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Homogenization and Orowan's law for anisotropic fractional operators of any order

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ABSTRACT. We consider an anisotropic Lévy operator $\mathcal{I}_s$ of any order $s \in (0,1)$ and we consider the homogenization properties of an evolution equation.

The scaling properties and the effective Hamiltonian that we obtain is different according to the cases $s < 1/2$ and $s > 1/2$.

In the isotropic onedimensional case, we also prove a statement related to the so-called Orowan's law, that is an appropriate scaling of the effective Hamiltonian presents a linear behavior.

1. INTRODUCTION

In this paper we study an evolutionary problem run by a fractional and possibly anisotropic operator of elliptic type.

These type of equations arise natural in crystallography, in which the solution of the equation has the physical meaning of the atom dislocation inside the crystal structure, see e.g. the detailed discussion of the Pierls-Nabarro crystal dislocation model in [12].

Due to their mathematical interest and in view of the concrete applications in physical models, these problems have been extensively studied in the recent literature, also using new methods coming from the analysis of fractional operators, see for instance [10, 11, 7, 5, 4] and references therein.

In particular, here we study an homogenization problem, related to long-time behaviors of the system at a macroscopic scale. The scaling of the system and the results obtained will be different according to the fractional parameter $s \in (0,1)$. Namely, when $s > 1/2$ the effective Hamiltonian "localizes" and it only depends on a first order differential operator. Conversely, when $s < 1/2$, the non-local features are predominant and the effective Hamiltonian will involve the fractional operator of order s. That is, roughly speaking, for any $s \in (0,1)$, the effective Hamiltonian is an operator of order $\min\{2s, 1\}$, which reveals the stronger non-local effects present in the case $s < 1/2$.

The strong non-local features of the case $s < 1/2$ are indeed quite typical in crystal dislocation dynamics, see [5] and [4]. Nevertheless, for any $s \in (0,1)$, we will be able to show that a suitably scaled effective Hamiltonian behaves linearly with respect to the leading operator, thus providing an extension of the so-called Orowan's law.

We now recall in further detail the state of the art for the homogenization of fractional problems in crystal dislocation, then we introduce the formal setting that we deal with and present in details our results.

In [10] Monneau and the first author study an homogenization problem for the evolutive Pierls-Nabarro model, which is a phase field model describing dislocation dynamics. They consider the following equation

$$\begin{cases} \partial_t u^\epsilon = \mathcal{I}_1[u^\epsilon(t,\cdot)] - W'\left(\frac{u^\epsilon}{\epsilon}\right) + \sigma\left(\frac{t}{\epsilon}, \frac{x}{\epsilon}\right) & \text{in} \quad \mathbb{R}^+ \times \mathbb{R}^N \\ u^\epsilon(0,x) = u_0(x) & \text{on} \quad \mathbb{R}^N, \end{cases} \tag{1.1}$$

where W is a periodic potential and $\mathcal{I}_1$ is an anisotropic Lévy operator of order 1, which includes as particular case the operator $-(-\Delta)^{\frac{1}{2}}$, and they prove that the solution u^ϵ of (1.1) converges as $\epsilon \to 0$ to the solution u^0 of the following homogenized problem

$$\begin{cases} \partial_t u = \overline{H}(\nabla_x u, \mathcal{I}_1[u(t,\cdot)]) & \text{in} \quad \mathbb{R}^+ \times \mathbb{R}^N \\ u(0,x) = u_0(x) & \text{on} \quad \mathbb{R}^N. \end{cases} \tag{1.2}$$

For $\epsilon = 1$, the solution u^ϵ has the physical meaning of an atom dislocation along a slip plane (the rest position of the atom lies on the lattice that is prescribed by the periodicity of the potential W). The number ϵ describes the ratio between the microscopic scale and the macroscopic scale and then it is a small number. After a suitable rescaling one gets equation (1.1). The limit u^0 can be interpreted as a macroscopic plastic strain satisfying the macroscopic plastic flow rule (1.2). The function $\overline{H}$, usually called effective Hamiltonian, is determined, as usual in homogenization, by a cell problem, which is in this case, for $p \in \mathbb{R}^N$ and $L \in \mathbb{R}$, the following:

$$(1.3)\qquad \begin{cases} \lambda + \partial_\tau v = \mathcal{I}_1[v(\tau,\cdot)] + L - W'(v + \lambda\tau + p\cdot y) + \sigma(\tau, y) & \text{in} \quad \mathbb{R}^+ \times \mathbb{R}^N \\ v(0,y) = 0 & \text{on} \quad \mathbb{R}^N. \end{cases}$$

For any $p \in \mathbb{R}^N$ and $L \in \mathbb{R}$, the quantity $\lambda = \lambda(p, L)$ is the unique number for which there exists a solution v of (1.3) which is bounded in $\mathbb{R}^+ \times \mathbb{R}^N$. Therefore, the function $\overline{H}(p, L) := \lambda(p, L)$ is well defined, and, in addition, this function turns out to be continuous and non-decreasing in L.

In a second paper [11], the authors consider, as a particular case, the one in which $N = 1$, $\mathcal{I}_1 = -(-\Delta)^{\frac{1}{2}}$ is the half Laplacian and $\sigma \equiv 0$, and they study the behavior of $\overline{H}(p, L)$ for small p and L. In this regime they recover the Orowan's law, which claims that

$$\overline{H}(p, L) \sim c_0|p|L$$

for some constant of proportionality $c_0 > 0$. To show this last result, estimates on the layer solution associated to $-(-\Delta)^{\frac{1}{2}}$, i.e. on the solution ϕ of

$$(1.4)\qquad \begin{cases} -(-\Delta)^{\frac{1}{2}}\phi = W'(\phi) & \text{in} \quad \mathbb{R} \\ \phi' > 0 & \text{in} \quad \mathbb{R} \\ \lim\limits_{x\to-\infty} \phi(x) = 0, \quad \lim\limits_{x\to+\infty} \phi(x) = 1, \quad \phi(0) = \dfrac{1}{2}, \end{cases}$$

are needed. Such estimates were proved in [7] under suitable assumptions on W, while the existence of a unique solution ϕ of (1.4) was proved in [3].

Recently, these kind of estimates have been proved for layer solutions associated to the fractional Laplacian $-(-\Delta)^s$ for $s \in (0, 1)$ by Palatucci, Savin and the second author in [13]. More general results on ϕ were obtained by Dipierro, Palatucci and the second author in [5] for the case $s \in \left[\frac{1}{2}, 1\right)$. See also [2] for related results.

In this paper, in view of these new estimates, we want to extend the results of [10] and [11] to the case where the non-local operator in (1.1) is an anisotropic Lévy operator of any order $s \in (0, 1)$. Precisely, given $\varphi \in C^2(\mathbb{R}^N) \cap L^\infty(\mathbb{R}^N)$, let us define

$$(1.5)\qquad \mathcal{I}_s[\varphi](x) := PV \int_{\mathbb{R}^N} \frac{\varphi(x+y) - \varphi(x)}{|y|^{N+2s}} g\left(\frac{y}{|y|}\right) dy,$$

where PV stands for the principal value of the integral and the function g satisfies

(H1) $g \in C(\mathbf{S}^{N-1})$, $g > 0, g$ even.

When $g \equiv C(N, s)$ with $C(N, s)$ suitable constant depending on the dimension N and on the exponent s, then (1.5) is the integral representation of $-(-\Delta)^s$.

In addition to (H1) we make the following assumptions:

(H2) $W \in C^{1,1}(\mathbb{R})$ and $W(v + 1) = W(v)$ for any $v \in \mathbb{R}$;
(H3) $\sigma \in C^{0,1}(\mathbb{R}^+ \times \mathbb{R}^N)$ and $\sigma(t + 1, x) = \sigma(t, x)$, $\sigma(t, x + k) = \sigma(t, x)$ for any $k \in \mathbb{Z}^N$ and $(t, x) \in \mathbb{R}^+ \times \mathbb{R}^N$;
(H4) $u_0 \in W^{2,\infty}(\mathbb{R}^N)$.

For $s > \frac{1}{2}$ we consider the following homogenization problem:

$$\begin{cases} \partial_t u^\epsilon = \epsilon^{2s-1}\mathcal{I}_s[u^\epsilon(t,\cdot)] - W'\left(\frac{u^\epsilon}{\epsilon}\right) + \sigma\left(\frac{t}{\epsilon},\frac{x}{\epsilon}\right) & \text{in} \quad \mathbb{R}^+ \times \mathbb{R}^N \\ u^\epsilon(0,x) = u_0(x) & \text{on} \quad \mathbb{R}^N, \end{cases} \tag{1.6}$$

and for $s < \frac{1}{2}$:

$$\begin{cases} \partial_t u^\epsilon = \mathcal{I}_s[u^\epsilon(t,\cdot)] - W'\left(\frac{u^\epsilon}{\epsilon^{2s}}\right) + \sigma\left(\frac{t}{\epsilon^{2s}},\frac{x}{\epsilon}\right) & \text{in} \quad \mathbb{R}^+ \times \mathbb{R}^N \\ u^\epsilon(0,x) = u_0(x) & \text{on} \quad \mathbb{R}^N. \end{cases} \tag{1.7}$$

Remark that the scalings for $s > \frac{1}{2}$ and $s < \frac{1}{2}$ are different. They formally coincide when $s = \frac{1}{2}$. We prove that the solution u^ϵ of (1.6) converges as $\epsilon \to 0$ to the solution u^0 of the homogenized problem

$$\begin{cases} \partial_t u = \overline{H}_1(\nabla_x u) & \text{in} \quad \mathbb{R}^+ \times \mathbb{R}^N \\ u(0,x) = u_0(x) & \text{on} \quad \mathbb{R}^N, \end{cases} \tag{1.8}$$

with an effective Hamiltonian $\overline{H}_1$ which does not depend on $\mathcal{I}_s$ anymore, while the solution u^ϵ of (1.7) converges as $\epsilon \to 0$ to u^0 solution of the following

$$\begin{cases} \partial_t u = \overline{H}_2(\mathcal{I}_s[u]) & \text{in} \quad \mathbb{R}^+ \times \mathbb{R}^N \\ u(0,x) = u_0(x) & \text{on} \quad \mathbb{R}^N, \end{cases} \tag{1.9}$$

with an effective Hamiltonian $\overline{H}_2$ not depending on the gradient. As we will see, the functions $\overline{H}_1$ and $\overline{H}_2$ are determined by the following cell problem:

$$\begin{cases} \lambda + \partial_\tau v = \mathcal{I}_s[v(\tau,\cdot)] + L - W'(v + \lambda\tau + p\cdot y) + \sigma(\tau,y) & \text{in} \quad \mathbb{R}^+ \times \mathbb{R}^N \\ v(0,y) = 0 & \text{on} \quad \mathbb{R}^N, \end{cases} \tag{1.10}$$

that is $\overline{H}_1$ and $\overline{H}_2$ are determined by the unique λ for which (1.10) possesses a bounded solution (according to the cases $s > \frac{1}{2}$ and $s < \frac{1}{2}$, respectively). We observe that the solutions of (1.8) and (1.9) may have quite different behaviors, since ∇u and $\mathcal{I}_s[u]$ may be very different at a given point, even in dimension 1 and when s is close to $\frac{1}{2}$ (see for instance [6]). Following [10], in order to solve (1.10), we show for any $p \in \mathbb{R}^N$ and $L \in \mathbb{R}$ the existence of a unique solution of

$$\begin{cases} \partial_\tau w = \mathcal{I}_s[w(\tau,\cdot)] + L - W'(w + p\cdot y) + \sigma(\tau,y) & \text{in} \quad \mathbb{R}^+ \times \mathbb{R}^N \\ w(0,y) = 0 & \text{on} \quad \mathbb{R}^N, \end{cases} \tag{1.11}$$

and we look for some λ such that $w - \lambda\tau$ is bounded. Precisely we have:

Theorem 1.1 (Ergodicity)**.** *Assume (H1)-(H4). For $L \in \mathbb{R}$ and $p \in \mathbb{R}^N$, there exists a unique viscosity solution $w \in C_b(\mathbb{R}^+ \times \mathbb{R}^N)$ of* (1.11) *and there exists a unique $\lambda \in \mathbb{R}$ such that w satisfies:*

$$\frac{w(\tau,y)}{\tau} \text{ converges towards } \lambda \text{ as } \tau \to +\infty, \text{ locally uniformly in } y.$$

The real number λ is denoted by $\overline{H}(p,L)$. The function $\overline{H}(p,L)$ is continuous on $\mathbb{R}^N \times \mathbb{R}$ and non-decreasing in L.

Once the cell problem is solved, we can prove the following convergence results:

Theorem 1.2 (Convergence for $s > \frac{1}{2}$)**.** *Assume (H1)-(H4). The solution u^ϵ of* (1.6) *converges towards the solution u^0 of* (1.8) *locally uniformly in (t,x), where*

$$\overline{H}_1(p) := \overline{H}(p,0)$$

and $\overline{H}(p,L)$ is defined in Theorem 1.1.

Theorem 1.3 (Convergence for $s<\frac{1}{2}$). *Assume (H1)-(H4). The solution u^ϵ of* (1.7) *converges towards the solution u^0 of* (1.9) *locally uniformly in (t,x), where*

$$\overline{H}_2(L) := \overline{H}(0,L)$$

and $\overline{H}(p,L)$ is defined in Theorem 1.1.

We point out that the effective Hamiltonians $\overline{H}_1$ and $\overline{H}_2$ represent the speed of propagation of the dislocation dynamics according to (1.8) and (1.9). In particular, due to Theorems 1.2 and 1.3, such speed only depends on the slope of the dislocation in the weakly non-local setting $s>\frac{1}{2}$ and only on an operator of order s of the dislocation in the strongly non-local setting $s<\frac{1}{2}$.

We will next consider the case: $N=1$, $\mathcal{I}_s=-(-\Delta)^s$ and $\sigma\equiv 0$, and we will make the further following assumptions on the potential W:

$$\begin{cases} W\in C^{4,\beta}(\mathbb{R}) & \text{for some } 0<\beta<1 \\ W(v+1)=W(v) & \text{for any } v\in\mathbb{R} \\ W=0 & \text{on } \mathbb{Z} \\ W>0 & \text{on } \mathbb{R}\setminus\mathbb{Z} \\ \alpha=W''(0)>0 & \\ W \text{ is even if } s\in\left(0,\frac{1}{2}\right). & \end{cases} \tag{1.12}$$

Under assumption (1.12), it is known, see [2] and [13], that there exists a unique function ϕ solution of

$$\begin{cases} \mathcal{I}_s[\phi]=W'(\phi) & \text{in} \quad \mathbb{R} \\ \phi'>0 & \text{in} \quad \mathbb{R} \\ \displaystyle\lim_{x\to-\infty}\phi(x)=0, \quad \lim_{x\to+\infty}\phi(x)=1, \quad \phi(0)=\frac{1}{2}. & \end{cases} \tag{1.13}$$

Then we can prove the following extension of the Orowan's law:

Theorem 1.4. *Assume* (1.12) *and let $p_0,\ L_0\in\mathbb{R}$ with $p_0\neq 0$. Then the function $\overline{H}$ defined in Theorem 1.1 satisfies*

$$\frac{\overline{H}(\delta p_0,\delta^{2s}L_0)}{\delta^{1+2s}}\to c_0|p_0|L_0 \quad \text{as } \delta\to 0^+ \quad \text{with} \quad c_0=\left(\int_{\mathbb{R}}(\phi')^2\right)^{-1}. \tag{1.14}$$

We notice that (1.14) can be rephrased using the notation $p:=\delta p_0$ and $L:=\delta L_0$, by saying

$$\overline{H}(p,L)=c_0|p|L + \text{higher order terms},$$

which in particular shows that $\overline{H}$ has a linear growth close to the origin. We observe that assumption (1.12) is stronger than (H2), since it requires the minima to be non-degenerate, it assumes further smoothness on the potential and the even property in the case $s<\frac{1}{2}$. This last property is natural for physical applications, since typically the effect of a dislocation in a given direction compensates with the one in the opposite direction (in particular it is satisfied in the classical Peierls-Nabarro model in which $W(u)=1-\cos(2\pi u)$). From the technical point of view, this property is needed only in the strongly non-local case $s<\frac{1}{2}$ since the first order asymptotic decay of the layer solution (1.13) lies below a critical threshold (the even property allows us to deduce a useful second order approximation).

The rest of the paper is organized as follows. First we recall some definitions and basic fact about viscosity solutions. Then, in Section 2 we imbed our problem into one in one dimension more, to keep track of all the homogenized quantities, and we state the ansatz on the solution we look for. The corrector equation will be studied in Section 3, where Theorem 1.1 will be proved. Thus, we will prove Theorems 1.2 and 1.3 in Sections 4 and 5, respectively. Then we present the extension of the Orowan's law and the proof of Theorem 1.4 in Section 6.

1.1. **Notations and definition of viscosity solution.** We denote by $B_r(x)$ the ball of radius r centered at x. The cylinder $(t-\tau, t+\tau) \times B_r(x)$ is denoted by $Q_{\tau,r}(t,x)$.

$\lfloor x \rfloor$ and $\lceil x \rceil$ denote respectively the floor and the ceiling integer part functions of a real number x.

It is convenient to introduce the singular measure defined on $\mathbb{R}^N \setminus \{0\}$ by

$$\mu(dz) = \frac{1}{|z|^{N+2s}} g\left(\frac{z}{|z|}\right) dz,$$

and to denote

$$\mathcal{I}_s^{1,r}[\varphi, x] = \int_{|z| \leqslant r} (\varphi(x+z) - \varphi(x) - \nabla\varphi(x) \cdot z) \mu(dz),$$

$$\mathcal{I}_s^{2,r}[\varphi, x] = \int_{|z|>r} (\varphi(x+z) - \varphi(x)) \mu(dz).$$

For a function u defined on $(0,T) \times \mathbb{R}^N$, $0 < T \leqslant +\infty$, for $0 < \alpha < 1$ we denote by $< u >_x^\alpha$ the seminorm defined by

$$< u >_x^\alpha := \sup_{\substack{(t,x''),\,(t,x') \in (0,T)\times\mathbb{R}^N \\ x'' \neq x'}} \frac{|u(t,x'') - u(t,x')|}{|x'' - x'|^\alpha}$$

and by $C_x^\alpha((0,T) \times \mathbb{R}^N)$ the space of continuous functions defined on $(0,T) \times \mathbb{R}^N$ that are bounded and with bounded seminorm $< u >_x^\alpha$.

Finally, we denote by $USC_b(\mathbb{R}^+ \times \mathbb{R}^N)$ (resp., $LSC_b(\mathbb{R}^+ \times \mathbb{R}^N)$) the set of upper (resp., lower) semicontinuous functions on $\mathbb{R}^+ \times \mathbb{R}^N$ which are bounded on $(0,T) \times \mathbb{R}^N$ for any $T > 0$ and we set $C_b(\mathbb{R}^+ \times \mathbb{R}^N) := USC_b(\mathbb{R}^+ \times \mathbb{R}^N) \cap LSC_b(\mathbb{R}^+ \times \mathbb{R}^N)$.

Let us conclude by recalling the definition of viscosity solution for a general first order non-local equation with associated initial condition:

$$\begin{cases} u_t = F(t,x,u,Du,\mathcal{I}_s[u]) & \text{in} \quad \mathbb{R}^+ \times \mathbb{R}^N \\ u(0,x) = u_0(x) & \text{on} \quad \mathbb{R}^N, \end{cases} \tag{1.15}$$

where $F(t,x,u,p,L)$ is continuous and non-decreasing in L. The definition relies on the following observation: if φ is a bounded C^2 function, then for any $r > 0$

$$\begin{aligned} \mathcal{I}_s[\varphi, x] &= \int_{|z|\leqslant r} (\varphi(x+z) - \varphi(x) - \nabla\varphi(x)\cdot z)\mu(dz) + \int_{|z|>r} (\varphi(x+z) - \varphi(x))\mu(dz) \\ &= \mathcal{I}_s^{1,r}[\varphi, x] + \mathcal{I}_s^{2,r}[\varphi, x] \end{aligned}$$

and this expression is independent of r because of the antisymmetry of $\nabla\varphi(x) \cdot z\mu(dz)$.

Definition 1.1 (viscosity solution). *A function $u \in USC_b(\mathbb{R}^+ \times \mathbb{R}^N)$ (resp., $u \in LSC_b(\mathbb{R}^+ \times \mathbb{R}^N)$) is a viscosity subsolution (resp., supersolution) of* (1.15) *if $u(0,x) \leqslant (u_0)^*(x)$ (resp., $u(0,x) \geqslant (u_0)_*(x)$) and for any $(t_0,x_0) \in \mathbb{R}^+ \times \mathbb{R}^N$, any $\tau \in (0,t_0)$ and any test function $\varphi \in C^2(\mathbb{R}^+ \times \mathbb{R}^N)$ such that $u - \varphi$ attains a local maximum (resp., minimum) at the point (t_0,x_0) on $Q_{(\tau,r)}(t_0,x_0)$, then we have*

$$\partial_t\varphi(t_0,x_0) - F(t_0,x_0,u(t_0,x_0),\nabla_x\varphi(t_0,x_0),\mathcal{I}_s^{1,r}[\varphi(t_0,\cdot),x_0] + \mathcal{I}_s^{2,r}[u(t_0,\cdot),x_0]) \leqslant 0$$

(resp., $\geqslant 0$),

for a positive number r. A function $u \in C_b(\mathbb{R}^+ \times \mathbb{R}^N)$ is a viscosity solution of (1.15) *if it is a viscosity sub and supersolution of* (1.15).

