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Asymptotic behaviour of a rigid body with a cavity filled by a viscous liquid

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Abstract. We consider the system of equations modeling the free motion of a rigid body with a cavity filled by a viscous (Navier-Stokes) liquid. We give a rigorous proof of Zhukovskiy's Theorem [24], which states that in the limit $t \to \infty$, the relative fluid velocity tends to 0 and the rigid velocity of the full structure tends to a steady rotation around one of the principle axes of inertia.

The existence of global weak solutions for this system was established in [20]. In particular, we prove that every weak solution of this type is subject to Zhukovskiy's Theorem. Independently of the geometry and of parameters, this shows that the presence of fluid prevents precession of the body in the limit. In general, we cannot predict which axis will be attained, but we show stability of the largest axis and provide criteria on the initial data which are decisive in special cases.

1. Introduction

We consider a system of equations describing the motion of a rigid body with a cavity filled by a viscous liquid. Let $\mathcal{S} \subset \mathbb{R}^3$ be a bounded closed domain which consists of a rigid body part $\mathcal{B}$, which is also closed, and an open connected cavity domain $\mathcal{F}$ which contains the fluid. In particular, $\mathcal{S} = \mathcal{B} \cup \mathcal{F}$ and there is no "leak", i.e. $\overline{\mathcal{F}} \cap \partial\mathcal{S} = \emptyset$. We assume that the boundary $\Gamma = \mathcal{B} \cap \overline{\mathcal{F}}$ of $\mathcal{F}$ is of class $C^{2,1}$. We impose no further restrictions on the geometry of $\mathcal{S}$.

Without loss of generality, we assume that the fluid has density $\rho_F = 1$ but the body's density is given by $\rho_B(y) > 0$, $y \in \mathcal{B}$. With $\mathcal{S}$ we associate the inertia tensor I given by

$$a^T \mathrm{I} b = \int_{\mathcal{F}} ((y - y_c) \times a) \cdot ((y - y_c) \times b)\,\mathrm{d}y + \int_{\mathcal{B}} ((y - y_c) \times a) \cdot ((y - y_c) \times b)\rho_B(y)\,\mathrm{d}y,$$

for all $a, b \in \mathbb{R}^3$, where y_c denotes the center of mass of $\mathcal{S}$. We provide more details on modeling in Section 2. In the absence of external forces or torques, the equations for the coupled motion of the fluid and the rigid body are given by

$$\text{(1.1)} \qquad \left\{ \begin{array}{rcll} \bar{u}' + \Omega' \times y - \nu\Delta\bar{u} + \nabla\bar{p} + 2\Omega \times \bar{u} + (\bar{u} \cdot \nabla)\bar{u} &=& 0, & \text{in } (0,\infty) \times \mathcal{F}, \\ \operatorname{div} \bar{u} &=& 0, & \text{in } (0,\infty) \times \mathcal{F}, \\ \bar{u} &=& 0, & \text{on } (0,\infty) \times \Gamma, \\ I\bar{\Omega}' + \Omega \times I\bar{\Omega} &=& 0, & t \in (0,\infty), \end{array} \right.$$

where ν is the viscosity, $\bar{u}$ is the *relative* velocity of the fluid, $\bar{p}$ a pressure potential and the angular velocity Ω of $\mathcal{B}$ and $\bar{\Omega}$ are related via

$$\text{(1.2)} \qquad \Omega = \bar{\Omega} - I^{-1} \int_{\mathcal{F}} y \times \bar{u}(y)\,\mathrm{d}y.$$

In this frame of reference, the fluid is driven by $\Omega' \times y$ and the Coriolis term $2\Omega \times \bar{u}$ and the rigid body dynamics are given by Euler's equations with a "fluid contribution".

In order to state our main result, we refer to Assumption 7.1 below, which asks that a weak solution for problem (1.1) satisfies conservation of momenta, the strong energy inequality and weak-strong uniqueness.

Theorem 1.1. *Let $(\bar{u}, \Omega)$ be a weak solution for* (1.1) *on $(0,\infty) \times \mathcal{S}$, satisfying Assumption 7.1. Then $\|\bar{u}(t)\|_{H^1(\mathcal{F})} \to 0$ and $\Omega(t) \to \Omega_\infty$ as $t \to \infty$, where $\Omega_\infty \in \mathbb{R}^3$ is a constant eigenvector of I.*

The claim of this result goes back to Zhukovskiy [24], and we recall his argument from [18, Chapter 2.2]: Since the relative fluid motion dissipates kinetic energy, it must come to rest as $t \to \infty$. If we plug $\bar{u} = 0$ into (1.1), then in the first line, only

$$\Omega' \times y = -\nabla\bar{p}$$

remains, where $\Omega' \times y$ has a potential only if $\Omega' = 0$. With $\Omega = \bar{\Omega}$ from (1.2), line 4 then implies that Ω is an eigenfunction of I.

This argument shows immediately that the liquid part cannot pretend to be rigid in general and that it excludes precession and the general Poinsot solutions to Euler's equations

$$I\bar{\Omega}' + \bar{\Omega} \times I\bar{\Omega} = 0$$

for the full structure. Note that in the presence of some dissipating mechanism, this reasoning would also hold for an inviscid fluid (i.e. the coupling of Euler's equation with the Euler equations). The aim of this paper is to give a rigorous proof of Zhukovskiy's argument for the viscous case.

There is a broad background on this problem in the engineering literature and we point to the monographs [18] and [11] for many more references, in which the main mathematical issue is the one of stability/instability of solutions for special geometries. A natural application of the model is the interpretation of the precession and nutation of planet earth, treated for example by Poincaré [19], and in [22], [10].

In mathematical analysis, there is an extensive literature on the *complement problem* of the movement of a free rigid body immersed in a fluid. We refer to [5] for a survey of this topic, and to [6], [4], [9] and [23] for additional existence and regularity theory. In the absence of external forces, this system is dissipative and both body and fluid must approach the rest state [3], but under the influence of external forces like gravity, many questions regarding asymptotics are still open.

For (1.1), global existence of weak solutions and local existence of mild and strong solutions were proved in [20]. We draw on their results and prove that these solutions satisfy the assumptions of Theorem 1.1. A result similar to ours as well as numerical studies of the problem were announced in [8].

In [15], local existence of regular solutions for the inviscid problem was proved. However, the main results of [15] and [16] are explicit criteria for the derivation of non-linear instability from linear instability in the inviscid limit. In particular, an instability result for uniform rotation around one axis is proved also for small viscosity. These results need a symmetry assumption for the structure around the unstable axis and they are based on spectral linear stability and instability for special geometries shown by Lyashenko [13], [14] and studied in [11]. Lyashenko and Friedlander's work directly addresses what is the main feature of this system from a mathematical point of view: It is given by the strong coupling of a non-dissipative and a dissipative part with limited access to the actual (Navier-Stokes) dissipation.

The general problem of specifying the limit angular velocity Ω_∞ from initial data thus seems to be very difficult and may not be solvable on the level of weak solutions. In Section 10, we combine the conservation of total angular momentum and dissipation of kinetic energy for this system in a very simple argument to define an open subset of initial data which will always approach the largest axis, proving stability in this sense. Making the set larger, we can still show that the smallest axis will not be attained from any of the initial data it contains. However, these estimates are crude, not depending on viscosity or the actual dissipation of energy and not sufficient for showing instability, e.g. of the "middle" axis, which might be expected from classical rigid body dynamics [17, Thm. 15.3.1].

The outline of the proof and the organization of the paper is as follows. Sections 2 to 4 mostly recount known results which are needed later on. In Section 2, we introduce the model, fix some notation and recall the change of coordinates to a Lagrangian formulation (with respect to the rigid body). In Section 3, we prove existence and continuous dependence on the data for local-in-time strong solutions. We recall the weak formulation and existence proof for global solutions given in [20] in Section 4.

In Section 5, we show that every weak solution given in [20] satisfies conservation of the total linear momentum and of the total angular momentum. Even though this shows that the kinetic energy

$$E(t) = \|\bar{u}(t, \cdot) + \Omega \times \cdot\|^2_{L^2(\mathcal{S})}$$

does not decay to zero in general in this system, we still want to show $\bar{u}(t) \to 0$. Since there is no stability, the usual uniform estimates (in the initial data) for the Navier-Stokes problem do not apply and we have to work "trajectory-wise". We provide a preparatory higher-order a priori estimate on $\bar{u}$ in Section 6. In Section 7, we prove that global weak solutions constructed in [20] satisfy the strong energy inequality and weak-strong uniqueness. In Section 8, these results are combined in order to prove that every weak solution becomes strong eventually and that the relative fluid velocity then goes to rest.

Section 9 concerns the second part of Zhukovskiy's argument and the asymptotics for $\Omega(t)$. We show that the kinetic energy is a strict Lyapunov functional on regular (large-time) trajectories and characterize the equilibrium set. We apply a version of LaSalle's invariance principle in order to prove Theorem 1.1.

Finally, in Section 10, we derive simple criteria which characterize the limit axis in special cases.

