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Affine LIBOR models with multiple curves: Theory, examples and calibration

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submitted: May 15, 2014

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1. INTRODUCTION

The recent financial crisis has led to paradigm shifting events in interest rate markets because substantial spreads have appeared between rates that used to be closely matched; see Figure 1.1 for an illustration. We can observe, for example, that before the credit crunch the spread between the three month LIBOR and the corresponding Overnight Indexed Swap (OIS) rate was non-zero, however it could be safely disregarded as negligible. The same is true for the three month vs six month basis swap spread. However, since August 2007 these spreads have been evolving randomly over time, are substantially too large to be neglected, and also depend on the tenor length. Therefore, the assumption of a single interest rate curve that could be used both for discounting and for generating future cash flows was seriously challenged, which led to the introduction of the so-called *multiple curve* interest rate models.

In the multiple curve framework, one curve is used for discounting purposes, where the usual choice is the OIS curve, and then as many LIBOR curves as market tenors (e.g. 1m, 3m, 6m and 1y) are built for generating future cash flows. The difference between the OIS and each LIBOR rate is usually called basis spread or simply spread. There are several ways of modeling the curves and different definitions of the spread. One approach is to model the OIS and LIBOR rates directly which leads to tractable pricing formulas, but the sign of the spread is more difficult to control and may become negative. Another approach is to model the OIS and the spread directly and infer the dynamics of the LIBOR; this grants the positivity of the spread, but pricing formulas are generally less tractable. We refer to Mercurio (2010b, pp. 11-12) for a detailed discussion of the advantages and disadvantages of each approach. Moreover, there exist various definitions of the spread: an additive spread is used e.g. by Mercurio (2010a), a multiplicative spread was proposed by Henrard (2010), while an instantaneous spread was used by Andersen and Piterbarg (2010); we refer to Mercurio and Xie (2012) for a discussion of the merits of each definition.

The literature on multiple curve models is growing rapidly and the different models proposed can be classified in one of the categories described above. Moreover, depending on the modeling approach, one can also distinguish between short rate models, Heath–Jarrow–Morton (HJM) models and LIBOR market models (LMM) with multiple curves. The spreads appearing as modeling quantities in the short rate and the HJM models are, by the very nature of these models, instantaneous and given in additive form. We refer to Bianchetti and Morini (2013) for a detailed overview of several multiple curve models. In the short rate framework, we mention Kenyon (2010), Kijima, Tanaka, and Wong (2009) and Morino and Runggaldier (2014), where the additive short rate spread is modeled, which leads to multiplicative adjustments for interest rate derivative prices. HJM-type models have been proposed e.g. by Fujii, Shimada, and Takahashi (2011), Crépey, Grbac, and Nguyen (2012), Moreni and Pallavicini (2014) and Cuchiero, Fontana, and Gnoatto (2014). The models by Mercurio (2009), Bianchetti (2010) (where an analogy with the cross-currency market has been exploited) and Henrard (2010) are developed in the LMM setup. Typically, multiple curve models address the issue of different interest rate curves under the same currency, however, the paper by Fujii et al. (2011) studies a multiple curve model in a cross-currency setup. Filipović and Trolle (2013) offer a thorough econometric analysis of the multiple curve phenomena and decompose the spread into a credit risk and a liquidity risk component. In
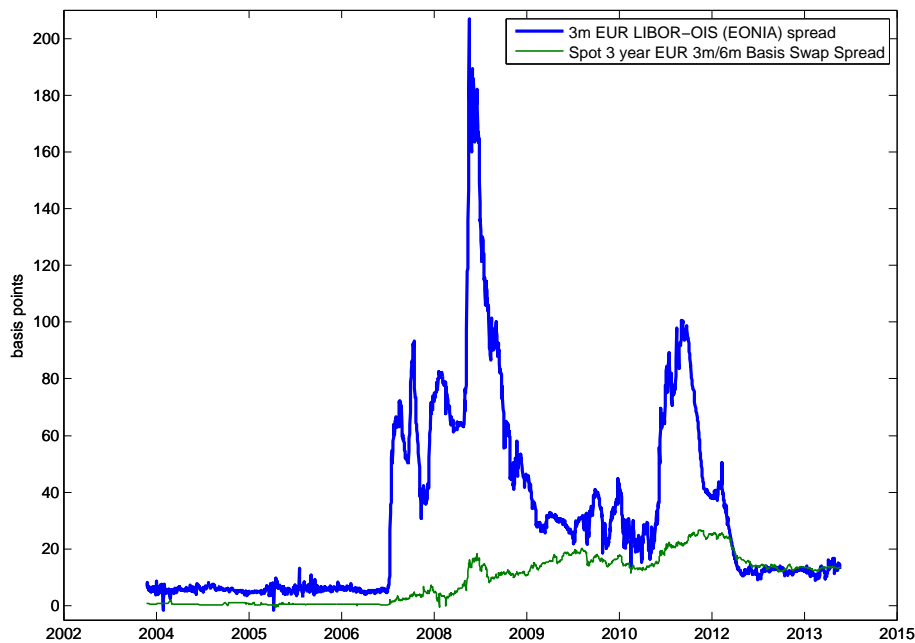
FIGURE 1.1. Spread Development from January 2004 to April 2014

recent work, Gallitschke, Müller, and Seifried (2014) construct a structural model for interbank rates, which provides an endogenous explanation for the emergence of basis spreads.

Let us also mention that there exist various other frameworks in the literature where different curves have been modeled simultaneously, for example when dealing with cross-currency markets (cf. e.g. Amin and Jarrow 1991) or when considering credit risk (cf. e.g. the book by Bielecki and Rutkowski 2002). The models in the multiple curve world often draw inspiration from these frameworks.

The aim of this paper is to develop a multiple curve LIBOR model that combines tractable model dynamics and semi-analytic pricing formulas with positive interest rates and basis spreads. The framework of the affine LIBOR models proposed by Keller-Ressel, Papapantoleon, and Teichmann (2013) turns out to be tailor-made for this task, since it allows us to model directly LIBOR rates that are greater than their OIS counterparts. In other words, the non-negativity of spreads is automatically ensured. Simultaneously, the dynamics are driven by the wide and flexible class of affine processes. Similar to the single curve case, the affine property is preserved under all forward measures, which leads to semi-analytical pricing formulas for liquid interest rate derivatives. In particular, the pricing of caplets is as easy as in the single curve setup, while the model structure allows to derive efficient and accurate approximations for the pricing of swaptions and basis swaptions using a linearization of the exercise boundary. In addition, the model offers adequate calibration results to a system of caplet prices for various strikes and maturities.

The paper is organized as follows: in Section 2 we review the main properties of affine processes and the construction of ordered martingales greater than one. Section 3 introduces the multiple curve interest rate setting. The multiple curve affine LIBOR model is presented in Section 4 and its main properties are discussed, in particular the ability to produce positive rates and spreads and the analytical tractability (i.e. the preservation of the affine property). In Section 5 we study the connection

between the class of affine LIBOR models and the class of LIBOR market models (driven by semi-martingales). Sections 6 and 7 are devoted to the valuation of the most liquid interest rate derivatives such as swaps, caps, swaptions and basis swaptions. In Section 8 we construct a multiple curve affine LIBOR model where rates are driven by common and idiosyncratic factors and calibrate this to market data. Moreover, we test numerically the accuracy of the swaption and basis swaption approximation formulas. Finally, Appendix A provides an explicit formula for the terminal correlation between LIBOR rates.

2. AFFINE PROCESSES

This section provides a brief review of the main properties of affine processes and the construction of ordered martingales greater than one. More details and proofs can be found in Keller-Ressel et al. (2013) and the references therein.

Let $(\Omega, \mathcal{F}, \mathbf{F}, \mathbb{P})$ denote a complete stochastic basis, where $\mathbf{F} = (\mathcal{F}_t)_{t \in [0,T]}$ and T denotes some finite time horizon. Consider a stochastic process X satisfying:

Assumption ($\mathbb{A}$). Let $X = (X_t)_{t \in [0,T]}$ be a conservative, time-homogeneous, stochastically continuous Markov process with values in $D = \mathbb{R}^d_{\geqslant 0}$, and $(\mathbb{P}_x)_{x \in D}$ a family of probability measures on $(\Omega, \mathcal{F})$, such that $X_0 = x$, $\mathbb{P}_x$-almost surely for every $x \in D$. Setting

$$\mathcal{I}_T := \left\{ u \in \mathbb{R}^d : \mathbb{E}_x\big[\mathrm{e}^{\langle u, X_T \rangle}\big] < \infty, \text{ for all } x \in D \right\}, \tag{2.1}$$

we assume that

(i) $0 \in \mathcal{I}_T^\circ$, where $\mathcal{I}_T^\circ$ denotes the interior of $\mathcal{I}_T$;

(ii) the conditional moment generating function of X_t under $\mathbb{P}_x$ has exponentially-affine dependence on x; that is, there exist functions $\phi_t(u) : [0,T] \times \mathcal{I}_T \to \mathbb{R}$ and $\psi_t(u) : [0,T] \times \mathcal{I}_T \to \mathbb{R}^d$ such that

$$\mathbb{E}_x\big[\exp\langle u, X_t \rangle\big] = \exp\big(\phi_t(u) + \langle \psi_t(u), x \rangle\big), \tag{2.2}$$

for all $(t, u, x) \in [0,T] \times \mathcal{I}_T \times D$.

Here $\langle \cdot, \cdot \rangle$ denotes the inner product on $\mathbb{R}^d$ and $\mathbb{E}_x$ the expectation with respect to $\mathbb{P}_x$.

The functions ϕ and ψ satisfy the following system of ODEs, known as *generalized Riccati equations*

$$\frac{\partial}{\partial t}\phi_t(u) = F(\psi_t(u)), \qquad \phi_0(u) = 0, \tag{2.3a}$$

$$\frac{\partial}{\partial t}\psi_t(u) = R(\psi_t(u)), \qquad \psi_0(u) = u, \tag{2.3b}$$

for $(t, u) \in [0,T] \times \mathcal{I}_T$. The functions F and R are of Lévy–Khintchine form:

$$F(u) = \langle b, u \rangle + \int_D \big(\mathrm{e}^{\langle \xi, u \rangle} - 1\rangle\big)m(\mathrm{d}\xi), \tag{2.4a}$$

$$R_i(u) = \langle \beta_i, u \rangle + \left\langle \frac{\alpha_i}{2}u, u \right\rangle + \int_D \big(\mathrm{e}^{\langle \xi, u \rangle} - 1 - \langle u, h_i(\xi) \rangle\big)\mu_i(\mathrm{d}\xi), \tag{2.4b}$$

where $(b, m, \alpha_i, \beta_i, \mu_i)_{1 \leq i \leq d}$ are *admissible parameters* and $h_i : \mathbb{R}^d_{\geqslant 0} \to \mathbb{R}^d$ are suitable truncation functions. The functions ϕ and ψ also satisfy the semi-flow equations

$$\phi_{t+s}(u) = \phi_t(u) + \phi_s(\psi_t(u)) \tag{2.5a}$$

$$\psi_{t+s}(u) = \psi_s(\psi_t(u)) \tag{2.5b}$$

for all $0 \leq t + s \leq T$ and $u \in \mathcal{I}_T$, with initial condition

$$\phi_0(u) = 0 \quad \text{and} \quad \psi_0(u) = u. \tag{2.6}$$

We refer to Duffie, Filipović, and Schachermayer (2003) for all the details.

The following results and definitions will be used in the sequel. Inequalities involving vectors are interpreted componentwise and $\mathbf{1} := (1, 1, \ldots, 1)$.

Lemma 2.1. *The functions ϕ and ψ satisfy the following:*

1. $\phi_t(0) = \psi_t(0) = 0$ *for all $t \in [0, T]$.*
2. $\mathcal{I}_T$ *is a convex set. Moreover, for each $t \in [0, T]$, the functions $\mathcal{I}_T \ni u \mapsto \phi_t(u)$ and $\mathcal{I}_T \ni u \mapsto \psi_t(u)$ are (componentwise) convex.*
3. $\phi_t(\cdot)$ *and $\psi_t(\cdot)$ are order-preserving: let $(t, u), (t, v) \in [0, T] \times \mathcal{I}_T$, with $u \leq v$. Then*

$$\phi_t(u) \leq \phi_t(v) \quad \text{and} \quad \psi_t(u) \leq \psi_t(v). \tag{2.7}$$

4. $\psi_t(\cdot)$ *is strictly order-preserving: let $(t, u), (t, v) \in [0, T] \times \mathcal{I}_T^\circ$, with $u < v$. Then $\psi_t(u) < \psi_t(v)$.*

Definition 2.2. Let X be a process satisfying Assumption $(\mathbb{A})$. Define

$$\gamma_X := \sup_{u \in \mathcal{I}_T \cap \mathbb{R}^d_{>0}} \mathbb{E}_\mathbf{1}\left[e^{\langle u, X_T \rangle}\right]. \tag{2.8}$$

An essential ingredient in affine LIBOR models is the construction of parametrized martingales which are greater than or equal to one and increasing in this parameter (see also Papapantoleon 2010).

Lemma 2.3. *Consider an affine process X satisfying Assumption ($\mathbb{A}$) and let $u \in \mathcal{I}_T \cap \mathbb{R}^d_{\geqslant 0}$. Then the process $M^u = (M^u_t)_{t \in [0, T]}$ with*

$$M^u_t = \exp\left(\phi_{T-t}(u) + \langle \psi_{T-t}(u), X_t \rangle\right), \tag{2.9}$$

is a martingale, greater than or equal to one, and the mapping $u \mapsto M^u_t$ is increasing, for every $t \in [0, T]$.

Proof. Consider the random variable $Y^u_T := e^{\langle u, X_T \rangle}$. Since $u \in \mathcal{I}_T \cap \mathbb{R}^d_{\geqslant 0}$ we have that Y^u_T is greater than one and integrable. Then, from the Markov property of X, (2.2) and the tower property of conditional expectations we deduce that

$$M^u_t = \mathbb{E}\left[e^{\langle u, X_T \rangle} | \mathcal{F}_t\right] = \exp\left(\phi_{T-t}(u) + \langle \psi_{T-t}(u), X_t \rangle\right), \tag{2.10}$$

is a martingale. Moreover, it is obvious that $M^u_t \geq 1$ for all $t \in [0, T]$, while the ordering

$$u \leq v \implies M^u_t \leq M^v_t \quad \forall t \in [0, T], \tag{2.11}$$

follows from the ordering of Y^u_T and the representation $M^u_t = \mathbb{E}[Y^u_T | \mathcal{F}_t]$. $\qquad \square$
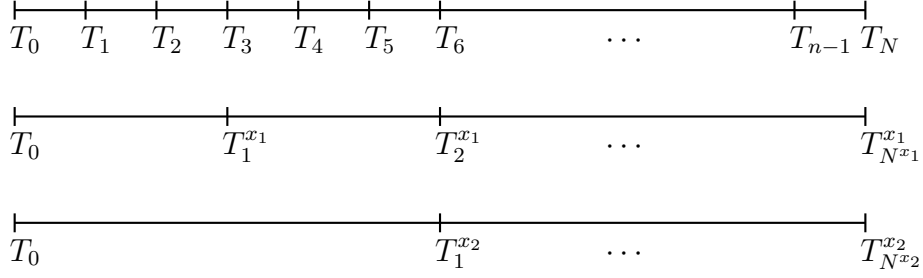
FIGURE 3.2. Illustration of different tenor structures.

3. A MULTIPLE CURVE LIBOR SETTING

We begin by introducing the notation and the main concepts of multiple curve LIBOR models. We will follow the approach introduced in Mercurio (2010a), which has become the industry standard in the meantime.

The fact that LIBOR-OIS spreads are now tenor-dependent means that we cannot work with a single tenor structure any longer. Hence, we start with a discrete, equidistant time structure $\mathcal{T} = \{0 = T_0 < T_1 < \cdots < T_N\}$, where T_k, $k \in \mathcal{K} := \{1, \ldots, N\}$, denote the maturities of the assets traded in the market. Next, we consider different subsets of $\mathcal{T}$ with equidistant time points, i.e. different tenor structures $\mathcal{T}^x = \{0 = T_0^x < T_1^x < \cdots < T_{N^x}^x\}$, where $x \in \mathcal{X} := \{x_1, x_2, \ldots, x_n\}$ is a label that indicates the tenor structure. Typically, we have $\mathcal{X} = \{1, 3, 6, 12\}$ months. We denote the tenor length by $\delta_x = T_k^x - T_{k-1}^x$, for every $x \in \mathcal{X}$. Let $\mathcal{K}^x := \{1, 2, \ldots, N^x\}$ denote the collection of all subscripts related to the tenor structure $\mathcal{T}^x$. We assume that $\mathcal{T}^x \subseteq \mathcal{T}$ and $T_{N^x}^x = T_N$ for all $x \in \mathcal{X}$. A graphical illustration of a possible relation between the different tenor structures appears in Figure 3.2.

Example 3.1. A natural construction of tenor structures is the following: Let $\mathcal{T} = \{0 = T_0 < T_1 < \cdots < T_N\}$ denote a discrete time structure, where $T_k = k\delta$ for $k = 1, \ldots, N$ and $\delta > 0$. Let $\mathcal{X} = \{1 = x_1, x_2, \ldots, x_n\} \subset \mathbb{N}$, where we assume that

$$x_k | N, \quad \text{for all } k = 1, \ldots, n.$$

Next, set for every $x \in \mathcal{X}$

$$T_k^x = k \cdot \delta \cdot x =: k\delta_x, \quad \text{for } k = 1, \ldots, N^x := N/x,$$

where obviously $T_k^x = T_{kx}$. Then, we can consider different subsets of $\mathcal{T}$, i.e. different tenor structures $\mathcal{T}^x = \{0 = T_0^x < T_1^x < \cdots < T_{N^x}^x\}$, which satisfy by construction $\mathcal{T}^x \subset \mathcal{T}^{x_1} = \mathcal{T}$ and also $T_{N^x}^x = N^x \cdot \delta \cdot x = T_N$, for all $x \in \mathcal{X}$.

