Weierstraß-Institut
für Angewandte Analysis und Stochastik
Leibniz-Institut im Forschungsverbund Berlin e. V.

Preprint ISSN 2198-5855

A functional limit theorem for limit order books

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submitted: May 20, 2014

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No. 1954
Berlin 2014

2010 *Mathematics Subject Classification.* 60B11, 90B22, 91B70.

Key words and phrases. Limit order book, scaling limit, averaging principle, queuing theory.

Ulrich Horst gratefully acknowledges financial support through the SFB 649 Economic Risk.

Edited by
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Abstract

We consider a stochastic model for the dynamics of the two-sided limit order book (LOB). For the joint dynamics of best bid and ask prices and the standing buy and sell volume densities, we derive a functional limit theorem, which states that our LOB model converges to a continuous-time limit when the order arrival rates tend to infinity, the impact of an individual order arrival on the book as well as the tick size tend to zero. The limits of the standing buy and sell volume densities are described by two linear stochastic partial differential equations, which are coupled with a two-dimensional reflected Brownian motion that is the limit of the best bid and ask price processes.

1 Introduction

In modern financial markets almost all transaction are settled through Limit Order Books (LOBs). An LOB is a record – maintained by an exchange or specialist – of unexecuted orders awaiting execution. Unexecuted (standing) orders are executed against incoming market orders according to a set of precedence rules. Orders standing at better prices have priority over orders submitted at less competitive price levels ("price priority") while orders with the same price-priority are executed on a first-in-first-out basis ("time-priority"). From a mathematical perspective, LOBs can thus be viewed as high-dimensional complex priority queuing systems. In this paper, we propose a queuing-theoretic LOB model whose dynamics converges to a coupled system of reflected Brownian motions and SPDEs after suitable scaling.

There is a substantial economic and econometric literature on LOBs [2, 9, 12, 21] that puts a lot of emphasis on the realistic modeling of the working of the LOB. At the same time, only few authors have analyzed LOB dynamics from a more probabilistic perspective. Kruk [17] studied a queuing theoretic LOB model with finitely many price levels. For the special case of two price levels, in his model the scaled number of standing buy and sell orders at the top of the book converges weakly to a semimartingale reflected two-dimensional Brownian motion in the first quadrant. Cont, Stoikov and Talreja [6] proposed an LOB model with finitely many submission price levels where the LOB dynamics follows an ergodic Markov process. Cont and DeLarrard [5] established a scaling limit for a Markovian limit order market in which the state of the book is represented by the best bid and ask prices along with the liquidity standing at these prices ("top of the book"). Under simplifying assumptions their price process converges to a Brownian motion with volatility. Recently, the same authors [4] studied the reduced state space model under weaker conditions and proved a refined diffusion limit by showing that under heavy traffic conditions the bid and ask queue lengths are given by a two-dimensional Brownian motion in the first quadrant with reflection to the interior at the boundaries, similar to the diffusion limit result for two price levels in [17].

When scaling limits of financial price fluctuations [1, 8, 10, 11, 14] or joint price and volume fluctuations at selected price levels [4, 5, 17] are studied, then the dynamics is finite-dimensional and its limit can naturally be described by ordinary differential equations or finite-dimensional diffusion processes, depending on the choice of scaling. The mathematical analysis is more challenging when the dynamics of the full book is considered. To the best of our knowledge, Osterrieder [20] was the first to model LOBs as measure-valued diffusions. Recently, Horst and Paulsen [13] proved a limit theorem for LOBs with an unbounded number of submission price levels when the tick size tends to zero and order arrival rates tend to infinity. With their choice of scaling the joint dynamics of volumes and prices converges to a coupled system of two PDEs that describe the limiting volume dynamics and two ODEs that describe the limiting price dynamics. In this paper, we consider a different scaling regime. With our choice scaling the best bid and ask price processes converge in distribution to a 2-dimensional reflected Brownian motion while volumes converge in distribution to an SPDE.

We assume that limit and market order arrivals and cancellations follow a Poisson dynamics and that incoming market orders match precisely against the standing liquidity at the best price. In particular, incoming market buy orders increase the best ask price by one tick while incoming market sell orders decrease best bid prices by one tick. Likewise, limit orders placed into the spread improve prices by one tick. In order to model order placements in the spread we follow an idea in [13] and introduce a “shadow book”. More precisely, limit order placements and cancellations occur at random distances from the best bid and ask prices. Placements and cancellations at non-negative distances change the (visible) state of the book while placements and cancellations at negative distances change the state of the shadow book. The shadow book becomes part of the visible book when price changes occur. A sell order placement in the spread, for instance, shifts in collection of the book in such a way that the volume that stood at one price level below the best ask in the shadow book before the price change now stands at the best ask price while the volume that previously stood at the best ask is now standing one tick into the book, i.e. at the new best ask price plus one price tick.

As in [13] our scaling limit requires two time scales: a fast time scale for cancellations and limit order placements outside the spread and a comparably slow time scale for market order arrivals and limit order placements in the spread. We assume that incoming limit orders and cancellations are subject to mean-zero noise. In the simplest case, an incoming limit order has a fixed size plus noise while cancellations are good for fixed proportions plus noise. In order to keep the analysis tractable we make three simplifying assumptions on the noise dynamics: (i) the noise terms share a common component that stays constant between prices changes; (ii) the impact of the common noise component on placements and cancellation is the same across all price levels; (iii) the impact of the noise is linear. Relaxing the first assumption is possible but it might lead to a different scaling limit, depending on the exact relaxation. Relaxing the second and third assumption is requires different mathematical techniques. Since we are primarily interested in establishing a benchmark framework within which to obtain an SPDE scaling limits for LOBs from first principles, i.e. order arrival dynamics, we believe that it is appropriate to work under these assumptions.

Our main result states that when the rates of market order arrivals scale by a factor n, the rates of limit order arrivals and cancellations scale by a factor n^2, the tick size scales by a factor $\frac{1}{\sqrt{n}}$, the sizes (proportions) of incoming orders (volumes cancelled) scale by a factor $\frac{1}{n^2}$ and the noise term scales by a factor $\frac{1}{n}$, then our LOB model converges to a diffusion limit as $n \to \infty$. The limiting model is such that the best bid and ask price dynamics can be described in terms of two-dimensional reflected Brownian motion, while the dynamics of the buy and sell volumes can be described in terms of two SPDEs. The convergence concept we chose is weak convergence in the class of càdlàg stochastic processes with sample bath in $\mathbb{R}^2 \times \mathcal{E}'$ where $\mathcal{E}'$ denotes the set of the tempered distributions. To justify our rather weak notion of convergence we note that the scaling is highly non-linear, due to the simultaneous increase in order arrivals and submission price levels.

The proof of convergence of the price process is standard and follows from established weak limit theorems for two-dimensional reflected Brownian motion, cf. [16]. The proof of convergence of volumes is more challenging. First, the volume process is not a Markov process, due to the nature of the noise. Second, the complex interaction of the various event dynamics renders the proof of tightness complex. In particular, limit order placements and cancellations follow a (spatial) Poisson dynamics on a Poisson time scale generated by market order arrivals. To prove tightness we decompose the volume processes into three components describing aggregate placements, cancellations and "noise contributions" at the various price levels, respectively.[1] We establish norm-bounds for each these processes from which we then deduce that the volume process as a whole satisfies a standard tightness criteria. Once tightness has been established, we modify a method of Kushner [19] to characterize the limit. To this end, we first prove joint convergence of prices and the "noise contributions" to the volume processes. Subsequently, we identify the limits of aggregate placements and cancellations and then use C-tightness to prove joint convergence of all the processes to the desired limit. It turns out that the limiting volume dynamics is essentially described by a family of diffusion processes (one for each price level) driven by two common Brownian motions (resulting from noisy placements and cancellations) evolving in a random environment (generated by the best bid and ask price processes).

The remainder of this paper is organized as follows. In Section 2 we define a sequence of limit order book in terms of our scaling parameters, state the main result and give an outline of the proof. In Section 3 we establish convergence of the bid/ask price dynamics to a 2-dimensional reflected Brownian motion. Section 4 is devoted to the analysis of the limiting volume dynamics. In Section 5, we give the conclusion. Selected results on tightness of stochastic processes as well as some technical proofs are collected in the appendix.

2 Model and main results

2.1 The discrete model

In this section we introduce a sequence of discrete order book models. The models are indexed by $n \in \mathbb{N}$. While our modeling framework closely follows [13] the choice of scaling and hence the limiting dynamics will be very different. Throughout, all random variables are defined on a common probability space $(\Omega, \mathcal{F}, \mathbb{P})$.

The set of price levels at which orders can be submitted in model n is $\{x_j^n\}_{j\in\mathbb{Z}}$. The assumption that there is no smallest price is made for analytical convenience; it avoids the introduction of an additional reflection term for the bid-ask price process at zero. We put $x_j^n := j \cdot \Delta x^n$ for each $j \in \mathbb{Z}$ where Δx^n is the *tick size*, i.e. the minimum difference between two consecutive price levels.

[1] It is the linearity of the noise that allows one to analyze aggregate "noise contributions" separately.

2.1.1 The initial state

The initial state of the book is given by a pair (B_0^n, A_0^n) with $B_0^n \leq A_0^n$ of best bid and ask prices together with the buy and sell limit order volumes at different price levels. We identify volumes at the bid (buy) and ask (sell) side of the book with the step functions:

$$v_b^n(x) := \sum_{j\in\mathbb{Z}} v_b^{n,j} \mathbb{1}_{[x_j^n, x_{j+1}^n)}(x), \quad v_a^n(x) := \sum_{j\in\mathbb{Z}} v_a^{n,j} \mathbb{1}_{[x_j^n, x_{j+1}^n)}(x) \quad (x \in \mathbb{R}).$$

Throughout, indices *b* and *a* indicate the bid and ask side of the book, respectively. The liquidity available for selling/buying $j \in \mathbb{N}$ ticks below/above the best bid/ask price is then given by, respectively:

$$\int_{B_0^n+j\Delta x^n}^{B_0^n+(j+1)\Delta x^n} v_b^n(x)dx = \Delta x^n \cdot v_b^{n,B_0^n/\Delta x^n+j}, \quad \int_{A_0^n+j\Delta x^n}^{A_0^n+(j+1)\Delta x^n} v_a^n(x)dx = \Delta x^n \cdot v_a^{n,A_0^n/\Delta x^n+j}.$$

The restriction of the functions $v_{b/a}^n$ to volumes standing at non-negative distances from the best bid/ask price will be called the *visible book*. The visible book collects the displayed volumes. The collection of volumes standing at negative distances will be referred to as the *shadow book*. The shadow book specifies the volumes that will be placed into the spread should such an event occurs next. The shadow book will undergo random fluctuations similar to those of the visible book. The role of the shadow book will become clear below; see also [13] for a detailed discussion of the shadow book.

Definition 2.1. The initial LOB state is given by a pair $\left(B_0^n, A_0^n\right)$ of bid and ask prices and two volume density functions $v_{b/a}^n(0, \cdot) : \mathbb{R} \to \mathbb{R}$.

There are eight events – labeled $\mathbf{M}_{b/a}, \mathbf{L}_{b/a}, \mathbf{C}_{b/a}, \mathbf{P}_{b/a}$ – that change the state of the book. The events $\mathbf{M}_b, \mathbf{L}_b, \mathbf{C}_b, \mathbf{P}_b$ affect the bid side of the book:

$\mathbf{M}_b$:= {market sell order} $\qquad$ $\mathbf{L}_b$:= {buy limit order placed in the spread}
$\mathbf{C}_b$:= {cancellation of buy volume} $\qquad$ $\mathbf{P}_b$:= {buy limit order not placed in spread}

The events $\mathbf{M}_a, \mathbf{L}_a, \mathbf{C}_a, \mathbf{P}_a$ affect the ask side of the book:

$\mathbf{M}_a$:= {market buy order} $\qquad$ $\mathbf{L}_a$:= {sell limit order placed in the spread}
$\mathbf{C}_a$:= {cancellation of sell volume} $\qquad$ $\mathbf{P}_a$:= {sell limit order not placed in the spread}.

In the sequel we describe how different events change the state of the book. To this end, we denote by $v_{b/a}^n(t, \cdot)$ the volume density function at the bid/ask side at time $t > 0$ and by $(B^n(t), A^n(t))$ the prevailing best bid and ask prices.

2.1.2 Active orders and price dynamics

Market order arrivals (Events $\mathbf{M}_{b/a}$) and placements of limit orders in the spread (Events $\mathbf{L}_{b/a}$) lead to price changes.[2] We refer to these order types as *active orders*.

[2]A market order that does not lead to a price change can be viewed as a cancellation of standing volume at the best bid/ask price.

Assumption 1. Active orders arrive according to independent Poisson processes $\widetilde{N}_b^n$ and $\widetilde{N}_a^n$ with intensities μ_b^n and μ_a^n at the bid and ask side of the book, respectively. The respective jump times are denoted $\left(\widetilde{\tau}_{b/a,i}^n\right)_{i=1}^{\infty}$.

We assume that market orders match precisely against the standing volume at the best available prices and that orders placed in the spread are placed at the first best price increment. In particular, active orders change prices by exactly one tick.

More precisely, a market sell order arriving at time $t > 0$ is good for $v_b^n(t-, B^n(t-)) \cdot \Delta x^n$ shares. A limit buy order placed in the spread at time $t \in (0, T)$ arrives at price level $B^n(t-) - \Delta x^n$ and is good for $v_b^n(t-, B^n(t-) - \Delta x^n) \cdot \Delta x^n$ shares. This illustrates the role of the shadow book. For simplicity we assume that prices increase and decrease with equal probability.

Since $\widetilde{N}_b^n$ and $\widetilde{N}_a^n$ are independent we may as well model price changes in terms of a single Poisson process $\widetilde{N}^n$ with suitable intensity μ^n and corresponding *active order jump times* $\left(\widetilde{\tau}_i^n\right)_{i=0}^{\infty}$ along with two independent sequences of i.i.d. random variables $(\xi_{b,i})_{i=0}^{\infty}$ and $(\xi_{a,i})_{i=0}^{\infty}$ where $\xi_{a,i}$ takes the values $+1$ and -1 with equal probability for each $i \in \mathbb{N}$. For instance, on the bid side, $\xi_{b,i} = \xi_{a,i} = +1$ increases the bid price by one tick, and hence corresponds to a placement of a limit order in the spread; $\xi_{b,i} = \xi_{a,i} = -1$ decreases the bid price by one tick and hence corresponds to a market sell order.

In order to guarantee that the best ask price never falls below the best bid price we introduce a reflection term. More precisely, we model the price dynamics as follows:

$$dB^n(t) = \frac{\Delta x^n}{2}(\xi_{b,\widetilde{N}^n(t)} + \xi_{a,\widetilde{N}^n(t)})\, d\widetilde{N}^n(t) - \Delta x^n \mathbf{1}_{A^n(t-)-B^n(t-)=\Delta x^n}\, d\widetilde{N}^n(t), \tag{1a}$$

$$dA^n(t) = \frac{\Delta x^n}{2}(\xi_{b,\widetilde{N}^n(t)} - \xi_{a,\widetilde{N}^n(t)})\, d\widetilde{N}^n(t) + \Delta x^n \mathbf{1}_{A^n(t-)-B^n(t-)=\Delta x^n} d\widetilde{N}^n(t). \tag{1b}$$

Remark 2.2. Of course, one could also introduce a reflection at zero for the bid price process. Such a reflection does not pose significant mathematical challenges but it leads to quite cumbersome dynamics as several cases have to be distinguished. We therefore choose to disregard the problem of negative prices.

2.1.3 Passive orders and volume changes

Limit order placements outside the spread and cancellations of standing volume do not change prices. We refer to these order types as *passive orders*. In our model cancellations (Events $\mathbf{C}_{b/a}$) occur for random *proportions* of the standing volume at random price levels while limit order placements outside the spread (Events $\mathbf{P}_{b/a}$) occur for random *volumes* at random price levels.

Assumption 2. Passive orders arrive according to independent Poisson processes N_b^n and N_a^n with intensities λ_b^n and λ_a^n at the buy and sell side of the book, respectively. The corresponding jump times $\left(\tau_{b/a,i}^n\right)_{i=1}^{\infty}$ will be called passive order times.

The *submission and cancellation price levels* are chosen relative to the best prices. Allowing for rather general placement and cancellation dynamics, we assume that the price levels are chosen according to a sequence of i.i.d. random variables $(\pi_i)_{i=0}^{\infty}$ where each π_i is of the form:

$$\pi_i = \left(\pi_i^{\mathbf{C}_b}, \pi_i^{\mathbf{C}_a}, \pi_i^{\mathbf{P}_b}, \pi_i^{\mathbf{P}_a}, \pi_i^{\mathbf{N}_b}, \pi_i^{\mathbf{N}_a}\right). \tag{2}$$

The entries takes values in an interval $[-M, M]$ for some $M > 0$; positive values indicate changes in the visible book while negative values indicate changes in the shadow book. Superscripts indicate event types and '$\mathbf{N}$' stands for 'noise'. The precise meaning of the entries will become clear below.

Passive order sizes are described by a sequence of i.i.d. random variables $(\omega_i)_{i=0}^{\infty}$ where each ω_i is of the form

$$\omega_i = \left(\omega_i^{\mathbf{C}_b}, \omega_i^{\mathbf{C}_a}, \omega_i^{\mathbf{P}_b}, \omega_i^{\mathbf{P}_a}, \omega_i^{\mathbf{N}_b}, \omega_i^{\mathbf{N}_a}\right). \tag{3}$$

The random variables $\omega_i^{\mathbf{P}_{b/a}}$ take values in $[0, \infty)$; they describe the *sizes* of order placements. Likewise, the random variables $\omega_i^{\mathbf{C}_{b/a}}$ take values in $[0, 1]$ and describe the *proportions* of cancellations. Placements and cancellations on both sides of the book are subject to noise. The impact of the noise is described by the non-negative random variables $\omega^{\mathbf{N}_{b/a}}$ and two sequences of i.i.d. random variables $\left(\widetilde{\xi}_{b/a,i}\right)_{i=0}^{\infty}$ where $\widetilde{\xi}_{b/a,i} \in \{-1, +1\}$ for each $i \in \mathbb{N}$.

