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# Concentration phenomena for the nonlocal Schrödinger equation with Dirichlet datum


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ABSTRACT. For a smooth, bounded domain $\Omega$, $s \in (0,1)$, $p \in \left(1, \frac{n+2s}{n-2s}\right)$ we consider the nonlocal equation

$$\varepsilon^{2s}(-\Delta)^s u + u = u^p \quad \text{in } \Omega$$

with zero Dirichlet datum and a small parameter $\varepsilon > 0$. We construct a family of solutions that concentrate as $\varepsilon \to 0$ at an interior point of the domain in the form of a scaling of the ground state in entire space. Unlike the classical case $s = 1$, the leading order of the associated reduced energy functional in a variational reduction procedure is of polynomial instead of exponential order on the distance from the boundary, due to the nonlocal effect. Delicate analysis is needed to overcome the lack of localization, in particular establishing the rather unexpected asymptotics for the Green function of $\varepsilon^{2s}(-\Delta)^s + 1$ in the expanding domain $\varepsilon^{-1}\Omega$ with zero exterior datum.

## 1. INTRODUCTION

Given $s \in (0,1)$, $n \in \mathbb{N}$ with $n > 2s$, $p \in \left(1, \frac{n+2s}{n-2s}\right)$ and a bounded smooth domain $\Omega \subset \mathbb{R}^n$, we consider the fractional Laplacian problem

$$\begin{cases} \varepsilon^{2s}(-\Delta)^s U + U = U^p & \text{in } \Omega, \\ U = 0 & \text{in } \mathbb{R}^n \setminus \Omega, \end{cases} \tag{1.1}$$

where $\varepsilon > 0$ is a small parameter.

As usual, the operator $(-\Delta)^s$ is the fractional Laplacian defined at any point $x \in \mathbb{R}^n$ as

$$(-\Delta)^s U(x) := c(n,s) \int_{\mathbb{R}^n} \frac{2U(x) - U(x+y) - U(x-y)}{|y|^{n+2s}}\, dy,$$

for a suitable positive normalizing constant $c(n,s)$. We refer to [13, 24, 34] for an introduction to the fractional Laplacian operator.

We provide in the appendix a heuristic physical motivation of the problem considered and of the relevance of our results in the light of a nonlocal quantum mechanics theory.

The goal of this paper is to construct solutions of problem (1.1) that concentrate at interior points of the domain for sufficiently small values of $\varepsilon$. More precisely, we shall establish the existence of a solution $U_\varepsilon$ that at main order looks like

$$U_\varepsilon(x) \approx w\left(\frac{x - \tilde{\xi}_\varepsilon}{\varepsilon}\right). \tag{1.2}$$

Here $\tilde{\xi}_\varepsilon$ is a point lying at a uniformly positive distance from the boundary $\partial\Omega$, and $w$ designates the unique radial positive *least energy solution* of the problem

$$(-\Delta)^s w + w = w^p, \quad w \in H^s(\mathbb{R}^n). \tag{1.3}$$

See for instance [17] for the existence of such a solution and its basic properties. See [3, 16, 20] for the (delicate) proof of uniqueness in special situations and [21] for the general case. The solution $w$ is smooth and has the asymptotic behavior

$$\alpha |x|^{-(n+2s)} \leqslant w(x) \leqslant \beta |x|^{-(n+2s)} \quad \text{for } |x| \geqslant 1, \tag{1.4}$$

for some positive constants $\alpha, \beta$, see Theorem 1.5 of [17], and the lower bound in formula (IV.6) of [6].

Our main result is the following.

**Theorem 1.1.** *If $\varepsilon$ is sufficiently small, there exist a point $\tilde{\xi}_\varepsilon \in \Omega$ and a solution $U_\varepsilon$ of problem (1.1) such that*

$$\left| U_\varepsilon(x) - w\left(\frac{x-\tilde{\xi}_\varepsilon}{\varepsilon}\right)\right| \leqslant C\varepsilon^{n+2s}, \tag{1.5}$$

*and $\operatorname{dist}(\tilde{\xi}_\varepsilon, \partial\Omega) \geqslant c$. Here, $c$ and $C$ are positive constants independent of $\varepsilon$ and $\Omega$.*

*Besides, the point $\xi_\varepsilon := \tilde{\xi}_\varepsilon/\varepsilon$ is such that*

$$\mathcal{H}_\varepsilon(\xi_\varepsilon) = \min_{\xi\in\Omega_\varepsilon} \mathcal{H}_\varepsilon(\xi) + O(\varepsilon^{n+4s}) \tag{1.6}$$

*for the functional $\mathcal{H}_\varepsilon(\xi)$ defined in (1.17) below, where*

$$\Omega_\varepsilon := \frac{\Omega}{\varepsilon} = \left\{\frac{x}{\varepsilon},\ x\in\Omega\right\}. \tag{1.7}$$

The basic idea of the proof, which also leads to the characterization (1.6) of the location of the point $\tilde{\xi}_\varepsilon$ goes as follows. Letting $u(x) := U(\varepsilon x)$, problem (1.1) becomes

$$\begin{cases} (-\Delta)^s u + u = u^p & \text{in } \Omega_\varepsilon,\\ u = 0 & \text{in } \mathbb{R}^n \setminus \Omega_\varepsilon, \end{cases} \tag{1.8}$$

where $\Omega_\varepsilon$ is defined in (1.7).

For a given $\xi \in \Omega_\varepsilon$, a first approximation $\bar{u}_\xi$ for the solution of problem (1.8) consistent with the desired form (1.2) and the Dirichlet exterior condition, can be taken as the solution of the linear problem

$$\begin{cases} (-\Delta)^s \bar{u}_\xi + \bar{u}_\xi = w_\xi^p & \text{in } \Omega_\varepsilon,\\ \bar{u}_\xi = 0 & \text{in } \mathbb{R}^n \setminus \Omega_\varepsilon, \end{cases} \tag{1.9}$$

where

$$w_\xi(x) := w(x-\xi).$$

The actual solution will be obtained as a small perturbation from $\bar{u}_\xi$ for a suitable point $\xi = \xi_\varepsilon$. Problem (1.8) is variational. It corresponds to the Euler-Lagrange equation for the functional

$$I_\varepsilon(u) = \frac{1}{2}\int_{\Omega_\varepsilon} \left((-\Delta)^s u(x)\, u(x) + u^2(x)\right) dx - \frac{1}{p+1}\int_{\Omega_\varepsilon} u^{p+1}(x)\, dx, \qquad u \in H_0^s(\Omega_\varepsilon), \tag{1.10}$$

where

$$H_0^s(\Omega_\varepsilon) = \{u \in H^s(\mathbb{R}^n) \text{ s.t. } u = 0 \text{ a.e. in } \mathbb{R}^n \setminus \Omega_\varepsilon\}.$$

Since the solution we look for should be close to $\bar{u}_\xi$ for $\xi = \xi_\varepsilon$, the functional $\xi \mapsto I_\varepsilon(\bar{u}_\xi)$ should have a critical point near the $\xi = \xi_\varepsilon$. We shall next argue that this functional actually has a global minimizer located at distance $\sim \frac{1}{\varepsilon}$ from $\partial\Omega_\varepsilon$.

The expansion of $I_\varepsilon(\bar{u}_\xi)$ involves the regular part of the Green function for the operator $(-\Delta)^s + 1$ in $\Omega_\varepsilon$, which we define next. In $\mathbb{R}^n$ the operator $(-\Delta)^s + 1$ has unique decaying fundamental solution $\Gamma$ which solves

$$(-\Delta)^s\Gamma + \Gamma = \delta_0. \tag{1.11}$$

The function $\Gamma$ is radially symmetric, positive and satisfies

$$\frac{\alpha}{|x|^{n+2s}} \leqslant \Gamma(x) \leqslant \frac{\beta}{|x|^{n+2s}} \tag{1.12}$$

for $|x| \geqslant 1$ and $\alpha, \beta > 0$ see for instance Lemma C.1 in [21].

The Green function $G_\varepsilon$ for $(-\Delta)^s + 1$ in $\Omega_\varepsilon$ solves

$$\begin{cases} (-\Delta)^s G_\varepsilon(x,y) + G_\varepsilon(x,y) = \delta_y & \text{if } x \in \Omega_\varepsilon, \\ G_\varepsilon(x,y) = 0 & \text{if } x \in \mathbb{R}^n \setminus \Omega_\varepsilon. \end{cases} \tag{1.13}$$

In other words

$$G_\varepsilon(x,y) := \Gamma(x-y) - H_\varepsilon(x,y) \tag{1.14}$$

where $H_\varepsilon(x,y)$, the regular part, satisfies, for fixed $y \in \mathbb{R}^n$,

$$\begin{cases} (-\Delta)^s H_\varepsilon(x,y) + H_\varepsilon(x,y) = 0 & \text{if } x \in \Omega_\varepsilon, \\ H_\varepsilon(x,y) = \Gamma(x-y) & \text{if } x \in \mathbb{R}^n \setminus \Omega_\varepsilon. \end{cases} \tag{1.15}$$

We will show in Theorem 4.1 that for $\operatorname{dist}(\xi, \partial\Omega_\varepsilon) \geqslant \delta/\varepsilon$, with $\delta > 0$ fixed and appropriately small, we have that

$$I_\varepsilon(\bar{u}_\xi) = I_0 + \frac{1}{2}\mathcal{H}_\varepsilon(\xi) + o(\varepsilon^{n+4s}), \tag{1.16}$$

where $I_0$ is the energy of $w$ computed in $\mathbb{R}^n$, and $\mathcal{H}_\varepsilon(\xi)$ is given by

$$\mathcal{H}_\varepsilon(\xi) := \int_{\Omega_\varepsilon} \int_{\Omega_\varepsilon} H_\varepsilon(x,y)\, w_\xi^p(x)\, w_\xi^p(y)\, dx\, dy. \tag{1.17}$$

We will show that $\mathcal{H}_\varepsilon$ satisfies

$$\frac{\alpha}{\operatorname{dist}(\xi, \partial\Omega_\varepsilon)^{n+4s}} \leqslant \mathcal{H}_\varepsilon(\xi) \leqslant \frac{\beta}{\operatorname{dist}(\xi, \partial\Omega_\varepsilon)^{n+4s}}, \tag{1.18}$$

where $\alpha, \beta > 0$, for all points $\xi \in \Omega_\varepsilon$ such that $\operatorname{dist}(\xi, \partial\Omega_\varepsilon) \in [5, \bar{\delta}/\varepsilon]$, for $\bar{\delta} > 0$ fixed, suitably small, and $\varepsilon \ll \bar{\delta}$.

From (1.18) and estimate (1.16), we deduce the existence of a global minimizer $\xi_\varepsilon$ for the functional $I_\varepsilon(\bar{u}_\xi)$ for all small $\varepsilon > 0$, which is located at distance $\sim \frac{1}{\varepsilon}$ from $\partial\Omega_\varepsilon$. The actual proof *reduces* the problem of finding a solution close to $w_\xi$ via a Lyapunov-Schmidt procedure, to that of finding a critical point $\xi_\varepsilon$ of a functional with a similar expansion to (1.16), as we will see in Section 7.

In the classical case (i.e. when $s = 1$ and the operator boils down to the classical Laplacian), there is a broad literature on concentration phenomena: we recall here the seminal papers [28, 29] and we refer to [2] for detailed discussions and more precise references. In particular, we recall that [11, 12, 28, 29] construct solutions of the classical Dirichlet problem that concentrate at points which maximize the distance from the boundary: in this sense, Theorem 1.1 may be seen as the nonlocal counterpart of these results. In our case, the determination of the concentrating point is less explicit than in the classical case, due to the nonlocal behavior of the energy expansion. More precisely, for $s = 1$ one gets the expansion parallel to (1.16),

$$I_\varepsilon(\bar{u}_\xi) = I_0 + \frac{1}{2}\mathcal{H}_\varepsilon(\xi) + O(e^{-\frac{(2+\sigma)\operatorname{dist}(\xi,\partial\Omega_\varepsilon)}{\varepsilon}}),$$

where now

$$\mathcal{H}_\varepsilon(\xi) \approx e^{-2\operatorname{dist}(\xi,\partial\Omega_\varepsilon)/\varepsilon} \tag{1.19}$$

see for instance Y.-Y. Li, L. Nirenberg [28] (compare (1.19) with (1.18)).

In the nonlocal case, much less is known. Multi-peak solutions of a fractional Schrödinger equation set in the whole of $\mathbb{R}^n$ were considered recently in [10]. The analysis needed in this paper is considerably more involved. Concentrating solutions for fractional problems involving critical or almost critical exponents were considered in [9]. See also [8] for some concentration phenomena in particular cases, and also [32] and references therein for related problems about Schrödinger-type equations in a fractional setting.

The paper is organized as follows. The rather delicate analysis of the behavior of the regular part of Green's function is contained in Section 2. We estimate the function $\bar{u}_\xi$ in Section 3, thus obtaining a first approximation of the energy expansion in Section 4.

The remainders of this expansion need to be carefully estimated: for this, we provide some decay and regularity estimates in Sections 5 and 6.

The Lyapunov-Schmidt method will be resumed in Section 7 where we discuss the linear theory and the bifurcation from it. A key ingredient is the linear nondegeneracy of the least energy solution $w$: this is an important result that was completely achieved only recently in [21], after preliminary works in particular cases discussed in [3, 16, 20]. Then we complete the proof of Theorem 1.1 in Section 8.

## 2. Estimates on the Robin function $H_\varepsilon$ and on the leading term of the energy functional

Given $\xi \in \Omega_\varepsilon$ and $x \in \mathbb{R}^n$, we define

$$\beta_\xi(x) := \int_{\mathbb{R}^n \setminus \Omega_\varepsilon} \Gamma(z-\xi)\,\Gamma(x-z)\,dz.$$

Notice that, for any $x \in \Omega_\varepsilon$ and $z \in \mathbb{R}^n \setminus \Omega_\varepsilon$ we have

$$\Big((-\Delta)^s + 1\Big)\Gamma(x-z) = \delta_0(x-z) = 0, \tag{2.1}$$

and so

$$\Big((-\Delta)^s + 1\Big)\beta_\xi(x) = \int_{\mathbb{R}^n \setminus \Omega_\varepsilon} \Gamma(z-\xi)\,\Big((-\Delta)^s + 1\Big)\Gamma(x-z)\,dz = 0 \tag{2.2}$$

for any $x \in \Omega_\varepsilon$. Our purpose is to use $\beta_\xi(x)$ as a barrier, from above and below, for the Robin function $H_\varepsilon(x,\xi)$, using (1.15), (2.2) and the Comparison Principle. For this scope, we estimate the behavior of $\beta_\xi$ outside $\Omega_\varepsilon$:

**Lemma 2.1.** *There exists $c \in (0,1)$ such that*

$$cH_\varepsilon(x,\xi) \leqslant \beta_\xi(x) \leqslant c^{-1}H_\varepsilon(x,\xi) \tag{2.3}$$

*for any $x \in \mathbb{R}^n \setminus \Omega_\varepsilon$ and $\xi \in \Omega_\varepsilon$ with*

$$\operatorname{dist}(\xi, \partial\Omega_\varepsilon) \geqslant 1. \tag{2.4}$$

*Proof.* First we observe that for any $x \in \mathbb{R}^n \setminus \Omega_\varepsilon$,

$$|B_{1/2}(x) \setminus \Omega_\varepsilon| \geqslant c_\star \tag{2.5}$$

for a suitable $c_\star > 0$. For concreteness, one can take $c_\star$ as the measure of the spherical segment

$$\Sigma := \Big\{ z = (z', z_n) \in \mathbb{R}^{n-1} \times \mathbb{R} \text{ s.t. } |z| < 1/2 \text{ and } z_n \geqslant 1/4 \Big\}.$$

To prove (2.5), we argue as follows. If $B_{1/2}(x) \subseteq (\mathbb{R}^n \setminus \Omega_\varepsilon)$ we are done. If not, let $p \in (\partial\Omega_\varepsilon) \cap B_{1/2}(x)$, with $\operatorname{dist}(x, \partial\Omega_\varepsilon) = |x-p|$. Notice that the ball centered at $x$ of radius $|x-p|$ is tangent to $\Omega_\varepsilon$ from the outside at $p$, and $|x-p| \leqslant 1/2$.

Up to a rigid motion, we suppose that $p = 0$ and $x = |x|e_n$. By scaling back, the ball of radius $|\hat{x}|$ centered at $\hat{x} := \varepsilon x = \varepsilon|x|e_n$ is tangent to $\Omega$ from the outside at the origin, and $|\hat{x}| = \varepsilon|x| = \varepsilon|x-p| \leqslant \varepsilon/2$.

From the regularity of $\Omega$, we have that there exists a ball of universal radius $r_o > 0$ touching $\Omega$ from the outside at any point, so in particular $B_{r_o}(r_o e_n)$ touches $\Omega$ from the outside at the origin, hence

$$B_{r_o}(r_o e_n) \subseteq \mathbb{R}^n \setminus \Omega. \tag{2.6}$$

We observe that

$$\hat{x} + \varepsilon\Sigma \subseteq B_{r_o}(r_o e_n). \tag{2.7}$$

Indeed, if $z = (z', z_n) \in \hat{x} + \varepsilon\Sigma$ then $\varepsilon\Sigma \ni z - \hat{x} = (z', z_n - |\hat{x}|)$ and so $z_n - |\hat{x}| \in [\varepsilon/4, \varepsilon/2]$ and $|z'| \leqslant |z| \leqslant \varepsilon/2$. Hence, for small $\varepsilon$, we have that $r_o - z_n \geqslant r_o - |\hat{x}| - (\varepsilon/2) \geqslant r_o - \varepsilon \geqslant 0$, and $r_o - z_n \leqslant r_o - |\hat{x}| - (\varepsilon/4) \leqslant r_o - (\varepsilon/4)$ and so

$$|z_n - r_o| = r_o - z_n \leqslant r_o - \frac{\varepsilon}{4}$$

that gives

$$\begin{aligned} |z - r_o e_n|^2 = |z'|^2 + |z_n - r_o|^2 &\leqslant \left(\frac{\varepsilon}{2}\right)^2 + \left(r_o - \frac{\varepsilon}{4}\right)^2 \\ &= r_o^2 + \frac{\varepsilon^2}{16} + \frac{\varepsilon^2}{4} - \frac{2r_o\varepsilon}{2} < r_o \end{aligned}$$

if $\varepsilon$ is sufficiently small. This proves (2.7).