We consider the OIS curve as discount curve, following the standard market practice of fully collateralized contracts. The market prices for caps and swaptions considered in the sequel for model calibration are indeed quoted under the assumption of full collateralization. A detailed discussion on the choice of the discount curve in the multiple curve setting can be found e.g. in Mercurio (2010a) and in Hull and White (2013). The discount factors $B(0, T)$ are stripped from market OIS rates and defined for

every possible maturity $T \in \mathcal{T}$ via

$$T \mapsto B(0, T) = B^{OIS}(0, T).$$

We denote by $B(t, T)$ the discount factor, i.e. the price of a zero coupon bond, at time t for maturity T, which is assumed to coincide with the corresponding OIS-based zero coupon bond for the same maturity.

We also assume that all our modeling objects live on a complete stochastic basis $(\Omega, \mathcal{F}, \mathbf{F}, \mathbb{P}_N)$, where $\mathbb{P}_N$ denotes the terminal forward measure, i.e. the martingale measure associated with the numeraire $B(\cdot, T_N)$. The corresponding expectation is denoted by $\mathbb{E}_N$. Then, we introduce forward measures $\mathbb{P}_k^x$ associated to the numeraire $B(\cdot, T_k^x)$ for every pair (x, k) with $x \in \mathcal{X}$ and $k \in \mathcal{K}^x$. The corresponding expectation is denoted by $\mathbb{E}_k^x$. The forward measures $\mathbb{P}_k^x$ are absolutely continuous with respect to $\mathbb{P}_N$, and defined in the usual way, i.e. via the Radon–Nikodym density

$$\frac{\mathrm{d}\mathbb{P}_k^x}{\mathrm{d}\mathbb{P}_N} = \frac{B(0, T_N)}{B(0, T_k^x)} \frac{1}{B(T_k^x, T_N)}. \tag{3.1}$$

Remark 3.2. Since $\mathcal{T}^x \subseteq \mathcal{T}$ there exists an $l \in \mathcal{K}$ and a $k \in \mathcal{K}^x$ such that $T_l = T_k^x$. Therefore, the corresponding numeraires and forward measures coincide, i.e. $B(\cdot, T_l) = B(\cdot, T_k^x)$ and $\mathbb{P}_l = \mathbb{P}_k^x$. See again Figure 3.2.

Next, we define the two rates that are the main modeling objects in the multiple curve LIBOR setting: the OIS forward rate and the LIBOR rate. We also define the additive and the multiplicative spread between these two rates. Let us denote by $L(T_{k-1}^x, T_k^x)$ the *spot LIBOR rate* at time T_{k-1}^x for the time interval $[T_{k-1}^x, T_k^x]$, which is an $\mathcal{F}_{T_{k-1}^x}$-measurable random variable on the given stochastic basis.

Definition 3.3. The time-t *OIS forward rate* for the time interval $[T_{k-1}^x, T_k^x]$ is defined by

$$F_k^x(t) := \frac{1}{\delta_x} \left(\frac{B(t, T_{k-1}^x)}{B(t, T_k^x)} - 1 \right). \tag{3.2}$$

Definition 3.4. The time-t *LIBOR rate* for the time interval $[T_{k-1}^x, T_k^x]$ is defined by

$$L_k^x(t) = \mathbb{E}_k^x \big[L(T_{k-1}^x, T_k^x) | \mathcal{F}_t \big]. \tag{3.3}$$

The LIBOR rate is the fixed rate that should be exchanged for the future spot LIBOR rate so that the forward rate agreement has zero initial value. Hence, this rate reflects the market expectations about the value of the future spot LIBOR rate. Notice that at time $t = T_{k-1}^x$ we have that

$$L_k^x(T_{k-1}^x) = \mathbb{E}_k^x \big[L(T_{k-1}^x, T_k^x) | \mathcal{F}_{T_{k-1}^x} \big] = L(T_{k-1}^x, T_k^x), \tag{3.4}$$

i.e. this rate coincides with the spot LIBOR rate at the corresponding tenor dates.

Remark 3.5. In the single curve setup, (3.2) is the definition of the *forward* LIBOR rate. However, in the multiple curve setup we have that

$$L(T_{k-1}^x, T_k^x) \neq \frac{1}{\delta_x} \left(\frac{1}{B(T_{k-1}^x, T_k^x)} - 1 \right),$$

hence the OIS and the LIBOR rates are no longer equal.

Definition 3.6. The *spread* between the LIBOR rate and the OIS forward rate is defined by

$$S_k^x(t) := L_k^x(t) - F_k^x(t). \tag{3.5}$$

Let us also provide an alternative definition of the spread based on a multiplicative, instead of an additive, decomposition.

Definition 3.7. The *multiplicative spread* between the LIBOR rate and the OIS forward rate is defined by

$$1 + \delta_x R_k^x(t) := \frac{1 + \delta_x L_k^x(t)}{1 + \delta_x F_k^x(t)}. \tag{3.6}$$

A model for the dynamic evolution of the OIS and LIBOR rates, and thus also of their spread, should satisfy certain conditions which stem from economic reasoning, arbitrage requirements and their respective definitions. These are, in general, consistent with market observations. We formulate them below as model requirements:

(M1) $F_k^x(t) \geq 0$ and $F_k^x \in \mathcal{M}(\mathbb{P}_k^x)$, for all $x \in \mathcal{X}$, $k \in \mathcal{K}^x$, $t \in [0, T_{k-1}^x]$.

(M2) $L_k^x(t) \geq 0$ and $L_k^x \in \mathcal{M}(\mathbb{P}_k^x)$, for all $x \in \mathcal{X}$, $k \in \mathcal{K}^x$, $t \in [0, T_{k-1}^x]$.

(M3) $S_k^x(t) \geq 0$, for all $x \in \mathcal{X}$, $k \in \mathcal{K}^x$, $t \in [0, T_{k-1}^x]$.

Here $\mathcal{M}(\mathbb{P}_k^x)$ denotes the set of $\mathbb{P}_k^x$-martingales.

Remark 3.8. If the additive spread is positive the multiplicative spread is also positive and vice versa.

4. THE MULTIPLE CURVE AFFINE LIBOR MODEL

We describe next the affine LIBOR model for the multiple curve interest rate setting and analyze its main properties. In particular, we show that it satisfies the modeling requirements (M1)–(M3) presented above and that it is analytically tractable. In this framework, OIS forward rates and LIBOR rates are modeled in the spirit of the affine LIBOR model introduced by Keller-Ressel et al. (2013).

Let X be an affine process defined on $(\Omega, \mathcal{F}, \mathbf{F}, \mathbb{P}_N)$, satisfying Assumption $(\mathbb{A})$ and starting at the canonical value $\mathbf{1}$. Consider a fixed $x \in \mathcal{X}$ and the associated tenor structure $\mathcal{T}^x$. We construct two families of parametrized martingales following Lemma 2.3: take two sequences of vectors $(u_k^x)_{k \in \mathcal{K}^x}$ and $(v_k^x)_{k \in \mathcal{K}^x}$, and define the $\mathbb{P}_N$-martingales $M^{u_k^x}$ and $M^{v_k^x}$ via

$$M_t^{u_k^x} = \exp\left(\phi_{T_N - t}(u_k^x) + \langle \psi_{T_N - t}(u_k^x), X_t \rangle\right), \tag{4.1}$$

and

$$M_t^{v_k^x} = \exp\left(\phi_{T_N - t}(v_k^x) + \langle \psi_{T_N - t}(v_k^x), X_t \rangle\right). \tag{4.2}$$

The *multiple curve affine LIBOR model* postulates that the OIS and the LIBOR rates associated with the x-tenor evolve according to

$$1 + \delta_x F_k^x(t) = \frac{M_t^{u_{k-1}^x}}{M_t^{u_k^x}} \quad \text{and} \quad 1 + \delta_x L_k^x(t) = \frac{M_t^{v_{k-1}^x}}{M_t^{u_k^x}}, \tag{4.3}$$

for every $k = 2, \ldots, N_x$ and $t \in [0, T_{k-1}^x]$.

In the following three propositions, we show how to construct a multiple curve affine LIBOR model from any given initial term structure of OIS and LIBOR rates.

Proposition 4.1. *Consider the time structure $\mathcal{T}$, let $B(0, T_l)$, $l \in \mathcal{K}$, be the initial term structure of non-negative OIS discount factors and assume that*

$$B(0, T_1) \geq \cdots \geq B(0, T_N).$$

Then the following statements hold:

1 *If $\gamma_X > B(0, T_1)/B(0, T_N)$, then there exists a decreasing sequence $u_1 \geq u_2 \geq \cdots \geq u_N = 0$ in $\mathcal{I}_T \cap \mathbb{R}^d_{\geqslant 0}$, such that*

$$M_0^{u_l} = \frac{B(0, T_l)}{B(0, T_N)} \qquad \text{for all } l \in \mathcal{K}. \tag{4.4}$$

In particular, if $\gamma_X = \infty$, the multiple curve affine LIBOR model can fit any initial term structure of OIS rates.

2 *If X is one-dimensional, the sequence $(u_l)_{l\in\mathcal{K}}$ is unique.*

3 *If all initial OIS rates are positive, the sequence $(u_l)_{l\in\mathcal{K}}$ is strictly decreasing.*

Proof. See Proposition 6.1 in Keller-Ressel et al. (2013). $\qquad\square$

After fitting the initial term structure of OIS discount factors, we want to fit the initial term structure of LIBOR rates, which is now tenor-dependent. Thus, for each $k \in \mathcal{K}^x$, we set

$$u_k^x := u_l, \tag{4.5}$$

where $l \in \mathcal{K}$ is such that $T_l = T_k^x$; see Remark 3.2. In general, we have that $l = kT_1^x/T_1$, while in the setting of Example 3.1 we simply have $l = kx$, i.e. $u_k^x = u_{kx}$.

Proposition 4.2. *Consider the setting of Proposition 4.1, the fixed $x \in \mathcal{X}$ and the corresponding tenor structure $\mathcal{T}^x$. Let $L_k^x(0)$, $k \in \mathcal{K}^x$, be the initial term structure of non-negative LIBOR rates and assume that for every $k \in \mathcal{K}^x$*

$$L_k^x(0) \geq \frac{1}{\delta_x} \left(\frac{B(0, T_{k-1}^x)}{B(0, T_k^x)} - 1 \right) = F_k^x(0). \tag{4.6}$$

The following statements hold:

1 *If $\gamma_X > \max_{k\in\mathcal{K}^x}(1+\delta_x L_k^x(0))B(0, T_k^x)/B(0, T_N^x)$, then there exists a sequence $v_1^x, v_2^x, \ldots, v_{N^x}^x = 0$ in $\mathcal{I}_T \cap \mathbb{R}^d_{\geqslant 0}$, such that $v_k^x \geq u_k^x$ and*

$$M_0^{v_k^x} = (1 + \delta_x L_{k+1}^x(0))M_0^{u_{k+1}^x}, \quad \text{for all } k \in \mathcal{K}^x\backslash\{N^x\}. \tag{4.7}$$

In particular, if $\gamma_X = \infty$, then the multiple curve affine LIBOR model can fit any initial term structure of LIBOR rates.

2 *If X is one-dimensional, the sequence $(v_k^x)_{k\in\mathcal{K}^x}$ is unique.*

3 *If all initial spreads are positive, then $v_k^x > u_k^x$, for all $k \in \mathcal{K}^x\backslash\{N^x\}$.*

Proof. Similarly to the previous proposition, by fitting the initial LIBOR rates we obtain a sequence $(v_k^x)_{k\in\mathcal{K}^x}$ which satisfies (1)–(3). The inequality $v_k^x \geq u_k^x$ follows directly from (4.6). $\qquad\square$

Proposition 4.3. *Consider the setting of the previous propositions. Then we have:*
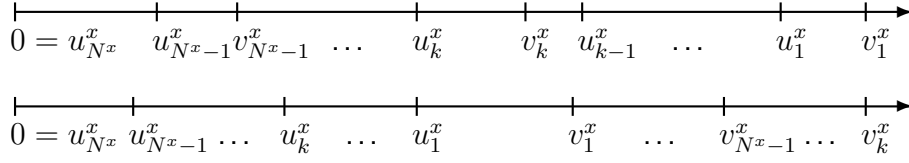
FIGURE 4.3. Two possible orderings of (u_k^x) and (v_k^x).

1 F_k^x and L_k^x are $\mathbb{P}_k^x$-martingales, for every $k \in \mathcal{K}^x$.
2 $L_k^x(t) \geq F_k^x(t) \geq 0$, for every $k \in \mathcal{K}^x$, $t \in [0, T_{k-1}^x]$.

Proof. Since $M^{u_k^x}$ and $M^{v_k^x}$ are $\mathbb{P}_N$-martingales and the density process relating the measures $\mathbb{P}_N$ and $\mathbb{P}_k^x$ is provided by

$$\frac{d\mathbb{P}_k^x}{d\mathbb{P}_N}\bigg|_{\mathcal{F}_t} = \frac{B(0, T_N)}{B(0, T_k^x)} \frac{B(t, T_k^x)}{B(t, T_N)} = \frac{M_t^{u_k^x}}{M_0^{u_k^x}}, \tag{4.8}$$

we get from (4.3)

$$1 + \delta_x F_k^x \in \mathcal{M}(\mathbb{P}_k^x) \text{ because } (1 + \delta_x F_k^x) M^{u_k^x} = M^{u_{k-1}^x} \in \mathcal{M}(\mathbb{P}_N). \tag{4.9}$$

Similarly,

$$1 + \delta_x L_k^x \in \mathcal{M}(\mathbb{P}_k^x) \text{ because } (1 + \delta_x L_k^x) M^{u_k^x} = M^{v_{k-1}^x} \in \mathcal{M}(\mathbb{P}_N). \tag{4.10}$$

The monotonicity of the sequence (u_k^x) together with (2.11) yields that $M^{u_{k-1}^x} \geq M^{u_k^x}$. Moreover, from the inequality $v_k^x \geq u_k^x$ together with (2.11) again, it follows that $M^{v_k^x} \geq M^{u_k^x}$, for all $k \in \mathcal{K}^x$. Hence,

$$1 + \delta_x L_k^x \geq 1 + \delta_x F_k^x \geq 1.$$

Therefore, the OIS forward rates, the LIBOR rates and the corresponding spreads are non-negative $\mathbb{P}_k^x$-martingales. $\square$

Remark 4.4. Let us now look more closely at the relationship between the sequences (v_k^x) and (u_k^x). Propositions 4.1 and 4.2 imply that $u_{k-1}^x \geq u_k^x$ and $v_k^x \geq u_k^x$ for all $k \in \mathcal{K}^x$. However, we do not know the ordering of v_k^x and u_{k-1}^x, or whether the sequence (v_k^x) is monotone or not. The market data for LIBOR spreads indicate that in a 'normal' market situation $v_k^x \in [u_k^x, u_{k-1}^x]$. More precisely, on the one side, we have $v_k^x \geq u_k^x$ because the LIBOR spreads are nonnegative. On the other side, if $v_k^x > u_{k-1}^x$, then the LIBOR rate would be more than two times higher than the OIS rate spanning an interval twice as long, starting at the same date. This contradicts normal market behavior, hence $v_k^x \in [u_k^x, u_{k-1}^x]$ and consequently the sequence (v_k^x) will also be decreasing. This ordering of the parameters (v_k^x) and (u_k^x) is illustrated in Figure 4.3 (top graph). However, the 'normal' market situation alternates with an 'extreme' situation, where the spread is higher than the OIS rate. In the bottom graph of Figure 4.3 we plot another possible ordering of the parameters (v_k^x) and (u_k^x) corresponding to such a case of very high spreads. Intuitively speaking, the value of the corresponding model spread depends on the distance between the parameters (v_k^x) and (u_k^x), although in a non-linear fashion.

The next result clarifies an important property of the multiple curve affine LIBOR model, namely its analytical tractability in the sense that the model structure is preserved under different forward measures. More precisely, the process X remains affine under any forward measure, although its 'characteristics'

are now time-dependent. We refer to Filipović (2005) for time-inhomogeneous affine processes. This property plays a crucial role in the derivation of tractable pricing formulas for interest rate derivatives in the forthcoming sections.