2. Model and Notation

In order to fix some notation and for technical reasons, we will first derive the model in Eulerian coordinates $x \in \mathbb{R}^3$, differing from (1.1). This implies that the positions $\mathcal{B}(t)$, $\mathcal{F}(t)$ and $\mathcal{S}(t)$ depend on time, with $\mathcal{S}(0) = \mathcal{S}$. The body's mass is given by $\mathrm{m}_B := \int_{\mathcal{B}} \rho_B(x)\,\mathrm{d}x$ and its inertia tensor $J_B(t)$ is given by

$$aJ_B(t)b = \int_{\mathcal{B}(t)} a \times (x - x_B(t)) \cdot b \times (x - x_B(t))\rho_{B(t)}(x)\,\mathrm{d}x \quad \text{for all } a, b \in \mathbb{R}^3.$$

The fluid motion is governed by the Navier-Stokes equations, driven by an initial velocity and a no-slip boundary condition at $\Gamma(t)$, where fluid and rigid body velocity must thus coincide. The rigid body's center of mass

$$x_B(t) = \frac{1}{\mathrm{m}_B} \int_{\mathcal{B}(t)} x\rho_{B(t)}(x)\,\mathrm{d}x$$

has a translational velocity η and the body rotates with an angular velocity ω with respect to x_B. It is driven by its initial velocity $\eta_0 + \omega_0 \times (x - x_B(0))$ and by the force $\int_{\Gamma(t)} \mathbf{T}(v,q)n(t)\,d\sigma$ and the torque $\int_{\Gamma(t)} (x - x_c(t)) \times \mathbf{T}(v,q)n(t)\,d\sigma$ exerted by the fluid velocity v and pressure q. Here,

$$\mathbf{T}(v,q) = 2\nu D(v) - q\mathrm{Id}$$

is the Newtonian fluid stress tensor given by a constant viscosity $\nu > 0$ and the symmetric part of the gradient $D(v) = \frac{1}{2}[(\nabla v) + (\nabla v)^T]$, where $n(t,x)$ denotes the outer normal of $\mathcal{B}(t)$ at $x \in \Gamma(t)$. The system may additionally be subject to external forces and torques l_0, l_1 and l_2 and in full, the equations read

$$\tag{2.1} \left\{ \begin{array}{rcll} v' - \operatorname{div}\mathbf{T}(v,q) + (v\cdot\nabla)v & = & l_0, & \text{in } \mathcal{Q}_{\mathcal{F}}, \\ \operatorname{div} v & = & 0, & \text{in } \mathcal{Q}_{\mathcal{F}}, \\ v(t,x) - \omega(t) \times (x - x_B(t)) - \eta(t) & = & 0, & \text{on } \mathcal{Q}_{\Gamma}, \\ v(0) & = & u_0 & \text{on } \mathcal{F}, \\ \mathrm{m}_B\eta' + \int_{\Gamma(t)} \mathbf{T}(v,q)n(t)\,d\sigma & = & l_1, & t > 0, \\ (J_B\omega)' + \int_{\Gamma(t)} (x - x_c(t)) \times \mathbf{T}(v,q)n(t)\,d\sigma & = & l_2, & t > 0, \\ \eta(0) = \xi_0 & \text{and} & \omega(0) = \Omega_0, & \end{array} \right.$$

where $\mathcal{Q}_{\mathcal{F}} := \{(t,x) \in (0,\infty) \times \mathbb{R}^3 : x \in \mathcal{F}(t)\}$ and $\mathcal{Q}_{\Gamma}$ is defined accordingly. In order to replace the non-cylindrical domain $\mathcal{Q}_{\mathcal{F}}$ with a cylindrical one, we change coordinates to a Lagrangian formulation with respect to the rigid body $\mathcal{B}$. In particular, x_B and J_B will become independent of time. Without loss of generality, we set $x_B(0) = 0$. This is a standard procedure for this type of problem, but we quickly repeat the construction as there is the technical detail of having to deal with three centers of mass, x_B, x_F and x_c and three inertia tensors I_B, I_F and I of body, fluid and the full structure. In particular, note that new coordinates are chosen with respect to x_B, but for $t \to \infty$, x_c and I are more relevant.

Let $m(t)$ denote the skew-symmetric matrix satisfying $m(t)x = \omega(t) \times x$. Note that in the following, we denote derivatives with respect to time by $\omega', v', \ldots$, regardless of whether they are full or partial. We consider the differential equation

$$\left\{ \begin{array}{rcll} X'(t,y) & = & m(t)(X(t,y) - x_B(t)) + \eta(t), & (t,y) \in (0,T) \times \mathbb{R}^3, \\ X(0,y) & = & y, & y \in \mathbb{R}^3. \end{array} \right.$$

As $\operatorname{div}(m(t)X(t,y)) = 0$, its solution is of the form $X(t,y) = Q(t)y + x_B(t)$, with some matrix $Q(t) \in \mathrm{SO}(3)$ for every $t \in (0,T)$. In particular, $Q \in H^2(0,T;\mathbb{R}^{3\times 3})$, if $\eta, \omega \in H^1(0,T)$, justifying this change of coordinates a posteriori for strong solutions. The corresponding inverse $Y(t)$ of $X(t)$ is given by

$$Y(t,x) = Q^T(t)(x - x_B(t)).$$

For $(t,y) \in [0,T) \times \mathbb{R}^3$, we thus define

$$
\begin{aligned}
u(t,y) &:= Q^T v(t, X(t,y)),\\
p(t,y) &:= q(t, X(t,y)),\\
\Omega(t) &:= Q^T(t)\omega(t),\\
\xi(t) &:= Q^T(t)\eta(t),\\
f_i(t,y) &:= Q^T l_i(t, X(t,y)).
\end{aligned}
\tag{2.2}
$$

It follows that the transformed inertia tensor $I_B = Q^T(t)J(t)Q(t)$ for the rigid body part $\mathcal{B}$ no longer depends on time and that for all $a, b \in \mathbb{R}^3$,

$$
aI_B b = \int_{\mathcal{B}} (a \times y) \cdot (b \times y)\, \rho_B(y) \mathrm{d}y. \tag{2.3}
$$

The definition

$$
T(u,p) := 2\nu D_y(u) - \nabla_y p,
$$

where D_y, ∇_y here explicitly indicate differentiation with respect to the new coordinates y, implies that

$$
\int_{\Gamma(t)} \mathbf{T}(v,q)n(t)\,\mathrm{d}\sigma = Q \int_\Gamma T(u,p)N\,\mathrm{d}\sigma
$$

and

$$
\int_{\Gamma(t)} (x - x_B(t)) \times \mathbf{T}(v,q)n(t)\,\mathrm{d}\sigma = Q \int_\Gamma y \times T(u,p)N\,\mathrm{d}\sigma.
$$

On the cylindrical domain $(0,T) \times \mathcal{F}$ with outer normal vector $-N = -Q^T(t)n(t)$, we obtain the system of equations

$$
\left\{
\begin{array}{rcll}
u' - \mu\Delta u + \nabla p + \Omega \times u + ((u - \xi - \Omega \times y)\cdot\nabla)u &=& f_0, & \text{in } (0,T)\times\mathcal{F},\\
\operatorname{div} u &=& 0, & \text{in } (0,T)\times\mathcal{F},\\
u(t,y) - \Omega(t)\times y - \xi(t) &=& 0, & \text{on } (0,T)\times\Gamma,\\
u(0) &=& u_0 & \text{in } \mathcal{F},\\
\mathrm{m}_B\xi' + \mathrm{m}_B(\Omega\times\xi) + \int_\Gamma T(u,p)N\,\mathrm{d}\sigma &=& f_1, & t \in (0,T),\\
I_B\Omega' + \Omega \times (I_B\Omega) + \int_\Gamma y \times T(u,p)N\,\mathrm{d}\sigma &=& f_2, & t \in (0,T),\\
\xi(0) = \xi_0 & \text{and} & \Omega(0) = \Omega_0, &
\end{array}
\right.
\tag{2.4}
$$

to be equivalent to (2.1) with the unknowns u, p the new fluid velocity and pressure and ξ, Ω the rigid body's translational and angular velocity. We denote the center of mass of the fluid part $\mathcal{F}$ by

$$
x_F(0) = y_F = \frac{1}{\mathrm{m}_F}\int_{\mathcal{F}} y\,\mathrm{d}y,
$$

and the center of mass of the full structure $\mathcal{S}$ by

$$
x_c(0) = y_c = \frac{1}{\mathrm{m}}\left[\int_{\mathcal{F}} y\,\mathrm{d}y + \int_{\mathcal{B}} y\rho_B(y)\,\mathrm{d}y\right] = \frac{m_F}{\mathrm{m}} y_F, \tag{2.5}
$$

where $m_F + m_B = \mathrm{m}$ is the total mass. We often use the calculation rules

$$
a \times (b \times c) = b(a\cdot c) - c(a \cdot b), \tag{2.6}
$$

$$
(a \times b)\cdot(c \times d) = (a\cdot c)(b \cdot d) - (a \cdot d)(b \cdot c), \tag{2.7}
$$

for all $a, b, c \in \mathbb{R}^3$. To the full structure, we associate the inertia tensor I calculated with respect to the center of mass y_c,

$$
a^T \mathrm{I} b = \int_{\mathcal{F}} ((y - y_c)\times a)\cdot((y-y_c)\times b)\,\mathrm{d}y + \int_{\mathcal{B}} ((y-y_c)\times a)\cdot((y-y_c)\times b)\rho_B(y)\,\mathrm{d}y
$$

as in Section 1. Using (2.3), (2.5) and (2.7), it follows that

$$
\mathrm{I}b = (I_B + I_F)b + \mathrm{m}y_c \times (y_c \times b), \tag{2.8}
$$

where I_F is the inertia tensor of $\mathcal{F}$, calculated with respect to the center of mass 0 of the rigid part.

In the following, to a triple u, ξ, Ω of solutions, we often associate the function

$$U(t,y) := \begin{cases} u(t,y), & y \in \mathcal{F}, \\ \Omega(t) \times y + \xi(t), & y \in \mathcal{B}, \end{cases} \tag{2.9}$$

and vice versa. To both, we associate the *relative fluid velocity*

$$\bar{u}(t,y) := u(t,y) - \xi(t) - \Omega(t).$$

Note that it is shown below that if U is a weak solution of (2.4), then

$$\bar{u}(t) \in H^1_0(\mathcal{F}) := \{u \in H^1(\mathcal{F}); u|_\Gamma = 0\}$$

for almost all t and, in addition, $\bar{u}(t) \in L^2_\sigma(\mathcal{F})$, where

$$L^2_\sigma(\mathcal{F}) := \{u \in L^2(\mathcal{F}); \operatorname{div} u = 0,\ u|_\Gamma \cdot n = 0 \,\text{in a weak sense}\}$$

is the usual space of solenoidal L^2-functions and $H^1(\mathcal{F})$ is the usual L^2-Sobolev space of order 1. We define

$$H := L^2_\sigma(\mathcal{F}) \cap H^1_0(\mathcal{F})$$

and for $1 \le q \le \infty$, $\|\cdot\|_q := \|\cdot\|_{L^q(\mathcal{F})}$ denotes the L^q-norm on $\mathcal{F}$. We often apply Poincaré's inequality to $\bar{u}$ with constant C_p, $\|\bar{u}\|_2 \le C_p \|\nabla \bar{u}\|_2$.

3. Local-in-time existence of strong solutions

For sufficiently regular solutions, it is required that the initial data u_0, ξ_0, Ω_0 satisfy the compatibility condition

$$U_0 \in \mathcal{W} := \{(u_0.\xi_0, \Omega_0) \in H^1(\mathcal{F}) \times \mathbb{R}^6; \operatorname{div} u_0 = 0,\ u_0|_\Gamma(y) = \Omega_0 \times y + \xi_0\},$$

where we refer to [9, Rem. 2.3c)] for a discussion of this constraint in this context. In particular, $\mathcal{W}$ is the time-trace space for the strong solution and it follows that $\bar{u}_0 \in H$.