One can prove that Definition 1.1 does not depend on r and if the inequality above is satisfied for a given $r>0$, then it is satisfied for any $r>0$, see [10] and the references therein.

2. The Ansatz

As explained in [10], because of the presence of the term $W'\left(\frac{u^\epsilon}{\epsilon}\right)$ in (1.6) and (1.7), in order to get the homogenization results, we need to imbed our problems into higher dimensional ones. Let us first assume $s>\frac{1}{2}$. Then we will consider:

$$\begin{cases}\partial_t U^\epsilon = \epsilon^{2s-1}\mathcal{I}_s[U^\epsilon(t,\cdot,x_{N+1})] - W'\left(\frac{U^\epsilon}{\epsilon}\right) + \sigma\left(\frac{t}{\epsilon},\frac{x}{\epsilon}\right) & \text{in } \mathbb{R}^+\times\mathbb{R}^{N+1}\\ U^\epsilon(0,x,x_{N+1}) = u_0(x)+x_{N+1} & \text{on } \mathbb{R}^{N+1}\end{cases} \tag{2.1}$$

and we will prove that U^ϵ converges as $\epsilon\to 0$ to the function

$$U^0(t,x,x_{N+1}) = u^0(t,x)+x_{N+1}$$

with u^0 the solution of (1.8). We remark that U^0 satisfies:

$$\begin{cases}\partial_t U = \overline{H}_1(\nabla_x U) & \text{in } \mathbb{R}^+\times\mathbb{R}^{N+1}\\ U(0,x,x_{N+1}) = u_0(x)+x_{N+1} & \text{on } \mathbb{R}^{N+1}.\end{cases} \tag{2.2}$$

The convergence of U^ϵ to U^0 will imply the converge of u^ϵ to u^0. In order to prove this result, we introduce the higher dimensional cell problem: for $P=(p,1)\in\mathbb{R}^{N+1}$ and $L\in\mathbb{R}$:

$$\begin{cases}\lambda+\partial_\tau V = L+\mathcal{I}_s[V(\tau,\cdot,y_{N+1})] - W'(V+P\cdot Y+\lambda\tau)+\sigma(\tau,y) & \text{in } \mathbb{R}^+\times\mathbb{R}^{N+1}\\ V(0,Y)=0 & \text{on } \mathbb{R}^{N+1}.\end{cases} \tag{2.3}$$

Here we use the notation $Y=(y,y_{N+1})$. The right Ansatz for U^ϵ solution of (2.1), turns out to be

$$U^\epsilon(t,x,x_{N+1}) \simeq \tilde{U}^\epsilon(t,x,x_{N+1}) := U^0(t,x,x_{N+1}) + \epsilon V\left(\frac{t}{\epsilon},\frac{x}{\epsilon},\frac{U^0(t,x,x_{N+1})-\lambda t - p\cdot x}{\epsilon}\right) \tag{2.4}$$

with V the bounded solution of (2.3), for suitable values of p and L. Let us verify it.

Fix $P_0=(t_0,x_0,x^0_{N+1})\in\mathbb{R}^+\times\mathbb{R}^{N+1}$ and let $\tilde{U}^\epsilon(t,x,x_{N+1})$ be defined as in (2.4). Let us denote

$$\lambda = \partial_t U^0(P_0), \quad p=\nabla_x U^0(P_0), \tag{2.5}$$

and

$$F(t,x,x_{N+1}) = U^0(t,x,x_{N+1})-\lambda t - p\cdot x, \quad \tau=\frac{t}{\epsilon},\ y=\frac{x}{\epsilon}, \quad y_{N+1}=\frac{F(t,x,x_{N+1})}{\epsilon}.$$

We remark that $P=(p,1)=\nabla_{(x,x_{N+1})}U^0(P_0)$ and

$$\frac{\tilde{U}^\epsilon(t,x,x_{N+1})}{\epsilon} = V(\tau,y,y_{N+1})+\lambda\tau+p\cdot y+y_{N+1} = V(\tau,Y)+P\cdot Y+\lambda\tau.$$

Here we assume for simplicity that U^0 and V are smooth. The proof of convergence consists in showing that $\tilde{U}^\epsilon$ is a solution of (2.1) in a cylinder $(t_0-r,t_0+r)\times B_r(x_0,x^0_{N+1})$ for $r>0$ small enough, up to an error that goes to 0 as $r\to 0^+$. This will allow us to compare U^ϵ with $\tilde{U}^\epsilon$ and, thanks to the boundedness of V, to conclude that U^ϵ converges to U^0 as $\epsilon\to 0$.

When we plug $\tilde{U}^\epsilon$ into (2.1), we find the equation

$$\begin{aligned}\lambda+\partial_\tau V(\tau,Y) &= \epsilon^{2s-1}\mathcal{I}_s[U^0(t,\cdot,x_{N+1}),x]+\mathcal{I}_s[V(\tau,\cdot,y_{N+1}),y]\\ &\quad - W'(V+PY+\lambda\tau)+\sigma(\tau,y)+\theta_r,\end{aligned}$$

where

$$\theta_r = (\partial_t U^0(P_0) - \partial_t U^0(t,x,x_{N+1}))(\partial_{y_{N+1}} V(\tau,Y)+1) \\ + \epsilon^{2s}\mathcal{I}_s\left[V\left(\frac{t}{\epsilon},\frac{\cdot}{\epsilon},\frac{F(t,\cdot,x_{N+1})}{\epsilon}\right),x\right] - \mathcal{I}_s[V(\tau,\cdot,y_{N+1}),y].$$

If V is solution of (2.3) with p as in (2.5) and $L=0$, and U^0 satisfies $\partial_t U^0(P_0)=\lambda=\overline{H}(\nabla_x U^0(P_0), 0)$, then $\tilde{U}^\epsilon$ will be a solution of (2.1) up to small errors $\epsilon^{2s-1}\mathcal{I}_s[U^0(t,\cdot,x_{N+1}),x]=o_\epsilon(1)$ as $\epsilon\to 0$ and $\theta_r = o_r(1)$ as $r\to 0^+$. As we will see in Section 4, this last property holds true if the corrector V satisfies: $|V|$, $|\partial_{y_{N+1}}V|\leqslant C$ in $\mathbb{R}^+\times\mathbb{R}^{N+1}$ for some $C>0$, and

$$\partial_{y_{N+1}}V(\tau,\cdot,\cdot) \quad \text{is Hölder continuous, uniformly in time.}$$

Since in (2.3) the quantity $\mathcal{I}_s[V(\tau,\cdot,y_{N+1})]$ is computed only in the y variable, we cannot expect this kind of regularity for the correctors. Nevertheless, following [10], we are able to construct regular approximated sub and supercorrectors, i.e., sub and supersolutions of approximate $N+1$-dimensional cell problems, and this is enough to conclude.

Similarly for $s<\frac{1}{2}$, we will consider:

(2.6)
$$\begin{cases}\partial_t U^\epsilon = \mathcal{I}_s[U^\epsilon(t,\cdot,x_{N+1})] - W'\left(\frac{U^\epsilon}{\epsilon^{2s}}\right)+\sigma\left(\frac{t}{\epsilon^{2s}},\frac{x}{\epsilon}\right) & \text{in } \mathbb{R}^+\times\mathbb{R}^{N+1}\\ U^\epsilon(0,x,x_{N+1}) = u_0(x)+x_{N+1} & \text{on } \mathbb{R}^{N+1},\end{cases}$$

and we will show that U^ϵ converges as $\epsilon\to 0$ to the function

$$U^0(t,x,x_{N+1}) = u^0(t,x)+x_{N+1}$$

with u^0 the solution of (1.9). Here U^0 is solution of

(2.7)
$$\begin{cases}\partial_t U = \overline{H}_2(\mathcal{I}_s[U(t,\cdot,x_{N+1})]) & \text{in } \mathbb{R}^+\times\mathbb{R}^{N+1}\\ U(0,x,x_{N+1}) = u_0(x)+x_{N+1} & \text{on } \mathbb{R}^{N+1}.\end{cases}$$

In this case, the right Ansatz turns out to be

$$U^\epsilon(t,x,x_{N+1}) \simeq U^0(t,x,x_{N+1}) + \epsilon^{2s}V\left(\frac{t}{\epsilon^{2s}},\frac{x}{\epsilon},\frac{U^0(t,x,x_{N+1})-\lambda t}{\epsilon^{2s}}\right)$$

where V is the bounded solution of (2.3) for $p=0$ and $L=\mathcal{I}_s[U^0(t,\cdot,x_{N+1}),x]$.

3. Correctors

In this section we prove Theorem 1.1 and the existence of smooth approximated sub and supersolutions of the higher dimensional cell problem (2.3) introduced in Section 2 which are needed to show the convergence Theorems 1.2 and 1.3. The proof of these results is given in [10] for the case $s=1$ and it is essentially based on the comparison principle and invariance under integer translations. Therefore it can be easily extended to the case $s\in(0,1)$ and for this reason, here we only give a sketch of it.

Step 1: Lipschitz correctors.

One introduces the problem: for $\eta\geqslant 0$, $a_0,\ L\in\mathbb{R}$, $p\in\mathbb{R}^N$ and $P=(p,1)$

(3.1)
$$\begin{cases}\partial_\tau U = L+\mathcal{I}_s[U(\tau,\cdot,y_{N+1})] - W'(U+P\cdot Y)+\sigma(\tau,y) & \\ \quad +\eta[a_0+\inf_{Y'}U(\tau,Y')-U(\tau,Y)]|\partial_{y_{N+1}}U+1| & \text{in } \mathbb{R}^+\times\mathbb{R}^{N+1}\\ U(0,Y)=0 & \text{on } \mathbb{R}^{N+1},\end{cases}$$

and show the existence of the viscosity solution $U_\eta \in C_b(\mathbb{R}^+ \times \mathbb{R}^{N+1})$. When $\eta > 0$ this solution turns out to be Lipschitz continuous in the variable y_{N+1} with

$$-1 \leqslant \partial_{y_{N+1}} U_\eta(\tau, Y) \leqslant \frac{\|W''\|_\infty}{\eta}.$$

See the proof of Propositions 6.2, 6.3 and 6.4 in [10] for details about the existence and regularity of the solution of (3.1). As we will explain in Step 5, choosing conveniently the number a_0 in (3.1), we obtain sub and supersolutions of the $N+1$-dimensional cell problem (2.3) which are Lipschitz continuous in y_{N+1}.

Step 2: Ergodicity.

Using the comparison principle, and the periodicity of σ and W, one can prove the following ergodic result:

Proposition 3.1 (Ergodic properties)**.** *There exists a unique $\lambda_\eta = \lambda_\eta(p, L)$ such that the viscosity solution $U_\eta \in C_b(\mathbb{R}^+ \times \mathbb{R}^{N+1})$ of* (3.1) *with $\eta \geqslant 0$, satisfies:*

(3.2) $$|U_\eta(\tau, Y) - \lambda_\eta \tau| \leqslant C \text{ for all } \tau > 0,\ Y \in \mathbb{R}^{N+1},$$

with C independent of η. Moreover

(3.3) $$L - \|W'\|_\infty - \|\sigma\|_\infty + \eta a_0 \leqslant \lambda_\eta \leqslant L + \|W'\|_\infty + \|\sigma\|_\infty + \eta a_0.$$

Proposition 3.1 can be proved like Proposition 6.4 in [10].

Step 3: Proof of Theorem 1.1.

Let U be the solution of (3.1) with $\eta = 0$, then the function

$$w(\tau, y) := U(\tau, y, 0)$$

is the solution of (1.11) and by Proposition 3.1, there exists a unique λ such that

(3.4) $$|w(\tau, y) - \lambda \tau| \leqslant C.$$

This property implies that λ is the unique number such that $w(\tau, y)/\tau$ converges towards λ as $\tau \to +\infty$, and Theorem 1.1 is proved.

The next two steps are only needed in the proof of Theorems 1.2 and 1.3. We first state some properties of the effective Hamiltonian, then in Step 5, we construct approximate sub and supersolutions of (2.3) which are smooth also in the additional variable y_{N+1}. This further regularity property is needed to control the error when we compare the solution U^ϵ of (2.1) and (2.6) with the corresponding ansatz, as explained in Section 2.

Step 4: Properties of the effective Hamiltonian

We have

Proposition 3.2 (Properties of the effective Hamiltonian)**.** *Let $p \in \mathbb{R}^N$ and $L \in \mathbb{R}$. Let $\overline{H}(p, L)$ be the constant defined by Theorem 1.1, then $\overline{H} : \mathbb{R}^N \times \mathbb{R} \to \mathbb{R}$ is a continuous function with the following properties:*

(i) $\overline{H}(p, L) \to \pm\infty$ *as* $L \to \pm\infty$ *for any* $p \in \mathbb{R}^N$*;*
(ii) $\overline{H}(p, \cdot)$ *is non-decreasing on* $\mathbb{R}$ *for any* $p \in \mathbb{R}^N$*;*
(iii) *If* $\sigma(\tau, y) = \sigma(\tau, -y)$ *then*
$$\overline{H}(p, L) = \overline{H}(-p, L);$$

(iv) *If* $W'(-s) = -W'(s)$ *and* $\sigma(\tau, -y) = -\sigma(\tau, y)$ *then*

$$\overline{H}(p, -L) = -\overline{H}(p, L).$$

For the proof of Proposition 3.2 see Proposition 5.4 in [10].

Step 5: Construction of smooth approximate sub and supercorrectors.

The ergodic property (3.1) of U_η implies that there exists $C_1 > 0$ such that

$$C_1 + \inf_{Y'} U_\eta(\tau, Y') - U_\eta(\tau, Y) > 0,$$

for any $\eta > 0$. Then, one take U_η^+ to be the solution of (3.1) with $a_0 = C_1$ and U_η^- to be the solution of (3.1) with $a_0 = 0$. We remark that U_η^+ and U_η^- are respectively super and subsolution of

$$\partial_\tau U = L + \mathcal{I}_s[U(\tau, \cdot, y_{N+1})] - W'(U + P \cdot Y) + \sigma(\tau, y).$$

Let $\lambda_\eta^+ = \lim_{\tau \to +\infty} \dfrac{U_\eta^+(\tau, Y)}{\tau}$ and $\lambda_\eta^- = \lim_{\tau \to +\infty} \dfrac{U_\eta^-(\tau, Y)}{\tau}$, whose the existence is guaranteed by Proposition 3.1. Stability results and the ergodic property (3.2) imply that $\lambda_\eta^+, \lambda_\eta^- \to \lambda$ as $\eta \to 0$, with λ given by Theorem 1.1.

Next, one set

$$W_\eta^+(\tau, Y) := U_\eta^+(\tau, Y) - \lambda_\eta^+ \tau$$

and

$$W_\eta^-(\tau, Y) := U_\eta^-(\tau, Y) - \lambda_\eta^- \tau.$$

Then W_η^+ and W_η^- are respectively super and subsolution of (2.3) with respectively $\lambda = \lambda_\eta^+$ and $\lambda = \lambda_\eta^-$, and are Lipschitz continuous in the variable y_{N+1}. One can in addition show that these functions are of class C^α with respect to y uniformly in y_{N+1}, for $0 < \alpha < \min\{1, 2s\}$. This comes from Proposition 4.7 in [10] that can be easily adapted to the case $s \in (0, 1)$.

The regularity properties of W_η^+ and W_η^- are not enough in order to prove the convergence results, Theorems 1.2 and 1.3, as pointed out in Section 2. Therefore, one introduces a positive smooth function $\rho : \mathbb{R} \to \mathbb{R}$, with support in $B_1(0)$ and mass 1 and defines a sequence of mollifiers $(\rho_\delta)_\delta$ by $\rho_\delta(r) = \frac{1}{\delta}\rho\left(\frac{r}{\delta}\right)$, $r \in \mathbb{R}$. Then, one finally defines

$$V_{\eta,\delta}^\pm(t, y, y_{N+1}) := W_\eta^\pm(t, y, \cdot) \star \rho_\delta(\cdot) = \int_{\mathbb{R}} W_\eta^\pm(t, y, z)\rho_\delta(y_{N+1} - z)dz.$$

Choosing properly $\delta = \delta(\eta)$, one can prove the following result:

Proposition 3.3 (Smooth approximate correctors). *Let λ be the constant defined by Theorem 1.1. For any fixed $p \in \mathbb{R}^N$, $P = (p, 1)$, $L \in \mathbb{R}$ and $\eta > 0$ small enough, there exist real numbers $\lambda_\eta^+(p, L)$, $\lambda_\eta^-(p, L)$, a constant $C > 0$ (independent of η, p and L) and bounded super and subcorrectors V_η^+, V_η^-, i.e. respectively a super and a subsolution of*

$$\begin{cases} \lambda_\eta^\pm + \partial_\tau V_\eta^\pm = L + \mathcal{I}_s[V_\eta^\pm(\tau, \cdot, y_{N+1})] \\ \qquad\qquad -W'(V_\eta^\pm + P \cdot Y + \lambda_\eta^\pm \tau) + \sigma(\tau, y) \mp o_\eta(1) & \text{in } \mathbb{R}^+ \times \mathbb{R}^{N+1} \\ V_\eta^\pm(0, Y) = 0 & \text{on } \mathbb{R}^{N+1}, \end{cases} \tag{3.5}$$

where $0 \leqslant o_\eta(1) \to 0$ as $\eta \to 0^+$, such that

$$\lim_{\eta \to 0^+} \lambda_\eta^+(p, L) = \lim_{\eta \to 0^+} \lambda_\eta^-(p, L) = \lambda(p, L), \tag{3.6}$$

locally uniformly in (p, L), $\lambda_\eta^\pm$ satisfy (i) and (ii) of Proposition 3.2 and for any $(\tau, Y) \in \mathbb{R}^+ \times \mathbb{R}^{N+1}$

$$|V_\eta^\pm(\tau, Y)| \leqslant C. \tag{3.7}$$

Moreover $V_\eta^\pm$ are of class C^2 w.r.t. y_{N+1}, and for any $0<\alpha<\min\{1,2s\}$

$$-1 \leqslant \partial_{y_{N+1}} V_\eta^\pm \leqslant \frac{\|W''\|_\infty}{\eta}, \tag{3.8}$$

$$\|\partial^2_{y_{N+1}y_{N+1}} V_\eta^\pm\|_\infty \leqslant C_\eta, \quad <\partial_{y_{N+1}} V_\eta^\pm>^\alpha_y, \leqslant C_{\eta,\alpha}. \tag{3.9}$$

4. Proof of Theorem 1.2

To prove Theorem 1.2, as explained in Section 2, we introduce the higher dimensional problem (2.1) and we prove the convergence of the solution U^ϵ to the solution U^0 of (2.2). Let us first state the following

Proposition 4.1. *For $\epsilon>0$ there exists $U^\epsilon \in C_b(\mathbb{R}^+\times\mathbb{R}^{N+1})$ (unique) viscosity solution of* (2.1)*. Moreover, there exists a constant $C>0$ independent of ϵ such that*

$$|U^\epsilon(t,x,x_{N+1}) - u_0(x) - x_{N+1}| \leqslant Ct. \tag{4.1}$$

Proposition 4.1 as well as the existence of a unique solution of problems (1.6), (1.8) and (2.2) is a consequence of the Perron's method and the comparison principle for these equations, see [10] and references therein. Let us exhibit the link between the problem in $\mathbb{R}^N$ and the problem in $\mathbb{R}^{N+1}$.

Lemma 4.2 (Link between the problems on $\mathbb{R}^N$ and on $\mathbb{R}^{N+1}$)**.** *If u^ϵ and U^ϵ denote respectively the solution of* (1.6) *and* (2.1)*, then we have*

$$\left|U^\epsilon(t,x,x_{N+1}) - u^\epsilon(t,x) - \epsilon\left\lfloor\frac{x_{N+1}}{\epsilon}\right\rfloor\right| \leqslant \epsilon,$$

$$U^\epsilon\left(t,x,x_{N+1}+\epsilon\left\lfloor\frac{a}{\epsilon}\right\rfloor\right) = U^\epsilon(t,x,x_{N+1}) + \epsilon\left\lfloor\frac{a}{\epsilon}\right\rfloor \quad \text{for any } a\in\mathbb{R}. \tag{4.2}$$

This lemma follows from the comparison principle for (2.1) and the invariance by ϵ-translations w.r.t. x_{N+1}.

Lemma 4.3. *Let u^0 and U^0 be respectively the solutions of* (1.8) *and* (2.2)*. Then, we have*

$$U^0(t,x,x_{N+1}) = u^0(t,x) + x_{N+1}.$$

Lemma 4.3 is a consequence of the comparison principle for (2.2) and the invariance by translations w.r.t. x_{N+1}.