Proposition 4.5. *The process X is a time-inhomogeneous affine process under the measure $\mathbb{P}_k^x$, for every $x \in \mathcal{X}$ and $k \in \mathcal{K}^x$. In particular*

$$\mathbb{E}_k^x\big[\mathrm{e}^{\langle w, X_t \rangle}\big] = \exp\left(\phi_t^{k,x}(w) + \langle \psi_t^{k,x}(w), X_0 \rangle\right), \tag{4.11}$$

where

$$\phi_t^{k,x}(w) := \phi_t\big(\psi_{T_N-t}(u_k^x) + w\big) - \phi_t\big(\psi_{T_N-t}(u_k^x)\big), \tag{4.12a}$$

$$\psi_t^{k,x}(w) := \psi_t\big(\psi_{T_N-t}(u_k^x) + w\big) - \psi_t\big(\psi_{T_N-t}(u_k^x)\big), \tag{4.12b}$$

for every $w \in \mathcal{I}^{k,x}$ with

$$\mathcal{I}^{k,x} := \big\{w \in \mathbb{R}^d : \psi_{T_N-t}(u_k^x) + w \in \mathcal{I}_T\big\}. \tag{4.13}$$

Proof. Using the density process between the forward measures, see (4.8), we have that

$$
\begin{aligned}
\mathbb{E}_k^x\big[\mathrm{e}^{\langle w, X_t \rangle}\big|\mathcal{F}_s\big] &= \mathbb{E}_N\big[\mathrm{e}^{\langle w, X_t \rangle} M_t^{u_k^x}/M_s^{u_k^x}\big|\mathcal{F}_s\big] \\
&= \mathbb{E}_N\Big[\exp\big(\phi_{T_N-t}(u_k^x) + \langle \psi_{T_N-t}(u_k^x) + w, X_t \rangle\big)\big|\mathcal{F}_s\Big]/M_s^{u_k^x} \\
&= \exp\Big(\phi_{T_N-t}(u_k^x) - \phi_{T_N-s}(u_k^x) + \phi_{t-s}(\psi_{T_N-t}(u_k^x) + w)\Big) \\
&\quad \times \exp\Big\langle \psi_{t-s}(\psi_{T_N-t}(u_k^x) + w) - \psi_{T_N-s}(u_k^x), X_s\Big\rangle, \tag{4.14}
\end{aligned}
$$

where the above expectation is finite for every $w \in \mathcal{I}^{k,x}$; recall (2.1). This shows that X is a time-inhomogeneous affine process under $\mathbb{P}_k^x$, while (4.11) follows by substituting $s = 0$ in (4.14) and using the flow equations (2.5). $\qquad\square$

Remark 4.6 (Single curve and deterministic spread)**.** The multiple curve affine LIBOR model easily reduces to its single curve counterpart (cf. Keller-Ressel et al. 2013) by setting $v_k^x = u_k^x$ for all $x \in \mathcal{X}$ and $k \in \mathcal{K}^x$. Another interesting question is whether the spread can be deterministic or, similarly, whether the LIBOR rate can be a deterministic transformation of the OIS rate.

Consider, for example, a 2-dimensional driving process $X = (X^1, X^2)$ where X^1 is an arbitrary affine process and X^2 the constant process (i.e. $X_t^2 \equiv X_0^2$). Then, by setting

$$u_{k-1}^x = (u_{1,k-1}^x, 0) \quad \text{and} \quad v_k^x = (u_{1,k-1}^x, v_{2,k-1}^x)$$

where $u_{1,k-1}^x, v_{2,k-1}^x > 0$ we arrive at

$$1 + \delta_x L_k^x(t) = (1 + \delta_x F_k^x(t))\,\mathrm{e}^{v_{2,k-1}^x \cdot X_0^2}.$$

Therefore, the LIBOR rate is a deterministic transformation of the OIS rate, although the spread S_k^x as defined in (3.5) is not deterministic. In that case, the multiplicative spread R_k^x defined in (3.6) is obviously deterministic.

Remark 4.7. The multiple curve affine LIBOR model fulfills requirements (M1)–(M3), which are consistent with the typical market observations of nonnegative interest rates and nonnegative spreads. However, negative rates and negative spreads have been observed for short time intervals, in particular when a tenor of one month is considered for the LIBOR. Since these occurrences are limited to spot rates of the shortest available tenor and occur for periods of a single day usually, we consider them to be of peripheral interest for our modeling framework. Nevertheless, negative interest rates and spreads can be easily accommodated in this setup by considering, for example, affine processes on $\mathbb{R}^d$ instead of $\mathbb{R}^d_{\geqslant 0}$ or 'shifted' positive affine processes where $\text{supp}(X) \in [a, \infty)^d$ with $a < 0$.

5. CONNECTION TO LIBOR MARKET MODELS

In this section, we will clarify the relationship between the affine LIBOR models and the 'classical' LIBOR market models, cf. Sandmann, Sondermann, and Miltersen (1995) and Brace, Gątarek, and Musiela (1997). More precisely, we will embed the multiple curve affine LIBOR model in the framework of Jamshidian (1997) and derive the dynamics of OIS forward rates and LIBOR rates. We will concentrate on affine diffusion processes for the sake of simplicity, in order to expose the ideas without too many technical details. The generalization to affine processes with jumps is straightforward and left to the interested reader.

An affine diffusion process on the state space $D = \mathbb{R}^d_{\geqslant 0}$ is the solution $X = X^{\mathrm{x}}$ of the SDE

$$\mathrm{d}X_t = (b + BX_t)\mathrm{d}t + \sigma(X_t)\mathrm{d}W_t, \quad X_0 = \mathrm{x}, \tag{5.1}$$

where $W = W^N$ is a d-dimensional $\mathbb{P}_N$-Brownian motion. The coefficients b, $B = (\beta_1, \dots, \beta_d)$ and σ have to satisfy the admissibility conditions for affine diffusions on $\mathbb{R}^d_{\geqslant 0}$, see Filipović (2009, Ch. 10). That is, the drift vectors satisfy

$$b \in \mathbb{R}^d_{\geqslant 0}, \quad \beta_{i(i)} \in \mathbb{R} \quad \text{and} \quad \beta_{i(j)} \in \mathbb{R}_{\geqslant 0} \quad \text{for all } 1 \leq i, j \leq d, \tag{5.2}$$

while the diffusion matrix $\sigma(x)$ satisfies

$$\sigma(x)\sigma(x)^{\mathsf{T}} = \sum_{i=1}^d \alpha_i x_i, \tag{5.3}$$

where α_i are symmetric, positive semidefinite matrices for all $1 \leq i \leq d$, such that

$$\alpha_{i(ii)} \in \mathbb{R}_{\geqslant 0} \quad \text{and} \quad \alpha_{i(jk)} = 0 \quad \text{for all } 1 \leq i, j, k \leq d. \tag{5.4}$$

Therefore, the affine diffusion process X is componentwise described by

$$\mathrm{d}X_t^i = (b + BX_t)^i \mathrm{d}t + \sqrt{X_t^i}\, \sigma_i^{\mathsf{T}}\, \mathrm{d}W_t, \tag{5.5}$$

for all $i = 1, \dots, d$, where $\sigma_i = \sqrt{\alpha_{i(ii)}} \cdot \mathrm{e}_i$ (with e_i the unit vector).

5.1. **OIS dynamics.** We start by computing the dynamics of OIS forward rates. As in the previous section, we consider a fixed $x \in \mathcal{X}$ and the associated tenor structure $\mathcal{T}^x$.

Using the structure of the $\mathbb{P}_N$-martingale $M^{u_k^x}$ in (4.1), we have that

$$\mathrm{d}M_t^{u_k^x} = M_t^{u_k^x} \psi_{T_N - t}(u_k^x)^{\mathsf{T}} \mathrm{d}X_t + \dots \mathrm{d}t. \tag{5.6}$$

Hence, applying Itô's product rule to (4.3) and using (5.6) yields that

$$\mathrm{d}F_k^x(t) = \frac{1}{\delta_x} \mathrm{d} \frac{M_t^{u_{k-1}^x}}{M_t^{u_k^x}} = \frac{1}{\delta_x} \frac{M_t^{u_{k-1}^x}}{M_t^{u_k^x}} \left(\psi_{T_N-t}(u_{k-1}^x) - \psi_{T_N-t}(u_k^x)\right)^\mathsf{T} \mathrm{d}X_t + \ldots \mathrm{d}t$$

$$= \frac{1}{\delta_x} \left(1 + \delta_x F_k^x(t)\right) \left(\psi_{T_N-t}(u_{k-1}^x) - \psi_{T_N-t}(u_k^x)\right)^\mathsf{T} \mathrm{d}X_t + \ldots \mathrm{d}t.$$

Therefore, the OIS rates satisfy the following SDE

$$\frac{\mathrm{d}F_k^x(t)}{F_k^x(t)} = \frac{1 + \delta_x F_k^x(t)}{\delta_x F_k^x(t)} \left(\psi_{T_N-t}(u_{k-1}^x) - \psi_{T_N-t}(u_k^x)\right)^\mathsf{T} \mathrm{d}X_t + \ldots \mathrm{d}t \tag{5.7}$$

for all $k = 2, ..., N^x$. Now, using the dynamics of the affine process X from (5.5) we arrive at

$$\frac{\mathrm{d}F_k^x(t)}{F_k^x(t)} = \frac{1 + \delta_x F_k^x(t)}{\delta_x F_k^x(t)} \sum_{i=1}^d \left(\psi_{T_N-t}^i(u_{k-1}^x) - \psi_{T_N-t}^i(u_k^x)\right) \mathrm{d}X_t^i + \ldots \mathrm{d}t$$

$$= \frac{1 + \delta_x F_k^x(t)}{\delta_x F_k^x(t)} \sum_{i=1}^d \left(\psi_{T_N-t}^i(u_{k-1}^x) - \psi_{T_N-t}^i(u_k^x)\right) \sqrt{X_t^i}\, \sigma_i^\mathsf{T} \mathrm{d}W_t^N + \ldots \mathrm{d}t$$

$$=: \Gamma_{x,k}^\mathsf{T}(t)\, \mathrm{d}W_t^N + \ldots \mathrm{d}t, \tag{5.8}$$

where we define the volatility structure

$$\Gamma_{x,k}(t) = \frac{1 + \delta_x F_k^x(t)}{\delta_x F_k^x(t)} \sum_{i=1}^d \left(\psi_{T_N-t}^i(u_{k-1}^x) - \psi_{T_N-t}^i(u_k^x)\right) \sqrt{X_t^i}\, \sigma_i \in \mathbb{R}_{\geqslant 0}^d. \tag{5.9}$$

On the other hand, we know from the general theory of discretely compounded forward rates (cf. Jamshidian 1997) that the OIS forward rate should satisfy the following SDE under the terminal measure $\mathbb{P}_N$

$$\frac{\mathrm{d}F_k^x(t)}{F_k^x(t)} = -\sum_{l=k+1}^{N^x} \frac{\delta_x F_l^x(t)}{1 + \delta_x F_l^x(t)} \Gamma_{x,l}^\mathsf{T}(t) \Gamma_{x,k}(t)\, \mathrm{d}t + \Gamma_{x,k}^\mathsf{T}(t)\, \mathrm{d}W_t^N, \tag{5.10}$$

for the volatility structure $\Gamma_{x,k}$ given in (5.9). Therefore, we get immediately that the $\mathbb{P}_k^x$-Brownian motion $W^{x,k}$ is related to the terminal Brownian motion W^N via the equality

$$W^{x,k} := W^N - \sum_{l=k+1}^{N^x} \int_0^\cdot \frac{\delta_x F_l^x(t)}{1 + \delta_x F_l^x(t)} \Gamma_{x,l}(t)\, \mathrm{d}t$$

$$= W^N - \sum_{l=k+1}^{N^x} \sum_{i=1}^d \int_0^\cdot \left(\psi_{T_N-t}^i(u_{l-1}^x) - \psi_{T_N-t}^i(u_l^x)\right) \sqrt{X_t^i}\, \sigma_i\, \mathrm{d}t. \tag{5.11}$$

Moreover, the dynamics of X under $\mathbb{P}_k^x$ take the form

$$
\begin{aligned}
\mathrm{d}X_t^i &= (b + BX_t)^i \, \mathrm{d}t + \sqrt{X_t^i} \, \sigma_i^{\mathsf{T}} \mathrm{d}W_t^{x,k} \\
&\quad + \sigma_i^{\mathsf{T}} \sqrt{X_t^i} \sum_{l=k+1}^{N^x} \sum_{j=1}^{d} \left(\psi_{T_N-t}^j \left(u_{l-1}^x \right) - \psi_{T_N-t}^j \left(u_l^x \right) \right) \sqrt{X_t^j} \sigma_j \mathrm{d}t \\
&= \left(b_i + (BX_t)^i + \sum_{l=k+1}^{N^x} \left(\psi_{T_N-t}^i (u_{l-1}^x) - \psi_{T_N-t}^i (u_l^x) \right) X_t^i \, |\sigma_i|^2 \right) \mathrm{d}t \\
&\quad + \sqrt{X_t^i} \, \sigma_i^{\mathsf{T}} \mathrm{d}W_t^{x,k},
\end{aligned}
\tag{5.12}
$$

for all $i = 1, \ldots, d$. The last equation provides an alternative proof to Proposition 4.5 in this setting, since it shows explicitly that X is a time-inhomogeneous affine diffusion process under $\mathbb{P}_k^x$. One should also note from (5.11), that the difference between the terminal and the forward Brownian motion does not depend on other forward rates as in 'classical' LIBOR market models. The same property is shared by forward price models, see e.g. Eberlein and Kluge (2007).

Thus, we arrive at the following $\mathbb{P}_k^x$-dynamics for the OIS forward rates

$$
\frac{\mathrm{d}F_k^x(t)}{F_k^x(t)} = \Gamma_{x,k}^{\mathsf{T}}(t) \, \mathrm{d}W_t^{x,k}
\tag{5.13}
$$

with the volatility structure $\Gamma_{x,k}$ provided by (5.9). The structure of $\Gamma_{x,k}$ shows that there is a built-in shift in the model, whereas the volatility structure is determined by ψ and σ.

5.2. **LIBOR dynamics.** Next, we derive the dynamics of the LIBOR rates associated to the same tenor. Using (4.3), (4.2) and repeating the same steps as above, we obtain the following

$$
\begin{aligned}
\frac{\mathrm{d}L_k^x(t)}{L_k^x(t)} &= \frac{1}{\delta_x L_k^x(t)} \mathrm{d}\frac{M_t^{v_{k-1}^x}}{M_t^{u_k^x}} \\
&= \frac{1}{\delta_x L_k^x(t)} \frac{M_t^{v_{k-1}^x}}{M_t^{u_k^x}} \left(\psi_{T_N-t}(v_{k-1}^x) - \psi_{T_N-t}(u_k^x) \right)^{\mathsf{T}} \mathrm{d}X_t + \ldots \mathrm{d}t \\
&= \frac{1 + \delta_x L_k^x(t)}{\delta_x L_k^x(t)} \sum_{i=1}^{d} \left(\psi_{T_N-t}^i(v_{k-1}^x) - \psi_{T_N-t}^i(u_k^x) \right) \sqrt{X_t^i} \, \sigma_i^{\mathsf{T}} \mathrm{d}W_t^N + \ldots \mathrm{d}t,
\end{aligned}
$$

for all $k = 2, \ldots, N^x$. Similarly to (5.9) we introduce the volatility structure

$$
\Lambda_{x,k}(t) := \frac{1 + \delta_x L_k^x(t)}{\delta_x L_k^x(t)} \sum_{i=1}^{d} \left(\psi_{T_N-t}^i(v_{k-1}^x) - \psi_{T_N-t}^i(u_k^x) \right) \sqrt{X_t^i} \, \sigma_i \in \mathbb{R}_{\geqslant 0}^d,
\tag{5.14}
$$

and then obtain for L_k^x the following $\mathbb{P}_k^x$-dynamics

$$
\frac{\mathrm{d}L_k^x(t)}{L_k^x(t)} = \Lambda_{x,k}^{\mathsf{T}}(t) \, \mathrm{d}W_t^{x,k},
\tag{5.15}
$$

where $W^{x,k}$ is the $\mathbb{P}_k^x$-Brownian motion given by (5.11), while the dynamics of X are provided by (5.12).

5.3. **Spread dynamics.** Using that $S_k^x = L_k^x - F_k^x$, the dynamics of LIBOR and OIS rates under the forward measure $\mathbb{P}_k^x$ in (5.13) and (5.15), as well as the structure of the volatilities in (5.9) and (5.14), after some straightforward calculations we arrive at

$$\mathrm{d}S_k^x(t) = \left\{ S_k^x(t)\Upsilon_t^\mathsf{T}(v_{k-1}^x, u_k^x) + \frac{1 + \delta_x F_k^x(t)}{\delta_x}\Upsilon_t^\mathsf{T}(v_{k-1}^x, u_{k-1}^x) \right\} \mathrm{d}W_t^{x,k},$$

where

$$\Upsilon_t(w, y) := \sum_{i=1}^d \left(\psi_{T_N-t}^i(w) - \psi_{T_N-t}^i(y) \right) \sqrt{X_t^i}\, \sigma_i. \tag{5.16}$$

5.4. **Instantaneous correlations.** The derivation of the SDEs that OIS and LIBOR rates satisfy allows to provide quickly formulas for various quantities of interest, such as the instantaneous correlations between OIS and LIBOR rates or LIBOR rates with different maturities or tenors. We have, for example, that the instantaneous correlation between the LIBOR rates maturing at T_k^x and T_l^x is heuristically described by

$$\mathsf{Corr}_t\big[L_k^x, L_l^x\big] = \frac{\frac{\mathrm{d}L_k^x(t)}{L_k^x(t)} \cdot \frac{\mathrm{d}L_l^x(t)}{L_l^x(t)}}{\sqrt{\frac{\mathrm{d}L_k^x(t)}{L_k^x(t)} \cdot \frac{\mathrm{d}L_k^x(t)}{L_k^x(t)}} \sqrt{\frac{\mathrm{d}L_l^x(t)}{L_l^x(t)} \cdot \frac{\mathrm{d}L_l^x(t)}{L_l^x(t)}}}$$

therefore we get that

$$\mathsf{Corr}_t\big[L_k^x, L_l^x\big] \stackrel{(5.15)}{=} \frac{\Lambda_{x,k}^\mathsf{T}\Lambda_{x,l}}{|\Lambda_{x,k}||\Lambda_{x,l}|}$$

$$= \frac{\sum_{i=1}^d \left(\psi_{T_N-t}^i\left(v_{k-1}^x\right) - \psi_{T_N-t}^i\left(u_k^x\right) \right) \left(\psi_{T_N-t}^i\left(v_{l-1}^x\right) - \psi_{T_N-t}^i\left(u_l^x\right) \right) X^i|\sigma_i|^2}{\sqrt{\sum_{i=1}^d \left(\psi_{T_N-t}^i\left(v_{k-1}^x\right) - \psi_{T_N-t}^i\left(u_k^x\right) \right)^2 X^i|\sigma_i|^2}}$$

$$\times \frac{1}{\sqrt{\sum_{i=1}^d \left(\psi_{T_N-t}^i\left(v_{l-1}^x\right) - \psi_{T_N-t}^i\left(u_l^x\right) \right)^2 X^i|\sigma_i|^2}}.$$

Similar expressions can be derived for other instantaneous correlations, e.g.

$$\mathsf{Corr}_t\big[F_k^x, L_k^x\big] \quad \text{or} \quad \mathsf{Corr}_t\big[L_k^{x_1}, L_k^{x_2}\big].$$

Instantaneous correlations are important for describing the (instantaneous) interdependencies between different LIBOR rates. In the LIBOR market model for instance, the rank of the instantaneous correlation matrix determines the number of factors (e.g. Brownian motions) that is needed to drive the model. Explicit expressions for terminal correlations between LIBOR rates are provided in Appendix A.