As a consequence of (2.6) and (2.7), we conclude that $\hat{x} + \varepsilon\Sigma \subseteq \mathbb{R}^n \setminus \Omega$, that is, by scaling back, $x + \Sigma \subseteq \mathbb{R}^n \setminus \Omega_\varepsilon$. Accordingly,

$$(B_{1/2}(x) \setminus \Omega_\varepsilon) \supseteq B_{1/2}(x) \cap (x + \Sigma) = x + \Sigma$$

and this ends the proof of (2.5).

Now we observe that if $a, b \in \mathbb{R}^n$ satisfy $|a - b| \leqslant |b - \xi|/2$, $\min\{|a - \xi|, |b - \xi|\} \geqslant 1$ then

$$\Gamma(a - \xi) \leqslant C\Gamma(b - \xi) \tag{2.8}$$

for some $C > 0$. Indeed,

$$|a - \xi| \geqslant |b - \xi| - |a - b| \geqslant \frac{|b - \xi|}{2}$$

and so, from (1.12),

$$\Gamma(a - \xi) \leqslant \frac{C}{|a - \xi|^{n+2s}} \leqslant \frac{2^{n+2s}C}{|b - \xi|^{n+2s}} \leqslant 2^{n+2s}C^2\Gamma(b - \xi).$$

This proves (2.8), up to relabeling the constants. As a consequence, given $x \in \mathbb{R}^n \setminus \Omega_\varepsilon$, we apply (2.8) with $a := x$ and $b := z \in B_{1/2}(x) \setminus \Omega_\varepsilon$, we recall (2.4) and (2.5) and we obtain that

$$\begin{aligned} \beta_\xi(x) &\geqslant \int_{B_{1/2}(x)\setminus\Omega_\varepsilon} \Gamma(z - \xi)\,\Gamma(x - z)\,dz \\ &\geqslant C^{-1} \int_{B_{1/2}(x)\setminus\Omega_\varepsilon} \Gamma(x - \xi)\,\Gamma(x - z)\,dz \\ &\geqslant C^{-1}\Gamma(x - \xi) \inf_{y \in B_{1/2}} \Gamma(y)\,|B_{1/2}(x) \setminus \Omega_\varepsilon| \\ &\geqslant c_\star C^{-1}\Gamma(x - \xi) \inf_{y \in B_{1/2}} \Gamma(y). \end{aligned}$$

This proves the first inequality in (2.3), since $x \in \mathbb{R}^n \setminus \Omega_\varepsilon$ and so

$$\Gamma(x-\xi) = H_\varepsilon(x,\xi). \tag{2.9}$$

Now we prove the second inequality in (2.3). For this, we use (2.8) once again (applied here with $a := z$, $b := x$ and recalling (2.4)) and the fact that

$$\int_{\mathbb{R}^n} \Gamma(z)\,dz \;=\; 1 \tag{2.10}$$

to see that

(2.11)

$$I_1 := \int_{B_{|x-\xi|/2}(x)\setminus\Omega_\varepsilon} \Gamma(z-\xi)\,\Gamma(x-z)\,dz \leqslant C \int_{B_{|x-\xi|/2}(x)} \Gamma(x-\xi)\,\Gamma(x-z)\,dz \leqslant C\,\Gamma(x-\xi).$$

On the other hand, if $z \notin B_{|x-\xi|/2}(x)$, we have that $|x-z| \geqslant |x-\xi|/2$ and so, by (1.12)

$$\Gamma(x-z) \leqslant \frac{C}{|x-z|^{n+2s}} \leqslant \frac{2^{n+2s}C}{|x-\xi|^{n+2s}} \leqslant 2^{n+2s}C^2\Gamma(x-\xi).$$

Consequently

$$I_2 := \int_{\mathbb{R}^n\setminus B_{|x-\xi|/2}(x)} \Gamma(z-\xi)\,\Gamma(x-z)\,dz \leqslant C' \int_{\mathbb{R}^n\setminus B_{|x-\xi|/2}(x)} \Gamma(z-\xi)\,\Gamma(x-\xi)\,dz \leqslant C'\Gamma(x-\xi),$$

for some $C' > 0$, thanks to (2.10). From this and (2.11) we obtain that

$$\beta_\xi(x) \leqslant I_1 + I_2 \leqslant C''\Gamma(x-\xi),$$

for some $C'' > 0$. This, together with (2.9), completes the proof of (2.3). ☐

**Corollary 2.2.** *There exists $c \in (0,1)$ such that*

$$cH_\varepsilon(x,\xi) \leqslant \beta_\xi(x) \leqslant c^{-1}H_\varepsilon(x,\xi)$$

*for any $x \in \mathbb{R}^n$ and $\xi \in \Omega_\varepsilon$ with $\operatorname{dist}(\xi,\partial\Omega_\varepsilon) \geqslant 1$.*

*Proof.* The desired estimate holds true outside $\Omega_\varepsilon$, thanks to (2.3). Then it holds true inside $\Omega_\varepsilon$ as well, in virtue of (2.2), (1.15) and the Comparison Principle. ☐

The above result implies an interesting lower bound on the symmetric version of the Robin function $H_\varepsilon(\xi,\xi)$, and in general for the values of the Robin function sufficiently close to the diagonal, according to the following

**Proposition 2.3.** *Let $\delta \in (0,1)$. Let $\xi \in \Omega_\varepsilon$ with*

$$d := \operatorname{dist}(\xi,\partial\Omega_\varepsilon) \in \left[2,\, \frac{\delta}{\varepsilon}\right].$$

*Let $x, y \in B_{d/2}(\xi)$. Then*

$$H_\varepsilon(x,y) \geqslant \frac{c_o}{d^{n+4s}}$$

*for a suitable $c_o \in (0,1)$, as long as $\delta$ is sufficiently small.*

*Proof.* Let $z \in \mathbb{R}^n \setminus \Omega_\varepsilon$. Notice that

$$|\xi - y| \leqslant \frac{d}{2} \leqslant \frac{|z-\xi|}{2} \tag{2.12}$$

and so

$$|z-y| \leqslant |z-\xi| + |\xi - y| \leqslant \frac{3}{2}|z-\xi|. \tag{2.13}$$

Similarly,

$$|z-x| \leqslant \frac{3}{2}|z-\xi|. \tag{2.14}$$

Another consequence of (2.12) is that

$$|z-y| \geqslant |z-\xi| - |\xi-y| \geqslant \frac{|z-\xi|}{2} \geqslant \frac{d}{2} \geqslant 1 \tag{2.15}$$

hence $\mathrm{dist}(y, \partial\Omega_\varepsilon) \geqslant 1$ (as a matter of fact, till now we only exploited that $d \geqslant 2$). Notice that in the same way, one has that

$$|z-x| \geqslant 1. \tag{2.16}$$

Therefore we can use Corollary 2.2 with $\xi$ replaced by $y$ and so, recalling (1.12), (2.15), (2.16), (2.13) and (2.14), we conclude that

$$\begin{aligned} H_\varepsilon(x,y) &\geqslant c\beta_y(x) \\ &= c\int_{\mathbb{R}^n\setminus\Omega_\varepsilon} \Gamma(z-y)\,\Gamma(x-z)\,dz \\ &\geqslant \frac{c}{C^2}\int_{\mathbb{R}^n\setminus\Omega_\varepsilon} \frac{1}{|y-z|^{n+2s}|x-z|^{n+2s}}\,dz \\ &\geqslant c'\int_{\mathbb{R}^n\setminus\Omega_\varepsilon} \frac{1}{|z-\xi|^{2n+4s}}\,dz \end{aligned} \tag{2.17}$$

for a suitable $c' > 0$.

Now we make some geometric considerations. By the smoothness of the domain, we can touch $\Omega$ from the outside at any point with balls of universal radius, say $r_o > 0$. By scaling, we can touch $\Omega_\varepsilon$ from the exterior by balls of radius $r_o\,\varepsilon^{-1}$, and so of radius $d$ (notice indeed that $d \leqslant \delta\varepsilon^{-1} \leqslant r_o\,\varepsilon^{-1}$ if $\delta$ is small enough). Let $\eta \in \partial\Omega_\varepsilon$ be such that $|\xi-\eta| = d$. By the above considerations, we can touch $\Omega_\varepsilon$ from the outside at $\eta$ with a ball $\mathcal{B}$ of radius $d$ (i.e. of diameter $2d$). We stress that $\mathcal{B} \subseteq \mathbb{R}^n \setminus \Omega_\varepsilon$, that $|\mathcal{B}| \geqslant \bar{c}d^n$ for some $\bar{c} > 0$ and that if $z \in \mathcal{B}$ then

$$|z-\xi| \leqslant |z-\eta| + |\eta-\xi| \leqslant 2d + d = 3d.$$

These observations and (2.17) yield that

$$H_\varepsilon(x,y) \geqslant c'\int_{\mathcal{B}} \frac{1}{|z-\xi|^{2n+4s}}\,dz \geqslant c'\,(3d)^{-(2n+4s)}\,|\mathcal{B}| = c'\,3^{-(2n+4s)}\,\bar{c}\,d^{-(n+4s)},$$

as desired. □

There is also an upper bound similar to the lower bound obtained in Proposition 2.3:

**Proposition 2.4.** *Let $\xi \in \Omega_\varepsilon$ with $d := \mathrm{dist}(\xi, \partial\Omega_\varepsilon) \geqslant 2$, and $x, y \in B_{d/2}(\xi)$. Then*

$$H_\varepsilon(x,y) \leqslant \frac{C_o}{d^{n+4s}}$$

*for a suitable $C_o > 0$.*

*Proof.* As noticed in the proof of Proposition 2.3 we can use Corollary 2.2 with $\xi$ replaced by $y$. Then, since $B_d(\xi) \subseteq \Omega_\varepsilon$, hence $(\mathbb{R}^n \setminus \Omega_\varepsilon) \subseteq (\mathbb{R}^n \setminus B_d(\xi))$, and therefore we obtain that

$$H_\varepsilon(x,y) \leqslant c^{-1}\beta_y(x) \leqslant c^{-1} \int_{\mathbb{R}^n \setminus B_d(\xi)} \Gamma(z-\xi)\,\Gamma(x-z)\,dz.$$

Also, if $z \in \mathbb{R}^n \setminus B_d(\xi)$, we have that

$$|z-x| \geqslant |z-\xi| - |\xi - x| \geqslant d - |\xi - x| \geqslant \frac{d}{2},$$

hence, by (1.12),

$$\begin{aligned} H_\varepsilon(x,y) &\leqslant c^{-1}C^2 \int_{\mathbb{R}^n \setminus B_d(\xi)} \frac{1}{|z-\xi|^{n+2s}|z-x|^{n+2s}}\,dz \\ &\leqslant c^{-1}C^2(2/d)^{n+2s} \int_{\mathbb{R}^n \setminus B_d(\xi)} \frac{1}{|z-\xi|^{n+2s}}\,dz. \end{aligned}$$

By computing the latter integral in polar coordinates, we obtain the desired result. □

It will be convenient to define for any $\xi \in \Omega_\varepsilon$,

$$\Pi_\varepsilon(x,\xi) := \int_{\Omega_\varepsilon} H_\varepsilon(x,y) w_\xi^p(y)\,dy. \tag{2.18}$$

As a consequence of Proposition 2.4, we have

**Lemma 2.5.** *Let $\xi \in \Omega_\varepsilon$ with $d := \operatorname{dist}(\xi, \partial\Omega_\varepsilon) \geqslant 2$. Let $x \in B_{d/8}(\xi)$. Then*

$$\Pi_\varepsilon(x,\xi) \leqslant \frac{C}{d^{n+4s}}$$

*for some $C > 0$, where $\Pi_\varepsilon(x,\xi)$ is defined in (2.18).*

*Proof.* We split the integral into two contributions, one in $B_{d/4}(\xi)$ and one outside such ball.

We can use Proposition 2.4 to obtain that, for $y \in \Omega_\varepsilon \cap B_{d/4}(\xi)$, it holds that $H_\varepsilon(x,y) \leqslant C_o d^{-n-4s}$ and so

$$\pi_1 := \int_{\Omega_\varepsilon \cap B_{d/4}(\xi)} H_\varepsilon(x,y) w_\xi^p(y)\,dy \leqslant C_o d^{-n-4s} \int_{\mathbb{R}^n} w_\xi^p(y)\,dy \leqslant C_1 d^{-n-4s},$$

for some $C_1 > 0$.

Now we consider the case in which $y \in \Omega_\varepsilon \setminus B_{d/4}(\xi)$. In this case, we use (1.4) to see that $w_\xi^p(y) \leqslant C_2|y-\xi|^{-p(n+2s)}$ for some $C_2 > 0$. Also, in this case,

$$|y-x| \geqslant |y-\xi| - |x-\xi| \geqslant \frac{d}{4} - \frac{d}{8} = \frac{d}{8}$$

hence, since by maximum principle

$$H_\varepsilon(x,y) \leqslant \Gamma(x-y) \leqslant \frac{C_3}{|x-y|^{n+2s}} \leqslant \frac{C_4}{d^{n+2s}}, \tag{2.19}$$

for some $C_3, C_4 > 0$. As a consequence

$$\pi_2 := \int_{\Omega_\varepsilon \setminus B_{d/4}(\xi)} H_\varepsilon(x,y) w_\xi^p(y)\,dy \leqslant \frac{C_2 C_4}{d^{n+2s}} \int_{\mathbb{R}^n \setminus B_{d/4}(\xi)} |y-\xi|^{-p(n+2s)}\,dz = \frac{C_5}{d^{2s+p(n+2s)}}$$

for some $C_5 > 0$. In particular, since $d \geqslant 1$ and $p > 1$, we see that $\pi_2 \leqslant C_5 d^{-n-4s}$ and therefore, recalling (2.18), we conclude that $\Pi_\varepsilon(x,\xi) \leqslant \pi_1 + \pi_2 \leqslant (C_1 + C_5)d^{-n-4s}$. □

The function $\mathcal{H}_\varepsilon$ defined in (1.17) will represent the first interesting order in the expansion of the reduced energy functional (see the forthcoming Theorem 4.1 for a precise statement). To show that this reduced energy functional has a local minimum, we will show that $\mathcal{H}_\varepsilon$ (and so the reduced energy functional itself) attains, in a certain domain, values that are smaller than the ones attained at the boundary (concretely, this domain will be given by the subset of $\Omega_\varepsilon$ with points of distance $\delta/\varepsilon$ from the boundary, for some $\delta \in (0,1)$ fixed suitably small possibly in dependence of $n$, $s$ and $\Omega$).

To this extent, a detailed statement will be given in Proposition 2.8 and the necessary bounds on $\mathcal{H}_\varepsilon$ will be given in the forthcoming Corollaries 2.6 and 2.7, which in turn follow from Propositions 2.3 and 2.4, respectively.

**Corollary 2.6.** *Let $\delta \in (0,1)$.*

*Let $\xi \in \Omega_\varepsilon$ with*

$$d := \operatorname{dist}(\xi, \partial\Omega_\varepsilon) \in \left[2, \frac{\delta}{\varepsilon}\right].$$

*Then*

$$\mathcal{H}_\varepsilon(\xi) \geqslant \frac{c}{d^{n+4s}},$$

*for a suitable $c > 0$, as long as $\delta$ is sufficiently small.*

*Proof.* Notice that $B_1(\xi) \subseteq B_{d/2}(\xi) \subseteq \Omega_\varepsilon$. So, by Proposition 2.3, $H_\varepsilon(x,y) \geqslant c_o d^{-(n+4s)}$ if $x, y \in B_1(\xi)$ and

$$\mathcal{H}_\varepsilon(\xi) \geqslant \int_{B_1(\xi)} \int_{B_1(\xi)} H_\varepsilon(x,y)\, w_\xi^p(x)\, w_\xi^p(y)\, dx\, dy \geqslant c_o d^{-(n+4s)} \left( \int_{B_1} w^p(z)\, dz \right)^2. \qquad \square$$

**Corollary 2.7.** *Let $\xi \in \Omega_\varepsilon$ with $d := \operatorname{dist}(\xi, \partial\Omega_\varepsilon) \geqslant 5$. Then*

$$\mathcal{H}_\varepsilon(\xi) \leqslant \frac{C}{d^{n+4s}},$$

*for a suitable $C > 0$.*

*Proof.* We split the integral in (1.17) into three contributions: first we treat the case in which $x, y \in B_{d/2}(\xi)$, then the case in which $x, y \in \mathbb{R}^n \setminus B_{d/2}(\xi)$, and finally the case in which $x \in B_{d/2}(\xi)$ and $y \in \mathbb{R}^n \setminus B_{d/2}(\xi)$ (the case in which $y \in B_{d/2}(\xi)$ and $x \in \mathbb{R}^n \setminus B_{d/2}(\xi)$ is, of course, symmetrical to this one).