More precisely, let us assume that the i-th passive order event occurs at time $t > 0$. If

$$\pi_i^{\mathbf{C}_b} = \gamma, \quad \pi_i^{\mathbf{P}_b} = \delta, \quad \pi_i^{\mathbf{N}_b} = \zeta$$

then the impact of a cancellation, order placement and noise is felt at the respective price levels

$$l := \left\lfloor \frac{B^n(t-) + \gamma}{\Delta x^n} \right\rfloor, \quad j := \left\lfloor \frac{B^n(t-) + \delta}{\Delta x^n} \right\rfloor, \quad r := \left\lfloor \frac{B^n(t-) + \zeta}{\Delta x^n} \right\rfloor,$$

and the *change* in the bid-side volume density function is given by:

$$\begin{aligned} v_b^n(t, \cdot) - v_b^n(t-, \cdot) = {} & \omega_i^{\mathbf{P}_b} \frac{\Delta v^n}{\Delta x^n} \mathbf{1}_{[x_j^n, x_{j+1}^n[}(\cdot) + \\ & - \omega_i^{\mathbf{C}_b} \frac{\Delta v^n}{\Delta x^n} v_b^n(t-, B^n(t-) + \gamma) \mathbf{1}_{[x_l^n, x_{l+1}^n[}(\cdot) + \omega_i^{\mathbf{N}_b} \widetilde{\xi}_{b, \widetilde{N}^n(t-)} \sqrt{\Delta v^n} \mathbf{1}_{[x_r^n, x_{r+1}^n[}(\cdot). \end{aligned} \tag{4}$$

Here Δv^n is a scaling parameter that measures the impact of an individual order on the state of the book. Similar considerations apply to the ask side. For our scaling limit it will be important that the noise parameters $\widetilde{\xi}_{b/a}$ stay constant between two price changes; this explains the random variable $\widetilde{\xi}_{a,\widetilde{N}^n(t-)}$ in (4). Note, however, that the constant fluctuation part $\widetilde{\xi}_{a/b}$ is further modulated by the non-negative random factor $\omega^{\mathbb{N}_{a/b}}$, which changes between consecutive passive orders.

Remark 2.3. The frequency of change of the 'common factor' $\widetilde{\xi}_{a/b}$ determines the structure of the martingale part of the limiting dynamics. In our case, the martingale part will be a an integral with respect to a Wiener process (resulting from the scaling of $\widetilde{\xi}_{a/b}$). If the common factor changes at the same rate at which passive orders arrive, we expect the martingale part to be space-time white noise. This case is left for future research.

Remark 2.4. In real-world markets only one event (market order arrival, cancellation, placement) happens at a time. Within our framework this corresponds to the special case where

$$\pi_k^{\mathbf{C}_{b/a}} = \pi_k^{\mathbf{P}_{b/a}} = \pi_k^{\mathbf{N}_{b/a}},$$

and only one of the random variables $\omega^{\mathbf{C}_{a/b}}, \omega^{\mathbf{P}_{a/b}}$ is different from zero. Our mathematical framework is flexible enough to allow for such correlation. For instance, if only $\omega_k^{\mathbf{C}_b}$ and $\omega_k^{\mathbf{N}_b}$ are different from zero and $\omega_k^{\mathbf{N}_b} = 1$, then the k-th event is a bid-side cancellation at the price level

$$\gamma = \left\lfloor \frac{B^n(t-) + \pi_k^{\mathbf{C}_b}}{\Delta x^n} \right\rfloor$$

and the volume cancelled is:

$$\omega_k^{\mathbf{C}_b} \cdot \Delta v^n \cdot v_b^n(t-, B^n(t-) + \gamma) + \omega_k^{\mathbf{N}_b} \widetilde{\xi}_{b,\widetilde{N}^n(t-)} \cdot \sqrt{\Delta v^n} \Delta x^n.$$

We acknowledge that volumes may become negative with our choice of scaling. This could be avoided by multiplying the noise term with the standing volume as well. However, additional technical arguments would be needed for such a multiplicative noise structure, which we postpone to future work.

Notation 2.5. We introduce the following important short-hand notations:

- For any (deterministic or random) function $u : [0, \infty) \times \mathbb{R} \to \mathbb{R}$ we denote by $u(t) : \mathbb{R} \to \mathbb{R}$ the function $x \mapsto u(t, x)$ for $t \in [0, \infty)$.
- In the nth model, we denote by $I^n(y)$ the sub-interval of the grid containing $y \in \mathbb{R}$. More precisely, its indicator function is given by

$$\mathbf{1}_{I^n(y)}(x) = \sum_{j \in \mathbb{Z}} \mathbf{1}_{[x_j^n, x_{j+1}^n[}(y) \mathbf{1}_{[x_j^n, x_{j+1}^n[}(x).$$

- Unless otherwise stated, $(L^p, \|\cdot\|_{L^p})$ $(p \in [1, \infty])$ refers to the space $L^p(\mathbb{R}, \mathcal{B}(\mathbb{R}), dx)$.
- For σ-algebras $\mathcal{G} \subset \mathcal{F}$ we shall write $E_{\mathcal{G}}[\cdot] := E[\cdot \,|\, \mathcal{G}]$.

For future use, we also introduce the filtration $\mathcal{F}^n$ generated by the n-th model. More precisely, we set

$$\begin{aligned}\mathcal{F}_t^n := \sigma\Big(&\big(\widetilde{N}_s^n\big)_{0 \le s \le t}, \big(\xi_{a,k}^n\big)_{k=1}^{\widetilde{N}^n(t)}, \big(\xi_{b,k}^n\big)_{k=1}^{\widetilde{N}^n(t)}, (N_a^n(s))_{0 \le s \le t}, \big(N_b^n(s)\big)_{0 \le s \le t}, \big(\omega_k^{\mathbf{C}_a}, \omega_k^{\mathbf{P}_a}, \omega_k^{\mathbf{N}_a}\big)_{k=1}^{N_a^n(t)}, \\ &\big(\omega_k^{\mathbf{C}_b}, \omega_k^{\mathbf{P}_b}, \omega_k^{\mathbf{N}_b}\big)_{k=1}^{N_b^n(t)}, \big(\pi_k^{\mathbf{C}_a}, \pi_k^{\mathbf{P}_a}, \pi_k^{\mathbf{N}_a}\big)_{k=1}^{N_a^n(t)}, \big(\pi_k^{\mathbf{C}_b}, \pi_k^{\mathbf{P}_b}, \pi_k^{\mathbf{N}_b}\big)_{k=1}^{N_a^b(t)}, \big(\widetilde{\xi}_{b,k}^n\big)_{k=1}^{N_a^n(t)}, \big(\widetilde{\xi}_{a,k}^n\big)_{k=1}^{N_b^n(t)}\Big).\end{aligned}$$

In terms of the independent Poisson processes $\widetilde{N}^n$ and $N_{b/a}^n$ governing the arrival of active and passive orders, respectively, and the active order arrival times $\big(\widetilde{\tau}_i^n\big)_{i=0}^{\infty}$, the dynamics of the buy and sell side volume density functions follows the dynamics

$$\begin{aligned}dv_b^n(t, \cdot) = \Big[&\mathbf{1}_{I^n\left(B^n(\widetilde{\tau}^n_{\widetilde{N}^n(t-)}) + \pi^{\mathbf{P}_b}_{N_b^n(t)}\right)}(\cdot)\,\omega^{\mathbf{P}_b}_{N_b^n(t-)} \frac{\Delta v^n}{\Delta x^n} \\ &- \mathbf{1}_{I^n\left(B^n(\widetilde{\tau}^n_{\widetilde{N}^n(t-)}) + \pi^{\mathbf{C}_b}_{N_b^n(t)}\right)}(\cdot)\,\omega^{\mathbf{C}_b}_{b,N_b^n(t-)} v_b^n(\tau^n_{b,N_b^n(t-)}, \cdot) \frac{\Delta v^n}{\Delta x^n} \\ &+ \mathbf{1}_{I^n\left(B^n(\widetilde{\tau}^n_{\widetilde{N}^n(t-)}) + \pi^{\mathbf{N}_b}_{N_b^n(t)}\right)}(\cdot)\,\omega^{\mathbf{N}_b}_{N_b^n(t-)} \widetilde{\xi}_{b,\widetilde{N}^n(t-)} \sqrt{\Delta v^n}\Big] dN_b^n(t),\end{aligned} \tag{5a}$$

$$\begin{aligned}dv_a^n(t, \cdot) = \Big[&\mathbf{1}_{I^n\left(A^n(\widetilde{\tau}^n_{\widetilde{N}^n(t-)}) + \pi^{\mathbf{P}_a}_{N_a^n(t)}\right)}(\cdot)\,\omega^{\mathbf{P}_a}_{N^n(t-)_a} \frac{\Delta v^n}{\Delta x^n} \\ &- \mathbf{1}_{I^n\left(A^n(\widetilde{\tau}^n_{\widetilde{N}^n(t-)}) + \pi^{\mathbf{C}_a}_{N_a^n(t)}\right)}(\cdot)\,\omega^{\mathbf{C}_a}_{N^n(t-)} v_a^n(\tau^n_{N_a^n(t-)}, \cdot) \frac{\Delta v^n}{\Delta x^n} \\ &+ \mathbf{1}_{I^n\left(A^n(\widetilde{\tau}^n_{\widetilde{N}^n(t-)}) + \pi^{\mathbf{N}_a}_{N_a^n(t)}\right)}(\cdot)\,\omega^{\mathbf{N}_a}_{N_a^n(t-)} \widetilde{\xi}_{a,\widetilde{N}^n(t-)} \sqrt{\Delta v^n}\Big] dN_a^n(t).\end{aligned} \tag{5b}$$

We see from the above equations that the random volume density functions evolve in a random environment described by the best bid and ask price processes. The specific structure of the dependence of the volume density functions on the bid and ask price as well as the random submission price levels reflects the fact the submission and cancellation price levels are chosen relative to the best bid/ask price.

2.2 The main result

We prove below that our LOB model converges to a continuous time limit if the order arrival rates tend to infinity and the impact of an individual order arrival on the book as well as the tick size tend to zero in a particular way. In order to make the convergence concept precise, and to state the main result, we need to introduce some notations.

2.2.1 Preliminaries

For $m \in (-\infty, \infty)$, we denote by $(H^m, \|\cdot\|_m)$ the space of Bessel potentials equipped with the usual Sobolev norm and inner product. Set

$$\mathcal{E}' = \cup_m H^{-m} \supset \cdots \supset H^{-1} \supset L^2 \supset H^1 \supset \cdots H^2 \supset \cdots \supset \cap_m H^m = \mathcal{E}.$$

It is well known that $H^0 = L^2$ and that $\mathcal{E}$ is a complete separable metric space. Sobolev's embedding theorem indicates that each element of $\mathcal{E}$ is an infinitely differentiable function. In what follows, denote the dual between $\mathcal{E}'$ and $\mathcal{E}$ by $\langle \cdot, \cdot \rangle$, which coincides with the inner product of $H^0 = L^2$.

The convergence concept we use is weak convergence in the Skorokhod space $\mathcal{D} := \mathcal{D}([0,\infty); \mathbb{R}^2 \times H^{-1} \times H^{-1})$ of all càdlàg functions on $[0,\infty)$ taking values in the space $\mathbb{R}^2 \times H^{-1} \times H^{-1}$. The space $\mathcal{D}$ is equipped with the usual Skorokhod metric (see Jacod and Shiryaev [15]).

2.2.2 The convergence result

In order to obtain our convergence result, we need the following assumptions. In particular, just as in [13], we need active and passive orders to arrive on different time scales.

Assumption 3.

- The scaling parameters $\lambda^n_{b/a}$ (arrival rate of passive orders), μ^n (arrival rate of active orders), Δv^n (order sizes) and Δx^n (tick size) satisfy the following conditions:

$$\lambda^n_{b/a} = n^2; \quad \mu^n = n; \quad \Delta v^n = n^{-2}; \quad \Delta x^n = n^{-1/2}.$$

- The Poisson processes N^n_a, N^n_b and $\tilde{N}^n$ are independent.
- For each 'event type' $\mathbf{T} = \mathbf{C}_{b/a}, \mathbf{P}_{b/a}, \mathbf{N}_{b/a}$ the random variables $\omega^{\mathbf{T}}_i$ $(i \in \mathbb{N})$ are i.i.d. with finite fourth moment, $\omega^{\mathbf{C}_{b/a}}_i \in [0,1]$, $\omega^{\mathbf{P}_{b/a}}_i, \omega^{\mathbf{N}_{b/a}}_i \in [0,\infty)$, and the random variables $\pi^{\mathbf{T}}_i$ have Lipschitz continuous and hence bounded densities $f^{\mathbf{T}}$ on some compact interval $[-M, M]$. The random variables $\omega^{\mathbf{T}}_i$ and $\pi^{\mathbf{T}}_i$ $(i \in \mathbb{N})$ are independent of the Poisson processes N^n_a, N^n_b and $\widetilde{N}^n$.

- The random variables $\widetilde{\xi}_{b/a,i}$ and $\xi_{b/a,i}$ are independent and independent of all other random variables and take the values ± 1 with equal probability.
- The sequence of initial data $(A_0^n, B_0^n, v_a^n(0,\cdot), v_b^n(0,\cdot))$ converges to $(a_0, b_0, v_{a,0}(\cdot), v_{b,0}(\cdot))$ in both $\mathbb{R}^2 \times L^2 \times L^2$ and $\mathbb{R}^2 \times L^\infty \times L^\infty$.

We are now ready to state the main result of this paper.

Theorem 2.6. *Let Assumptions 1-3 be satisfied. There are four independent Wiener processes β_a, β_b, W_a and W_b such that the sequence (A^n, B^n, v_a^n, v_b^n) of stochastic processes converges in distribution in $\mathcal{D}([0,\infty); \mathbb{R}^2 \times H^{-1} \times H^{-1})$ to (A, B, v_a, v_b). Here (A, B) is a two-dimensional reflected Brownian motion:*

$$\begin{aligned} dA_t =& \frac{1}{\sqrt{2}} d\beta_t^a + dL_t; \quad A_0 = a_0; \\ dB_t =& \frac{1}{\sqrt{2}} d\beta_t^b - dL_t; \quad B_0 = b_0; \\ dL_t =& \mathbf{1}_{A_t = B_t} dL_t; \quad L_0 = 0 \end{aligned}$$

and the volume density processes satisfies the infinite-dimensional SDE

$$\begin{aligned} v_b(t,\cdot) =& v_{b,0}(\cdot) + \int_0^t \left(E[\omega_1^{\mathbf{P}_b}] f^{\mathbf{P}_b}(\cdot + B_s) - E[\omega_1^{\mathbf{C}_b}] f^{\mathbf{C}_b}(\cdot + B_s) v_b(s,\cdot) \right) ds \\ & + \sqrt{2} E\left[\omega_1^{\mathbf{N}_b}\right] \int_0^t f^{\mathbf{N}_b}(\cdot + B_s)\, dW_b(s), \quad t \geq 0; \\ v_a(t,\cdot) =& v_{a,0}(\cdot) + \int_0^t \left(E[\omega_1^{\mathbf{P}_a}] f^{\mathbf{P}_a}(\cdot + A_s) - E[\omega_1^{\mathbf{C}_a}] f^{\mathbf{C}_a}(\cdot + A_s) v_a(s,\cdot) \right) ds \\ & + \sqrt{2} E\left[\omega_1^{\mathbf{N}_a}\right] \int_0^t f^{\mathbf{N}_a}(\cdot + A_s)\, dW_a(s), \quad t \geq 0. \end{aligned}$$

For any $T \in (0, \infty)$, the existence and uniqueness of the adapted solution of the above infinite-dimensional SDE in $L^2(\Omega \times [0,T] \times \mathbb{R})$ is obvious; see [7] for a general theory on stochastic equations in infinite dimensions. If the model parameters are sufficiently smooth, then the density functions are smooth as well. The proof of the following corollary is an immediate consequence of Itô's formula.

Corollary 2.7. *If $v_{b/a,0}$ and the densities $f^{\mathbf{T}}$ belong to H^m with $m > 3$, then $v_{b/a}(t)$ take values in H^m and hence by embedding, in $C^2(\mathbb{R})$. Furthermore, the relative volume processes*

$$(U_a, U_b)(t,x) = (v_a(t, x - A_t), v_b(t, x - B_t)),$$

satisfies the stochastic partial differential equation:

$$\begin{aligned} dU_a(t,x) =& \left[\frac{1}{4} \Delta U_a(t,x) + E[\omega_1^{\mathbf{P}_a}] f^{\mathbf{P}_a}(x) - E[\omega_1^{\mathbf{C}_a}] f^{\mathbf{C}_a}(x) U_a(t,x) \right] dt - \partial_x U_a(t,x) dL_t \\ & - \partial_x U_a(t,x) d\beta_t^a + \sqrt{2} E\left[\omega_1^{\mathbf{N}_a}\right] f^{\mathbf{N}_a}(x) dW_a(t), \quad t \geq 0; \\ U_a(0,x) =& v_{a,0}(x - a_0); \end{aligned}$$

$$
\begin{aligned}
dU_b(t,x) =& \left[\frac{1}{4}\Delta U_b(t,x) + E[\omega_1^{\mathbf{P}_b}]f^{\mathbf{P}_b}(x) - E[\omega_1^{\mathbf{C}_b}]f^{\mathbf{C}_b}(x)U_b(t,x)\right] dt + \partial_x U_b(t,x)dL_t \\
& - \partial_x U_b(t,x)d\beta_t^b + \sqrt{2}E\left[\omega_1^{\mathbf{N}_b}\right] f^{\mathbf{N}_b}(x)dW_b(t), \quad t \geq 0; \\
U_b(0,x) =& v_{b,0}(x - b_0).
\end{aligned}
$$

2.3 Outline of the proof

The proof of Theorem 2.6 is carried out in the following sections. The proof of convergence of the bid and ask price processes draws on established results on weak limits of reflected random walks and is carried out in Section 3. The proof of convergence of the volume density processes on the bid and ask sides of the limit order book is recalled in Section 4. For the convenience of the reader we now give an outline of our strategy for the convergence proof for the volume densities.

2.3.1 Some auxiliary processes

We split the dynamics of the volume density functions into three processes, which we are going to handle separately, before finally pasting them back together to obtain the result for the full dynamics.

From equation (5a) we identify the following three processes which drive the evolution of the bid-side volume density function:

$$
V_b^{n,1}(t,x) = \sum_{i=1}^{N_b^n(t)} \mathbf{1}_{I^n\left(B^n(\widetilde{\tau}^n_{\widetilde{N}^n(\tau^n_{b,i})}) + \pi_i^{\mathbf{P}_b}\right)}(x)\omega_i^{\mathbf{P}_b}\frac{\Delta v^n}{\Delta x^n}, \tag{6a}
$$

$$
V_b^{n,2}(t,x) = \sum_{i=1}^{N_b^n(t)} \mathbf{1}_{I^n\left(A^n(\widetilde{\tau}^n_{\widetilde{N}^n(\tau^n_{b,i})}) + \pi_i^{\mathbf{C}_b}\right)}(x)\omega_i^{\mathbf{C}_b}\frac{\Delta v^n}{\Delta x^n}, \tag{6b}
$$

$$
V_b^{n,3}(t,x) = \sum_{i=1}^{N_b^n(t)} \mathbf{1}_{I^n\left(B^n(\widetilde{\tau}^n_{\widetilde{N}^n(\tau^n_{b,i})}) + \pi_i^{\mathbf{N}_b}\right)}(x)\omega_i^{\mathbf{N}_b}\widetilde{\xi}_{b,\widetilde{N}^n(\tau^n_{b,i})+1}\sqrt{\Delta v^n}, \tag{6c}
$$

corresponding to the volume changes due to incoming order placements ($V_b^{n,1}$), the proportional cancellations of standing volume ($V_b^{n,2}$) and aggregated random fluctuations ($V_b^{3,n}$). We notice that $V_b^{n,1}$ and $V_b^{n,2}$ are increasing functions in time for each n. The process $V_b^{n,3}$ will contribute the martingale part in the continuous scaling limit.[3] We introduce similar processes $V_a^{n,1}$, $V_a^{n,2}$ and $V_a^{n,3}$ for the ask side.

