Kac equation (18). First we restrict the expectation at the r.h.s. additionally to the event $\{\tau^c \le T\}$ (with τ^c the hitting time of the catalytic center) and want to show that this results in a negligible term as $K \to \infty$. In fact, this part of the expectation can be bounded from above by restricting the integral in the exponent additionally to $r > \tau^c$. Next we use the strong Markov property at time τ^c, and the uniqueness of the solutions to (18). Then the resulting upper bound of this part of the r.h.s. of (18) can be written as

$$\widetilde{\Pi}_{s,\omega_s} 1\{\tau^c \le T\}\, v_K\big(\tau^c, \widetilde{W}_{\tau^c}\big).$$

By monotone convergence this tends to $\widetilde{\Pi}_{s,\omega_s} 1\{\tau^c \le T\}\, v\big(\tau^c, \widetilde{W}_{\tau^c}\big)$ as $K \to \infty$. However, this vanishes, since by $(\widetilde{W}_{\tau^c})_{\tau^c} = W_{\tau^c} = c$ the latter v-expression disappears on the event $\{\tau^c < T\}$ according to Lemma 16, and since τ^c has a continuous law.

It remains to show that

$$\widetilde{\Pi}_{s,\omega_s}\,1\{\tau^c > T\}\,\varphi(\widetilde{W}_T)\exp\Big[-\int_s^T \mathrm{d}r\;(\varrho_\sigma \wedge K)(W_r)\,v_K\big(r,\widetilde{W}_r\big)\Big] \tag{27}$$

converges as $K \to \infty$ to the analogous expression without involving the K (recall the identity (14)). This we will provide via two-sided estimates.

First of all, to estimate from above, switch in (27) from v_K to v, and let K tend to ∞. Then the desired limit term will come out by monotone convergence based on $(\varrho_\sigma \wedge K) \uparrow \varrho_\sigma$.

Concerning the other direction, pass from $(\varrho_\sigma \wedge K)$ to ϱ_σ in (27). If we assume for the moment that a.s. we still have a finite integral in the exponent, than again by monotone convergence we will be done. To demonstrate the mentioned finiteness, it suffices to show that the (weighted) expectation of the new integral in the exponent is finite:

$$\widetilde{\Pi}_{s,\omega_s}\,1\{\tau^c > T\}\,\varphi(\widetilde{W}_T)\int_s^T \mathrm{d}r\;\varrho_\sigma(W_r)\,v_K\big(r,\widetilde{W}_r\big) < +\infty.$$

But since from (18) always

$$v_K(r,\cdot) \le \widetilde{\Pi}_{r,\cdot}\,\varphi(\widetilde{W}_T)$$

follows, and $0 \le \varphi \le \|\varphi\|_\infty$ trivially holds, it is sufficient to show that

$$\widetilde{\Pi}_{s,\omega_s}\,1\{\tau^c > T\}\int_s^T \mathrm{d}r\;\varrho_\sigma(W_r) < +\infty.$$

Recall that $(\omega_s)_s \ne c$. By time-homogeneity of the Brownian motion we are left with the claim

$$\Pi_a\,1\{\tau^c > t\}\int_0^t \mathrm{d}r\;\varrho_\sigma(W_r) < +\infty, \qquad t > 0,\ a \ne c. \tag{28}$$

For this purpose, without loss of generality we may assume that $a > c = 0$. Changing the order of expectation and integration, we get

$$= \int_{(0,t)} \mathrm{d}r\;\Pi_a\,1\{\tau^c > t\}\,\varrho_\sigma(W_r).$$

Using the Markov property at time r, we can rewrite this as

$$= \int_{(0,t)} \mathrm{d}r \int_{(0,\infty)} \mathrm{d}b\;p^c(r,a,b)\,\frac{1}{b^\sigma}\,\Pi_b\big\{\tau^c > t-r\big\} \tag{29}$$

with the transition density p^c of Brownian motion killed at $c = 0$ (recall (4)). We may additionally restrict the internal integral to $0 < b < \frac{a}{2}$ (since for the opposite case the finiteness is obvious). For the internal probability expression we have

$$\Pi_b\{\tau^c > t-r\} = \int_{t-r}^{\infty} \mathrm{d}r' \; q(b,r') = \int_{(t-r)/b^2}^{\infty} \mathrm{d}r' \; q(1,r') \tag{30}$$

with q defined in (7). Insert this together with the upper estimate of Lemma 6 into (29) (with the restricted b), and substitute $b \longmapsto (t-r)^{1/2}b$ to arrive at

$$\int_0^t \mathrm{d}r \; (t-r)^{1-\sigma/2} \int_0^{a/(2\sqrt{t-r})} \mathrm{d}b \; b^{1-\sigma} \, a \, r^{-3/2} \exp\big[-\tfrac{a^2}{8r}\big] \int_{1/b^2}^{\infty} \mathrm{d}r' \; q(1,r'),$$

except a positive constant. As $b \to 0$, the most internal integral is of order b, and $b^{2-\sigma}$ is integrable (under $\sigma < 3$). On the other hand, as $b \to \infty$, the internal integral remains bounded by 1, and as $r \uparrow t$ the upper part of the b-integral is of order $(t-r)^{-1+\sigma/2}$ which is compensated. But the remaining r-integral as $r \downarrow 0$ is certainly finite since $a \neq 0$ by assumption. This finishes the proof. ■

3.9 Uniqueness of solutions under $\sigma < 2$

In the previous subsection we showed that the limiting function $v = V\varphi$ introduced in (23) satisfies the limiting Feynman-Kac equation. In the case of a moderate hyperbolic branching rate we are able to determine v as the *unique* solution of that equation:

Proposition 19 (uniqueness in the moderate case) *Suppose* $1 \leq \sigma < 2$. *For* $\varphi \in b\mathcal{B}_+[\mathbf{C}, \mathbb{R}]$ *fixed, the limiting function* $v = V\varphi$ *of* (23) *is the unique element in* $b\mathcal{B}\big[I \widehat{\times} \mathbf{C}^\bullet, \mathbb{R}_+\big]$ *which solves the Feynman-Kac equation* (25).