6. VALUATION OF SWAPS AND CAPS

6.1. **Interest rate and basis swaps.** We start by presenting a fixed-for-floating payer interest rate swap on a notional amount normalized to 1, where fixed payments are exchanged for floating payments linked to the LIBOR rate. The LIBOR rate is set in advance and the payments are made in arrears. The swap is initiated at time $T_p^x \geq 0$, where $x \in \mathcal{X}$ and $p \in \mathcal{K}^x$. The collection of payment

dates is denoted by $\mathcal{T}_{pq}^x := \{T_{p+1}^x < \cdots < T_q^x\}$, and the fixed rate is denoted by K. Then, the time-t value of the swap, for $t \leq T_p^x$, is given by

$$\mathbb{S}_t(K, \mathcal{T}_{pq}^x) = \sum_{k=p+1}^{q} \delta_x B(t, T_k^x) \, \mathbb{E}_k^x \big[L(T_{k-1}^x, T_k^x) - K | \mathcal{F}_t \big]$$

$$= \delta_x \sum_{k=p+1}^{q} B(t, T_k^x) \left(L_k^x(t) - K \right). \tag{6.1}$$

Thus, the *fair swap rate* $K_t(\mathcal{T}_{pq}^x)$ is given by

$$K_t(\mathcal{T}_{pq}^x) = \frac{\sum_{k=p+1}^{q} B(t, T_k^x) L_k^x(t)}{\sum_{k=p+1}^{q} B(t, T_k^x)}. \tag{6.2}$$

Basis swaps are new products in interest rate markets, whose value reflects the discrepancy between the LIBOR rates of *different* tenors. A basis swap is a swap where two streams of floating payments linked to the LIBOR rates of different tenors are exchanged. For example, in a 3m–6m basis swap, a 3m-LIBOR is paid (received) quarterly and a 6m-LIBOR is received (paid) semiannually. We assume in the sequel that both rates are set in advance and paid in arrears; of course, other conventions regarding the payments on the two legs of a basis swap also exist. A more detailed account on basis swaps can be found in Mercurio (2010b, Section 5.2) or in Filipović and Trolle (2013, Section 2.4 and Appendix F). Note that in the pre-crisis setup the value of such a product would have been zero at any time point, due to the no-arbitrage relation between the LIBOR rates of different tenors; see e.g. Crépey, Grbac, and Nguyen (2012).

Let us consider a basis swap associated with two tenor structures denoted by $\mathcal{T}_{pq}^{x_1} := \{T_{p_1}^{x_1} < \ldots < T_{q_1}^{x_1}\}$ and $\mathcal{T}_{pq}^{x_2} := \{T_{p_2}^{x_2} < \ldots < T_{q_2}^{x_2}\}$, where $T_{p_1}^{x_1} = T_{p_2}^{x_2} \geq 0$, $T_{q_1}^{x_1} = T_{q_2}^{x_2}$ and $\mathcal{T}_{pq}^{x_2} \subset \mathcal{T}_{pq}^{x_1}$. The notional amount is again assumed to be 1 and the swap is initiated at time $T_{p_1}^{x_1}$, while the first payments are due at times $T_{p_1+1}^{x_1}$ and $T_{p_2+1}^{x_2}$ respectively. The basis swap spread is a fixed rate S, which is added to the payments on the shorter tenor length. More precisely, for the x_1-tenor, the floating interest rate $L(T_{i-1}^{x_1}, T_i^{x_1})$ at tenor date $T_i^{x_1}$ is replaced by $L(T_{i-1}^{x_1}, T_i^{x_1}) + S$, for every $i \in \{p_1 + 1, \ldots, q_1\}$. The time-$t$ value of such an agreement is given, for $0 \leq t \leq T_{p_1}^{x_1} = T_{p_2}^{x_2}$, by

$$\mathbb{BS}_t(S, \mathcal{T}_{pq}^{x_1}, \mathcal{T}_{pq}^{x_2}) = \sum_{i=p_2+1}^{q_2} \delta_{x_2} B(t, T_i^{x_2}) \, \mathbb{E}_i^{x_2} \big[L(T_{i-1}^{x_2}, T_i^{x_2}) | \mathcal{F}_t \big]$$

$$- \sum_{i=p_1+1}^{q_1} \delta_{x_1} B(t, T_i^{x_1}) \, \mathbb{E}_i^{x_1} \big[L(T_{i-1}^{x_1}, T_i^{x_1}) + S | \mathcal{F}_t \big]$$

$$= \sum_{i=p_2+1}^{q_2} \delta_{x_2} B(t, T_i^{x_2}) L_i^{x_2}(t) - \sum_{i=p_1+1}^{q_1} \delta_{x_1} B(t, T_i^{x_1}) \big(L_i^{x_1}(t) + S \big). \tag{6.3}$$

We also want to compute the *fair basis swap spread* $S_t(\mathcal{T}_{pq}^{x_1}, \mathcal{T}_{pq}^{x_2})$. This is the spread that makes the value of the basis swap equal zero at time t, i.e. it is obtained by solving $\mathbb{BS}_t(S, \mathcal{T}_{pq}^{x_1}, \mathcal{T}_{pq}^{x_2}) = 0$. We

get that

$$S_t(\mathcal{T}_{pq}^{x_1}, \mathcal{T}_{pq}^{x_2}) = \frac{\sum_{i=p_2+1}^{q_2} \delta_{x_2} B(t, T_i^{x_2}) L_i^{x_2}(t) - \sum_{i=p_1+1}^{q_1} \delta_{x_1} B(t, T_i^{x_1}) L_i^{x_1}(t)}{\sum_{i=p_1+1}^{q_1} \delta_{x_1} B(t, T_i^{x_1})}. \tag{6.4}$$

The formulas for the fair swap rate and basis spread can be used to bootstrap the initial values of LIBOR rates from market data, see Mercurio (2010b, §2.4).

6.2. **Caps.** The valuation of caplets, and thus caps, in the multiple curve affine LIBOR model is an easy task, which has complexity equal to the complexity of the valuation of caplets in the single-curve affine LIBOR model. There are two reasons for this: on the one hand, the LIBOR rate is modeled directly—compare e.g. with Mercurio (2010a) where the LIBOR rate is modeled implicitly as the sum of the OIS rate and the spread. On the other hand, the driving process remains affine under any forward measure, cf. Proposition 4.5, which allows the application of Fourier methods for option pricing.

Proposition 6.1. *Consider an x-tenor caplet with strike K that pays out $\delta_x(L(T_{k-1}^x, T_k^x) - K)^+$ at T_k^x. The time-0 price is provided by*

$$\mathbb{C}_0(K, T_k^x) = \frac{B(0, T_k^x)}{2\pi} \int_{\mathbb{R}} K_x^{1-R+iw} \frac{\Theta_{\mathcal{W}_{k-1}^x}(R - iw)}{(R - iw)(R - 1 - iw)} \mathrm{d}w, \tag{6.5}$$

for $R \in (1, \infty) \cap \widetilde{\mathcal{I}}^{k,x}$, where $K_x = 1 + \delta_x K$, $\Theta_{\mathcal{W}_{k-1}^x}$ is given by (6.7), while the set $\widetilde{\mathcal{I}}^{k,x}$ is defined as

$$\widetilde{\mathcal{I}}^{k,x} = \left\{ z \in \mathbb{R} : (1 - z)\psi_{T_N - T_{k-1}^x}(u_k^x) + z\psi_{T_N - T_{k-1}^x}(v_{k-1}^x) \in \mathcal{I}_T \right\}.$$

Proof. Using (3.3) and (4.3) the time-0 price of the caplet equals

$$\begin{aligned}
\mathbb{C}_0(K, T_k^x) &= \delta_x \, B(0, T_k^x) \, \mathbb{E}_k^x \big[(L(T_{k-1}^x, T_k^x) - K)^+ \big] \\
&= \delta_x \, B(0, T_k^x) \, \mathbb{E}_k^x \big[(L_k^x(T_{k-1}^x) - K)^+ \big] \\
&= B(0, T_k^x) \, \mathbb{E}_k^x \Big[\big(M_{T_{k-1}^x}^{v_{k-1}^x} / M_{T_{k-1}^x}^{u_k^x} - K_x \big)^+ \Big] \\
&= B(0, T_k^x) \, \mathbb{E}_k^x \Big[\big(\mathrm{e}^{\mathcal{W}_{k-1}^x} - K_x \big)^+ \Big],
\end{aligned}$$

where

$$\begin{aligned}
\mathcal{W}_{k-1}^x &= \log \Big(M_{T_{k-1}^x}^{v_{k-1}^x} / M_{T_{k-1}^x}^{u_k^x} \Big) \\
&= \phi_{T_N - T_{k-1}^x}(v_{k-1}^x) - \phi_{T_N - T_{k-1}^x}(u_k^x) \\
&\quad + \big\langle \psi_{T_N - T_{k-1}^x}(v_{k-1}^x) - \psi_{T_N - T_{k-1}^x}(u_k^x), X_{T_{k-1}^x} \big\rangle \\
&=: A + \langle B, X_{T_{k-1}^x} \rangle. \tag{6.6}
\end{aligned}$$

Now, using Eberlein, Glau, and Papapantoleon (2010, Thm 2.2, Ex. 5.1), we arrive directly at (6.5), where $\Theta_{\mathcal{W}_{k-1}^x}$ denotes the $\mathbb{P}_k^x$-moment generating function of the random variable $\mathcal{W}_{k-1}^x$, i.e. for

$z \in \widetilde{\mathcal{I}}^{k,x}$,

$$\begin{aligned}
\Theta_{\mathcal{W}_{k-1}^x}(z) &= \mathbb{E}_k^x \big[e^{z\mathcal{W}_{k-1}^x} \big] \\
&= \mathbb{E}_k^x \big[\exp \big(z(A + \langle B, X_{T_{k-1}^x} \rangle) \big) \big] \\
&= \exp \Big(zA + \phi_{T_{k-1}^x}^{k,x}(zB) + \big\langle \psi_{T_{k-1}^x}^{k,x}(zB), X_0 \big\rangle \Big).
\end{aligned} \qquad (6.7)$$

The last equality follows from Proposition 4.5, noting that $z \in \widetilde{\mathcal{I}}^{k,x}$ implies $zB \in \mathcal{I}^{k,x}$. $\qquad \square$

7. Valuation of swaptions and basis swaptions

This section is devoted to the pricing of options on interest rate and basis swaps, in other words, to the pricing of swaptions and basis swaptions. In the first part, we provide general expressions for the valuation of swaptions and basis swaptions making use of the structure of multiple curve affine LIBOR models. In the following two parts, we derive efficient and accurate approximations for the pricing of swaptions and basis swaptions by further utilizing the model properties—in particular, the preservation of the affine structure under forward measures—and applying the linear boundary approximation developed by Singleton and Umantsev (2002).

Let us consider first a payer swaption with strike rate K and exercise date T_p^x on a fixed-for-floating interest rate swap starting at T_p^x and maturing at T_q^x; this was defined in Section 6.1. A swaption can be regarded as a sequence of fixed payments $\delta_x (K_{T_p^x}(\mathcal{T}_{pq}) - K)^+$ that are received at the payment dates $T_{p+1}^x, \dots, T_q^x$; see Musiela and Rutkowski (2005, Section 13.1.2, p. 524). Here $K_{T_p^x}(\mathcal{T}_{pq}^x)$ is the swap rate of the underlying swap at time T_p^x, cf. (6.2). Note that the classical transformation of a payer (resp. receiver) swaption into a put (resp. call) option on a coupon bond is not valid in the multiple curve setup, since LIBOR rates cannot be expressed in terms of zero coupon bonds; see Remark 3.5.

The value of the swaption at time $t \le T_p^x$ is given by

$$\mathbb{S}_t^+(K, \mathcal{T}_{pq}^x) = B(t, T_p^x) \sum_{i=p+1}^q \delta_x \, \mathbb{E}_p^x \Big[B(T_p^x, T_i^x) \big(K_{T_p^x}(\mathcal{T}_{pq}^x) - K \big)^+ \big| \mathcal{F}_t \Big].$$

Therefore, at time $t = 0$ we have

$$\begin{aligned}
\mathbb{S}_0^+(K, \mathcal{T}_{pq}^x) &= B(0, T_p^x) \, \mathbb{E}_p^x \left[\sum_{i=p+1}^q \delta_x B(T_p^x, T_i^x) \big(K_{T_p^x}(\mathcal{T}_{pq}^x) - K \big)^+ \right] \\
&= B(0, T_p^x) \, \mathbb{E}_p^x \left[\left(\sum_{i=p+1}^q \delta_x L_i^x(T_p^x) B(T_p^x, T_i^x) - \sum_{i=p+1}^q \delta_x K B(T_p^x, T_i^x) \right)^+ \right]
\end{aligned}$$

since the swap rate $K_{T_p^x}(\mathcal{T}_{pq}^x)$ is given by (6.2) for $t = T_p^x$. Using (3.2), (4.3) and a telescoping product, we get that

$$B(T_p^x, T_i^x) = M_{T_p^x}^{u_i^x} / M_{T_p^x}^{u_p^x}.$$

Moreover, using (4.3) and (4.8), we obtain that

$$\mathbb{S}_0^+(K, \mathcal{T}_{pq}^x) = B(0, T_p^x) \, \mathbb{E}_p^x \left[\left(\sum_{i=p+1}^{q} \frac{M_{T_p^x}^{v_{i-1}^x}}{M_{T_p^x}^{u_p^x}} - \sum_{i=p+1}^{q} K_x \frac{M_{T_p^x}^{u_i^x}}{M_{T_p^x}^{u_p^x}} \right)^+ \right]$$

$$= B(0, T_N) \, \mathbb{E}_N \left[\left(\sum_{i=p+1}^{q} M_{T_p^x}^{v_{i-1}^x} - \sum_{i=p+1}^{q} K_x M_{T_p^x}^{u_i^x} \right)^+ \right], \qquad (7.1)$$

where $K_x := 1 + \delta_x K$.

Next, we move on to the pricing of basis swaptions. A basis swaption is an option to enter a basis swap with spread S. We consider a basis swap as defined in section 6.1, which starts at $T_{p_1}^{x_1} = T_{p_2}^{x_2}$ and ends at $T_{q_1}^{x_1} = T_{q_2}^{x_2}$, while we assume that the exercise date is $T_{p_1}^{x_1}$. The payoff of a basis swap at time $T_{p_1}^{x_1}$ is given by (6.3) for $t = T_{p_1}^{x_1}$. Therefore, the price of a basis swaption at time $t = 0$ is provided by

$$\mathbb{BS}_0^+(S, \mathcal{T}_{pq}^{x_1}, \mathcal{T}_{pq}^{x_2}) = B(0, T_{p_1}^{x_1}) \, \mathbb{E}_{p_1}^{x_1} \left[\left(\sum_{i=p_2+1}^{q_2} \delta_{x_2} L_i^{x_2}(T_{p_2}^{x_2}) B(T_{p_2}^{x_2}, T_i^{x_2}) \right. \right.$$

$$\left. \left. - \sum_{i=p_1+1}^{q_1} \delta_{x_1} \left(L_i^{x_1}(T_{p_1}^{x_1}) + S \right) B(T_{p_1}^{x_1}, T_i^{x_1}) \right)^+ \right].$$

Along the lines of the derivation for swaptions and using $M_{T_{p_2}^{x_2}}^{u_{p_2}^{x_2}} = M_{T_{p_1}^{x_1}}^{u_{p_1}^{x_1}}$ (cf. (4.5)), we arrive at

$$\mathbb{BS}_0^+(S, \mathcal{T}_{pq}^{x_1}, \mathcal{T}_{pq}^{x_2}) =$$

$$= B(0, T_{p_1}^{x_1}) \, \mathbb{E}_{p_1}^{x_1} \left[\left(\sum_{i=p_2+1}^{q_2} \left(M_{T_{p_2}^{x_2}}^{v_{i-1}^{x_2}} / M_{T_{p_2}^{x_2}}^{u_{p_2}^{x_2}} - M_{T_{p_2}^{x_2}}^{u_i^{x_2}} / M_{T_{p_2}^{x_2}}^{u_{p_2}^{x_2}} \right) \right. \right.$$

$$\left. \left. - \sum_{i=p_1+1}^{q_1} \left(M_{T_{p_1}^{x_1}}^{v_{i-1}^{x_1}} / M_{T_{p_1}^{x_1}}^{u_{p_1}^{x_1}} - S_{x_1} M_{T_{p_1}^{x_1}}^{u_i^{x_1}} / M_{T_{p_1}^{x_1}}^{u_{p_1}^{x_1}} \right) \right)^+ \right]$$

$$= B(0, T_N) \, \mathbb{E}_N \left[\left(\sum_{i=p_2+1}^{q_2} \left(M_{T_{p_2}^{x_2}}^{v_{i-1}^{x_2}} - M_{T_{p_2}^{x_2}}^{u_i^{x_2}} \right) - \sum_{i=p_1+1}^{q_1} \left(M_{T_{p_1}^{x_1}}^{v_{i-1}^{x_1}} - S_{x_1} M_{T_{p_1}^{x_1}}^{u_i^{x_1}} \right) \right)^+ \right], \quad (7.2)$$

where $S_{x_1} := 1 - \delta_{x_1} S$.