In the first case, we use Proposition 2.4, obtaining that

$$\begin{aligned} &\int_{B_{d/2}(\xi)} dx \int_{B_{d/2}(\xi)} dy\, H_\varepsilon(x,y)\, w_\xi^p(x)\, w_\xi^p(y) \\ &\leqslant C_o d^{-(n+4s)} \left( \int_{B_{d/2}} w^p(z)\, dz \right)^2 \\ &\leqslant C_o d^{-(n+4s)} \left( \int_{\mathbb{R}^n} w^p(z)\, dz \right)^2. \end{aligned} \tag{2.20}$$

In the second case, we use twice the decay of $w$ given in (1.4), (2.19) and (2.10), obtaining that

$$\begin{aligned}
&\int_{\mathbb{R}^n\setminus B_{d/2}(\xi)} dx \int_{\mathbb{R}^n\setminus B_{d/2}(\xi)} dy\, H_\varepsilon(x,y)\, w_\xi^p(x)\, w_\xi^p(y)\\
&\leqslant C^{2p} \int_{\mathbb{R}^n\setminus B_{d/2}(\xi)} dx \int_{\mathbb{R}^n\setminus B_{d/2}(\xi)} dy\, |x-\xi|^{-p(n+2s)} |y-\xi|^{-p(n+2s)}\, \Gamma(x-y)\\
&\leqslant C^{2p}\, (d/2)^{-p(n+2s)} \int_{\mathbb{R}^n\setminus B_{d/2}(\xi)} dx \int_{\mathbb{R}^n\setminus B_{d/2}(\xi)} dy\, |x-\xi|^{-p(n+2s)}\, \Gamma(x-y)\\
&\leqslant C^{2p}\, (d/2)^{-p(n+2s)} \int_{\mathbb{R}^n\setminus B_{d/2}} d\eta \int_{\mathbb{R}^n} d\theta\, |\eta|^{-p(n+2s)}\, \Gamma(\theta)\\
&\leqslant C' d^{-2p(n+2s)+n}\\
&\leqslant C' d^{-(n+4s)},
\end{aligned} \tag{2.21}$$

for some $C' > 0$.

As for the third case, we take $x \in B_{d/2}(\xi)$ and $y \in \mathbb{R}^n \setminus B_{d/2}(\xi)$ and we distinguish two subcases: either $|x-y| \leqslant d/6$ or $|x-y| > d/6$.

In the first subcase, we use a translated version of Proposition 2.4. Namely, if $x \in B_{d/2}(\xi)$, $y \in \mathbb{R}^n \setminus B_{d/2}(\xi)$ and $|x-y| \leqslant d/6$ we take $\hat\xi := (x+y)/2$. Notice that

$$|\xi - y| \leqslant |\xi - x| + |x-y| \leqslant \frac{d}{2} + \frac{d}{6}$$

and therefore

$$2|\hat\xi - \xi| = |(x+y) - 2\xi| \leqslant |x-\xi| + |y-\xi| \leqslant \frac{d}{2} + \left(\frac{d}{2} + \frac{d}{6}\right) = \frac{7d}{6}.$$

As a consequence

$$\hat d := \operatorname{dist}(\hat\xi, \partial\Omega_\varepsilon) \geqslant \operatorname{dist}(\xi, \partial\Omega_\varepsilon) - |\hat\xi - \xi| \geqslant d - \frac{7d}{12} = \frac{5d}{12}. \tag{2.22}$$

In particular

$$\hat d \geqslant 2. \tag{2.23}$$

Also, by construction $x - \hat\xi = \hat\xi - y = (x-y)/2$, and so

$$|x - \hat\xi| = |\hat\xi - y| = \frac{1}{2}|x-y| \leqslant \frac{d}{12}.$$

This and (2.22) say that

$$x, y \in B_{d/12}(\hat\xi) \subseteq B_{\hat d/2}(\hat\xi). \tag{2.24}$$

Thanks to (2.23) and (2.24) we can now use Proposition 2.4 with $\xi$ and $d$ replaced by $\hat\xi$ and $\hat d$, respectively. So we obtain that, in this case,

$$H_\varepsilon(x,y) \leqslant \frac{C_o}{\hat d^{n+4s}} \leqslant \frac{\hat C}{d^{n+4s}}, \tag{2.25}$$

for some $\hat C > 0$, where (2.22) was used again in the last inequality.

So, we make use of (1.4) and (2.25) to obtain that

$$\begin{aligned}
&\int_{B_{d/2}(\xi)} dx \int_{B_{d/6}(x)\setminus B_{d/2}(\xi)} dy\, H_\varepsilon(x,y)\, w_\xi^p(x)\, w_\xi^p(y)\\
&\leqslant C^p\, \hat{C}\, d^{-(n+4s)} \int_{B_{d/2}(\xi)} dx \int_{B_{d/6}(x)\setminus B_{d/2}(\xi)} dy\, w_\xi^p(x)\, |y-\xi|^{-p(n+2s)}\\
&\leqslant C^p\, \hat{C}\, d^{-(n+4s)} \int_{\mathbb{R}^n} dx \int_{\mathbb{R}^n\setminus B_{d/2}} dz\, w_\xi^p(x)\, |z|^{-p(n+2s)}\\
&\leqslant \tilde{C}\, d^{-4s-p(n+2s)}\\
&\leqslant \tilde{C}\, d^{-(n+4s)}\ldots
\end{aligned} \tag{2.26}$$

Finally, we consider the subcase in which $x \in B_{d/2}(\xi)$, $y \in \mathbb{R}^n \setminus B_{d/2}(\xi)$ and $|x-y| > d/6$. In this circumstance we use (1.4), (2.19) and (1.12) to conclude that

$$\begin{aligned}
&\int_{B_{d/2}(\xi)} dx \int_{\substack{\mathbb{R}^n\setminus B_{d/2}(\xi)\\ \{|x-y|>d/6\}}} dy\, H_\varepsilon(x,y)\, w_\xi^p(x)\, w_\xi^p(y)\\
&\leqslant C^p \int_{B_{d/2}(\xi)} dx \int_{\substack{\mathbb{R}^n\setminus B_{d/2}(\xi)\\ \{|x-y|>d/6\}}} dy\, \Gamma(x-y)\, w_\xi^p(x)\, |y-\xi|^{-p(n+2s)}\\
&\leqslant \underline{C} \int_{B_{d/2}(\xi)} dx \int_{\substack{\mathbb{R}^n\setminus B_{d/2}(\xi)\\ \{|x-y|>d/6\}}} dy\, |x-y|^{-(n+2s)}\, w_\xi^p(x)\, |y-\xi|^{-p(n+2s)}\\
&\leqslant \underline{C}(d/6)^{-(n+2s)} \int_{\mathbb{R}^n} dx \int_{\mathbb{R}^n\setminus B_{d/2}(\xi)} dy\, w_\xi^p(x)\, |y-\xi|^{-p(n+2s)}\\
&\leqslant \overline{C} d^{-2s-p(n+2s)}\\
&\leqslant \overline{C} d^{-(n+4s)}
\end{aligned} \tag{2.27}$$

for suitable $\underline{C}$, $\overline{C} > 0$. From (2.26) and (2.27) we complete the third case, namely when $x \in B_{d/2}(\xi)$ and $y \in \mathbb{R}^n \setminus B_{d/2}(\xi)$, by obtaining that

$$\int_{B_{d/2}(\xi)} dx \int_{\mathbb{R}^n\setminus B_{d/2}(\xi)} dy H_\varepsilon(x,y)\, w_\xi^p(x)\, w_\xi^p(y) \leqslant (\tilde{C}+\overline{C}) d^{-(n+4s)}. \tag{2.28}$$

The desired result follows from (1.17), (2.20), (2.21) and (2.28). □

For concreteness, we summarize the results of Corollaries 2.6 and 2.7 in the following

**Proposition 2.8.** *Let $\delta > 0$ be suitably small and*

$$\Omega_{\varepsilon,\delta} := \{x \in \Omega_\varepsilon \text{ s.t. } \operatorname{dist}(x, \partial\Omega_\varepsilon) > \delta/\varepsilon\}. \tag{2.29}$$

*Then $\mathcal{H}_\varepsilon$ attains an interior minimum in $\Omega_{\varepsilon,\delta}$, namely there exist $c_1$, $c_2 > 0$ such that*

$$\min_{\Omega_{\varepsilon,\delta}} \mathcal{H}_\varepsilon \leqslant c_1 \varepsilon^{n+4s} < c_2 \left(\frac{\varepsilon}{\delta}\right)^{n+4s} \leqslant \min_{\partial\Omega_{\varepsilon,\delta}} \mathcal{H}_\varepsilon.$$

*Proof.* Let $\delta_\star$ to be the maximal distance that a point of $\Omega$ may attain from the boundary of $\Omega$. By scaling, the maximal distance that a point of $\Omega_\varepsilon$ may attain from the boundary of $\Omega_\varepsilon$ is $\delta_\star/\varepsilon$. Let $\xi_\star$ be such a point, i.e.

$$d_\star := \operatorname{dist}(\xi_\star, \partial\Omega_\varepsilon) = \frac{\delta_\star}{\varepsilon}.$$

For $\delta$ sufficiently small we have that $\xi_\star \in \Omega_{\varepsilon,\delta}$. So, by Corollary 2.7,

$$\min_{\Omega_{\varepsilon,\delta}} \mathcal{H}_\varepsilon \leqslant \mathcal{H}_\varepsilon(\xi_\star) \leqslant \frac{C}{d_\star^{n+4s}} = \frac{C\varepsilon^{n+4s}}{\delta_\star^{n+4s}} = c_1 \varepsilon^{n+4s},$$

for a suitable $c_1 > 0$. On the other hand, by Corollary 2.6,

$$\min_{\partial\Omega_{\varepsilon,\delta}} \mathcal{H}_\varepsilon \geqslant \frac{c\varepsilon^{n+4s}}{\delta^{n+4s}},$$

which implies the desired result for $\delta$ appropriately small. □

## 3. Estimates on $\bar{u}_\xi$ and first approximation of the solution

Now we make some estimates on the function $\bar{u}_\xi$ introduced in (1.9), by using in particular the auxiliary function $\Pi_\varepsilon$ in (2.18). For this, we define, for any $\xi \in \Omega_\varepsilon$,

$$\Lambda_\xi(x) := \int_{\mathbb{R}^n \setminus \Omega_\varepsilon} w_\xi^p(y) \Gamma(x-y)\, dy. \tag{3.1}$$

We have the following estimate for $\Lambda_\xi$:

**Lemma 3.1.** *Let $x, \xi \in \Omega_\varepsilon$. Assume that $d := \operatorname{dist}(\xi, \partial\Omega_\varepsilon) \geqslant 1$. Then*

$$0 \leqslant \Lambda_\xi(x) \leqslant \frac{C}{d^{(n+2s)p}},$$

*where $C > 0$ depends on $n$, $p$, $s$ and $\Omega$.*

*Proof.* If $y \in \mathbb{R}^n \setminus \Omega_\varepsilon$ then $|y-\xi| \geqslant \operatorname{dist}(\xi, \partial\Omega_\varepsilon) \geqslant 1$ therefore, by (1.4),

$$|w_\xi(y)| = |w(y-\xi)| \leqslant C|y-\xi|^{-(n+2s)} \leqslant C d^{-(n+2s)}.$$

As a consequence of this, and recalling (2.10), we deduce that

$$\int_{\mathbb{R}^n \setminus \Omega_\varepsilon} w_\xi^p(y)\Gamma(x-y)\, dy \leqslant \left(C d^{-(n+2s)}\right)^p \int_{\mathbb{R}^n \setminus \Omega_\varepsilon} \Gamma(x-y)\, dy \leqslant \left(C d^{-(n+2s)}\right)^p.$$ □

**Lemma 3.2.** *Let $x, \xi \in \Omega_\varepsilon$. Assume that $d := \operatorname{dist}(\xi, \partial\Omega_\varepsilon) \geqslant 1$. Then*

$$\bar{u}_\xi(x) = w_\xi(x) - \Lambda_\xi(x) - \Pi_\varepsilon(x,\xi) \tag{3.2}$$

*and*

$$0 \leqslant w_\xi(x) - \bar{u}_\xi(x) - \Pi_\varepsilon(x,\xi) \leqslant \frac{C}{d^{(n+2s)p}}, \tag{3.3}$$

*for a suitable $C > 0$ that depends on $n$, $p$, $s$ and $\Omega$.*

*Proof.* First of all, notice that $w = w^p * \Gamma$, since they both satisfy (1.3), thanks to (1.11), and uniqueness holds. As a consequence

$$w_\xi(x) = w(x-\xi) = \int_{\mathbb{R}^n} w^p(x-\xi-y)\Gamma(y)\, dy = \int_{\mathbb{R}^n} w_\xi^p(y)\Gamma(x-y)\, dy. \tag{3.4}$$

Similarly, recalling (1.9), (1.13) and the symmetry of $G_\varepsilon$, we see that

$$\begin{aligned}
\bar u_\xi(x) &= \int_{\Omega_\varepsilon} \bar u_\xi(z)\delta_x(z)\,dz \\
&= \int_{\Omega_\varepsilon} \bar u_\xi(z)((-\Delta)^s+1)G_\varepsilon(z,x)\,dz \\
&= \int_{\Omega_\varepsilon} w_\xi^p(z)G_\varepsilon(x,z)\,dz \\
&= \int_{\Omega_\varepsilon} w_\xi^p(z)\Gamma(x-z)\,dz - \int_{\Omega_\varepsilon} w_\xi^p(z)H_\varepsilon(x,z)\,dz \\
&= \int_{\mathbb{R}^n} w_\xi^p(z)\Gamma(x-z)\,dz - \int_{\mathbb{R}^n\setminus\Omega_\varepsilon} w_\xi^p(z)\Gamma(x-z)\,dz - \int_{\Omega_\varepsilon} w_\xi^p(z)H_\varepsilon(x,z)\,dz.
\end{aligned}$$

This, (2.18), (3.1) and (3.4) imply (3.2), which, together with Lemma 3.1, implies (3.3). $\square$

**Lemma 3.3.** *Let $\xi\in\Omega_\varepsilon$. Assume that $d:=\operatorname{dist}(\xi,\partial\Omega_\varepsilon)\geqslant 2$. Then*

$$\int_{\Omega_\varepsilon} w_\xi^{p-1}(x)\,\Lambda_\xi(x)\,\Pi_\varepsilon(x,\xi)\,dx \leqslant \frac{C}{d^{(n+2s)p+2s}},$$

*for a suitable $C>0$ that depends on $n$, $p$, $s$ and $\Omega$.*

*Proof.* First of all, we notice that for $y\in\mathbb{R}^n\setminus\Omega_\varepsilon$ we have $|y-\xi|\geqslant d>1$, and therefore, thanks to (1.4),

$$|w_\xi(y)| = |w(y-\xi)| \leqslant C|y-\xi|^{-(n+2s)} \leqslant C\,d^{-(n+2s)}.$$

Hence, recalling (3.1),

$$\begin{aligned}
&\int_{\Omega_\varepsilon} w_\xi^{p-1}(x)\Lambda_\xi(x)\,\Pi_\varepsilon(x,\xi)\,dx \\
&=\int_{\Omega_\varepsilon} w_\xi^{p-1}(x)\left(\int_{\mathbb{R}^n\setminus\Omega_\varepsilon} w_\xi^p(y)\Gamma(x-y)\,dy\right)\Pi_\varepsilon(x,\xi)\,dx \\
&\leqslant Cd^{-(n+2s)p}\int_{\Omega_\varepsilon} dx\int_{\mathbb{R}^n\setminus\Omega_\varepsilon} dy\,w_\xi^{p-1}(x)\Gamma(x-y)\Pi_\varepsilon(x,\xi) \\
&\leqslant Cd^{-(n+2s)p}\int_{\{|x-\xi|\leqslant d/4\}} dx\int_{\mathbb{R}^n\setminus\Omega_\varepsilon} dy\,w_\xi^{p-1}(x)\Gamma(x-y)\Pi_\varepsilon(x,\xi) \\
&\qquad + Cd^{-(n+2s)p}\int_{\{|x-\xi|>d/4\}} dx\int_{\mathbb{R}^n\setminus\Omega_\varepsilon} dy\,w_\xi^{p-1}(x)\Gamma(x-y)\Pi_\varepsilon(x,\xi) \\
&=: I_1+I_2.
\end{aligned}\tag{3.5}$$

Now, thanks to (3.3), we have that $\Pi_\varepsilon(x,\xi)\leqslant w_\xi(x)$, and so

$$w_\xi^{p-1}(x)\,\Pi_\varepsilon(x,\xi)\leqslant w_\xi^p(x).\tag{3.6}$$

Therefore, $I_1$ can be estimated as follows:

$$\begin{aligned}
I_1 &\leqslant Cd^{-(n+2s)p}\int_{\{|x-\xi|\leqslant d/4\}} dx\int_{\mathbb{R}^n\setminus\Omega_\varepsilon} dy\,w_\xi^p(x)\,\Gamma(x-y) \\
&\leqslant Cd^{-(n+2s)p}\int_{\{|x-\xi|\leqslant d/4\}} dx\int_{\mathbb{R}^n\setminus B_{d/2}(\xi)} dy\,w_\xi^p(x)\,\Gamma(x-y).
\end{aligned}$$

We notice that, in the above domain,

$$|x-y| \geqslant |y-\xi| - |x-\xi| \geqslant \frac{d}{2} - \frac{d}{4} = \frac{d}{4},$$

hence

$$\Gamma(x-y) \leqslant \frac{C}{|x-y|^{n+2s}}.$$

Now, we can compute in polar coordinates the following integral

$$\int_{\mathbb{R}^n \setminus B_{d/2}(\xi)} \frac{1}{|x-y|^{n+2s}} dy \leqslant \frac{C}{d^{2s}},$$

up to renaming the constant $C$. This and the fact that $w_\xi^p$ is integrable give

$$\begin{aligned} I_1 &\leqslant C_1 d^{-(n+2s)p} \int_{\{|x-\xi|\leqslant d/4\}} dx \int_{\mathbb{R}^n \setminus B_{d/2}(\xi)} dy \, \frac{w_\xi^p(x)}{|x-y|^{n+2s}} \\ &\leqslant C_2 d^{-(n+2s)p} d^{-2s} \int_{\{|x-\xi|\leqslant d/4\}} w_\xi^p(x)\, dx \\ &\leqslant C_3 d^{-(n+2s)p} d^{-2s}, \end{aligned} \tag{3.7}$$

for suitable $C_1, C_2, C_3 > 0$. Now, if $|x-\xi| > d/4$ then, thanks to (1.4),

$$|w_\xi(x)| = |w(x-\xi)| \leqslant C|x-\xi|^{-(n+2s)}.$$

This together with (3.6) and (2.10) implies that

$$\begin{aligned} I_2 &\leqslant C d^{-(n+2s)p} \int_{\{|x-\xi|>d/4\}} dx \int_{\mathbb{R}^n \setminus \Omega_\varepsilon} dy \, w_\xi^p(x) \Gamma(x-y) \\ &\leqslant C' d^{-(n+2s)p} \int_{\{|x-\xi|>d/4\}} dx \int_{\mathbb{R}^n \setminus \Omega_\varepsilon} dy \, \frac{\Gamma(x-y)}{|x-\xi|^{(n+2s)p}} \\ &\leqslant C'' d^{-(n+2s)p} d^{-(n+2s)p+n}, \end{aligned} \tag{3.8}$$

for suitable $C', C'' > 0$, where in the last inequality we have computed the integral in $dx$ in polar coordinates and used (2.10). Putting together (3.7) and (3.8) and recalling (3.5) we get the desired estimate. $\square$