Theorem 3.1. *Let $U_0 \simeq (u_0, \xi_0, \Omega_0) \in \mathcal{W}$ and*

$$F := (f_0, f_1, f_2) \in L^2(0, T_0; L^2(\mathcal{F})) \times L^2(0, T_0; \mathbb{R}^6) =: \mathcal{V}^{T_0}$$

be given. Then there exists $0 < T_{max} \le T_0$ such that problem (2.1) *admits a unique strong solution*

$$\begin{aligned} u &\in L^2(0,T; H^2(\mathcal{F})) \cap H^1(0,T; L^2(\mathcal{F})) =: X^T_{2,2} \\ \nabla p &\in L^2(0,T; L^2(\mathcal{F})), \\ (\Omega, \xi) &\in H^1(0,T; \mathbb{R}^6), \end{aligned}$$

for all $0 < T < T_{max}$. Moreover, the solution in these spaces depends continuously on the data (U_0, F) in $\mathcal{W} \times \mathcal{V}^{T_{max}}$.

We prove this result almost exactly as in the "complement case" of a rigid body immersed in a viscous liquid (filling a bounded or exterior domain). Note that for our exact situation, a proof was given already in [20], however, we need to recall some arguments in order to justify continuous dependence on the data and Corollary 6.1 below. The proof here uses maximal regularity-type estimates of the linearized problem in $L^2(\mathcal{F}) \times \mathbb{R}^3 \times \mathbb{R}^3$ and the contraction mapping principle. A suitable linearization of (2.4) is exactly the same as for the complement problem, except that here, we do not need to include an additional boundary condition at $\partial\mathcal{S}$, i.e. it is given by

$$\left\{ \begin{array}{rcll} u_t - \Delta u + \nabla p &=& f_0, & \text{in } (0,T) \times \mathcal{F}, \\ \operatorname{div} u &=& 0, & \text{in } (0,T) \times \mathcal{F}, \\ \bar{u} &=& 0, & \text{on } (0,T) \times \Gamma, \\ u(0) &=& u_0 & \text{in } \mathcal{F}, \\ \mathrm{m}_B \xi' + \int_\Gamma T(u,p) N \,\mathrm{d}\sigma &=& f_1, & t \in (0,T), \\ I_B \Omega' + \int_\Gamma y \times T(u,p) N \,\mathrm{d}\sigma &=& f_2, & t \in (0,T), \\ \xi(0) = \xi_0 & \text{and} & \Omega(0) = \Omega_0. & \end{array} \right. \tag{3.1}$$

Thus, we cite the following result from [9, Thm. 4.1].

Proposition 3.2. *For all $(U_0, F) \in \mathcal{W} \times \mathcal{V}^{T_0}$, there is a unique solution*

$$\begin{array}{rcl} u & \in & X_{2,2}^{T_0} \\ \nabla p & \in & L^2(0,T_0;L^2(\mathcal{F})), \\ (\xi,\Omega) & \in & H^1(0,T_0;\mathbb{R}^6) \end{array}$$

to (3.1)*, which satisfies*

$$\|u\|_{X_{2,2}^{T_0}} + \|\nabla p\|_{L^2(0,T_0;L^2(\mathcal{F}))} + \|\xi\|_{H^1(0,T_0)} + \|\Omega\|_{H^1(0,T_0)} \le C_{MR}(\|F\|_{\mathcal{V}^{T_0}} + \|U_0\|_{\mathcal{W}}),$$

where the constant C_{MR} depends on geometry and material parameters and on T_0.

In order to prove Theorem 3.1, we rewrite (2.4) as a fixed point equation in U. We denote by $u^*, p^*, \xi^*, \Omega^*$ the unique solution of (3.1). Let $\hat{u} = u - u^*$,$\hat{p} = p - p^*$, $\hat{\xi} = \xi - \xi^*$, $\hat{\Omega} = \Omega - \Omega^*$ and $\hat{\bar{u}} = \bar{u} - \bar{u}^*$. Then (2.4) is equivalent to

$$(3.2)\qquad \left\{ \begin{array}{rcll} \hat{u}_t - \Delta\hat{u} + \nabla\hat{p} & = & \mathcal{R}_0(\hat{u},\hat{\xi},\hat{\Omega}), & \text{in } (0,T)\times\mathcal{F}, \\ \operatorname{div}\hat{u} & = & 0, & \text{in } (0,T)\times\mathcal{F}, \\ \hat{\bar{u}} & = & 0, & \text{on } (0,T)\times\Gamma, \\ \hat{u}(0) & = & 0, & \text{in } \mathcal{F}, \\ \mathrm{m}_B(\hat{\xi})' + \int_\Gamma T(\hat{u},\hat{p})N\,\mathrm{d}\sigma & = & \mathcal{R}_1(\hat{\xi},\hat{\Omega}), & \text{in } (0,T), \\ I_B(\hat{\Omega})' + \int_\Gamma y \times T(\hat{u},\hat{p})N\,\mathrm{d}\sigma & = & \mathcal{R}_2(\hat{\Omega}), & \text{in } (0,T), \\ \hat{\xi}(0) = 0 & \text{and} & \hat{\Omega}(0) = 0, & \end{array} \right.$$

where

$$\begin{array}{rcl} \mathcal{R}_0(\hat{u},\hat{\xi},\hat{\Omega}) & = & (\hat{\Omega}+\Omega^*)\times(\hat{u}+u^*) - ((\hat{\bar{u}}+\bar{u}^*)\cdot\nabla)(\hat{u}+u^*), \\ \mathcal{R}_1(\hat{\xi},\hat{\Omega}) & = & \mathrm{m}(\hat{\xi}+\xi^*)\times(\hat{\Omega}+\Omega^*), \\ \mathcal{R}_2(\hat{\Omega}) & = & (\hat{\Omega}+\Omega^*)\times I(\hat{\Omega}+\Omega^*). \end{array}$$

Given a fixed T_0, we define

$$C_* := \|U^*\|_{\mathcal{X}^{T_0}} = C_{MR}(\|U_0\|_{\mathcal{W}} + \|f\|_{\mathcal{V}^{T_0}}).$$

The solution should satisfy, for some $T \le T_0$,

$$\begin{array}{rcl} \hat{u} & \in & X_{2,2,0}^T := \{\hat{u} \in X_{2,2}^T : \hat{u}|_{t=0} = 0\}, \\ \nabla\hat{p} & \in & L^2(0,T;L^2(\mathcal{F})) \text{ and} \\ \hat{\xi},\hat{\Omega} & \in & H_0^1(0,T) := \{\hat{r} \in H^1(0,T) : \hat{r}|_{t=0} = 0\}, \end{array}$$

so we choose the ball $\mathcal{X}_R^T$ as

$$\mathcal{X}_R^T := \{U \in X_{2,2,0}^T \times H_0^1(0,T;\mathbb{R}^3) \times H_0^1(0,T;\mathbb{R}^3) : \|U\|_{\mathcal{X}^T} \le R\}.$$

In (3.6) below, we see that $R := C_*$ is optimal. Let

$$\phi_R^T : U^R \mapsto \begin{pmatrix} \mathcal{R}_0(u^R,\xi^R,\Omega^R) \\ \mathcal{R}_1(\xi^R,\Omega^R) \\ \mathcal{R}_2(\Omega^R) \end{pmatrix} \mapsto U$$

be the function which maps $U^R \in \mathcal{X}_R^T$ to the solution U of the linear problem (3.1) with right hand sides $\mathcal{R}_0(u^R,\xi^R,\Omega^R), \mathcal{R}_1(\xi^R,\Omega^R), \mathcal{R}_2(\Omega^R)$ and initial value $U_0 = 0$.

In the following, $C > 0$ denotes a generic constant which may depend on T_0, but can be chosen independently of T, R for $0 < T \le T_0$. Assume $U^R, U_1^R, U_2^R \in \mathcal{X}_R^T$ and set $U = U^R + U^*$, $U_1 = U_1^R + U_1^*$ etc.

We use the estimates

$$\|U\|_{\mathcal{X}^T} \le R + C_* \text{ and } \|\bar{u}\|_{X_{2,2}^T} \le C\|U\|_{\mathcal{X}^T}$$

and that $u \in X_{2,2}^T$ satisfies

$$(3.3)\qquad u \in BUC([0,T];H^1(\mathcal{F})),$$

which follows from [1, III.4.10] and by the construction of u in [9, Section 4]. Here, $BUC([0,T)$ denotes the space of bounded uniformly continuous functions on $[0,T]$ and we note that the embedding constant does not depend on $T \le T_0$ if it is restricted to the subspace $X^T_{2,2,0}$ of $X^T_{2,2}$. It follows that

$$\|\nabla u\|_{2,\infty} + \|\nabla \bar{u}\|_{2,\infty} \le C(R + C_*). \tag{3.4}$$

The function ϕ^T_R maps $\mathcal{X}^T_R$ into itself, if it is shown that

$$\|\phi^T_R(U^R)\|_{\mathcal{X}^T} \quad \le \quad C_{MR}\big(\|\mathcal{R}_0(u^R,\xi^R,\Omega^R), \mathcal{R}_1(\xi^R,\Omega^R), \mathcal{R}_2(\Omega^R)\|_{\mathcal{V}^T}\big) \le R.$$

The term $((\hat{\bar{u}} + \bar{u}^*)\cdot\nabla)(\hat{u} + u^*)$ in this estimate is treated as usual for the Navier-Stokes problem, i.e. note that $\bar{u} \in X^T_{2,2}$ if

$$U \simeq (u,\Omega,\xi) \in X^T_{2,2} \times H^1(0,T) \times H^1(0,T) := \mathcal{X}^T$$

and that in addition, $\bar{u} = 0$ on Γ. Given $u, \bar{u} \in X^T_{2,2}$ such that $\bar{u} \in H^1_0(\mathcal{F})$, we have by Hölder's inequality,

$$\|(\bar{u}\cdot\nabla)u\|_{2,2} \le \|\bar{u}\|_{2,\infty}\|\nabla u\|_{\infty,2}.$$