7.1. **Approximation formula for swaptions.** We will now derive an efficient approximation formula for the pricing of swaptions. The main ingredients in this formula are the affine property of the driving process under forward measures and the linearization of the exercise boundary. Numerical results for this approximation will be reported in Section 8.3.

We start by presenting some technical tools and assumptions that will be used in the sequel. We define the probability measures $\overline{\mathbb{P}}_k^x$, for every $k \in \mathcal{K}^x$, via the Radon–Nikodym density

$$\frac{\mathrm{d}\overline{\mathbb{P}}_k^x}{\mathrm{d}\mathbb{P}_N}\bigg|_{\mathcal{F}_t} = \frac{M_t^{v_k^x}}{M_0^{v_k^x}}. \tag{7.3}$$

The process X is obviously a time-inhomogeneous affine process under every $\overline{\mathbb{P}}_k^x$. More precisely, we have the following result which follows directly from Proposition 4.5.

Corollary 7.1. *The process X is a time-inhomogeneous affine process under the measure $\overline{\mathbb{P}}_k^x$, for every $x \in \mathcal{X}, k \in \mathcal{K}^x$, with*

$$\overline{\mathbb{E}}_k^x\big[\mathrm{e}^{\langle w, X_t \rangle}\big] = \exp\left(\overline{\phi}_t^{k,x}(w) + \langle \overline{\psi}_t^{k,x}(w), X_0 \rangle\right), \tag{7.4}$$

where

$$\overline{\phi}_t^{k,x}(w) := \phi_t\big(\psi_{T_N-t}(v_k^x) + w\big) - \phi_t\big(\psi_{T_N-t}(v_k^x)\big), \tag{7.5a}$$

$$\overline{\psi}_t^{k,x}(w) := \psi_t\big(\psi_{T_N-t}(v_k^x) + w\big) - \psi_t\big(\psi_{T_N-t}(v_k^x)\big), \tag{7.5b}$$

for every $w \in \overline{\mathcal{I}}^{k,x}$ with

$$\overline{\mathcal{I}}^{k,x} := \big\{w \in \mathbb{R}^d : \psi_{T_N-t}(v_k^x) + w \in \mathcal{I}_T\big\}. \tag{7.6}$$

Next, we define the function $f : \mathbb{R}_{\geqslant 0}^d \to \mathbb{R}$ by

$$\begin{aligned}
f(y) = &\sum_{i=p+1}^{q} \exp\big(\phi_{T_N-T_p^x}(v_{i-1}^x) + \langle \psi_{T_N-T_p^x}(v_{i-1}^x), y \rangle\big) \\
&- \sum_{i=p+1}^{q} K_x \exp\big(\phi_{T_N-T_p^x}(u_i^x) + \langle \psi_{T_N-T_p^x}(u_i^x), y \rangle\big).
\end{aligned} \tag{7.7}$$

This function determines the exercise boundary for the price of the swaption, as will become clear below. Since we cannot compute the characteristic function of $f(X_{T_p^x})$ explicitly, we will follow Singleton and Umantsev (2002) and approximate f by a linear function.

Approximation ($\mathbb{S}$). We approximate

$$f(X_{T_p^x}) \approx \widetilde{f}(X_{T_p^x}) := \mathcal{A} + \langle \mathcal{B}, X_{T_p^x} \rangle, \tag{7.8}$$

where the constants $\mathcal{A}, \mathcal{B}$ are determined according to the linear regression procedure described in Singleton and Umantsev (2002, pp. 432-434). The line $\langle \mathcal{B}, X_{T_p^x} \rangle = -\mathcal{A}$ approximates the exercise boundary, hence $\mathcal{A}, \mathcal{B}$ are strike-dependent.

The following assumption will be used for the pricing of swaptions and basis swaptions.

Assumption ($\mathbb{CD}$). The cumulative distribution function of X_t is continuous for all $t \in [0, T_N]$.

Let $\Im(z)$ denote the imaginary part of a complex number $z \in \mathbb{C}$. Now, we state the main result of this subsection.

Proposition 7.2. *Assume that* $\mathcal{A}, \mathcal{B}$ *are determined by Approximation* $(\mathbb{S})$ *and that Assumption* $(\mathbb{CD})$ *is satisfied. The price of a payer swaption with strike* K, *option maturity* T_p^x, *and swap maturity* T_q^x, *is approximated by*

$$
\widetilde{\mathbb{S}}_0^+(K, \mathcal{T}_{pq}^x) = B(0, T_N) \sum_{i=p+1}^q M_0^{v_{i-1}^x} \left[\frac{1}{2} + \frac{1}{\pi} \int_0^\infty \frac{\Im\big(\widetilde{\xi}_{i-1}^x(z)\big)}{z} \mathrm{d}z \right]
$$

$$
- K_x \sum_{i=p+1}^q B(0, T_i^x) \left[\frac{1}{2} + \frac{1}{\pi} \int_0^\infty \frac{\Im\big(\widetilde{\zeta}_i^x(z)\big)}{z} \mathrm{d}z \right], \qquad (7.9)
$$

where $\widetilde{\zeta}_i^x$ *and* $\widetilde{\xi}_i^x$ *are defined by* (7.13) *and* (7.14) *respectively.*

Proof. Recall that the price of a swaption is provided by (7.1). Using (4.1) and (4.2) and the definition of f in (7.7), we can express the swaption price as follows

$$
\mathbb{S}_0^+(K, \mathcal{T}_{pq}^x) = B(0, T_N) \, \mathbb{E}_N \left[\left(\sum_{i=p+1}^q M_{T_p^x}^{v_{i-1}^x} - \sum_{i=p+1}^q K_x M_{T_p^x}^{u_i^x} \right) \mathbf{1}_{\{f(X_{T_p^x}) \geq 0\}} \right]
$$

$$
= B(0, T_N) \left(\sum_{i=p+1}^q \mathbb{E}_N \left[M_{T_p^x}^{v_{i-1}^x} \mathbf{1}_{\{f(X_{T_p^x}) \geq 0\}} \right] \right.
$$

$$
\left. - K_x \sum_{i=p+1}^q \mathbb{E}_N \left[M_{T_p^x}^{u_i^x} \mathbf{1}_{\{f(X_{T_p^x}) \geq 0\}} \right] \right).
$$

Moreover, using the relation between the terminal measure $\mathbb{P}_N$ and the measures $\mathbb{P}_k^x$ and $\overline{\mathbb{P}}_k^x$ in (4.8) and (7.3), we get that

$$
\mathbb{S}_0^+(K, \mathcal{T}_{pq}^x) = B(0, T_N) \sum_{i=p+1}^q M_0^{v_{i-1}^x} \overline{\mathbb{E}}_{i-1}^x \left[\mathbf{1}_{\{f(X_{T_p^x}) \geq 0\}} \right]
$$

$$
- K_x \sum_{i=p+1}^q B(0, T_i^x) \, \mathbb{E}_i^x \left[\mathbf{1}_{\{f(X_{T_p^x}) \geq 0\}} \right]. \quad (7.10)
$$

In addition, from the inversion formula of Gil-Pelaez (1951) and using Assumption $(\mathbb{CD})$, we get that

$$
\mathbb{E}_i^x \big[\mathbf{1}_{\{f(X_{T_p^x}) \geq 0\}} \big] = \frac{1}{2} + \frac{1}{\pi} \int_0^\infty \frac{\Im(\zeta_i^x(z))}{z} \mathrm{d}z, \qquad (7.11)
$$

$$
\overline{\mathbb{E}}_i^x \big[\mathbf{1}_{\{f(X_{T_p^x}) \geq 0\}} \big] = \frac{1}{2} + \frac{1}{\pi} \int_0^\infty \frac{\Im(\xi_i^x(z))}{z} \mathrm{d}z, \qquad (7.12)
$$

for each $i \in \mathcal{K}^x$, where we define

$$
\zeta_i^x(z) := \mathbb{E}_i^x \big[\exp\big(\mathrm{i}z f(X_{T_p^x})\big) \big] \quad \text{and} \quad \xi_i^x(z) := \overline{\mathbb{E}}_i^x \big[\exp\big(\mathrm{i}z f(X_{T_p^x})\big) \big].
$$

However, the above characteristic functions cannot be computed explicitly, in general, thus we will linearize the exercise boundary as described by Approximation $(\mathbb{S})$. Therefore, we approximate the unknown characteristic functions with ones that admit an explicit expression due to the affine property of X under the forward measures. Indeed, from (7.8), Proposition 4.5 and Corollary 7.1 we get that

$$
\begin{aligned}
\zeta_k^x(z) \approx \widetilde{\zeta}_k^x(z) &:= \mathbb{E}_k^x\big[\exp\big(\mathrm{i}z\widetilde{f}(X_{T_p^x})\big)\big] \\
&= \exp\Big(\mathrm{i}z\mathcal{A} + \phi_{T_p^x}^{k,x}(\mathrm{i}z\mathcal{B}) + \big\langle \psi_{T_p^x}^{k,x}(\mathrm{i}z\mathcal{B}), X_0 \big\rangle\Big),
\end{aligned}
\tag{7.13}
$$

$$
\begin{aligned}
\xi_k^x(z) \approx \widetilde{\xi}_k^x(z) &:= \overline{\mathbb{E}}_k^x\big[\exp\big(\mathrm{i}z\widetilde{f}(X_{T_p^x})\big)\big] \\
&= \exp\Big(\mathrm{i}z\mathcal{A} + \overline{\phi}_{T_p^x}^{k,x}(\mathrm{i}z\mathcal{B}) + \big\langle \overline{\psi}_{T_p^x}^{k,x}(\mathrm{i}z\mathcal{B}), X_0 \big\rangle\Big).
\end{aligned}
\tag{7.14}
$$

After inserting (7.11) and (7.12) into (7.10) and using (7.13) and (7.14) we arrive at the approximation formula for swaptions (7.9). $\qquad\square$

Remark 7.3. The pricing of swaptions is inherently a high-dimensional problem. The expectation in (7.1) corresponds to a d-dimensional integral, where d is the dimension of the driving process. However, the exercise boundary is non-linear and hard to compute, in general. See, e.g. Brace et al. (1997), Eberlein and Kluge (2006) or Keller-Ressel et al. (2013, §7.2, §8.3) for some exceptional cases that admit explicit solutions. Alternatively, one could express a swaption as a zero strike basket option written on $2(q-p)$ underlying assets and use Fourier methods for pricing; see Hubalek and Kallsen (2005) or Hurd and Zhou (2010). This leads to a $2(q-p)$-dimensional numerical integration. Instead, the approximation derived in this section requires only the evaluation of $2(q-p)$ *univariate* integrals together with the computation of the constants $\mathcal{A}, \mathcal{B}$. This reduces the complexity of the problem considerably.

7.2. **Approximation formula for basis swaptions.** In this subsection, we derive an analogous approximate pricing formula for basis swaptions. Numerical results for this approximation will be reported in section 8.4.

Similar to the case of swaptions, we define the function

$$
\begin{aligned}
g(y) =\ &\sum_{i=p_2+1}^{q_2} \exp\big(\phi_{T_N-T_{p_2}^{x_2}}(v_{i-1}^{x_2}) + \langle \psi_{T_N-T_{p_2}^{x_2}}(v_{i-1}^{x_2}), y \rangle\big) \\
&- \sum_{i=p_2+1}^{q_2} \exp\big(\phi_{T_N-T_{p_2}^{x_2}}(u_i^{x_2}) + \langle \psi_{T_N-T_{p_2}^{x_2}}(u_i^{x_2}), y \rangle\big) \\
&- \sum_{i=p_1+1}^{q_1} \exp\big(\phi_{T_N-T_{p_1}^{x_1}}(v_{i-1}^{x_1}) + \langle \psi_{T_N-T_{p_1}^{x_1}}(v_{i-1}^{x_1}), y \rangle\big) \\
&+ \sum_{i=p_1+1}^{q_1} S_{x_1} \exp\big(\phi_{T_N-T_{p_1}^{x_1}}(u_i^{x_1}) + \langle \psi_{T_N-T_{p_1}^{x_1}}(u_i^{x_1}), y \rangle\big),
\end{aligned}
\tag{7.15}
$$

which determines the exercise boundary for the price of the basis swaption. This will be approximated by a linear function following again Singleton and Umantsev (2002).

Approximation ($\mathbb{BS}$). We approximate

$$g(X_{T_{p_1}^{x_1}}) \approx \widetilde{g}(X_{T_{p_1}^{x_1}}) := \mathcal{C} + \langle \mathcal{D}, X_{T_{p_1}^{x_1}} \rangle, \tag{7.16}$$

where $\mathcal{C}$ and $\mathcal{D}$ are determined via a linear regression.

Proposition 7.4. *Assume that $\mathcal{C}, \mathcal{D}$ are determined by Approximation $(\mathbb{BS})$ and that Assumption $(\mathbb{CD})$ is satisfied. The price of a basis swaption with spread S, option maturity $T_{p_1}^{x_1} = T_{p_2}^{x_2}$, and swap maturity $T_{q_1}^{x_1} = T_{q_2}^{x_2}$, is approximated by*

$$\begin{aligned}
\widetilde{\mathbb{BS}}_0^+(S, \mathcal{T}_{pq}^{x_1}, \mathcal{T}_{pq}^{x_2}) = {} & B(0, T_N) \sum_{i=p_2+1}^{q_2} M_0^{v_{i-1}^{x_2}} \left[\frac{1}{2} + \frac{1}{\pi} \int_0^\infty \frac{\Im(\widetilde{\xi}_{i-1}^{x_2}(z))}{z} \mathrm{d}z \right] \\
& - \sum_{i=p_2+1}^{q_2} B(0, T_i^{x_2}) \left[\frac{1}{2} + \frac{1}{\pi} \int_0^\infty \frac{\Im(\widetilde{\zeta}_i^{x_2}(z))}{z} \mathrm{d}z \right] \\
& - B(0, T_N) \sum_{i=p_1+1}^{q_1} M_0^{v_{i-1}^{x_1}} \left[\frac{1}{2} + \frac{1}{\pi} \int_0^\infty \frac{\Im(\widetilde{\xi}_{i-1}^{x_1}(z))}{z} \mathrm{d}z \right] \\
& + S_{x_1} \sum_{i=p_1+1}^{q_1} B(0, T_i^{x_1}) \left[\frac{1}{2} + \frac{1}{\pi} \int_0^\infty \frac{\Im(\widetilde{\zeta}_i^{x_1}(z))}{z} \mathrm{d}z \right],
\end{aligned} \tag{7.17}$$

where $\widetilde{\zeta}_i^{x_l}$ and $\widetilde{\xi}_i^{x_l}$ are defined by (7.18) and (7.19) for $l = 1, 2$.

Proof. Using the definition of g in (7.15) we can rewrite the price of a basis swaption (7.2) as follows:

$$\begin{aligned}
\mathbb{BS}_0^+(S, \mathcal{T}_{pq}^{x_1}, \mathcal{T}_{pq}^{x_2}) = {} & \\
= {} & B(0, T_N) \Bigg\{ \sum_{i=p_2+1}^{q_2} \left(\mathbb{E}_N \left[M_{T_{p_2}^{x_2}}^{v_{i-1}^{x_2}} \mathbf{1}_{\{g(X_{T_{p_2}^{x_2}}) \geq 0\}} \right] - \mathbb{E}_N \left[M_{T_{p_2}^{x_2}}^{u_i^{x_2}} \mathbf{1}_{\{g(X_{T_{p_2}^{x_2}}) \geq 0\}} \right] \right) \\
& - \sum_{i=p_1+1}^{q_1} \left(\mathbb{E}_N \left[M_{T_{p_1}^{x_1}}^{v_{i-1}^{x_1}} \mathbf{1}_{\{g(X_{T_{p_1}^{x_1}}) \geq 0\}} \right] - S_{x_1} \mathbb{E}_N \left[M_{T_{p_1}^{x_1}}^{u_i^{x_1}} \mathbf{1}_{\{g(X_{T_{p_1}^{x_1}}) \geq 0\}} \right] \right) \Bigg\}.
\end{aligned}$$

Then we follow the same steps as in the previous section: First, we use the relation between the terminal measure $\mathbb{P}_N$ and the measures $\mathbb{P}_k^x, \overline{\mathbb{P}}_k^x$ to arrive at an expression similar to (7.10). Second, we approximate g by $\widetilde{g}$ in (7.16). Third, we define the approximate characteristic functions, which can be computed explicitly:

$$\begin{aligned}
\widetilde{\zeta}_i^{x_l}(z) & := \mathbb{E}_i^{x_l} \left[\exp\left(\mathrm{i}z\widetilde{g}(X_{T_{p_l}^{x_l}}) \right) \right] \\
& = \exp\left(\mathrm{i}z\mathcal{C} + \phi_{T_{p_l}}^{i,x_l}(\mathrm{i}z\mathcal{D}) + \langle \psi_{T_{p_l}}^{i,x_l}(\mathrm{i}z\mathcal{D}), X_0 \rangle \right), \tag{7.18} \\
\widetilde{\xi}_i^{x_l}(z) & := \overline{\mathbb{E}}_i^{x_k} \left[\exp\left(\mathrm{i}z\widetilde{g}(X_{T_{p_l}^{x_l}}) \right) \right] \\
& = \exp\left(\mathrm{i}z\mathcal{C} + \overline{\phi}_{T_{p_l}}^{i,x_l}(\mathrm{i}z\mathcal{D}) + \langle \overline{\psi}_{T_{p_l}}^{i,x_l}(\mathrm{i}z\mathcal{D}), X_0 \rangle \right), \tag{7.19}
\end{aligned}$$

for $l = 1, 2$. Finally, putting all the pieces together we arrive at the approximation formula (7.17) for the price of a basis swaption. $\qquad \square$

8. Numerical examples and calibration

The aim of this section is twofold: on the one hand, we demonstrate how the multiple curve affine LIBOR model can be calibrated to market data and, on the other hand, we test the accuracy of the swaption and basis swaption approximation formulas. We start by discussing how to build a model which can simultaneously fit caplet volatilities when the options have different underlying tenors. Next, we test numerically the swaption and basis swaption approximation formulas (7.9) and (7.17) using the calibrated models and parameters. In the last subsection, we build a simple model and compute exact and approximate swaption and basis swaption prices in a setup which can be easily reproduced by interested readers.