**Lemma 3.4.** *Let $\xi \in \Omega_\varepsilon$ and $\tilde{p} := \min\{p, 2\}$. Assume that $d := \mathrm{dist}(\xi, \partial\Omega_\varepsilon) \geqslant 2$. Then*

$$\int_{\Omega_\varepsilon} w_\xi^{p-1}(x)\, \Pi_\varepsilon^2(x,\xi)\, dx \leqslant \frac{C}{d^{\tilde{p}(n+2s)+2s}},$$

*for a suitable $C > 0$ that depends on $n$, $p$, $s$ and $\Omega$.*

*Proof.* First we observe that

$$\begin{aligned} &\text{if } p \geqslant 2 \text{ then } 2(n+4s) \geqslant \tilde{p}(n+4s) = \tilde{p}(n+2s+2s) > \tilde{p}(n+2s) + 2s, \\ &\text{if } 1 < p < 2 \text{ then } p(n+4s) \geqslant \tilde{p}(n+4s) = \tilde{p}(n+2s+2s) > \tilde{p}(n+2s) + 2s, \\ &\qquad (n+2s)(p+1) - n = p(n+2s) + 2s \geqslant \tilde{p}(n+2s) + 2s. \end{aligned} \tag{3.9}$$

Now, we can write the integral that we want to estimate as

$$\begin{aligned}
&\int_{\Omega_\varepsilon} w_\xi^{p-1}(x)\,\Pi_\varepsilon^2(x,\xi)\,dx \\
&= \int_{\{|x-\xi|\leqslant d/8\}} w_\xi^{p-1}(x)\,\Pi_\varepsilon^2(x,\xi)\,dx + \int_{\{|x-\xi|>d/8\}} w_\xi^{p-1}(x)\,\Pi_\varepsilon^2(x,\xi)\,dx \\
&=: I_1 + I_2.
\end{aligned} \tag{3.10}$$

If $p \geqslant 2$, to estimate $I_1$ we use Lemma 2.5 together with the fact that $w_\xi^{p-1}$ is integrable to get

$$I_1 \leqslant \frac{C}{d^{2(n+4s)}}. \tag{3.11}$$

If $1 < p < 2$, we notice that, thanks to (3.3), $\Pi_\varepsilon(x,\xi) \leqslant w_\xi(x)$ and so

$$w_\xi^{p-1}(x)\,\Pi_\varepsilon^2(x,\xi) = w_\xi^{p-1}(x)\,\Pi_\varepsilon^{2-p}(x,\xi)\,\Pi_\varepsilon^{p}(x,\xi) \leqslant w_\xi^{p-1}(x)\,w_\xi^{2-p}(x)\,\Pi_\varepsilon^{p}(x,\xi) = w_\xi(x)\,\Pi_\varepsilon^{p}(x,\xi).$$

Therefore, using again Lemma 2.5 and the fact that $w_\xi$ is integrable, we obtain

$$\begin{aligned}
I_1 &\leqslant \int_{\{|x-\xi|\leqslant d/8\}} w_\xi(x)\,\Pi_\varepsilon^p(x,\xi)\,dx \\
&\leqslant \frac{C}{d^{p(n+4s)}} \int_{\{|x-\xi|\leqslant d/8\}} w_\xi(x)\,dx \\
&\leqslant \frac{C}{d^{p(n+4s)}}.
\end{aligned} \tag{3.12}$$

To estimate $I_2$, we use (3.3) to obtain that $\Pi_\varepsilon(x,\xi) \leqslant w_\xi(x)$, and so $w_\xi^{p-1}(x)\Pi_\varepsilon^2(x,\xi) \leqslant w_\xi^{p+1}(x)$. This implies that

$$I_2 \leqslant \int_{\{|x-\xi|>d/8\}} w_\xi^{p+1}(x)\,dx.$$

Since $|x-\xi| > d/8$, thanks to (1.4), we have that

$$|w_\xi(x)| = |w(x-\xi)| \leqslant C|x-\xi|^{-(n+2s)}.$$

Therefore, computing the integral in polar coordinates,

$$I_2 \leqslant \int_{\{|x-\xi|>d/8\}} \frac{C}{|x-\xi|^{(n+2s)(p+1)}}\,dx \leqslant \frac{C}{d^{(n+2s)(p+1)-n}}. \tag{3.13}$$

Putting together (3.11), (3.12) and (3.13) and recalling (3.10), we obtain the result (one can use (3.9) to obtain a simpler common exponent). $\square$

**Lemma 3.5.** *Let $\delta \in (0,1)$. Let $\xi \in \Omega_\varepsilon$ such that $d := \mathrm{dist}(\xi, \partial\Omega_\varepsilon) \geqslant \delta/\varepsilon$. Then*

$$\int_{\Omega_\varepsilon} w_\xi^{p-1}(x)\,\Lambda_\xi^2(x)\,dx \leqslant C\,\varepsilon^{2p(n+2s)-n},$$

*for a suitable $C > 0$ that depends on $n$, $p$, $s$, $\delta$ and $\Omega$.*

*Proof.* We use Lemma 3.1 and the fact that $\Omega$ is bounded to obtain that

$$\begin{aligned}
\int_{\Omega_\varepsilon} w_\xi^{p-1}(x)\,\Lambda_\xi^2(x)\,dx &\leqslant \frac{C}{d^{2p(n+2s)}} \int_{\Omega_\varepsilon} w_\xi^{p-1}(x)\,dx \\
&\leqslant \frac{C'}{d^{2p(n+2s)}}|\Omega_\varepsilon| \leqslant \frac{C''}{d^{2p(n+2s)}\varepsilon^n} \leqslant \frac{C''\,\varepsilon^{2p(n+2s)}}{\delta^{2p(n+2s)}\varepsilon^n},
\end{aligned}$$

for suitable $C', C'' > 0$. This implies the desired estimate. $\square$

## 4. ENERGY ESTIMATES AND FUNCTIONAL EXPANSION IN $\bar{u}_\xi$

In this section we make some estimates for the energy functional (1.10). For this, we consider the functional associated to problem (1.3):
(4.1)
$$I(u) = \frac{1}{2}\int_{\mathbb{R}^n} \big((-\Delta)^s u(x)\, u(x) + u^2(x)\big)\, dx - \frac{1}{p+1}\int_{\mathbb{R}^n} u^{p+1}(x)\, dx, \qquad u \in H^s(\mathbb{R}^n).$$

**Theorem 4.1.** *Fix $\delta \in (0,1)$ and $\xi \in \Omega_\varepsilon$ such that $d := \operatorname{dist}(\xi, \partial\Omega_\varepsilon) \geqslant \delta/\varepsilon$. Then, we have*

$$I_\varepsilon(\bar{u}_\xi) = I(w) + \frac{1}{2}\mathcal{H}_\varepsilon(\xi) + o(\varepsilon^{n+4s}), \tag{4.2}$$

*as $\varepsilon \to 0$, where $I$ is given by (4.1), $w$ is the solution to (1.3) and $\mathcal{H}_\varepsilon(\xi)$ is defined in (1.17), as long as $\delta$ is sufficiently small.*

The following simple observation will be used often in the sequel:

**Lemma 4.2.** *Let $\delta \geqslant 1$ and $q > 1$. Then*

$$\int_{\mathbb{R}^n \setminus B_\delta(\xi)} w_\xi^q(z)\, dz \leqslant \frac{C}{\delta^{n(q-1)+2sq}},$$

*for some $C > 0$.*

*Proof.* First of all we observe that

$$n - 1 - (n+2s)q < n - 1 - (n+2s) = -1 - 2s < -1$$

and therefore

$$\int_\delta^{+\infty} \rho^{n-1-(n+2s)q}\, d\rho = \frac{\delta^{n-(n+2s)q}}{(n+2s)q - n}. \tag{4.3}$$

Now, we use (1.4) to see that

$$\int_{\mathbb{R}^n \setminus B_\delta(\xi)} w_\xi^q(z)\, dz \leqslant \int_{\mathbb{R}^n \setminus B_\delta(\xi)} \frac{C}{|x-\xi|^{(n+2s)q}}\, dz = C' \int_\delta^{+\infty} \rho^{n-1-(n+2s)q}\, d\rho$$

for some $C' > 0$. This and (4.3) imply the desired result. □

**Corollary 4.3.** *Let $\xi \in \Omega_\varepsilon$, with $d := \operatorname{dist}(\xi, \partial\Omega_\varepsilon) \geqslant 1$. Then*

$$\int_{\mathbb{R}^n \setminus \Omega_\varepsilon} w_\xi^{p+1}(z)\, dz \leqslant \frac{C}{d^{np+2s(p+1)}},$$

*for some $C > 0$.*

*Proof.* Notice that $(\mathbb{R}^n \setminus \Omega_\varepsilon) \subseteq (\mathbb{R}^n \setminus B_d(\xi))$ and exploit Lemma 4.2. □

*Proof of Theorem 4.1.* Using (1.9) and (3.2), we have

$$
\begin{aligned}
I_\varepsilon(\bar u_\xi) &= \frac{1}{2}\int_{\Omega_\varepsilon} \left((-\Delta)^s \bar u_\xi(x) + \bar u_\xi(x)\right) \bar u_\xi(x)\,dx - \frac{1}{p+1}\int_{\Omega_\varepsilon} \bar u_\xi^{p+1}(x)\,dx \\
&= \frac{1}{2}\int_{\Omega_\varepsilon} w_\xi^p(x)\,\bar u_\xi(x)\,dx - \frac{1}{p+1}\int_{\Omega_\varepsilon} \bar u_\xi^{p+1}(x)\,dx \\
&= \frac{1}{2}\int_{\Omega_\varepsilon} w_\xi^p(x)\left(w_\xi(x) - \Lambda_\xi(x) - \Pi_\varepsilon(x,\xi)\right)dx \\
&\qquad - \frac{1}{p+1}\int_{\Omega_\varepsilon} \left(w_\xi(x) - \Lambda_\xi(x) - \Pi_\varepsilon(x,\xi)\right)^{p+1} dx \\
&= \left(\frac{1}{2} - \frac{1}{p+1}\right)\int_{\Omega_\varepsilon} w_\xi^{p+1}(x)\,dx - \frac{1}{2}\int_{\Omega_\varepsilon} w_\xi^p(x)\left(\Lambda_\xi(x) + \Pi_\varepsilon(x,\xi)\right)dx \\
&\qquad + \frac{1}{p+1}\int_{\Omega_\varepsilon} \left[w_\xi^{p+1}(x) - \left(w_\xi(x) - \Lambda_\xi(x) - \Pi_\varepsilon(x,\xi)\right)^{p+1}\right] dx.
\end{aligned}
\tag{4.4}
$$

We notice that the first term in the right hand side of (4.4) can be written as

$$
\begin{aligned}
&\left(\frac{1}{2} - \frac{1}{p+1}\right)\int_{\Omega_\varepsilon} w_\xi^{p+1}(x)\,dx \\
= &\left(\frac{1}{2} - \frac{1}{p+1}\right)\int_{\mathbb{R}^n} w_\xi^{p+1}(x)\,dx - \left(\frac{1}{2} - \frac{1}{p+1}\right)\int_{\mathbb{R}^n\setminus\Omega_\varepsilon} w_\xi^{p+1}(x)\,dx \\
= &\; I(w) - \left(\frac{1}{2} - \frac{1}{p+1}\right)\int_{\mathbb{R}^n\setminus\Omega_\varepsilon} w_\xi^{p+1}(x)\,dx,
\end{aligned}
$$

since $w$ is a solution to (1.3). Therefore,

$$
\begin{aligned}
I_\varepsilon(\bar u_\xi) &= I(w) - \left(\frac{1}{2} - \frac{1}{p+1}\right)\int_{\mathbb{R}^n\setminus\Omega_\varepsilon} w_\xi^{p+1}(x)\,dx \\
&\qquad - \frac{1}{2}\int_{\Omega_\varepsilon} w_\xi^p(x)\left(\Lambda_\xi(x) + \Pi_\varepsilon(x,\xi)\right)dx \\
&\qquad + \frac{1}{p+1}\int_{\Omega_\varepsilon} \left[w_\xi^{p+1}(x) - \left(w_\xi(x) - \Lambda_\xi(x) - \Pi_\varepsilon(x,\xi)\right)^{p+1}\right] dx \\
&=: I(w) - \left(\frac{1}{2} - \frac{1}{p+1}\right) J_1 - \frac{1}{2} J_2 + \frac{1}{p+1} J_3,
\end{aligned}
\tag{4.5}
$$

where

$$
\begin{aligned}
J_1 &:= \int_{\mathbb{R}^n\setminus\Omega_\varepsilon} w_\xi^{p+1}(x)\,dx, \\
J_2 &:= \int_{\Omega_\varepsilon} w_\xi^p(x)\left(\Lambda_\xi(x) + \Pi_\varepsilon(x,\xi)\right)dx \\
\text{and} \qquad J_3 &:= \int_{\Omega_\varepsilon} \left[w_\xi^{p+1}(x) - \left(w_\xi(x) - \Lambda_\xi(x) - \Pi_\varepsilon(x,\xi)\right)^{p+1}\right] dx...
\end{aligned}
$$

Now, we estimate separately $J_1$, $J_2$ and $J_3$. Thanks to Corollary 4.3, we have that

$$
J_1 = \int_{\mathbb{R}^n\setminus\Omega_\varepsilon} w_\xi^{p+1}(x)\,dx \leqslant \frac{C}{d^{np+2s(p+1)}} \leqslant \frac{C}{\delta^{np+2s(p+1)}}\varepsilon^{np+2s(p+1)}.
\tag{4.6}
$$

Concerning $J_2$, we write it as

$$J_2 = J_{21} + J_{22},$$

where

$$\begin{aligned} J_{21} &:= \int_{\Omega_\varepsilon} w_\xi^p(x)\,\Lambda_\xi(x)\,dx, \\ J_{22} &:= \int_{\Omega_\varepsilon} w_\xi^p(x)\,\Pi_\varepsilon(x,\xi)\,dx. \end{aligned} \tag{4.7}$$

Recalling the definition of $\Lambda_\xi$ in (3.1) and the estimate in (1.4), we have that

(4.8)

$$\begin{aligned} J_{21} &= \int_{\Omega_\varepsilon} dx \int_{\mathbb{R}^n\setminus\Omega_\varepsilon} dy\, w_\xi^p(x)\, w_\xi^p(y)\,\Gamma(x-y) \\ &\leqslant C \int_{\Omega_\varepsilon} dx \int_{\mathbb{R}^n\setminus\Omega_\varepsilon} dy\, w_\xi^p(x) \frac{\Gamma(x-y)}{|y-\xi|^{(n+2s)p}} \\ &\leqslant \frac{C}{d^{(n+2s)p}} \int_{\Omega_\varepsilon} dx \int_{\mathbb{R}^n\setminus\Omega_\varepsilon} dy\, w_\xi^p(x)\,\Gamma(x-y) \\ &= \frac{C}{d^{(n+2s)p}} \left( \int_{\Omega_\varepsilon} dx \int_{\substack{\mathbb{R}^n\setminus\Omega_\varepsilon \\ \{|x-y|\leqslant d/2\}}} dy\, w_\xi^p(x)\,\Gamma(x-y) + \int_{\Omega_\varepsilon} dx \int_{\substack{\mathbb{R}^n\setminus\Omega_\varepsilon \\ \{|x-y|> d/2\}}} dy\, w_\xi^p(x)\,\Gamma(x-y) \right). \end{aligned}$$

We notice that, if $x \in \Omega_\varepsilon$ and $y \in \mathbb{R}^n \setminus \Omega_\varepsilon$ with $|x-y| \leqslant d/2$, then

$$|x-\xi| \geqslant |y-\xi| - |x-y| \geqslant d - \frac{d}{2} = \frac{d}{2}.$$

Therefore, using (1.4), (2.10) and the fact that $\Omega$ is bounded, we have

$$\begin{aligned} \int_{\Omega_\varepsilon} dx \int_{\substack{\mathbb{R}^n\setminus\Omega_\varepsilon \\ \{|x-y|\leqslant d/2\}}} dy\, w_\xi^p(x)\,\Gamma(x-y) &\leqslant C' \int_{\Omega_\varepsilon} dx \int_{\substack{\mathbb{R}^n\setminus\Omega_\varepsilon \\ \{|x-y|\leqslant d/2\}}} dy\, \frac{\Gamma(x-y)}{|x-\xi|^{(n+2s)p}} \\ &\leqslant C' \int_{\Omega_\varepsilon} dx \int_{\mathbb{R}^n} d\tilde{y} \frac{\Gamma(\tilde{y})}{|x-\xi|^{(n+2s)p}} \\ &\leqslant C''(1/d)^{(n+2s)p} |\Omega_\varepsilon| \\ &\leqslant \frac{C'''}{d^{(n+2s)p}\varepsilon^n} \\ &\leqslant \frac{C'''}{\delta^{(n+2s)p}} \varepsilon^{(n+2s)p-n}, \end{aligned} \tag{4.9}$$

for suitable constants $C', C'', C''' > 0$. Moreover, if $|x-y| > d/2$, we use (1.12) to get

$$\begin{aligned} \int_{\Omega_\varepsilon} dx \int_{\substack{\mathbb{R}^n\setminus\Omega_\varepsilon \\ \{|x-y|> d/2\}}} dy\, w_\xi^p(x)\,\Gamma(x-y) &\leqslant C \int_{\mathbb{R}^n} dx \int_{\substack{\mathbb{R}^n\setminus\Omega_\varepsilon \\ \{|x-y|> d/2\}}} dy\, \frac{w_\xi^p(x)}{|x-y|^{n+2s}} \\ &\leqslant \tilde{C} d^{-2s} \leqslant \frac{\tilde{C}}{\delta^{2s}} \varepsilon^{2s}, \end{aligned} \tag{4.10}$$

for some $\tilde{C} > 0$. Putting together (4.9) and (4.10) and recalling (4.8), we obtain

$$
\begin{aligned}
J_{21} &\leqslant \frac{C}{d^{(n+2s)p}} \left( \frac{C'''}{\delta^{(n+2s)p}} \varepsilon^{(n+2s)p-n} + \frac{\tilde{C}}{\delta^{2s}} \varepsilon^{2s} \right) \\
&\leqslant \frac{C}{\delta^{(n+2s)p}} \varepsilon^{(n+2s)p} \left( \frac{C'''}{\delta^{(n+2s)p}} \varepsilon^{(n+2s)p-n} + \frac{\tilde{C}}{\delta^{2s}} \varepsilon^{2s} \right) \leqslant \hat{C} \varepsilon^{np+2s(p+1)},
\end{aligned} \tag{4.11}
$$

for suitable $\hat{C} > 0$. Therefore,

$$
J_2 = J_{22} + o(\varepsilon^{n+4s}) = \int_{\Omega_\varepsilon} w_\xi^p(x)\, \Pi_\varepsilon(x,\xi)\, dx + o(\varepsilon^{n+4s}). \tag{4.12}
$$