2.3.2 'Markovization'

The previously introduced processes are not convenient for characterizing the limit process. The reason is that the discrete processes $\left(V_{b/a}^{3,n}(\tau_i^n,\cdot)\right)_{i\in\mathbb{N}}$ which capture the fluctuations are not Markov

[3]Note that $V_b^{n,3}$ itself is *not* a martingale (in the filtration $\mathcal{F}^n$ generated by the full model), as the fluctuations $\widetilde{\xi}$ are constant between two active order times.

chains because part of the fluctuations change only at active order times. As a result, we cannot directly use existing results on the scaling of Markov processes.

The necessary 'markovization' is achieved by registering changes to the processes $V^{i,n}_{b/a}$ and $v^n_{b/a}$ only at active order times. To this end, we introduce the following processes (making use of our short-hand notations):

$$\widehat{v}^n_{b/a}(t) := v^n_{b/a}(\widetilde{\tau}^n_{k-1}), \quad \widehat{V}^{i,n}_{b/a}(t) := V^{i,n}_{b/a}(\widetilde{\tau}^n_{k-1}), \quad \widetilde{\tau}^n_{k-1} \leq t < \widetilde{\tau}^n_k, \tag{7}$$

for $i = 1, 2, 3$. Note that we have, for instance,

$$\widehat{V}^{n,1}_b(t,x) = \sum_{k=1}^{\widetilde{N}^n_t} \sum_{i=N^n_b(\widetilde{\tau}^n_{k-1})+1}^{N^n_b(\widetilde{\tau}^n_k)} \omega^{\mathbf{P}_b}_i \mathbf{1}_{I^n\left(B^n(\widetilde{\tau}_{k-1})+\pi^{\mathbf{P}_b}_i\right)}(x) \frac{\Delta v^n}{\Delta x^n}.$$

Obviously, together with $\left((A^n_{\widetilde{\tau}^n_i}, B^n_{\widetilde{\tau}^n_i})\right)_{i\in\mathbb{N}}$ the processes $\left(\widehat{v}^n_{b/a}(\widetilde{\tau}^n_i, \cdot)\right)_{i\in\mathbb{N}}$, $\left(\widehat{V}^{n,1/2/3}_{b/a}(\widetilde{\tau}^n_i, \cdot)\right)_{i\in\mathbb{N}}$ are Markov processes, and $\widehat{V}^{n,3}_{b/a}$ is, in fact, a martingale. Thus, the methods of, e.g., [19], are, in principle, applicable to these processes. Nonetheless, we find it useful to add yet another layer of auxiliary processes, this time by separating out active order times, i.e., by considering the process as if active orders arrive at deterministic points in time. More precisely, we define the time-change $\overline{\eta}$ together with its inverse η by

$$\begin{aligned} \overline{\eta}^n_u &:= \widetilde{\tau}^n_{\lfloor nu \rfloor}, \quad u \in [0,\infty); \\ \eta^n_u &:= \inf\{t : t > 0, \overline{\eta}^n_t > u\} - \frac{1}{n}, \quad u \in [0,\infty). \end{aligned} \tag{8}$$

Then new processes are defined, which correspond to the "hat" processes when evaluated on the time-scale η^n. More precisely, we put:

$$\overline{A}^n_u := A^n_0 + \frac{\Delta x^n}{2} \sum_{i=1}^{\lfloor nu \rfloor} (\xi_{b,i} + \xi_{a,i}) + \Delta x^n \sum_{0\leq t\leq u} \mathbf{1}_{\overline{A}^n(t-)-\overline{B}^n(t-)=\Delta x^n}, \tag{9a}$$

$$\overline{B}^n_u := B^n_0 + \frac{\Delta x^n}{2} \sum_{i=1}^{\lfloor nu \rfloor} (\xi_{b,i} - \xi_{a,i}) - \Delta x^n \sum_{0\leq t\leq u} \mathbf{1}_{\overline{A}^n(t-)-\overline{B}^n(t-)=\Delta x^n}, \tag{9b}$$

$$\overline{V}^{n,1}_a(u,x) := \sum_{i=1}^{N^n_a(\widetilde{\tau}^n_{\lfloor nu \rfloor})} \omega^{\mathbf{P}_a}_i \mathbf{1}_{I^n\left(\overline{A}^n\left(\eta^n_{\tau^n_{a,i}}\right)+\pi^{\mathbf{P}_a}_i\right)}(x) \frac{\Delta v^n}{\Delta x^n}, \tag{9c}$$

$$\overline{V}^{n,2}_a(u,x) := \sum_{i=1}^{N^n_a(\widetilde{\tau}^n_{\lfloor nu \rfloor})} \omega^{\mathbf{C}_a}_i \mathbf{1}_{I^n\left(\overline{A}^n\left(\eta^n_{\tau^n_{a,i}}\right)+\pi^{\mathbf{C}_a}_i\right)}(x) \frac{\Delta v^n}{\Delta x^n}, \tag{9d}$$

$$\overline{V}^{n,3}_a(u,x) := \sum_{i=1}^{N^n_a(\widetilde{\tau}^n_{\lfloor nu \rfloor})} \omega^{\mathbf{N}_a}_i \mathbf{1}_{I^n\left(\overline{A}^n\left(\eta^n_{\tau^n_{a,i}}\right)+\pi^{\mathbf{N}_a}_i\right)}(x) \widetilde{\xi}_{a,\widetilde{N}^n(\tau^n_{a,i})+1} \sqrt{\Delta v^n}, \tag{9e}$$

$$\begin{aligned} \overline{v}^n_a(u,x) := {} & v^n_a(0,x) + \overline{V}^{n,1}_a(u,x) + \overline{V}^{n,3}_a(u,x) \\ & - \sum_{i=1}^{N^n_a(\widetilde{\tau}^n_{\lfloor nu \rfloor})} \omega^{\mathbf{C}}_i \mathbf{1}_{I^n\left(\overline{A}^n\left(\eta^n_{\tau^n_{a,i}}\right)+\pi^{\mathbf{C}_a}_i\right)}(x) v^n_a(\tau^n_{a,i}, x) \frac{\Delta v^n}{\Delta x^n}, \end{aligned} \tag{9f}$$

and similarly for the processes on the bid side of the limit order book. Thus, we have the desired property that, for instance,

$$(A^n, B^n, \widehat{v}^n_a)(u) = (\overline{A}^n, \overline{B}^n, \overline{v}^n_a)(\eta^n_u).$$

Note that $\left((\overline{A}^n_{i/n}, \overline{B}^n_{i/n}, \overline{v}^n_a(i/n, \cdot))\right)_{i\in\mathbb{N}}$ is a Markov chain.

2.3.3 Structure of the proof

With these preparations we can now describe the structure of the proof. We first prove tightness of each of the processes $\overline{V}^{n,i}_{b/a}$ $(i = 1, 2, 3)$ and of $\overline{v}^n_{b/a}$ in the distributional sense indicated above. For this part, we heavily rely on Mitoma's theorem (Theorem B.2) together with Kurtz's criterion (Theorem B.1). This first part is presented in Section 4.1, see Proposition 4.7.

The natural next step would be to extend the tightness result for $\overline{v}^n_{b/a}$ to $\widehat{v}^n_{b/a}$ and, subsequently, to $v^n_{b/a}$. However, it turns out that this extension requires C-tightness of $\overline{v}^n_{b/a}$. Hence, in Section 4.2, we instead characterize the limit $\overline{v}_{b/a}$ of $\overline{v}^n_{b/a}$.

Finally, in the third step presented in Section 4.3 we extend the tightness to $\widehat{v}^n_{b/a}$ and prove tightness of $v^n_{b/a}$. In fact, we thereby also obtain the limits for these processes; as it turns out, the processes $\overline{v}^n_{b/a}$, $\widehat{v}^n_{b/a}$ and $v^n_{b/a}$ must all have the same limit. In some more detail, we first show the joint convergence (in a weak sense) of

$$\left(\overline{v}^n_{b/a}\,,\,\eta^n\right) \xrightarrow{n\to\infty} (\overline{v}_{b/a}\,,\,\mathrm{id})\,.$$

By a theorem of Billingsley ([3, Lemma on p. 151]), this implies that (in the appropriate weak sense)

$$\lim_{n\to\infty} \widehat{v}^n_{b/a} = \lim_{n\to\infty} \overline{v}^n_{b/a} \circ (\eta^n) = \overline{v}_{b/a}.$$

Note that for this implication we need C-tightness of the sequence $\overline{v}^n_{b/a}$. Then we prove the tightness of $v^n_{b/a}$ and further verify that $\widehat{v}^n_{b/a} - v^n_{b/a}$ converges to 0 in an $L^2(\Omega; L^2(\mathbb{R}))$-sense, thereby implying that

$$\lim_{n\to\infty} v^n_{b/a} = \lim_{n\to\infty} \widehat{v}^n_{b/a} = \overline{v}_{b/a}.$$

At this stage, we have only treated the convergence of each of the individual sequences of processes (A^n, B^n, v^n_b) and (A^n, B^n, v^n_a) to some limiting processes. However, as all these limiting processes are actually continuous, joint tightness and, finally, joint weak convergence of $\left(A^n, B^n, v^n_b, v^n_a\right)$ therefore follows by Corollary B.3.

3 The scaling limit of the price process

In this section we prove convergence in law of the bid/ask price process to a 2-dimensional reflected Brownian motion. We start with an auxiliary observation on the convergence of the time-change process η^n. According to a strong approximation result, due to Kurtz [18], a standard Poisson process (N_t) can be realized on the same probability space as a Brownian motion (W_t) in such a way that the random variable

$$Y := \sup_{t\geq 0} \frac{|N_t - t - W_t|}{\log(\max\{2, t\})}$$

has finite moment generating function in the neighborhood of the origin and hence finite mean. In particular, Y is almost surely finite. In view of the law of iterated logarithm for Brownian motion, this means that

$$\lim_{n\to\infty} \eta_t^n = t$$

almost surely, uniformly on compact time intervals. Hence, we have the following result.

Lemma 3.1. *The sequence of processes η^n converges almost surely to the identity function uniformly on compact time intervals.*

We are now ready to state the main result of this section.

Proposition 3.2. *As a sequence of processes whose sample paths lie in $\mathcal{D}([0,\infty);\mathbb{R}^2)$, both (A^n, B^n) and $(\overline{A}^n, \overline{B}^n)$ are C-tight and converge to the two-dimensional reflected Brownian motion (A, B) satisfying*

$$dA_t = \frac{1}{\sqrt{2}} d\beta_t^a + dL_t; \quad A_0 = a_0;$$
$$dB_t = \frac{1}{\sqrt{2}} d\beta_t^b - dL_t; \quad B_0 = b_0;$$

where $a_0 > b_0$, β^a and β^b are two independent Wiener processes and L is a non-decreasing process satisfying

$$L_t = \int_0^t \mathbf{1}_{\{A_s = B_s\}}\, dL_s.$$

Proof. It follows from a result on semimartingale reflecting Brownian motion by Kang and Williams [16, Theorem 4.3] that $(\overline{A}^n, \overline{B}^n)$ is C-tight and converges weakly to (A, B) which is the two-dimensional reflected Brownian motion given above. Recalling that

$$(A_u^n, B_u^n) = (\overline{A}^n, \overline{B}^n) \circ (\eta_u^n), \quad u \geq 0, \tag{10}$$

the assertion follows from Lemmas 3.1 and B.4. □

4 The scaling limit of the volume density

In this section, we prove weak convergence in a distributional sense of the volume density function.Throughout, we use the symbol C for deterministic constants which may change from occurrence to occurrence.

4.1 Tightness of the auxiliary process $\overline{v}_{b/a}^n$

We first prove tightness of the processes $\overline{v}_{b/a}^n$. The arguments are the same for the bid and ask side of the book. To ease notation we therefore drop the index indicating bid/ask side volumes in what follows.

Notation 4.1. Where appropriate (i.e., when there is only a negligible chance of confusion and where all considerations can be trivially generalized to all relevant processes), we shall adopt the following notations:

- We drop the superscript "n" (referring to the place in the model hierarchy) as well as the subscript "a" or "b" and any other indices, which are not essential in the respective context. E.g., we may write $\overline{V}^2$ or even just $\overline{V}$ instead of $\overline{V}_b^{n,2}$.
- We may denote by A or $\overline{A}$ either the ask or the bid price.
- We may denote the random location of any activity in the book by π or π_i and its size by ω or ω_i, disregarding the type of activity and whether the sell or buy sides are involved.

We start with an elementary auxiliary lemma on the distribution of a Poisson process as seen from a second, independent Poisson process. The lemma will be key to compute the distribution of passive order arrivals between two consecutive active order times.

Lemma 4.2. *Let N_1 and N_2 be two independent Poisson processes with intensities λ_1 and λ_2, respectively. Moreover, let T_i, $i = 1, \ldots$, denote the jump times of the Poisson process N_1. For any $\alpha = 1, 2, \ldots$, the random variable $N_2(T_\alpha)$ has a negative binomial (NB) distribution with parameters $r = \alpha$ and $p = \frac{\lambda_2}{\lambda_1+\lambda_2}$, i.e., we have*

$$P\left(N_2(T_\alpha) = l\right) = \binom{l+\alpha-1}{\alpha-1}\left(\frac{\lambda_2}{\lambda_1+\lambda_2}\right)^l\left(\frac{\lambda_1}{\lambda_1+\lambda_2}\right)^\alpha, \quad l = 0, 1, \ldots$$

In particular, the moment-generating function reads

$$Ee^{tN_2(T_\alpha)} = \left(\frac{1-p}{1-pe^t}\right)^\alpha, \quad \textit{for } t < -\log\, p,$$

and

$$\begin{aligned}
&E\left[N_2(T_\alpha)\right] = \alpha\frac{\lambda_2}{\lambda_1},\\
&E\left[N_2(T_\alpha)\left(N_2(T_\alpha)-1\right)\right] = \alpha(1+\alpha)\frac{\lambda_2^2}{\lambda_1^2},\\
&E\left[N_2(T_\alpha)\left(N_2(T_\alpha)-1\right)\left(N_2(T_\alpha)-2\right)\right] = \alpha(1+\alpha)(2+\alpha)\frac{\lambda_2^3}{\lambda_1^3},\\
&E\left[N_2(T_\alpha)\left(N_2(T_\alpha)-1\right)\left(N_2(T_\alpha)-2\right)\left(N_2(T_\alpha)-3\right)\right] = \alpha(1+\alpha)(2+\alpha)(3+\alpha)\frac{\lambda_2^4}{\lambda_1^4}.
\end{aligned}$$

In the next lemma, we provide growth estimates for the processes $\overline{V}^{n,1/2}$. As the growth mechanism for these processes (but not for $\overline{V}^{n,3}$) work in the same way, we merge the discussion into one lemma. Denote by $\overline{\mathcal{F}}^n$ the filtration generated by the processes $\overline{V}_{a/b}^{n,1/2/3}$ and $\overline{v}_{a/b}^n$.

Lemma 4.3. *There is a constant $C > 0$ (independent of n, s, t) such that we can bound*

$$E_{\overline{\mathcal{F}}_s^n}\left[\left\|\overline{V}^{n,1/2}(t,\cdot) - \overline{V}^{n,1/2}(s,\cdot)\right\|_{L^2}^2\right] \le C\left((t-s)^2 + \frac{|t-s|}{n}\right),$$

$$\sup_{x\in\mathbb{R}} E_{\overline{\mathcal{F}}^n_s}\left[\left(\overline{V}^{n,1/2}(t,x) - \overline{V}^{n,1/2}(s,x)\right)^2\right] \le C\left((t-s)^2 + \frac{|t-s|}{n}\right),$$
$$E_{\overline{\mathcal{F}}^n_s}\left[\left\|\overline{V}^{n,1/2}(t,\cdot) - \overline{V}^{n,1/2}(s,\cdot)\right\|^4_{L^4}\right] \le C\left((t-s)^4 + \frac{|t-s|^3}{n} + \frac{|t-s|^2}{n^2} + \frac{|t-s|}{n^3}\right),$$
$$\sup_{x\in\mathbb{R}} E_{\overline{\mathcal{F}}^n_s}\left[\left(\overline{V}^{n,1/2}(t,x) - \overline{V}^{n,1/2}(s,x)\right)^4\right] \le C\left((t-s)^4 + \frac{|t-s|^3}{n} + \frac{|t-s|^2}{n^2} + \frac{|t-s|}{n^3}\right).$$

Proof. Without loss of generality, we can choose $s = 0$. Moreover, following the notation convention adopted above, we drop the super-scripts from all the processes and random variables and denote by "A" either bid or ask price, respectively. Let $\alpha := \lfloor nt \rfloor$ and consider

$$E\left[\overline{V}(t,x)^2\right] = E\left[\left(\sum_{i=1}^{N(\widetilde{\tau}_\alpha)} \mathbf{1}_{I\left(\overline{A}(\eta_{\tau_i})+\pi_i\right)}(x)\omega_i\right)^2\right]\left(\frac{\Delta v}{\Delta x}\right)^2.$$