Proof Assume that v^1 and v^2 are *different* solutions of (25). Let $s_0 \leq T$ denote the supremum over all $s < T$ such that $v^1(s,\cdot) \neq v^2(s,\cdot)$. Fix for the moment $0 \leq s < s_0$ and $\omega_s \in \mathbf{C}^s$. Next we search for an upper bound for $|v^1(s,\omega_s) - v^2(s,\omega_s)|$ by using the equation (25). To this aim, split up the common part of the integral in the exponent, estimate the related exponential term by 1, and use the elementary inequality $|\mathrm{e}^{-x} - \mathrm{e}^{-y}| \leq |x-y|, \;\; x,y \geq 0$, to get the bound

$$\widetilde{\Pi}^c_{s,\omega_s} \, \varphi(\widetilde{W}^c_T) \int_s^{s_0} \mathrm{d}r \; \varrho_\sigma(W^c_r) \left| v^1(r, \widetilde{W}^c_r) - v^2(r, \widetilde{W}^c_r) \right|.$$

Denoting (in this proof) by $\|\cdot\|$ the supremum norm on $[0,s_0)\widehat{\times}\mathbf{C}^\bullet$, and using the time-homogeneity of the Brownian motion killed at c, we conclude for

$$\|v^1 - v^2\| \;\leq\; \|\varphi\|_\infty \, \|v^1 - v^2\| \, \sup_a \Pi^c_a \int_0^{s_0 - s} \mathrm{d}r \; \varrho_\sigma(W^c_r).$$

To get a contradiction, it suffices to show that

$$\sup_a \int_0^\varepsilon \mathrm{d}r\ \Pi_a^c \varrho_\sigma(W_r^c) \underset{\varepsilon\downarrow 0}{\longrightarrow} 0.$$

Using the Brownian scaling, this supremum expression is of order $\varepsilon^{1-\sigma/2}$ converging to 0 as $\varepsilon \downarrow 0$ (since $\sigma < 2$ by assumption) provided that

$$\sup_a \int_{(0,1)} \mathrm{d}r\ \Pi_a^c \varrho_\sigma(W_r^c) < +\infty.$$

Without loss of generality we may restrict the supremum to the case $a > c = 0$. Then we have to show

$$\sup_{a>0} \int_{(0,1)} \mathrm{d}r \int_0^\infty \mathrm{d}b\ p^c(r,a,b)\, b^{-\sigma} \ < +\infty. \tag{31}$$

Using the substitutions $b \mapsto r^{1/2}b$, for (31) we get the upper bound

$$\leq \int_0^1 \mathrm{d}r\ r^{-\sigma/2} \sup_{a>0} \int_0^\infty \mathrm{d}b\ p^c(1,a,b)\, b^{-\sigma}.$$

The first integral is finite by our assumption $\sigma < 2$. It remains to show that the latter supremum expression is also finite. If in addition $b \geq 1$, then simply use that all the $p^c(1,a,\cdot)$ are probability densities. In the other case $b < 1$, apply the inequality $p^c(1,a,b) \leq \mathit{const}\ b$ (recall Lemma 6) to finish the proof. ■

4 Historical hyperbolic super-Brownian motion

The purpose of this section is the construction of the hyperbolic SBM X in $\mathbb{R}$ as claimed in the existence Theorem 3. Actually we will construct the related *historical* superprocess $\widetilde{X}$. This can be done by starting from the historical SBM $\widetilde{X}^K$ with truncated branching rate $\varrho_\sigma \wedge K$ and passing to the limit as $K \to \infty$.

4.1 Semigroup structure of limiting functions

In the previous section, all paths w ended at time T. Now we write t instead of T and think of t as a variable. To have again a unified reference space, we replace $\mathcal{C}\big[[0,t],\mathbb{R}\big]$ by $\mathcal{C}[\mathbb{R}_+,\mathbb{R}] =: \mathbf{C}$ endowed with the topology of uniform convergence on bounded intervals (Polish space). $\mathbf{C}^s$ is again the closed subspace of all continuous paths stopped at time s. Also the other notations of the previous section are modified in the obvious way.

Take $\varphi \in b\mathcal{B}_+[\mathbf{C}^t,\mathbb{R}]$. Recall the solution $V^K\varphi$ of the truncated cumulant equation (16) with T replaced by t, and similarly $V\varphi$ for its limit as $K \uparrow \infty$. We write now more carefully

$$V_{s,t}^K\varphi\,(\omega_s) := V^K\varphi\,(s,\omega_s), \qquad 0 \leq s \leq t, \quad \omega_s \in \mathbf{C}^s, \tag{32}$$

and define $V_{s,t}\,\varphi\,(\omega_s)$ analogously based on $V\varphi\,(s,\omega_s)$.