Let us proceed with the proof of Theorem 1.2. In what follows we will use the notation $X=(x,x_{N+1})$. By (4.1), we know that the family of functions $\{U^\epsilon\}_{\epsilon>0}$ is locally bounded, then

$$U^+(t,X) := \limsup_{\epsilon\to0}{}^* U^\epsilon(t,X) := \limsup_{\substack{\epsilon\to0\\(t',X')\to(t,X)}} U^\epsilon(t',X')$$

is everywhere finite, so it becomes classical to prove that U^+ is a subsolution of (2.2).

Similarly, we can prove that

$$U^-(t,X) := \liminf_{\epsilon\to0}{}_* U^\epsilon(t,X) := \liminf_{\substack{\epsilon\to0\\(t',X')\to(t,X)}} U^\epsilon(t',X')$$

is a supersolution of (2.2). Moreover $U^+(0,X)=U^-(0,X)=u_0(x)+x_{N+1}$. The comparison principle for (2.2) then implies that $U^+\leqslant U^-$. Since the reverse inequality $U^-\leqslant U^+$ always holds true, we conclude that the two functions coincide with U^0, the unique viscosity solution of (2.2).

By Lemmata 4.2 and 4.3, the convergence of U^ϵ to U^0 proves in particular that u^ϵ converges towards u^0 viscosity solution of (1.8).

To prove that U^+ is a subsolution of (2.2), we argue by contradiction. We consider a test function ϕ such that $U^+ - \phi$ attains a zero maximum at (t_0, X_0) with $t_0 > 0$ and $X_0 = (x_0, x^0_{N+1})$. Without loss of generality we may assume that the maximum is strict and global. Suppose that there exists $\theta > 0$ such that

$$\partial_t \phi(t_0, X_0) = \overline{H}_1(\nabla_x \phi(t_0, X_0)) + \theta.$$

By Proposition 3.2, we know that there exists $L_1 > 0$ (that we take minimal) such that

$$\overline{H}_1(\nabla_x \phi(t_0, X_0)) + \theta = \overline{H}(\nabla_x \phi(t_0, X_0), 0) + \theta = \overline{H}(\nabla_x \phi(t_0, X_0), L_1).$$

By Propositions 3.3 and 3.2, we can consider a sequence $L_\eta \to L_1$ as $\eta \to 0^+$, such that $\lambda^+_\eta(\nabla_x \phi(t_0, X_0), L_\eta) = \lambda(\nabla_x \phi(t_0, X_0), L_1)$. We choose η so small that $L_\eta - o_\eta(1) \geqslant L_1/2 > 0$, where $o_\eta(1)$ is defined in Proposition 3.3. Let V^+_η be the approximate supercorrector given by Proposition 3.3 with

$$p = \nabla_x \phi(t_0, X_0), \quad L = L_\eta$$

and

$$\lambda^+_\eta = \lambda^+_\eta(p, L_\eta) = \lambda(p, L_1) = \partial_t \phi(t_0, X_0).$$

For simplicity of notations, in the following we denote $V = V^+_\eta$. We consider the function

$$F(t, X) = \phi(t, X) - p \cdot x - \lambda^+_\eta t,$$

and as in [10] we introduce the "x_{N+1}-twisted perturbed test function" ϕ^ϵ defined by:

$$\phi^\epsilon(t, X) := \begin{cases} \phi(t, X) + \epsilon V\left(\frac{t}{\epsilon}, \frac{x}{\epsilon}, \frac{F(t,X)}{\epsilon}\right) + \epsilon k_\epsilon & \text{in} \quad (\frac{t_0}{2}, 2t_0) \times B_{\frac{1}{2}}(X_0) \\ U^\epsilon(t, X) & \text{outside,} \end{cases} \tag{4.3}$$

where $k_\epsilon \in \mathbb{Z}$ will be chosen later.

We are going to prove that ϕ^ϵ is a supersolution of (2.1) in $Q_{r,r}(t_0, X_0)$ for some $r < \frac{1}{2}$ properly chosen and such that $Q_{r,r}(t_0, X_0) \subset (\frac{t_0}{2}, 2t_0) \times B_{\frac{1}{2}}(X_0)$. First, we observe that since $U^+ - \phi$ attains a strict maximum at (t_0, X_0) with $U^+ - \phi = 0$ at (t_0, X_0) and V is bounded, we can ensure that there exists $\epsilon_0 = \epsilon_0(r) > 0$ such that for $\epsilon \leqslant \epsilon_0$

$$U^\epsilon(t, X) \leqslant \phi(t, X) + \epsilon V\left(\frac{t}{\epsilon}, \frac{x}{\epsilon}, \frac{F(t, X)}{\epsilon}\right) - \gamma_r, \quad \text{in} \left(\frac{t_0}{3}, 3t_0\right) \times B_1(X_0) \setminus Q_{r,r}(t_0, X_0) \tag{4.4}$$

for some $\gamma_r = o_r(1) > 0$. Hence choosing $k_\epsilon = \lceil \frac{-\gamma_r}{\epsilon} \rceil$ we get

$$U^\epsilon \leqslant \phi^\epsilon \quad \text{outside } Q_{r,r}(t_0, X_0).$$

Let us next study the equation satisfied by ϕ^ϵ. For this, we observe that

$$\frac{a}{\epsilon} - 1 \leqslant \left\lfloor \frac{a}{\epsilon} \right\rfloor \leqslant \frac{a}{\epsilon}$$

and so, from (4.2), we deduce that

$$U^\epsilon(t, x, x_{N+1}) + a - \epsilon \ \leqslant \ U^\epsilon\left(t, x, x_{N+1} + \epsilon \left\lfloor \frac{a}{\epsilon} \right\rfloor\right) \ \leqslant \ U^\epsilon(t, x, x_{N+1}) + a.$$

Consequently, passing to the limit, we obtain that $U^+(t, x, x_{N+1} + a) = U^+(t, x, x_{N+1}) + a$ for any $a \in \mathbb{R}$.

From this, we derive that $\partial_{x_{N+1}} F(t_0, X_0) = \partial_{x_{N+1}} \phi(t_0, X_0) = 1$. Then, there exists $r_0 > 0$ such that the map

$$\begin{array}{rccc} Id\times F: & Q_{r_0,r_0}(t_0,X_0) & \longrightarrow & \mathcal{U}_{r_0} \\ & (t,x,x_{N+1}) & \longmapsto & (t,x,F(t,x,x_{N+1})) \end{array}$$

is a C^1-diffeomorphism from $Q_{r_0,r_0}(t_0,X_0)$ onto its range $\mathcal{U}_{r_0}$. Let $G:\mathcal{U}_{r_0}\to\mathbb{R}$ be the map such that

$$\begin{array}{rccc} Id\times G: & \mathcal{U}_{r_0} & \longrightarrow & Q_{r_0,r_0}(t_0,X_0) \\ & (t,x,\xi_{N+1}) & \longmapsto & (t,x,G(t,x,\xi_{N+1})) \end{array}$$

is the inverse of $Id\times F$. Let us introduce the variables $\tau=t/\epsilon$, $Y=(y,y_{N+1})$ with $y=x/\epsilon$ and $y_{N+1}=F(t,X)/\epsilon$. Let us consider a test function ψ such that $\phi^\epsilon-\psi$ attains a global zero minimum at $(\bar t,\overline{X})\in Q_{r_0,r_0}(t_0,X_0)$ and define

$$\Gamma^\epsilon(\tau,Y)=\frac{1}{\epsilon}[\psi(\epsilon\tau,\epsilon y,G(\epsilon\tau,\epsilon y,\epsilon y_{N+1}))-\phi(\epsilon\tau,\epsilon y,G(\epsilon\tau,\epsilon y,\epsilon y_{N+1}))]-k_\epsilon.$$

Then

$$\psi(t,X)=\phi(t,X)+\epsilon\Gamma^\epsilon\left(\frac{t}{\epsilon},\frac{x}{\epsilon},\frac{F(t,X)}{\epsilon}\right)+\epsilon k_\epsilon$$

and Γ^ϵ is a test function for V:

$$\text{(4.5)}\qquad \Gamma^\epsilon(\overline{\tau},\overline{Y})=V(\overline{\tau},\overline{Y})\quad\text{and}\quad\Gamma^\epsilon(\tau,Y)\leqslant V(\tau,Y)\quad\text{for all }(\epsilon\tau,\epsilon Y)\in Q_{r_0,r_0}(t_0,X_0),$$

where $\overline{\tau}=\bar t/\epsilon$, $\overline{y}=\overline{x}/\epsilon$, $\overline{y}_{N+1}=F(\bar t,\overline{X})/\epsilon$, $\overline{Y}=(\overline{y},\overline{y}_{N+1})$. From Proposition 3.3, we know that V is Lipschitz continuous w.r.t. y_{N+1} with Lipschitz constant M_η depending on η. This implies that

$$\text{(4.6)}\qquad |\partial_{y_{N+1}}\Gamma^\epsilon(\overline{\tau},\overline{Y})|\leqslant M_\eta.$$

Simple computations yield with $P=(p,1)\in\mathbb{R}^{N+1}$:

$$\text{(4.7)}\qquad \begin{cases}\lambda_\eta^++\partial_\tau\Gamma^\epsilon(\overline{\tau},\overline{Y})=\partial_t\psi(\bar t,\overline{X})+\left(1+\partial_{y_{N+1}}\Gamma^\epsilon(\overline{\tau},\overline{Y})\right)(\partial_t\phi(t_0,X_0)-\partial_t\phi(\bar t,\overline{X})),\\ \lambda_\eta^+\overline{\tau}+P\cdot\overline{Y}+V(\overline{\tau},\overline{Y})=\frac{\phi^\epsilon(\bar t,\overline{X})}{\epsilon}-k_\epsilon.\end{cases}$$

Using (4.7) and (4.6), equation (3.5) yields for any $\rho>0$

$$\text{(4.8)}\qquad \begin{aligned}\partial_t\psi(\bar t,\overline{X})+o_r(1)\geqslant L_\eta&+\mathcal{I}_s^{1,\rho}[\Gamma^\epsilon(\overline{\tau},\cdot,\overline{y}_{N+1}),\overline{y}]+\mathcal{I}_s^{2,\rho}[V(\overline{\tau},\cdot,\overline{y}_{N+1}),\overline{y}]\\&-W'\left(\frac{\phi^\epsilon(\bar t,\overline{X})}{\epsilon}\right)+\sigma\left(\frac{\bar t}{\epsilon},\frac{\overline{x}}{\epsilon}\right)-o_\eta(1).\end{aligned}$$

Now, to complete the proof of Theorem 1.2, we state the following lemma (which will be proved in the next subsection):

Lemma 4.4. (Supersolution property for ϕ^ϵ)
For $\epsilon\leqslant\epsilon_0(r)<r\leqslant r_0$, we have

$$\text{(4.9)}\qquad \begin{aligned}\partial_t\psi(\bar t,\overline{X})\geqslant\epsilon^{2s-1}&\left(\mathcal{I}_s^{1,1}\left[\psi(\bar t,\cdot,\overline{x}_{N+1}),\overline{x}\right]+\mathcal{I}_s^{2,1}\left[\phi^\epsilon(\bar t,\cdot,\overline{x}_{N+1}),\overline{x}\right]\right)\\&-W'\left(\frac{\phi^\epsilon(\bar t,\overline{X})}{\epsilon}\right)+\sigma\left(\frac{\bar t}{\epsilon},\frac{\overline{x}}{\epsilon}\right)-o_\eta(1)+o_r(1)+L_\eta.\end{aligned}$$

The proof of Lemma 4.4 is postponed to the next subsection, for the convenience of the reader, so we complete now the proof of Theorem 1.2. For this, let $r\leqslant r_0$ be so small that $o_r(1)\geqslant-L_1/4$. Then, recalling that $L_\eta-o_\eta(1)\geqslant L_1/2$, for $\epsilon\leqslant\epsilon_0(r)$ we have

$$\begin{aligned}\partial_t\psi(\bar t,\overline{X})\geqslant\epsilon^{2s-1}\left(\mathcal{I}_s^{1,1}\left[\psi(\bar t,\cdot,\overline{x}_{N+1}),\overline{x}\right]+\mathcal{I}_s^{2,1}\left[\phi^\epsilon(\bar t,\cdot,\overline{x}_{N+1}),\overline{x}\right]\right)-W'\left(\frac{\phi^\epsilon(\bar t,\overline{X})}{\epsilon}\right)\\+\sigma\left(\frac{\bar t}{\epsilon},\frac{\overline{x}}{\epsilon}\right)+\frac{L_1}{4},\end{aligned}$$

and therefore ϕ^ϵ is a supersolution of (2.1) in $Q_{r,r}(t_0, X_0)$.
Since $U^\epsilon \leqslant \phi^\epsilon$ outside $Q_{r,r}(t_0, X_0)$, by the comparison principle, we conclude that

$$U^\epsilon(t,X) \leqslant \phi(t,X) + \epsilon V\left(\frac{t}{\epsilon}, \frac{x}{\epsilon}, \frac{F(t,X)}{\epsilon}\right) + \epsilon k_\epsilon \quad \text{in } Q_{r,r}(t_0, X_0)$$

and we obtain the desired contradiction by passing to the upper limit as $\epsilon \to 0$ at (t_0, X_0) using the fact that $U^+(t_0, X_0) = \phi(t_0, X_0)$: $0 \leqslant -\gamma_r$.
This ends the proof of Theorem 1.2.

4.1. **Proof of Lemma 4.4.** The result will follow from (4.8) and the following inequality

$$\begin{aligned} &\mathcal{I}_s^{1,\rho}[\Gamma^\epsilon(\overline{\tau}, \cdot, \overline{y}_{N+1}), \overline{y}] + \mathcal{I}_s^{2,\rho}[V(\overline{\tau}, \cdot, \overline{y}_{N+1}), \overline{y}] \\ &\geqslant \epsilon^{2s-1}\left(\mathcal{I}_s^{1,1}\left[\psi(\overline{t}, \cdot, \overline{x}_{N+1}), \overline{x}\right] + \mathcal{I}_s^{2,1}\left[\phi^\epsilon(\overline{t}, \cdot, \overline{x}_{N+1}), \overline{x}\right]\right) + o_r(1) \end{aligned} \tag{4.10}$$

Keep in mind that $\overline{y}_{N+1} = \frac{F(\overline{t},\overline{X})}{\epsilon}$. Since $\psi(t,X) = \phi(t,X) + \epsilon\Gamma^\epsilon\left(\frac{t}{\epsilon}, \frac{x}{\epsilon}, \frac{F(t,X)}{\epsilon}\right) + \epsilon k_\epsilon$, we have

$$\mathcal{I}_s^{1,1}\left[\psi(\overline{t}, \cdot, \overline{x}_{N+1}), \overline{x}\right] = I_1 + I_2, \tag{4.11}$$

where

$$\left\{\begin{array}{lcl} I_1 &=& \displaystyle\int_{|x|\leqslant 1} \epsilon \left(\begin{array}{l} \Gamma^\epsilon\left(\frac{\overline{t}}{\epsilon}, \frac{\overline{x}+x}{\epsilon}, \frac{F(\overline{t},\overline{x}+x,\overline{x}_{N+1})}{\epsilon}\right) - \Gamma^\epsilon(\overline{\tau}, \overline{Y}) \\ -\nabla_y\Gamma^\epsilon(\overline{\tau},\overline{Y})\cdot\frac{x}{\epsilon} - \partial_{y_{N+1}}\Gamma^\epsilon(\overline{\tau},\overline{Y})\nabla_x F(\overline{t},\overline{X})\cdot\frac{x}{\epsilon} \end{array}\right)\mu(dx), \\ I_2 &=& \displaystyle\int_{|x|\leqslant 1} \left(\phi(\overline{t}, \overline{x}+x, \overline{x}_{N+1}) - \phi(\overline{t},\overline{X}) - \nabla\phi(\overline{t},\overline{X})\cdot x\right)\mu(dx). \end{array}\right.$$

In order to show (4.10), we show successively in Steps 1, 2 and 3:

$$\left\{\begin{array}{l} \epsilon^{2s-1} I_1 \leqslant \mathcal{I}_s^{1,\rho}[\Gamma^\epsilon(\overline{\tau}, \cdot, \overline{y}_{N+1}), \overline{y}] + \mathcal{I}_s^{2,\rho}[V(\overline{\tau}, \cdot, \overline{y}_{N+1}), \overline{y}] + o_r(1) + C_\epsilon \rho^{2-2s} \\ \epsilon^{2s-1} I_2 \leqslant o_r(1) \\ \epsilon^{2s-1}\mathcal{I}_s^{2,1}\left[\phi^\epsilon(\overline{t}, \cdot, \overline{x}_{N+1}), \overline{x}\right] \leqslant o_r(1) \end{array}\right.$$

Because the expressions are non linear and non-local and with a singular kernel, there is no simple computation and we have to carefully check those inequalities sometimes splitting terms in easier parts to estimate.

Step 1: We can choose ϵ_0 so small that for any $\epsilon \leqslant \epsilon_0$ and any $\rho > 0$ small enough

$$\epsilon^{2s-1} I_1 \leqslant \mathcal{I}_s^{1,\rho}[\Gamma^\epsilon(\overline{\tau}, \cdot, \overline{y}_{N+1}), \overline{y}] + \mathcal{I}_s^{2,\rho}[V(\overline{\tau}, \cdot, \overline{y}_{N+1}), \overline{y}] + o_r(1) + C_\epsilon \rho^{2-2s}.$$

Take $\rho > 0$, $\delta > \rho$ small and $R > 0$ large and such that $\epsilon R < 1$. Since g is even, we can write

$$I_1 = I_1^0 + I_1^1 + I_1^2 + I_1^3,$$

where

$$\begin{aligned} I_1^0 = \int_{|x|\leqslant\epsilon\rho} \epsilon &\left(\Gamma^\epsilon\left(\frac{\overline{t}}{\epsilon}, \frac{\overline{x}+x}{\epsilon}, \frac{F(\overline{t},\overline{x}+x,\overline{x}_{N+1})}{\epsilon}\right) - \Gamma^\epsilon(\overline{\tau},\overline{Y}) - \nabla_y\Gamma^\epsilon(\overline{\tau},\overline{Y})\cdot\frac{x}{\epsilon}\right. \\ &\left.-\partial_{y_{N+1}}\Gamma^\epsilon(\overline{\tau},\overline{Y})\nabla_x F(\overline{t},\overline{X})\cdot\frac{x}{\epsilon}\right)\mu(dx), \end{aligned}$$

$$I_1^1 = \int_{\epsilon\rho \leqslant |x| \leqslant \epsilon\delta} \epsilon \left(\Gamma^\epsilon \left(\frac{\bar{t}}{\epsilon}, \frac{\overline{x}+x}{\epsilon}, \frac{F(\bar{t},\overline{x}+x,\overline{x}_{N+1})}{\epsilon} \right) - \Gamma^\epsilon(\overline{\tau},\overline{Y}) \right) \mu(dx),$$

$$I_1^2 = \int_{\epsilon\delta \leqslant |x| \leqslant \epsilon R} \epsilon \left(\Gamma^\epsilon \left(\frac{\bar{t}}{\epsilon}, \frac{\overline{x}+x}{\epsilon}, \frac{F(\bar{t},\overline{x}+x,\overline{x}_{N+1})}{\epsilon} \right) - \Gamma^\epsilon(\overline{\tau},\overline{Y}) \right) \mu(dx),$$

$$I_1^3 = \int_{\epsilon R \leqslant |x| \leqslant 1} \epsilon \left(\Gamma^\epsilon \left(\frac{\bar{t}}{\epsilon}, \frac{\overline{x}+x}{\epsilon}, \frac{F(\bar{t},\overline{x}+x,\overline{x}_{N+1})}{\epsilon} \right) - \Gamma^\epsilon(\overline{\tau},\overline{Y}) \right) \mu(dx).$$

Moreover

$$\mathcal{I}_s^{2,\rho}[V(\overline{\tau},\cdot,\overline{y}_{N+1}),\overline{y}] = J_1 + J_2 + J_3,$$

where

$$J_1 = \int_{\rho<|z|\leqslant\delta} (V(\overline{\tau},\overline{y}+z,\overline{y}_{N+1}) - V(\overline{\tau},\overline{Y}))\mu(dz),$$

$$J_2 = \int_{\delta<|z|\leqslant R} (V(\overline{\tau},\overline{y}+z,\overline{y}_{N+1}) - V(\overline{\tau},\overline{Y}))\mu(dz),$$

$$J_3 = \int_{|z|>R} (V(\overline{\tau},\overline{y}+z,\overline{y}_{N+1}) - V(\overline{\tau},\overline{Y}))\mu(dz).$$

STEP 1.1: *Estimate of* $\epsilon^{2s-1}I_1^0$ *and* $\mathcal{I}_s^{1,\rho}[\Gamma^\epsilon(\overline{\tau},\cdot,\overline{y}_{N+1}),\overline{y}]$.

Since Γ^ϵ is of class C^2, we have

(4.12)
$$|\epsilon^{2s-1}I_1^0|,\ |\mathcal{I}_s^{1,\rho}[\Gamma^\epsilon(\overline{\tau},\cdot,\overline{y}_{N+1}),\overline{y}]| \leqslant C_\epsilon \rho^{2-2s},$$

where C_ϵ depends on the second derivatives of Γ^ϵ. Notice that if we knew that V is smooth in y too, we could choose $\rho = 0$.