To estimate $J_3$ we expand $w_\xi^{p+1}(x)$ in the following way

$$
w_\xi^{p+1}(x) = \bar{u}_\xi^{p+1}(x) + (p+1) w_\xi^p(x)(w_\xi(x) - \bar{u}_\xi(x)) + c_p \alpha_\xi^{p-1}(x)(w_\xi(x) - \bar{u}_\xi(x))^2,
$$

where $0 \leqslant \bar{u}_\xi \leqslant \alpha_\xi \leqslant w_\xi$ and $c_p$ is a positive constant depending only on $p$. Therefore, recalling (3.2) and (4.7),

(4.13)

$$
\begin{aligned}
J_3 &= \int_{\Omega_\varepsilon} \left[ w_\xi^{p+1}(x) - (w_\xi(x) - \Lambda_\xi(x) - \Pi_\varepsilon(x,\xi))^{p+1} \right] dx \\
&= \int_{\Omega_\varepsilon} \left[ w_\xi^{p+1}(x) - \bar{u}_\xi^{p+1}(x) \right] dx \\
&= (p+1) \int_{\Omega_\varepsilon} w_\xi^p(x) \left( \Lambda_\xi(x) + \Pi_\varepsilon(x,\xi) \right) dx + c_p \int_{\Omega_\varepsilon} \alpha_\xi^{p-1}(x) \left( \Lambda_\xi(x) + \Pi_\varepsilon(x,\xi) \right)^2 dx \\
&= (p+1)(J_{21} + J_{22}) + c_p \int_{\Omega_\varepsilon} \alpha_\xi^{p-1}(x) \left( \Lambda_\xi(x) + \Pi_\varepsilon(x,\xi) \right)^2 dx.
\end{aligned}
$$

Since $\alpha_\xi(x) \leqslant w_\xi(x)$, we have that

$$
\begin{aligned}
&\int_{\Omega_\varepsilon} \alpha_\xi^{p-1}(x) \left( \Lambda_\xi(x) + \Pi_\varepsilon(x,\xi) \right)^2 dx \\
\leqslant &\int_{\Omega_\varepsilon} w_\xi^{p-1}(x) \left( \Lambda_\xi(x) + \Pi_\varepsilon(x,\xi) \right)^2 dx \\
= &\int_{\Omega_\varepsilon} w_\xi^{p-1}(x)\, \Lambda_\xi^2(x)\, dx + \int_{\Omega_\varepsilon} w_\xi^{p-1}(x)\, \Pi_\varepsilon^2(x,\xi)\, dx + 2 \int_{\Omega_\varepsilon} w_\xi^{p-1}(x)\, \Lambda_\xi(x)\, \Pi_\varepsilon(x,\xi)\, dx.
\end{aligned}
$$

Hence, from Lemmata 3.3, 3.4 and 3.5 (together with the fact that $d \geqslant \delta/\varepsilon$) we deduce that

$$
\int_{\Omega_\varepsilon} \alpha_\xi^{p-1}(x) \left( \Lambda_\xi(x) + \Pi_\varepsilon(x,\xi) \right)^2 dx \leqslant C_\delta (\varepsilon^{2p(n+2s)-n} + \varepsilon^{(n+2s)\tilde{p}+2s} + \varepsilon^{(n+2s)p+2s}),
$$

for some $C_\delta$, where $\tilde{p} = \min\{p, 2\}$. The last estimate, (4.11) and (4.13) give

$$
J_3 = (p+1) J_{22} + o(\varepsilon^{n+4s}) = (p+1) \int_{\Omega_\varepsilon} w_\xi^p(x)\, \Pi_\varepsilon(x,\xi)\, dx + o(\varepsilon^{n+4s}). \tag{4.14}
$$

Putting together (4.6), (4.12) and (4.14) and using (4.5), we get

$$
I_\varepsilon(\bar{u}_\xi) = I(w) + \frac{1}{2} \int_{\Omega_\varepsilon} w_\xi^p(x)\, \Pi_\varepsilon(x,\xi)\, dx + o(\varepsilon^{n+4s}).
$$

Thus, recalling the definitions of $\Pi_\varepsilon$ and $\mathcal{H}_\varepsilon$ in (2.18) and (1.17) respectively, we obtain (4.2). $\square$

## 5. Decay of the ground state $w$

In this section we recall some basic (though not optimal) decay properties of the ground state and of its derivatives.

For this, we start with a general convolution result:

**Lemma 5.1.** *Let $a, b > n$, $C_a, C_b > 0$ and $f, g \in L^\infty(\mathbb{R}^n)$, with*

$$|f(x)| \leqslant C_a\,(1+|x|)^{-a} \text{ and } |g(x)| \leqslant C_b\,(1+|x|)^{-b}.$$

*Then there exists $C > 0$ such that*

$$\big|(f * g)(x)\big| \leqslant C\,(1+|x|)^{-c},$$

*with $c := \min\{a, b\}$.*

*Proof.* Fix $x \in \mathbb{R}^n$ and $r := |x|/2$, and observe that if $y \in B_r(x)$ then

$$|y| \geqslant |x| - |x-y| \geqslant |x| - r = \frac{|x|}{2}.$$

As a consequence

$$\begin{aligned}\int_{B_r(x)} \frac{C_a}{(1+|x-y|)^a}\,\frac{C_b}{(1+|y|)^b}\,dy &\leqslant \int_{B_r(x)} \frac{C_a}{(1+|x-y|)^a}\,\frac{C_b}{(1+(|x|/2))^b}\,dy \\ &\leqslant \frac{C}{(1+|x|)^b}\int_{\mathbb{R}^n} \frac{1}{(1+|x-y|)^a}\,dy \leqslant \frac{C}{(1+|x|)^b},\end{aligned} \tag{5.1}$$

up to renaming constants. On the other hand, if $y \in \mathbb{R}^n \setminus B_r(x)$ then $|x-y| \geqslant r = |x|/2$, thus

$$\begin{aligned}\int_{\mathbb{R}^n\setminus B_r(x)} \frac{C_a}{(1+|x-y|)^a}\,\frac{C_b}{(1+|y|)^b}\,dy &\leqslant \int_{\mathbb{R}^n\setminus B_r(x)} \frac{C_a}{(1+(|x|/2))^a}\,\frac{C_b}{(1+|y|)^b}\,dy \\ &\leqslant \frac{C}{(1+|x|)^a}\int_{\mathbb{R}^n} \frac{1}{(1+|y|)^b}\,dy \leqslant \frac{C}{(1+|x|)^a}.\end{aligned} \tag{5.2}$$

Putting together (5.1) and (5.2) we obtain the desired result. $\square$

Now we fix $\xi \in \Omega_\varepsilon$ and we define, for any $i \in \{1, \dots, n\}$,

$$Z_i := \frac{\partial w_\xi}{\partial x_i}, \tag{5.3}$$

where $w_\xi$ is the ground state solution centered at $\xi$. Moreover, we denote by $\mathcal{Z}$ the linear space spanned by the functions $Z_i$.

We prove first the following lemmata:

**Lemma 5.2.** *There exists a positive constant $C$ such that, for any $i = 1, \dots, n$,*

$$|Z_i| \leqslant C|x-\xi|^{-\nu_1}, \text{ for any } |x-\xi| \geqslant 1,$$

*where $\nu_1 := \min\{(n+2s+1),\, p(n+2s)\}$.*

*Proof.* Given $R > 0$, we take $\Gamma_{1,R} \in C^\infty(\mathbb{R}^n)$, with $0 \leqslant \Gamma_{1,R} \leqslant \Gamma$ in $\mathbb{R}^n$ and $\Gamma_{1,R} = \Gamma$ outside $B_R$, and we define $\Gamma_{2,R} := \Gamma - \Gamma_{1,R}$. We use (1.11) to write

$$w = \Gamma * w^p = \Gamma_{1,R} * w^p + \Gamma_{2,R} * w^p. \tag{5.4}$$

We assume, up to translation, that $\xi = 0$. Then, our goal is to prove that, for any $k \in \mathbb{N}$ we have that

$$\left|\frac{\partial w}{\partial x_i}(x)\right| \leqslant C_k(1+|x|)^{-\nu(k)}, \tag{5.5}$$

where

$$\nu(k) := \min\{(n+2s+1),\ p(n+2s),\ k(p-1)(n+2s)\} = \min\{\nu_1,\ k(p-1)(n+2s)\},$$

for some $C_k > 0$. Indeed, the desired claim would follows from (5.5) simply by taking the smallest $k$ for which $k(p-1) > p$.

To prove (5.5) we perform an inductive argument. So, we first check (5.5) when $k = 0$. For this, we use the fact that $w \in L^\infty(\mathbb{R}^n)$ and that $\Gamma \in L^1(\mathbb{R}^n)$ to find $R > 0$ sufficiently small that

$$\int_{B_R} \Gamma(y)\,dy \leqslant \frac{1}{2p\,\|w\|_{L^\infty(\mathbb{R}^n)}}.$$

This fixes $R$ once and for all for the proof of (5.5) when $k = 0$. Hence, we use the sign of $\Gamma_{1,R}$ and the fact that $\Gamma_{2,R} = 0$ outside $B_R$ to obtain that

$$\int_{\mathbb{R}^n} \Gamma_{2,R}(y)\,dy = \int_{B_R} \Gamma_{2,R}(y)\,dy \leqslant \int_{B_R} \Gamma_{1,R}(y) + \Gamma_{2,R}(y)\,dy \leqslant \frac{1}{2p\,\|w\|_{L^\infty(\mathbb{R}^n)}}. \tag{5.6}$$

Then, for any $t \in (0,1)$, we define $D_t w(x) := \big(w(x+te_i) - w(x)\big)/t$ and we infer from (5.4) that

$$D_t w = (D_t\Gamma_{1,R}) * w^p + \Gamma_{2,R} * (D_t w^p). \tag{5.7}$$

Also, from formula (3.2) of [17] we know that

$$|\nabla\Gamma(x)| \leqslant C\,|x|^{-(n+2s+1)}, \qquad \text{for any } |x| \geqslant 1. \tag{5.8}$$

As a consequence, if $|x| > 2$ and $\eta \in B_1(x)$, we have that

$$|\eta| \geqslant |x| - |x-\eta| \geqslant \frac{|x|}{2} > 1,$$

hence

$$|\Gamma(x+te_1) - \Gamma(x)| \leqslant t \sup_{\eta\in B_1(x)} |\nabla\Gamma(\eta)| \leqslant C\,t\,|x|^{-(n+2s+1)},$$

up to renaming $C$. This gives that $|D_t\Gamma(x)| \leqslant C\,(1+|x|)^{-(n+2s+1)}$, and so $|D_t\Gamma_{1,R}(x)| \leqslant C\,(1+|x|)^{-(n+2s+1)}$. Accordingly, we have that

$$|(D_t\Gamma_{1,R}) * w^p| \leqslant \|w\|^p_{L^\infty(\mathbb{R}^n)} \int_{\mathbb{R}^n} |D_t\Gamma_{1,R}(y)|\,dy \leqslant C. \tag{5.9}$$

Also

$$|w^p(x+te_i) - w^p(x)| \leqslant p\,\|w\|^{p-1}_{L^\infty(\mathbb{R}^n)}\,|w(x+te_i) - w(x)|.$$

This says that

$$|D_t w^p(x)| \leqslant p\,\|w\|^{p-1}_{L^\infty(\mathbb{R}^n)}\,|D_t w(x)|.$$

Moreover

$$|D_t w(x)| \leqslant \frac{2\,\|w\|_{L^\infty(\mathbb{R}^n)}}{t},$$

hence we can define

$$M(t) := \sup_{x\in\mathbb{R}^n} |D_t w(x)|,$$

so we obtain that

$$|D_t w^p(x)| \leqslant p\,\|w\|^{p-1}_{L^\infty(\mathbb{R}^n)}\,M(t),$$

for every $x \in \mathbb{R}^n$, and thus

$$|\Gamma_{2,R} * (D_t w^p)(x)| \leqslant \int_{\mathbb{R}^n} \Gamma_{2,R}(y)\,|D_t w^p(x-y)|\,dy$$
$$\leqslant p\,\|w\|^{p-1}_{L^\infty(\mathbb{R}^n)}\,M(t)\int_{\mathbb{R}^n} \Gamma_{2,R}(y)\,dy \leqslant \frac{M(t)}{2},$$

thanks to (5.6). Using this and (5.9) into (5.7), we conclude that

$$D_t w \leqslant C + \frac{M(t)}{2}.$$

By taking the supremum, we obtain that

$$M(t) \leqslant C + \frac{M(t)}{2},$$

and this gives, up to renaming $C$, that $M(t) \leqslant C$. By sending $t \searrow 0$, we complete the proof of (5.5) when $k = 0$.

Now we suppose that (5.5) holds true for some $k$ and we prove it for $k+1$. The proof is indeed similar to the case $k = 0$: here we take $R := 1$ and use the short notation $\Gamma_1 := \Gamma_{1,R}$ and $\Gamma_2 := \Gamma_{2,R}$. By (5.5) for $k = 0$ and the regularity theory (applied to the equation for $D_t w$), we know that $w \in C^1(\mathbb{R})$, hence we can differentiate (5.4) and obtain that

$$\frac{\partial w}{\partial x_i} = \frac{\partial \Gamma_1}{\partial x_i} * w^p + \Gamma_2 * \left( p w^{p-1} \frac{\partial w}{\partial x_i} \right). \tag{5.10}$$

So, we use (1.4), (5.8) and Lemma 5.1 to obtain

$$\left| \frac{\partial \Gamma_1}{\partial x_i} * w^p(x) \right| \leqslant C(1+|x|)^{-\min\{(n+2s+1),\, p(n+2s)\}}. \tag{5.11}$$

Moreover, we notice that

$$(p-1)(n+2s) + \nu(k) = \min\{(p-1)(n+2s) + \nu_1,\ (k+1)(p-1)(n+2s)\}$$
$$\geqslant \min\{\nu_1,\ (k+1)(p-1)(n+2s)\} = \nu(k+1).$$

Hence, using (5.5) for $k$ and (1.4) we see that

$$\left| p w^{p-1} \frac{\partial w}{\partial x_i}(x) \right| \leqslant C(1+|x|)^{-(p-1)(n+2s)-\nu(k)} \leqslant C(1+|x|)^{-\nu(k+1)}, \tag{5.12}$$

up to renaming constants (possibly depending on $p$). Now, we observe that

$$\text{if } x \in \mathbb{R}^n \text{ and } |y| < 1 \text{ then } 1 + |x-y| \geqslant \frac{1}{3}(1+|x|). \tag{5.13}$$

Indeed, if $|x| \geqslant 2$ and $|y| < 1$, then

$$|x-y| \geqslant |x| - |y| \geqslant \frac{|x|}{2},$$

which implies (5.13) in this case. If instead $|x| < 2$ and $|y| < 1$, we have that

$$1 + |x| < 3 < 3(1+|x-y|),$$

and this finishes the proof of (5.13).

Therefore, since $\Gamma_2$ vanishes outside $B_1$, using (5.12) and (5.13), we have

$$\begin{aligned}\left|\Gamma_2 * \left(pw^{p-1}\frac{\partial w}{\partial x_i}\right)(x)\right| &\leqslant C\int_{B_1}\frac{\Gamma_2(y)}{(1+|x-y|)^{\nu(k+1)}}\,dy\\ &\leqslant C\int_{B_1}\frac{\Gamma_2(y)}{(1+|x|)^{\nu(k+1)}}\,dy \leqslant \frac{C}{(1+|x|)^{\nu(k+1)}}\int_{\mathbb{R}^n}\Gamma(y)\,dy = \frac{C}{(1+|x|)^{\nu(k+1)}}.\end{aligned}$$

This and (5.11) establish (5.5) for $k+1$, thus completing the inductive argument. □

**Lemma 5.3.** *There exists a positive constant $C$ such that, for any $i=1,\dots,n$,*

$$|\nabla Z_i| \leqslant C|x-\xi|^{-\nu_2}, \text{ for any } |x-\xi|\geqslant 1,$$

*where $\nu_2 := \min\{(n+2s+2),\, p(n+2s)\}$.*

*Proof.* From formula (3.2) of [17] we know that

$$|D^2\Gamma(x)| \leqslant C|x|^{-(n+2s+2)}, \quad |x|\geqslant 1. \tag{5.14}$$

Hence the proof of Lemma 5.3 follows as the one of Lemma 5.2 by using (5.14) instead of (5.8). □

**Lemma 5.4.** *For any $k\in\mathbb{N}$ there exists a positive constant $C_k$ such that, for any $i=1,\dots,n$,*

$$|D^k Z_i| \leqslant C_k|x-\xi|^{-n}, \text{ for any } |x-\xi|\geqslant 1.$$

*Proof.* From Lemma C.1(ii) of [21], we have that

$$|D^{k+1}\Gamma(x)| \leqslant C_k|x|^{-n}, \quad |x|\geqslant 1. \tag{5.15}$$

The proof of Lemma 5.4 follows as the one of Lemma 5.2 by using (5.15) instead of (5.8). □

We notice that

$$\int_{\mathbb{R}^n} Z_i^2\,dx = \int_{\mathbb{R}^n} Z_j^2\,dx \text{ for any } i,j=1,\dots,n. \tag{5.16}$$

We set

$$\alpha := \int_{\mathbb{R}^n} Z_1^2\,dx, \tag{5.17}$$

and so, thanks to (5.16), we observe that

$$\int_{\mathbb{R}^n} Z_i^2\,dx = \alpha \text{ for any } i=1,\dots,n. \tag{5.18}$$

**Lemma 5.5.** *The $Z_i$'s satisfy the following condition*

$$\int_{\mathbb{R}^n} Z_i\,Z_j\,dx = \alpha\,\delta_{ij}. \tag{5.19}$$

*Also, if $\tau_o\in L^\infty([0,+\infty))$, $\tau(x) := \tau_o(|x-\xi|)$ for any $x\in\mathbb{R}^n$ and $\tilde Z_i := \tau Z_i$, then*

$$\int_{\mathbb{R}^n} \tilde Z_i\,Z_j\,dx = \tilde\alpha\,\delta_{ij}, \tag{5.20}$$

*where*[1]

$$\tilde\alpha := \int_{\mathbb{R}^n} \tilde Z_1\,Z_1\,dx... \tag{5.21}$$

[1] In particular, we note that, if $\tau_o$ has a sign and does not vanish identically then $\tilde\alpha\neq 0$ (and we will often implicitly assume that this is so in the sequel).