Lemma 20 (semigroup structure) *The limiting functions of* (23) *satisfy*

$$V_{s,r}\,V_{r,t}\,\varphi = V_{s,t}\,\varphi, \qquad 0 \le s \le r \le t, \quad \varphi \in b\mathcal{B}_+[\mathbf{C}^t, \mathbb{R}].$$

Proof Fix s,r,t,φ as in the lemma. From equation (16) or (18) we get

$$V^K_{s,r}\,V^K_{r,t}\,\varphi = V^K_{s,t}\,\varphi, \qquad K > 1, \quad 0 \le s \le r \le t, \quad \varphi \in b\mathcal{B}_+[\mathbf{C}^t, \mathbb{R}].$$

Then the claim follows from the limit assertion (23) and the independence of choice Lemma 13. ∎

4.2 Existence of the historical hyperbolic SBM $\widetilde{X}$

Now we are ready to formulate the following existence theorem.

Theorem 21 (existence of the historical hyperbolic SBM $\widetilde{X}$) *There is a (time-inhomogeneous) Markov process* $\widetilde{X} = [\widetilde{X}, \widetilde{P}_{s,\mu},\, s \ge 0,\, \mu \in \mathcal{M}[\mathbf{C}^s]]$ *with states* $\widetilde{X}_t \in \mathcal{M}[\mathbf{C}^t]$, $t \ge s$, *having the following Laplace transition functional*

$$\widetilde{P}_{s,\mu} \exp\big\langle \widetilde{X}_t\,, -\varphi\big\rangle \;=\; \exp\big\langle \mu, -V_{s,t}\,\varphi\big\rangle, \tag{33}$$

$0 \le s \le t,\ \mu \in \mathcal{M}[\mathbf{C}^s],\ \varphi \in b\mathcal{B}_+[\mathbf{C}^t, \mathbb{R}]$. *Here* $V_{\cdot,t}\,\varphi \ge 0$ *is the limiting function of* (23) (*with* T *replaced by* t), *which solves the Feynman-Kac equation*

$$v(s,\omega_s) = \widetilde{\Pi}^c_{s,\omega_s}\, \varphi(\widetilde{W}^c_t) \exp\Big[-\int_s^t \mathrm{d}r\ \varrho_\sigma(W^c_r)\, v\big(r, \widetilde{W}^c_r\big)\Big], \tag{34}$$

$[s,\omega_s] \in [0,t)\widehat{\times}\mathbf{C}^\bullet$. *The following expectation and variance formulas hold:*

$$\widetilde{P}_{s,\mu}\big\langle \widetilde{X}_t\,, \varphi\big\rangle \;=\; \widetilde{\Pi}^c_{s,\mu}\, \varphi\big(\widetilde{W}^c_t\big), \tag{35}$$

$$\widetilde{V}\mathrm{ar}_{s,\mu}\big\langle \widetilde{X}_t\,, \varphi\big\rangle = 2\,\widetilde{\Pi}^c_{s,\mu} \int_s^t \mathrm{d}r\ \varrho_\sigma(W^c_r)\Big[\widetilde{\Pi}^c_{r,\widetilde{W}^c_r}\, \varphi\big(\widetilde{W}^c_t\big)\Big]^2, \tag{36}$$

$0 \le s < t,\ \ \mu \in \mathcal{M}[\mathbf{C}^s],\ \ \varphi \in b\mathcal{B}_+[\mathbf{C}^t, \mathbb{R}]$.

$\widetilde{X}$ is called the *historical hyperbolic super-Brownian motion* in $\mathbb{R}$. Note that $\widetilde{X}_t$ is a measure on continuous paths w on $\mathbb{R}_+$ stopped at time t. It describes the ancestry of all particles alive at time t.

As a rough interpretation of the expectation formula (35) one could say: The "expectation" of the historical superprocess $\widetilde{X}$ is given by the Brownian path process $\widetilde{W}^c$ with killing at c.

Recall also (Proposition 19) that in the moderate case $\sigma < 2$, the limiting function $v = V_{\cdot,t}\,\varphi,\ t > 0$, was characterized as *unique* solution to the *Feynman-Kac equation* (34).

Before we turn to the proof of this theorem in the next subsection, we want to show how it implies the *existence Theorem* 3.

Proof of Theorem 3 By the *projection*

$$X_t(B) := \widetilde{X}_t\Big(\big\{w \in \mathbf{C}^t;\ w_t \in B\big\}\Big), \qquad B \text{ Borel} \subseteq \mathsf{IR}, \tag{37}$$

we define the *hyperbolic super-Brownian motion* $X = \big[X, P_\mu, \mu \in \mathcal{M}\big]$ in IR (statement (a) of Theorem 3, except the claimed non-degeneration). Note that X (as opposed to $\widetilde{X}$) is a time-homogeneous $\mathcal{M}$-valued Markov process.

From the expectation formula (35) follows

$$P_\mu \langle X_t, \varphi\rangle = \int \mu(\mathrm{d}a)\ S_t^c\varphi\,(a), \qquad \mu \in \mathcal{M}, \quad t > 0, \quad \varphi \in b\mathcal{B}_+\,, \tag{38}$$

in particular,

$$E\big\{X_t(\mathsf{IR})\big|\, X_0\big\} \;=\; \int X_0(\mathrm{d}a)\ \Pi_a(\tau^c > t) \;<\; X_0(\mathsf{IR}), \qquad X_0 \neq 0, \quad t > 0.$$

Hence, the *total mass process* $t \mapsto X_t(\mathsf{IR})$ is a supermartingale but no longer a martingale (part (b) of Theorem 3), as opposed to the critical superprocesses with locally finite catalytic mass. In fact, since the underlying Brownian motion is killed at the center c of the catalytic medium, no mass can be born at c and the expectation of the process is not preserved (except the zero mass), despite the otherwise criticality of the branching mechanism.