STEP 1.2 *Estimate of* $\epsilon^{2s-1}I_1^1 - J_1$.

Using (4.5) and the fact that g is even, we can estimate $\epsilon^{2s-1}I_1^1 - J_1$ as follows

$$\begin{aligned}
\epsilon^{2s-1}I_1^1 - J_1 &\leqslant \int_{\rho<|z|\leqslant\delta} \left[V\left(\overline{\tau},\overline{y}+z,\frac{F(\bar{t},\overline{x}+\epsilon z,\overline{x}_{N+1})}{\epsilon}\right) - V\left(\overline{\tau},\overline{y}+z,\frac{F(\bar{t},\overline{X})}{\epsilon}\right)\right]\mu(dz) \\
&= \int_{\rho<|z|\leqslant\delta} \left\{ \left[V\left(\overline{\tau},\overline{y}+z,\frac{F(\bar{t},\overline{x}+\epsilon z,\overline{x}_{N+1})}{\epsilon}\right) - V\left(\overline{\tau},\overline{y}+z,\frac{F(\bar{t},\overline{X})}{\epsilon}\right)\right.\right. \\
&\left. -\partial_{y_{N+1}}V\left(\overline{\tau},\overline{y}+z,\frac{F(\bar{t},\overline{X})}{\epsilon}\right)\nabla_x F(\bar{t},\overline{X})\cdot z \right] \\
&\left. + \left[\partial_{y_{N+1}}V(\overline{\tau},\overline{y}+z,\overline{y}_{N+1}) - \partial_{y_{N+1}}V(\overline{\tau},\overline{Y})\right]\nabla_x F(\bar{t},\overline{X})\cdot z \right\}\mu(dz).
\end{aligned}$$

Next, using (3.9), we get

(4.13)
$$\epsilon^{2s-1}I_1^1 - J_1 \leqslant C\int_{|z|\leqslant\delta}(|z|^2+|z|^{1+\alpha})\mu(dz) \leqslant C\delta^{\alpha+1-2s},$$

for $2s-1<\alpha<1$.

STEP 1.3 *Estimate of* $\epsilon^{2s-1}I_1^2 - J_2$.

If M_η is the Lipschitz constant of V w.r.t. y_{N+1}, then

$$\begin{aligned}\epsilon^{2s-1}I_1^2 - J_2 &\leqslant \int_{\delta<|z|\leqslant R}\left(V\left(\overline{\tau},\overline{y}+z,\frac{F(\overline{t},\overline{x}+\epsilon z,\overline{x}_{N+1})}{\epsilon}\right)-V\left(\overline{\tau},\overline{y}+z,\frac{F(\overline{t},\overline{X})}{\epsilon}\right)\right)\mu(dz)\\ &\leqslant M_\eta\int_{\delta<|z|\leqslant R}\left|\frac{F(\overline{t},\overline{x}+\epsilon z,\overline{x}_{N+1})}{\epsilon}-\frac{F(\overline{t},\overline{X})}{\epsilon}\right|\mu(dz)\\ &\leqslant M_\eta\int_{\delta<|z|\leqslant R}\sup_{|z|\leqslant R}|\nabla_x F(\overline{t},\overline{x}+\epsilon z,\overline{x}_{N+1})||z|\mu(dz).\end{aligned}$$

Then

$$\epsilon^{2s-1}I_1^2 - J_2 \leqslant C\sup_{|z|\leqslant R}|\nabla_x F(\overline{t},\overline{x}+\epsilon z,\overline{x}_{N+1})|\left(\frac{1}{\delta^{2s-1}}-\frac{1}{R^{2s-1}}\right). \tag{4.14}$$

STEP 1.4: *Estimate of* $\epsilon^{2s-1}I_1^3$ *and* J_3*.*

Since V is uniformly bounded on $\mathbb{R}^+\times\mathbb{R}^{N+1}$, we have

$$\begin{aligned}\epsilon^{2s-1}I_1^3 &\leqslant \int_{R<|z|\leqslant\frac{1}{\epsilon}}\left(V\left(\overline{\tau},\overline{y}+z,\frac{F(\overline{t},\overline{x}+\epsilon z,\overline{x}_{N+1})}{\epsilon}\right)-V(\overline{\tau},\overline{Y})\right)\mu(dz)\\ &\leqslant \int_{|z|>R}2\|V\|_\infty\mu(dz)\leqslant\frac{C}{R^{2s}}.\end{aligned} \tag{4.15}$$

Similarly

$$|J_3|\leqslant\frac{C}{R^{2s}}. \tag{4.16}$$

Now, from (4.12), (4.13), (4.14), (4.15) and (4.16), we infer that

$$\begin{aligned}\epsilon^{2s-1}I_1 &\leqslant \mathcal{I}_s^{1,\rho}[\Gamma^\epsilon(\overline{\tau},\cdot,\overline{y}_{N+1}),\overline{y}]+\mathcal{I}_s^{2,\rho}[V(\overline{\tau},\cdot,\overline{y}_{N+1}),\overline{y}]+2C_\epsilon\rho^{2-2s}+C\delta^{\alpha+1-2s}\\ &\quad + C\sup_{|z|\leqslant R}|\nabla_x F(\overline{t},\overline{x}+\epsilon z,\overline{x}_{N+1})|\left(\frac{1}{\delta^{2s-1}}-\frac{1}{R^{2s-1}}\right)+\frac{C}{R^{2s}}.\end{aligned}$$

We remark that, from the definition of F, we have

$$\begin{aligned}\sup_{|z|\leqslant R}|\nabla_x F(\overline{t},\overline{x}+\epsilon z,\overline{x}_{N+1})| &\leqslant \sup_{|z|\leqslant R}|\nabla\phi(\overline{t},\overline{x}+\epsilon z,\overline{x}_{N+1})-\nabla\phi(t_0,X_0)|\\ &\leqslant \sup_{|z|\leqslant R}|\nabla\phi(\overline{t},\overline{x}+\epsilon z,\overline{x}_{N+1})-\nabla\phi(\overline{t},\overline{X})|\\ &\quad+|\nabla\phi(\overline{t},\overline{X})-\nabla\phi(t_0,X_0)|\\ &\leqslant C(\epsilon R+r).\end{aligned}$$

Now, we choose $R=R(r)$ such $R\to+\infty$ as $r\to 0^+$, $\epsilon_0=\epsilon_0(r)$ such that $R\epsilon_0(r)\leqslant r$ and $\delta=\delta(r)>0$ such that $\delta\to 0$ as $r\to 0^+$ and $r/\delta^{2s-1}\to 0$ as $r\to 0^+$. With this choice, for any $\epsilon\leqslant\epsilon_0$ and any $\rho<\delta$

$$C\delta^{\alpha+1-2s}+C\sup_{|z|\leqslant R}|\nabla_x F(\overline{t},\overline{x}+\epsilon z,\overline{x}_{N+1})|\left(\frac{1}{\delta^{2s-1}}-\frac{1}{R^{2s-1}}\right)+\frac{C}{R^{2s}}=o_r(1)\quad\text{as } r\to 0^+,$$

and Step 1 is proved.

The next two steps are trivial.

Step 2: $\epsilon^{2s-1}I_2\leqslant C\epsilon^{2s-1}$**.**

Step 3: $\epsilon^{2s-1}\mathcal{I}_s^{2,1}\left[\phi^\epsilon(\overline{t},\cdot,\overline{x}_{N+1}),\overline{x}\right]\leqslant C\epsilon^{2s-1}$**.**

Finally Steps 1, 2 and 3 give

$$\begin{aligned}&\epsilon^{2s-1}\mathcal{I}_s^{1,1}\left[\psi(\overline{t},\cdot,\overline{x}_{N+1}),\overline{x}\right]+\epsilon^{2s-1}\mathcal{I}_s^{2,1}\left[\phi^\epsilon(\overline{t},\cdot,\overline{x}_{N+1}),\overline{x}\right]\\&\leqslant\mathcal{I}_s^{1,\rho}[\Gamma^\epsilon(\overline{\tau},\cdot,\overline{y}_{N+1}),\overline{y}]+\mathcal{I}_s^{2,\rho}[V(\overline{\tau},\cdot,\overline{y}_{N+1}),\overline{y}]+o_r(1)+C_\epsilon\rho^{2-2s}.\end{aligned}$$

from which, using inequality (4.8) and letting $\rho\to 0^+$, we get (4.9).

5. Proof of Theorem 1.3

The proof of Theorem 1.3 is similar to the proof of Theorem 1.2, therefore we only give a sketch of it. As in Theorem 1.2, we argue by contradiction, assuming that there is a test function ϕ such that $U^+-\phi$ attains a strict zero maximum at (t_0,X_0) with $t_0>0$ and $X_0=(x_0,x_{N+1}^0)$, and

$$\partial_t\phi(t_0,X_0)=\overline{H}_2(L_0)+\theta$$

for some $\theta>0$, where

$$\begin{aligned}L_0=&\int_{|x|\leqslant 1}(\phi(t_0,x_0+x,x_{N+1}^0)-\phi(t_0,X_0)-\nabla_x\phi(t_0,X_0)\cdot x)\mu(dx)\\&+\int_{|x|>1}(U^+(t_0,x_0+x,x_{N+1}^0)-U^+(t_0,X_0))\mu(dx).\end{aligned}\tag{5.1}$$

Then, we choose $L_1>0$ and a sequence $L_\eta\to L_1$ as $\eta\to 0^+$, such that

$$\lambda_\eta^+(0,L_\eta+L_0)=\lambda(0,L_1+L_0)=\lambda(0,L_0)+\theta=\overline{H}_2(L_0)+\theta.$$

Let V be the approximate supercorrector given by Proposition 3.3 with

$$p=0,\quad L=L_0+L_\eta$$

and

$$\lambda_\eta^+=\lambda_\eta^+(0,L_0+L_\eta)=\partial_t\phi(t_0,X_0).$$

Let us introduce the "x_{N+1}-twisted perturbed test function" ϕ^ϵ defined by:

$$\phi^\epsilon(t,X):=\begin{cases}\phi(t,X)+\epsilon^{2s}V\left(\frac{t}{\epsilon^{2s}},\frac{x}{\epsilon},\frac{F(t,X)}{\epsilon^{2s}}\right)+\epsilon^{2s}k_\epsilon & \text{in}\quad(\frac{t_0}{2},2t_0)\times B_{\frac12}(X_0)\\ U^\epsilon(t,X) & \text{outside},\end{cases}$$

where $F(t,X)=\phi(t,X)-\lambda_\eta^+t$ and $k_\epsilon\in\mathbb{Z}$ is opportunely chosen. As in Section 4, we can prove that ϕ^ϵ is a supersolution of (2.6) in a neighborhood $Q_{r,r}(t_0,X_0)$ of (t_0,X_0), for some small r properly chosen. Moreover

$$U^\epsilon\leqslant\phi^\epsilon\quad\text{outside } Q_{r,r}(t_0,X_0).$$

The contradiction follows by comparison.

6. Proof of Theorem 1.4

In this section we restrict ourself to the case: $N=1,\mathcal{I}_s=-(-\Delta)^s$ and $\sigma\equiv 0$. For fixed $p,L\in\mathbb{R}$, let us introduce the corrector

$$u(\tau,y):=w(\tau,y)+py$$

where w is the solution of (1.11) given by Theorem 1.1. Then u is solution of

(6.1) $$\begin{cases} \partial_\tau u = L + \mathcal{I}_s[u(\tau,\cdot)] - W'(u) & \text{in} \quad \mathbb{R}^+ \times \mathbb{R} \\ u(0,y) = py & \text{on} \quad \mathbb{R}, \end{cases}$$

and by the ergodic property (3.4) it satisfies

(6.2) $$|u(\tau,y) - py - \lambda\tau| \leqslant C.$$

The idea underlying the proof of Theorem 1.4 is related to a fine asymptotics of equation (6.1). We want to show that if u solves (6.1) with $p = \delta|p_0|$ and $L = \delta^{2s}L_0$, i.e.

(6.3) $$\partial_\tau u = \delta^{2s}L_0 + \mathcal{I}_s[u(\tau,\cdot)] - W'(u)$$

and $u(0,y) = \delta p_0 y$, then

$$u(\tau,y) \sim \delta p_0 y + \lambda\tau + \text{bounded} \quad \text{with } \lambda \sim \delta^{1+2s}c_0|p_0|L_0.$$

We deduce that we should have

$$\frac{u(\tau,y)}{\tau} \to \lambda = \delta^{1+2s}c_0|p_0|L_0 \quad \text{as } \tau \to +\infty.$$

We see that this $\lambda = \overline{H}(\delta p_0, \delta^{2s}L_0)$ is exactly the one we expect asymptotically in Theorem 1.4.

Following the idea of [11], one may expect to find particular solutions u of (6.3) that we can write

$$u(\tau,y) = h(\delta p_0 y + \lambda\tau)$$

for some $\lambda \in \mathbb{R}$ and a function h (called hull function) satisfying

$$|h(z) - z| \leqslant C.$$

This means that h solves

$$\lambda h' = \delta^{2s}L_0 + \delta^{2s}|p_0|^{2s}\mathcal{I}_s[h] - W'(h).$$

Then it is natural to introduce the non-linear operator:

(6.4) $$NL^\lambda_{L_0}[h] := \lambda h' - \delta^{2s}L_0 - \delta^{2s}|p_0|^{2s}\mathcal{I}_s[h] + W'(h)$$

and for the ansatz for λ:

$$\overline{\lambda}^{L_0}_\delta = \delta^{1+2s}c_0|p_0|L_0$$

it is natural to look for an ansatz $h^{L_0}_\delta$ for h. We define (see Proposition 6.1)

$$h^{L_0}_\delta(x) = \lim_{n\to+\infty} s^{L_0}_{\delta,n}(x)$$

where for $s \geqslant \frac{1}{2}$ and for all $p_0 \neq 0$, $L_0 \in \mathbb{R}$, $\delta > 0$ and $n \in \mathbb{N}$ we define the sequence of functions $\{s^{L_0}_{\delta,n}(x)\}_n$ by

(6.5) $$s^{L_0}_{\delta,n}(x) = \frac{\delta^{2s}L_0}{\alpha} + \sum_{i=-n}^{n} \phi\left(\frac{x-i}{\delta|p_0|}\right) - n + \delta^{2s}\sum_{i=-n}^{n}\psi\left(\frac{x-i}{\delta|p_0|}\right)$$

where $\alpha = W''(0) > 0$ and ϕ is the solution of (1.13). The corrector ψ is the solution of the following problem

(6.6) $$\begin{cases} \mathcal{I}_s[\psi] = W''(\phi)\psi + \frac{L_0}{W''(0)}(W''(\phi) - W''(0)) + c\phi' & \text{in} \quad \mathbb{R} \\ \lim\limits_{x\to\pm\infty} \psi(x) = 0 \\ c = \frac{L_0}{\int_{\mathbb{R}}(\phi')^2}. \end{cases}$$

For $s < \frac{1}{2}$, the function ψ defined above may not decay fast enough so that the sequence

$$\sum_{i=-n}^{n} \psi\left(\frac{x-i}{\delta|p_0|}\right)$$

converges. Therefore, in this case we define

$$s_{\delta,n}^{L_0}(x) = \frac{\delta^{2s}L_0}{\alpha} + \sum_{i=-n}^{n} \phi\left(\frac{x-i}{\delta|p_0|}\right) - n + \delta^{2s}\sum_{i=-n}^{n} \psi\left(\frac{x-i}{\delta|p_0|}\right)\tau\left(\frac{x-i}{\delta|p_0|}\right) \tag{6.7}$$

where $\tau = \tau_R$, is a smooth function satisfying

$$\begin{cases} 0 \leqslant \tau(x) \leqslant 1 & \text{for any } x \in \mathbb{R} \\ \tau_R(x) = 1 & \text{if } |x| \leqslant R \\ \tau_R(x) = 0 & \text{if } |x| \geqslant 2R. \end{cases} \tag{6.8}$$

The number R is a large parameter that will be chosen depending on δ.

Proposition 6.1. (Good ansatz)
Assume (1.12) *and* $R = \frac{1}{2\delta|p_0|}$ *in* (6.8). *Then, for any* $L \in \mathbb{R}$, $\delta > 0$ *and* $x \in \mathbb{R}$, *there exists the finite limit*

$$h_\delta^L(x) = \lim_{n\to+\infty} s_{\delta,n}^L(x).$$

Moreover h_δ^L *has the following properties:*

(i) $h_\delta^L \in C^2(\mathbb{R})$ *and satisfies*

$$NL_L^{\overline{\lambda}_\delta^L}[h_\delta^L](x) = o(\delta^{2s}), \tag{6.9}$$

where $\lim_{\delta\to 0} \dfrac{o(\delta^{2s})}{\delta^{2s}} = 0$, *uniformly for* $x \in \mathbb{R}$ *and locally uniformly in* $L \in \mathbb{R}$*; Here*

$$\overline{\lambda}_\delta^L = \delta^{1+2s}c_0|p_0|L$$

and NL_L^λ *is defined in (6.4).*

(ii) *There exists a constant* $C > 0$ *such that* $|h_\delta^L(x) - x| \leqslant C$ *for any* $x \in \mathbb{R}$.

6.1. **Proof of Theorem 1.4.** We will show that Theorem 1.4 follows from Proposition 6.1 and the comparison principle.

Fix $\eta > 0$ and let $L = L_0 - \eta$. By (i) of Proposition 6.1, there exists $\delta_0 = \delta_0(\eta) > 0$ such that for any $\delta \in (0, \delta_0)$ we have

$$NL_{L_0}^{\overline{\lambda}_\delta^L}[h_\delta^L] = NL_L^{\overline{\lambda}_\delta^L}[h_\delta^L] - \delta^{2s}\eta < 0 \quad \text{in } \mathbb{R}. \tag{6.10}$$

Let us consider the function $\widetilde{u}(\tau, y)$, defined by

$$\widetilde{u}(\tau, y) = h_\delta^L(\delta p_0 y + \overline{\lambda}_\delta^L \tau).$$

By (ii) of Proposition 6.1, we have

$$|\widetilde{u}(\tau, y) - \delta p_0 y - \overline{\lambda}_\delta^L \tau| \leqslant \lceil C \rceil, \tag{6.11}$$

where $\lceil C \rceil$ is the ceil integer part of C. Moreover, by (6.10) and (6.11), $\widetilde{u}$ satisfies

$$\begin{cases} \widetilde{u}_\tau \leqslant \delta^{2s}L_0 + \mathcal{I}_s[\widetilde{u}] - W'(\widetilde{u}) & \text{in } \mathbb{R}^+ \times \mathbb{R} \\ \widetilde{u}(0, y) \leqslant \delta p_0 y + \lceil C \rceil & \text{on } \mathbb{R}. \end{cases}$$

Let $u(\tau, y)$ be the solution of (6.1), with $p = \delta p_0$ and $L = \delta^{2s} L_0$, whose existence is ensured by Theorem 1.1. Then from the comparison principle and the periodicity of W, we deduce that

$$\widetilde{u}(\tau, y) \leqslant u(\tau, y) + \lceil C \rceil.$$

By the previous inequality and (6.11), we get

$$\overline{\lambda}_\delta^L \tau \leqslant u(\tau, y) - \delta p_0 y + 2\lceil C \rceil,$$

and dividing by τ and letting τ go to $+\infty$, we finally obtain

$$\delta^{1+2s} c_0 |p_0| (L_0 - \eta) = \overline{\lambda}_\delta^L \leqslant \overline{H}(\delta p_0, \delta^{2s} L_0).$$

Similarly, it is possible to show that

$$\overline{H}(\delta p_0, \delta^{2s} L_0) \leqslant \delta^{1+2s} c_0 |p_0| (L_0 + \eta).$$

We have proved that for any $\eta > 0$ there exists $\delta_0 = \delta_0(\eta) > 0$ such that for any $\delta \in (0, \delta_0)$ we have

$$\left| \frac{\overline{H}(\delta p_0, \delta^{2s} L_0)}{\delta^{1+2s}} - c_0 |p_0| L_0 \right| \leqslant c_0 |p_0| \eta,$$

i.e. (1.14), as desired.

6.2. **Preliminary results.**

Under the assumptions (1.12) on W, there exists a unique solution of (1.13) which is of class $C^{2,\beta}$, as shown in [2], see also [13]. When $s < \frac{1}{2}$ we suppose in addition that W is even. This implies that the function

$$\phi - \frac{1}{2}$$

is odd. The existence of a solution of class $C^{1,\beta}_{loc}(\mathbb{R}) \cap L^\infty(\mathbb{R})$ of the problem (6.6) is proved in [13]. Actually, the regularity of W implies that $\phi \in C^{4,\beta}(\mathbb{R})$ and $\psi \in C^{3,\beta}(\mathbb{R})$.