8.1. **A specification with dependent rates.** There are numerous ways of constructing models and the trade-off is usually between parsimony and fitting ability. We opt here for a heavily parametrized approach that focuses on the fitting ability, as we believe it best demonstrates the utility of our model. In addition, it is usually easier to move from a complex specification towards a simpler one, than the converse.

More precisely, we provide below a model specification where LIBOR rates are driven by common and idiosyncratic factors, which is suitable for sequential calibration to market data. The starting point is to revisit the expression for the LIBOR rates in (4.3):

$$
\begin{aligned}
1 + \delta_x L_k^x(t) &= M_t^{v_{k-1}^x}/M_t^{u_k^x} \\
&= \exp\left(\phi_{T_N-t}(v_{k-1}^x) - \phi_{T_N-t}(u_k^x) + \left\langle \psi_{T_N-t}(v_{k-1}^x) - \psi_{T_N-t}(u_k^x), X_t \right\rangle\right).
\end{aligned}
\tag{8.1}
$$

According to Proposition 4.2, when the dimension of the driving process is greater than one, then the vectors v_{k-1}^x and u_k^x are not fully determined by the initial term structure. Therefore, we can navigate through different model specifications by altering the structure of the sequences (u_k^x) and (v_k^x).

Remark 8.1. The following observation allows to create an (exponential) linear factor structure for the LIBOR rates with as many common and idiosyncratic factors as desired. Consider an $\mathbb{R}_{\geqslant 0}^d$-valued affine process

$$
X = (X^1, \dots, X^d),
\tag{8.2}
$$

and denote the vectors $v_{k-1}^x, u_k^x \in \mathbb{R}_{\geqslant 0}^d$ by

$$
v_{k-1}^x = (v_{1,k-1}^x, \dots, v_{d,k-1}^x) \quad \text{and} \quad u_k^x = (u_{1,k}^x, \dots, u_{d,k}^x).
\tag{8.3}
$$

Select a subset $\mathcal{J}_k \subset \{1, \dots, d\}$, set $v_{i,k-1}^x = u_{i,k}^x$ for all $i \in \mathcal{J}_k$, and assume that $\{X^i\}_{i \in \mathcal{J}_k}$ are independent of $\{X^j\}_{j \in \{1,\dots,d\} \setminus \mathcal{J}_k}$. Then, it follows from (8.1) and Keller-Ressel (2008, Prop. 4.7) that L_k^x will also be independent of $\{X^i\}_{i \in \mathcal{J}_k}$ and will depend only on $\{X^j\}_{j \in \{1,\dots,d\} \setminus \mathcal{J}_k}$.

Let $x_1, x_2 \in \mathcal{X}$ and consider the tenor structures $\mathcal{T}^{x_1}, \mathcal{T}^{x_2}$ where $\mathcal{T}^{x_2} \subset \mathcal{T}^{x_1}$. The dataset under consideration contains caplets maturing on M different dates for each tenor, where M is less than

the number of tenor points in $\mathcal{T}^{x_1}$ and $\mathcal{T}^{x_2}$. In other words, only M maturities are relevant for the calibration. The dynamics of OIS and LIBOR rates are driven by tuples of affine processes

$$\mathrm{d}X_t^i = -\lambda_i(X_t^i - \theta_i)\mathrm{d}t + 2\eta_i\sqrt{X_t^i}\mathrm{d}W_t^i + \mathrm{d}Z_t^i, \tag{8.4}$$

$$\mathrm{d}X_t^c = -\lambda_c(X_t^c - \theta_c)\mathrm{d}t + 2\eta_c\sqrt{X_t^c}\mathrm{d}W_t^c, \tag{8.5}$$

for $i = 1, \ldots, M$, where X^c denotes the common and X^i the idiosyncratic factor (for the i-th maturity). Here $X_0^i \in \mathbb{R}_{\geqslant 0}, \lambda_i, \theta_i, \eta_i \in \mathbb{R}_{\geqslant 0}$ for $i = c, 1, \ldots, M$, and $W^c, W^1, \ldots, W^M$, are independent Brownian motions. Moreover, Z^i are independent compound Poisson processes with constant intensity ν_i and exponentially distributed jumps with mean values μ_i, for $i = 1, \ldots, M$. Therefore, the full process has dimension $M + 1$:

$$X = \left(X^c, X^1, \ldots, X^M\right). \tag{8.6}$$

The affine processes $X^c, X^1, \ldots, X^M$ are mutually independent hence, using Proposition 4.7 in Keller-Ressel (2008), the functions $\phi_{(X^c, X^i)}$, respectively $\psi_{(X^c, X^i)}$, are known in terms of the functions ϕ_{X^c} and ϕ_{X^i}, respectively ψ_{X^c} and ψ_{X^i}, for all $i \in \{1, \ldots, M\}$. The latter are provided, for example, by Grbac and Papapantoleon (2013, Ex. 2.3).

In order to create a 'diagonal plus common' factor structure, where each rate for each tenor is driven by the common factor X^c and an idiosyncratic factor X^i, we will use Remark 8.1. The proposed structures for the u^{x_1}'s and v^{x_1}'s are described in Figures 8.4 and 8.5, where elements of u^{x_1} below a certain 'diagonal' are copied into v^{x_1}. In particular, we start from the longest maturity and add one independent factor at each caplet maturity date, i.e. at each $\ell_1(k)$, $k = 1, \ldots, M$. Here $\ell_1(k) := k/\delta_{x_1}$ for $k = 1, \ldots, M$, i.e. this function maps caplet maturities into tenor points. Simultaneously, the values of u^{x_1} for the subsequent factor are 'frozen' to the latest-set value. The construction of v^{x_1} is analogous, where now the 'frozen' values are copied from u^{x_1}; see again Remark 8.1. Moreover, all elements in these matrices are non-negative and $u_{N^{x_1}}^{x_1} = v_{N^{x_1}}^{x_1} = 0 \in \mathbb{R}^{M+1}$.

The boxed elements are the only ones that matter in terms of pricing caplets, when these are not available at every tenor date of $\mathcal{T}^{x_1}$. The role of the common factor is determined by the difference between $\tilde{v}_{k-1}^{x_1}$ and $\tilde{u}_k^{x_1}$. If we set $\tilde{v}_{k-1}^{x_1} = \tilde{u}_k^{x_1}$, it follows from (8.1) that $L_k^{x_1}$ will be independent of the common factor X^c and thus determined solely by the corresponding idiosyncratic factor X^i, with $k = \ell_1(i)$. If the values of $\tilde{v}_k^{x_1}$ and $\tilde{u}_k^{x_1}$ are fixed a priori, the remaining values $(\bar{u}_k^{x_1})_{k=1,\ldots,N^{x_1}}$ and $(\bar{v}_k^{x_1})_{k=1,\ldots,N^{x_1}}$ are determined uniquely by the initial term structure of OIS and LIBOR rates; see again Propositions 4.1 and 4.2. This model structure is consistent with $v_{k-1}^{x_1} \geq u_{k-1}^{x_1} \geq u_k^{x_1}$ if and only if the sequences $\tilde{u}^{x_1}$ and $\bar{u}^{x_1}$ are decreasing, $\tilde{v}_k^{x_1} \geq \tilde{u}_k^{x_1}$ and $\bar{v}_k^{x_1} \geq \bar{u}_k^{x_1}$ for every $k = 1, \ldots, N^{x_1}$. Moreover, this structure will be consistent with the 'normal' market situation described in Remark 4.4 if, in addition, $\tilde{v}_k^{x_1} \in [\tilde{u}_k^{x_1}, \tilde{u}_{k-1}^{x_1}]$ and $\bar{v}_k^{x_1} \in [\bar{u}_k^{x_1}, \bar{u}_{k-1}^{x_1}]$ for every $k = 1, \ldots, N^{x_1}$.

The corresponding matrices for the x_2 tenor are constructed in a similar manner. More precisely, u^{x_2} is constructed by simply copying the relevant rows from u^{x_1}. Simultaneously, for v^{x_2} the elements $(\bar{v}_k^{x_2})_{k=0,\ldots,N^{x_2}}$ are introduced in order to fit the x_2 initial LIBOR term structure, as well as the elements $(\tilde{v}_k^{x_2})_{k=0,\ldots,N^{x_2}}$ which determine the role of the common factor. We present only four rows from these matrices in Figures 8.6 and 8.7, for the sake of brevity.

8.2. Calibration to caplet data.
The data we use for calibration are from the EUR market on 27 May 2013 collected from Bloomberg. Bloomberg provides synthetic zero coupon bond prices for EURIBOR

$$
\begin{aligned}
\boxed{u^{x_1}_{\ell_1(M)}} &= \begin{pmatrix} \tilde{u}^{x_1}_{\ell_1(M)} & 0 & \cdots & 0 & 0 & 0 & \bar{u}^{x_1}_{\ell_1(M)} \end{pmatrix} \\
u^{x_1}_{\ell_1(M)-1} &= \begin{pmatrix} \tilde{u}^{x_1}_{\ell_1(M)-1} & 0 & \cdots & 0 & 0 & 0 & \bar{u}^{x_1}_{\ell_1(M)-1} \end{pmatrix} \\
u^{x_1}_{\ell_1(M)-2} &= \begin{pmatrix} \tilde{u}^{x_1}_{\ell_1(M)-2} & 0 & \cdots & 0 & 0 & 0 & \bar{u}^{x_1}_{\ell_1(M)-2} \end{pmatrix} \\
u^{x_1}_{\ell_1(M)-3} &= \begin{pmatrix} \tilde{u}^{x_1}_{\ell_1(M)-3} & 0 & \cdots & 0 & 0 & 0 & \bar{u}^{x_1}_{\ell_1(M)-3} \end{pmatrix} \\
\boxed{u^{x_1}_{\ell_1(M-1)}} &= \begin{pmatrix} \tilde{u}^{x_1}_{\ell_1(M-1)} & 0 & \cdots & 0 & 0 & \bar{u}^{x_1}_{\ell_1(M-1)} & \bar{u}^{x_1}_{\ell_1(M)-3} \end{pmatrix} \\
u^{x_1}_{\ell_1(M-1)-1} &= \begin{pmatrix} \tilde{u}^{x_1}_{\ell_1(M-1)-1} & 0 & \cdots & 0 & 0 & \bar{u}^{x_1}_{\ell_1(M-1)-1} & \bar{u}^{x_1}_{\ell_1(M)-3} \end{pmatrix} \\
u^{x_1}_{\ell_1(M-1)-2} &= \begin{pmatrix} \tilde{u}^{x_1}_{\ell_1(M-1)-2} & 0 & \cdots & 0 & 0 & \bar{u}^{x_1}_{\ell_1(M-1)-2} & \bar{u}^{x_1}_{\ell_1(M)-3} \end{pmatrix} \\
u^{x_1}_{\ell_1(M-1)-3} &= \begin{pmatrix} \tilde{u}^{x_1}_{\ell_1(M-1)-3} & 0 & \cdots & 0 & 0 & \bar{u}^{x_1}_{\ell_1(M-1)-3} & \bar{u}^{x_1}_{\ell_1(M)-3} \end{pmatrix} \\
\boxed{u^{x_1}_{\ell_1(M-2)}} &= \begin{pmatrix} \tilde{u}^{x_1}_{\ell_1(M-2)} & 0 & \cdots & 0 & \bar{u}^{x_1}_{\ell_1(M-2)} & \bar{u}^{x_1}_{\ell_1(M-1)-3} & \bar{u}^{x_1}_{\ell_1(M)-3} \end{pmatrix} \\
&\ \vdots \\
\boxed{u^{x_1}_{\ell_1(1)}} &= \begin{pmatrix} \tilde{u}^{x_1}_{\ell_1(1)} & \bar{u}^{x_1}_{\ell_1(1)} & \bar{u}^{x_1}_{\ell_1(2)-3} & \cdots & \bar{u}^{x_1}_{\ell_1(M-2)-3} & \bar{u}^{x_1}_{\ell_1(M-1)-3} & \bar{u}^{x_1}_{\ell_1(M)-3} \end{pmatrix} \\
&\ \vdots \\
u^{x_1}_{1} &= \begin{pmatrix} \tilde{u}^{x_1}_{1} & \bar{u}^{x_1}_{1} & \bar{u}^{x_1}_{\ell_1(2)-3} & \cdots & \bar{u}^{x_1}_{\ell_1(M-2)-3} & \bar{u}^{x_1}_{\ell_1(M-1)-3} & \bar{u}^{x_1}_{\ell_1(M)-3} \end{pmatrix}
\end{aligned}
$$

FIGURE 8.4. The sequence u^{x_1} encompasses the proposed 'diagonal plus common' factor structure. In this particular example, $x_1 = 3$ months and caplets mature on entire years.

$$
\begin{aligned}
v^{x_1}_{\ell_1(M)} &= \begin{pmatrix} \tilde{v}^{x_1}_{\ell_1(M)} & 0 & \cdots & 0 & 0 & 0 & \bar{v}^{x_1}_{\ell_1(M)} \end{pmatrix} \\
\boxed{v^{x_1}_{\ell_1(M)-1}} &= \begin{pmatrix} \tilde{v}^{x_1}_{\ell_1(M)-1} & 0 & \cdots & 0 & 0 & 0 & \bar{v}^{x_1}_{\ell_1(M)-1} \end{pmatrix} \\
v^{x_1}_{\ell_1(M)-2} &= \begin{pmatrix} \tilde{v}^{x_1}_{\ell_1(M)-2} & 0 & \cdots & 0 & 0 & 0 & \bar{v}^{x_1}_{\ell_1(M)-2} \end{pmatrix} \\
v^{x_1}_{\ell_1(M)-3} &= \begin{pmatrix} \tilde{v}^{x_1}_{\ell_1(M)-3} & 0 & \cdots & 0 & 0 & 0 & \bar{v}^{x_1}_{\ell_1(M)-3} \end{pmatrix} \\
v^{x_1}_{\ell_1(M-1)} &= \begin{pmatrix} \tilde{v}^{x_1}_{\ell_1(M-1)} & 0 & \cdots & 0 & 0 & \bar{v}^{x_1}_{\ell_1(M-1)} & \bar{u}^{x_1}_{\ell_1(M)-3} \end{pmatrix} \\
\boxed{v^{x_1}_{\ell_1(M-1)-1}} &= \begin{pmatrix} \tilde{v}^{x_1}_{\ell_1(M-1)-1} & 0 & \cdots & 0 & 0 & \bar{v}^{x_1}_{\ell_1(M-1)-1} & \bar{u}^{x_1}_{\ell_1(M)-3} \end{pmatrix} \\
v^{x_1}_{\ell_1(M-1)-2} &= \begin{pmatrix} \tilde{v}^{x_1}_{\ell_1(M-1)-2} & 0 & \cdots & 0 & 0 & \bar{v}^{x_1}_{\ell_1(M-1)-2} & \bar{u}^{x_1}_{\ell_1(M)-3} \end{pmatrix} \\
v^{x_1}_{\ell_1(M-1)-3} &= \begin{pmatrix} \tilde{v}^{x_1}_{\ell_1(M-1)-3} & 0 & \cdots & 0 & 0 & \bar{v}^{x_1}_{\ell_1(M-1)-3} & \bar{u}^{x_1}_{\ell_1(M)-3} \end{pmatrix} \\
v^{x_1}_{\ell_1(M-2)} &= \begin{pmatrix} \tilde{v}^{x_1}_{\ell_1(M-2)} & 0 & \cdots & 0 & \bar{v}^{x_1}_{\ell_1(M-2)} & \bar{u}^{x_1}_{\ell_1(M-1)-3} & \bar{v}^{x_1}_{\ell_1(M)-3} \end{pmatrix} \\
\boxed{v^{x_1}_{\ell_1(M-2)-1}} &= \begin{pmatrix} \tilde{v}^{x_1}_{\ell_1(M-2)-1} & 0 & \cdots & 0 & \bar{v}^{x_1}_{\ell_1(M-2)-1} & \bar{u}^{x_1}_{\ell_1(M-1)-3} & \bar{v}^{x_1}_{\ell_1(M)-3} \end{pmatrix} \\
&\ \vdots \\
v^{x_1}_{\ell_1(1)} &= \begin{pmatrix} \tilde{v}^{x_1}_{\ell_1(1)} & \bar{v}^{x_1}_{\ell_1(1)} & \bar{u}^{x_1}_{\ell_1(2)-3} & \cdots & \bar{u}^{x_1}_{\ell_1(M-2)-3} & \bar{u}^{x_1}_{\ell_1(M-1)-3} & \bar{u}^{x_1}_{\ell_1(M)-3} \end{pmatrix} \\
\boxed{v^{x_1}_{\ell_1(1)-1}} &= \begin{pmatrix} \tilde{v}^{x_1}_{\ell_1(1)-1} & \bar{v}^{x_1}_{\ell_1(1)} & \bar{u}^{x_1}_{\ell_1(2)-3} & \cdots & \bar{u}^{x_1}_{\ell_1(M-2)-3} & \bar{u}^{x_1}_{\ell_1(M-1)-3} & \bar{u}^{x_1}_{\ell_1(M)-3} \end{pmatrix} \\
&\ \vdots \\
v^{x_1}_{1} &= \begin{pmatrix} \tilde{v}^{x_1}_{1} & \bar{v}^{x_1}_{1} & \bar{u}^{x_1}_{\ell_1(2)-3} & \cdots & \bar{u}^{x_1}_{\ell_1(M-2)-3} & \bar{u}^{x_1}_{\ell_1(M-1)-3} & \bar{u}^{x_1}_{\ell_1(M)-3} \end{pmatrix}
\end{aligned}
$$