The variance formula (36) specializes for X as follows:

$$\mathrm{Var}_\mu \langle X_t, \varphi\rangle = 2\int_0^t \mathrm{d}r \int_{a\neq c} \mu(\mathrm{d}a) \int \mathrm{d}b\ \varrho_\sigma(b)\, p^c(r,a,b)\, \big(S_{t-r}^c\varphi\big)^2(b), \tag{39}$$

$\mu \in \mathcal{M},\ \varphi \in b\mathcal{B}_+\,,\ t > 0$. In particular,

$$\mathrm{Var}_{\delta_a} X_t(\mathsf{IR}) = 2\int_0^t \mathrm{d}r \int \mathrm{d}b\ \varrho_\sigma(b)\, p^c(r,a,b) > 0, \qquad a \neq c, \quad t > 0. \tag{40}$$

Note that the latter expression is *finite*, provided that $\sigma < 2$ (moderate case), whereas it is *infinite* for $\sigma = 2$ (critical hyperbolic branching rate). In fact, only the influence of the singularity for $b \to c$ of the branching rate ϱ_σ has to be checked, for $a \neq c$. But for this we can apply the bounds in Lemma 6. (Consequently, the total mass process $X(\mathsf{IR})$ has finite variance if and only if the branching functional $K(\mathrm{d}r) = \varrho_\sigma(W_r^c)\,\mathrm{d}r$ has finite characteristic; recall Remark 4.)

Since the variance (40) is not zero, the hyperbolic super-Brownian motion X is *non-degenerate*. This completes the proof of Theorem 3. ∎

Remark 22 (stochastic equation) It can be expected that the hyperbolic super-Brownian motion X lives on the set of absolutely continuous measures, and that there is a density field jointly continuous on $\{t > 0\} \times \mathbb{R}$ satisfying the stochastic equation (1). Setting formally $\varphi = \delta_c$ in (38) and (39) suggests that this density field vanishes identically at the catalytic center (as opposed to the single point-catalytic super-Brownian motion where the variance of the density field blows up approaching the catalyst and the density field is non-zero at the catalyst's position at some random times). $\diamond$

4.3 Proof of the existence theorem

Now we are ready for the *Proof of Theorem* 21. For $K > 1$, let

$$\widetilde{X}^K = \left[\widetilde{X}^K,\, \widetilde{P}^K_{s,\mu},\, s \geq 0,\, \mu \in \mathcal{M}[\mathbf{C}^s]\right]$$

denote the *historical SBM* related to the *truncated* branching rate $\varrho_\sigma \wedge K$, that is, a Markov process with Laplace transition functionals

$$\widetilde{P}^K_{s,\mu} \exp \langle \widetilde{X}^K_t, -\varphi\rangle = \exp\langle \mu, -V^K_{s,t}\varphi\rangle, \tag{41}$$

$0 \leq s \leq t$, $\mu \in \mathcal{M}[\mathbf{C}^s]$, $\varphi \in b\mathcal{B}_+[\mathbf{C}^t, \mathbb{R}]$. Here $V^K_{\cdot,t}\varphi \geq 0$ uniquely solves the truncated equation (16) (with T replaced by t). For a detailed exposition we refer e.g. to Dawson and Perkins [DP91, Chapter 2] or to Dynkin [Dyn91b]; see also Mueller and Perkins [MP92].

The *interpretation* is that $\widetilde{X}^K_t(\mathrm{d}w)$ describes the mass of all particles at time t with location w_t but only those which (or whose ancestors) moved during the time interval $[s,t]$ along the curve $\{w_r;\ s \leq r \leq t\}$. In this sense, $\widetilde{X}^K$ is a refinement of the usual continuous super-Brownian motion X^K with truncated branching rate $\varrho_\sigma \wedge K$.

Passing in (41) to the monotone limit $V^K_{s,t}\varphi \downarrow V_{s,t}\varphi$ as $K \uparrow \infty$, we get limiting Laplace functionals of (proper) random measures since $0 \leq V_{s,t}\varphi \leq \|\varphi\|_\infty$. Actually, from the (non-linear) semi-group property of V according to Lemma 20, we obtain limiting Laplace transition functionals which determine the finite-dimensional distributions of a time-inhomogeneous Markov process $\widetilde{X}$, the historical hyperbolic SBM, as formulated in Theorem 21.

Note that for $t > 0$ the limiting function $V_{\cdot,t}\varphi$ solves the Feynman-Kac equation (34). Based on the representation (33) of Laplace transition functionals, by standard arguments, the mentioned moment formulas can be derived. $\blacksquare$

Immediately from the previous proof, by bounded convergence we conclude for the following result.

Corollary 23 (convergence of truncated processes) *The historical SBM $\widetilde{X}^K$ with truncated branching rate $\varrho_\sigma \wedge K$ (and non-killed Brownian motion*

W as motion law) tends to the historical hyperbolic SBM $\widetilde{X}$ (with the killed Brownian motion W^c as motion law) as $K \to \infty$ in the sense of convergence of all finite-dimensional distributions.

4.4 Stopped historical hyperbolic super-Brownian motion

There is a further refinement of the historical super-Brownian motion $\widetilde{X}^K$ with truncated branching rate $\varrho_\sigma \wedge K$, which goes back to Dynkin [Dyn91a, § 1.5]. In fact, for $s \geq 0$, let $\mathcal{T}_s$ denote the set of all *finite s-stopping times* τ. Then there is a family

$$\{\widetilde{X}^K_\tau;\ \tau \in \mathcal{T}_s,\ \ s \geq 0\}$$

called the *stopped historical super-Brownian motion* with truncated branching rate $\varrho_\sigma \wedge K$. The principal idea here is that $\widetilde{X}^K_\tau$ is a random measure on paths stopped at time τ instead of t, illustrating the historical population picture at the random moment τ.