To prove Proposition 6.1 we need several preliminary results. We first state the following two lemmata about the behavior of the functions ϕ and ψ at infinity. We denote by $H(x)$ the Heaviside function defined by

$$H(x) = \begin{cases} 1 & \text{for } x \geqslant 0 \\ 0 & \text{for } x < 0. \end{cases}$$

Then we have

Lemma 6.2 (Behavior of ϕ)**.** *Assume* (1.12). *Let ϕ be the solution of* (1.13)*, then there exists a constant $K_1 > 0$ such that*

(6.12) $$\left| \phi(x) - H(x) + \frac{1}{2s\alpha} \frac{x}{|x|^{1+2s}} \right| \leqslant \frac{K_1}{|x|^{1+2s}}, \quad \text{for } |x| \geqslant 1, \quad \text{if } s \geqslant \frac{1}{2},$$

(6.13) $$|\phi(x) - H(x)| \leqslant \frac{K_1}{|x|^{2s}}, \quad \text{for } |x| \geqslant 1, \quad \text{if } s < \frac{1}{2},$$

and for any $x \in \mathbb{R}$, $s \in (0, 1)$,

(6.14) $$0 < \phi'(x) \leqslant \frac{K_1}{1 + |x|^{1+2s}},$$

(6.15) $$|\phi''(x)| \leqslant \frac{K_1}{1 + |x|^{1+2s}},$$

$$|\phi'''(x)| \leqslant \frac{K_1}{1+|x|^{1+2s}}. \tag{6.16}$$

Proof. Estimate (6.12) is proved in [5], while estimates (6.13) and (6.14) are proved in [13].

Since the proof of (6.15) and (6.16) is an adaptation of the one given in [11] for the same estimates in the case $s = \frac{1}{2}$, we only sketch it.

To get (6.15), as in the proof of Lemma 3.1 in [11] one looks to the equations satisfied by $\overline{\phi} := \phi'' - C\phi'_a(x)$, where $\phi'_a(x) := \phi'\left(\frac{x}{a}\right)$, $a > 0$:

$$\mathcal{I}_s[\overline{\phi}] - W''(\phi)\overline{\phi} = C\phi'_a\left(W''(\phi) - \frac{1}{a^{2s}}W''(\phi_a)\right) + W'''(\phi)(\phi')^2.$$

For a and R_1 large enough, we can prove that in $\mathbb{R}\setminus[-R_1, R_1]$ we have

$$\mathcal{I}_s[\overline{\phi}] - W''(\phi)\overline{\phi} \geqslant 0 \quad \text{and} \quad W''(\phi) > 0.$$

Choosing C so large that $\overline{\phi} \leqslant 0$ on $[-R_1, R_1]$, the comparison principle implies $\overline{\phi} \leqslant 0$ in $\mathbb{R}$, therefore $\phi'' \leqslant C\phi'_a(x)$ in $\mathbb{R}$. Similarly one can prove that $\phi'' \geqslant -C\phi'_a(x)$ in $\mathbb{R}$, and using (6.14), (6.15) follows.

In the same way, comparing ϕ''' with $C\phi'_a(x)$, we get estimate (6.16). □

Lemma 6.3 (Behavior of ψ). *Assume* (1.12)*. Let ψ be the solution of* (6.6)*, then for any $L \in \mathbb{R}$ there exist K_2 and $K_3 > 0$, depending on L such that*

$$\left|\psi(x) - K_2\frac{x}{|x|^{1+2s}}\right| \leqslant \frac{K_3}{|x|^{1+2s}}, \quad \text{for } |x| \geqslant 1, \text{ if } s \geqslant \frac{1}{2} \tag{6.17}$$

and for any $s \in (0,1)$ and $x \in \mathbb{R}$

$$|\psi'(x)| \leqslant \frac{K_3}{1+|x|^{1+2s}}, \tag{6.18}$$

$$|\psi''(x)| \leqslant \frac{K_3}{1+|x|^{1+2s}}. \tag{6.19}$$

Proof. We follow the proof of Lemma 3.2 in [11]. Let us start with the proof of (6.17). Since we want to point out where we use $s \geqslant \frac{1}{2}$, we give it in the details. For $a > 0$ we denote $\phi_a(x) := \phi\left(\frac{x}{a}\right)$, which is solution of

$$\mathcal{I}_s[\phi_a] = \frac{1}{a^{2s}}W'(\phi_a) \quad \text{in } \mathbb{R}.$$

In what follows, we denote $\widetilde{\phi}(x) = \phi(x) - H(x)$. Let a and b be positive numbers, then making a Taylor expansion of the derivatives of W (remind $W'(0) = 0$), we get

$$\begin{aligned}
\mathcal{I}_s[\psi - (\phi_a - \phi_b)] &= W''(\phi)\psi + \frac{L}{\alpha}(W''(\phi) - W''(0)) + c\phi' + \left(\frac{1}{b^{2s}}W'(\phi_b) - \frac{1}{a^{2s}}W'(\phi_a)\right)\\
&= W''(\phi)(\psi - (\phi_a - \phi_b)) + W''(\widetilde{\phi})(\phi_a - \phi_b) + \frac{L}{\alpha}(W''(\widetilde{\phi}) - W''(0))\\
&+ c\phi' + \left(\frac{1}{b^{2s}}W'(\widetilde{\phi}_b) - \frac{1}{a^{2s}}W'(\widetilde{\phi}_a)\right)\\
&= W''(\phi)(\psi - (\phi_a - \phi_b)) + W''(0)(\phi_a - \phi_b) + \frac{L}{\alpha}W'''(0)\widetilde{\phi} + c\phi'\\
&+ W''(0)\left(\frac{1}{b^{2s}}\widetilde{\phi}_b - \frac{1}{a^{2s}}\widetilde{\phi}_a\right) + (\phi_a - \phi_b)O(\widetilde{\phi}) + O(\widetilde{\phi})^2 + O(\widetilde{\phi}_a)^2 + O(\widetilde{\phi}_b)^2.
\end{aligned}$$

Then the function $\overline{\psi} = \psi - (\phi_a - \phi_b)$ satisfies

$$\mathcal{I}_s[\overline{\psi}] - W''(\phi)\overline{\psi} = \alpha(\phi_a - \phi_b) + \frac{L}{\alpha}W'''(0)\widetilde{\phi} + c\phi' + \alpha\left(\frac{1}{b^{2s}}\widetilde{\phi}_b - \frac{1}{a^{2s}}\widetilde{\phi}_a\right) + (\phi_a - \phi_b)O(\widetilde{\phi}) + O(\widetilde{\phi})^2 + O(\widetilde{\phi}_a)^2 + O(\widetilde{\phi}_b)^2.$$

We want to estimate the right-hand side of the last equality. By Lemma 6.2, for $|x| \geqslant \max\{1, |a|, |b|\}$ we have

$$\alpha(\phi_a - \phi_b) + \frac{L}{\alpha}W'''(0)\widetilde{\phi} \geqslant -\frac{x}{2s|x|^{1+2s}}\left[(a^{2s} - b^{2s}) + \frac{L}{\alpha^2}W'''(0)\right] - \frac{K_1\alpha}{|x|^{1+2s}}\left(a^{1+2s} + b^{1+2s} + \frac{|L|}{\alpha^2}|W'''(0)|\right).$$

Choose $a, b > 0$ such that $(a^{2s} - b^{2s}) + \frac{L}{\alpha^2}W'''(0) = 0$, then

$$\alpha(\phi_a - \phi_b) + \frac{L}{\alpha}W'''(0)\widetilde{\phi} \geqslant -\frac{C}{|x|^{1+2s}},$$

for $|x| \geqslant \max\{1, |a|, |b|\}$. Here and in what follows, as usual C denotes various positive constants. From Lemma 6.2 we also derive that

$$\alpha\left(\frac{1}{b^{2s}}\widetilde{\phi}_b - \frac{1}{a^{2s}}\widetilde{\phi}_a\right) \geqslant -\frac{C}{|x|^{1+2s}},$$

$$\text{and} \qquad c\phi' \geqslant -\frac{C}{1 + |x|^{1+2s}}.$$

Moreover, since $s \geqslant \frac{1}{2}$, we have

$$(\phi_a - \phi_b)O(\widetilde{\phi}) + O(\widetilde{\phi})^2 + O(\widetilde{\phi}_a)^2 + O(\widetilde{\phi}_b)^2 \geqslant -\frac{C}{1 + |x|^{4s}} \geqslant -\frac{C}{1 + |x|^{1+2s}},$$

for $|x| \geqslant \max\{1, |a|, |b|\}$. Then we conclude that there exists $R_1 > 0$ such that for $|x| \geqslant R_1$ we have

$$\mathcal{I}_s[\overline{\psi}] - W''(\phi)\overline{\psi} \geqslant -\frac{C}{1 + |x|^{1+2s}}.$$

Now, let us consider the function $\phi'_d(x) = \phi'\left(\frac{x}{d}\right)$, $d > 0$, which is solution of

$$\mathcal{I}_s[\phi'_d] = \frac{1}{d^{2s}}W''(\phi_d)\phi'_d \quad \text{in } \mathbb{R},$$

and denote

$$\overline{\overline{\psi}} = \overline{\psi} - \widetilde{C}\phi'_d,$$

with $\widetilde{C} > 0$. Then, for $|x| \geqslant R_1$ we have

$$\begin{aligned}\mathcal{I}_s[\overline{\overline{\psi}}] &\geqslant W''(\phi)\overline{\psi} - \frac{\widetilde{C}}{d^{2s}}W''(\phi_d)\phi'_d - \frac{C}{1 + |x|^{1+2s}} \\ &= W''(\phi)\overline{\overline{\psi}} + \widetilde{C}\phi'_d\left(W''(\phi) - \frac{1}{d^{2s}}W''(\phi_d)\right) - \frac{C}{1 + |x|^{1+2s}}.\end{aligned}$$

Let us choose $d > 0$ and $R_2 > R_1$ such that

$$\begin{cases} W''(\phi) - \frac{1}{d^{2s}}W''(\phi_d) > \frac{1}{2}W''(0) > 0 & \text{in } \mathbb{R} \setminus [-R_2, R_2]; \\ W''(\phi) > 0 & \text{on } \mathbb{R} \setminus [-R_2, R_2], \end{cases}$$

then from (6.14), for $\widetilde{C}$ large enough we get

$$\mathcal{I}_s[\overline{\overline{\psi}}] - W''(\phi)\overline{\overline{\psi}} \geqslant 0 \quad \text{on } \mathbb{R} \setminus [-R_2, R_2].$$

Choosing $\widetilde{C}$ such that moreover

$$\overline{\overline{\psi}} < 0 \quad \text{on } [-R_2, R_2],$$

we can ensure that $\overline{\overline{\psi}} \leqslant 0$ on $\mathbb{R}$. Indeed, assume by contradiction that there exists $x_0 \in \mathbb{R}\backslash[-R_2, R_2]$ such that

$$\overline{\overline{\psi}}(x_0) = \sup_{\mathbb{R}} \overline{\overline{\psi}} > 0.$$

Then

$$\begin{cases} \mathcal{I}_s[\overline{\overline{\psi}}, x_0] \leqslant 0; \\ \mathcal{I}_s[\overline{\overline{\psi}}, x_0] - W''(\phi(x_0))\overline{\overline{\psi}}(x_0) \geqslant 0; \\ W''(\phi(x_0)) > 0, \end{cases}$$

from which

$$\overline{\overline{\psi}}(x_0) \leqslant 0,$$

a contradiction. Therefore, $\overline{\overline{\psi}} \leqslant 0$ on $\mathbb{R}$ which implies, with together (6.12) and (6.14),

$$\psi \leqslant \frac{K_2 x}{|x|^{1+2s}} + \frac{K_3}{|x|^{1+2s}} \quad \text{for } |x| \geqslant 1.$$

Looking at the function $\psi - (\phi_a - \phi_b) + \widetilde{C}\phi'_d$, we conclude similarly that

$$\psi \geqslant \frac{K_2 x}{|x|^{1+2s}} - \frac{K_3}{|x|^{1+2s}} \quad \text{for } |x| \geqslant 1,$$

and (6.17) is proved.

Now let us turn to (6.18). By deriving the first equation in (6.6), we see that the function ψ' which is bounded and of class $C^{2,\beta}$, is a solution of

$$\mathcal{I}_s[\psi'] = W''(\phi)\psi' + W'''(\phi)\phi'\psi + \frac{L}{\alpha}W'''(\phi)\phi' + c\phi'' \quad \text{in } \mathbb{R}.$$

Then the function $\overline{\psi}' = \psi' - C\phi'_a$, satisfies

$$\begin{aligned} \mathcal{I}_s[\overline{\psi}'] - W''(\phi)\overline{\psi}' &= C\phi'_a\left(W''(\phi) - \frac{1}{a^{2s}}W''(\phi_a)\right) + W'''(\phi)\phi'\psi + \frac{L}{\alpha}W'''(\phi)\phi' + c\phi'' \\ &= C\phi'_a\left(W''(\phi) - \frac{1}{a^{2s}}W''(\phi_a)\right) + O\left(\frac{1}{1+|x|^{1+2s}}\right), \end{aligned}$$

by (6.14) and (6.15) and as before we deduce that for C and a large enough $\overline{\psi}' \leqslant 0$ on $\mathbb{R}$, which implies that $\psi' \leqslant \frac{K_3}{1+|x|^{1+2s}}$. The inequality $\psi' \geqslant -\frac{K_3}{1+|x|^{1+2s}}$ is obtained similarly by proving that $\overline{\psi}' + C\phi'_a \geqslant 0$ on $\mathbb{R}$.

Similarly, estimate (6.19) is obtained by comparing ψ'' with $C\phi'_a$ for some large a and C and using (6.14), (6.15) and (6.16). □

6.3. **Proof of Proposition 6.1.**

For simplicity of notation we denote (for the rest of the paper)

$$x_i = \frac{x - i}{\delta|p_0|}, \quad \widetilde{\phi}(z) = \phi(z) - H(z).$$

Then we have the following six claims (whose proofs are postponed to the end of the section).

Claim 1: *Let $x = i_0 + \gamma$, with $i_0 \in \mathbb{Z}$ and $\gamma \in \left(-\frac{1}{2}, \frac{1}{2}\right]$, then there exist numbers $\theta_i \in (-1, 1)$ such that*

$$\sum_{\substack{i=-n \\ i \neq i_0}}^{n} \frac{x-i}{|x-i|^{1+2s}} \to -4s\gamma \sum_{i=1}^{+\infty} \frac{(i-\theta_i\gamma)^{2s-1}}{(i+\gamma)^{2s}(i-\gamma)^{2s}} \quad \text{as } n \to +\infty,$$

$$\sum_{i=-n}^{i_0-1} \frac{1}{|x-i|^{1+2s}} \to \sum_{i=1}^{+\infty} \frac{1}{(i+\gamma)^{1+2s}} \quad \text{as } n \to +\infty,$$

$$\sum_{i=i_0+1}^{n} \frac{1}{|x-i|^{1+2s}} \to \sum_{i=1}^{+\infty} \frac{1}{(i-\gamma)^{1+2s}} \quad \text{as } n \to +\infty.$$

We remark that the three series on the right hand side above converge uniformly for $\gamma \in \left(-\frac{1}{2}, \frac{1}{2}\right]$ and $\theta_i \in (-1, 1)$ since behave like the series $\displaystyle\sum_{i=1}^{+\infty} \frac{1}{i^{1+2s}}$.

Claim 2: *Assume $s < \frac{1}{2}$. Let $x = i_0 + \gamma$, with $i_0 \in \mathbb{Z}$ and $\gamma \in \left(-\frac{1}{2}, \frac{1}{2}\right]$, then*

(6.20)
$$\lim_{n\to\infty} \left| \sum_{\substack{i=-n \\ i \neq i_0}}^{n} [\widetilde{\phi}(x_i)]^{2k-1} \right| \leqslant Ck\delta^{2s(2k-1)}|\gamma|$$

and

(6.21)
$$\sum_{\substack{i=-n \\ i \neq i_0, i_0 \pm 1}}^{n} |\mathcal{I}_s[\tau, x_i]| \leqslant C\delta^{2s}.$$

Claim 3: *For any $x \in \mathbb{R}$ the sequence $\{s^L_{\delta,n}(x)\}_n$ converges as $n \to +\infty$.*

Claim 4: *The sequence $\{(s^L_{\delta,n})'\}_n$ converges on $\mathbb{R}$ as $n \to +\infty$, uniformly on compact sets.*

Claim 5: *The sequence $\{(s^L_{\delta,n})''\}_n$ converges on $\mathbb{R}$ as $n \to +\infty$, uniformly on compact sets.*

Claim 6: *For any $x \in \mathbb{R}$ the sequence $\displaystyle\sum_{i=-n}^{n} \mathcal{I}_s[s^L_{\delta,n}, x_i]$ converges as $n \to +\infty$.*

With these claims, we are in the position of completing the proof of Proposition 6.1, by arguing as follows.

Proof of ii)
When $s \geqslant \frac{1}{2}$, (ii) is a consequence of (6.50) in the proof of Claim 3.

Next, let us assume $s < \frac{1}{2}$. Let $x = i_0 + \gamma$ with $i_0 \in \mathbb{Z}$ and $\gamma \in \left(-\frac{1}{2}, \frac{1}{2}\right]$. For $n > |i_0|$, we have

$$\begin{aligned}\sum_{i=-n}^{i=n} \phi(x_i) - n - x &= \sum_{i=-n}^{i=n} \phi(x_i) - n - i_0 - \gamma \\ &= \sum_{i=-n}^{i_0-1} (\phi(x_i) - 1) + \phi(x_{i_0}) + \sum_{i=i_0+1}^{n} \phi(x_i) - \gamma \\ &= \sum_{\substack{i=-n \\ i \neq i_0}}^{i=n} \widetilde{\phi}(x_i) + \phi(x_{i_0}) - \gamma.\end{aligned}$$

Then by (6.20) with $k = 1$

$$\left| \sum_{i=-n}^{i=n} \phi(x_i) - n - x \right| \leqslant C, \tag{6.22}$$

with C independent of x. Finally, for $i \neq i_0 - 1, i_0, i_0 + 1$ and $R = \frac{1}{2\delta|p_0|}$

$$|x_i| = \frac{|i_0 + \gamma - i|}{\delta|p_0|} \geqslant \frac{3}{2\delta|p_0|} > 2R,$$

therefore $\tau(x_i) = 0$. This implies that $\sum_{i=-n}^{n} \psi(x_i)\tau(x_i)$ is actually the sum of only three terms and therefore

$$\left| \sum_{i=-n}^{i=n} \psi(x_i)\tau(x_i) \right| \leqslant 3\|\psi\|_\infty. \tag{6.23}$$

Estimates (6.22) and (6.23) imply (ii).

Proof of i)
The function $h_\delta^L(x) = \lim_{n\to+\infty} s_{\delta,n}^L(x)$ is well defined for any $x \in \mathbb{R}$ by Claim 3. Moreover, by Claims 4 and 5 and classical analysis results, it is of class C^2 on $\mathbb{R}$ with

$$(h_\delta^L)'(x) = \lim_{n\to+\infty} (s_{\delta,n}^L)'(x),$$

$$(h_\delta^L)''(x) = \lim_{n\to+\infty} (s_{\delta,n}^L)''(x),$$

and the convergence of $\{s_{\delta,n}^L\}_n$, $\{(s_{\delta,n}^L)'\}_n$ and $\{(s_{\delta,n}^L)''\}_n$ is uniform on compact sets.
Finally, as in [11] (see Section 4), we have for any $x \in \mathbb{R}$

$$\mathcal{I}_s[h_\delta^L, x] = \lim_{n\to+\infty} \mathcal{I}_s[s_{\delta,n}^L, x]. \tag{6.24}$$

To conclude the proof of Proposition 6.1, we only have to prove (6.9), which is a consequence of the estimates above and the following lemma.

Lemma 6.4. (First asymptotics) *We have*

$$\lim_{n\to+\infty} NL_L^{\overline{\lambda}_\delta^L}[s_{\delta,n}^L](x) = o(\delta^{2s}) \quad \textit{as } \delta \to 0$$

where $\lim_{\delta\to 0} \frac{o(\delta^{2s})}{\delta^{2s}} = 0$*, uniformly for* $x \in \mathbb{R}$*.*

Now we can conclude the proof of (i). Indeed, by Claim 3, Claim 4 and (6.24), for any $x \in \mathbb{R}$

$$NL_L^{\overline{\lambda}_\delta^L}[h_\delta^L](x) = \lim_{n\to+\infty} NL_L^{\overline{\lambda}_\delta^L}[s_{\delta,n}^L](x),$$

and Lemma 6.4 implies that

$$NL_L^{\overline{\lambda}_\delta^L}[h_\delta^L](x) = o(\delta^{2s}), \quad \text{as } \delta \to 0,$$

where $\lim_{\delta\to 0} \frac{o(\delta^{2s})}{\delta^{2s}} = 0$, uniformly for $x \in \mathbb{R}$.

Proof of Lemma 6.4.

Let us first assume $s \geqslant \frac{1}{2}$.