FIGURE 8.5. The sequence v^{x_1} is constructed analogously to u^{x_1}. In this particular example, $x_1 = 3$ months and caplets mature on entire years.

$$
\begin{array}{rcl}
u^{x_2}_{\ell_2(M)} & = & \begin{pmatrix} \tilde{u}^{x_1}_{\ell_1(M)} & 0 & \ldots & 0 & 0 & \bar{u}^{x_1}_{\ell_1(M)} \end{pmatrix} \\
u^{x_2}_{\ell_2(M)-1} & = & \begin{pmatrix} \tilde{u}^{x_1}_{\ell_1(M)-2} & 0 & \ldots & 0 & 0 & \bar{u}^{x_1}_{\ell_1(M)-2} \end{pmatrix} \\
u^{x_2}_{\ell_2(M-1)} & = & \begin{pmatrix} \tilde{u}^{x_1}_{\ell_1(M-1)} & 0 & \ldots & 0 & \bar{u}^{x_1}_{\ell_1(M-1)} & \bar{u}^{x_1}_{\ell_1(M)-3} \end{pmatrix} \\
u^{x_2}_{\ell_2(M-1)-1} & = & \begin{pmatrix} \tilde{u}^{x_1}_{\ell_1(M-1)-2} & 0 & \ldots & 0 & \bar{u}^{x_1}_{\ell_1(M-1)-2} & \bar{u}^{x_1}_{\ell_1(M)-3} \end{pmatrix}
\end{array}
$$

FIGURE 8.6. The first four rows of u^{x_2}. In this particular example, $x_2 = 6$ months and caplets mature on entire years.

$$
\begin{array}{rcl}
v^{x_2}_{\ell_2(M)} & = & \begin{pmatrix} \tilde{v}^{x_2}_{\ell_2(M)} & 0 & \ldots & 0 & 0 & \bar{v}^{x_2}_{\ell_2(M)} \end{pmatrix} \\
v^{x_2}_{\ell_2(M)-1} & = & \begin{pmatrix} \tilde{v}^{x_2}_{\ell_2(M)-1} & 0 & \ldots & 0 & 0 & \bar{v}^{x_2}_{\ell_2(M)-1} \end{pmatrix} \\
v^{x_2}_{\ell_2(M-1)} & = & \begin{pmatrix} \tilde{v}^{x_2}_{\ell_2(M-1)} & 0 & \ldots & 0 & \bar{v}^{x_2}_{\ell_2(M-1)} & \bar{u}^{x_1}_{\ell_1(M)-3} \end{pmatrix} \\
v^{x_2}_{\ell_2(M-1)-1} & = & \begin{pmatrix} \tilde{v}^{x_2}_{\ell_2(M-1)-1} & 0 & \ldots & 0 & \bar{v}^{x_2}_{\ell_2(M-1)-1} & \bar{u}^{x_1}_{\ell_1(M)-3} \end{pmatrix}
\end{array}
$$

FIGURE 8.7. The first four rows of v^{x_2}. In this particular example, $x_2 = 6$ months and caplets mature on entire years.
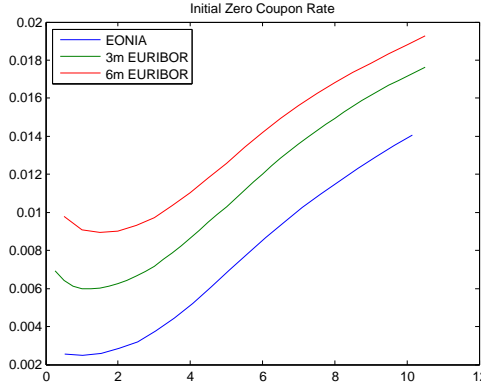

FIGURE 8.8. Zero coupon rates, EUR market, 27 May 2013.

rates, as well as OIS rates constructed in a manner described in Akkara (2012). In our example, we will focus on the 3 and 6 month tenors only. The zero coupon bond prices are converted into zero coupon rates and plotted in Figure 8.8. Cap prices are converted into caplet implied volatilities using the algorithm described in Levin (2012). The implied volatility is calculated using OIS discounting when inverting the Black (1976) formula. The caplet data we have at our disposal correspond to 3- and 6-month tenor structures. More precisely, in the EUR market caps written on the 3-month tenor are quoted only up to a maturity of 2 years, while 6-month tenor caps are quoted from maturity 3 years and onwards. Moreover, we have option prices only for the maturities corresponding to entire years and not for every tenor point. We have a fixed grid of 14 strikes ranging from 1% to 10%. We calibrate to caplet data for maturities $1, 2, \ldots, 10$ years and the OIS zero coupon bond $B(\cdot, 10.5)$ defines the terminal measure[1]. We fix in advance the values of the parameters $(\tilde{u}^{x_l})$ and $(\tilde{v}^{x_l})$, $l = 1, 2$, as well

---

[1]We found that the model performs slightly better in calibration using this numeraire, than when choosing 10 years as the terminal maturity.

as the parameters of X^c. The role of X^c is determined by the spread between $\tilde{u}^{x_1}_{\ell_1(i)}$ and $\tilde{v}^{x_1}_{\ell_1(i)-1}$, and $\tilde{u}^{x_2}_{\ell_2(i)}$ and $\tilde{v}^{x_2}_{\ell_2(i)-1}$ respectively for the 3m and 6m tenor caplets, and we will simplify by setting $\tilde{u}^{x_1}_1 = \cdots = \tilde{u}^{x_1}_{N^{x_1}-1} = u_c$ constant. Although u_c, $\tilde{v}^{x_1}_1, \ldots, \tilde{v}^{x_1}_{N^{x_1}}$ and $\tilde{v}^{x_2}_1, \ldots, \tilde{v}^{x_2}_{N^{x_2}}$ are determined by us, they cannot be chosen completely freely and one has to validate that the values of $u^{x_1}_k, u^{x_2}_k$ and $v^{x_1}_k, v^{x_2}_k$ stemming from fitting the initial term structures satisfy the necessary inequalities, i.e. $v^x_{k-1} \geq u^x_{k-1} \geq u^x_k$. Having this in mind, we chose these values in a manner such that X^c accounts for approximately 50% of the total variance of LIBOR rates from maturity 4 till 10, and about 10% of the total variance for maturities 1 till 3. We have verified through experimentation that this ad-hoc choice of dependence structure does not have a qualitative impact on the results of the following sections. Alternatively, these parameters could be calibrated to derivatives such as swaptions, basis swaptions or other derivatives partly determined by the dependence structure of the LIBOR rates. However, since interest rate derivative markets remain highly segmented and joint calibration of caplets and swaptions is a perennial challenge, see e.g. Brigo and Mercurio (2006) or Ladkau, Schoenmakers, and Zhang (2013), we will leave this issue for future research.

The model construction summarized in Figures 8.4–8.7 has the advantage that caplets can be calibrated sequentially one maturity at a time starting at the longest maturity and then moving backwards. In the calibration procedure we fit the parameters of each idiosyncratic process X^i to caplet prices with maturity T^x_i while simultaneously choosing $u^{x_1}_i, u^{x_2}_i$ and $v^{x_1}_i, v^{x_2}_i$ to match the corresponding values of the initial OIS and LIBOR rates. Caplets are priced using formula (6.5) and the parameters are found using standard least-squares minimization between market and model implied volatility. The results[2] from fitting the caplets are shown in Figure 8.9. We can observe that the model performs very well for different types of volatility smiles across the whole term structure, with only minor problems for extreme strikes in maturities 1-3.

8.3. **Swaption price approximation.** The next two sections are devoted to numerically testing the validity of the swaption and basis swaption price approximation formulas derived in Sections 7.1 and 7.2. We will run a Monte Carlo study comparing the true price with the linear boundary approximation formula. The model parameters used stem from the calibration to the market data described in the previous section.

Let us denote the true and the approximate prices as follows:

$$
\mathbb{S}^+_0(K, \mathcal{T}^x_{pq}) = B(0, T_N)\, \mathbb{E}_N\left[\left(\sum_{i=p+1}^{q} M^{v^x_{i-1}}_{T^x_p} - \sum_{i=p+1}^{q} K_x M^{u^x_i}_{T^x_p}\right) \mathbf{1}_{\{f(X_{T^x_p}) \geq 0\}}\right],
$$

$$
\widetilde{\mathbb{S}}^+_0(K, \mathcal{T}^x_{pq}) = B(0, T_N)\, \mathbb{E}_N\left[\left(\sum_{i=p+1}^{q} M^{v^x_{i-1}}_{T^x_p} - \sum_{i=p+1}^{q} K_x M^{u^x_i}_{T^x_p}\right) \mathbf{1}_{\{\widetilde{f}(X_{T^x_p}) \geq 0\}}\right],
$$

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[2]All calibrated and chosen parameter values as well as the calibrated matrices u^{x_j}, v^{x_j} for $j = 1, 2$ are available from the authors upon request.
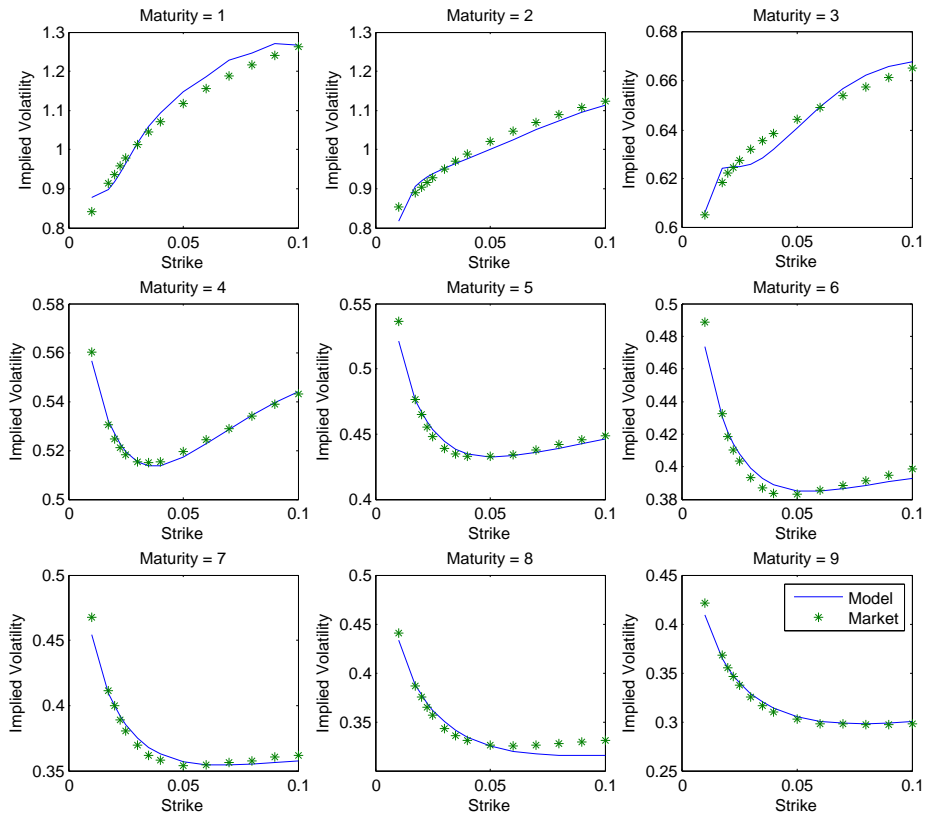
FIGURE 8.9. Market and model implied volatility for caplets written on 3 (1–2 year maturity) and 6 (2–9 year maturity) month tenor EURIBOR.

where f and $\widetilde{f}$ were defined in (7.7) and (7.8) respectively. The Monte Carlo (MC) estimator[3] of $\mathbb{S}_0^+(K, \mathcal{T}_{pq}^x)$ is denoted by $\hat{\mathbb{S}}_0^+(K, \mathcal{T}_{pq}^x)$ and we will refer to it as the 'true price'. Instead of computing $\widetilde{\mathbb{S}}_0^+(K, \mathcal{T}_{pq}^x)$ using Fourier methods, we will form another MC estimator $\hat{\widetilde{\mathbb{S}}}_0^+(K, \mathcal{T}_{pq}^x)$. This has the advantage that, when the same realizations are used to calculate both MC estimators, the difference $\hat{\mathbb{S}}_0^+(K, \mathcal{T}_{pq}^x) - \hat{\widetilde{\mathbb{S}}}_0^+(K, \mathcal{T}_{pq}^x)$ will be an estimate of the error induced by the linear boundary approximation which is minimally affected by simulation bias.

Swaption prices vary considerably across strike and maturity, thus we will express the difference between the true and the approximate price in terms of implied volatility (using OIS discounting), which better demonstrates the economic significance of any potential errors. We price swaptions on three different underlying swaps. The results for the 3m underlying tenor are exhibited in Figure 8.10. The corresponding results for the 6m tenor swaptions have errors which are approximately one half the level in the graphs shown here and have been omitted for brevity.

On the left hand side of Figure 8.10, implied volatility levels are plotted for the true and the approximate prices. The strikes are chosen to range from 60% to 200% of the spot value of the underlying fair swap

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[3]We construct the Monte Carlo estimate using 5 million paths of X with 10 discretization steps per year. In each discritization step the continuous part is simulated using the algorithm in Glasserman (2003, §3.4.1) while the jump part is handled using Glasserman (2003, pp. 137–139) with jump size distribution changed from log-normal to exponential.
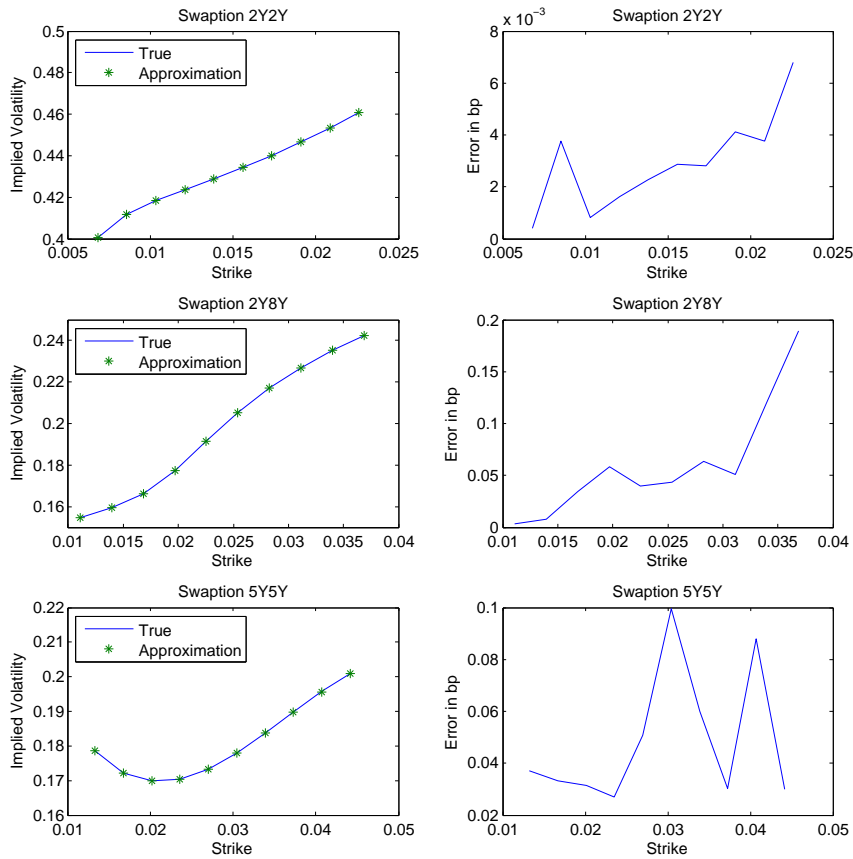
FIGURE 8.10. Implied volatility and absolute errors for 3m swaptions.

rate, which is the normal range the products are quoted. The right hand side shows the difference between the two implied volatilities in basis points (i.e. multiplied by 10^4). As was also documented in Schrager and Pelsser (2006), the errors of the approximation usually increase with the number of payments in the underlying swap. This is also the case here, however the level of the errors is in all cases very low. In normal markets, bid-ask spreads typically range from 10 to 300 bp (at the at-the-money level) thus even the highest errors are too small to be of any economic significance. This is true even in the case of the 2Y8Y swaption which contains 32 payments.