Let $\mathcal{G}$ denote the σ-algebra generated by all sources of randomness *except* $(\omega_i)_{i\in\mathbb{N}}$. Using the fact that the random variables ω_i are i.i.d. and independent from all the other random terms above, we get

$$\begin{aligned} E\left[\overline{V}(t,x)^2\right] &= E\Bigg[\sum_{i\neq j=1}^{N(\widetilde{\tau}_\alpha)} E_{\mathcal{G}}\left[\omega_i\omega_j\right]\mathbf{1}_{I\left(\overline{A}(\eta_{\tau_i})+\pi_i\right)}(x)\mathbf{1}_{I\left(\overline{A}(\eta_{\tau_j})+\pi_j\right)}(x) + \\ &\qquad + \sum_{i=1}^{N(\widetilde{\tau}_\alpha)} E_{\mathcal{G}}\left[\omega_i^2\right]\mathbf{1}_{I\left(\overline{A}(\eta_{\tau_i})+\pi_i\right)}(x)\Bigg]\left(\frac{\Delta v}{\Delta x}\right)^2 \\ &= E\Bigg[\sum_{i\neq j=1}^{N(\widetilde{\tau}_\alpha)} E\left[\omega_1\right]^2\mathbf{1}_{I\left(\overline{A}(\eta_{\tau_i})+\pi_i\right)}(x)\mathbf{1}_{I\left(\overline{A}(\eta_{\tau_j})+\pi_j\right)}(x) + \\ &\qquad + \sum_{i=1}^{N(\widetilde{\tau}_\alpha)} E\left[\omega_1^2\right]\mathbf{1}_{I\left(\overline{A}(\eta_{\tau_i})+\pi_i\right)}(x)\Bigg]\left(\frac{\Delta v}{\Delta x}\right)^2. \end{aligned}$$

Again using independence of $\pi_i, \pi_{i'}$ and all the other random variables, we can bound

$$E\left[\mathbf{1}_{I(y+\pi_i)}(x)\right] \le \|f\|_{L^\infty}\,\Delta x\,\mathbf{1}_{[y-M,y+M]}(x), \tag{11a}$$
$$E\left[\mathbf{1}_{I(y+\pi_i)}(x)\mathbf{1}_{I(y'+\pi_{i'})}(x)\right] \le \|f\|^2_{L^\infty}\,\Delta x^2\,\mathbf{1}_{[\max(y,y')-M,\min(y,y')+M]}(x). \tag{11b}$$

Conditioning on the σ-algebra generated by all sources of randomness *except* $(\pi_i)_{i\in\mathbb{N}}$, these bounds enable us to estimate:

$$\begin{aligned} E\left[\overline{V}(t,x)^2\right] \le E\Bigg[&E\left[\omega_1\right]^2\|f\|^2_{L^\infty}\,\Delta x^2\sum_{i\neq j=1}^{N(\widetilde{\tau}_\alpha)}\mathbf{1}_{\left[\max\left(\overline{A}(\eta_{\tau_i}),\overline{A}(\eta_{\tau_j})\right)-M,\min\left(\overline{A}(\eta_{\tau_i}),\overline{A}(\eta_{\tau_j})\right)+M\right]}(x) + \\ &+ E\left[\omega_1^2\right]\|f\|_{L^\infty}\,\Delta x\sum_{i=1}^{N(\widetilde{\tau}_\alpha)}\mathbf{1}_{\left[\overline{A}(\eta_{\tau_i})-M,\overline{A}(\eta_{\tau_i})+M\right]}(x)\Bigg]\left(\frac{\Delta v}{\Delta x}\right)^2. \end{aligned}$$

At this stage, we can easily bound $\overline{V}$ both in $L^2(\mathbb{R})$ and as a supremum in x. More precisely, we have

$$E\left[\left\|\overline{V}(t)\right\|_{L^2}^2\right] + \sup_{x\in\mathbb{R}} E\left[\overline{V}(t,x)^2\right] \le (2M+1)\Big(E[\omega_1]^2 \|f\|_{L^\infty}^2 \Delta x^2 E\left[N(\widetilde{\tau}_\alpha)\left(N(\widetilde{\tau}_\alpha)-1\right)\right] +$$

$$+ E\left[\omega_1^2\right] \|f\|_{L^\infty} \Delta x E\left[N(\widetilde{\tau}_\alpha)\right]\Big)\left(\frac{\Delta v}{\Delta x}\right)^2.$$

Finally, inserting the moment formulas given in Lemma 4.2 and applying the trivial estimate $\alpha = \lfloor nt \rfloor \le nt$ together with Assumption 3, we arrive at

$$E\left[\left\|\overline{V}(t)\right\|_{L^2}^2\right] + \sup_{x\in\mathbb{R}} E\left[\overline{V}(t,x)^2\right] \le Cn^{-7/2}\left\{n^{-1/2} nt(1+nt)\frac{n^4}{n^2} + nt\frac{n^2}{n}\right\} = C\left(t^2 + (n^{-1} + n^{-3/2})t\right) \le C\left(t^2 + \frac{t}{n}\right).$$

The estimate for the fourth moment follows analogously and is therefore skipped. □

The growth bound for $\overline{V}^{n,3}$ works, in principle, similarly. Note, however, that the scaling for $\overline{V}^{n,3}$ is much smaller. Hence, we need to take advantage of the martingale-difference structure in order to avoid the mixed terms in the proof of Lemma 4.3.

Lemma 4.4. *There is a constant C (independent of n, s, t) such that*

$$E_{\overline{\mathcal{F}}_s^n}\left[\sup_{s\le u\le t}\left\|\overline{V}^{n,3}(u) - \overline{V}^{n,3}(s)\right\|_{L^2}^2\right] + \sup_{x\in\mathbb{R}} E_{\overline{\mathcal{F}}_s^n}\left[\sup_{s\le u\le t}\left|\overline{V}^{n,3}(u) - \overline{V}^{n,3}(s)\right|^2\right] \le C|t-s|, \tag{12}$$

$$E_{\overline{\mathcal{F}}_s^n}\left[\sup_{s\le u\le t}\left\|\overline{V}^{n,3}(u) - \overline{V}^{n,3}(s)\right\|_{L^4}^4\right] + \sup_{x\in\mathbb{R}} E_{\overline{\mathcal{F}}_s^n}\left[\sup_{s\le u\le t}\left|\overline{V}^{n,3}(u) - \overline{V}^{n,3}(s)\right|^4\right] \le C\left((t-s)^2 + \frac{|t-s|}{n}\right). \tag{13}$$

Proof. Again, we restrict ourselves to proving the case $s = 0$, and we drop all indices from the notation. Re-writing $\overline{V}$ in a form more clearly expressing its martingale structure, we consider

$$\overline{V}(t) = \sum_{i=1}^{N(\widetilde{\tau}_\alpha)} \mathbf{1}_{I\left(\overline{A}(\eta^n_{\tau^n_i}) + \pi_i\right)}(x)\omega_i \widetilde{\xi}_{\widetilde{N}(\tau_i)}\sqrt{\Delta v} = \sum_{j=0}^{\alpha-1}\sum_{i=N(\widetilde{\tau}_j)+1}^{N(\widetilde{\tau}_{j+1})} \mathbf{1}_{I\left(\overline{A}(j/n)+\pi_i\right)}(x)\omega_i\widetilde{\xi}_j\sqrt{\Delta v},$$

where we again use the short-hand notation $\alpha = \lfloor tn \rfloor$. Using Doob's inequality and the fact that $E\left[\widetilde{\xi}_i\widetilde{\xi}_j\right] = \delta_{ij}$ with $\widetilde{\xi}_i^2 = 1$, we have

$$\begin{aligned} E\left[\sup_{0\le u\le t}|\overline{V}(u,x)|^2\right] &\le 4E\left[|\overline{V}(t,x)|^2\right] \\ &= 4\Delta v E\left[\left(\sum_{j=0}^{\alpha-1}\widetilde{\xi}_j \sum_{i=N(\widetilde{\tau}_j)+1}^{N(\widetilde{\tau}_{j+1})} \mathbf{1}_{I\left(\overline{A}(j/n)+\pi_i\right)}(x)\omega_i\right)^2\right] \\ &= 4\Delta v E\left[\sum_{j=0}^{\alpha-1}\left(\sum_{i=N(\widetilde{\tau}_j)+1}^{N(\widetilde{\tau}_{j+1})} \mathbf{1}_{I\left(\overline{A}(j/n)+\pi_i\right)}(x)\omega_i\right)^2\right]. \end{aligned}$$

In the next step, we shall again estimate the contribution of the random locations π in a similar way as in (11). To this end, let $\mathcal{G}$ denote the σ-algebra generated by all the sources of randomness *except* $(\pi_i)_{i=1}^\infty$, which is then by construction independent from $\mathcal{G}$. Hence, we have

$$E\left[\sup_{0\le u\le t}\left\|\overline{V}(u)\right\|_{L^2}^2\right] \le 4\Delta v E\left[\sum_{j=0}^{\alpha-1}\left\{\sum_{i\ne i'=N(\widetilde{\tau}_j)+1}^{N(\widetilde{\tau}_{j+1})} \omega_i\omega_{i'}\int_{\mathbb{R}} E_{\mathcal{G}}\left[\mathbf{1}_{I\left(\overline{A}(j/n)+\pi_i\right)}(x)\mathbf{1}_{I\left(\overline{A}(j/n)+\pi_{i'}\right)}(x)\right]\right\}\right]dx +$$

$$+\sum_{i=N(\widetilde{\tau}_j)+1}^{N(\widetilde{\tau}_{j+1})}\omega_i^2\int_{\mathbb{R}}E_{\mathcal{G}}\left[\mathbf{1}_{I\left(\overline{A}(j/n)+\pi_i\right)}(x)\right]dx\Bigg]$$

$$\leq 4\Delta v E\left[\sum_{j=0}^{\alpha-1}\left\{\sum_{i\neq i'=N(\widetilde{\tau}_j)+1}^{N(\widetilde{\tau}_{j+1})}\omega_i\omega_{i'}\|f\|_{L^\infty}^2\Delta x^2(2M)+\sum_{i=N(\widetilde{\tau}_j)+1}^{N(\widetilde{\tau}_{j+1})}\omega_i^2\|f\|_{L^\infty}\Delta x(2M)\right\}\right],$$

and similarly,

$$\sup_{x\in\mathbb{R}}E\left[\sup_{0\leq u\leq t}|\overline{V}(u,x)|^2\right]\leq 4\Delta v E\left[\sum_{j=0}^{\alpha-1}\left\{\sum_{i\neq i'=N(\widetilde{\tau}_j)+1}^{N(\widetilde{\tau}_{j+1})}\omega_i\omega_{i'}\|f\|_{L^\infty}^2\Delta x^2+\sum_{i=N(\widetilde{\tau}_j)+1}^{N(\widetilde{\tau}_{j+1})}\omega_i^2\|f\|_{L^\infty}\Delta x\right\}\right].$$

By independence of the Poisson processes N and $\widetilde{N}$ from ω_i and by the fact that the distribution of the increments $N(\widetilde{\tau}_{j+1})-N(\widetilde{\tau}_j)$ of one Poisson process as seen from the other does not depend on j, we see that

$$E\left[\sup_{0\leq u\leq t}\left\|\overline{V}(u)\right\|_{L^2}^2\right]+\sup_{x\in\mathbb{R}}E\left[\sup_{0\leq u\leq t}|\overline{V}(u,x)|^2\right]$$
$$\leq C\,\Delta v E\left[\alpha\left\{E[\omega_1]^2\left\|f^\pi\right\|_{L^\infty}^2(\Delta x)^2N(\widetilde{\tau}_1)\left(N(\widetilde{\tau}_1)-1\right)+E\left[\omega_1^2\right]\left\|f^\pi\right\|_{L^\infty}\Delta x N(\widetilde{\tau}_1)\right\}\right].$$

Again appealing to Lemma 4.2 (with $\alpha=1$) together with Assumption 3, we obtain

$$E\left[\sup_{0\leq u\leq t}\left\|\overline{V}(u)\right\|_{L^2}^2\right]+\sup_{x\in\mathbb{R}}E\left[\sup_{0\leq u\leq t}|\overline{V}(u,x)|^2\right]\leq C\frac{1}{n^2}nt\left\{\frac{1}{n}2\frac{n^4}{n^2}+\frac{1}{\sqrt{n}}\frac{n^2}{n}\right\}=Ct\{2+1/\sqrt{n}\}\leq Ct.$$

As in the proof of Lemma 4.3, the estimate for the fourth moment follows by the similar arguments.

□

At this stage we can patch together the growth bounds of Lemmas 4.3 and 4.4 to obtain a similar growth bound for the process $\overline{v}^n$. The proof is based on an event-by-event decomposition of the limit order book dynamics. More precisely, in terms of the increments

$$h_{a,i}^{n,1}(x):=\omega_i^{\mathbf{P}_a}\mathbf{1}_{I^n\left(\overline{A}^n\left(\eta^n_{\tau^n_{a,i}}\right)+\pi_i^{\mathbf{P}_a}\right)}(x)\frac{\Delta v^n}{\Delta x^n},$$
$$h_{a,i}^{n,2}(x):=\omega_i^{\mathbf{C}_a}\mathbf{1}_{I^n\left(\overline{A}^n\left(\eta^n_{\tau^n_{a,i}}\right)+\pi_i^{\mathbf{C}_a}\right)}(x)\frac{\Delta v^n}{\Delta x^n},$$
$$h_{a,i}^{n,3}(x):=\mathbf{1}_{I^n\left(\overline{A}^n\left(\eta^n_{\tau^n_{a,i}}\right)+\pi_i^{\mathbf{N}_a}\right)}(x)\omega_i^{\mathbf{N}_a}\widetilde{\xi}_{a,\widetilde{N}^n(\tau^n_{a,i})+1}\sqrt{\Delta v^n}$$

of the processes $\overline{V}_a^{n,j}$ $(j=1,2,3)$ – and similarly for the buy-side – one has the following decomposition of the LOB dynamics:

$$\overline{v}^n(t,x)=\prod_{i=1}^{N^n(\overline{\tau}^n_{\lfloor nt\rfloor})}\left(1-h_i^{n,2}(x)\right)\overline{v}^n(0,x)+$$

$$+\prod_{i=1}^{N^n_{a/b}(\overline{\tau}^n_{\lfloor nt\rfloor})}\left(1-h^{n,2}_i(x)\right)\left[\sum_{i=1}^{N^n(\overline{\tau}^n_{\lfloor nt\rfloor})}\frac{1}{\prod_{m=1}^i\left(1-h^{n,2}_m(x)\right)}\left(h^{n,1}_i(x)+h^{n,3}_i(x)\right)\right] \quad (14)$$

Clearly, $h^{n,1}_i$ is the effect of the placement at the i'th passive order event, $h^{n,3}_i$ the fluctuation effect, whereas $h^{n,2}_i$ is the proportion of standing volume canceled.

Lemma 4.5. *There is a sequence of non-negative adapted process C^n_t and a deterministic constant C such that for $p \in \{2,4\}$*

$$E_{\overline{\mathcal{F}}^n_s}\left[\sup_{s\le r\le t}\left\|\overline{v}^n(r)-\overline{v}^n(s)\right\|^p_{L^p}\right]+\sup_{x\in\mathbb{R}}E_{\overline{\mathcal{F}}^n_s}\left[\sup_{s\le r\le t}\left|\overline{v}^n(r,x)-\overline{v}^n(s,x)\right|^p\right]\le C^n_s\left((t-s)^p+(t-s)\right)$$

with

$$\sup_n E\left[\sup_{0\le s\le t}C^n_s\right]\le C(t^p+t). \quad (15)$$

Proof. We may again drop the dependence on n from the notation and w.l.o.g. assume $s=0$. Note that $0\le 1-h^2_i(x)\le 1$ and

$$\left|\prod_{i=1}^{N(\overline{\tau}_{\lfloor nt\rfloor})}(1-h^2_i(x))-1\right|\le\sum_{i=1}^{N(\overline{\tau}_{\lfloor nt\rfloor})}h^2_i(x)=\overline{V}^2(t,x).$$

Hence, (14) together with Lemma 4.3 and 4.4 implies that for $p\in\{2,4\}$,

$$\begin{aligned}|\overline{v}(t,x)-\overline{v}(0,x)|^p&=\left|\left(\prod_{i=1}^{N(\overline{\tau}_{\lfloor nt\rfloor})}\left(1-h^2_i\right)-1\right)\overline{v}(0,x)+\prod_{i=1}^{N(\overline{\tau}_{\lfloor nt\rfloor})}\left(1-h^2_i\right)\left(\sum_{i=1}^{N(\overline{\tau}_{\lfloor nt\rfloor})}\frac{1}{\prod_{m=1}^i\left(1-h^2_i\right)}\left(h^1_i+h^3_i\right)\right)\right|^p\\&\le C\left\{|\overline{v}(0,x)|^p\sup_{x\in\mathbb{R}}\left(\overline{V}^2(t,x)\right)^p+\left|\overline{V}^1(t,x)\right|^p+\sup_{0\le s\le t}\left|\overline{V}^3(s,x)\right|^p\right\}.\end{aligned}$$

It then follows that for $p\in\{2,4\}$,

$$\sup_{x\in\mathbb{R}}E\left[\sup_{0\le u\le t}|\overline{v}(u,x)-\overline{v}(0,x)|^p\right]+E\left[\sup_{0\le u\le t}\|\overline{v}(u)-\overline{v}(0)\|^p_{L^p}\right]\le C\left(\sup_{x\in\mathbb{R}}|\overline{v}(0,x)|^p+E\left[\|\overline{v}(0)\|^p_{L^p}\right]+1\right)(t^p+t).$$

For a general $s\in[0,t]$, this only proves the estimate for a $\overline{\mathcal{F}}^n_s$-measurable random variable C^n_s which depends in an affine way on $\|\overline{v}(s)\|_{L^2}$. Note, however, that the above estimate also implies that for $p\in\{2,4\}$

$$\sup_{n\in\mathbb{N}}\left(E\left[\sup_{0\le s\le t}\left\|\overline{v}^n_{a/b}(s)\right\|^p_{L^p}\right]+\sup_{x\in\mathbb{R}}E\left[\sup_{0\le s\le t}\left|\overline{v}^n_{a/b}(s,x)\right|^p\right]\right)<C(t^p+t),$$

so that we can, indeed, find a deterministic constant C which is independent of s, t and n and bounds $E\left[\sup_{0\le s\le t}C^n_s\right]\le C(t^p+t)$. □

Remark 4.6. In a similar way to the above proof, we obtain for $p\in\{2,4\}$ and $k=0,1,2,\cdots$,

$$E\left[\sup_{i\in[N^n_a(\overline{\tau}^n_k),N^n_a(\overline{\tau}^n_{k+1})]\cap\mathbb{N}}\|v^n_a(\tau_{a,i})-v^n_a(\eta^n_k)\|^p_{L^p}\right]+\sup_{x\in\mathbb{R}}E\left[\sup_{i\in[N^n_a(\overline{\tau}^n_k),N^n_a(\overline{\tau}^n_{k+1})]\cap\mathbb{N}}\left|v^n_a(\tau_{a,i},x)-v^n_a(\overline{\tau}^n_k,x)\right|^p\right]\le C\frac{t+t^p}{n},$$

where the constant C is independent of n, k and t.

We are now ready to state and prove the main result of this section.