This family satisfies the so-called *special Markov property,* which roughly says that at any $\tau \in \mathcal{T}_s$, the stopped historical SBM starts anew (see Dynkin [Dyn91a, Theorem 1.6]).

As in (35) and (36), we have the following expectation and variance formulas:

$$\widetilde{P}^K_{s,\mu}\langle \widetilde{X}^K_\tau, \Phi\rangle = \widetilde{\Pi}_{s,\mu}\Phi(\widetilde{W}_\tau), \tag{42}$$

$$\widetilde{V}\mathrm{ar}^K_{s,\mu}\langle \widetilde{X}^K_\tau, \Phi\rangle = 2\,\widetilde{\Pi}_{s,\mu}\int_s^\tau \mathrm{d}r\ (\varrho_\sigma \wedge K)(W_r)\Big[\Pi_{\tau,\widetilde{W}_r}\,\Phi(\widetilde{W}_\tau)\Big]^2, \tag{43}$$

$s \geq 0$, $\tau \in \mathcal{T}_s$, $\Phi \in b\mathcal{B}[\mathbf{C}^\tau, \mathbb{R}_+]$, and $\mu \in \mathcal{M}(\mathbf{C}^s)$; see e.g. Dynkin [Dyn91a, (1.50a)].

Finally, we mention that for $\tau \in \mathcal{T}_s$ which is an exit time from a set $A \subseteq \mathbb{R}$ such that $\mathbb{R}_+ \times A$ is finely open in $\mathbb{R}_+ \times \mathbb{R}$, the so-called *CB-property* holds:

$$\log \widetilde{P}^K_{s,\mu}(\widetilde{X}^K_\tau = 0) = \int \mu(\mathrm{d}w)\, \widetilde{P}^K_{s,w}(\widetilde{X}^K_\tau = 0) \tag{44}$$

(cf. Dynkin [Dyn93, (4.18) and (3.19)]).

As with the existence Theorem 21 and Corollary 23, there is also a refinement of the historical hyperbolic SBM $\widetilde{X}$ as described above in the truncated case. In fact, for $s \geq 0$, let $\mathcal{T}^c_s$ denote the set of all *finite s-stopping times* τ strictly smaller than τ^c. Then there is a family

$$\{\widetilde{X}_\tau;\ \tau \in \mathcal{T}^c_s,\ \ s \geq 0\}$$

called the *stopped historical hyperbolic super-Brownian motion.* Of course, $\widetilde{X}_\tau$ is again a random measure on paths stopped at time $\tau < \tau^c$, illustrating the historical population picture at the random moment τ.

Also this family satisfies the *special Markov property,* saying that at any $\tau \in \mathcal{T}_s^c$, the stopped historical hyperbolic SBM starts anew.

The *expectation* formula (42) again reads as

$$\widetilde{P}_{s,\mu}\langle \widetilde{X}_\tau \,, \Phi \rangle = \widetilde{\Pi}_{s,\mu} \Phi\big(\widetilde{W}_\tau\big), \tag{45}$$

$s \geq 0,\ \tau \in \mathcal{T}_s^c,\ \Phi \in b\mathcal{B}\big[\mathbf{C}^\tau, \mathbb{R}_+\big]$, but where the measure $\mu \in \mathcal{M}(\mathbf{C}^s)$ satisfies $\mu\{w \in \mathbf{C}^s;\ w_s = c\} = 0$.

As before, for $\tau \in \mathcal{T}_s^c$ which is an exit time from a set $A \subseteq \mathbb{R} \setminus \{0\}$ such that $\mathbb{R}_+ \times A$ is finely open in $\mathbb{R}_+ \times \mathbb{R} \setminus \{0\}$, the *CB-property* holds:

$$\log \widetilde{P}_{s,\mu}\big(\widetilde{X}_\tau = 0\big) = \int \mu(\mathrm{d}w)\, \widetilde{P}_{s,w}\big(\widetilde{X}_\tau = 0\big). \tag{46}$$

5 Killing around the critical hyperbolic pole

In this section we want to show that for the super-Brownian motion X in the critical hyperbolic medium ϱ_2 no mass will ever reach the catalytic center. That is, the particles hidden in the clouds die already *before* they hit c. This will be based on some methods involving historical superprocesses developed in Chapter 8 of Dawson and Perkins [DP91] to estimate the modulus of continuity of the support of superprocesses (see also Mueller and Perkins [MP92]).

5.1 Reformulation of the strong killing theorem

Actually, we restate Theorem 5 at the level of historical superprocesses in the following way.

Theorem 24 (strong killing in the case of the critical ϱ_2) *There exists a stopping time $\tau < \tau^c$ such that*

$$\widetilde{P}_{0,\mu}\big(\widetilde{X}_\tau = 0\big) = 1$$

for each $\mu \in \mathcal{M}$ with $\mu(\{c\}) = 0$.

For the proof of this theorem, without loss of generality we may set $c = 0$. Moreover, by the CB-property (46) and symmetry, we may restrict our attention to $\mu = \delta_a$ for an $a > 0$ *fixed*, which is assumed for the remainder of this section. That is, the historical catalytic superprocess starts off at time 0 with a unit mass concentrated at $a > 0$. To simplify notation, we identify δ_a with a and write $\widetilde{P}_a$ instead of $\widetilde{P}_{0,\delta_a}$.