8.4. **Basis swaption price approximation.** In order to test the approximation formula for basis swaptions, we will follow the same methodology as in the previous subsection. That is, we calculate MC estimators for the following two expectations:

$$\mathbb{BS}_0^+(S, \mathcal{T}_{pq}^{x_1}, \mathcal{T}_{pq}^{x_2}) = B(0, T_N)\, \mathbb{E}_N\Bigg[\Bigg(\sum_{i=p_2+1}^{q_2}\Big(M_{T_{p_2}^{x_2}}^{v_{i-1}^{x_2}} - M_{T_{p_2}^{x_2}}^{u_i^{x_2}}\Big) \\ - \sum_{i=p_1+1}^{q_1}\Big(M_{T_{p_1}^{x_1}}^{v_{i-1}^{x_1}} - S_{x_1} M_{T_{p_1}^{x_1}}^{u_i^{x_1}}\Big)\Bigg)\mathbf{1}_{\{g(X_{T_{p_2}})\geq 0\}}\Bigg],$$
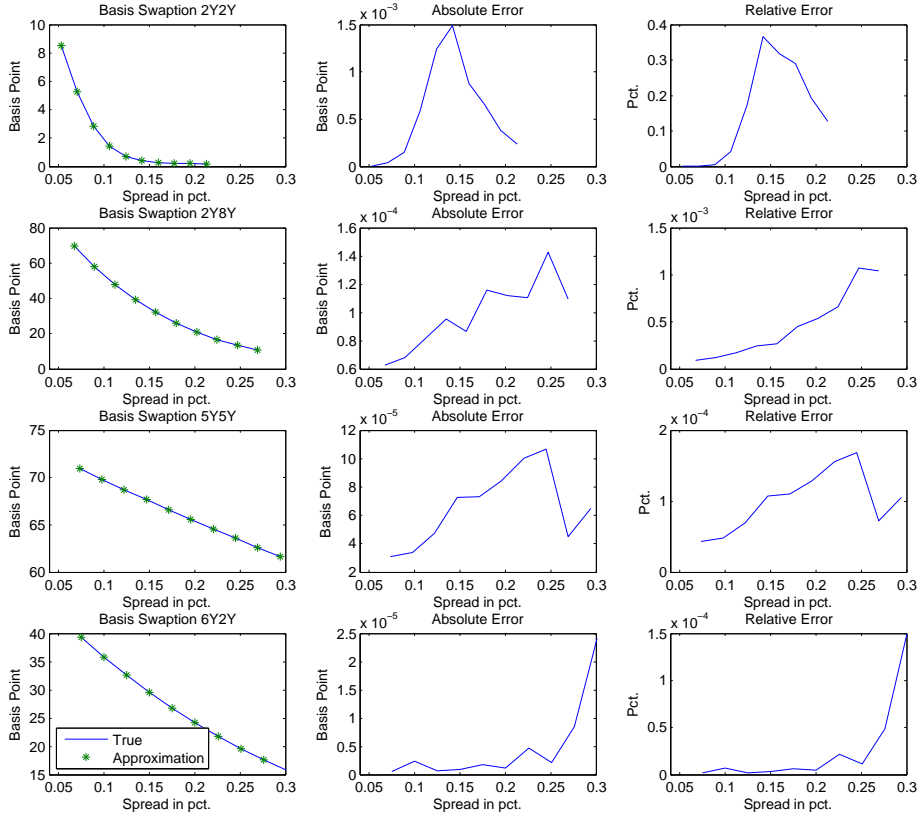
FIGURE 8.11. Prices in basis points, absolute and relative errors for 3m–6m basis swaptions.

$$\widetilde{\mathbb{BS}}_0^+\left(S, \mathcal{T}_{pq}^{x_1}, \mathcal{T}_{pq}^{x_2}\right) = B(0, T_N)\, \mathbb{E}_N\Bigg[\Bigg(\sum_{i=p_2+1}^{q_2}\left(M_{T_{p_2}^{x_2}}^{v_{i-1}^{x_2}} - M_{T_{p_2}^{x_2}}^{u_i^{x_2}}\right)$$

$$- \sum_{i=p_1+1}^{q_1}\left(M_{T_{p_1}^{x_1}}^{v_{i-1}^{x_1}} - S_{x_1} M_{T_{p_1}^{x_1}}^{u_i^{x_1}}\right)\Bigg)\mathbf{1}_{\{\widetilde{g}(X_{T_{p_2}})\geq 0\}}\Bigg],$$

where $S_{x_1} = 1 - \delta_{x_1}S$, while g and $\widetilde{g}$ were defined in (7.15) and (7.16). Using the same realizations, we plot the level, absolute and relative differences between both prices measured in basis points as a function of the spread for three different underlying basis swaps. The spreads are chosen to range from 50% to 200% of the at-the-money level, i.e. the spread that sets the underlying basis swap to a value of zero, see again (6.4):

$$S_{ATM} := S_0\big(\mathcal{T}_{pq}^{x_1}, \mathcal{T}_{pq}^{x_2}\big).$$

The numerical results can be seen in Figure 8.11. We have chosen these maturities to be representative of two general patterns. The first is that the errors tend to increase with the length of the basis swap, which is exemplified by comparing errors for the 2Y8Y, 5Y5Y and 6Y2Y contracts. The second pattern relates to when the majority of payments in the contract are paid out. We can notice that the errors for the 2Y2Y contract are much larger than for the corresponding 6Y2Y, even though both contain the same number of payments. Furthermore, we can also see that the 2Y2Y contract has larger errors than the 2Y8Y even though both have the same maturity and the latter has more payments. This

anomalous result can be explained by the convexity of the term structure of interest rates. In Figure 8.8 we can notice that the majority of the payments of the 2Y2Y contract fall in a particularly curved region of the term structure. This will result in an exercise boundary which is also more nonlinear, thus leading to the relative deterioration of the linear boundary approximation. However, it must be emphasized that the errors are still at a level easily deemed economically insignificant with a maximum relative error of 0.4% in a spread region where the price levels are particularly low.

8.5. **A simple example.** The purpose of this section is to test the approximation for swaptions and basis swaptions in a fully constructed and more manageable example than the one in Sections 8.3 and 8.4. We start by choosing a simple two factor model $X = (X^1, X^2)$ with

$$\mathrm{d}X_t^i = -\lambda_i(X_t^i - \theta_i)\mathrm{d}t + 2\eta_i\sqrt{X_t^i}\mathrm{d}W_t^i + \mathrm{d}Z_t^i,, \quad i = 1, 2, \tag{8.7}$$

where we set

i	X_0^i	λ_i	θ_i	η_i	ν_i	μ_i
1	0.5000	0.1000	1.5300	0.2660	0	0
2	9.4531	0.0407	0.0591	0.4640	0.0074	0.2499

The initial term structures are constructed from a Nelson–Siegel parametrization of the zero coupon rate $R(t)$

$$R(T) = \beta_0 + \beta_1\frac{1 - \mathrm{e}^{-\gamma T}}{\gamma T} + \beta_2\left(\frac{1 - \mathrm{e}^{-\gamma T}}{\gamma T} - \mathrm{e}^{-\tau T}\right). \tag{8.8}$$

We limit ourselves to two tenors, x_1 corresponding to 3 months and x_2 corresponding to 6 months. We construct the initial curves from the following parameters

Curve	β_0	β_1	β_0	γ
OIS	0.0003	0.01	0.07	0.06
3m	0.0032	0.01	0.07	0.06
6m	0.0050	0.01	0.07	0.06

The zero coupon rates can then be converted to zero coupon bond prices and LIBOR rates through the relation $B(0, T) = \exp(-R(T))$. We construct the matrices u^{x_j} and v^{x_j} in the following simple manner

$$u_k^{x_1} = \begin{pmatrix} u_c & \bar{u}_k^{x_1} \end{pmatrix}, \quad k = 1, \ldots, N^{x_1} \tag{8.9}$$

$$u_k^{x_2} = u_{k\delta_{x_2}/\delta_{x_1}}^{x_1}, \quad k = 1, \ldots, N^{x_2} \tag{8.10}$$

$$v_k^{x_1} = \begin{pmatrix} \tilde{v}_c^{x_1} & \bar{v}_k^{x_1} \end{pmatrix}, \quad k = 0, \ldots, N^{x_1} - 1 \tag{8.11}$$

$$v_k^{x_2} = \begin{pmatrix} \tilde{v}_c^{x_2} & \bar{v}_k^{x_2} \end{pmatrix}, \quad k = 0, \ldots, N^{x_2} - 1 \tag{8.12}$$

and $u_{N^{x_1}}^{x_1} = u_{N^{x_2}}^{x_2} = 0$, where $\bar{u}_k^{x_j}, \bar{v}_k^{x_j} \in \mathbb{R}_{\geqslant 0}$ for $j = 1, 2$. The bond $B(\cdot, 4.5)$ defines the terminal measure, thus $N^{x_1} = 18$ and $N^{x_2} = 9$. We set $u_c = 0.0065$, $\tilde{v}_c^{x_1} = 0.007$ and $\tilde{v}_c^{x_2} = 0.0075$. The remaining values can then be determined uniquely using equations (4.4) and (4.6), i.e. by fitting the initial term structures. We get that

k	$u_k^{x_1}$	$v_k^{x_1}$	$u_k^{x_2}$	$v_k^{x_2}$
0	-	0.008966	-	0.009035
1	0.008638	0.008641	0.008286	0.008358
2	0.008286	0.008289	0.007505	0.007577
3	0.007908	0.007911	0.006625	0.006697
4	0.007505	0.007507	0.005652	0.005725
5	0.007077	0.007079	0.004591	0.004664
6	0.006625	0.006627	0.003447	0.003520
7	0.006150	0.006152	0.002225	0.002298
8	0.005652	0.005654	0.000929	0.001003
9	0.005132	0.005135	0	-
10	0.004591	0.004594		
11	0.004029	0.004032		
12	0.003447	0.003450		
13	0.002847	0.002848		
14	0.002225	0.002228		
15	0.001586	0.001589		
16	0.000929	0.000932		
17	0.000254	0.000257		
18	0	-		

We can observe that all sequences u^{x_j}, v^{x_j} for $j = 1, 2$ are decreasing, which corresponds to the 'normal' market situation; see again Remark 4.4.

8.5.1. *Swaption approximation.* Let us consider a 2Y2Y swaption on 3 month LIBOR rates, i.e. a swaption in the notation of Section 7.1 with $p = 8$ and $q = 16$. We run a Monte Carlo study equivalent to the one in Section 8.3 and the results are reported for four different strikes:

Strike (K)	$\hat{\mathbb{S}}_0^+$	Error	IV (%)	IV Error	$\mathcal{A}$	$\mathcal{B}$
0.013238	176.17	2.06e-08	30.38	2.326e-10	-5.5403	(1.1596 1)
0.023535	52.214	4.31e-08	26.78	1.818e-10	-10.2982	(1.1605 1)
0.033831	9.7898	4.09e-08	24.82	2.971e-10	-15.0481	(1.1615 1)
0.044128	1.4016	7.90e-09	23.72	2.016e-10	-19.7899	(1.1625 1)

where

- $\hat{\mathbb{S}}_0^+ = \hat{\mathbb{S}}_0^+(K, \mathcal{T}_{8,16}^{x_1})$ and IV denote the MC estimator of the price (in basis points) and the implied volatility (with OIS discounting) using the true exercise boundary defined in (7.7).
- Error $= |\hat{\mathbb{S}}_0^+(K, \mathcal{T}_{8,16}^{x_1}) - \tilde{\hat{\mathbb{S}}}_0^+(K, \mathcal{T}_{8,16}^{x_1})|$, where $\tilde{\hat{\mathbb{S}}}_0^+(K, \mathcal{T}_{8,16}^{x_1})$ denotes the MC estimator of the price (in basis points) using the approximate exercise boundary defined in (7.8).
- IV Error $= |\text{IV} - \widetilde{\text{IV}}|$, where $\widetilde{\text{IV}}$ denotes the implied volatility (with OIS discounting) calculated from $\tilde{\hat{\mathbb{S}}}_0^+(K, \mathcal{T}_{8,16}^{x_1})$.
- $\mathcal{A} \in \mathbb{R}$ and $\mathcal{B} \in \mathbb{R}^2$ are calculated using the linear regression procedure described in Singleton and Umantsev (2002, pp. 432–434).

8.5.2. *Basis swaption approximation.* Let us also consider a 2Y2Y basis swaption. This is an option to enter into a basis swap paying 3 month LIBOR plus spread S and receiving 6 month LIBOR, which starts at year 2 and ends at year 4. Once again we conduct a Monte Carlo study equivalent to Section 8.4, and get that

Spread (S)	$\hat{\mathbb{BS}}_0^+$	Price Error	$\mathcal{C}$	$\mathcal{D}$
0.0010945	13.778	2.103e-06	-7.7191	(1 5.7514)
0.0019458	3.7972	4.784e-05	-14.0029	(1 5.7694)
0.0027971	0.64406	9.364e-05	-20.2158	(1 5.7868)
0.0036484	0.080951	5.852e-05	-26.3597	(1 5.8037)

where

- $\hat{\mathbb{BS}}_0^+ = \hat{\mathbb{BS}}_0^+(S, \mathcal{T}_{8,16}^{x_1}, \mathcal{T}_{4,8}^{x_2})$ denotes the MC estimator of the price (in basis points) using the true exercise boundary defined in (7.15).
- Price Error $= |\hat{\mathbb{BS}}_0^+(S, \mathcal{T}_{8,16}^{x_1}, \mathcal{T}_{4,8}^{x_2}) - \widetilde{\hat{\mathbb{BS}}}_0^+(S, \mathcal{T}_{8,16}^{x_1}, \mathcal{T}_{4,8}^{x_2})|$, where similarly $\widetilde{\hat{\mathbb{BS}}}_0^+(K, \mathcal{T}_{8,16}^{x_1}, \mathcal{T}_{4,8}^{x_2})$ denotes the MC estimator of the price (in basis points) using the approximate exercise boundary defined in (7.16).
- $\mathcal{C} \in \mathbb{R}$ and $\mathcal{D} \in \mathbb{R}^2$ are calculated using the linear regression procedure described in Singleton and Umantsev (2002, pp. 432–434).

These simple examples highlight once again the accuracy of the linear boundary approximations developed in Sections 7.1 and 7.2.

Appendix A. Terminal correlations

This appendix is devoted to the computation of terminal correlations. The expression 'terminal correlation' is used in the same sense as in Brigo and Mercurio (2006, §6.6), i.e. it summarizes the degree of dependence between two LIBOR rates at a fixed, terminal time point. Here the driving process is a general affine process and not just an affine diffusion as in section 5.4.

We start by introducing some shorthand notation

$$\Phi_k^x(t) := \phi_{T_N-t}(v_{k-1}^x) - \phi_{T_N-t}(u_k^x),$$
$$\Psi_k^x(t) := \psi_{T_N-t}(v_{k-1}^x) - \psi_{T_N-t}(u_k^x),$$
$$\Phi_{k_1,k_2}^{x_1,x_2}(t) := \Phi_{k_1}^{x_1}(t) + \Phi_{k_2}^{x_2}(t),$$
$$\Psi_{k_1,k_2}^{x_1,x_2}(t) := \Psi_{k_1}^{x_1}(t) + \Psi_{k_2}^{x_2}(t),$$

where $k \in \mathcal{K}^x$ and $k_l \in \mathcal{K}^{x_l}$ for $l = 1, 2$. Then, we have from (4.3) that

$$1 + \delta_{x_l} L_{k_l}^{x_l}(T_i) = M_{T_i}^{v_{k_l-1}^{x_l}}/M_{T_i}^{u_{k_l}^{x_l}} = \exp\left(\Phi_{k_l}^{x_l}(T_i) + \left\langle \Psi_{k_l}^{x_l}(T_i), X_{T_i} \right\rangle\right), \tag{A.1}$$

for $l = 1, 2$ and $T_i \leq T_{k_1-1}^{x_1} \vee T_{k_2-1}^{x_2}$. We also denote the moment generating function of X_{T_i} under the measure $\mathbb{P}_N$ as follows

$$\Theta_{T_i}(z) = \mathbb{E}_N\left[e^{\langle z, X_{T_i} \rangle}\right] = \exp\left(\phi_{T_i}(z) + \langle \psi_{T_i}(z), X_0 \rangle\right). \tag{A.2}$$

Therefore we get that

$$\mathbb{E}_N\left[M_{T_i}^{v_{k-1}^x}/M_{T_i}^{u_k^x}\right] = \mathrm{e}^{\Phi_k^x(T_i)}\Theta_{T_i}\big(\Psi_k^x(T_i)\big), \tag{A.3}$$

$$\mathbb{E}_N\left[\left(M_{T_i}^{v_{k-1}^x}/M_{T_i}^{u_k^x}\right)^2\right] = \mathrm{e}^{2\Phi_k^x(T_i)}\Theta_{T_i}\big(2\Psi_k^x(T_i)\big), \tag{A.4}$$

$$\mathbb{E}_N\left[M_{T_i}^{v_{k_1-1}^{x_1}}/M_{T_i}^{u_{k_1}^{x_1}}\cdot M_{T_i}^{v_{k_2-1}^{x_2}}/M_{T_i}^{u_{k_2}^{x_2}}\right] = \mathrm{e}^{\Phi_{k_1,k_2}^{x_1,x_2}(T_i)}\Theta_{T_i}\big(\Psi_{k_1,k_2}^{x_1,x_2}(T_i)\big). \tag{A.5}$$

Finally, the formula for terminal correlations follows after inserting the expressions above in the definition of correlation and doing some tedious, but straightforward, computations

$$\mathrm{Corr}_{T_i}\big[L_{k_1}^{x_1}, L_{k_2}^{x_2}\big] \overset{\text{(A.1)}}{=} \mathrm{Corr}\left[M_{T_i}^{v_{k_1-1}^{x_1}}/M_{T_i}^{u_{k_1}^{x_1}}, M_{T_i}^{v_{k_2-1}^{x_2}}/M_{T_i}^{u_{k_2}^{x_2}}\right]$$

$$= \frac{\Theta_{T_i}\big(\Psi_{k_1,k_2}^{x_1,x_2}(T_i)\big) - \Theta_{T_i}\big(\Psi_{k_1}^{x_1}(T_i)\big)\Theta_{T_i}\big(\Psi_{k_2}^{x_2}(T_i)\big)}{\sqrt{\Theta_{T_i}\big(2\Psi_{k_1}^{x_1}(T_i)\big) - \Theta_{T_i}\big(\Psi_{k_1}^{x_1}(T_i)\big)^2}\sqrt{\Theta_{T_i}\big(2\Psi_{k_2}^{x_2}(T_i)\big) - \Theta_{T_i}\big(\Psi_{k_2}^{x_2}(T_i)\big)^2}}.$$

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