Proposition 4.7. *The processes $\overline{v}^n_{b/a}$, $\overline{V}^{n,1}_{b/a}$, $\overline{V}^{n,2}_{b/a}$ and $\overline{V}^{n,3}_{b/a}$ are tight as processes with paths in $\mathcal{D}\big([0,\infty);H^{-1}\big)$.*

Proof. For $X^n \in \{\overline{v}^n_{b/a}, \overline{V}^{n,1}_{b/a}, \overline{V}^{n,2}_{b/a}, \overline{V}^{n,3}_{b/a}\}$, according to the definition it is obvious that the tightness of X^n is equivalent to that of $((1+t)^{-1}X^n(t))_{t\in[0,\infty)}$ which we denote again by X^n. The reason we scale the processes this way is estimate (15) which prevents us from applying Theorem B.1 directly to the original processes.

By Mitoma's theorem (see Theorem B.2), we need to prove tightness of the processes $\langle X^n\,,\phi\rangle$ for any test function $\phi \in \mathcal{E} \subset L^2(\mathbb{R}, dx)$, for which we, in turn, will appeal to Kurtz's criterion (see Theorem B.1). Hence, we need to estimate

$$E_{\mathcal{F}^n_s}\left[\langle X^n(t) - X^n(s)\,,\phi\rangle^2\right].$$

As each of the processes $\overline{v}^n, \overline{V}^{n,1}, \overline{V}^{n,2}, \overline{V}^{n,3}$ takes values in L^2, the bracket $\langle X^n\,,\phi\rangle$ is equal to the L^2 inner product $\langle X^n\,,\phi\rangle_{L^2}$, and so we can estimate

$$E_{\mathcal{F}^n_s}\left[\langle X^n(t) - X^n(s)\,,\phi\rangle^2\right] \le E_{\mathcal{F}^n_s}\left[\left\|X^n(t) - X^n(s)\right\|^2_{L^2}\right]\|\phi\|^2_{L^2} \le \sup_{\tau\in[0,\infty)}(1+\tau)^{-2}C^n_\tau\left[(t-s)^2 + (t-s)\right]\|\phi\|^2_{L^2}\,.$$

The second condition of Theorem B.1 follows with $\gamma_n(\delta) = \sup_{\tau\in[0,\infty)}(1+\tau)^{-2}C^n_\tau(\delta^2+\delta)$ by Lemmas 4.3, 4.4 and 4.5.

For the first condition, i.e., tightness of the sequence of random variables $\langle X^n(t)\,,\phi\rangle$ for each (rational) t, we note that this trivially follows from uniform boundedness of the sequence of random variables $\langle X^n(t)\,,\phi\rangle$ in $L^2(\Omega,\mathcal{F},P)$. Moreover, for any $T \in (0,\infty)$, we have by Lemmas 4.3, 4.4 and 4.5,

$$\sup_n E \sup_{t\in[0,T]} \|X^n(t)\|^2_{L^2} \le C(T+T^2),$$

with the constant C being independent of n and T. It follows that for $N \in (0,\infty)$,

$$P\left(\sup_{t\in[0,T]}\|X^n(t)\|^2_{L^2} > N\right) \le \frac{C(T+T^2)}{N} \to 0, \quad \text{as } N\to\infty.$$

By Theorems B.1 and B.2, X^n and hence $\overline{v}^n_{b/a}$, $\overline{V}^{n,1}_{b/a}$, $\overline{V}^{n,2}_{b/a}$ and $\overline{V}^{n,3}_{b/a}$ are tight as sequences of processes with paths in $\mathcal{D}\big([0,\infty);H^{-1}\big)$. □

Remark 4.8. This proof *almost* gives us tightness in $\mathcal{D}\big([0,\infty);L^2(\mathbb{R})\big)$ for $L^2(\mathbb{R})$ equipped with the weak topology. Note, however, that $L^2(\mathbb{R})$ is not a metric space when equipped with the weak topology. Hence we cannot use Kurtz's criterion as it does not apply to non-metric state spaces.

4.2 Characterization of the limit of $\overline{v}^n_{b/a}$

In this section, we characterize the limit of the sequence $\overline{v}^n_{b/a}$. We give the arguments for the ask side and write **P**, **C** and $\mathbf{N}$ for $\mathbf{P}_a$, $\mathbf{C}_a$ and $\mathbf{N}_a$. The arguments for the bid side are identical. We start with establishing joint convergence in distribution of bid/ask prices along with the aggregate fluctuations of standing volumes on the ask side of the book.

Proposition 4.9. *There is a Wiener process* $\{W_a(t); t \in [0,\infty)\}$ *such that* $(\overline{A}^n, \overline{B}^n, \overline{V}_a^{n,3}) \Rightarrow (A, B, V_a^3)$, *with* (A, B) *the two-dimensional reflected Brownian motion given in Theorem 2.6 and*

$$V_a^3(t) = \sqrt{2}E\left[\omega_1^{\mathbf{N}_a}\right]\int_0^t f^{\mathbf{N}_a}(x + A_s)\, dW_a(s), \quad t \in [0,\infty).$$

Proof. Combining Proposition 4.7, Corollary B.3 and Proposition 3.2, we conclude that $(\overline{A}^n, \overline{B}^n, \overline{V}_a^{n,3})$ is tight as a sequence of processes with paths lying in $\mathcal{D}([0,\infty); \mathbb{R}^2 \times H^{-1})$ and that $(\overline{A}^n, \overline{B}^n)$ converges in distribution to the two-dimensional reflected Brownian motion (A, B). By Skorohod's lemma, we may assume that all random variables and processes are defined on a common probability space, and – restricting to a subsequence if necessary – that the sequence $(\overline{A}^n, \overline{B}^n, \overline{V}_a^{n,3})$ converges with probability 1 to $(\overline{A}, \overline{B}, \overline{V}_a^3)$ as a sequence of processes whose sample paths belong to $\mathcal{D}(0,\infty; \mathbb{R}^2 \times H^{-1})$.

Since the sequence of price processes is C-tight and converges to the 2-dimensional reflected Brownian motion, it is sufficient to characterize the weak accumulation point $\overline{V}_a^3$. To this end, we define for any $\phi \in \mathcal{E}$

$$\overline{Y}_t^n = \langle \phi, \overline{V}_a^{n,3}(t) \rangle, \quad t \in [0,\infty),$$

and denote by $\mathcal{G}^n$ the filtration generated by the processes $(\overline{A}_t^n, \overline{B}_t^n, \overline{V}_t^{n,3})$. Note that the sequence $(\overline{A}^n, \overline{B}^n, \overline{Y}^n)$ converges with probability 1 to $(\overline{A}, \overline{B}, \langle \phi, \overline{V}_a^3 \rangle)$ as a sequence of processes whose sample paths belong to $\mathcal{D}(0,\infty; \mathbb{R}^3)$. Let

$$\begin{aligned}
a_0^n(\cdot) &:= \Big(\sum_j \int_{x_j^n}^{x_{j+1}^n} f(x+\cdot)\, dx \int_{x_j^n}^{x_{j+1}^n} \phi(x)\, dx\Big)^2 (\Delta x^n)^{-2} E\left[\omega_1^{\mathbf{N}}\right]^2, \\
a_1^n(\cdot) &:= \sum_j \int_{x_j^n}^{x_{j+1}^n} f(x+\cdot)\, dx \,\Big| \int_{x_j^n}^{x_{j+1}^n} \phi(x)\, dx \Big|^2 (\Delta x^n)^{-2} E\left[(\omega_1^{\mathbf{N}})^2\right] \\
\sigma^n(\cdot) &:= \left\{(2a_0^n + \frac{1}{n} a_1^n)(\cdot)\right\}^{1/2},
\end{aligned}$$

with $f = f^{\mathbf{N}_a}$. Since the number of passive order arrivals $\left(N^n_{\overline{\tau}^n_{a,k}} - N^n_{\overline{\tau}^n_{a,k-1}}\right)$ on $[\frac{k-1}{n}, \frac{k}{n})$ follows a negative binomial distribution $\mathrm{NB}\left(1, \frac{\lambda^n}{\lambda^n + \mu^n}\right)$ (see Lemma 4.2), we have:

$$\begin{aligned}
&E_{\mathcal{G}^n_{\frac{k-1}{n}}}\left[|\overline{Y}^n_{k/n} - \overline{Y}^n_{(k-1)/n}|^2\right] \\
&= \Delta v^n \Big\{ \left(N^n_{\overline{\tau}^n_{a,k}} - N^n_{\overline{\tau}^n_{a,k-1}}\right)\left(N^n_{\overline{\tau}^n_{a,k}} - N^n_{\overline{\tau}^n_{a,k-1}} - 1\right)\Big(\sum_j \int_{x_j^n}^{x_{j+1}^n} f(x + \overline{A}^n_{\frac{k-1}{n}})\, dx \int_{x_j^n}^{x_{j+1}^n} \phi(x)\, dx\Big)^2 E\left[\omega_1^{\mathbf{N}}\right]^2 \\
&\quad + \left(N^n_{\overline{\tau}^n_{a,k}} - N^n_{\overline{\tau}^n_{a,k-1}}\right) \sum_j \int_{x_j^n}^{x_{j+1}^n} f(x + \overline{A}^n_{\frac{k-1}{n}})\, dx \Big| \int_{x_j^n}^{x_{j+1}^n} \phi(x)\, dx \Big|^2 E\left[\left(\omega_1^{\mathbf{N}}\right)^2\right] \Big\} \\
&= \Delta v^n (\Delta x^n)^2 \Big\{ \left(N^n_{\overline{\tau}^n_{a,k}} - N^n_{\overline{\tau}^n_{a,k-1}}\right)\left(N^n_{\overline{\tau}^n_{a,k}} - N^n_{\overline{\tau}^n_{a,k-1}} - 1\right) a_0^n(\overline{A}^n_{\frac{k-1}{n}}) + \left(N^n_{\overline{\tau}^n_{a,k}} - N^n_{\overline{\tau}^n_{a,k-1}}\right) a_1^n(\overline{A}^n_{\frac{k-1}{n}}) \Big\} \\
&= \frac{1}{n^3}\left(2n^2 a_0^n + n a_1^n\right)(\overline{A}^n_{\frac{k-1}{n}})
\end{aligned}$$

$$= \frac{1}{n}\left(\sigma^n(\overline{A}^n_{\frac{k-1}{n}})\right)^2.$$

Set

$$\sigma(\cdot) = \sqrt{2}\int_{\mathbb{R}} f_a^{\pi}(x+\cdot)\phi(x)\,dxE\left[\omega_1^{\mathbf{N}}\right], \quad t \in [0,\infty).$$

In order to conclude, we apply a result on the convergence of interpolated Markov chains to a diffusion due to Kushner [19]. For this we need to verify the following conditions for any $t > 0$:

$$E\left[\sum_{k=1}^{\lfloor nt \rfloor} |\sigma^n(\overline{A}^n_{\frac{k-1}{n}}) - \sigma(\overline{A}^n_{\frac{k-1}{n}})|^2\right]\frac{1}{n} \to 0, \tag{C1}$$

$$E\sum_{k=1}^{\lfloor nt \rfloor+1} |\overline{Y}^n_{k/n} - \overline{Y}^n_{(k-1)/n}|^4 \to 0. \tag{C2}$$

Indeed, under (C1) and (C2), an easy extension of [19, Theorem 1, Page 44–48] gives convergence (with probability 1) of $(\overline{A}^n, \overline{B}^n, \overline{Y}^n)$ (or a proper subsequence thereof) to $(A, B, \overline{Y})$ as a sequence of processes whose paths belong to $\mathcal{D}(0,\infty;\mathbb{R}^3)$ with

$$d\overline{Y}_t = \sigma(A_t)dW_t, \ \ t \in [0,\infty); \quad \overline{Y}_0 = Y_0. \tag{16}$$

Condition (C1) can be verified easily. Hence, we concentrate on the condition (C2) (specialized from assumption (A4) of [19, Page 42]):

$$\begin{aligned}
E_{\mathcal{G}^n_{\frac{k-1}{n}}}\left[|\overline{Y}^n_{k/n} - \overline{Y}^n_{(k-1)/n}|^4\right] &\le C(\Delta v^n)^2(\Delta x^n)^4 E\left[\left|N^n_{\overline{\tau}^n_{a,k}} - N^n_{\overline{\tau}^n_{a,k-1}}\right|^4\right] \\
&\le C\frac{1}{n^6}\left[n^4 + n\right] \\
&\le C\frac{1}{n^2},
\end{aligned}$$

where C is a positive constant which is independent n and may vary from line to line. Thus, for any $t \in (0,\infty)$,

$$E\sum_{k=1}^{\lfloor nt \rfloor+1} |\overline{Y}^n_{k/n} - \overline{Y}^n_{(k-1)/n}|^4 \le C(nt+1)\frac{1}{n^2} \to 0 \quad \text{as} \quad n \to \infty.$$

Hence, $\overline{Y}^n$ converges with probability 1 to $\overline{Y}$ of (16) being valued in $\mathcal{D}(0,\infty;\mathbb{R})$. Since $\mathcal{E}$ is dense in H^1 and the limit does not depend on the selected subsequence, this proves our assertion. □

Remark 4.10. In the above proof, the triples $\left(\overline{A}^n_{\frac{k}{n}}, \overline{B}^n_{\frac{k}{n}}, \overline{Y}^n_{\frac{k}{n}}\right)$ can be seen as a sequence of interpolated Markov chains but fall beyond the framework of Kushner [19] as the limit of $(\overline{A}^n_t, \overline{B}^n_t)$ turns out to be the two-dimensional reflected Brownian motion. However, after verifying the tightness of $(\overline{A}^n_t, \overline{B}^n_t, \overline{Y}^n_t)$ and characterizing the limit of $(\overline{A}^n_t, \overline{B}^n_t)$, we use directly the method of [19, Theorem 1, Page 44–48] to identify the limit of $\overline{Y}^n_t$ and the proof is so similar that we just verify the sufficient conditions listed therein.

The previous proposition characterizes the diffusion part of the limiting ask-side volume density process. Next we are going to study the limiting structures of aggregate order placements and cancellations, disregarding the random fluctuations. As we expect order placements and cancellations to contribute the drift part of the limiting model, we find it helpful to re-write their dynamics in the form of an integral in time. That is, if we write

$$\overline{V}_a^{n,2}(t,x) = \int_0^{\frac{\lfloor nt \rfloor}{n}} g^n(s,x)ds,$$
$$\overline{V}_a^{n,1}(t,x) = \int_0^{\frac{\lfloor nt \rfloor}{n}} \widetilde{g}^n(s,x)ds,$$

it is clear that we can identify the limiting drift term by studying the limits of g^n and $\widetilde{g}^n$. Comparing with (9), we have

$$g^n(t,x) := \sum_{k=1}^{\infty} \sum_{i=N_a^n(\overline{\tau}_{k-1}^n)+1}^{N_a^n(\overline{\tau}_k^n)} \mathbf{1}_{I^n\left(\pi_i^{\mathbf{C}}+\overline{A}_{\frac{k-1}{n}}^n\right)}(x)\omega_i^{\mathbf{C}} \mathbf{1}_{[\frac{k}{n},\frac{k+1}{n})}(t) \frac{\Delta v^n}{\Delta x^n} n,$$
$$\widetilde{g}^n(t,x) := \sum_{k=1}^{\infty} \sum_{i=N_a^n(\overline{\tau}_{k-1}^n)+1}^{N_a^n(\overline{\tau}_k^n)} \mathbf{1}_{\left(\pi_i^{\mathbf{P}}+\overline{A}_{\frac{k-1}{n}}^n\right)}(x)\omega_i^{\mathbf{P}} \mathbf{1}_{[\frac{k}{n},\frac{k+1}{n})}(t) \frac{\Delta v^n}{\Delta x^n} n.$$

With regards to aggregate cancellations, g^n only captures the proportionality of cancellations in terms of present volume. Therefore, we need to introduce one more term $\overline{g}^n$ describing the actual cancellations, i.e.,

$$\overline{v}_a^n(t,x) - v_a(0,x) - \overline{V}_a^{n,1}(t,x) - \overline{V}_a^{n,3}(t,x) = \int_0^{\frac{\lfloor nt \rfloor}{n}} \overline{g}^n(s,x)ds.$$

Clearly, $\overline{g}^n$ is given by

$$\overline{g}^n(t,x) := \sum_{k=1}^{\infty} \sum_{i=N_a^n(\overline{\tau}_{k-1}^n)+1}^{N_a^n(\overline{\tau}_k^n)} \mathbf{1}_{I^n\left(\pi_i^{\mathbf{C}}+\overline{A}_{\frac{k-1}{n}}^n\right)}(x)\omega_i^{\mathbf{C}} v_a^n(\tau_{a,i-1}^n,x) \mathbf{1}_{[\frac{k}{n},\frac{k+1}{n})}(t) \frac{\Delta v^n}{\Delta x^n} n.$$

We will analyze the impact of order cancellations in the limit in two steps: first we show that we can replace $\overline{g}^n$ by the (much simpler) expression $g^n \overline{v}_a^n$ in the limit (see Lemma 4.12), and then we characterize the limit of the latter term in the appropriate sense (see Lemma 4.13, where we also characterize the limiting object of the order placements).

Remark 4.11. From Lemma 4.3, it follows that for $p \in \{2,4\}$,

$$E\left[\left\|g^n(t)\right\|_{L^p}^p\right] + \sup_{x\in\mathbb{R}} E_{\mathcal{F}_s^n} E\left[\left|g^n(t,x)\right|^p\right] \le C,$$

which implies that

$$\sup_{x\in\mathbb{R}} E \int_0^t \left|g^n(s,x)\right|^p ds + E \int_{\mathbb{R}} \int_0^t \left|g^n(s,x)\right|^p dsdx \le Ct,$$

with the constants C being independent of n and t.