*Proof.* We first observe that the function $w$ is radial (see, for instance, [17]) and therefore, recalling the definition of $Z_i$ in (5.3), we have that

$$Z_i = \frac{\partial w}{\partial x_i}(x-\xi) = w'_\xi(|x-\xi|)\,\frac{x_i-\xi_i}{|x-\xi|}.$$

Hence, using the change of variable $y = x - \xi$, for any $i, j = 1, \dots, n$, we have

$$\begin{aligned}\int_{\mathbb{R}^n} \tilde{Z}_i\, Z_j\, dx &= \int_{\mathbb{R}^n} \tau_o(|x-\xi|)\, |w'(|x-\xi|)|^2 \frac{(x_i-\xi_i)(x_j-\xi_j)}{|x-\xi|^2}\, dx \\ &= \int_{\mathbb{R}^n} \tau_o(|y|)\, |w'(|y|)|^2 \frac{y_i\, y_j}{|y|^2}\, dy.\end{aligned} \tag{5.22}$$

Therefore, if $i \neq j$,

$$\int_{\mathbb{R}^n} \tilde{Z}_i\, Z_j\, dx = \int_{\mathbb{R}^{n-1}} y_j \left( \int_{\mathbb{R}} \tau_o(|y|)\, |w'(|y|)|^2 \frac{y_i}{|y|^2}\, dy_i \right) dy' = 0,$$

since the function $\tau_o(|y|)\, |w'(|y|)|^2 \frac{y_i}{|y|^2}$ is odd. This proves (5.20) when $i \neq j$. On the other hand, if $i = j$, formula (5.22) becomes

$$\int_{\mathbb{R}^n} \tilde{Z}_i\, Z_i\, dx = \int_{\mathbb{R}^n} \tau_o(|y|)\, |w'(|y|)|^2 \frac{y_i^2}{|y|^2}\, dy.$$

We observe that the latter integral is invariant under rotation, hence

$$\int_{\mathbb{R}^n} \tilde{Z}_i\, Z_i\, dx = \int_{\mathbb{R}^n} \tau_o(|y|)\, |w'(|y|)|^2 \frac{y_1^2}{|y|^2}\, dy = \tilde{\alpha}.$$

This establishes (5.20) also when $i = j$. Then, (5.19) follows from (5.20) by choosing $\tau_o := 1$ and comparing (5.18) and (5.21). □

**Corollary 5.6.** *The $Z_i$'s satisfy the following condition*

$$\int_{\Omega_\varepsilon} Z_i\, Z_j\, dx = \alpha \delta_{ij} + O(\varepsilon^\nu),$$

*with $\nu > n + 4s$.*

*Proof.* From Lemma 5.5,we have that

$$\begin{aligned}\int_{\Omega_\varepsilon} Z_i\, Z_j\, dx &= \int_{\mathbb{R}^n} Z_i\, Z_j\, dx - \int_{\mathbb{R}^n \setminus \Omega_\varepsilon} Z_i\, Z_j\, dx \\ &= \alpha \delta_{ij} - \int_{\mathbb{R}^n \setminus \Omega_\varepsilon} Z_i\, Z_j\, dx.\end{aligned}$$

Moreover, from Lemma 5.2, we obtain that

$$\int_{\mathbb{R}^n \setminus \Omega_\varepsilon} Z_i\, Z_j\, dx \leqslant C \int_{\mathbb{R}^n \setminus \Omega_\varepsilon} \frac{1}{|x-\xi|^{2\nu_1}}\, dx \leqslant C\, \varepsilon^{2\nu_1 - n},$$

which implies the desired result. □

## 6. SOME REGULARITY ESTIMATES

Here we perform some uniform estimates on the solutions of our differential equations. For this, we introduce some notation: given $\xi \in \Omega_\varepsilon$ with

$$\operatorname{dist}(\xi, \partial\Omega_\varepsilon) \geqslant \frac{c}{\varepsilon}, \text{ for some } c \in (0,1), \tag{6.1}$$

and $\frac{n}{2} < \mu < n + 2s$, we define, for any $x \in \mathbb{R}^n$,

$$\rho_\xi(x) := \frac{1}{(1 + |x - \xi|)^\mu}. \tag{6.2}$$

Moreover, we set

$$\|\psi\|_{\star,\xi} := \|\rho_\xi^{-1}\psi\|_{L^\infty(\mathbb{R}^n)}.$$

**Lemma 6.1.** *Let $g \in L^2(\mathbb{R}^n) \cap L^\infty(\mathbb{R}^n)$ and let $\psi \in H^s(\mathbb{R}^n)$ be a solution to the problem*

$$\begin{cases} (-\Delta)^s\psi + \psi + g = 0 & \text{in } \Omega_\varepsilon, \\ \psi = 0 & \text{in } \mathbb{R}^n \setminus \Omega_\varepsilon. \end{cases} \tag{6.3}$$

*Then, there exists a positive constant $C$ such that*

$$\|\psi\|_{L^\infty(\mathbb{R}^n)} + \sup_{x \neq y} \frac{|\psi(x) - \psi(y)|}{|x - y|^s} \leqslant C \left( \|g\|_{L^\infty(\mathbb{R}^n)} + \|g\|_{L^2(\mathbb{R}^n)} \right).$$

*Proof.* From Theorem $8.2$ in [14] we have that $\psi \in L^\infty(\mathbb{R}^n)$ and there exists a constant $C > 0$ such that

$$\|\psi\|_{L^\infty(\mathbb{R}^n)} \leqslant C \left( \|g\|_{L^\infty(\mathbb{R}^n)} + \|\psi\|_{L^2(\mathbb{R}^n)} \right). \tag{6.4}$$

Now, we show that

$$\|\psi\|_{L^2(\mathbb{R}^n)} \leqslant \|g\|_{L^2(\mathbb{R}^n)}. \tag{6.5}$$

Indeed, we multiply the equation in (6.3) by $\psi$ and we integrate over $\Omega_\varepsilon$, obtaining that

$$\int_{\Omega_\varepsilon} (-\Delta)^s\psi\,\psi + \psi^2 + g\,\psi\,dx = 0. \tag{6.6}$$

We notice that, thanks to formula $(1.5)$ in [30],

$$\int_{\Omega_\varepsilon} (-\Delta)^s\psi\,\psi\,dx = \int_{\Omega_\varepsilon} \left|(-\Delta)^s\psi\right|^2\,dx \geqslant 0.$$

Hence, from (6.6) we have

$$\int_{\Omega_\varepsilon} \psi^2\,dx \leqslant \int_{\Omega_\varepsilon} -g\,\psi\,dx.$$

So, using Hölder inequality, we get

$$\int_{\Omega_\varepsilon} \psi^2\,dx \leqslant \left( \int_{\Omega_\varepsilon} g^2\,dx \right)^{1/2} \left( \int_{\Omega_\varepsilon} \psi^2\,dx \right)^{1/2},$$

and therefore, dividing by $\left( \int_{\Omega_\varepsilon} \psi^2\,dx \right)^{1/2}$, we obtain (6.5).

From (6.4) and (6.5), we have that

$$\|\psi\|_{L^\infty(\mathbb{R}^n)} \leqslant C \left( \|g\|_{L^\infty(\mathbb{R}^n)} + \|g\|_{L^2(\mathbb{R}^n)} \right).$$

Now, since both $\psi$ and $g$ are bounded, from the regularity results in [35] we have that $\psi$ is $C^\alpha$ in the interior of $\Omega_\varepsilon$, for some $\alpha \in (0, 2s)$.

It remains to prove that $\psi$ is $C^\alpha$ near the boundary of $\Omega_\varepsilon$. For this, we fix a point $p \in \partial\Omega_\varepsilon$ and we look at the equation in the ball $B_1(p)$.

We notice that $|(-\Delta)^s \psi|$ is bounded, since both $\psi$ and $g$ are in $L^\infty(\mathbb{R}^n)$, and therefore we can apply Proposition $3.5$ in [31], obtaining that, for any $x, y \in B_1(p) \cap \Omega_\varepsilon$,

$$\frac{\psi(x)}{d^s(x)} - \frac{\psi(y)}{d^s(y)} \leqslant C_1 \left( \|\psi\|_{L^\infty(\mathbb{R}^n)} + \|g\|_{L^\infty(\mathbb{R}^n)} \right), \tag{6.7}$$

where $d(x) := \operatorname{dist}(x, \partial\Omega_\varepsilon)$. In particular, we can fix $y \in B_1(p) \cap \Omega_\varepsilon$, such that $d(y) = 1/2$. Since $\psi$ is bounded, from (6.7) we have that

$$\frac{\psi(x)}{d^s(x)} \leqslant C_2 \left( \|\psi\|_{L^\infty(\mathbb{R}^n)} + \|g\|_{L^\infty(\mathbb{R}^n)} \right),$$

which gives that

$$\psi(x) \leqslant C_2 \left( \|\psi\|_{L^\infty(\mathbb{R}^n)} + \|g\|_{L^\infty(\mathbb{R}^n)} \right) d^s(x).$$

This implies that $\psi$ is $C^s$ also near the boundary and concludes the proof of the lemma. $\square$

**Lemma 6.2.** *Let $\xi \in \Omega_\varepsilon$, $\mathcal{B}$ be a bounded subset of $\mathbb{R}^n$, and $R_0 > 0$ be such that*

$$B_{R_0}(\xi) \supseteq \mathcal{B}. \tag{6.8}$$

*Let $\mathcal{W} \in L^\infty(\mathbb{R}^n)$ be such that*

$$m := \inf_{\mathbb{R}^n \setminus \mathcal{B}} \mathcal{W} > 0. \tag{6.9}$$

*Let also $g \in L^2(\mathbb{R}^n)$, with $\|g\|_{\star,\xi} < +\infty$, and let $\psi \in H^s(\mathbb{R}^n)$ be a solution to*

$$\begin{cases} (-\Delta)^s \psi + \mathcal{W}\psi + g = 0 & \text{in } \Omega_\varepsilon, \\ \psi = 0 & \text{in } \mathbb{R}^n \setminus \Omega_\varepsilon. \end{cases}$$

*Then, there exists a positive constant $C$, possibly depending on $m$, $R_0$ and $\|\mathcal{W}\|_{L^\infty(\mathbb{R}^n)}$ (and also on $n$, $s$, and $\Omega$), such that*[2]

$$\|\psi\|_{\star,\xi} \leqslant C \left( \|\psi\|_{L^\infty(\mathcal{B})} + \|g\|_{\star,\xi} \right). \tag{6.10}$$

*Proof.* We define

$$\begin{aligned} W &:= m\chi_{\mathcal{B}} + \mathcal{W}\,\chi_{\mathbb{R}^n \setminus \mathcal{B}} \\ \text{and } \; G &:= (m - \mathcal{W})\,\chi_{\mathcal{B}}\,\psi - g. \end{aligned} \tag{6.11}$$

We observe that

$$\begin{aligned} \|G\|_{\star,\xi} &\leqslant \sup_{x \in \mathcal{B}} (1 + |x - \xi|)^\mu \,(m + \mathcal{W}(x))\,\psi(x) + \|g\|_{\star,\xi} \\ &\leqslant 2\,(1 + R_0)^\mu \,\|\mathcal{W}\|_{L^\infty(\mathbb{R}^n)}\,\|\psi\|_{L^\infty(\mathcal{B})} + \|g\|_{\star,\xi} \\ &\leqslant C_0\,\|\psi\|_{L^\infty(\mathcal{B})} + \|g\|_{\star,\xi}, \end{aligned} \tag{6.12}$$

[2] In (6.10) we use the standard convention that $\|\psi\|_{L^\infty(\mathcal{B})} := 0$ when $\mathcal{B} := \varnothing$ (equivalently, if $\mathcal{B} = \varnothing$, the term $\|\psi\|_{L^\infty(\mathcal{B})}$ can be neglected in the proof of Lemma 6.2, since, in this case, $G$ and $g$ are the same from (6.11) on).

for a suitable $C_0 > 0$ possibly depending on $R_0$ and $\|\mathcal{W}\|_{L^\infty(\mathbb{R}^n)}$ (notice that (6.8) was used here). Also $\psi$ is a solution of

$$\begin{aligned}(-\Delta)^s\psi + W\psi &= (W-\mathcal{W})\psi - g\\ &= (W - \mathcal{W}\chi_{\mathbb{R}^n\setminus\mathcal{B}} - \mathcal{W}\chi_{\mathcal{B}})\psi - g\\ &= (m\chi_{\mathcal{B}} - \mathcal{W}\chi_{\mathcal{B}})\psi - g\\ &= G\end{aligned} \tag{6.13}$$

and, in virtue of (6.9),

$$W \geqslant m\chi_{\mathcal{B}} + m\chi_{\mathbb{R}^n\setminus\mathcal{B}} = m. \tag{6.14}$$

We take $\rho_0 := (1+|x|)^{-\mu}$ and $\eta \in H^s(\mathbb{R}^n)$ to be a solution of

$$(-\Delta)^s\eta + m\eta = \rho_0. \tag{6.15}$$

We refer to formula (2.4) in [10] for the existence of such solution and to Lemma 2.2 there for the following estimate:

$$\sup_{x\in\mathbb{R}^n}(1+|x|)^\mu\eta(x) \leqslant C_1 \sup_{x\in\mathbb{R}^n}(1+|x|)^\mu\rho_0(x) = C_1, \tag{6.16}$$

for some $C_1 > 0$, possibly depending on $m$. Also, by Lemma 2.4 in [10], we have that $\eta \geqslant 0$, and so, recalling (6.14), we obtain that

$$(W(x)-m)\ \eta(x-\xi) \geqslant 0. \tag{6.17}$$

Now we define $\eta_\xi(x) := \eta(x-\xi)$,

$$C_\star := \|G\|_{\star,\xi} \tag{6.18}$$

and $\omega := C_\star\eta_\xi \pm \psi$. We remark that the quantity $C_\star$ plays a different role from the other constants $C_0$, $C_1$ and $C_2$: indeed, while $C_0$, $C_1$ and $C_2$ depend only on $m$, $R_0$ and $\|\mathcal{W}\|_{L^\infty(\mathbb{R}^n)}$ (as well on $n$, $s$ and $\Omega$), the quantity $C_\star$ also depend on $G$, and this will be made explicit at the end of the proof.

Notice also that $\rho_0(x-\xi) = \rho_\xi(x)$, due to the definition in (6.2), and

$$\begin{aligned}C_\star\rho_\xi(x) \pm G(x) &\geqslant \rho_\xi(x)\Big(C_\star - \rho_\xi^{-1}(x)|G(x)|\Big)\\ &\geqslant \rho_\xi(x)\Big(C_\star - \|G\|_{\star,\xi}\Big)\\ &\geqslant 0.\end{aligned} \tag{6.19}$$

Thus we infer that

$$\begin{aligned}(-\Delta)^s\omega + W\omega &= C_\star\Big((-\Delta)^s\eta_\xi + W\eta_\xi\Big) \pm \Big((-\Delta)^s\psi + W\psi\Big)\\ &= C_\star\rho_\xi + C_\star\ (W-m)\ \eta_\xi \pm G\\ &\geqslant 0,\end{aligned} \tag{6.20}$$

in $\Omega_\varepsilon$, thanks to (6.13), (6.15), (6.17) and (6.19). Furthermore, in $\mathbb{R}^n \setminus \Omega_\varepsilon$ we have that $\omega = C_\star\eta_\xi \geqslant 0$. As a consequence of this, (6.20) and the maximum principle (see e.g. Lemma 6 in [33]), we conclude that $\omega \geqslant 0$ in the whole of $\mathbb{R}^n$.