By this choice of the initial state, we may restrict to particle paths w in

$$\mathbf{C}_a := \big\{w \in \mathbf{C} = \mathcal{C}[\mathbb{R}_+ \,, \mathbb{R}];\ w_0 = a > 0\big\}.$$

We will observe these paths at most until they reach the catalytic center $c = 0$. More precisely, for $w \in \mathbf{C}_a$, we pay attention to the following increasing sequence of *hitting times*

$$\tau_n := \tau(2^{-n}a), \qquad n \geq 0, \tag{47}$$

where $\tau(b) := \tau^b$ denotes the (first) time the path w hits the point $b \in \mathbb{R}$. (Recall that by the recurrence of one-dimensional Brownian motion each hitting time $\tau(b)$ is finite Π_a–a.s.) In § 5.2 we will actually use this monotone sequence $\tau_0, \tau_1, \ldots$ of hitting times smaller than τ^0 to prove Theorem 24. For this purpose it *suffices* to show that

$$\widetilde{P}_a\big(\widetilde{X}_{\tau_n} \neq 0\big) \underset{n\to\infty}{\longrightarrow} 0. \tag{48}$$

In fact, then these probabilities are summable along a subsequence, and from Borel-Cantelli we conclude for the existence of a smallest (random) integer N (concerning this subsequence) such that $\widetilde{X}_{\tau_N} = 0$ with $\widetilde{P}_a$–probability one. Since 0 is an absorbing state, we found a stopping time $\tau := \tau_N$ strictly smaller than τ^c at which the stopped historical process does not have any mass, then proving Theorem 24.

As a technical preparation for the proof of (48), in this subsection we still state the following simple property of the Brownian motion.

Lemma 25 (infinite accumulated branching rate) *Along a Brownian path W until it reaches 0, the accumulated rate of branching is infinite:*

$$\int_0^{\tau^0} \mathrm{d}t\ \varrho_2(W_t) = \infty \quad \Pi_a\text{–}a.s., \qquad a > 0.$$

Proof First of all, using the hitting times τ_n from (47),

$$\int_0^{\tau^0} \mathrm{d}t\ \varrho_2(W_t) \geq \sum_{n=0}^{\infty} R_n,$$

where

$$R_n := \int_{\tau_n}^{\tau_{n+1}} \mathrm{d}t\ 1\Big\{W_t \leq 2^{-n}a\Big\}\, \varrho_2(2^{-n}a), \qquad n \geq 0. \tag{49}$$

Since $R_0 > 0$ with Π_a–probability one, it suffices to show that the $R_0, R_1, \ldots$ are independent and identically distributed (with respect to Π_a).

The independence immediately follows from the strong Markov property. We want to calculate $\Pi_a\big(R_n \leq r\big)$, $r > 0$. Again by the strong Markov property but also time- and space-homogeneity as well as symmetry, it equals

$$\Pi_0\bigg(\theta 2^{2n} a^{-2} \int_0^{\tau(2^{-n-1}a)} \mathrm{d}t\ 1\big\{W_t \geq 0\big\} \leq r\bigg).$$

Now we can use the self-similarity of standard Brownian motion starting from the origin to continue with

$$= \Pi_0\left(\theta a^{-2}\int_0^{\tau(a/2)} \mathrm{d}t\, 1\{W_t \ge 0\} \le r\right) = \Pi_a(R_0 \le r)$$

finishing the proof. ∎

Now we restrict our attention to paths $w \in \mathbf{C}_a$ with infinite accumulated rate of branching as in Lemma 25.

5.2 Proof of the killing theorem

Recall that for the proof of Theorem 24 it suffices to show the convergence statement (48).

From Lemma 25 we conclude for the existence of positive sequences $\varepsilon_n \to 0$ and $\xi_n \to \infty$ as $n \to \infty$ such that

$$\Pi_a(E_n) \ge 1-\varepsilon_n^2 \quad \text{where} \quad E_n := \left\{w \in \mathbf{C}_a\,;\ \int_0^{\tau_n} \mathrm{d}t\ \varrho_2(w_t) \ge \xi_n\right\}. \tag{50}$$

Note that E_n belongs to $\mathcal{F}_{\tau_n}$, the σ-field generated by w_t for $t \le \tau_n$.

Here is the *intuitive picture* for the further procedure. For a fixed large n, according to (50) there is only a small Π_a-chance that a Brownian path belongs to the complement E_n^c of E_n (in $\mathbf{C}_a$). Therefore, E_n^c has a small stopped historical $\widetilde{X}_{\tau_n}$-measure, with a high $\widetilde{P}_a$-probability. Then using Iscoe's [Isc88] techniques, we will show that the set of related particles is likely to die out before time τ_{n+1}. Also, for the original set E_n of paths we will show that with high probability, the related particles have died out before time τ_n, by the huge accumulated rate of branching.

Now we give the details along these lines to arrive at the claim (48).

Step 1° Recall that $\mathbf{C}^t$ denotes the set of all paths stopped at time t. We set $\mathbf{C}_a^t := \mathbf{C}_a \cap \mathbf{C}^t$ and

$$\lambda_n := \widetilde{X}_{\tau_n}\big(\mathbf{C}_a^{\tau_n} \setminus E_n\big). \tag{51}$$

By the expectation formula (45) with $s = 0,\ \tau = \tau_n$ and $\Phi = 1\{\mathbf{C}_a^{\tau_n} \setminus E_n\}$, we have

$$\widetilde{P}_a\lambda_n = \widetilde{P}_a\widetilde{X}_{\tau_n}\big(\mathbf{C}_a^{\tau_n} \setminus E_n\big) = \widetilde{\Pi}_{0,a}\big(\widetilde{W}_{\tau_n} \notin E_n\big) \le \Pi_a(E_n^c) \le \varepsilon_n^2.$$

Using Markov's inequality, we therefore conclude

$$\widetilde{P}_a\big(\lambda_n \ge \varepsilon_n\big) \le \varepsilon_n \underset{n\to\infty}{\longrightarrow} 0. \tag{52}$$

Consequently, the set E_n^c has small $\widetilde{X}_{\tau_n}$-measure λ_n with high $\widetilde{P}_a$-probability, as desired.