Lemma 4.12. *For any $t > 0$, we have*

$$\lim_{n\to\infty} E\left[\int_{\mathbb{R}}\int_0^{\frac{\lfloor nt\rfloor}{n}} \left|\overline{g}^n(s,x) - g^n(s,x)\overline{v}_a^n(s,x)\right|^2 dsdx\right] = 0. \tag{17}$$

Proof. Using Fubini's theorem and Remark 4.6, we have

$$\begin{aligned}
&E\int_{\mathbb{R}}\int_0^{\frac{\lfloor nt\rfloor}{n}} \left|\overline{g}^n(s,x) - g^n(s,x)\overline{v}_a^n(s,x)\right|^2 dsdx \\
&= \int_0^{\frac{\lfloor nt\rfloor}{n}} E\int_{\mathbb{R}}\Big|\sum_{k\in\mathbb{N}}\sum_{i=N_a^n(\overline{\tau}_{k-1}^n)+1}^{N_a^n(\overline{\tau}_k^n)} \mathbf{1}_{I^n\left(\pi_i^{\mathbf{C}}+\overline{A}_{\frac{k}{n}}\right)}(x)\omega_i^{\mathbf{C}}\Big(v_a^n(\tau_{a,i-1}^n,x) - \overline{v}_a^n(s,x)\Big)\mathbf{1}_{[\frac{k}{n},\frac{k+1}{n})}(s)\frac{\Delta v^n}{\Delta x^n n^{-1}}\Big|^2 dxds \\
&\le \int_0^{\frac{\lfloor nt\rfloor}{n}} \sum_{k\in\mathbb{N}\cup\{0\}} \mathbf{1}_{[\frac{k}{n},\frac{k+1}{n})}(s)\Big(E\int_{\mathbb{R}}|g^n(s,x)|^4dx\Big)^{1/2}\Big(E\sup_{i\in[N_a^n(\overline{\tau}_{k-1}^n),\,N_a^n(\overline{\tau}_k^n)]\cap\mathbb{N}}\|v_a^n(\tau_{a,i}) - v_a^n(\overline{\tau}_{k-1}^n)\|_{L^4}^4\Big)^{1/2}ds \\
&\le C\frac{1}{\sqrt{n}}\int_0^{\frac{\lfloor nt\rfloor}{n}}\Big(E\int_{\mathbb{R}}|g^n(s,x)|^4dx\Big)^{1/2}ds \\
&\le C\frac{1}{\sqrt{n}}\Big(E\int_0^{\frac{\lfloor nt\rfloor}{n}}\int_{\mathbb{R}}|g^n(s,x)|^4dx\Big)^{1/2},
\end{aligned}$$

which by Remark 4.11 converges to zero as n tends to infinity. □

We can now analyze the limiting objects obtained from order placements and cancellations. The proof of Lemma 4.13 is technical and rather long and hence postponed to Appendix A.

Lemma 4.13. *For any $t = \frac{\lfloor nt\rfloor}{n}$ with $n \in \mathbb{N}$,*

$$\lim_{n\to\infty}\sup_{x\in\mathbb{R}} E\left[\left|\int_0^t \Big(g^n(s,x) - E[\omega_1^{\mathbf{C}}]f^{\mathbf{C}}(A_s+x)\Big)(1-\alpha+\alpha\overline{v}_a^n(s,x))\,ds\right|^2\right] = 0, \quad \forall\,\alpha\in\{0,1\}, \tag{18}$$

$$\lim_{n\to\infty}\sup_{x\in\mathbb{R}} E\left[\left|\int_0^t \Big(\overline{g}^n(s,x) - E[\omega_1^{\mathbf{P}}]f^{\mathbf{P}}(A_s+x)\Big)\,ds\right|^2\right] = 0. \tag{19}$$

Combining the characterization of the limit of the fluctuation part of $\overline{v}_{a/b}^n$ obtained in Proposition 4.9 with the characterization of the limits of order cancellations and placements obtained in Lemma 4.13 together with Lemma 4.12, we are in the position to characterize the limit of $\overline{v}^n$ itself.

Theorem 4.14. *There is a Wiener process $\{W_a(t); t\in[0,\infty)\}$ such that $(\overline{A}^n, \overline{B}^n, \overline{V}_a^{n,3}, \overline{v}_a^n) \Rightarrow (A, B, V_a^3, v_a)$, with (A,B) the two-dimensional reflected Brownian motion, V_a^3 the limit obtained in Proposition 4.9 and*

$$\begin{aligned}
v_a(t,x) = v_a(0,x) &+ \int_0^t \Big(E[\omega_1^{\mathbf{P}}]f^{\mathbf{P}}(x+A_s) - E[\omega_1^{\mathbf{C}}]f^{\mathbf{C}}(x+A_s)v_a(s,x)\Big)\,ds + \\
&+ \sqrt{2}\int_0^t E\left[\omega_1^{\mathbf{N}}\right]f^{\mathbf{N}}(x+A_s)\,dW_a(s), \quad t\ge 0.
\end{aligned}$$

Proof. The sequence of bid/ask prices is C-tight as a process taking values in $\mathbb{R}^2$ and converges to a 2-dimensional reflected Brownian motion. The processes $\overline{V}_a^{n,3}$ and $\overline{V}_a^{n,3}$ are tight taking values in H^{-1}, due to Proposition 4.7. Furthermore, $\overline{V}_a^{n,3}$ is C-tight, due to Proposition 4.9. Hence, the sequence $(\overline{A}^n, \overline{B}^n, \overline{V}_a^{n,3}, \overline{v}_a^n)$ is tight with paths in $\mathcal{D}(0, \infty; \mathbb{R}^2 \times H^{-1} \times H^{-1})$. Thus, to characterize the limit of $(\overline{A}^n, \overline{B}^n, \overline{V}_a^{n,3}, \overline{v}_a^n)$, it is sufficient to identify the limit of $\overline{v}_a^n$.

In view of Skorohod's theorem we may assume that all processes are defined on a common probability space and that the sequence $(\overline{A}^n, \overline{B}^n, \overline{V}_a^{n,3}, \overline{v}_a^n)$ converges to (A, B, V_a^3, v_a) w.p.1 (for some process v_a to be determined) along a subsequence as a sequence of processes whose sample paths lie in $\mathcal{D}(0, \infty; \mathbb{R}^2 \times H^{-1} \times H^{-1})$. In particular, this implies that $(\overline{A}^n, \overline{B}^n, \overline{V}_a^{n,3}, \overline{v}_a^n)$ converges to (A, B, V_a^3, v_a) in $\mathbb{R}^2 \times H^{-1} \times H^{-1}$ for almost every (ω, t) along this subsequence.

To prove our convergence result, we analyze each term of the following additive decomposition separately:

$$\overline{v}_a^n(t,x) - v_a^n(0,x) = \overline{V}_a^{n,1}(t,x) + \widetilde{V}_a^{n,2}(t,x) + \overline{V}^{n,3}(t,x), \quad (t,x) \in [0,\infty) \times \mathbb{R}, \tag{20}$$

where

$$\widetilde{V}_a^{n,2}(t,x) := \int_0^{\frac{[nt]}{n}} \overline{g}^n(s,x)ds.$$

Moreover, we restrict our processes to an interval $[0,T]$ with arbitrary fixed $T > 0$. Let us next show that v_a is actually a weak limit of the sequence $\overline{v}_a^n$ in the Hilbert space $L^2(\Omega \times [0,T] \times \mathbb{R})$, where $(\Omega, \mathcal{F}, P)$ denotes the probability space obtained by Skorohod's theorem. The reason we work with L^2 here is that below we want to test against L^2 functions, not just Schwartz functions, as the latter might not contain the density $f^{\mathbf{C}}$.

Let us now recall that the sequence of processes $\{\overline{v}_a^n\}$ is uniformly bounded in $\mathcal{L}^2(\Omega \times [0,T] \times \mathbb{R})$ by Lemma 4.5, and thus admits a weakly converging subsequence, say with a limit $\widetilde{v}_a$. By the Banach-Saks theorem, $\widetilde{v}_a$ is a strong limit in Cesaro sense of a subsequence (of the chosen subsequence) of $\overline{v}_a^n$ in $\mathcal{L}^2(\Omega \times [0,T] \times \mathbb{R}) \subset L^2\left(\Omega \times [0,T]; H^{-1}\right)$. Hence, its limit $\widetilde{v}_a$ must coincide with v_a, as a weak limit in $L^2(\Omega \times [0,T] \times \mathbb{R})$.

Due to Lemma 4.13, the process $\overline{V}_a^{n,1}(t)$ converges (along the selected subsequence) $\mathbb{P} \otimes dx$-a.e. to the process $\overline{V}^1$ defined by

$$\overline{V}^1(t) := E[\omega_a^{\mathbf{P}}] \int_0^t f_a^{\mathbf{P}}(\cdot + A_s)ds.$$

In view of (20), we may assume that $\widetilde{V}_a^{n,2}$ converges to some process K taking values in H^{-1} for almost all (ω, t). In view of the boundedness estimates of Lemma 4.5, combining Lemmas 4.12 and 4.13, we are allowed to take K as the weak limit of $\widetilde{V}_a^{n,2}$ as well as of

$$\int_o^{\cdot} E[\omega_1^{\mathbf{C}}] f^{\mathbf{C}}(A_s + \cdot)\overline{v}_a^n(s,\cdot)\,ds$$

in the Hilbert space $\mathcal{L}^2(\Omega \times [0,T] \times \mathbb{R}, \mathcal{F}_T \otimes \mathcal{B}([0,T] \times \mathbb{R}))$.

In order to identify the process K we test against test functions $\psi \in L^\infty(\Omega \times [0,T], \mathcal{F} \otimes \mathcal{B}([0,T]))$ and $\phi \in L^2(\mathbb{R})$. Weak convergence of $\overline{v}_a^n$ and $\widetilde{V}^{n,2}$ in the Hilbert space $L^2(\Omega \times [0,T] \times \mathbb{R}, \mathcal{F} \otimes \mathcal{B}([0,T] \times \mathbb{R}))$

yields that

$$
\begin{aligned}
E\int_0^T\int_{\mathbb{R}}\psi(t)K(t,x)\phi(x)\,dx\,dt &= \lim_{n\to\infty}E\int_0^T\psi(t)\langle\widetilde{V}_a^{n,2}(t),\,\phi\rangle\,dt\\
&= \lim_{n\to\infty}E\int_0^T\psi(t)\int_0^{\frac{[nt]}{n}}\int_{\mathbb{R}}\overline{g}^n(s,x)\phi(x)\,dxds\,dt\\
&\quad\text{(by Lemma 4.12)}\\
&= \lim_{n\to\infty}E\int_0^T\psi(t)\int_0^{\frac{[nt]}{n}}\int_{\mathbb{R}}g^n(s,x)\overline{v}_a^n(s,x)\phi(x)\,dxds\,dt\\
&\quad\text{(by Lemma 4.13)}\\
&=E[\omega_{a,1}^{\mathbf{C}}]\lim_{n\to\infty}E\int_0^T\psi(t)\int_0^{\frac{[nt]}{n}}\int_{\mathbb{R}}f^{\mathbf{C}}(x+A_s)\overline{v}_a^n(s,x)\phi(x)\,dxds\,dt\\
&=E[\omega_{a,1}^{\mathbf{C}}]\lim_{n\to\infty}E\int_0^T\int_{\mathbb{R}}f^{\mathbf{C}}(x+A_s)\overline{v}_a^n(s,x)\phi(x)\,dxE_{\overline{\mathcal{F}}_s}\Big[\int_s^T\psi(t)\,dt\Big]\,ds\\
&\quad\text{(by the weak covergence in Hilbert space)}\\
&=E[\omega_{a,1}^{\mathbf{C}}]E\int_0^T\int_{\mathbb{R}}f^{\mathbf{C}}(x+A_s)v_a(s,x)\phi(x)\,dxE_{\overline{\mathcal{F}}_s}\Big[\int_s^T\psi(t)\,dt\Big]\,ds\\
&=E[\omega_{a,1}^{\mathbf{C}}]E\int_0^T\psi(t)\int_0^t\int_{\mathbb{R}}f^{\mathbf{C}}(x+A_s)v_a(s,x)\phi(x)\,dxds\,dt,
\end{aligned}
$$

where $\overline{\mathcal{F}}_t$ denotes the filtration generated by all the processes A, B, $\overline{v}_a^n$ and v_a. Since $\phi\in L^2$ and $\psi\in L^\infty(\Omega\times[0,T],\mathcal{F}\otimes\mathcal{B}([0,T]))$ are arbitrary, we get

$$K(t,x)=E[\omega_1^{\mathbf{C}}]\int_0^t f^{\mathbf{C}}(x+A_s)v_a(s,x)\,ds$$

for almost every $(t,\omega,x)\in[0,T]\times\Omega\times\mathbb{R}$. Hence, the limit v_a satisfies

$$
\begin{aligned}
v_a(t,x)=v_a(0,x)+\int_0^t\Big(E[\omega_1^{\mathbf{P}}]f_a^{\mathbf{P}}(x+A_s)-E[\omega_1^{\mathbf{C}}]f^{\mathbf{C}}(x+A_s)v_a(s,x)\Big)\,ds+\\
+\int_0^t f^{\mathbf{N}}(x+A_s)\,dW_a(s),\quad t\geq 0.\quad\square
\end{aligned}
$$

4.3 The limit of the volume density

With tightness of the sequence of auxiliary processes $\overline{v}_{a/b}^n$ established in Proposition 4.7, we can now turn to the actual volume densities $v_{a/b}^n$. Recall that

$$\widehat{v}_{a/b}^n(u)=\overline{v}_{a/b}^n(\eta_u^n),$$

where $\widehat{v}_{a/b}^n$ is a piece-wise constant right-continuous process obtained by registering all order cancellations and placements at the next price-change, see (7), and η_u^n was defined in (8). Then Lemmas 3.1, B.4 and Theorem 4.14 implies that the limit of $(A^n,B^n,\widehat{v}_a^n)$ coincides with that of $(\overline{A}^n,\overline{B}^n,\overline{v}_a^n)$, namely (A,B,v_a) of Theorem 4.14.

Let $\delta v_{a/b}^n:=v_{a/b}^n-\widehat{v}_{a/b}^n$ and define analogously $\delta V_{a/b}^{n,i}:=V_{a/b}^{n,i}-\widehat{V}_{a/b}^{n,i}$, $i=1,2,3$. Below, we shall prove that $\delta v^n_{a/b}$ converges weakly to 0 as $n \to \infty$. Obviously, this implies (see Theorem 4.18 below) that if v^n_a converges then the limit must coincide with that of $\widehat{v}^n_a$ as well as of $\overline{v}^n_a$, namely v_a.

The first step for proving $\delta v^n_{a/b} \to 0$ is to establish bounds for second moments of the increments, in a similar way to Lemmas 4.4 and 4.5. In fact, analogous to Proposition 4.7 these estimates indicate the tightness of v^n_a and thus the tightness of (A^n, B^n, v^n_a). The rather technical proof is deferred to Appendix A.

Lemma 4.15. *There holds*

$$E_{\mathcal{F}^n_s}\left[\sum_{i=1}^{3}\left\|V^{n,i}_{a/b}(t) - V^{n,i}_{a/b}(s)\right\|^2_{L^2}\right] \leq C\left[(t-s)+(t-s)^2\right], \quad 0 \leq s \leq t < \infty,$$
$$E_{\mathcal{F}^n_s}\left[\left\|v^n_{a/b}(t) - v^n_{a/b}(s)\right\|^2_{L^2}\right] \leq C^n_s\left[(t-s)+(t-s)^2\right], \quad 0 \leq s \leq t < \infty,$$

with $\sup_n E\left[\sup_{s\in[0,t]} C^n_s\right] \leq C(t^2+t)$, $t \in [0,\infty)$*, where the constants* C *are independent of* n*,* s *and* t*.*

Furthermore, we will show that $\delta v^n_{a/b}(t)$ converges point-wise to 0, for which we need some elementary results on Poisson processes.

Lemma 4.16. *Let* N_1 *and* N_2 *be two independent Poisson processes with intensities* λ_1 *and* λ_2*, respectively. Moreover, let* T_i*,* $i = 1, \ldots,$ *denote the jump times of the Poisson process* N_1*. Then we have*

$$E\left[N_2(t) - N_2(T_{N_1(t)})\right] = \frac{\lambda_2}{\lambda_1}\left(1 - e^{-\lambda_1 t}\right),$$
$$E\left[(N_2(t) - N_2(T_{N_1(t)}))\,(N_2(t) - N_2(T_{N_1(t)}) - 1)\right] = 4\frac{\lambda_2^2}{\lambda_1^2}\left(1 - (1+t\lambda_1)e^{-\lambda_1 t}\right).$$

Proof. Notice that conditional on $N_1(t) = l$, the relative difference $(t - T_l)/t$ has a beta distribution with parameters 1 and l, as this is the distribution of the differences in the order statistics of l random variables distributed uniformly on $[0,1]$. Hence, elementary calculations give

$$E\left[N_2(t) - N_2(T_l)\,|\,N_1(t) = l\right] = \sum_{k=0}^{\infty} k \int_0^1 e^{-\lambda_2 t x}\frac{(\lambda_2 t x)^k}{k!}\frac{1-x)^{l-1}}{B(1,l)}dx = \frac{\lambda_2 t}{1+l}$$

and

$$E\left[(N_2(t) - N_2(T_l))\,(N_2(t) - N_2(T_l) - 1)\;|\,N_1(t) = l\right] = \sum_{k=0}^{\infty} k(k-1)\int_0^1 e^{-\lambda_2 t x}\frac{(\lambda_2 t x)^k}{k!}\frac{1-x)^{l-1}}{B(1,l)}dx = \frac{2\lambda_2^2 t^2}{2+3l+l^2}.$$

Multiplying these terms with $P(N_1(t) = l) = e^{-\lambda_1 t}\frac{(\lambda_1 t)^l}{l!}$ and summing over l gives the formulas from above. □

Lemma 4.17. *Let* $u = u(t) = u(t,x)$ *denote any of the processes* $\delta v^n_{a/b}$*,* $\delta V^{n,i}_{a/b}$*,* $i = 1,2,3$*. Moreover, assume that the sequence* $v^n_{a/b}(0)$ *is uniformly bounded in* L^2*. Then there is a constant* C *independent of* n *or* t *such that*

$$E\left[\|u(t)\|^2_{L^2}\right] \leq C(1+t+t^2)/n, \quad \forall\, t \in [0,\infty).$$

Proof. Let us first consider $u = \delta V^{n,i}_{a/b}$ for some $i = 1,2,3$ and a or b. Note that for some random variables ω_i and π_i we have for some scaling constant ϵ (either equal to $\Delta v/\Delta x$ or equal to $\sqrt{\Delta v}$)

$$u(t,x)^2 = \left(\sum_{i=N\left(\widetilde{\tau}_{\widetilde{N}(t)}\right)}^{N(t)} \mathbf{1}_{I\left(A^n(\widetilde{\tau}^n_{\widetilde{N}(t)})+\pi_i\right)}(x)\omega_i \right)^2 \epsilon^2,$$

as $\widetilde{\xi}_{a/b,i}$ is constant in i and $\widetilde{\xi}^2_{a/b,i} = 1$. Letting $\mathcal{G}$ denote the σ-algebra generated by all sources of randomness *except* $(\omega_i)_{i\in\mathbb{N}}$, we have

$$\begin{aligned} E\left[u(t,x)^2\right] &= E\left[\left\{ \sum_{i\neq i'=N\left(\widetilde{\tau}_{\widetilde{N}(t)}\right)}^{N(t)} E_{\mathcal{G}}\left[\omega_i\omega_{i'}\right] \mathbf{1}_{I\left(A^n(\widetilde{\tau}^n_{\widetilde{N}(t)})+\pi_i\right)}(x)\mathbf{1}_{I\left(A^n(\widetilde{\tau}^n_{\widetilde{N}(t)})+\pi_{i'}\right)}(x) + \sum_{i=N\left(\widetilde{\tau}_{\widetilde{N}(t)}\right)}^{N(t)} E_{\mathcal{G}}\left[\omega_i^2\right] \mathbf{1}_{I\left(A^n(\widetilde{\tau}^n_{\widetilde{N}(t)})+\pi_i\right)}(x) \right\}\right] \epsilon^2 \\ &= E\left[\left\{ \sum_{i\neq i'=N\left(\widetilde{\tau}_{\widetilde{N}(t)}\right)}^{N(t)} \mathbf{1}_{I\left(A^n(\widetilde{\tau}^n_{\widetilde{N}(t)})+\pi_i\right)}(x)\mathbf{1}_{I\left(A^n(\widetilde{\tau}^n_{\widetilde{N}(t)})+\pi_{i'}\right)}(x)E[\omega_1]^2 + \sum_{i=N\left(\widetilde{\tau}_{\widetilde{N}(t)}\right)}^{N(t)} \mathbf{1}_{I\left(A^n(\widetilde{\tau}^n_{\widetilde{N}(t)})+\pi_i\right)}(x)E\left[\omega_1^2\right] \right\}\right] \epsilon^2. \end{aligned}$$