Since $\mathcal{F}$ is bouded, we have $\|\bar{u}(t)\|_\infty \le C(1+C_p)\|\nabla\bar{u}\|_{3+\varepsilon}$ for every $\varepsilon > 0$. The Riesz-Thorin Theorem yields $\|\nabla\bar{u}\|_{3+\varepsilon} \le 2\|\nabla\bar{u}\|_2^\theta\|\nabla\bar{u}\|_6^{(1-\theta)}$ where $\theta = \frac{1}{2}\frac{3-\varepsilon}{3+\varepsilon}$. In conclusion,

$$\begin{aligned}
\|(\bar{u}\cdot\nabla)u\|_{2,2} \quad &\le \quad \|\bar{u}\|_{2,\infty}\|\nabla u\|_{\infty,2} \\
&\le \quad C\Big(\int_0^T \|\nabla\bar{u}(t)\|_2^{2\theta}\|\nabla\bar{u}(t)\|_6^{2(1-\theta)}\,\mathrm{d}t\Big)^{1/2}\|\nabla u\|_{\infty,2} \\
&\le \quad C\|\nabla u\|_{\infty,2}\|\nabla\bar{u}\|^\theta_{\infty,2}\|\nabla\bar{u}\|^{(1-\theta)}_{2(1-\theta),6} \\
&\le \quad CT^{\theta/2}\|\nabla u\|_{\infty,2}\|\nabla\bar{u}\|^\theta_{\infty,2}\|D^2\bar{u}\|^{(1-\theta)}_{2,2}. \qquad (3.5)
\end{aligned}$$

In addition (not optimally),

$$\|\Omega\times u\|_{2,2} \le \|\Omega\|_{2,\infty}\|u\|_{\infty,2} \le CT^{1/2}(R+C_*)^2,$$

so that by (3.5), (3.3) and (3.4), we obtain

$$\|\mathcal{R}_0(u^R,\xi^R,\Omega^R)\|_{L^2(0,T;L^2(\mathcal{F}))} \quad \le \quad C(T^{1/2} + T^{\theta/2})(R+C_*)^2,$$

and (again not optimally),

$$\|\mathcal{R}_1(\xi^R,\Omega^R)\|_{L^2(0,T)} + \|\mathcal{R}_2(\Omega^R)\|_{L^2(0,T)} \quad \le \quad CT^{1/2}(R+C_*)^2.$$

Thus, ϕ^T_R maps into $\mathcal{X}^T_R$ if

$$T^{1/2} + T^{\theta/2} \le \frac{R}{CC_{MR}(R+C_*)^2}. \tag{3.6}$$

A simple argument shows that $R = C_*$ maximizes T with $T(C_*)^{1/2} + T(C)_*^{\theta/2} = \frac{1}{4CC_{MR}C_*}$. Via the estimate

$$\|\phi^T_R(U^R_1) - \phi^T_R(U^R_2)\|_{\mathcal{X}^T} \quad \le \quad CC_*(T^{1/2} + T^{\theta/2})\|U^R_1 - U^R_2\|_{\mathcal{X}^T},$$

the map $\phi^T_{C_*}$ is contractive for small T. The contraction mapping theorem yields a unique fixed point $\hat{U}$ of $\phi^T_{C_*}$ which gives a corresponding pressure $\hat{p}$ and a strong solution $(\hat{u},\hat{p},\hat{\xi},\hat{\Omega})$ of problem (3.2). At the same time, we can deduce continuous dependence of the solution on the data in the following sense. Given $U_0, V_0 \in \mathcal{W}$ and $F, G \in \mathcal{V}^{T_0}$, there are solutions $U^*, V^* \in \mathcal{X}^{T_0}$ of the linear problem (3.1) and solutions $U \in \mathcal{X}^{T(C_*(U_0,f))}$, $V \in \mathcal{X}^{T(C_*(V_0,g))}$ of (2.4). Their difference can be estimated by

$$\begin{aligned}
\|U - V\|_{\mathcal{X}^T} \quad &\le \quad C_{MR}\|\mathcal{R}(U+U^*) - \mathcal{R}(V+V^*)\|_{2,2\times L^2(0,T)\times L^2(0,T)} \\
&\le \quad CC_{MR}\max(C_*)(T^{1/2}+T^{\theta/2})(\|U-V\|_{\mathcal{X}^T} + \|U^*-V^*\|_{\mathcal{X}^T}),
\end{aligned}$$

where $T = \min(T(C_*))$. Since $CC_{MR}\max(C_*)(T^{1/2}+T^{\theta/2}) < 1$, and U^* and V^* depend linearly on the data, we obtain

$$\|U - V\|_{\mathcal{X}^T} \le C(T,\max(C_*))\left(\|U_0 - V_0\|_{\mathcal{W}} + \|F - G\|_{\mathcal{V}^{T_0}}\right).$$

The maximal time T_{max} of existence for these solutions is characterized as follows. Either $T_{max} = \infty$ or one of the functions

$$t \mapsto \|u(t)\|_{H^1}, \quad t \mapsto |\Omega(t)|, \quad t \mapsto |\xi(t)| \tag{3.7}$$

blows up as $t \to T_{max}$, because otherwise, the solution could be extended. In (**??**) below, we show that this condition is equivalent to the blow-up of $t \mapsto \|\bar{u}(t)\|_H$. Note that continuous dependence on the data extends to T_{max}. This proves Theorem 3.1.

4. Global-in-time existence of weak solutions

From here, we assume that there are no external forces or torques driving the system, i.e. $f_0, f_1, f_2 = 0$. We cite from [20, Def. 5.5, Thm. 5.6] the global existence of weak solutions for (2.4). Let us first introduce some notation. We write

$$\begin{aligned}
\mathcal{L} &:= \{U \in L^2(\mathcal{S}) : \operatorname{div} U = 0 \text{ in } \mathcal{S}, D(U) = 0 \text{ in } \mathcal{B}\} \\
\mathcal{H} &:= \{U \in H^1(\mathcal{S}) : \operatorname{div} U = 0 \text{ in } \mathcal{S}, D(U) = 0 \text{ in } \mathcal{B}\}.
\end{aligned}$$

By $\langle \cdot, \cdot \rangle_{\mathcal{H}}$, we denote the duality product between $\mathcal{H}$ and $\mathcal{H}'$. For a detailed discussion and characterization of these spaces, we refer to [20, Sections 3,4]. Here, we note that each $U \in \mathcal{L}$ can be characterized as

$$U|_{\mathcal{B}}(y) = \Omega_U \times y + \xi_U \tag{4.1}$$

for some $\Omega_U, \xi_U \in \mathbb{R}^3$, so that the identifications in (2.9) apply. Moreover, we note that $L^2(\mathcal{S})$ is endowed with the measure $\rho \, \mathrm{d}y$, where

$$\rho(y) = \begin{cases} \rho_B(y), & y \in \mathcal{B}, \\ 1, & y \in \mathcal{F}. \end{cases}$$

The symmetric continuous bilinear form $a : \mathcal{H} \times \mathcal{H} \to \mathbb{R}$ is given by

$$a(U, W) := 2\nu \int_{\mathcal{F}} D(U) : D(W) \, \mathrm{d}y$$

and the trilinear form $b : \mathcal{H} \times \mathcal{H} \times \mathcal{H} \to \mathbb{R}$ is given by

$$b(U, V, W) := m_B(\Omega_V \times \xi_U) \cdot \xi_W + \Omega_V \times I_B \Omega_U \cdot \Omega_W + \int_{\mathcal{F}} (\bar{v} \cdot \nabla u) \cdot w \, \mathrm{d}y + \int_{\mathcal{F}} \Omega_V \times u \cdot w \, \mathrm{d}y.$$

Theorem 4.1. *Let $U_0 \in \mathcal{L}$. Then there exists a $U \in L^\infty(0, \infty; \mathcal{L}) \cap L^2_{loc}([0, \infty); \mathcal{H}) \cap C_w([0, \infty); \mathcal{L})$ with $U' \in L^{4/3}([0, \infty); \mathcal{H}')$ such that for all $\phi \in \mathcal{H}$,*

$$\langle U', \phi \rangle + a(U, \phi) + b(U, U, \phi) = 0 \quad \text{a.e. in } (0, \infty), \tag{4.2}$$

and $U(0) = U_0$ is attained in the weak sense. In particular, for all $t > 0$, U satisfies the energy inequality

$$\|U(t)\|^2_{L^2(\mathcal{S})} + 4\nu \int_0^t \|D(u)(s)\|_2^2 \, \mathrm{d}s \le \|U_0\|^2_{L^2(\mathcal{S})}. \tag{4.3}$$

Note that by a direct calculation and the identifications in (2.9), every strong solution u, ξ, Ω given by Theorem 3.1 provides a weak solution U on the interval $(0, T_{max})$.

5. Conservation of Momenta

Let $U_0 \in \mathcal{L}$ and U be a weak solution given by Theorem 4.1. We define

$$L(t) := \mathrm{m}\xi(t) + \mathrm{m}_F \Omega(t) \times y_F$$

to be the *total linear momentum* of the system, where the second term is due to the fact that Ω is calculated with respect to the center of mass y_B of the rigid body and not with respect to the center of mass of the full structre, y_c. We denote the *total angular momentum* of the system by

$$A(t) := \int_{\mathcal{F}} y \times \bar{u}(t, y) \, \mathrm{d}y + I\Omega(t). \tag{5.1}$$

Lemma 5.1. *Let U be a weak solution given by Theorem 4.1. Then*

$$L'(t) + \Omega(t) \times L(t) = 0, \quad L(0) = \mathrm{m}\xi_0 + \mathrm{m}_F \Omega_0 \times y_F =: L_0, \tag{5.2}$$

and

$$A'(t) + \Omega(t) \times A(t) = 0, \quad A(0) = \int_{\mathcal{F}} y \times \bar{u}_0(y)\,\mathrm{d}y + I\Omega_0 =: A_0. \tag{5.3}$$

In particular, for all $t \geq 0$,

$$\frac{\mathrm{d}}{\mathrm{dt}}|A(t)|^2 = \frac{\mathrm{d}}{\mathrm{d}t}|L(t)|^2 = 0, \qquad |A(t)| = |A_0|, |L(t)| = |L_0|. \tag{5.4}$$

Proof. In the weak formulation (4.2), for $j \in \{1, 2, 3\}$, we take the j-th unit vector e_j as well as the functions $y \times e_j$ as test functions ϕ. In [20, p. 18], it is shown that, by integration by parts and an approximation argument, it follows that

$$\frac{\mathrm{d}}{\mathrm{dt}}(\mathrm{m}_B\xi(t) + \int_{\mathcal{F}} u(t)\,\mathrm{d}y) = -\Omega(t) \times (\mathrm{m}_B\xi(t) + \int_{\mathcal{F}} u(t)\,\mathrm{d}y)$$

and

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(I_B\Omega(t) + \int_{\mathcal{F}} y \times u(t,y)\,\mathrm{d}y\right) = -\Omega(t)\times\left(I_B\Omega(t) + \int_{\mathcal{F}} y \times u(t,y)\,\mathrm{d}y\right) - \xi(t)\times\int_{\mathcal{F}} u(t)\,\mathrm{d}y. \tag{5.5}$$