Step 1: First computation
Fix $x \in \mathbb{R}$, let $i_0 \in \mathbb{Z}$ and $\gamma \in \left(-\frac{1}{2}, \frac{1}{2}\right]$ be such that $x = i_0 + \gamma$, let $\frac{1}{\delta|p_0|} \geqslant 2$ and $n > |i_0|$. Then we have

$$\begin{aligned}
A := \quad & NL_L^{\overline{\lambda}_\delta^L}[s_{\delta,n}^L](x) \\
= \quad & \frac{\overline{\lambda}_\delta^L}{\delta|p_0|} \sum_{i=-n}^{n} \left[\phi'(x_i) + \delta^{2s}\psi'(x_i)\right] - \sum_{i=-n}^{n} \left[\mathcal{I}_s[\phi, x_i] + \delta^{2s}\mathcal{I}_s[\psi, x_i]\right] \\
& + W'\left(\frac{L\delta^{2s}}{\alpha} + \sum_{i=-n}^{n} \left[\phi(x_i) + \delta^{2s}\psi(x_i)\right]\right) - \delta^{2s}L
\end{aligned}$$

where we have used the definitions and the periodicity of W. Using the equation (1.13) satisfied by ϕ, we can rewrite it as

$$\begin{aligned}
A = \quad & \frac{\overline{\lambda}_\delta^L}{\delta|p_0|} \left\{ \phi'(x_{i_0}) + \delta^{2s}\psi'(x_{i_0}) + \sum_{\substack{i=-n \\ i\neq i_0}}^{n} \left[\phi'(x_i) + \delta^{2s}\psi'(x_i)\right] \right\} - \sum_{\substack{i=-n \\ i\neq i_0}}^{n} W'(\widetilde{\phi}(x_i)) \\
& - \delta^{2s} \sum_{\substack{i=-n \\ i\neq i_0}}^{n} \mathcal{I}_s[\psi, x_i] - \delta^{2s}\mathcal{I}_s[\psi, x_{i_0}] \\
& + W'\left(\frac{L\delta^{2s}}{\alpha} + \sum_{i=-n}^{n} \left[\widetilde{\phi}(x_i) + \delta^{2s}\psi(x_i)\right]\right) - W'(\widetilde{\phi}(x_{i_0})) - \delta^{2s}L.
\end{aligned}$$

Using the Taylor expansion of W' (remind that $W'(0)=0$) and the definition of $\overline{\lambda}_\delta^L$, we get

$$\begin{aligned} A = \quad & \delta^{2s} c_0 L \left\{ \phi'(x_{i_0}) + \delta^{2s}\psi'(x_{i_0}) + \sum_{\substack{i=-n\\ i\neq i_0}}^{n} \left[\phi'(x_i) + \delta^{2s}\psi'(x_i)\right] \right\} \\ & - W''(0) \sum_{\substack{i=-n\\ i\neq i_0}}^{n} \widetilde{\phi}(x_i) - \delta^{2s} \sum_{\substack{i=-n\\ i\neq i_0}}^{n} \mathcal{I}_s[\psi, x_i] - \delta^{2s}\mathcal{I}_s[\psi, x_{i_0}] \\ & + W''(\phi(x_{i_0})) \left(\frac{L\delta^{2s}}{\alpha} + \delta^{2s}\psi(x_{i_0}) + \sum_{\substack{i=-n\\ i\neq i_0}}^{n} \left[\widetilde{\phi}(x_i) + \delta^{2s}\psi(x_i)\right] \right) - \delta^{2s} L + E \end{aligned}$$

with the error term

$$E = E_1 + E_2,$$

where

$$E_1 = - \sum_{\substack{i=-n\\ i\neq i_0}}^{n} W'(\widetilde{\phi}(x_i)) + W''(0) \sum_{\substack{i=-n\\ i\neq i_0}}^{n} \widetilde{\phi}(x_i)$$

and

$$E_2 = O\left(\frac{L\delta^{2s}}{\alpha} + \delta^{2s}\psi(x_{i_0}) + \sum_{\substack{i=-n\\ i\neq i_0}}^{n} \left[\widetilde{\phi}(x_i) + \delta^{2s}\psi(x_i)\right] \right)^2 .$$

Simply reorganizing the terms, we get with $c = c_0 L$:

$$\begin{aligned} A = \quad & \delta^{2s} c \left\{ \delta^{2s}\psi'(x_{i_0}) + \sum_{\substack{i=-n\\ i\neq i_0}}^{n} \left[\phi'(x_i) + \delta^{2s}\psi'(x_i)\right] \right\} - W''(0) \sum_{\substack{i=-n\\ i\neq i_0}}^{n} \widetilde{\phi}(x_i) - \delta^{2s} \sum_{\substack{i=-n\\ i\neq i_0}}^{n} \mathcal{I}_s[\psi, x_i] \\ & + W''(\phi(x_{i_0})) \left(\sum_{\substack{i=-n\\ i\neq i_0}}^{n} \left[\widetilde{\phi}(x_i) + \delta^{2s}\psi(x_i)\right] \right) \\ & + \delta^{2s}\Big(- \mathcal{I}_s[\psi, x_{i_0}] + W''(\phi(x_{i_0}))\psi(x_{i_0}) + \frac{L}{\alpha} W''(\phi(x_{i_0})) - L + c\phi'(x_{i_0}) \Big) + E. \end{aligned}$$

Using equation (6.6) satisfied by ψ, we get

$$\begin{aligned} A = \quad & \delta^{2s} c \left\{ \delta^{2s}\psi'(x_{i_0}) + \sum_{\substack{i=-n\\ i\neq i_0}}^{n} \left[\phi'(x_i) + \delta^{2s}\psi'(x_i)\right] \right\} + \left(W''(\phi(x_{i_0})) - W''(0)\right) \sum_{\substack{i=-n\\ i\neq i_0}}^{n} \widetilde{\phi}(x_i) \\ & - \delta^{2s} \sum_{\substack{i=-n\\ i\neq i_0}}^{n} \mathcal{I}_s[\psi, x_i] + W''(\phi(x_{i_0}))\delta^{2s} \sum_{\substack{i=-n\\ i\neq i_0}}^{n} \psi(x_i) + E. \end{aligned}$$

Let us bound the second term of the last equality, uniformly in x.

Step 2: Bound on $\sum_{\substack{i=-n\\ i\neq i_0}}^{n} \left[\phi'(x_i)+\delta^{2s}\psi'(x_i)\right]$

From (6.14) and (6.18) it follows that

$$
\begin{aligned}
&-\delta^{1+4s}|p_0|^{1+2s}K_3 \sum_{\substack{i=-n\\ i\neq i_0}}^{n} \frac{1}{|x-i|^{1+2s}} \leqslant \sum_{\substack{i=-n\\ i\neq i_0}}^{n} \left[\phi'(x_i)+\delta^{2s}\psi'(x_i)\right]\\
&\leqslant \delta^{1+2s}|p_0|^{1+2s}(K_1+\delta^{2s}K_3) \sum_{\substack{i=-n\\ i\neq i_0}}^{n} \frac{1}{|x-i|^{1+2s}},
\end{aligned}
$$

and then by Claim 1 we get

$$
-C\delta^{1+4s} \leqslant \lim_{n\to+\infty} \sum_{\substack{i=-n\\ i\neq i_0}}^{n} \left[\phi'(x_i)+\delta^{2s}\psi'(x_i)\right] \leqslant C\delta^{1+2s}. \tag{6.25}
$$

Here and henceforth, C denotes various positive constants independent of x.

Step 3: Bound on $(W''(\phi(x_{i_0}))-W''(0)) \sum_{\substack{i=-n\\ i\neq i_0}}^{n} \widetilde{\phi}(x_i)$

Let us prove that

$$
\lim_{n\to+\infty} \left| (W''(\phi(x_{i_0}))-W''(0)) \sum_{\substack{i=-n\\ i\neq i_0}}^{n} \widetilde{\phi}(x_i) \right| \leqslant C\delta^{1+2s}. \tag{6.26}
$$

By (6.12) we have

$$
\left| \sum_{\substack{i=-n\\ i\neq i_0}}^{n} \widetilde{\phi}(x_i) + \frac{\delta^{2s}|p_0|^{2s}}{2s\alpha} \sum_{\substack{i=-n\\ i\neq i_0}}^{n} \frac{x-i}{|x-i|^{1+2s}} \right| \leqslant K_1\delta^{1+2s}|p_0|^{1+2s} \sum_{\substack{i=-n\\ i\neq i_0}}^{n} \frac{1}{|x-i|^{1+2s}}. \tag{6.27}
$$

If $|\gamma| \geqslant \delta|p_0|$ then $|x_{i_0}| = \frac{|\gamma|}{\delta|p_0|} \geqslant 1$ and again from (6.12),

$$
|\widetilde{\phi}(x_{i_0}) + \frac{\delta^{2s}|p_0|^{2s}}{2s\alpha}\frac{\gamma}{|\gamma|^{1+2s}}| \leqslant K_1 \frac{\delta^{1+2s}|p_0|^{1+2s}}{|\gamma|^{1+2s}}
$$

which implies that

$$
|W''(\widetilde{\phi}(x_{i_0}))-W''(0)| \leqslant |W'''(0)\widetilde{\phi}(x_{i_0})| + O(\widetilde{\phi}(x_{i_0}))^2 \leqslant C\frac{\delta^{2s}}{|\gamma|^{2s}} + C\frac{\delta^{1+2s}}{|\gamma|^{1+2s}}.
$$

By the previous inequality, (6.27) and Claim 1 we deduce that

$$
\begin{aligned}
\lim_{n\to+\infty} \left| (W''(\phi(x_{i_0}))-W''(0)) \sum_{\substack{i=-n\\ i\neq i_0}}^{n} \widetilde{\phi}(x_i) \right| &\leqslant C\left(\frac{\delta^{2s}}{|\gamma|^{2s}} + \frac{\delta^{1+2s}}{|\gamma|^{1+2s}}\right)(\delta^{2s}|\gamma| + \delta^{1+2s})\\
&\leqslant C\delta^{1+2s}
\end{aligned}
$$

where C is independent of γ.

Finally, if $|\gamma| < \delta|p_0|$, from (6.27) and Claim 1 we conclude that

$$\left|\lim_{n\to+\infty}(W''(\phi(x_{i_0})) - W''(0)) \sum_{\substack{i=-n \\ i\neq i_0}}^{n} \widetilde{\phi}(x_i)\right| \leqslant C\delta^{2s}|\gamma| + C\delta^{1+2s} \leqslant C\delta^{1+2s},$$

and (6.26) is proved.

Step 4: Bound on $\delta^{2s} \sum_{\substack{i=-n \\ i\neq i_0}}^{n} \mathcal{I}_s[\psi, x_i]$

We compute

$$\begin{aligned} \mathcal{I}_s[\psi] &= W''(\widetilde{\phi})\psi + \frac{L}{\alpha}(W''(\widetilde{\phi}) - W''(0)) + c\phi' \\ &= W''(0)\psi + \frac{L}{\alpha}W'''(0)\widetilde{\phi} + O(\widetilde{\phi})\psi + O(\widetilde{\phi})^2 + c\phi'. \end{aligned} \tag{6.28}$$

Estimates (6.12) and (6.17) implies that the sequences

$$\sum_{\substack{i=-n \\ i\neq i_0}}^{n} O(\widetilde{\phi}(x_i))\psi(x_i), \quad \sum_{\substack{i=-n \\ i\neq i_0}}^{n} O(\widetilde{\phi}(x_i))^2$$

behave like the series $\sum_{i=1}^{\infty} \frac{1}{i^{4s}}$, therefore they are convergent since $s \geqslant \frac{1}{2}$. Moreover, by (6.28), (6.12), (6.14), (6.17) and Claim 1, we have

$$\left|\lim_{n\to+\infty} \delta^{2s} \sum_{\substack{i=-n \\ i\neq i_0}}^{n} \mathcal{I}_s[\psi, x_i]\right| \leqslant C(\delta^{4s} + \delta^{1+2s}) \leqslant C\delta^{1+2s}. \tag{6.29}$$

Step 5: Bound on $W''(\phi(x_{i_0}))\delta^{2s} \sum_{\substack{i=-n \\ i\neq i_0}}^{n} \psi(x_i)$

Similarly, from (6.17) and Claim 1 we get

$$\left|\lim_{n\to+\infty} W''(\phi(x_{i_0}))\delta^{2s} \sum_{\substack{i=-n \\ i\neq i_0}}^{n} \psi(x_i)\right| \leqslant C\delta^{1+2s}. \tag{6.30}$$

Step 6: Bound on the error E

Finally, again from (6.12), (6.17) and Claim 1 it follows that

(6.31)

$$\left|\lim_{n\to+\infty} E_2\right| = \left|\lim_{n\to+\infty} O\left(\frac{L\delta^{2s}}{\alpha} + \delta^{2s}\psi(x_{i_0}) + \sum_{\substack{i=-n \\ i\neq i_0}}^{n} \left[\widetilde{\phi}(x_i) + \delta^{2s}\psi(x_i)\right]\right)^2\right| \leqslant C\delta^{4s} \leqslant C\delta^{1+2s}.$$

Next, let us estimate E_1. From (6.12) and using $s \geqslant \frac{1}{2}$, we have

$$|E_1| \leqslant \sum_{\substack{i=-n \\ i \neq i_0}}^{n} |W'(\widetilde{\phi}(x_i)) - W''(0)\widetilde{\phi}(x_i)| = \sum_{\substack{i=-n \\ i \neq i_0}}^{n} |O(\widetilde{\phi}(x_i))^2| \leqslant C\delta^{4s} \leqslant C\delta^{1+2s}. \tag{6.32}$$

Step 7: Conclusion

Therefore, from (6.25), (6.26), (6.29), (6.30), (6.31) and (6.32) we conclude that

$$-C\delta^{1+2s} \leqslant \lim_{n\to+\infty} NL_L^{\overline{\lambda}_\delta^L}[s_{\delta,n}^L] \leqslant C\delta^{1+2s}$$

with C independent of x and Lemma 6.4 for $s \geqslant \frac{1}{2}$ is proved.

Now, let us turn to the case $s < \frac{1}{2}$.

Step 1': First computation

Making computations like in Step 1, we get

$$\begin{aligned}
A &:= NL_L^{\overline{\lambda}_\delta^L}[s_{\delta,n}^L](x) \\
&= \delta^{2s} c \left\{ \delta^{2s}(\psi\tau)'(x_{i_0}) + \sum_{\substack{i=-n \\ i \neq i_0}}^{n} \left[\phi'(x_i) + \delta^{2s}(\psi\tau)'(x_i)\right] \right\} \\
&+ (W''(\phi(x_{i_0})) - W''(0)) \sum_{\substack{i=-n \\ i \neq i_0}}^{n} \widetilde{\phi}(x_i) \\
&- \delta^{2s} \sum_{\substack{i=-n \\ i \neq i_0}}^{n} \mathcal{I}_s[\psi\tau, x_i] + W''(\phi(x_{i_0}))\delta^{2s} \sum_{\substack{i=-n \\ i \neq i_0}}^{n} (\psi\tau)(x_i) \\
&+ \delta^{2s}\Big(-\mathcal{I}_s[\psi\tau, x_{i_0}] + W''(\phi(x_{i_0}))(\psi\tau)(x_{i_0}) + \frac{L}{\alpha} W''(\phi(x_{i_0})) - L + c\phi'(x_{i_0}) \Big) + E,
\end{aligned}$$

where again $E = E_1 + E_2$ with E_1 the error term coming from in the approximation of $\sum_{\substack{i=-n \\ i \neq i_0}}^{n} W'(\widetilde{\phi}(x_i))$

with $W''(0) \sum_{\substack{i=-n \\ i \neq i_0}}^{n} \widetilde{\phi}(x_i)$, and

$$E_2 = O\left(\frac{L\delta^{2s}}{\alpha} + \delta^{2s}\psi(x_{i_0}) + \sum_{\substack{i=-n \\ i \neq i_0}}^{n} \left[\widetilde{\phi}(x_i) + \delta^{2s}\psi(x_i)\tau(x_i)\right] \right)^2.$$

To control the term $\mathcal{I}_s[\psi\tau, x_i]$, we use the following formula which can be found for instance in [1] page 7:

$$\mathcal{I}_s[\psi\tau, x_i] = \tau(x_i)\mathcal{I}_s[\psi, x_i] + \psi(x_i)\mathcal{I}_s[\tau, x_i] - B(\psi,\tau)(x_i), \tag{6.33}$$

where

$$B(\psi,\tau)(x_i) = C(s) \int_{\mathbb{R}} \frac{(\psi(y) - \psi(x_i))(\tau(y) - \tau(x_i))}{|x_i - y|^{1+2s}} dy.$$

Therefore the quantity A can be rewritten in the following way:

$$\begin{aligned} A = \;& \delta^{2s} c \left\{ \delta^{2s}(\psi\tau)'(x_{i_0}) + \sum_{\substack{i=-n\\ i\neq i_0}}^{n} \left[\phi'(x_i) + \delta^{2s}(\psi\tau)'(x_i)\right] \right\} \\ & + (W''(\phi(x_{i_0})) - W''(0)) \sum_{\substack{i=-n\\ i\neq i_0}}^{n} \widetilde{\phi}(x_i) \\ & - \delta^{2s} \sum_{\substack{i=-n\\ i\neq i_0}}^{n} \mathcal{I}_s[\psi\tau, x_i] + W''(\phi(x_{i_0}))\delta^{2s} \sum_{\substack{i=-n\\ i\neq i_0}}^{n} (\psi\tau)(x_i) \\ & + \delta^{2s}\Big(- \tau(x_{i_0})\mathcal{I}_s[\psi, x_{i_0}] + W''(\phi(x_{i_0}))(\psi\tau)(x_{i_0}) + \frac{L}{\alpha} W''(\phi(x_{i_0})) - L + c\phi'(x_{i_0})\Big) \\ & + \delta^{2s} B(\psi,\tau)(x_{i_0}) - \delta^{2s}\psi(x_{i_0})\mathcal{I}_s[\tau, x_{i_0}] + E. \end{aligned}$$

Now, we remark that

$$|x_{i_0}| = \frac{|\gamma|}{\delta|p_0|} \leqslant \frac{1}{2\delta|p_0|} = R,$$

then by (6.8) $\tau(x_{i_0}) = 1$. Therefore, using the equation satisfied by ψ (6.6), we get

$$-\tau(x_{i_0})\mathcal{I}_s[\psi, x_{i_0}] + W''(\phi(x_{i_0}))(\psi\tau)(x_{i_0}) + \frac{L}{\alpha} W''(\phi(x_{i_0})) - L + c\phi'(x_{i_0}) = 0$$

and consequently

$$\begin{aligned} A = \;& \delta^{2s} c \left\{ \delta^{2s}(\psi\tau)'(x_{i_0}) + \sum_{\substack{i=-n\\ i\neq i_0}}^{n} \left[\phi'(x_i) + \delta^{2s}(\psi\tau)'(x_i)\right] \right\} \\ & + (W''(\phi(x_{i_0})) - W''(0)) \sum_{\substack{i=-n\\ i\neq i_0}}^{n} \widetilde{\phi}(x_i) \\ & - \delta^{2s} \sum_{\substack{i=-n\\ i\neq i_0}}^{n} \mathcal{I}_s[\psi\tau, x_i] + W''(\phi(x_{i_0}))\delta^{2s} \sum_{\substack{i=-n\\ i\neq i_0}}^{n} (\psi\tau)(x_i) \\ & + \delta^{2s} B(\psi,\tau)(x_{i_0}) - \delta^{2s}\psi(x_{i_0})\mathcal{I}_s[\tau, x_{i_0}] + E. \end{aligned}$$

Let us proceed to the estimate of A.