Furthermore, conditioning on the σ-algebra generated by all sources of randomness except for $(\pi_i)_{i\in\mathbb{N}}$, we can bound in a similar way to (11)

$$\begin{aligned} E\left[u(t,x)^2\right] \leq E\Bigg[& E\left[\omega_1\right]^2 \|f\|^2_\infty \Delta x^2 \left(N(t) - N\left(\widetilde{\tau}_{\widetilde{N}(t)}\right)\right)\left(N(t) - N\left(\widetilde{\tau}_{\widetilde{N}(t)}\right) - 1\right) \mathbf{1}_{\left[A(\widetilde{\tau}_{\widetilde{N}(t)})-M, A(\widetilde{\tau}_{\widetilde{N}(t)})+M\right]}(x) + \\ & + E\left[\omega_1^2\right] \|f\|^2_\infty \Delta x \left(N(t) - N\left(\widetilde{\tau}_{\widetilde{N}(t)}\right)\right) \mathbf{1}_{\left[A(\widetilde{\tau}_{\widetilde{N}(t)})-M, A(\widetilde{\tau}_{\widetilde{N}(t)})+M\right]}(x) \Bigg] \epsilon^2. \end{aligned}$$

Hence, plugging in Lemma 4.16, we obtain

$$\begin{aligned} E\left[\|u(t)\|^2_{L^2}\right] &\leq C\left(\Delta x^2 E\left[\left(N(t) - N\left(\widetilde{\tau}_{\widetilde{N}(t)}\right)\right)\left(N(t) - N\left(\widetilde{\tau}_{\widetilde{N}(t)}\right) - 1\right)\right] + \Delta x E\left[\left(N(t) - N\left(\widetilde{\tau}_{\widetilde{N}(t)}\right)\right)\right]\right)\epsilon^2 \\ &= C\left(\Delta x^2 4\frac{\lambda^2}{\mu^2}\left[1 - (1+t\mu)e^{-\mu t}\right] + \Delta x\frac{\lambda}{\mu}\left[1 - e^{-\mu t}\right]\right)\epsilon^2 \\ &\leq C\left(\frac{1}{n}\frac{n^4}{n^2} + \frac{1}{\sqrt{n}}\frac{n^2}{n}\right)\epsilon^2 \\ &= C\left(n + \sqrt{n}\right)\epsilon^2. \end{aligned}$$

Now we recall that $\epsilon^2 = \frac{\Delta v^2}{\Delta x^2} = n^{-3}$ in case $i = 1,2$ and $\epsilon^2 = \Delta v = n^{-2}$ in case $i = 3$.

The proof for the estimate of $\delta v^n_{a/b}$ works in precisely the same way as the proof of Lemma 4.5, taking into account the appropriate estimates for $\delta V^{n,i}_{a/b}$ derived above. □

Combining these lemmas with the results in Theorem 4.14 we can now prove the last part of the main Theorem 2.6, namely the convergence of the volume densities.

Theorem 4.18. *There are two independent Wiener processes* $\{W_a(t); t \in [0,T]\}$ *and* $\{W_b(t); t \in [0,T]\}$ *such that* $(A^n, B^n, v^n_a, v^n_b) \Rightarrow (A, B, v_a, v_b)$, *with* (A,B) *the two-dimensional reflected Brownian motion and the volume processes* v_a *and* v_b *satisfying the infinite-dimensional SDE*

$$v_b(t,\cdot) = v_{b,0}(\cdot) + \int_0^t \left(E[\omega_1^{\mathbf{P}_b}]f^{\mathbf{P}_b}(\cdot + B_s) - E[\omega_1^{\mathbf{C}_b}]f^{\mathbf{C}_b}(\cdot + B_s)v_b(s,\cdot)\right) ds$$

$$+ \sqrt{2}E[\omega_1^{\mathbf{N}_b}] \int_0^t f^{\mathbf{N}_b}(\cdot + B_s)\, dW_b(s), \quad t \geq 0;$$

$$v_a(t,\cdot) = v_{a,0}(\cdot) + \int_0^t \Big(E[\omega_1^{\mathbf{P}_a}] f^{\mathbf{P}_a}(\cdot + A_s) - E[\omega_1^{\mathbf{C}_a}] f^{\mathbf{C}_a}(\cdot + A_s) v_a(s,\cdot) \Big)\, ds$$

$$+ \sqrt{2}E[\omega_1^{\mathbf{N}_a}] \int_0^t f^{\mathbf{N}_a}(\cdot + A_s)\, dW_a(s), \quad t \geq 0.$$

Proof. Recall that

$$(A^n, B^n, \widehat{v}_a^n)(u) = (\overline{A}^n, \overline{B}^n, \overline{v}_a^n)(\eta_u^n).$$

Combining Lemmas 3.1, B.4 and Theorem 4.14, we conclude that $(A^n, B^n, \widehat{v}_a^n) \Rightarrow (A, B, v_a)$. On the other hand, in a similar way to Proposition 4.7 we derive from Lemma 4.15 the tightness of (A^n, B^n, v_a^n). Additionally, Lemma 4.17 indicates that the limit of (A^n, B^n, v_a^n) coincides with that of $(A^n, B^n, \widehat{v}_a^n)$, namely (A, B, v_a). This implies the C-tightness of (A^n, B^n, v_a^n) and thus the tightness of (A^n, B^n, v_a^n, v_b^n) by Corollary B.3. Furthermore, in a similar way to the ask side we verify that $(A^n, B^n, v_a^n, v_b^n) \Rightarrow (A, B, v_a, v_b)$. □

5 Conclusion

This paper establishes a functional limit theorem for limit order books. The limiting dynamics are derived from individual order arrival, placement and cancellation dynamics. With our choice of scaling, the limiting dynamics converges in distribution to a coupled system of reflected Brownian motions and linear SPDEs. We essentially assumed that all random variables were independent. It should be too difficult, though, to establish a similar limiting result for a model where the intensities of active and passive orders arrivals depend on the prevailing prices (or the spread) and hence to obtain a mean-reverting dynamics for the spread. Allowing for a non-linear impact of the noise terms is more challenging. For instance, it would certainly be desirable to allow for multiplicative noise to avoid negative volumes. This case, as well as a limiting result where the martingale part is driven by a random measures (rather then Brownian motions) is left for future research.

A Technical proofs

Proof of Lemma 4.13. We prove (18); the second assertion follows similarly. Without any loss of generality, we assume $E[\omega_1^{\mathbf{C}}] = 1$. For each $s \in (\frac{1}{n}, t)$ with $n \in \mathbb{N}$, we choose $k_s^n \in \mathbb{Z}$ such that $s \in [\frac{k_s^n+1}{n}, \frac{k_s^n+2}{n})$. For $s \in (0, \frac{1}{n})$, put $k_s^n = 0$. For notational simplicity, we set $\tilde{v}_a^n(s,x) = 1 - \alpha + \alpha \overline{v}_a^n(s,x)$, with $\alpha \in \{1, 0\}$. Then

$$\sup_{x\in\mathbb{R}} E\left| \int_0^t \Big(g^n(s,x) - E[\omega_1^{\mathbf{C}}] f^{\mathbf{C}}(A_s + x) \Big) \tilde{v}_a^n(s,x)\, ds \right|^2$$

$$\leq 2 \sup_{x\in\mathbb{R}} E\left| \int_0^t \Big(f^{\mathbf{C}}(x + \overline{A}^n_{\frac{k_s^n}{n}}) - f^{\mathbf{C}}(x + A_s) \Big) \tilde{v}_a^n(s,x)\, ds \right|^2 + 2 \sup_{x\in\mathbb{R}} E\left| \int_0^t \Big(g^n(s,x) - f^{\mathbf{C}}(x + \overline{A}^n_{\frac{k_s^n}{n}}) \Big) \tilde{v}_a^n(s,x) ds \right|^2$$

$$:= 2(\Gamma_1 + \Gamma_2).$$

Since $f^{\mathbf{C}}$ is Lipschitz continuous and vanishes outside a compact interval there exists a constant $C < \infty$ such that

$$\Gamma_1 = \sup_{x\in\mathbb{R}} E\left|\int_0^t \Big(f^{\mathbf{C}}(x+\overline{A}^n_{\frac{k^n_s}{n}}) - f^{\mathbf{C}}(x+A_s)\Big)\tilde{v}^n_a(s,x)\,ds\right|^2$$
$$\leq C\sup_{x\in\mathbb{R}} E\int_0^t |\tilde{v}^n_a(s,x)|^2 ds\, E\int_0^t |A_s - \overline{A}^n_{\frac{k^n_s}{n}}|^2 \wedge 1\, ds.$$

Hence, by Lemma 4.5, $\Gamma_1 \to 0$ as $n\to\infty$ by dominated convergence, due to the a.s. continuity of the reflected Brownian motion A. Using independence of cancellation price levels and volumes, a direct computation yields:

$$\begin{aligned}
\Gamma_2 &= \sup_{x\in\mathbb{R}} E\left|\int_0^t \Big(g^n(s,x) - f^{\mathbf{C}}(x+\overline{A}^n_{\frac{k^n_s}{n}})\Big)\tilde{v}^n_a(s,x)\,ds\right|^2\\
&= \sup_{x\in\mathbb{R}} E\left|\int_0^t\left(\sum_{i=N^n_a(\overline{\tau}^n_{k^n_s})+1}^{N^n_a(\overline{\tau}^n_{k^n_s+1})}\sum_{j\in\mathbb{Z}} \mathbf{1}_{[x^n_j,x^n_{j+1})}(\pi^{\mathbf{C}}_i+\overline{A}^n_{\frac{k^n_s}{n}})\omega^{\mathbf{C}}_i \mathbf{1}_{[x^n_j,x^n_{j+1})}(x)\frac{\Delta v^n n}{\Delta x^n} - f^{\mathbf{C}}(x+\overline{A}^n_{\frac{k^n_s}{n}})\right)\tilde{v}^n_a(s,x)\,ds\right|^2\\
&\leq 3\sup_{x\in\mathbb{R}} E\left|\int_0^t \sum_{i=N^n_a(\overline{\tau}^n_{k^n_s})+1}^{N^n_a(\overline{\tau}^n_{k^n_s+1})}\sum_{j\in\mathbb{Z}}\Big(\mathbf{1}_{[x^n_j,x^n_{j+1})}(\pi^{\mathbf{C}}_i+\overline{A}^n_{\frac{k^n_s}{n}})\omega^{\mathbf{C}}_i - \int_{[x^n_j,x^n_{j+1})} f^{\mathbf{C}}(y+\overline{A}^n_{\frac{k^n_s}{n}})\,dy\Big)\mathbf{1}_{[x^n_j,x^n_{j+1})}(x)\frac{\Delta v^n\tilde{v}^n_a(s,x)n}{\Delta x^n}ds\right|^2\\
&\quad+3\sup_{x\in\mathbb{R}} E\left|\int_0^t \sum_{i=N^n_a(\overline{\tau}^n_{k^n_s})+1}^{N^n_a(\overline{\tau}^n_{k^n_s+1})}\left(\sum_{j\in\mathbb{Z}}\frac{1}{\Delta x^n}\int_{[x^n_j,x^n_{j+1})} f^{\mathbf{C}}(y+\overline{A}^n_{\frac{k^n_s}{n}})\,dy\mathbf{1}_{[x^n_j,x^n_{j+1})}(x) - f^{\mathbf{C}}(x+\overline{A}^n_{\frac{k^n_s}{n}})\right)n\Delta v^n\tilde{v}^n_a(s,x)ds\right|^2\\
&\quad+3\sup_{x\in\mathbb{R}} E\left|\int_0^t\Big(\Big(N^n_a(\overline{\tau}^n_{k^n_s+1}) - N^n_a(\overline{\tau}^n_{k^n_s})\Big)n\Delta v^n - 1\Big)f^{\mathbf{C}}(x+\overline{A}^n_{\frac{k^n_s}{n}})\tilde{v}^n_a(s,x)\,ds\right|^2\\
&:= 3\Big(\gamma_0+\gamma_1+\gamma_2\Big).
\end{aligned}$$

To estimate γ_0 we use again independence of involved random variables, the fact that

$$E_{\mathcal{F}^n_{\frac{k^n_s}{n}}}\Big[\mathbf{1}_{[x^n_j,x^n_{j+1})}(\pi^{\mathbf{C}}_i+\overline{A}^n_{\frac{k^n_s}{n}})\omega^{\mathbf{C}}_i\Big] = \int_{[x^n_j,x^n_{j+1})} f^{\mathbf{C}}(y+\overline{A}^n_{\frac{k^n_s}{n}})\,dy$$

along with Lemmas 4.2 and 4.5 and the properties of the scaling constants to conclude that:

$$\gamma_0 \leq Ct^2\sup_{x\in\mathbb{R}} E\sup_{s\in[0,t]}|\tilde{v}^n_a(s,x)|^2\frac{\lambda^n}{\mu^n}\Big(\frac{n\Delta v^n}{\Delta x^n}\Big)^2\|f^{\mathbf{C}}\|_{L^\infty}\Delta x^n \leq\ Ct^2\Big(t^2+t+1\Big)\Delta x^n \longrightarrow 0,\quad \text{as } n\to\infty.$$

To estimate γ_1 we first deduce from Lipschitz continuity of $f^{\mathbf{C}}$ for $x\in[x^n_j,x^n_{j+1})$ that

$$\frac{1}{\Delta x^n}\int_{[x^n_j,x^n_{j+1})}\Big|f^{\mathbf{C}}(y+\overline{A}^n_{\frac{k^n_s}{n}}) - f^{\mathbf{C}}(x+\overline{A}^n_{\frac{k^n_s}{n}})\Big|\,dy \leq\ L\frac{1}{\Delta x^n}\int_{[x^n_j,x^n_{j+1})}|\Delta x^n|\,dy = L\Delta x^n.$$

Thus, using again Lemmas 4.2 and 4.5, the properties of the scaling constants and the fact that $f^{\mathbf{C}}$ vanishes outside a compact interval we find a constant $C<\infty$ such that:

$$\gamma_1 \leq Ct^2n^{-1}\sup_{x\in\mathbb{R}} E\sup_{s\in[0,t]}|\tilde{v}^n_a(s,x)|^2 \leq\ Ct^2\Big(t^2+t+1\Big)n^{-1}\longrightarrow 0,\quad \text{as } n\to\infty.$$

In view of Lemma 4.2, boundedness of $f^{\mathbf{C}}$ and independence of involved random variables, we have

$$\begin{aligned}
\gamma_2 &= \sup_{x\in\mathbb{R}} E\left|\int_0^t \left(\left(N_a^n(\overline{\tau}^n_{k_s^n+1}) - N_a^n(\overline{\tau}^n_{k_s^n})\right) n\Delta v^n - 1\right) f^{\mathbf{C}}(x+\overline{A}^n_{\frac{k_s^n}{n}})\tilde{v}_a^n(s,x)\,ds\right|^2 \\
&\le 2\sup_{x\in\mathbb{R}} E\left|\sum_{l=1}^{\lfloor nt\rfloor} \left(\left(N_a^n(\overline{\tau}^n_{l}) - N_a^n(\overline{\tau}^n_{l-1})\right) n\Delta v^n - 1\right) f^{\mathbf{C}}(x+\overline{A}^n_{\frac{l-1}{n}})\tilde{v}_a^n(\frac{l-1}{n},x)\frac{1}{n}\right|^2 \\
&\quad + 2\sup_{x\in\mathbb{R}} E\left|\int_0^{\frac{1}{n}} f^{\mathbf{C}}(x+\overline{A}^n_0)\tilde{v}_a^n(0,x)\,ds\right|^2 \\
&\le 2(\Delta v^n)^2 E\sup_{x\in\mathbb{R}} \sum_{l=1}^{\lfloor nt\rfloor} \left|\left(N_a^n(\overline{\tau}^n_{l}) - N_a^n(\overline{\tau}^n_{l-1}) - E[N_a^n(\overline{\tau}^n_{l}) - N_a^n(\overline{\tau}^n_{l-1})]\right) f^{\mathbf{C}}(x+\overline{A}^n_{\frac{l-1}{n}})\tilde{v}_a^n(\frac{l-1}{n},x)\right|^2 + C/n \\
&\le \frac{C}{n}\left(1 + t\sup_{x\in\mathbb{R}} E\sup_{s\in[0,t]}|\tilde{v}_a^n(s,x)|^2\right) \longrightarrow 0, \quad \text{as } n\to\infty.\square
\end{aligned}$$

Proof of Lemma 4.15. Without any loss of generality, we take $s = 0$ and prove the assertions for the ask side. To this end, let $\mathcal{G}$ denote the σ-algebra generated by all the sources of randomness *except* $(\pi_i)_{i=1}^{\infty}$ and $(\omega_i)_{i=1}^{\infty}$, which is then by construction, independent from $\mathcal{G}$.