Since $\bar{u}(t) \in L^2_\sigma(\mathcal{F})$ for every weak solution, $\int_{\mathcal{F}} \bar{u}(t)\,\mathrm{d}y = 0$ and (5.2) follows directly. By (2.8),

$$I_B\Omega(t) + \int_{\mathcal{F}} y \times u(t,y)\,\mathrm{d}y = A(t) + \frac{\mathrm{m}_F}{\mathrm{m}} y_F \times L(t)$$

and by (2.6),

$$\Omega \times (A + \frac{\mathrm{m}_F}{\mathrm{m}} y_F \times L) = \Omega \times A + \frac{\mathrm{m}_F}{\mathrm{m}} y_F \times (\Omega \times L) - \frac{\mathrm{m}_F}{\mathrm{m}} L \times (\Omega \times y_F),$$

so that by (5.5) and (5.2),

$$\frac{\mathrm{d}}{\mathrm{d}t}A(t) = -\Omega(t) \times A(t) + \mathrm{m}_F\xi \times (\Omega \times y_F) - \xi(t) \times \int_{\mathcal{F}} u(t)\,\mathrm{d}y.$$

Again since $\bar{u}(t) \in L^2_\sigma(\mathcal{F})$, $\xi \times \int_{\mathcal{F}} u\,\mathrm{d}y = \xi \times (\Omega \times \mathrm{m}_F y_F)$, which shows that equation (5.3) holds. (5.4) follows by multiplying (5.2) by L and (5.3) by A, respectively. □

Remark 5.2. Lemma 5.1 shows that in the inertial frame, the total momenta $l(t) := Q(t)L(t)$ and $a(t) := Q(t)A(t)$ are conserved, i.e. $l(t) = L_0$ and $a(t) = A_0$ by (2.2). In particular, if $y_F = x_F(0) = x_B(0)$, we obtain a constant translational movement of the system in the inertial frame, $\eta'(t) = 0$.

Since $|A(t)|$ is conserved by the system, for the study of asymptotic behavior of solutions it is convenient to associate a rigid angular velocity $\bar{\Omega}(t) := I^{-1}A(t)$ to A for all $t \geq 0$. It follows that

$$\mathrm{I}\bar{\Omega}' + \Omega \times \mathrm{I}\bar{\Omega} = 0$$

and that

$$|\mathrm{I}\bar{\Omega}(t)| = |I\bar{\Omega}(0)| = |A_0|$$

for all $t \geq 0$. We define the rigid angular velocity of the system relative to $\bar{\Omega}$ as

$$\tilde{\Omega} := \Omega - \bar{\Omega},$$

and note that this implies

$$\int_{\mathcal{F}} y \times \bar{u}(y)\,\mathrm{d}y = -\mathrm{I}\tilde{\Omega} \tag{5.6}$$

by (5.1).

In order to state equations (2.4) in terms of $\bar{u}$ and $\bar{\Omega}$, we define a *relative* pressure

$$\bar{p}(y) := p(y) + (\xi' + \Omega \times \xi) \cdot y + \frac{1}{2}|\Omega \times y|^2$$

and note that the first line in (2.4) can be expressed as

$$\bar{u}' + \Omega' \times y - \nu\Delta\bar{u} + \nabla\bar{p} + 2\Omega \times \bar{u} + (\bar{u} \cdot \nabla)\bar{u} \;\; = \;\; 0, \quad \text{in } (0,T) \times \mathcal{F}, \tag{5.7}$$

where $\bar{p}$ has absorbed all dependence on ξ. Moreover, Lemma 5.1 shows that given Ω, the translational velocity ξ can be calculated a posteriori and that the translational movement of the center of mass can be decoupled from the remaining system in both the weak and the strong setting. An equivalent formulation of (2.4) in terms of $\bar{u}$ and $\bar{\Omega}$ is thus given by

$$
(5.8) \qquad \begin{cases} \bar{u}' + \Omega' \times y - \nu\Delta\bar{u} + \nabla\bar{p} + 2\Omega \times \bar{u} + (\bar{u}\cdot\nabla)\bar{u} &= 0, \quad \text{in } (0,T)\times\mathcal{F}, \\ \operatorname{div}\bar{u} &= 0, \quad \text{in } (0,T)\times\mathcal{F}, \\ \bar{u} &= 0, \quad \text{on } (0,T)\times\Gamma, \\ I\bar{\Omega}' + \Omega\times I\bar{\Omega} &= 0, \quad t\in(0,T), \end{cases}
$$

with initial conditions $\bar{u}(0) = \bar{u}_0$, $I\bar{\Omega}(0) = A_0$. This reduction naturally also shows in the kinetic energy $E(t) := \|U(t)\|^2_{L^2(\mathcal{S})}$. For strong solutions of the full system (2.4), corresponding to (4.3), we obtain the energy equation

$$
(5.9) \qquad E(t) + \int_s^t \|\nabla\bar{u}(\tau)\|_2^2 \,\mathrm{d}\tau = E(s) \quad \text{for all } 0 \le s < t \le T_{max}.
$$

Using (5.6) and $\int_{\mathcal{F}} \bar{u}(t)\,\mathrm{d}y = 0$, we calculate, both for weak and strong solutions,

$$
\begin{aligned}
\|U(t)\|^2_{L^2(\mathcal{S})} &= \|\bar{u}(t) + \Omega(t)\times\cdot + \xi(t)\|^2_{L^2(\mathcal{S})} \\
&= \|\bar{u}(t)\|_2^2 - 2\tilde{\Omega}I\tilde{\Omega}(t) + 2\bar{\Omega}I\tilde{\Omega}(t) + \|\Omega(t)\times\cdot\|^2_{L^2(\mathcal{S})} + \mathrm{m}|\xi(t)|^2 + 2\xi(t)\cdot(\Omega(t)\times y_c).
\end{aligned}
$$

Note that by (2.8),

$$
\|\Omega\times\cdot\|^2_{L^2(\mathcal{S})} = \Omega I\Omega + \mathrm{m}|\Omega\times y_c|^2,
$$

so $E(t) = \|U(t)\|^2 = \|\bar{u}(t)\|_2^2 - \tilde{\Omega}I\tilde{\Omega}(t) + \bar{\Omega}I\bar{\Omega}(t) + \frac{1}{\mathrm{m}}|L(t)|^2$. By Lemma 5.1, $\frac{\mathrm{d}}{\mathrm{d}t}|L(t)|^2 = 0$, so that we can ignore this contribution in (4.3) and (5.9) and abuse notation by referring to the kinetic energy

$$
(5.10) \qquad E(t) := \|\bar{u}(t)\|_2^2 - \tilde{\Omega}I\tilde{\Omega}(t) + \bar{\Omega}I\bar{\Omega}(t)
$$

in the following.

Remark 5.3. The kinetic energy $E(t) = \bar{E}(t) + \tilde{E}(t)$ splits into a rigid part

$$
\bar{E}(t) = \bar{\Omega}I\bar{\Omega}(t)
$$

and a *positive* “relative” fluid part

$$
\tilde{E}(t) = \|w(t)\|^2_{L^2(\mathcal{S})} = \|\bar{u}(t)\|_2^2 - \tilde{\Omega}I\tilde{\Omega} \ge 0,
$$

where $w \in L^2(\mathcal{S})$ is given by

$$
(5.11) \qquad \begin{cases} \bar{u}(t,y) + \tilde{\Omega}(t)\times y, & y\in\mathcal{F}, \\ \tilde{\Omega}(t)\times y, & y\in\mathcal{B}. \end{cases}
$$

As I is a positive matrix, we also have

$$
(5.12) \qquad \tilde{E}(t) \le \|\bar{u}(t)\|_2^2.
$$

Remark 5.4. There are two special cases in which the rigid part $\bar{E}(t)$ is constant in time. By (5.9), Gronwall's lemma, (5.12) and Poincaré's inequality, both cases imply exponential decay of $\tilde{E}(t)$. The first case is that of $|A_0| = 0$, which implies $\bar{\Omega}(t) = I^{-1}A(t) = 0$ for all $t \ge 0$. This is treated in [20] as the *orthogonality condition.* The second case is the one of $\mathcal{S}$ essentially being a sphere, i.e. $I = \mathrm{Id}_{\mathbb{R}^3}$ and $\bar{\Omega}(t) = I^{-1}A_0$ for all $t \ge 0$ by Lemma 5.1. In this situation, the full structure does not have any preferred direction of rotation, so that the fluid is only driven by its own inertia. We briefly consider this special case again in Section 10.

6. Properties of the semidynamical system $(\bar{u}, \bar{\Omega})$

By Theorem 3.1, given $U_0 \in \mathcal{W}$, there exists a strong solution $U \in \mathcal{X}^{T_{max}}$, which implies $\bar{u}, \bar{\Omega} \in X_{2,2}^{T_{max}} \times H^1(\mathbb{R}^3)$. From the embedding (3.3) and Lemma 5.1, we deduce

$$(\bar{u}, \bar{\Omega}) \in BUC([0,T]; H \times \Phi_{|A_0|}) \qquad \text{for every } 0 < T < T_{max},$$

where we recall $H = H_0^1(\mathcal{F}) \cap L_\sigma^2(\mathcal{F})$ and we define the ellipsoid

$$\Phi_{|A_0|} := \{r \in \mathbb{R}^3 : |Ir| = |A_0|\}.$$

For the remainder of the proof of Theorem 1.1, we choose an arbitrary but fixed $A_0 \in \mathbb{R}^3$ and, for every $\delta > 0$, we define

$$\mathcal{Z} := H \times \Phi_{|A_0|}, \quad \mathcal{Z}_\delta := \{z = (\bar{v}, a) \in \mathcal{Z} : \|\bar{v}\|_H \le \delta\},$$

endowed with the metric of $H \times \mathbb{R}^3$. For all $\delta > 0$, we define the semiflow $S_t^\delta : \mathcal{Z}_\delta \to \mathcal{Z}$ by $S_t^\delta(\bar{u}_0, I^{-1}A_0) = (\bar{u}(t), \bar{\Omega}(t))$ for $0 \le t < T_{max}$. In the following, we write S_t if no confusion is possible, as only the domain, but not the map itself depends on δ.