Step 2° Let us examine the further fate of the mass λ_n defined in (51) we need to study in the case $\lambda_n \to 0$. Fix n. By the definition of τ_n, at the moment τ_n the (projected) population X_{τ_n} will be concentrated at the space point $2^{-n}a$. Based on the special Markov property, we can start X anew after the time τ_n, namely with the mass λ_n attached to the point $2^{-n}a$.

Now we adapt Iscoe's [Isc88] analysis in the constant branching rate case to our situation of a critical hyperbolic medium ϱ_2. We estimate the probability that, starting with the mass λ_n, completely concentrated at the point $2^{-n}a$, the *range* of the arising superprocess will be contained in the surrounding space interval $I_n := (2^{-n-1}a,\, 3\cdot 2^{-n-1}a)$ (at which the branching rate ϱ_2 is bounded since a and n are fixed). This probability, denoted by p_n, is given by

$$p_n = \exp\big[-\lambda_n\, u(2^{-n}a)\big] \tag{53}$$

where u satisfies

$$\tfrac{1}{2}\,\Delta u = \varrho_2\, u^2 \quad \text{on} \quad I_n\,, \qquad u\big|_{\partial I_n} = \infty,$$

(cf. Dynkin [Dyn93, Corollary II.8.2]). Since on I_n the critical hyperbolic branching rate ϱ_2 is not smaller than $\theta\, 3^{-2}2^{2n+2}a^{-2}$, we conclude that $u \le \overline{u}$, where $\overline{u}$ solves

$$\tfrac{1}{2}\,\Delta \overline{u} = \theta\, 3^{-2}2^{2n+2}a^{-2}\,\overline{u}^{\,2} \quad \text{on} \quad I_n\,, \qquad \overline{u}\big|_{\partial I_n} = \infty,$$

(recall that $a > 0$ is the fixed starting point, and $\theta > 0$ is an additional weight of the branching rate ϱ_2). By scaling, we find that $v(b) := \overline{u}(2^{-n}b)$, $b \in I_0$, satisfies

$$\tfrac{1}{2}\,\Delta v = \theta\, 3^{-2}2^{2}a^{-2}\, v^2 \quad \text{on} \quad I_0 = \big(a/2,\, 3a/2\big), \qquad v\big|_{\partial I_0} = \infty. \tag{54}$$

But then $u(2^{-n}a) \le \overline{u}(2^{-n}a) = v(a)$, which by uniqueness of the (maximal) solution to equation (54) does not depend on n. So finally we get

$$p_n \ge \exp\big[-\lambda_n\, v(a)\big] \tag{55}$$

converging to 1 if $\lambda_n \to 0$. Consequently, the small starting mass λ_n at time τ_n concentrated at $2^{-n}a$ will essentially not hit $2^{-n-1}a$ in the further development.

Step 3° Assume for the moment that

$$\widetilde{P}_a\Big(\widetilde{X}_{\tau_n}(E_n) > 0\Big) \underset{n\to\infty}{\longrightarrow} 0. \tag{56}$$

Then we would have together all ingredients to show (48). Indeed, recalling the definition (51) of λ_n, by the special Markov property we have

$$\begin{aligned} \widetilde{P}_a\Big(\widetilde{X}_{\tau_{n+1}} \neq 0\Big) \;\le\; & \widetilde{P}_a\Big(\widetilde{X}_{\tau_n}(E_n) > 0\Big) + \widetilde{P}_a\big(\lambda_n \ge \varepsilon_n\big) \\ & + \widetilde{P}_a\Big\{P_{\tau_n,\lambda_n\delta_{2^{-n}a}}\big(X_{\tau_{n+1}} \neq 0\big)\,\Big|\,\lambda_n < \varepsilon_n\Big\} \end{aligned} \tag{57}$$

In fact, in order to be non-extinct at time τ_{n+1}, either at time τ_n we have some particles with path in E_n, or we do not have such particles. In the latter case, we must have particles with a path in E_n^c. But then their mass λ_n is either larger than ε_n or smaller. If their mass is smaller than ε_n, we use the fact that the superprocess X starts anew at time τ_n with this mass λ_n concentrated at $2^{-n}a$ and has to survive by time τ_{n+1}.

By the preliminary assumption (56), the first term at the r.h.s. converges to 0 as $n \to \infty$, whereas the second one tends to 0 by (52). Concerning the third, conditional expectation term, estimate the interior probability from above by using the definition of p_n given before (53), and its estimate (55) to obtain the bound

$$P_{\tau_n, \lambda_n \delta_{2^{-n}a}}\big(X_{\tau_{n+1}} \neq 0\big) \leq 1 - p_n \leq 1 - \exp\big[-\lambda_n v(a)\big] \leq \lambda_n v(a) \leq \varepsilon_n v(a).$$

But the latter expression bounds the total third term in (57) and converges to 0 as $n \to \infty$.

Consequently, (48) is true, provided we know (56) which is all what remains to verify.

Step 4° In order to prove (56) we intend to estimate the probability expression in (56) from above by a term converging to 0 as $n \to \infty$. For the purpose of getting such an estimate, we will *fix* an n, and we will study $\widetilde{X}$ only until τ_n, that is until the particles reach $2^{-n}a$. Until this time, we may read our hyperbolic branching rate ϱ_2 as a truncated rate $\varrho_2 \wedge K$, for a suitable K we fix from now on.