Accordingly, for any $x \in \mathbb{R}^n$

$$\begin{aligned}
\mp\rho_\xi^{-1}(x)\psi(x) &= \rho_\xi^{-1}(x)\Big(C_\star\eta_\xi(x) - \omega(x)\Big) \\
&\leqslant C_\star\rho_\xi^{-1}(x)\eta_\xi(x) \\
&\leqslant C_\star \sup_{y\in\mathbb{R}^n} \rho_\xi^{-1}(y)\eta_\xi(y) \\
&= C_\star \sup_{y\in\mathbb{R}^n} \rho_\xi^{-1}(y+\xi)\eta_\xi(y+\xi) \\
&= C_\star \sup_{y\in\mathbb{R}^n} (1+|y|)^\mu \eta(y) \\
&\leqslant C_1\, C_\star,
\end{aligned}$$

where (6.16) was used in the last step. Hence, recalling (6.18) and (6.12),

$$|\rho_\xi^{-1}(x)\psi(x)| \leqslant C_1\,\|G\|_{\star,\xi} \leqslant C_1\Big(C_0\,\|\psi\|_{L^\infty(\mathcal{B})} + \|g\|_{\star,\xi}\Big),$$

which implies (6.10). □

As a consequence of Lemma 6.2, we obtain the following two corollaries:

**Corollary 6.3.** *Let $g \in L^2(\mathbb{R}^n)$, with $\|g\|_{\star,\xi} < +\infty$, and let $\psi \in H^s(\mathbb{R}^n)$ be a solution to*

$$\begin{cases} (-\Delta)^s\psi + \psi - pw_\xi^{p-1}\psi + g = 0 & \text{in } \Omega_\varepsilon, \\ \psi = 0 & \text{in } \mathbb{R}^n\setminus\Omega_\varepsilon. \end{cases}$$

*Then, there exist positive constants $C$ and $R$ such that*

$$\|\psi\|_{\star,\xi} \leqslant C\left(\|\psi\|_{L^\infty(B_R(\xi))} + \|g\|_{\star,\xi}\right). \tag{6.21}$$

*Proof.* We apply Lemma 6.2 with $\mathcal{W} := 1 - pw_\xi^{p-1}$ and $\mathcal{B} := B_R(\xi)$ (notice that, with this notation (6.21) would follow from (6.10)). So, we only need to check that (6.9) holds true with a suitable choice of $R$. For this, we use that $w$ decays at infinity (recall (1.4)), hence we can fix $R$ large enough such that

$$pw^{p-1}(x) \leqslant \frac{1}{2} \text{ for every } x \in \mathbb{R}^n\setminus B_R.$$

accordingly $\mathcal{W} \geqslant 1-(1/2) = 1/2$, which establishes (6.9) with $m := 1/2$. □

**Corollary 6.4.** *Let $g \in L^2(\mathbb{R}^n)$, with $\|g\|_{\star,\xi} < +\infty$, and let $\psi \in H^s(\mathbb{R}^n)$ be a solution to*

$$\begin{cases} (-\Delta)^s\psi + \psi + g = 0 & \text{in } \Omega_\varepsilon, \\ \psi = 0 & \text{in } \mathbb{R}^n\setminus\Omega_\varepsilon. \end{cases}$$

*Then, there exists a positive constant $C$ such that*

$$\|\psi\|_{\star,\xi} \leqslant C\|g\|_{\star,\xi}.$$

*Proof.* We use for this Lemma 6.2 with $\mathcal{W} := 1$ and $\mathcal{B} := \varnothing$ (recall footnote 2). □

## 7. The Lyapunov-Schmidt reduction

In this section we deal with the linear theory associated to the scaled problem (1.8). For this, we introduce the functional space

$$\Psi := \left\{ \psi \in H^s(\mathbb{R}^n) \text{ s.t. } \psi = 0 \text{ in } \mathbb{R}^n \setminus \Omega_\varepsilon \text{ and } \int_{\Omega_\varepsilon} \psi\, Z_i\, dx = 0 \text{ for any } i = 1, \dots, n \right\},$$

where the $Z_i$'s are introduced in (5.3). We remark that the condition

$$\int_{\Omega_\varepsilon} \psi\, Z_i\, dx = 0 \text{ for any } i = 1, \dots, n$$

means that $\psi$ is orthogonal to the space $\mathcal{Z}$ (that is the space spanned by $Z_i$) with respect to the scalar product in $L^2(\Omega_\varepsilon)$.

We look for solution to (1.8) of the form

$$u = u_\xi := \bar{u}_\xi + \psi, \tag{7.1}$$

where $\bar{u}_\xi$ is the solution to (1.9) and $\psi$ is a small function (for $\varepsilon$ sufficiently small) which belongs to $\Psi$.

Inserting $u$ (given in (7.1)) into (1.8) and recalling that $\bar{u}_\xi$ is a solution to (1.9), we have that, in order to obtain a solution to (1.8), $\psi$ must satisfy

$$(-\Delta)^s \psi + \psi - p w_\xi^{p-1} \psi = E(\psi) + N(\psi) \text{ in } \Omega_\varepsilon, \tag{7.2}$$

where[3]

$$\begin{aligned} E(\psi) &:= (\bar{u}_\xi + \psi)^p - (w_\xi + \psi)^p \\ \text{and } N(\psi) &:= (w_\xi + \psi)^p - w_\xi^p - p w_\xi^{p-1} \psi. \end{aligned} \tag{7.3}$$

Instead of solving (7.2), we will consider a projected version of the problem. Namely we will look for a solution $\psi \in H^s(\mathbb{R}^n)$ of the equation

$$(-\Delta)^s \psi + \psi - p w_\xi^{p-1} \psi = E(\psi) + N(\psi) + \sum_{i=1}^{n} c_i\, Z_i \text{ in } \Omega_\varepsilon, \tag{7.4}$$

for some coefficients $c_i \in \mathbb{R}$, $i \in \{1, \dots, n\}$. Moreover, we require that $\psi$ satisfies the conditions

$$\psi = 0 \text{ in } \mathbb{R}^n \setminus \Omega_\varepsilon, \tag{7.5}$$

and

$$\int_{\Omega_\varepsilon} \psi\, Z_i\, dx = 0 \text{ for any } i = 1, \dots, n. \tag{7.6}$$

We will prove that problem (7.4)-(7.6) admits a unique solution, which is small if $\varepsilon$ is sufficiently small, and then we will show that the coefficients $c_i$ are equal to zero for every $i \in \{1, \dots, n\}$ for a suitable $\xi$. This will give us a solution $\psi \in \Psi$ to (7.2), and therefore a solution $u$ of (1.8), thanks to the definition in (7.1).

---

[3]As a matter of fact, one should write the positive parts in (7.3), namely set $E(\psi) := (\bar{u}_\xi + \psi)_+^p - (w_\xi + \psi)_+^p$ and $N(\psi) := (w_\xi + \psi)_+^p - w_\xi^p - p w_\xi^{p-1} \psi$, but, a posteriori, this is the same by maximum principle. So we preferred, with a slight abuse of notation, to drop the positive parts for simplicity of notation.

7.1. **Linear theory.** In this subsection we develop a general theory that will give us the existence result for the linear problem (7.4)-(7.6).

**Theorem 7.1.** *Let $g \in L^2(\mathbb{R}^n)$ with $\|g\|_{\star,\xi} < +\infty$. If $\varepsilon > 0$ is sufficiently small, there exist a unique $\psi \in \Psi$ and numbers $c_i \in \mathbb{R}$, for any $i \in \{1, \dots, n\}$, such that*

$$(-\Delta)^s \psi + \psi - p w_\xi^{p-1} \psi + g = \sum_{i=1}^{n} c_i Z_i \text{ in } \Omega_\varepsilon. \tag{7.7}$$

*Moreover, there exists a constant $C > 0$ such that*

$$\|\psi\|_{\star,\xi} \leqslant C \|g\|_{\star,\xi}. \tag{7.8}$$

Before proving Theorem 7.1 we need some preliminary lemmata. In the next lemma we show that we can uniquely determine the coefficients $c_i$ in (7.7) in terms of $\psi$ and $g$. Actually, we will show that the estimate on the $c_i$'s holds in a more general case, that is we do not need the orthogonality condition in (7.6).

**Lemma 7.2.** *Let $g \in L^2(\mathbb{R}^n)$ with $\|g\|_{\star,\xi} < +\infty$. Suppose that $\psi \in H^s(\mathbb{R}^n)$ satisfies*

$$\begin{cases} (-\Delta)^s \psi + \psi - p w_\xi^{p-1} \psi + g = \displaystyle\sum_{i=1}^{n} c_i Z_i & \text{in } \Omega_\varepsilon, \\ \psi = 0 & \text{in } \mathbb{R}^n \setminus \Omega_\varepsilon, \end{cases} \tag{7.9}$$

*for some $c_i \in \mathbb{R}$, $i = 1, \dots, n$.*

*Then, for $\varepsilon > 0$ sufficiently small and for any $i \in \{1, \dots, n\}$, the coefficient $c_i$ is given by*

$$c_i = \frac{1}{\alpha} \int_{\mathbb{R}^n} g\, Z_i \, dx + f_i, \tag{7.10}$$

*where $\alpha$ is defined in (5.17), for suitable $f_i \in \mathbb{R}$ that satisfies*

$$|f_i| \leqslant C\, \varepsilon^{n/2} \left( \|\psi\|_{L^2(\mathbb{R}^n)} + \|g\|_{L^2(\mathbb{R}^n)} \right), \tag{7.11}$$

*for some positive constant $C$.*

*Proof.* We start with some considerations in Fourier space on a function $T \in C^2(\mathbb{R}^n) \cap H^2(\mathbb{R}^n)$. First of all, for any $j \in \{1, \dots, n\}$

$$\|\partial_j^2 T\|^2_{L^2(\mathbb{R}^n)} = \|\mathcal{F}(\partial_j^2 T)\|^2_{L^2(\mathbb{R}^n)} = \|\xi_j^2 \hat{T}\|^2_{L^2(\mathbb{R}^n)} = \int_{\mathbb{R}^n} \xi_j^4 \, |\hat{T}(\xi)|^2 \, d\xi.$$

Moreover, by convexity,

$$|\xi|^4 = \left( \sum_{j=1}^{n} \xi_j^2 \right)^2 \leqslant 2 \sum_{j=1}^{n} \xi_j^4$$

and therefore

$$2\|D^2 T\|^2_{L^2(\mathbb{R}^n)} = \sum_{j=1}^{n} 2\|\partial_j^2 T\|^2_{L^2(\mathbb{R}^n)} \geqslant \int_{\mathbb{R}^n} |\xi|^4 \, |\hat{T}(\xi)|^2 \, d\xi.$$

As a consequence

$$\begin{aligned} \|(-\Delta)^s T\|^2_{L^2(\mathbb{R}^n)} &= \|\mathcal{F}((-\Delta)^s T)\|^2_{L^2(\mathbb{R}^n)} = \||\xi|^{2s} \hat{T}\|^2_{L^2(\mathbb{R}^n)} \\ &= \int_{\mathbb{R}^n} |\xi|^{4s} \, |\hat{T}(\xi)|^2 \, d\xi \leqslant \int_{\mathbb{R}^n} (1 + |\xi|^4) \, |\hat{T}(\xi)|^2 \, d\xi \\ &\leqslant \|T\|^2_{L^2(\mathbb{R}^n)} + 2\|D^2 T\|^2_{L^2(\mathbb{R}^n)} \leqslant C \|T\|_{H^2(\mathbb{R}^n)}, \end{aligned} \tag{7.12}$$

for some $C > 0$.

Now, without loss of generality, we may suppose that

$$B_{c/\varepsilon}(\xi) \subseteq \Omega_\varepsilon, \tag{7.13}$$

for some $c > 0$. Fix $\varepsilon > 0$, and let $\tau_\varepsilon \in C^\infty(\mathbb{R}^n, [0,1])$, with $\tau_\varepsilon = 1$ in $B_{(c/\varepsilon)-1}(\xi)$, $\tau_\varepsilon = 0$ outside $B_{c/\varepsilon}(\xi)$ and $|\nabla\tau_\varepsilon| \leqslant C$. We set $T_{\varepsilon,j} := Z_j \tau_\varepsilon$. Hence, from (7.12) and Lemmata 5.3 and 5.4,

$$\|(-\Delta)^s T_{\varepsilon,j}\|^2_{L^2(\mathbb{R}^n)} \leqslant C, \tag{7.14}$$

for some $C > 0$, independent of $\varepsilon$ and $j$.

Moreover, the function $T_{\varepsilon,j}$ belongs to $H^s(\mathbb{R}^n)$ and vanishes outside $B_{c/\varepsilon}(\xi)$, and so in particular outside $\Omega_\varepsilon$, thanks to (7.13).

Thus (see e.g. formula (1.5) in [30])

$$\int_{\Omega_\varepsilon} (-\Delta)^s\psi\, T_{\varepsilon,j}\, dx = \int_{\Omega_\varepsilon} (-\Delta)^{s/2}\psi\, (-\Delta)^{s/2} T_{\varepsilon,j}\, dx = \int_{\Omega_\varepsilon} \psi\, (-\Delta)^s T_{\varepsilon,j}\, dx. \tag{7.15}$$

As a consequence, recalling (7.14)

$$\left| \int_{\Omega_\varepsilon} (-\Delta)^s\psi\, T_{\varepsilon,j}\, dx \right| \leqslant \|\psi\|_{L^2(\Omega_\varepsilon)}\, \|(-\Delta)^s T_{\varepsilon,j}\|_{L^2(\Omega_\varepsilon)} \leqslant C\, \|\psi\|_{L^2(\mathbb{R}^n)}. \tag{7.16}$$

Now, we fix $j \in \{1, \dots, n\}$, we multiply the equation in (7.9) by $T_{\varepsilon,j}$ and we integrate over $\Omega_\varepsilon$. We obtain

$$\sum_{i=1}^n c_i \int_{\Omega_\varepsilon} Z_i\, T_{\varepsilon,j}\, dx = \int_{\Omega_\varepsilon} T_{\varepsilon,j}\, \left((-\Delta)^s\psi + \psi - p w_\xi^{p-1}\psi + g\right) dx. \tag{7.17}$$

Now, we observe that, thanks to (7.15), we can write

$$\int_{\Omega_\varepsilon} (-\Delta)^s\psi\, T_{\varepsilon,j}\, dx = \int_{\Omega_\varepsilon} \psi(-\Delta)^s T_{\varepsilon,j}\, dx = \int_{\Omega_\varepsilon} \psi(-\Delta)^s \left(T_{\varepsilon,j} - Z_j\right) dx + \int_{\Omega_\varepsilon} \psi(-\Delta)^s Z_j\, dx. \tag{7.18}$$

Using Hölder inequality and (7.12), we have that

$$\begin{aligned} \left| \int_{\Omega_\varepsilon} \psi(-\Delta)^s \left(T_{\varepsilon,j} - Z_j\right) dx \right| &\leqslant \|\psi\|_{L^2(\mathbb{R}^n)} \|(-\Delta)^s \left(T_{\varepsilon,j} - Z_j\right)\|_{L^2(\mathbb{R}^n)} \\ &\leqslant C\|\psi\|_{L^2(\mathbb{R}^n)} \|T_{\varepsilon,j} - Z_j\|_{H^2(\mathbb{R}^n)}. \end{aligned} \tag{7.19}$$

Let us estimate the $H^2$-norm of $T_{\varepsilon,j} - Z_j$. First we have that

$$\|T_{\varepsilon,j} - Z_j\|^2_{L^2(\mathbb{R}^n)} = \int_{\mathbb{R}^n} (\tau_\varepsilon - 1)^2\, Z_j^2\, dx \leqslant \int_{B^c_{(c/\varepsilon)-1}(\xi)} Z_j^2\, dx,$$

since $\tau_\varepsilon = 1$ in $B_{(c/\varepsilon)-1}(\xi)$ and takes values in $(0,1)$. Hence, from Lemma 5.2 we deduce that

$$\|T_{\varepsilon,j} - Z_j\|^2_{L^2(\mathbb{R}^n)} \leqslant C \int_{B^c_{(c/\varepsilon)-1}(\xi)} \frac{1}{|x-\xi|^{2\nu_1}}\, dx \leqslant C\,\varepsilon^n,$$

up to renaming $C$. Therefore,

$$\|T_{\varepsilon,j} - Z_j\|_{L^2(\mathbb{R}^n)} \leqslant C\,\varepsilon^{n/2}. \tag{7.20}$$

Moreover, we have that

$$\begin{aligned}\|\nabla\left(T_{\varepsilon,j}-Z_j\right)\|^2_{L^2(\mathbb{R}^n)} &= \int_{\mathbb{R}^n} |(\tau_\varepsilon-1)\nabla Z_j+\nabla\tau_\varepsilon Z_j|^2\,dx\\ &= \int_{\mathbb{R}^n} (\tau_\varepsilon-1)^2|\nabla Z_j|^2+|\nabla\tau_\varepsilon|^2Z_j^2+2(\tau_\varepsilon-1)Z_j\nabla Z_j\cdot\nabla\tau_\varepsilon\,dx.\end{aligned}$$

Using the fact that both $\tau_\varepsilon-1$ and $\nabla\tau_\varepsilon$ have support outside $B_{(c/\varepsilon)-1}(\xi)$ and Lemmata 5.2 and 5.3, we obtain that

$$\|\nabla\left(T_{\varepsilon,j}-Z_j\right)\|_{L^2(\mathbb{R}^n)} \leqslant C\,\varepsilon^{n/2}. \tag{7.21}$$

Finally, using again the fact that $\tau_\varepsilon-1$, $\nabla\tau_\varepsilon$ and $D^2\tau_\varepsilon$ have support outside $B_{(c/\varepsilon)-1}(\xi)$ and Lemmata 5.2, 5.3 and 5.4, we obtain that

$$\|D^2\left(T_{\varepsilon,j}-Z_j\right)\|_{L^2(\mathbb{R}^n)} \leqslant C\,\varepsilon^{n/2}.$$

Using this, (7.20) and (7.21) we have that

$$\|T_{\varepsilon,j}-Z_j\|_{H^2(\mathbb{R}^n)} \leqslant C\,\varepsilon^{n/2},$$

and so from (7.19) we obtain

$$\left|\int_{\Omega_\varepsilon}\psi(-\Delta)^s\left(T_{\varepsilon,j}-Z_j\right)\,dx\right| \leqslant C\,\varepsilon^{n/2}\|\psi\|_{L^2(\mathbb{R}^n)}. \tag{7.22}$$

Now, using (7.18), we have that

$$\begin{aligned}&\int_{\Omega_\varepsilon} T_{\varepsilon,j}\left((-\Delta)^s\psi+\psi-pw_\xi^{p-1}\psi\right)dx\\ &\quad= \int_{\Omega_\varepsilon}\psi(-\Delta)^sZ_j+T_{\varepsilon,j}\,\psi-pw_\xi^{p-1}\psi\,T_{\varepsilon,j}\,dx+\int_{\Omega_\varepsilon}\psi(-\Delta)^s(T_{\varepsilon,j}-Z_j)\,dx.\end{aligned}$$

Since $w_\xi$ is a solution to (1.3), we have that $Z_j$ solves

$$(-\Delta)^sZ_j+Z_j=pw_\xi^{p-1}Z_j,$$

and this implies that

$$\begin{aligned}&\int_{\Omega_\varepsilon} T_{\varepsilon,j}\left((-\Delta)^s\psi+\psi-pw_\xi^{p-1}\psi\right)dx\\ &\quad= \int_{\Omega_\varepsilon}\psi(T_{\varepsilon,j}-Z_j)-pw_\xi^{p-1}\psi(T_{\varepsilon,j}-Z_j)\,dx+\int_{\Omega_\varepsilon}\psi(-\Delta)^s(T_{\varepsilon,j}-Z_j)\,dx.\end{aligned}$$

Hence, using the fact that $w_\xi$ is bounded (see (1.4)) and Hölder inequality, we have that

$$\begin{aligned}&\left|\int_{\Omega_\varepsilon} T_{\varepsilon,j}\left((-\Delta)^s\psi+\psi-pw_\xi^{p-1}\psi\right)dx\right|\\ &\quad\leqslant C\Big(\|\psi\|_{L^2(\mathbb{R}^n)}\|T_{\varepsilon,j}-Z_j\|_{L^2(\mathbb{R}^n)}+\left|\int_{\Omega_\varepsilon}\psi\,(-\Delta)^s(T_{\varepsilon,j}-Z_j)\,dx\right|\Big)\\ &\quad\leqslant C\,\varepsilon^{n/2}\|\psi\|_{L^2(\mathbb{R}^n)},\end{aligned} \tag{7.23}$$

where we have used (7.20) and (7.22) in the last step.