Step 2': Bound on $\displaystyle\sum_{\substack{i=-n\\ i\neq i_0}}^{n} \left[\phi'(x_i) + \delta^{2s}(\psi\tau)'(x_i)\right]$

As in Step 2, using (6.14) and Claim 1, we get

$$0 \leqslant \lim_{n\to+\infty} \sum_{\substack{i=-n\\ i\neq i_0}}^{n} \phi'(x_i) \leqslant C\delta^{1+2s}. \tag{6.34}$$

Next, for $i \neq i_0 - 1, i_0, i_0 + 1$, and $R = \frac{1}{2\delta|p_0|}$

$$|x_i| = \frac{|i_0 + \gamma - i|}{\delta|p_0|} \geqslant \frac{3}{2\delta|p_0|} > 2R,$$

therefore $\tau(x_i) = \tau'(x_i) = 0$. Then, using (6.18) and the fact that $\lim\limits_{x\to\pm\infty} \psi(x) = 0$, we get

$$\begin{aligned}\delta^{2s} \sum_{\substack{i=-n\\ i\neq i_0}}^{n} (\psi\tau)'(x_i) &= \delta^{2s}(\psi\tau)'(x_{i_0-1}) + \delta^{2s}(\psi\tau)'(x_{i_0+1}) \\ &= \delta^{2s}(\psi\tau)'\left(\frac{-1+\gamma}{\delta|p_0|}\right) + \delta^{2s}(\psi\tau)'\left(\frac{1+\gamma}{\delta|p_0|}\right) \\ &= o(\delta^{2s}).\end{aligned} \tag{6.35}$$

Step 3': Bound on $(W''(\phi(x_{i_0})) - W''(0)) \sum\limits_{\substack{i=-n\\ i\neq i_0}}^{n} \widetilde{\phi}(x_i)$

From (6.20) with $k = 1$ we know that

$$\lim_{n\to+\infty} \left| \sum_{\substack{i=-n\\ i\neq i_0}}^{n} \widetilde{\phi}(x_i) \right| \leqslant C\delta^{2s}|\gamma|.$$

As in Step 3 if $|\gamma| \geqslant \delta|p_0|$, then (6.13) implies

$$|\widetilde{\phi}(x_{i_0})| \leqslant C\frac{\delta^{2s}}{|\gamma|^{2s}},$$

and so, using that $W'''(0) = 0$

$$|W''(\widetilde{\phi}(x_{i_0})) - W''(0)| \leqslant C\frac{\delta^{4s}}{|\gamma|^{4s}}.$$

Then we have

$$\lim_{n\to+\infty} \left| (W''(\phi(x_{i_0})) - W''(0)) \sum_{\substack{i=-n\\ i\neq i_0}}^{n} \widetilde{\phi}(x_i) \right| \leqslant C\frac{\delta^{4s}}{|\gamma|^{4s}}\delta^{2s}|\gamma| \leqslant C\delta^{4s}.$$

Finally, if $|\gamma| < \delta|p_0|$, then

$$\lim_{n\to\infty} \left| \sum_{\substack{i=-n\\ i\neq i_0}}^{n} \widetilde{\phi}(x_i) \right| \leqslant C\delta^{2s}|\gamma| \leqslant C\delta^{1+2s}.$$

We conclude that

$$\lim_{n\to+\infty} \left| (W''(\phi(x_{i_0})) - W''(0)) \sum_{\substack{i=-n\\ i\neq i_0}}^{n} \widetilde{\phi}(x_i) \right| \leqslant C\delta^{4s}. \tag{6.36}$$

Step 4': Bound on $\delta^{2s}\sum_{\substack{i=-n\\ i\neq i_0}}^{n}\mathcal{I}_s[\psi\tau,x_i]$

Using formula (6.33), we see that

$$\delta^{2s}\sum_{\substack{i=-n\\ i\neq i_0}}^{n}\mathcal{I}_s[\psi\tau,x_i]=\delta^{2s}\sum_{\substack{i=-n\\ i\neq i_0}}^{n}\left\{\tau(x_i)\mathcal{I}_s[\psi,x_i]+\psi(x_i)\mathcal{I}_s[\tau,x_i]-B(\psi,\tau)(x_i)\right\}. \tag{6.37}$$

As we have already pointed out in Step 2', for $i\neq i_0-1,i_0,i_0+1$, $\tau(x_i)=0$, therefore

$$\delta^{2s}\sum_{\substack{i=-n\\ i\neq i_0}}^{n}\tau(x_i)\mathcal{I}_s[\psi,x_i]=\delta^{2s}\tau(x_{i_0-1})\mathcal{I}_s[\psi,x_{i_0-1}]+\delta^{2s}\tau(x_{i_0+1})\mathcal{I}_s[\psi,x_{i_0+1}].$$

We point out that

$$x_{i_0-1}=\frac{-1+\gamma}{\delta|p_0|}\to-\infty\quad\text{as }\delta\to 0$$

and

$$x_{i_0+1}=\frac{1+\gamma}{\delta|p_0|}\to+\infty\quad\text{as }\delta\to 0.$$

Then from the equation (6.6), estimates (6.12), (6.14) and $\lim_{x\to\pm\infty}\psi(x)=0$, we deduce that $\mathcal{I}_s[\psi,x_{i_0-1}]$ and $\mathcal{I}_s[\psi,x_{i_0+1}]$ are $o(1)$ as $\delta\to 0$, this implies that

$$\delta^{2s}\sum_{\substack{i=-n\\ i\neq i_0}}^{n}\tau(x_i)\mathcal{I}_s[\psi,x_i]=o(\delta^{2s})\quad\text{as }\delta\to 0. \tag{6.38}$$

Similarly, from the behavior of ψ at infinity we infer that

$$\delta^{2s}\psi(x_{i_0-1})\mathcal{I}_s[\tau,x_{i_0-1}],\ \delta^{2s}\psi(x_{i_0+1})\mathcal{I}_s[\tau,x_{i_0+1}]\quad\text{are }o(\delta^{2s})\quad\text{as }\delta\to 0.$$

This and (6.21) imply that

$$\delta^{2s}\sum_{\substack{i=-n\\ i\neq i_0}}^{n}\psi(x_i)\mathcal{I}_s[\tau,x_i]=o(\delta^{2s})\quad\text{as }\delta\to 0. \tag{6.39}$$

Let us now consider the term $\delta^{2s}\sum_{\substack{i=-n\\ i\neq i_0}}^{n}B(\psi,\tau)(x_i)$. For $i\neq i_0-1,i_0,i_0+1$, using that $\tau(x_i)=0$, we have

$$|B(\psi,\tau)(x_i)|\leqslant C(s)\int_{\mathbb{R}}\frac{|\psi(y)-\psi(x_i)|\tau(y)}{|x_i-y|^{1+2s}}dy\leqslant C\int_{\mathbb{R}}\frac{\tau(y)}{|x_i-y|^{1+2s}}dy=C\mathcal{I}_s[\tau,x_i].$$

Therefore, from (6.21) we infer that

$$\delta^{2s}\sum_{\substack{i=-n\\ i\neq i_0,i_0\pm1}}^{n}|B(\psi,\tau)(x_i)|\leqslant C\delta^{4s}. \tag{6.40}$$

Next, we remark that for $\gamma\in\left(-\frac{1}{2},\frac{1}{2}\right]$ and $R=\frac{1}{2\delta|p_0|}$, either $x_{i_0-1}\in[-2R,-R]$ or $x_{i_0+1}\in(R,2R]$. Suppose for instance that $x_{i_0-1}\in[-2R,-R]$ (i.e. $0\leqslant\gamma\leqslant\frac{1}{2}$). We have

$$B(\psi,\tau)(x_{i_0-1}) = C(s)\int_{\mathbb{R}} \frac{(\psi(y)-\psi(x_{i_0-1}))(\tau(y)-\tau(x_{i_0-1}))}{|x_{i_0-1}-y|^{1+2s}}dy$$
$$= C(s)\int_{-2R}^{2R} \frac{(\psi(y)-\psi(x_{i_0-1}))\tau(y)}{|x_{i_0-1}-y|^{1+2s}}dy - C\tau(x_{i_0-1}))\mathcal{I}_s[\psi, x_{i_0-1}].$$

We have already pointed out that $\mathcal{I}_s[\psi, x_{i_0-1}] = o(1)$ as $\delta \to 0$. Let us consider the first term of the right-hand side of the last equality. Using that $R = \frac{1}{2\delta|p_0|}$, $x_{i_0-1} = \frac{-1+\gamma}{2\delta|p_0|} \in [-2R, -R]$ and estimate (6.18), we get

$$\left|\int_{-2R}^{2R} \frac{(\psi(y)-\psi(x_{i_0-1}))\tau(y)}{|x_{i_0-1}-y|^{1+2s}}dy\right| \leqslant \max_{[-2R,-R/2]} \psi' \int_{-2R}^{-\frac{R}{2}} \frac{1}{|x_{i_0-1}-y|^{2s}}dy$$
$$+ C\int_{-\frac{R}{2}}^{2R} \frac{1}{|x_{i_0-1}-y|^{1+2s}}dy$$
$$= C \max_{[-2R,-R/2]} \psi' \left[(x_{i_0-1}+2R)^{1-2s} + \left(-\frac{R}{2} - x_{i_0-1}\right)^{1-2s}\right]$$
$$+ C\left[\frac{1}{(2R - x_{i_0-1})^{2s}} - \frac{1}{(-\frac{R}{2} - x_{i_0-1})^{2s}}\right]$$
$$\leqslant C\delta^{2s}.$$

We conclude that

$$\delta^{2s} B(\psi,\tau)(x_{i_0-1}) = o(\delta^{2s}) \quad \text{as } \delta \to 0. \tag{6.41}$$

Similarly we can prove that

$$\delta^{2s} B(\psi,\tau)(x_{i_0+1}) = o(\delta^{2s}) \quad \text{as } \delta \to 0. \tag{6.42}$$

Estimates (6.40), (6.41) and (6.42) imply

$$\delta^{2s} \sum_{\substack{i=-n \\ i\neq i_0}}^{n} B(\psi,\tau)(x_i) = o(\delta^{2s}) \quad \text{as } \delta \to 0. \tag{6.43}$$

In conclusion, putting together (6.37), (6.38), (6.39) and (6.43) we get

$$\delta^{2s} \sum_{\substack{i=-n \\ i\neq i_0}}^{n} \mathcal{I}_s[\psi\tau, x_i] = o(\delta^{2s}) \quad \text{as } \delta \to 0. \tag{6.44}$$

Step 5': Bound on $W''(\phi(x_{i_0}))\delta^{2s} \sum_{\substack{i=-n \\ i\neq i_0}}^{n} (\psi\tau)(x_i)$

As in Step 2', using that $\tau(x_i) = 0$ for $i \neq i_0 - 1, i_0, i_0 + 1$ and that $\lim_{x\to\pm\infty} \psi(x) = 0$, we get

$$\delta^{2s} \sum_{\substack{i=-n \\ i\neq i_0}}^{n} (\psi\tau)(x_i) = \delta^{2s}(\psi\tau)(x_{i_0-1}) + \delta^{2s}(\psi\tau)(x_{i_0+1}) = o(\delta^{2s}) \quad \text{as } \delta \to 0. \tag{6.45}$$

Step 6': Bound on $\delta^{2s}B(\psi,\tau)(x_{i_0})-\delta^{2s}\psi(x_{i_0})\mathcal{I}_s[\tau,x_{i_0}]$ Remember that

$$x_{i_0}=\frac{\gamma}{\delta|p_0|},\quad |\gamma|\leqslant\frac{1}{2}.$$

Let us first assume $|\gamma|\leqslant\frac{1}{4}$, then

$$|x_{i_0}|\leqslant\frac{1}{4\delta|p_0|}=\frac{R}{2},$$

and

$$\begin{aligned}|\mathcal{I}_s[\tau,x_{i_0}]|&=C\left|\int_{\mathbb{R}}\frac{\tau(y)-1}{|y-x_{i_0}|^{1+2s}}dy\right|\\&=C\left|\int_{|y|>R}\frac{\tau(y)-1}{|y-x_{i_0}|^{1+2s}}dy\right|\\&\leqslant C\int_{|y|>R}\frac{1}{|y-x_{i_0}|^{1+2s}}dy\\&=\frac{C}{(x_{i_0}+R)^{2s}}+\frac{C}{(R-x_{i_0})^{2s}}\\&\leqslant\frac{C}{R^{2s}}\\&=C\delta^{2s}.\end{aligned}$$

Then

$$\delta^{2s}|\psi(x_{i_0})\mathcal{I}_s[\tau,x_{i_0}]|\leqslant C\delta^{4s}.$$

Now let us assume $|\gamma|>\frac{1}{4}$. In this case $\psi(x_{i_0})=o(1)$ as $\delta\to0$, with $o(1)$ independent of γ and then $\delta^{2s}\psi(x_{i_0})\mathcal{I}_s[\tau,x_{i_0}]=o(\delta^{2s})$ as $\delta\to0$. We conclude that for any $\gamma\in\left(-\frac{1}{2},\frac{1}{2}\right]$ we have

$$\delta^{2s}\psi(x_{i_0})\mathcal{I}_s[\tau,x_{i_0}]=o(\delta^{2s})\quad\text{as }\delta\to0. \tag{6.46}$$

Finally, let us consider the term $\delta^{2s}B(\psi,\tau)(x_{i_0})$. Again, if $|\gamma|\leqslant\frac{1}{4}$, then

$$\begin{aligned}\delta^{2s}|B(\psi,\tau)(x_{i_0})|&=\delta^{2s}C(s)\left|\int_{\mathbb{R}}\frac{(\psi(y)-\psi(x_{i_0}))(\tau(y)-1)}{|y-x_{i_0-1}|^{1+2s}}dy\right|\\&\leqslant\delta^{2s}C\int_{|y|>R}\frac{1}{|y-x_{i_0}|^{1+2s}}dy\\&\leqslant C\delta^{4s}.\end{aligned}$$

If $|\gamma|>\frac{1}{4}$, then either $x_{i_0}\in\left[-R,-\frac{R}{2}\right]$ or $x_{i_0}\in\left[\frac{R}{2},R\right]$. Suppose for instance $x_{i_0}\in\left[-R,-\frac{R}{2}\right]$, then computations similar to those done in Step 5' for $B(\psi,\tau)(x_{i_0-1})$, show that $B(\psi,\tau)(x_{i_0})=o(1)$ as $\delta\to0$. We conclude that for any $\gamma\in\left(-\frac{1}{2},\frac{1}{2}\right]$ we have

$$\delta^{2s}B(\psi,\tau)(x_{i_0})=o(\delta^{2s})\quad\text{as }\delta\to0. \tag{6.47}$$

Step 6": Bound on the error E

From (6.20) with $k=1$, and the fact that $\tau(x_i)=0$ for $i\neq i_0-1,i_0,i_0+1$ it follows that
(6.48)

$$\left|\lim_{n\to+\infty}E_2\right|=\left|\lim_{n\to+\infty}O\left(\frac{L\delta^{2s}}{\alpha}+\delta^{2s}\psi(x_{i_0})+\sum_{\substack{i=-n\\i\neq i_0}}^{n}\left[\widetilde{\phi}(x_i)+\delta^{2s}\psi(x_i)\tau(x_i)\right]\right)^2\right|\leqslant C\delta^{4s}.$$

Next, let us estimate E_1. Remember that for $s<\frac{1}{2}$ we assume W even, this implies $W^{2k-1}(0)=0$ for any integer $k\geqslant 1$. Therefore

$$
\begin{aligned}
-E_1 &= \sum_{\substack{i=-n\\ i\neq i_0}}^{n} W'(\widetilde{\phi}(x_i)) - W''(0)\widetilde{\phi}(x_i)\\
&= W^{IV}(0)\sum_{\substack{i=-n\\ i\neq i_0}}^{n}(\widetilde{\phi}(x_i))^3 + W^{VI}(0)\sum_{\substack{i=-n\\ i\neq i_0}}^{n}(\widetilde{\phi}(x_i))^5+...+W^{2k_0}(0)\sum_{\substack{i=-n\\ i\neq i_0}}^{n}(\widetilde{\phi}(x_i))^{2k_0-1}\\
&+\sum_{\substack{i=-n\\ i\neq i_0}}^{n} O((\widetilde{\phi}(x_i))^{2k_0+1}).
\end{aligned}
$$

Fix k_0 such that $2s(2k_0+1)>1$, then by (6.13) the sequence $\sum_{\substack{i=-n\\ i\neq i_0}}^{n} O((\widetilde{\phi}(x_i))^{2k_0+1})$ is convergent since behaves like the series $\sum_{i=1}^{\infty}\frac{1}{i^{2s(2k_0+1)}}$ and

$$\sum_{\substack{i=-n\\ i\neq i_0}}^{n} |O((\widetilde{\phi})^{2k_0+1})| \leqslant C\delta^{2s(2k_0+1)}.$$

This estimate, together with (6.20) imply that

$$|E_1|\leqslant C\delta^{4s}. \tag{6.49}$$

Step 7': Conclusion

Therefore, from (6.34), (6.35), (6.36), (6.44), (6.45), (6.46), (6.47), (6.48) and (6.49) we conclude that

$$\lim_{n\to+\infty} NL_L^{\overline{\lambda}_\delta^L}[s^L_{\delta,n}] = o(\delta^{2s}) \quad \text{as } \delta\to 0$$

where $\lim_{\delta\to 0}\frac{o(\delta^{2s})}{\delta^{2s}}=0$, uniformly for $x\in\mathbb{R}$ and Lemma 6.4 for $s<\frac{1}{2}$ is proved.

6.4. **Proof of Claims 1-6.**

Proof of Claim 1.

We have for $n>|i_0|$

$$
\begin{aligned}
\sum_{\substack{i=-n\\ i\neq i_0}}^{n}\frac{x-i}{|x-i|^{1+2s}} &= \sum_{i=-n}^{i_0-1}\frac{i_0+\gamma-i}{(i_0+\gamma-i)^{1+2s}} + \sum_{i=i_0+1}^{n}\frac{i_0+\gamma-i}{(i-i_0-\gamma)^{1+2s}}\\
&= \sum_{i=1}^{n+i_0}\frac{1}{(i+\gamma)^{2s}} - \sum_{i=1}^{n-i_0}\frac{1}{(i-\gamma)^{2s}}
\end{aligned}
$$

Using that, for some $\theta_i\in(-1,1)$

$$\frac{(i-\gamma)^{2s}-(i+\gamma)^{2s}}{(i+\gamma)^{2s}(i-\gamma)^{2s}} = \frac{4s\gamma(i-\theta_i\gamma)^{2s-1}}{(i+\gamma)^{2s}(i-\gamma)^{2s}},$$

we get

$$\sum_{\substack{i=-n\\ i\neq i_0}}^{n} \frac{x-i}{|x-i|^{1+2s}} = \begin{cases} \displaystyle\sum_{i=1}^{n} \frac{4s\gamma(i-\theta_i\gamma)^{2s-1}}{(i+\gamma)^{2s}(i-\gamma)^{2s}}, & \text{if } i_0 = 0 \\ \displaystyle\sum_{i=1}^{n-i_0} \frac{4s\gamma(i-\theta_i\gamma)^{2s-1}}{(i+\gamma)^{2s}(i-\gamma)^{2s}} + \sum_{i=n-i_0+1}^{n+i_0} \frac{1}{(i+\gamma)^{2s}}, & \text{if } i_0 > 0 \\ \displaystyle\sum_{i=1}^{n+i_0} \frac{4s\gamma(i-\theta_i\gamma)^{2s-1}}{(i+\gamma)^{2s}(i-\gamma)^{2s}} - \sum_{i=n+i_0+1}^{n-i_0} \frac{1}{(i-\gamma)^{2s}}, & \text{if } i_0 < 0 \end{cases}$$

$$\to -\sum_{i=1}^{+\infty} \frac{4s\gamma(i-\theta_i\gamma)^{2s-1}}{(i+\gamma)^{2s}(i-\gamma)^{2s}} \quad \text{as } n \to +\infty.$$

Let us prove the second limit of the claim.

$$\sum_{i=-n}^{i_0-1} \frac{1}{|x-i|^{1+2s}} = \sum_{i=1}^{n+i_0} \frac{1}{(i+\gamma)^{1+2s}} \to \sum_{i=1}^{+\infty} \frac{1}{(i+\gamma)^{1+2s}} \quad \text{as } n \to +\infty.$$

Finally

$$\sum_{i=i_0+1}^{n} \frac{1}{|x-i|^{1+2s}} = \sum_{i=1}^{n-i_0} \frac{1}{(i-\gamma)^{1+2s}} \to \sum_{i=1}^{+\infty} \frac{1}{(i-\gamma)^{1+2s}} \quad \text{as } n \to +\infty,$$

and the claim is proved.

Proof of Claim 2.