Corollary 6.1. *The family $(S_t)_t$ has the following properties.*

1. $(S_t)_{0 \le t < T_{\delta,max}}$ *defines a semidynamical system (cf.* [2, Def. 9.1.1]*) from $\mathcal{Z}_\delta$ to $\mathcal{Z}$, i.e.*
   (a) *for all $0 \le t \le T_{\delta,max}$, $S_t \in C(\mathcal{Z}_\delta, \mathcal{Z})$,*
   (b) $S_0 = \mathrm{Id}_{\mathcal{Z}}$,
   (c) *for all $0 \le t + s \le T_{\delta,max}$, $S_{t+s} = S_t \circ S_s$,*
   (d) *the function $t \mapsto S_t z$ is in $C([0, T_{\delta,max}); \mathcal{Z})$ for all $z \in \mathcal{Z}_\delta$.*
2. *For every $\delta > 0$, there is a time $0 < T_\delta \le T_{max}$ such that for all $z \in \mathcal{Z}_\delta$ and for all $0 \le t \le T_\delta$,*
$$\|\bar{u}(t)\|_H \le 2\delta.$$
3. *Given $|A_0|, \delta > 0$, we can choose an interval of existence of strong solutions uniformly in $\mathcal{Z}_\delta$, i.e. $0 < T_{\delta,max} := \inf_{(\bar{u}_0, I^{-1}A_0) \in \mathcal{Z}_\delta} T_{max}(\bar{u}_0, I^{-1}A_0)$ exists.*

Proof. (1a) follows from continuous dependence of strong solutions on their data, where $T_{\delta,max}$ is chosen as in (3). Given two initial data $z_0^1, z_0^2 \in \mathcal{Z}_\delta$ and corresponding solutions U^1 and U^2, by (3.3),

$$\sup_{t \in (0, T_{\delta,max})} \|S_t z_0^1 - S_t z_0^2\|_{\mathcal{Z}} \le C(T_{\delta,max}) \|U^1 - U^2\|_{\mathcal{X}_{2,2}^{T_{\delta,max}}} \to 0 \text{ as } z_0^1 \to z_0^2$$

by Theorem 3.1. Thus, for all $0 \le t \le T_{\delta,max}$, $S_t z_0^1 \to S_t z_0^2$ in $\mathcal{Z}$ as $z_0^1 \to z_0^2$ in $\mathcal{Z}$. (1d), (1b) follow directly from (3.3) and (1c) follows by definition.

In order to prove (2), we multiply the first line in (5.8) by $\bar{u}'$ and integrate over $\mathcal{F}$. Using integration by parts on $-\nu \int_{\mathcal{F}} \Delta \bar{u} \cdot \bar{u}' \, \mathrm{d}y$, which can be justified by approximation in the strong setting, we obtain:

$$\frac{\nu}{2} \frac{\mathrm{d}}{\mathrm{d}t} \|\nabla \bar{u}(t)\|_2^2 + \|\bar{u}'(t)\|_2^2 = \Omega' \cdot I\tilde{\Omega}'(t) - \int_{\mathcal{F}} (((\bar{u} \cdot \nabla)\bar{u}) \cdot \bar{u}')(t) \, \mathrm{d}y - 2 \int_{\mathcal{F}} ((\Omega \times \bar{u}) \cdot \bar{u}')(t) \, \mathrm{d}y. \tag{6.1}$$

The second term on the right-hand side satisfies $\int_{\mathcal{F}} ((\bar{u} \cdot \nabla)\bar{u}) \cdot \bar{u}' \, \mathrm{d}y \le \frac{\mu}{4} \|\bar{u}'\|_2^2 + C\|(\bar{u} \cdot \nabla)\bar{u}\|_2^2$, where $\mu > 0$ has to be a small constant which will be determined below in (6.4). We obtain

$$\begin{aligned} \|((\bar{u} \cdot \nabla)\bar{u})(t)\|_2^2 &\le \|\bar{u}(t)\|_6^2 \|\nabla \bar{u}(t)\|_3^2 \\ &\le C\|\bar{u}(t)\|_{H^1(\mathcal{F})}^2 \|\nabla \bar{u}(t)\|_2 \|\nabla \bar{u}(t)\|_6 \\ &\le C\|\nabla \bar{u}(t)\|_2^3 (\|\nabla \bar{u}(t)\|_2 + \|D^2 \bar{u}(t)\|_2), \end{aligned}$$

by Hölder's inequality, the Sobolev embedding $H^1(\mathcal{F}) \hookrightarrow L^6(\mathcal{F})$, interpolation for $L^3(\mathcal{F})$ and Poincaré's inequality. Moreover, $\bar{u}(t)$ satisfies the stationary Stokes problem

$$\left\{ \begin{array}{rcll} -\mu \Delta \bar{u} + \nabla \bar{p} &=& -\bar{u}' - \Omega' \times y - 2\Omega \times \bar{u} - (\bar{u} \cdot \nabla)\bar{u}, & \text{in } \mathcal{F}, \\ \operatorname{div} \bar{u} &=& 0, & \text{in } \mathcal{F}, \\ \bar{u} &=& 0, & \text{on } \Gamma, \end{array} \right.$$

almost everywhere in time. Thus, by properties of the Stokes operator (cf. e.g. [21]),

$$\|D^2 \bar{u}\|_2 \le C\left(\|\bar{u}'\|_2 + \|\Omega' \times y\|_2 + \|\Omega \times \bar{u}\|_2 + \|(\bar{u} \cdot \nabla)\bar{u}\|_2 + \|\nabla \bar{u}\|_2\right).$$

It follows that

$$\|(\bar u\cdot\nabla)\bar u\|_2^2 \le C\|\nabla\bar u\|_2^3\Big(\|\bar u'\|_2+|\tilde\Omega'|^2+|\bar\Omega'|^2+|\Omega|\|\bar u\|_2+\|(\bar u\cdot\nabla)\bar u\|_2+\|\nabla\bar u\|_2\Big)\,. \tag{6.2}$$

Note that $|\tilde\Omega|\le C\|\bar u\|_2$, $|\bar\Omega|\le C|A_0|$ and

$$\bar\Omega' I\bar\Omega' \le C(|A_0|^2+\|\bar u\|_2^2)$$

by (5.3). Thus, by Young's and Poincaré's inequalities, (6.2) implies

$$\frac12\|(\bar u\cdot\nabla)\bar u\|_2^2 \le C\|\nabla\bar u\|_2^6+\frac{\mu}{4}\|\bar u'\|_2^2+C\|\nabla\bar u\|^3\left(|A_0|^2+(1+|A_0|)\|\nabla\bar u\|_2+\|\nabla\bar u\|_2^2\right).$$

In conclusion, the second term on the right-hand side of (6.1) satisfies

$$|\int_{\mathcal F}((\bar u\cdot\nabla)\bar u)\cdot\bar u'\,\mathrm dy| \le \frac{\mu}{2}\|\bar u'\|_2^2+C\|\nabla\bar u\|_2^3\left(|A_0|^2+(1+|A_0|)\|\nabla\bar u\|_2+\|\nabla\bar u\|_2^2+\|\nabla\bar u\|_2^3\right).$$

The last term on the right-hand side of (6.1) satisfies

$$|\int_{\mathcal F}(\Omega\times\bar u)\cdot\bar u'\,\mathrm dy| \le C(|A_0|+\|\bar u\|_2^2)\|\bar u\|_2^2+\frac{\mu}{4}\|\bar u'\|_2^2$$

and for the first term on the right-hand side of (6.1), we obtain

$$\begin{aligned}\bar\Omega' I\tilde\Omega' &\le C\bar\Omega' I\bar\Omega'+\frac{\mu}{4}\|\bar u'\|_2^2\\ &\le C(|A_0|^2+\|\bar u\|_2^2)+\frac{\mu}{4}\|\bar u'\|_2^2,\end{aligned}$$

so that (6.1) becomes

$$\begin{aligned}&\frac12\frac{\mathrm d}{\mathrm dt}\|\nabla\bar u\|_2^2+(1-\mu)\|\bar u'\|_2^2-\tilde\Omega' I\tilde\Omega' \\ \le\ & C|A_0|^2+C\|\nabla\bar u\|_2^2\left(1+|A_0|+|A_0|^2\|\nabla\bar u\|_2+(1+|A_0|)\|\nabla\bar u\|_2^2+\|\nabla\bar u\|_2^3+\|\nabla\bar u\|_2^4\right).\end{aligned}\tag{6.3}$$

Note that by (5.11), we obtain

$$(1-\mu)\|\bar u'\|_2^2-\tilde\Omega' I\tilde\Omega' = (1-\mu)\|w'\|_2^2+(1-2\mu)\tilde\Omega' I_B\tilde\Omega'-\mu\tilde\Omega' I_F\tilde\Omega'\ge 0, \tag{6.4}$$

if μ is chosen sufficiently small, depending on the "ratio" of I_B and I_F. Integrating (6.3) in time yields

$$\|\nabla\bar u\|_2^2(t)\le\|\nabla\bar u_0\|_2^2+Ct|A_0|^2+\sum_{i=2}^{6}C(i,|A_0|)\int_0^t\|\nabla\bar u\|_2^i(s)\,\mathrm ds. \tag{6.5}$$

Now it is clear that given $\|\bar u_0\|_H\le\delta$, $\|\bar u(t)\|_H^2\le 4\delta^2$ holds as long as $t\le T_\delta:=\frac{C\delta^2}{|A_0|^2+\sum_{i=2}^6\delta^i}$, where C is a constant depending on $\mathcal B$, $\mathcal F$ and ρ_B. Moreover, since the moduli of A and L are conserved along solutions, the blow-up criterion given in (3.7) reduces to

$$\|\bar u(t)\|_H\to\infty \text{ for } t\to T_{max}$$

and thus estimate (6.5) shows (3). □

7. Properties of the weak solution

There may be several different methods of constructing weak solutions for (2.4), but of course, we do not show uniqueness of global solutions here, so let us state the requirements needed of weak solutions in general in order to make our subsequent arguments work. We then show that the solutions constructed in [20] satisfy these requirements.