$$\begin{aligned}
&E\left\|V_a^{n,1}(t)\right\|^2_{L^2} \\
&= \left(\frac{\Delta v^n}{\Delta x^n}\right)^2 \int_{\mathbb{R}} E\left|\sum_{i=1}^{N_a^n(t)}\sum_{j\in\mathbb{Z}} \omega_i^{\mathbf{P}_a}\mathbf{1}_{[x_j^n,x_{j+1}^n)}(x)\mathbf{1}_{[x_j^n,x_{j+1}^n)}(A^n(\tilde{\tau}^n_{\widetilde{N}^n(\tau^n_{a,i})}) + \pi_i^{\mathbf{P}_a})\right|^2 dx \\
&= \left(\frac{\Delta v^n}{\Delta x^n}\right)^2 \int_{\mathbb{R}} \sum_{l=1}^{\infty} \frac{(\lambda^n t)^l}{l!} e^{-\lambda^n t} E_{N_a^n(t)=l}\Bigg[\sum_{i>i';i,i'=1}^{l} \\
&E_{\mathcal{G}}\left[\left(\sum_{j\in\mathbb{Z}} \omega_i^{\mathbf{P}_a}\mathbf{1}_{[x_j^n,x_{j+1}^n)}(x)\mathbf{1}_{[x_j^n,x_{j+1}^n)}(A^n(\tilde{\tau}^n_{\widetilde{N}^n(\tau^n_{a,i})}) + \pi_i^{\mathbf{P}_a})\right)\left(\sum_{j\in\mathbb{Z}} \omega_{i'}^{\mathbf{P}_a}\mathbf{1}_{[x_j^n,x_{j+1}^n)}(x)\mathbf{1}_{[x_j^n,x_{j+1}^n)}(A^n(\tilde{\tau}^n_{\widetilde{N}^n(\tau^n_{a,i'})}) + \pi_{i'}^{\mathbf{P}_a})\right)\right] \\
&+ \sum_{i=1}^{l}\sum_{j\in\mathbb{Z}} E|\omega_i^{\mathbf{P}_a}|^2\mathbf{1}_{[x_j^n,x_{j+1}^n)}(x)\mathbf{1}_{[x_j^n,x_{j+1}^n)}(A^n(\tilde{\tau}^n_{\widetilde{N}^n(\tau^n_{a,i})}) + \pi_i^{\mathbf{P}_a})\Bigg] dx \\
&\le C\left(\frac{\Delta v^n}{\Delta x^n}\right)^2 \int_{\mathbb{R}} \sum_{l=1}^{\infty} \frac{(\lambda^n t)^l}{l!} e^{-\lambda^n t} E_{N_a^n(t)=l}\Bigg[\sum_{i>i';i,i'=1}^{l} (E\omega_1^{\mathbf{P}_a})^2\mathbf{1}_{[-M+A^n(\tilde{\tau}^n_{\widetilde{N}^n(\tau^n_{a,i})}),M+A^n(\tilde{\tau}^n_{\widetilde{N}^n(\tau^n_{a,i})})]}(x)\|f^{\mathbf{P}_a}\|^2_{L^\infty}(\Delta x^n)^2 \\
&+ E|\omega_1^{\mathbf{P}_a}|^2\sum_{i=1}^{l}\mathbf{1}_{[-M+A^n\tilde{\tau}^n_{\widetilde{N}^n(\tau^n_{a,i})}),M+A^n(\tilde{\tau}^n_{\widetilde{N}^n(\tau^n_{a,i})}))}(x)\|f^{\mathbf{P}_a}\|_{L^\infty}\Delta x^n\Bigg] dx \\
&\le C\left(\frac{\Delta v^n}{\Delta x^n}\right)^2 \sum_{l=1}^{\infty} \frac{(\lambda^n t)^l}{l!} e^{-\lambda^n t}\left[l(l-1)\|f^{\mathbf{P}_a}\|^2_{L^\infty}(\Delta x^n)^2 + l\|f^{\mathbf{P}_a}\|_{L^\infty}\Delta x^n\right] \\
&\le C\left(\frac{\Delta v^n}{\Delta x^n}\right)^2\left[(\lambda^n t\Delta x^n)^2 + \lambda^n t\Delta x^n\right]
\end{aligned}$$

$\leq C(t^2 + t),$

and similarly, we have $E\left\|V_a^{n,2}(t)\right\|_{L^2}^2 \leq C(t^2+t)$, where the constants Cs are independent of n. Taking the supremum norm $\|\cdot\|_{L^\infty}$ instead, we obtain

$$\sup_{x\in\mathbb{R}} E_{\mathcal{F}_s^n}|V_a^{n,1}(t) - V_a^{n,1}(s)|^2 + \sup_{x\in\mathbb{R}} E_{\mathcal{F}_s^n}|V_a^{n,2}(t) - V_a^{n,2}(s)|^2 \leq C[t-s+(t-s)^2], \quad 0 \leq s \leq t < \infty.$$

On the other hand,

$$\begin{aligned}
&E\sup_{s\in[0,t]}\|V_a^{n,3}(s)\|_{L^2}^2\\
&= E\sup_{s\in[0,t]}\left\|\sum_{i=1}^{N_a^n(s)}\sum_{j\in\mathbb{Z}}\mathbf{1}_{[x_j^n,x_{j+1}^n)}(\cdot)\mathbf{1}_{[x_j^n,x_{j+1}^n)}(A^n(\tilde\tau^n_{\widetilde{N}^n(\tau^n_{a,i})}) + \pi_i^{\mathsf{P}_a})\widetilde{\xi}_{a,\widetilde{N}^n(\tau^n_{a,i})+1}\sqrt{\Delta v^n}\right\|_{L^2}^2\\
&= E\sup_{s\in[0,t]}\left\|\sum_{k=1}^{\widetilde{N}^n(s)}\sum_{i=N_a^n(\tilde\tau^n_{k-1})+1}^{N_a^n(\tilde\tau^n_k)}\sum_{j\in\mathbb{Z}}\mathbf{1}_{[x_j^n,x_{j+1}^n)}(\pi_i^{\mathsf{P}_a} + A^n(\tilde\tau^n_{k-1}))\mathbf{1}_{[x_j^n,x_{j+1}^n)}(\cdot)\,\widetilde{\xi}_{a,k}\sqrt{\Delta v^n}\right.\\
&\quad\left.+\sum_{i=N_a^n(\tilde\tau^n_{\widetilde{N}^n(s)})+1}^{N_a^n(s)}\sum_{j\in\mathbb{Z}}\mathbf{1}_{[x_j^n,x_{j+1}^n)}(\pi_i^{\mathsf{P}_a} + A^n(\tilde\tau^n_{\widetilde{N}^n(s)}))\mathbf{1}_{[x_j^n,x_{j+1}^n)}(\cdot)\widetilde{\xi}_{a,\widetilde{N}^n(s)+1}\sqrt{\Delta v^n}\right\|_{L^2}^2\\
&\leq C\Delta v^n E\left[\sum_{k=1}^{\widetilde{N}^n(t)}\left\|\sum_{i=N_a^n(\tilde\tau^n_{k-1})+1}^{N_a^n(\tilde\tau^n_k)}\sum_{j\in\mathbb{Z}}\mathbf{1}_{[x_j^n,x_{j+1}^n)}(\pi_i^{\mathsf{P}_a} + A^n(\tilde\tau^n_{k-1}))\mathbf{1}_{[x_j^n,x_{j+1}^n)}(\cdot)\right\|_{L^2}^2\right.\\
&\quad\left.+\left\|\sum_{i=N_a^n(\tilde\tau^n_{\widetilde{N}^n(t)})+1}^{N_a^n(t)}\sum_{j\in\mathbb{Z}}\mathbf{1}_{[x_j^n,x_{j+1}^n)}(\pi_i^{\mathsf{P}_a} + A^n(\tilde\tau^n_{\widetilde{N}^n(t)}))\mathbf{1}_{[x_j^n,x_{j+1}^n)}(\cdot)\right\|_{L^2}^2\right]\\
&= C\Delta v^n\sum_{l=0}^\infty\frac{(\mu^n t)^l}{l!}e^{-\mu^n t}E_{\widetilde{N}^n(t)=l}\left[\sum_{k=1}^l\left\|\sum_{j\in\mathbb{Z}}\sum_{i=N_a^n(\tilde\tau^n_{k-1})+1}^{N_a^n(\tilde\tau^n_k)}\mathbf{1}_{[x_j^n,x_{j+1}^n)}(\pi_i^{\mathsf{P}_a} + A^n(\tilde\tau^n_{k-1}))\mathbf{1}_{[x_j^n,x_{j+1}^n)}(\cdot)\right\|_{L^2}^2\right.\\
&\quad\left.+\left\|\sum_{i=N_a^n(\tilde\tau^n_l)+1}^{N_a^n(t)}\sum_{j\in\mathbb{Z}}\mathbf{1}_{[x_j^n,x_{j+1}^n)}(\pi_i^{\mathsf{P}_a} + A^n(\tilde\tau^n_l))\mathbf{1}_{[x_j^n,x_{j+1}^n)}\right\|_{L^2}^2\right]\\
&\leq C\Delta v^n\sum_{l=0}^\infty\frac{(\mu^n t)^l}{l!}e^{-\mu^n t}E_{\widetilde{N}^n(t)=l}\left[\sum_{k=1}^l(N_a^n(\tilde\tau^n_k) - N_a^n(\tilde\tau^n_{k-1}))(N_a^n(\tilde\tau^n_k) - N_a^n(\tilde\tau^n_{k-1}) - 1)\|f^{\mathsf{P}_a}\|_{L^\infty}^2(\Delta x^n)^2\|\mathbf{1}_{[-M+A^n(\tilde\tau^n_l),M+A^n(\tilde\tau^n_l))}\|_{L^2}^2\right.\\
&\quad+\sum_{k=1}^l(N_a^n(\tilde\tau^n_k) - N_a^n(\tilde\tau^n_{k-1}))\|f^{\mathsf{P}_a}\|_{L^\infty}\Delta x^n\|\mathbf{1}_{[-M+A^n(\tilde\tau^n_l),M+A^n(\tilde\tau^n_l)]}\|_{L^2}^2\\
&\quad+(N_a^n(t) - N_a^n(\tilde\tau^n_l))(N_a^n(t) - N_a^n(\tilde\tau^n_l) - 1)\|f^{\mathsf{P}_a}\|_{L^\infty}^2(\Delta x^n)^2\|\mathbf{1}_{[-M+A^n(\tilde\tau^n_l),M+A^n(\tilde\tau^n_l)]}\|_{L^2}^2\\
&\quad\left.+(N_a^n(t) - N_a^n(\tilde\tau^n_l))\|f^{\mathsf{P}_a}\|_{L^\infty}\Delta x^n\|\mathbf{1}_{[-M+A^n(\tilde\tau^n_l),M+A^n(\tilde\tau^n_l)]}\|_{L^2}^2\right]\\
&\leq C\Delta v^n\sum_{l=0}^\infty\frac{(\mu^n t)^l}{l!}e^{-\mu^n t}E_{\widetilde{N}^n(t)=l}\left[\sum_{k=1}^l(N_a^n(\tilde\tau^n_k) - N_a^n(\tilde\tau^n_{k-1}))(N_a^n(\tilde\tau^n_k) - N_a^n(\tilde\tau^n_{k-1}) - 1)\,(\Delta x^n)^2\right.\\
&\quad\left.+\sum_{k=1}^l(N_a^n(\tilde\tau^n_k) - N_a^n(\tilde\tau^n_{k-1})) + (N_a^n(t) - N_a^n(\tilde\tau^n_l))(N_a^n(t) - N_a^n(\tilde\tau^n_l) - 1)(\Delta x^n)^2 + (N_a^n(t) - N_a^n(\tilde\tau^n_l))\right]
\end{aligned}$$

$$
\begin{aligned}
&= C\Delta v^n \sum_{l=0}^{\infty} \frac{(\mu^n t)^l}{l!} e^{-\mu^n t} E\Bigg[\\
&\quad lN_a^n(\beta(1,l))(N_a^n(\beta(1,l)) - 1)\ (\Delta x^n)^2 + lN_a^n(\beta(1,l)) + N_a^n(\beta(1,l))(\Delta x^n)^2 + N_a^n(\beta(1,l))\Bigg] \\
&= C\Delta v^n \sum_{l=0}^{\infty} \frac{(\mu^n t)^l}{l!} e^{-\mu^n t} \sum_{m=0}^{\infty} \Big[lm(m-1)(\Delta x^n)^2 + m^2(\Delta x^n)^2 + (l+1)m\Big] \int_0^1 \frac{(\lambda^n tz)^m}{m!} e^{-\lambda^n tz} \frac{(1-z)^{l-1}}{B(1,l)} dz \\
&\le Ct\Delta v^n \left[\frac{(\lambda^n \Delta x^n)^2}{\mu^n} + \lambda^n\right] \\
&\le Ct,
\end{aligned}
$$

with the constant C independent of n and t.

The estimate of $v_{a/b}^n$ follows in precisely the same way as the proof of Lemma 4.5, taking into account the appropriate estimates for $V_{a/b}^{n,i}$ derived above. □

B Classical tightness results

For the convenience of the reader, we recall some classical results on tightness which the derivations of Section 4 are based on. We first note that though the following theorems and lemmas may be originally established on finite time intervals, we state them on the half line $[0,\infty)$ since there is no essential difficulty to make such extensions in the spirit of Jacod and Shiryaev [15].

The first result is a sufficient condition for tightness in the Skorokhod space $\mathcal{D}([0,\infty);E)$ for a complete separable metric state space (E,ρ) due to Aldous and Kurtz. We take it from [22, Th. 6.8].

Theorem B.1. *Let X_n be a sequence of processes taking values in $\mathcal{D}([0,\infty);E)$ such that the family $(X_n(t))_{n\in\mathbb{N}}$ of random variables is tight (in E) for any rational t. Moreover, assume that there is a number $p>0$ and processes $(\gamma_n(\delta))_{\delta\in[0,\infty)}$, $n\in\mathbb{N}$, such that*

$$
\begin{aligned}
E\left[\rho\left(X_n(t+\delta), X_n(t)\right)^p \,\middle|\, \mathcal{F}_t^n\right] &\le E\left[\gamma_n(\delta) \,|\, \mathcal{F}_t^n\right], \\
\lim_{\delta\to 0} \limsup_{n\to\infty} E\left[\gamma_n(\delta)\right] &= 0,
\end{aligned}
$$

where the filtration $\mathcal{F}^n$ is generated by X^n. Then $(X_n)_{n\in\mathbb{N}}$ is tight in $\mathcal{D}([0,\infty);E)$.

Proof. See [22, Th. 6.8]. Note that Walsh assumes one joint filtration $\mathcal{F}_t$, whereas we allow for filtrations depending on n. This difference is, however, inconsequential, e.g., by choosing X^n to be defined on a common probability space in an independent way and then choosing $\mathcal{F}_t$ to be the filtration generated by all the filtrations $\mathcal{F}_t^n$. □

The main theoretical tool in this paper is Mitoma's theorem, on basis of [22, Th. 6.13, Lem. 6.14, Note on p. 365], which relates tightness of distribution-valued processes to real-valued processes obtained by applying test-functions. We specialize the general formulation given in [22] so that the theorem can be directly applied to our setting.

Theorem B.2 (Mitoma's theorem)**.** *For any positive integer d, let $X^n := (X_1^n, \cdots, X_d^n)$ be a sequence of processes with sample paths lying in $\mathcal{D}\left([0,\infty);(\mathcal{E}')^d\right)$. The sequence X^n is tight as processes with paths in $\mathcal{D}\left([0,\infty);(\mathcal{E}')^d\right)$, if and only if for any $\phi_1, \cdots, \phi_d \in \mathcal{E}$ we have tightness of the sequence of $\mathcal{D}([0,\infty);\mathbb{R})$-valued processes $\sum_{i=1}^d \left\langle X_i^n, \phi_i \right\rangle$. In particular, if for any $\epsilon, N \in (0,\infty)$ there exists $\widetilde{N} \in (0,\infty)$ such that $\sup_n \mathbb{P}(\sup_{t\in[0,N]} \sum_{i=1}^d \|X_i^n(t)\|_{L^2} > \widetilde{N}) < \epsilon$, then X^n is tight as a sequence of processes with paths in $\mathcal{D}\left([0,\infty);\left(H^{-1}\right)^d\right)$.*

Here we choose H^{-1} for convenience. Indeed, in view of the arguments in [22, Page 335, Example 1a], we can replace the space H^{-1} by H^{-m} for any $m > 1/2$. On the other hand, an immediate application of Theorem B.2 is the following corollary, which states that joint tightness of a pair of sequences of stochastic processes follows from individual tightness assuming that at least one of the involved sequences is C-tight, i.e., all its accumulation points are continuous processes.

Corollary B.3. *Let Y^n and Z^n be sequences of stochastic processes taking values in $(\mathcal{E}')^d$ and $(\mathcal{E}')^l$ respectively, with $d, l \in \mathbb{N}$. If Y^n is C-tight with paths in $\mathcal{D}\left([0,\infty);(\mathcal{E}')^d\right)$ and Z^n is tight with paths in $\mathcal{D}\left([0,\infty);(\mathcal{E}')^l\right)$, then the pair of processes (Y^n, Z^n) is tight with paths in $\mathcal{D}\left([0,\infty);(\mathcal{E}')^{d+l}\right)$.*

Proof. We fist note that for the finite-dimensional case where $(\mathcal{E}')^d$ and $(\mathcal{E}')^l$ are replaced by Euclidean spaces, Corollary B.3 coincides with [15, Cor. VI.3.33]. Obviously the C-tightness of Y^n with paths in $\mathcal{D}\left([0,\infty);(\mathcal{E}')^d\right)$ implies that of $\sum_{i=1}^d \langle Y_i^n, \phi_i \rangle$ with paths in $\mathcal{D}([0,\infty);\mathbb{R})$ for any $\phi_1, \cdots, \phi_d \in \mathcal{E}$. As Theorem B.2 allows us to prove the tightness of distribution-valued processes by verifying that of the real-valued processes obtained by applying test-functions, there follows the tightness of pair of processes (Y^n, Z^n) with paths in $\mathcal{D}\left([0,\infty);(\mathcal{E}')^{d+l}\right)$. □

We remark that the method of proof for the finite-dimensional case (see [15, Page 353, Cor. VI.3.33]) can not directly be applied to Corollary B.3, as the compactness of the unit ball is key to their proof of the finite-dimensional case. On the other hand, if we replace $(\mathcal{E}')^d$ for Y^n by $\mathbb{R}^m \times (\mathcal{E}')^d$ with $m \in \mathbb{N}$, then Corollary B.3 still holds, since the finite-dimensional space is isomorphic as well as homeomorphic to some subspace of $\mathcal{E}'$.

Finally, we use a lemma of Billingsley about weak limits under time-changes.

Lemma B.4. *Let X^n be a sequence of processes taking values in $\mathcal{D}([0,\infty);E)$ for some separable metric space E and let Φ^n be a sequence of non-decreasing processes with paths in $\mathcal{D}([0,\infty);[0,\infty))$. Assume that (X^n, Φ^n) converge weakly to a pair of processes $(X, \Phi) \in \mathcal{D}([0,\infty);E \times [0,\infty))$ such that $X \in C([0,\infty);E)$ with probability 1. Then*

$$X^n \circ \Phi^n \Rightarrow X \circ \Phi.$$

Proof. The proof in Billingsley [3, p. 151] (for the special case $E = \mathbb{R}$) can be immediately adapted to this more general setting. □

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