When $s < \frac{1}{2}$, we assume that W is even and this implies that the function

$$\phi(x) - \frac{1}{2}$$

is odd, which means that ϕ satisfies

$$\phi(-x) = -\phi(x) + 1,$$

and therefore for any integer $k \geqslant 1$

$$[\phi(-x)]^{2k-1} = [-\phi(x)+1]^{2k-1} = -[\phi(x)-1]^{2k-1}.$$

For simplicity, let us assume $i_0 > 0$. We have

$$\begin{aligned}
\sum_{\substack{i=-n \\ i\neq i_0}}^{n} [\widetilde{\phi}(x_i)]^{2k-1} &= \sum_{i=-n}^{i_0-1} [\phi(x_i)-1]^{2k-1} + \sum_{i=i_0+1}^{n} [\phi(x_i)]^{2k-1} \\
&= \sum_{i=1}^{n+i_0} \left[\phi\left(\frac{i+\gamma}{\delta|p_0|}\right)-1\right]^{2k-1} + \sum_{i=1}^{n-i_0} \left[\phi\left(-\frac{i-\gamma}{\delta|p_0|}\right)\right]^{2k-1} \\
&= \sum_{i=1}^{n+i_0} \left[\phi\left(\frac{i+\gamma}{\delta|p_0|}\right)-1\right]^{2k-1} - \sum_{i=1}^{n-i_0} \left[\phi\left(\frac{i-\gamma}{\delta|p_0|}\right)-1\right]^{2k-1} \\
&= \sum_{i=1}^{n-i_0} \left\{\left[\phi\left(\frac{i+\gamma}{\delta|p_0|}\right)-1\right]^{2k-1} - \left[\phi\left(\frac{i-\gamma}{\delta|p_0|}\right)-1\right]^{2k-1}\right\} \\
&+ \sum_{i=n-i_0+1}^{n+i_0} \left[\phi\left(\frac{i+\gamma}{\delta|p_0|}\right)-1\right]^{2k-1} \\
&= \sum_{i=1}^{n-i_0} \left\{\left[\phi\left(\frac{i+\gamma}{\delta|p_0|}\right)-\phi\left(\frac{i-\gamma}{\delta|p_0|}\right)\right]\right. \\
&\cdot \left.\sum_{l=0}^{2k-2} \left(\phi\left(\frac{i+\gamma}{\delta|p_0|}\right)-1\right)^{l} \left(\phi\left(\frac{i-\gamma}{\delta|p_0|}\right)-1\right)^{2k-2-l}\right\} \\
&+ \sum_{i=n-i_0+1}^{n+i_0} \left[\phi\left(\frac{i+\gamma}{\delta|p_0|}\right)-1\right]^{2k-1} \\
&= \sum_{i=1}^{n-i_0} \phi'\left(\frac{i+\theta_i\gamma}{\delta|p_0|}\right) \frac{2\gamma}{\delta|p_0|} \sum_{l=0}^{2k-2} \left(\phi\left(\frac{i+\gamma}{\delta|p_0|}\right)-1\right)^{l} \left(\phi\left(\frac{i-\gamma}{\delta|p_0|}\right)-1\right)^{2k-2-l} \\
&+ \sum_{i=n-i_0+1}^{n+i_0} \left[\phi\left(\frac{i+\gamma}{\delta|p_0|}\right)-1\right]^{2k-1}
\end{aligned}$$

for some $\theta_i \in (-1,1)$. Therefore, using (6.14) and (6.13), we get

$$\begin{aligned}
\left|\sum_{\substack{i=-n \\ i\neq i_0}}^{n} [\widetilde{\phi}(x_i)]^{2k-1}\right| &\leqslant C\delta^{2s(2k-1)}|\gamma| \sum_{i=1}^{n-i_0} \frac{1}{(i-|\gamma|)^{1+2s}} \sum_{l=0}^{2k-2} \frac{1}{(i-|\gamma|)^{2s(2k-2)}} \\
&+ C \sum_{i=n-i_0+1}^{n+i_0} \frac{1}{|i+\gamma|^{2s(2k-1)}} \\
&\leqslant Ck\delta^{2s(2k-1)}|\gamma| \sum_{i=1}^{n-i_0} \frac{1}{(i-|\gamma|)^{1+2s(2k-1)}} + C \sum_{i=n-i_0+1}^{n+i_0} \frac{1}{|i+\gamma|^{2s(2k-1)}}.
\end{aligned}$$

Passing to the limit as $n \to +\infty$, we get (6.20)

Next, let us turn to the proof of (6.21). For $i \neq i_0-1, i_0, i_0+1$, and $R = \frac{1}{2\delta|p_0|}$

$$|x_i| = \frac{|i_0+\gamma-i|}{\delta|p_0|} \geqslant \frac{3}{2\delta|p_0|} > 2R,$$

therefore $\tau(x_i) = 0$ and

$$
\begin{aligned}
0 \leqslant \mathcal{I}_s[\tau, x_i] &= \int_{\mathbb{R}} \frac{\tau(y)}{|x_i - y|^{1+2s}} dy \\
&= \int_{-2R}^{2R} \frac{\tau(y)}{|x_i - y|^{1+2s}} dy \\
&\leqslant \int_{-2R}^{2R} \frac{1}{|x_i - y|^{1+2s}} dy \\
&= \int_{|x_i|-2R}^{|x_i|+2R} \frac{1}{y^{1+2s}} dy \\
&= 2s \left[\frac{1}{(|x_i| - 2R)^{2s}} - \frac{1}{(|x_i| + 2R)^{2s}} \right] \\
&= 16s^2 R \frac{(|x_i| + 2R\theta_i)^{2s-1}}{(|x_i| - 2R)^{2s}(|x_i| + 2R)^{2s}},
\end{aligned}
$$

for some $\theta_i \in (-1, 1)$. Therefore, for $R = \frac{1}{2\delta|p_0|}$ we have

$$
\begin{aligned}
0 \leqslant \sum_{\substack{i=-n \\ i \neq i_0, i_0 \pm 1}}^{n} \mathcal{I}_s[\tau, x_i] &\leqslant 8s^2 \delta^{2s} |p_0|^{2s} \sum_{i=-n}^{i_0 - 2} \frac{(i_0 + \gamma - i + \theta_i)^{2s-1}}{(i_0 + \gamma - i - 1)^{2s}(i_0 + \gamma - i + 1)^{2s}} \\
&\quad + 8s^2 \delta^{2s} |p_0|^{2s} \sum_{i=i_0+2}^{n} \frac{(-i_0 - \gamma + i + \theta_i)^{2s-1}}{(-i_0 - \gamma + i - 1)^{2s}(-i_0 - \gamma + i + 1)^{2s}} \\
&= C\delta^{2s} \sum_{i=2}^{n+i_0} \frac{(i + \gamma + \theta_i)^{2s-1}}{(i + \gamma - 1)^{2s}(i + \gamma + 1)^{2s}} \\
&\quad + C\delta^{2s} \sum_{i=2}^{n-i_0} \frac{(i - \gamma + \theta_i)^{2s-1}}{(i - \gamma - 1)^{2s}(i - \gamma + 1)^{2s}} \\
&\leqslant C\delta^{2s} \sum_{i=2}^{n+|i_0|} \frac{1}{\left(i + \frac{1}{2}\right)^{2s} \left(i - \frac{3}{2}\right)},
\end{aligned}
$$

which implies (6.20).

Proof of Claim 3.
Fix $x \in \mathbb{R}$ and let $i_0 \in \mathbb{Z}$ be the closest integer to x such that $x = i_0 + \gamma$, with $\gamma \in \left(-\frac{1}{2}, \frac{1}{2}\right]$ and $|x - i| \geqslant \frac{1}{2}$ for $i \neq i_0$. Let δ be so small that $\frac{1}{\delta|p_0|} \geqslant 2$, then $\frac{|x-i|}{\delta|p_0|} \geqslant 1$ for $i \neq i_0$. Let us first assume

$s \geqslant \frac{1}{2}$. Then, for $n > |i_0|$ using (6.12) and (6.17) we get

$$
\begin{aligned}
s^L_{\delta,n}(x) &= \phi(x_{i_0}) + \delta^{2s}\psi(x_{i_0}) + i_0 + \sum_{i=-n}^{i_0-1} \left[\phi(x_i) - 1 + \delta^{2s}\psi(x_i)\right] \\
&\quad + \sum_{i=i_0+1}^{n} \left[\phi(x_i) + \delta^{2s}\psi(x_i)\right] \\
&\leqslant C + i_0 - \left(\frac{1}{2s\alpha} - \delta^{2s}K_2\right)\delta^{2s}|p_0|^{2s} \sum_{\substack{i=-n \\ i\neq i_0}}^{n} \frac{x-i}{|x-i|^{1+2s}} \\
&\quad + (K_1 + \delta^{2s}K_3)\delta^{1+2s}|p_0|^{1+2s} \sum_{\substack{i=-n \\ i\neq i_0}}^{n} \frac{1}{|x-i|^{1+2s}},
\end{aligned}
$$

and

$$
\begin{aligned}
s^L_{\delta,n}(x) &\geqslant C + i_0 - \left(\frac{1}{2s\alpha} - \delta^{2s}K_2\right)\delta^{2s}|p_0|^{2s} \sum_{\substack{i=-n \\ i\neq i_0}}^{n} \frac{x-i}{|x-i|^{1+2s}} \\
&\quad - (K_1 + \delta^{2s}K_3)\delta^{1+2s}|p_0|^{1+2s} \sum_{\substack{i=-n \\ i\neq i_0}}^{n} \frac{1}{|x-i|^{1+2s}}.
\end{aligned}
$$

Then from Claim 1 we conclude that the sequence $\{s^L_{\delta,n}(x)\}_n$ is convergent as $n \to +\infty$, moreover for $x = i_0 + \gamma$, we have

$$
|s^L_{\delta,n}(x) - x| \leqslant C. \tag{6.50}
$$

When $s < \frac{1}{2}$, the convergence of $\sum_{i=-n}^{n} \phi(x_i) - n$ follows from (6.20) for $k = 1$. The sum $\sum_{i=-n}^{n} \psi(x_i)\tau(x_i)$ is actually the sum of only three terms, since as we have seen in the proof of Claim 2, $\tau(x_i) = 0$ for $i \neq i_0 - 1, i_0, i_0 + 1$. This concludes the proof of Claim 3.

Proof of Claim 4.
To prove the uniform convergence, it suffices to show that $\{(s^L_{\delta,n})'(x)\}_n$ is a Cauchy sequence uniformly on compact sets. Let us consider a bounded interval $[a,b]$ and let $x \in [a,b]$. Let us first assume $s \geqslant \frac{1}{2}$. For $\frac{1}{\delta|p_0|} \geqslant 2$ and $k > m > 1/2 + \max\{|a|,|b|\}$, by (6.14) and (6.18) we have

$$
\begin{aligned}
(s^L_{\delta,k})'(x) - (s^L_{\delta,m})'(x) &= \frac{1}{\delta|p_0|} \sum_{i=-k}^{-m-1} \left[\phi'(x_i) + \delta^{2s}\psi'(x_i)\right] + \frac{1}{\delta|p_0|} \sum_{i=m+1}^{k} \left[\phi'(x_i) + \delta^{2s}\psi'(x_i)\right] \\
&\leqslant (K_1 + \delta^{2s}K_3)\delta^{2s}|p_0|^{2s} \left[\sum_{i=-k}^{-m-1} \frac{1}{|x-i|^{1+2s}} + \sum_{i=m+1}^{k} \frac{1}{|x-i|^{1+2s}}\right] \\
&\leqslant (K_1 + \delta^{2s}K_3)\delta^{2s}|p_0|^{2s} \left[\sum_{i=-k}^{-m-1} \frac{1}{|a-i|^{1+2s}} + \sum_{i=m+1}^{k} \frac{1}{|b-i|^{1+2s}}\right],
\end{aligned}
$$

and

$$
(s^L_{\delta,k})'(x) - (s^L_{\delta,m})'(x) \geqslant -K_3\delta^{4s}|p_0|^{2s} \left[\sum_{i=-k}^{-m-1} \frac{1}{|a-i|^{1+2s}} + \sum_{i=m+1}^{k} \frac{1}{|b-i|^{1+2s}}\right].
$$

Then by Claim 1

$$\sup_{x\in[a,b]} |(s^L_{\delta,k})'(x) - (s^L_{\delta,m})'(x)| \to 0 \quad \text{as } k, m \to +\infty.$$

When $s < \frac{1}{2}$, the convergence of $\sum_{\substack{i=-n\\ i\neq i_0}}^{n} \phi'(x_i)$ is again consequence of estimate (6.14), and the convergence of $\sum_{\substack{i=-n\\ i\neq i_0}}^{n} (\psi\tau)'(x_i)$ comes from the fact that this is actually the sum of three terms, being $\tau(x_i) = \tau'(x_i) = 0$ for $i \neq i_0 - 1, i_0, i_0 + 1$. Claim 4 is therefore proved.

Proof of Claim 5.
Claim 5 can be proved like Claim 4, using (6.15), (6.19) and the properties of τ.

Proof of Claim 6.
Let us first assume $s \geqslant \frac{1}{2}$. We have

$$\mathcal{I}_s[\phi] = W'(\phi) = W'(\widetilde{\phi}) = W''(0)\widetilde{\phi} + O(\widetilde{\phi})^2.$$

We note that, since $s \geqslant \frac{1}{2}$, if $|x| \geqslant 1$ then

$$(\phi(x))^2 \leqslant \frac{C}{|x|^{4s}} \leqslant \frac{C}{|x|^{1+2s}}.$$

Let $x = i_0 + \gamma$ with $\gamma \in \left(-\frac{1}{2}, \frac{1}{2}\right]$, and $n > |i_0|$. From (6.12) we get

$$\begin{aligned}
\sum_{i=-n}^{n} \mathcal{I}_s[\phi, x_i] &= I[\phi, x_{i_0}] + \sum_{\substack{i=-n\\ i\neq i_0}}^{n} \mathcal{I}_s[\phi, x_i] \\
&= I[\phi, x_{i_0}] + \sum_{\substack{i=-n\\ i\neq i_0}}^{n} [\alpha\widetilde{\phi}(x_i) + O(\widetilde{\phi}(x_i))^2] \\
&\leqslant I[\phi, x_{i_0}] - \frac{\delta^{2s}|p_0|^{2s}}{2s} \sum_{\substack{i=-n\\ i\neq i_0}}^{n} \frac{x-i}{|x-i|^{1+2s}} + C \sum_{\substack{i=-n\\ i\neq i_0}}^{n} \frac{1}{|x-i|^{1+2s}},
\end{aligned}$$

for some $C > 0$ and

$$\sum_{i=-n}^{n} \mathcal{I}_s[\phi, x_i] \geqslant I[\phi, x_{i_0}] - \frac{\delta^{2s}|p_0|^{2s}}{2s} \sum_{\substack{i=-n\\ i\neq i_0}}^{n} \frac{x-i}{|x-i|^{1+2s}} - C \sum_{\substack{i=-n\\ i\neq i_0}}^{n} \frac{1}{|x-i|^{1+2s}}.$$

Then, by Claim 1 $\sum_{i=-n}^{n} \mathcal{I}_s[\phi, x_i]$ converges as $n \to +\infty$.

Let us consider now $\sum_{i=-n}^{n} \mathcal{I}_s[\psi, x_i]$. From the following estimate

$$\begin{aligned}
\mathcal{I}_s[\psi] &= W''(\widetilde{\phi})\psi + \frac{L}{\alpha}(W''(\widetilde{\phi}) - W''(0)) + c\phi' \\
&= W''(0)\psi + \frac{L}{\alpha}W'''(0)\widetilde{\phi} + O(\widetilde{\phi})\psi + O(\widetilde{\phi})^2 + c\phi',
\end{aligned}$$

(6.12), (6.14) and (6.17) we get

$$\sum_{i=-n}^{n} \mathcal{I}_s[\psi, x_i] \leqslant I[\psi, x_{i_0}] + \widetilde{C} \sum_{\substack{i=-n \\ i\neq i_0}}^{n} \frac{x-i}{|x-i|^{1+2s}} + C \sum_{\substack{i=-n \\ i\neq i_0}}^{n} \frac{1}{|x-i|^{1+2s}},$$

and

$$\sum_{i=-n}^{n} \mathcal{I}_s[\psi, x_i] \geqslant I[\psi, x_{i_0}] + \widetilde{C} \sum_{\substack{i=-n \\ i\neq i_0}}^{n} \frac{x-i}{|x-i|^{1+2s}} - C \sum_{\substack{i=-n \\ i\neq i_0}}^{n} \frac{1}{|x-i|^{1+2s}},$$

for some $\widetilde{C} \in \mathbb{R}$ and $C > 0$, which ensures the convergence of $\sum_{i=-n}^{n} \mathcal{I}_s[\psi, x_i]$.

Now, let us assume $s < \frac{1}{2}$. Fix k_0 such that $2s(2k_0+1) > 1$. Since W is even, $W^{2k+1}(0) = 0$ for any integer $k \geqslant 1$. Then

$$\mathcal{I}_s[\phi] = W'(\widetilde{\phi}) = W''(0)\widetilde{\phi} + W^{IV}(0)(\widetilde{\phi})^3 + ... + W^{2k_0}(0)(\widetilde{\phi})^{2k_0-1} + O((\widetilde{\phi})^{2k_0+1}).$$

Therefore, for $x = i_0 + \gamma$

$$\begin{aligned}\sum_{i=-n}^{n} \mathcal{I}_s[\phi, x_i] &= \mathcal{I}_s[\phi, x_{i_0}] + \sum_{\substack{i=-n \\ i\neq i_0}}^{n} \mathcal{I}_s[\phi, x_i] \\ &= \mathcal{I}_s[\phi, x_{i_0}] + W''(0) \sum_{\substack{i=-n \\ i\neq i_0}}^{n} \widetilde{\phi}(x_i) + W^{IV}(0) \sum_{\substack{i=-n \\ i\neq i_0}}^{n} (\widetilde{\phi}(x_i))^3 + ... \\ &+ W^{2k_0}(0) \sum_{\substack{i=-n \\ i\neq i_0}}^{n} (\widetilde{\phi}(x_i))^{2k_0-1} + \sum_{\substack{i=-n \\ i\neq i_0}}^{n} O((\widetilde{\phi}(x_i))^{2k_0+1}).\end{aligned}$$

The sequence $\sum_{\substack{i=-n \\ i\neq i_0}}^{n} O((\widetilde{\phi}(x_i))^{2k_0+1})$ is convergent since, by (6.13) behaves like $\sum_{i=1}^{n} \frac{1}{i^{2s(2k_0+1)}}$ which is convergent being the exponent $2s(2k_0+1)$ greater than 1. The convergence of the remaining sequences is assured by (6.20).

Finally, let us consider $\sum_{i=-n}^{n} \mathcal{I}_s[\psi\tau, x_i]$. The following formula, which can be found for instance in [1] page 7, holds true

$$\mathcal{I}_s[\psi\tau, x_i] = \tau(x_i)\mathcal{I}_s[\psi, x_i] + \psi(x_i)\mathcal{I}_s[\tau, x_i] - B(\psi, \tau)(x_i),$$

where

$$B(\psi, \tau)(x_i) = C(s) \int_{\mathbb{R}} \frac{(\psi(y) - \psi(x_i))(\tau(y) - \tau(x_i))}{|x-y|^{1+2s}} dy.$$

We remark that

$$|B(\psi, \tau)(x_i)| \leqslant C \int_{\mathbb{R}} \frac{|\tau(y) - \tau(x_i)|}{|x-y|^{1+2s}} dy = C \int_{\mathbb{R}} \frac{\tau(y)}{|x-y|^{1+2s}} dy = C\mathcal{I}_s[\tau, x_i],$$

for $i \neq i_0-1, i_0, i_0+1$. Indeed, as we have already pointed out $\tau(x_i) = 0$ for these indices. Therefore the sequences $\sum_{i=-n}^{n} \psi(x_i)\mathcal{I}_s[\tau, x_i]$ and $\sum_{i=-n}^{n} B(\psi, \tau)(x_i)$ converge by (6.21). Also $\sum_{i=-n}^{n} \tau(x_i)\mathcal{I}_s[\psi, x_i]$

is the sum of only three terms and then we can conclude that $\sum_{i=-n}^{n} \mathcal{I}_s[\psi\tau, x_i]$ is convergent as $n \to +\infty$. This concludes the proof of Claim 6.

REFERENCES

[1] B. BARRIOS, I. PERAL, F. SORIA AND E. VALDINOCI, A Widder's type theorem for the heat equation with nonlocal diffusion, *Arch. Ration. Mech. Anal.*, to appear.

[2] X. CABRÉ AND Y. SIRE, Nonlinear equations for fractional Laplacians II: existence, uniqueness, and qualitative properties of solutions. *Trans. Amer. Math. Soc.*, to appear.

[3] X. CABRÉ AND J. SOLÀ-MORALES, Layer solutions in a half-space for boundary reactions, *Comm. Pure Appl. Math.*, **58** (2005) no. 12, 1678-1732.

[4] S. DIPIERRO, A. FIGALLI AND E. VALDINOCI, Strongly nonlocal dislocation dynamics in crystals, *Comm. Partial Differential Equations*, to appear.

[5] S. DIPIERRO, G. PALATUCCI AND E. VALDINOCI, Dislocation dynamics in crystals: a macroscopic theory in a fractional Laplace setting, *Comm. Math. Phys.*, to appear.

[6] S. DIPIERRO, O. SAVIN AND E. VALDINOCI, All functions are locally s-harmonic up to a small error, *preprint*.

[7] M. GONZÁLEZ AND R. MONNEAU, Slow motion of particle systems as a limit of a reaction-diffusion equation with half-Laplacian in dimension one, *Discrete Contin. Dyn. Syst.*, **32** (2012), no. 4, 1255-1286.

[8] C. IMBERT AND R. MONNEAU, Homogenization of first order equations with u/ϵ-periodic Hamiltonians. Part I: local equations, *Arch. Ration. Mech. Anal.*, **187** (2008), no. 1, 49-89.

[9] C. IMBERT, R. MONNEAU AND E. ROUY, Homogenization of first order equations with u/ϵ-periodic Hamiltonians. Part II: application to dislocations dynamics, *Comm. Partial Differential Equations*, **33** (2008), no. 1-3, 479-516.

[10] R. MONNEAU AND S. PATRIZI, Homogenization of the Peierls-Nabarro model for dislocation dynamics, *J. Differential Equations*, **253** (2012), no. 7, 2064-2015.

[11] R. MONNEAU AND S. PATRIZI, Derivation of the Orowan's law from the Peierls-Nabarro model, *Comm. Partial Differential Equations*, **37** (2012), no. 10, 1887-1911.

[12] F. R. N. NABARRO, Fifty-year study of the Peierls-Nabarro stress, *Mat. Sci. Eng. A*, **234-236** (1997), 67-76.

[13] G. PALATUCCI, O. SAVIN AND E. VALDINOCI, Local and global minimizers for a variational energy involving a fractional norm, *Ann. Mat. Pura Appl.*, (4) **192** (2013), no. 4, 673-718.