Assumption 7.1. *Given $U_0\in\mathcal L$, there is a weak solution $U\in C_w([0,\infty;\mathcal L)\cap L^2_{loc}(0,\infty;\mathcal H)$ of* (2.4) *which satisfies*

1. *for all $t\ge0$, $|I\bar\Omega|^2(t)=|A_0|^2$,*
2. *the* strong energy inequality, *i.e. for almost all $0\le s<t\le\infty$,*

$$E(t)+2\nu\int_s^t\|\nabla\bar u(\tau)\|_2^2\,\mathrm d\tau\le E(s), \tag{7.1}$$

where $E(s)$ is defined in (5.10).

(3) weak-strong-uniqueness, *i.e. if* $U_0 \in \mathcal{W}$*, then* U *is unique on* $[0, T_{max}(U_0))$ *and it is equal to the strong solution* $W \in \mathcal{X}^{T_{max}}$*, also emenating from* U_0*, given on* $[0, T_{max})$ *by Theorem 3.1. In particular, every strong solution is a weak solution.*

Corollary 7.2. *The weak solution* U *given in Theorem 4.1 satisfies Assumption 7.1.*

Proof. Property (1) was proved in Lemma 5.1. Since $\nabla \bar{u} = D(U)$ on $\mathcal{F}$, for $s = 0$, (2) is a consequence of (4.3) and the discussion of (5.10) in Section 5 and we note that for a general weak solution, (2) in this form implicitly gives conservation of linear momentum. In (4.3), U is constructed by a Galerkin approximation where the approximants U_k satisfy the energy equality (5.9), and they converge strongly to U in the norm $L^2_{loc}(0, \infty; \mathcal{L})$ for a subsequence [20, p. 16]. This implies $\|U_k(s)\|_2^2 \to \|U(s)\|_2^2$ for almost all $s \in (0, \infty)$ for a subsequence, so that (7.1) also follows by passing to the limit and weak lower semicontinuity of the norm.

In order to prove (3), we apply W as a test function for U and obtain

$$\langle U', W\rangle + a(U, W) + b(U, U, W) = 0, \qquad \text{a.e. in } (0, T_{max}). \tag{7.2}$$

At the same time, we can apply the approximants U_k of U as test functions for W to get

$$\langle W', U_k\rangle + a(W, U_k) + b(W, W, U_k) = 0, \qquad \text{a.e. in } (0, T_{max}). \tag{7.3}$$

We integrate both equations in time and note that

$$\int_0^t \langle W', U_k\rangle(s)\,\mathrm{d}s = -\int_0^t \langle W, U_k'\rangle(s)\,\mathrm{d}s + \langle W, U\rangle(t) - \|U_0\|^2_{L^2(\mathcal{S})}.$$

We add up the (original version of the) energy inequality for U,

$$\|U(t)\|^2_{L^2(\mathcal{S})} + 2\int_0^t a(U, U)(s)\,\mathrm{d}s \leq \|U_0\|^2_{L^2(\mathcal{S})}$$

and the energy equality for W,

$$\|W(t)\|^2_{L^2(\mathcal{S})} + 2\int_0^t a(W, W)(s)\,\mathrm{d}s = \|U_0\|^2_{L^2(\mathcal{S})}$$

and subtract twice the time integrals of (7.3) and (7.2) and pass to the limit in the linear terms to obtain

$$\|(U - W)(t)\|^2_{L^2(\mathcal{S})} + 2\int_0^t a(U - W, U - W)\,\mathrm{d}s \leq \int_0^t b(W, W, U_k)(s) + b(U, U, W)(s)\,\mathrm{d}s. \tag{7.4}$$

Manipulation and passage to the limit on the right-hand side is critical only in the terms of type $\int_0^t \int_{\mathcal{F}} ((\bar{w} \cdot \nabla)\bar{w}) \cdot \bar{u}_k \,\mathrm{d}y\mathrm{d}t$, which work as usual for the Navier-Stokes problem and are justified also in [20, p. 13]. In particular, we can show here that

$$\lim_{k\to\infty} \int_0^t b(W, W, U_k)(s) + b(U, U, W)(s)\,\mathrm{d}s = \int_0^t b(U - W, U - W, W)(s)\,\mathrm{d}s,$$

using $b(W, W, U_k) = -b(U_k, W, W) - \Omega_W \cdot (\Omega_U \times I_B\Omega_W)$ and $b(W, U, W) = \Omega_W \cdot (\Omega_U \times I_B\Omega_W)$. Let $Z := U - W$. It remains to estimate

$$\begin{aligned}
\int_0^t b(Z, Z, W)\,\mathrm{d}s \quad &\leq \quad C\int_0^t \|W\|_{L^\infty(\mathcal{S})}(\|z\|_2^2 + \Omega_Z I_B \Omega_Z + m_B|\xi_Z|^2)\,\mathrm{d}s \\
&\quad\; +\nu \int_0^t \|\nabla \bar{z}\|_2^2\,\mathrm{d}s + C\int_0^T \|w\|_\infty^2 \|z\|_2^2\,\mathrm{d}s \\
&\leq \quad \int_0^t a(Z, Z)\,\mathrm{d}s + C\int_0^t \|W\|^2_{L^\infty(\mathcal{S})}\|Z\|^2_{L^2(\mathcal{S})}(s)\,\mathrm{d}s
\end{aligned} \tag{7.5}$$

by Hölder's and Young's inequalities. Since $s \mapsto \|W(s)\|^2_{L^\infty(\mathcal{S})}$ is integrable on $(0, T)$, $T < T_{max}$ by the assumption $W \in \mathcal{X}^T_{2,2}$, by Gronwall's Lemma, (7.4) and (7.5) imply $U = W$ on $(0, T)$. $\square$

8. Strong solutions for large time

For the Navier-Stokes equations, the strong energy inequality and weak-strong uniqueness imply a Leray Structure Theorem ([12] and cf. [7, Sect. 6] for a survey on known results depending on the fluid domain). In particular, every weak solution can be shown to remain regular after some (possibly large) time.

Proposition 8.1. *Given $A_0 \in \mathbb{R}^3$ and $\bar{u}_0 \in L^2_\sigma(\mathcal{F})$, for every weak solution U satisfying Assumption 7.1,*

1. *there is a time $T_*(\bar{u}_0, A_0)$ such that $(\bar{u}, \bar{\Omega})(T_*) \in H \times \mathbb{R}^3$ and $(\bar{u}, \bar{\Omega})$ is the unique strong solution of (5.8) on (T_*, ∞) with initial values $(\bar{u}, \bar{\Omega})(T_*)$.*
2. *$\|\bar{u}(t)\|_H \to 0$ as $t \to \infty$.*

Proof. By assumption, we have (7.1) and thus the total dissipation

$$2\nu \int_0^\infty \|\nabla \bar{u}(\tau)\|_2^2 \,\mathrm{d}\tau \le E(0) \tag{8.1}$$

is bounded. It follows that for every $d > 0$, there is a time $t_d \ge 0$ such that

$$\int_{t_d}^\infty \|\nabla \bar{u}(s)\|_2^2 \,\mathrm{d}s < d.$$

Let $\delta > 0$ and $d \le \frac{T_\delta}{2}\delta(1+C_p)$, where T_δ is the constant from Corollary 6.1. Again by (8.1), there is a time $T_*(\delta) \ge t_d$ such that $\|\bar{u}(T_*)\|_H \le \delta$ and this choice is reasonable since $\bar{u} \in C_w([0,\infty; L^2(\mathcal{F}))$. By Theorem 3.1 and Assumption 7.1.3, $(\bar{u}, \bar{\Omega})$ is unique and strong on $(T_*, T_* + T_\delta)$. Moreover, there must be a point in time $t_\delta \in (T_* + \frac{T_\delta}{2}, T_* + T_\delta)$, such that again $\|\bar{u}(t_\delta)\|_H \le \delta$, because if we assume the contrary, then

$$\int_{T_*+\frac{T_\delta}{2}}^{T_*+T_\delta} \|\bar{u}(s)\|_H \,\mathrm{d}s \ge (1+\frac{1}{C_p}) \int_{T_*+\frac{T_\delta}{2}}^{T_*+T_\delta} \|\nabla \bar{u}(s)\|_2 \,\mathrm{d}s \ge (1+\frac{1}{C_p})\frac{T_\delta}{2}\delta > d.$$

This shows that for every $\delta > 0$, the strong solution starting from T_* can be extended indefinitely by glueing together intervals of length $> \frac{T_\delta}{2}$. Moreover, for all $t \ge T_*(\delta)$, $\|\bar{u}(t)\|_H \le 2\delta$ by Corollary 6.1. This shows that $\|\bar{u}(t)\|_H \to 0$ as $t \to \infty$. □

9. Application of LaSalle's invariance principle and proof of the main result

It remains to show the asymptotics for Ω in Theorem 1.1. We use a modification of LaSalle's Invariance Principle. Following along the lines of the proof of [2, Thm. 9.2.7], we reprove this principle in the present situation as small adjustments have to be made due to the facts that we cannot a priori work with a globally defined semiflow on a complete metric space and that we have to single out trajectories. Throughout this section, let $A_0 \in \mathbb{R}^3$ and $\bar{u}_0 \in L^2_\sigma(\mathcal{F})$ be fixed. Let U be a weak solution corresponding to these inital values and choose $\delta > 0$, e.g. $\delta = 1$, such that $T_* = T_*(\delta)$ from the proof of Proposition 8.1 is given with $\|\bar{u}(t)\|_H \le 2\delta$ for all $t \ge T_*$. We set

$$z_0 := (\bar{u}(T_*), \bar{\Omega}(T_*))$$

and start a new time $t \in \mathbb{R}_+$ at T_*. We define $\mathcal{O}(z_0) := \bigcup_{t\ge 0}\{S_t z_0\}$ to be the orbit of z_0 and note that for all $t \ge 0$, $S_t : \mathcal{O}(z_0) \subset \mathcal{Z}_{2\delta} \to \mathcal{O}(z_0)$ is well-defined. Let

$$\omega(z_0) := \{z \in \mathcal{Z} : \exists t_n \overset{n\to\infty}{\to} \infty \text{ such that } S_{t_n} z_0 \to z\}$$

be the ω-limit set of z_0.