Step 5° We next intend to define a *new time scale* denoted by r. Given for the moment $w \in \mathbf{C}_a$, set

$$R(t) := \int_0^t \mathrm{d}s\, \big[\varrho_2(w_s) \wedge K\big], \qquad t \geq 0. \tag{58}$$

Note that with Π_a–probability one, $R(t) \to \infty$ as $t \uparrow \infty$. Since $R = R(w)$ is strictly increasing Π_a–a.s., define finite stopping times $\sigma(r)$ (converging to infinity as $r \to \infty$) by

$$R\big(\sigma(r)\big) = r, \qquad r \geq 0.$$

Note that

$$\frac{\mathrm{d}\sigma(r)}{\mathrm{d}r} = \frac{1}{\varrho_2\big(w_{\sigma(r)}\big) \wedge K} \tag{59}$$

for almost all r. Therefore the time change to the scale r has the advantage that the branching rate $\varrho_2 \wedge K$ will be "eliminated". This will enable us to use the well-know fact, that the total mass process of the continuous super-Brownian motion with uniform rate satisfies the simple stochastic equation (62) below.

Step 6° Define

$$Y_n := \widetilde{X}^K_{\tau_n}(E_n)$$

and set

$$Z_r := \langle \widetilde{X}^K_{\sigma(r)}, 1 \rangle, \qquad r \geq 0. \tag{60}$$

Note that

$$E_n = \{\tau_n \geq \sigma(\xi_n)\}, \quad \text{hence } E_n \text{ belongs to } \mathcal{F}_{\sigma(\xi_n)\wedge\tau_n}. \tag{61}$$

Assume for the moment that under the probability law $\widetilde{P}^K_a$ the following two statements hold:

(i) If $Z_{\xi_n} = 0$ then $Y_n = 0$.

(ii) Z satisfies

$$\mathrm{d}Z_r = \sqrt{2Z_r}\,\mathrm{d}B_r\,, \quad Z_0 = 1, \tag{62}$$

for some Brownian motion B.

Then,

$$\widetilde{P}^K_a\left(\widetilde{X}^K_{\tau_n}(E_n) > 0\right) = \widetilde{P}^K_a(Y_n > 0) \leq \widetilde{P}^K_a(Z_{\xi_n} > 0),$$

and from the well-know survival probability formula for solutions Z of (62) we continue with

$$= 1 - \mathrm{e}^{-1/\xi_n} \leq 1/\xi_n\,.$$

Since the fixed n was arbitrary, we can let n tend to ∞ to arrive at (56).

Step 7° We are left with proving the statements **(i)** and **(ii)**. The first one is easy. Indeed, if $\langle \widetilde{X}^K_{\sigma(\xi_n)}, 1\rangle = Z_{\xi_n} = 0$ then $\widetilde{X}^K_{\sigma(\xi_n)}(E_n) = 0$. But since $\sigma(\xi_n) \leq \tau_n$ on E_n, it follows that $Y_n = \widetilde{X}^K_{\tau_n}(E_n) = 0$ (recall (61)).

Now we can concentrate on proving **(ii)**. The initial condition is trivially fulfilled. Following Dynkin's terminology, we let $\mathcal{G}_{\sigma(r)}$ denote the pre–$\sigma(r)$ σ–field for the historical superprocess $\widetilde{X}^K$, for each $r \geq 0$. It suffices to show that Z is a $(\widetilde{P}^K_a, \mathcal{G}_{\sigma(r)})$–martingale with square variation

$$\langle\!\langle Z\rangle\!\rangle_r = 2\int_0^r \mathrm{d}s\; Z_s\,, \qquad r \geq 0.$$

This claim would be verified if we could demonstrate that for $0 \leq r < r'$,

$$\begin{aligned} \widetilde{P}^K_a\left(Z_{r'} \,\middle|\, \mathcal{G}_{\sigma(r)}\right) &= Z_r\,,\\ \widetilde{P}^K_a\left(Z^2_{r'} - 2\textstyle\int_r^{r'}\mathrm{d}s\; Z_s \,\middle|\, \mathcal{G}_{\sigma(r)}\right) &= Z^2_r. \end{aligned} \tag{63}$$

Now we claim that it is enough to show that for each fixed $T > 0$ and finite measure μ on $\mathbb{R}$

$$\widetilde{P}^K_{0,\mu}(Z_T) = \mu(\mathbb{R}), \tag{64}$$

$$\widetilde{P}^K_{0,\mu}\left(Z^2_T - 2\textstyle\int_0^T \mathrm{d}s\; Z_s\right) = \left(\mu(\mathbb{R})\right)^2. \tag{65}$$

In fact, by projection as in (37), Z_r of (60) coincides with $\langle \widetilde{X}^K_{\sigma(r)}, 1\rangle$. Thus, indeed we can use the time-homogeneity of the super-Brownian motion X^K, in conjunction with Dynkin's special Markov property to see the equivalence of (63) with (64) and (65).

Step 8° The expectation formula (64) directly follows from (42). Using this, the statement (65) is equivalent to

$$\widetilde{V}\mathrm{ar}^K_{0,\mu} Z_T = 2\,T\mu(\mathbb{R}). \tag{66}$$

But from the variance formula (43) we get

$$\widetilde{V}\mathrm{ar}^K_{0,\mu} Z_T = \widetilde{V}\mathrm{ar}^K_{0,\mu}\langle \widetilde{X}^K_{\sigma(T)}, 1\rangle = 2\widetilde{\Pi}_{0,\mu}\int_0^{\sigma(T)} \mathrm{d}s\ \big(\varrho_2 \wedge K\big)(W_s).$$

Substituting $s = \sigma(r)$ and recalling (59), we arrive at (66).

This completes the proof of Theorem 24. ■

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