Proposition 9.1. *We collect the following properties of $\mathcal{O}(z_0)$.*

1. *The closure of $\mathcal{O}(z_0)$ in $\mathcal{Z}$ satisfies $\overline{\mathcal{O}(z_0)}^{\mathcal{Z}} = \mathcal{O}(z_0) \cup \omega(z_0) \subset \mathcal{Z}_{2\delta}$.*
2. *$\mathcal{O}(z_0)$ is relatively compact in $\mathcal{Z}$.*
3. *We have $\lim_{t\to\infty} d(S_t z_0, \omega(z_0)) = 0$, where $d(\cdot,\cdot)$ is the distance induced by the $\mathcal{Z}$-norm on $\mathcal{Z}_{2\delta}$.*
4. *For $0 \le t \le T_{2\delta,max}$, $\omega(z_0)$ is invariant under S_t.*

Proof. The first property (1) follows by defnition. By Proposition 8.1,

$$\lim_{t\to\infty} S_t z_0 \subset \{0\} \times \Phi_{|A_0|},$$

which implies (2). In order to show (3), we assume that to the contrary, there is an $\varepsilon > 0$ and a sequencce $t_n \overset{n\to\infty}{\to} \infty$ such that $d(S_{t_n} z_0; \omega(z_0)) \geq \varepsilon$. Then by relative compactness, there is a subsequence $t_{n_k} \overset{k\to\infty}{\to} \infty$ such that $S_{t_{n_k}} z_0 \to \bar{z} \in \omega(z_0)$, yielding a contradiction. Finally, for all $\bar{z} \in \omega(z_0) \subset Z_{2\delta}$,

$$S_t \bar{z} = S_t(\lim_{n\to\infty} S_{t_n} z_0) = \lim_{n\to\infty} S_{t+t_n} z_0 \subset \omega(z_0) \subset \mathcal{Z}_{2\delta}$$

by the continuity of S_t on $\mathcal{Z}_{2\delta}$, cf. Corollary 6.1. This proves (4). □

Proposition 9.2. *The total kinetic energy $E(\bar{u}(t), \bar{\Omega}(t)) = E(t)$ is a strict Lyapunov functional for $\mathcal{O}(z_0)$ and the equilibrium set $\mathcal{E} := \{z \in \mathcal{Z}_{2\delta} : \exists 0 \leq t < T_{2\delta,max}, S_t z = z\}$ is characterized by*

$$\mathcal{E} = \{0\} \times \{\Omega_\infty \in \Phi_{|A_0|} : \Omega_\infty \textit{ is an eigenvector of } I\}.$$

Proof. The function $E : \mathcal{Z} \to \mathbb{R}_+$ is continuous by definition and decreasing along the trajectory of z_0 by the energy equality (5.9). If we assume that for some $z \in \mathcal{Z}_{2\delta}$ and $0 < t \leq T_{2\delta,max}$ we have $E(S_t z) = E(z)$, then by (5.9), $\int_0^t \|\nabla \bar{u}\|_2^2(s)\, \mathrm{d}s = 0$ and thus $\bar{u}(s) \equiv 0$ on $(0,t)$ by Poincaré's inequality. It follows that $\tilde{\Omega} = 0$ and $\Omega = \bar{\Omega}$ on $(0,t)$. Since $S_t z$ gives a strong solution of (5.8), we obtain

$$\bar{\Omega}'(s) \times y + \nabla \bar{p}(s,y) = 0 \text{ for a.e. } (s,y) \in (0,t) \times \mathcal{F}$$

from the first line. But the linear function $\bar{\Omega}' \times \cdot$ cannot be the gradient of a function $\bar{p} \in H^1(\mathcal{F})$, except if $\bar{\Omega}' = 0$. It follows that $S_s z = z$ for all $s \in [0,t]$, i.e. $z \in \mathcal{E}$ and thus, E is a strict Lyapunov functional. Line 4 in (5.8) shows that in this situation, $\bar{\Omega} \times I\bar{\Omega} = 0$, so that the vector $\bar{\Omega}$ constant on $(0,t)$ must be an eigenvector of I. This proves the claim on $\mathcal{E}$. □

It remains to show that $\omega(z_0) \subset \mathcal{E}$. Since for all $t_n \overset{n\to\infty}{\to} \infty$, $(E(S_{t_n} z_0))_{n\in\mathbb{N}}$ is monotone and bounded from below,

$$E_\infty := \lim_{t\to\infty} E(S_t z_0)$$

exists and for all $\bar{z} \in \omega(z_0)$,

$$E(\bar{z}) = E_\infty. \tag{9.1}$$

Let $\mathcal{E}_\infty := \{(0, \Omega_\infty) \in \mathcal{E} : \Omega_\infty I \Omega_\infty = E_\infty)$. Since E is constant on $\omega(z_0)$ by (9.1) and E is a strict Lyapunov functional, $\omega(z_0) \subset \mathcal{E}_\infty$. By Proposition 9.1, we conclude that $\lim_{t\to\infty} d(S_t z_0, \mathcal{E}_\infty) = 0$. Clearly, if the eigenvalues $\lambda_j, j \in \{1,2,3\}$ of I are distinct, then $\mathcal{E}_\infty$ contains six isolated vectors in $\mathbb{R}^3$ and thus $S_t z_0$ must converge to one of them as $t \to \infty$. This proves Theorem 1.1.

10. A priori characterizations of Ω_∞

From the initial data, we extract some information about which vector Ω_∞ is finally attained, using the elementary relations of A_0, E_0 and E_∞. Note that if $|A_0| = 0$, then always $\Omega_\infty = 0$, cf. Remark 5.4. Without loss of generality, we assume that I is given by a diagonal matrix.

(1) The first case is $I = \lambda \mathrm{Id}_{\mathbb{R}^3}$ for some $\lambda > 0$. This makes $\mathcal{S}$ essentially a sphere but $\mathcal{B}$ and $\mathcal{F}$ individually may still have a much more complicated geometries. By Lemma 5.1,

$$\bar{\Omega}(t) = \bar{\Omega}(0) = A_0 \text{ for all } t \geq 0$$

and by Theorem 1.1, $\Omega(t) \to A_0$ as $t \to \infty$. By Remark 5.4, the rate of convergence of $\bar{u}$ in the $L^2(\mathcal{F})$-norm is exponetial.

(2) The second case is that of $\mathcal{S}$ essentially being an *egg*, i.e. $I = \mathrm{diag}(\lambda_s, \lambda_s, \lambda_l)$ where $0 < \lambda_s < \lambda_l$. We show the following.

Proposition 10.1. *If*

$$\tilde{E}(0) < \lambda_l\left(\frac{\lambda_l}{\lambda_s} - 1\right)(\bar{\Omega}_0)_3^2, \tag{10.1}$$

then $\Omega_\infty = \mu_l e_3$, where μ_l is determined by $\mu_l^2 \lambda_l^2 = |I\Omega_\infty|^2 = |A_0|^2$.

Proof. We use a contradiction argument and assume that $\Omega_\infty = \mu_1 e_1 + \mu_2 e_2$ for some $\mu_1, \mu_2 \in \mathbb{R}$. We show that this can only occur if the energy initially stored in the e_3-axis is smaller than the initial kinetic energy provided by the fluid, which is the interpretation of (10.1). The initial kinetic energy for our problem is given by

$$E(0) = \lambda_s[(\bar{\Omega}_0)_1^2 + (\bar{\Omega}_0)_2^2] + \lambda_l(\bar{\Omega}_0)_3^2 + \tilde{E}(0)$$

and the initial absolute value of angular momentum is given by

$$|A_0|^2 = \lambda_s^2[(\bar{\Omega}_0)_1^2 + (\bar{\Omega}_0)_2^2] + \lambda_l^2(\bar{\Omega}_0)_3^2.$$

Thus,

$$E(0) = \frac{|A_0|^2}{\lambda_s} - \lambda_l(\frac{\lambda_l}{\lambda_s} - 1)(\bar{\Omega}_0)_3^2 + \tilde{E}(0).$$

The final absolute value of the angular momentum is equal to the initial one and in this case it is given by

$$|A_0|^2 = |I\Omega_\infty|^2 = \lambda_s^2[(\Omega_\infty)_1^2 + (\Omega_\infty)_2^2].$$

The final kinetic energy is thus given by

$$E_\infty = \lambda_s[(\Omega_\infty)_1^2 + (\Omega_\infty)_2^2] = \frac{|A_0|^2}{\lambda_s}.$$

For all weak solutions satisfying Assumption 7.1, by the energy inequality, we may crudely estimate

$$E_\infty \leq E(0), \tag{10.2}$$

which, by the above calculations, implies

$$\lambda_l(\frac{\lambda_l}{\lambda_s} - 1)(\bar{\Omega}_0)_3^2 \leq \tilde{E}(0).$$

□

(3) Analogous arguments apply in the general case $I = \operatorname{diag}(\lambda_s, \lambda_m, \lambda_l)$, where $0 < \lambda_s < \lambda_m < \lambda_l$. We obtain the following two characterizations: If initially

$$\lambda_l(\frac{\lambda_l}{\lambda_m} - 1)(\bar{\Omega}_0)_3^2 > \tilde{E}(0) + \lambda_s(1 - \frac{\lambda_s}{\lambda_m})(\bar{\Omega}_0)_1^2,$$

and

$$\lambda_m(\frac{\lambda_m}{\lambda_s} - 1)(\bar{\Omega}_0)_2^2 + \lambda_l(\frac{\lambda_l}{\lambda_s} - 1)(\bar{\Omega}_0)_3^2 > \tilde{E}(0), \tag{10.3}$$

then $\Omega_\infty = \mu_l e_3$ with $\mu_l^2 \lambda_l^2 = |A_0|$. If only (10.3) holds, then still $\Omega_\infty = \mu_s e_1$ cannot be attained for any $\mu_s \in \mathbb{R}$.

Remark 10.2. A priori information about the size of the dissipation $2\nu \int_0^\infty \|\nabla \bar{u}(\tau)\|_2^2 \,\mathrm{d}\tau$ improves the estimate in (10.2) and would yield better criteria. This information may not be available in general for weak solutions. It is shown in [16] that the viscosity parameter ν is decisive for the asymptotics and this can also be seen in numerical simulations [8].

Remark 10.3. In this context, it may be relevant that the system has a scaling: for every solution $\bar{u}, \bar{\Omega}$, and every $\lambda \in \mathbb{R}$,

$$\bar{u}_\lambda(s, x) = \lambda \bar{u}(\lambda^2 s, \lambda x),\ (\bar{\Omega})_\lambda(s) = \lambda^2 \bar{\Omega}(\lambda^2 s)$$

also gives a solution.

Remark 10.4. The extension of Theorem 1.1 to the case of external forcing $F \neq 0$, which could for example be given by a gravitational field, is open. We expect the result to still hold if the forcing vanishes suitably as $t \to \infty$, e.g. $F \in \mathcal{V}^\infty \cap L^1(0, \infty; L^2(\mathcal{F}) \times \mathbb{R}^6)$.

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