Now, we can write

$$\int_{\Omega_\varepsilon} T_{\varepsilon,j}\,g\,dx = \int_{\Omega_\varepsilon}(T_{\varepsilon,j}-Z_j)g\,dx+\int_{\Omega_\varepsilon}Z_j\,g\,dx = \int_{\Omega_\varepsilon}(T_{\varepsilon,j}-Z_j)g\,dx+\int_{\mathbb{R}^n}Z_j\,g\,dx-\int_{\mathbb{R}^n\setminus\Omega_\varepsilon}Z_j\,g\,dx. \tag{7.24}$$

Using Hölder inequality and (7.20), we can estimate

$$\left|\int_{\Omega_\varepsilon}(T_{\varepsilon,j}-Z_j)g\,dx\right|\leqslant\|T_{\varepsilon,j}-Z_j\|_{L^2(\mathbb{R}^n)}\|g\|_{L^2(\mathbb{R}^n)}\leqslant C\varepsilon^{n/2}\|g\|_{L^2(\mathbb{R}^n)}.$$

Moreover, from Hölder inequality and Lemma 5.2 (and recalling that $\mathrm{dist}(\xi,\partial\Omega_\varepsilon)\geqslant c/\varepsilon$), we obtain that

$$\left|\int_{\mathbb{R}^n\setminus\Omega_\varepsilon}Z_j\,g\,dx\right|\leqslant\|g\|_{L^2(\mathbb{R}^n)}\left(\int_{\mathbb{R}^n\setminus\Omega_\varepsilon}\frac{C}{|x-\xi|^{2\nu_1}}\,dx\right)^{1/2}\leqslant C\,\varepsilon^{n/2}\|g\|_{L^2(\mathbb{R}^n)}.$$

The last two estimates and (7.24) imply that

$$\int_{\Omega_\varepsilon}T_{\varepsilon,j}\,g\,dx=\int_{\mathbb{R}^n}Z_j\,g\,dx+\tilde f_j,$$

where

$$|\tilde f_j|\leqslant C\,\varepsilon^{n/2}\|g\|_{L^2(\mathbb{R}^n)}.$$

From this, (7.17) and (7.23), we have that

$$\sum_{i=1}^n c_i\int_{\Omega_\varepsilon}Z_i\,T_{\varepsilon,j}\,dx=\int_{\mathbb{R}^n}g\,Z_j\,dx+\bar f_j, \tag{7.25}$$

where

$$|\bar f_j|\leqslant C\,\varepsilon^{n/2}\left(\|\psi\|_{L^2(\mathbb{R}^n)}+\|g\|_{L^2(\mathbb{R}^n)}\right) \tag{7.26}$$

up to renaming the constants.

On the other hand, we can write

$$\int_{\Omega_\varepsilon}Z_i\,T_{\varepsilon,j}\,dx=\int_{\Omega_\varepsilon}Z_i\,(T_{\varepsilon,j}-Z_j)\,dx+\int_{\Omega_\varepsilon}Z_i\,Z_j\,dx. \tag{7.27}$$

From Hölder inequality, (7.20) and Lemma 5.2, we have that

$$\left|\int_{\Omega_\varepsilon}Z_i\,(T_{\varepsilon,j}-Z_j)\,dx\right|\leqslant\left(\int_{\Omega_\varepsilon}Z_i^2\,dx\right)^{1/2}\|T_{\varepsilon,j}-Z_j\|_{L^2(\mathbb{R}^n)}\leqslant C\,\varepsilon^{n/2}.$$

Using this and Corollary 5.6 in (7.27), we obtain that

$$\int_{\Omega_\varepsilon}Z_i\,T_{\varepsilon,j}\,dx=\alpha\,\delta_{ij}+O(\varepsilon^{n/2}). \tag{7.28}$$

So, we consider the matrix $A\in$ Mat$(n\times n)$ defined as

$$A_{ji}:=\int_{\Omega_\varepsilon}Z_i\,T_{\varepsilon,j}\,dx. \tag{7.29}$$

Thanks to (7.28), the matrix $\alpha^{-1}A$ is a perturbation of the identity and so it is invertible for $\varepsilon$ sufficiently small, with inverse equal to the identity plus smaller order term of size $\varepsilon^{n/2}$. Hence, the matrix $A$ is invertible too, with inverse

$$(A^{-1})_{ji}=\alpha^{-1}\delta_{ij}+O(\varepsilon^{n/2}). \tag{7.30}$$

So we consider the vector $d=(d_1,\dots,d_n)$ defined by

$$d_j:=\int_{\mathbb{R}^n}g\,Z_j\,dx+\bar f_j. \tag{7.31}$$

We observe that

$$\left|\int_{\mathbb{R}^n}g\,Z_j\,dx\right|\leqslant\|g\|_{L^2(\mathbb{R}^n)}\|Z_j\|_{L^2(\mathbb{R}^n)}\leqslant C\,\|g\|_{L^2(\mathbb{R}^n)},$$

thanks Lemma 5.2. As a consequence, recalling (7.26), we obtain that

$$|d| \leqslant C \left( \|\psi\|_{L^2(\mathbb{R}^n)} + \|g\|_{L^2(\mathbb{R}^n)} \right), \tag{7.32}$$

up to renaming $C$.

With the setting above, (7.25) reads

$$\sum_{i=1}^{n} c_i\, A_{ji} = \int_{\mathbb{R}^n} g\, Z_j\, dx + \bar{f}_j = d_j$$

that is, in matrix notation, $Ac = d$. We can invert such relation using (7.30) and write

$$c = A^{-1}d = \alpha^{-1}d + f^\sharp$$

with

$$|f^\sharp| \leqslant C\,\varepsilon^{n/2}\,|d| \leqslant C\,\varepsilon^{n/2} \left( \|\psi\|_{L^2(\mathbb{R}^n)} + \|g\|_{L^2(\mathbb{R}^n)} \right), \tag{7.33}$$

in virtue of (7.32). So, using (7.31),

$$c_i = \alpha^{-1}d_i + f_i^\sharp = \alpha^{-1} \int_{\mathbb{R}^n} g\, Z_i\, dx + \alpha^{-1}\bar{f}_i + f_i^\sharp.$$

This proves (7.10) with

$$f_i := \alpha^{-1}\bar{f}_i + f_i^\sharp,$$

and then (7.11) follows from (7.26) and (7.33). □

Now, we show that solutions to (7.7) satisfy an a priori estimate. We remark that the result in the following lemma is different from the one in Corollary 6.4, since here also a combination of $Z_i$, for $i = 1, \dots, n$, appears in the equation satisfied by $\psi$.

**Lemma 7.3.** *Let $g \in L^2(\mathbb{R}^n)$ with $\|g\|_{\star,\xi} < +\infty$. Let $\psi \in \Psi$ be a solution to (7.7) for some coefficients $c_i \in \mathbb{R}$, $i = 1, \dots, n$ and for $\varepsilon$ sufficiently small.*

*Then,*

$$\|\psi\|_{\star,\xi} \leqslant C\|g\|_{\star,\xi}.$$

*Proof.* Suppose by contradiction that there exists a sequence $\varepsilon_j \searrow 0$ as $j \to +\infty$ such that, for any $j \in \mathbb{N}$, the function $\psi_j$ satisfies

$$\begin{cases} (-\Delta)^s\psi_j + \psi_j - p w_{\xi_j}^{p-1}\psi_j + g_j = \displaystyle\sum_{i=1}^{n} c_i^j\, Z_i^j & \text{in } \Omega_{\varepsilon_j}, \\ \psi_j = 0 & \text{in } \mathbb{R}^n \setminus \Omega_{\varepsilon_j}, \\ \displaystyle\int_{\Omega_{\varepsilon_j}} \psi_j\, Z_i^j\, dx = 0 & \text{for any } i = 1, \dots, n, \end{cases} \tag{7.34}$$

for suitable $g_j \in L^2(\mathbb{R}^n)$ and $\xi_j \in \Omega_{\varepsilon_j}$, where

$$Z_i^j := \frac{\partial w_{\xi_j}}{\partial x_i}.$$

Moreover,

$$\|\psi_j\|_{\star,\xi_j} = 1 \text{ for any } j \in \mathbb{N} \tag{7.35}$$

$$\text{and} \quad \|g_j\|_{\star,\xi_j} \searrow 0 \text{ as } j \to +\infty. \tag{7.36}$$

Notice that the fact that the equation in (7.34) is linear with respect to $\psi_j$, $g_j$ and $Z_i^j$ allows us to take the sequences $\psi_j$ and $g_j$ as in (7.35) and (7.36).

We claim that, for any given $R>0$,

(7.37) $$\|\psi_j\|_{L^\infty(B_R(\xi_j))} \to 0 \text{ as } j\to+\infty.$$

For this, we argue by contradiction and we assume that there exists $\delta>0$ and $j_0\in\mathbb{N}$ such that, for any $j\geqslant j_0$, we have that $\|\psi_j\|_{L^\infty(B_R(\xi_j))}\geqslant\delta$.

Thanks to Lemmata 7.2 and 5.2, we have that

$$|c_i^j| \leqslant \frac{C_1}{\alpha_i}\|g_j\|_{\star,\xi_j} + C_2\,\varepsilon_j^{n/2},$$

for suitable positive constants $C_1$ and $C_2$. Hence, from (7.36) we obtain that

(7.38) $$c_i^j \searrow 0 \text{ as } j\to+\infty \text{ for any } i\in\{1,\dots,n\}.$$

Now, from Lemma 6.1 we have that

(7.39)
$$\begin{aligned}&\sup_{x\neq y}\frac{|\psi_j(x)-\psi_j(y)|}{|x-y|^s}\\ &\quad\leqslant C\left(\left\|g_j+\sum_{i=1}^n c_i^j\,Z_i^j + pw_{\xi_j}^{p-1}\psi_j\right\|_{L^\infty(\mathbb{R}^n)} + \left\|g_j+\sum_{i=1}^n c_i^j\,Z_i^j + pw_{\xi_j}^{p-1}\psi_j\right\|_{L^2(\mathbb{R}^n)}\right).\end{aligned}$$

We observe that

$$\left\|g_j+\sum_{i=1}^n c_i^j\,Z_i^j + pw_{\xi_j}^{p-1}\psi_j\right\|_{L^\infty(\mathbb{R}^n)} \leqslant C\left(\|g\|_{\star,\xi_j}+\sum_{i=1}^n|c_i^j| + \|\psi_j\|_{\star,\xi_j}\right),$$

thanks to the decay of $Z_i^j$ in Lemma 5.2 and the fact that $w_{\xi_j}^{p-1}$ is bounded (recall (1.4)). So, from (7.36), (7.38) and (7.35), we obtain that

(7.40) $$\left\|g_j+\sum_{i=1}^n c_i^j\,Z_i^j + pw_{\xi_j}^{p-1}\psi_j\right\|_{L^\infty(\mathbb{R}^n)} \leqslant C,$$

for a suitable constant $C>0$ independent of $j$.

We claim that

(7.41) $$\left\|g_j+\sum_{i=1}^n c_i^j\,Z_i^j + pw_{\xi_j}^{p-1}\psi_j\right\|_{L^2(\mathbb{R}^n)} \leqslant C,$$

where $C>0$ does not depend on $j$. Indeed,

$$\begin{aligned}\|g_j\|_{L^2(\mathbb{R}^n)} &= \left(\int_{\mathbb{R}^n} g_j^2\,dx\right)^{1/2} \leqslant \|g_j\|_{\star,\xi_j}\left(\int_{\mathbb{R}^n}\rho_{\xi_j}^2\,dx\right)^{1/2}\\ &\leqslant \|g_j\|_{\star,\xi_j}\left(\int_{\mathbb{R}^n}\frac{1}{(1+|x-\xi_j|)^{2\mu}}\,dx\right)^{1/2} \leqslant C\|g_j\|_{\star,\xi_j}\leqslant C,\end{aligned}$$

since $2\mu>n$ and (7.36) holds. Moreover,

$$\left\|\sum_{i=1}^n c_i^j\,Z_i^j\right\|_{L^2(\mathbb{R}^n)} \leqslant \sum_{i=1}^n |c_i^j|\,\|Z_i^j\|_{L^2(\mathbb{R}^n)} \leqslant C,$$

thanks to (7.38) and Lemma 5.2. Finally, using (1.4), the fact that $2\mu > n$ and (7.35), we have that

$$\begin{aligned}\left\| p w_{\xi_j}^{p-1} \psi_j \right\|_{L^2(\mathbb{R}^n)} &\leqslant p\|\psi_j\|_{\star,\xi_j} \left( \int_{\mathbb{R}^n} w_{\xi_j}^{2(p-1)} \rho_{\xi_j}^2 \, dx \right)^{1/2} \\ &\leqslant C\|\psi_j\|_{\star,\xi_j} \left( \int_{\mathbb{R}^n} \frac{1}{(1+|x-\xi_j|)^{2(p-1)(n+2s)+2\mu}} \, dx \right)^{1/2} \leqslant C.\end{aligned}$$

Putting together the above estimates, we obtain (7.41).

Hence, from (7.39), (7.40) and (7.41), we have that the $\psi_j$'s are equicontinuous.

For any $j = 1, \dots, n$, we define the function

$$\tilde{\psi}_j(x) := \psi_j(x+\xi_j),$$

and the set

$$\tilde{\Omega}_j := \{x = y - \xi_j \text{ with } y \in \Omega_{\varepsilon_j}\}.$$

We notice that $\tilde{\psi}_j$ satisfies

$$(-\Delta)^s \tilde{\psi}_j + \tilde{\psi}_j - p w^{p-1} \tilde{\psi}_j + \tilde{g}_j = \sum_{i=1}^{n} c_i^j \, \tilde{Z}_i \text{ in } \tilde{\Omega}_j, \tag{7.42}$$

where $\tilde{g}_j(x) := g(x+\xi_j)$ and $\tilde{Z}_i := \frac{\partial w}{\partial x_i}$. Moreover

$$\tilde{\psi}_j = 0 \text{ in } \mathbb{R}^n \setminus \tilde{\Omega}_j. \tag{7.43}$$

Now, thanks to (6.1) we have that $B_{c/\varepsilon_j}(\xi_j) \subset \Omega_{\varepsilon_j}$. Hence, $B_{c/\varepsilon_j} \subset \tilde{\Omega}_j$, which means that $\tilde{\Omega}_j$ converges to $\mathbb{R}^n$ when $j \to +\infty$.

Furthermore, we have that

$$\|\tilde{\psi}_j\|_{L^\infty(B_R)} \geqslant \delta \quad \text{and} \quad \|(1+|x|)^\mu \tilde{\psi}_j\|_{L^\infty(\mathbb{R}^n)} = 1. \tag{7.44}$$

Now, since the $\psi_j$'s are equicontinuous, the $\tilde{\psi}_j$'s are equicontinuous too, and therefore there exists a function $\bar{\psi}$ such that, up to subsequences, $\tilde{\psi}_j$ converge to $\bar{\psi}$ uniformly on compact sets.

The function $\bar{\psi} \in L^2(\mathbb{R}^n)$. Indeed, by Fatou's Theorem and (7.35) and recalling that $2\mu > n$, we have

$$\int_{\mathbb{R}^n} \bar{\psi}^2 \, dx \leqslant \liminf_{j\to+\infty} \int_{\mathbb{R}^n} \psi_j^2 \, dx \leqslant \liminf_{j\to+\infty} \|\psi_j\|_{\star,\xi_j}^2 \int_{\mathbb{R}^n} \frac{1}{(1+|x-\xi_j|)^{2\mu}} \, dx \leqslant C.$$

Moreover, $\bar{\psi}$ satisfies the conditions

$$\|\bar{\psi}\|_{L^\infty(B_R)} \geqslant \delta \tag{7.45}$$

$$\text{and} \quad \|(1+|x|)^\mu \bar{\psi}\|_{L^\infty(\mathbb{R}^n)} \leqslant 1. \tag{7.46}$$

We prove that $\bar{\psi}$ solves the equation

$$(-\Delta)^s \bar{\psi} + \bar{\psi} = p w^{p-1} \bar{\psi} \text{ in } \mathbb{R}^n. \tag{7.47}$$

Indeed, we multiply the equation in (7.42) by a function $\eta \in C_0^\infty(\tilde{\Omega}_j)$ and we integrate over $\mathbb{R}^n$. We notice that both $\eta$ and $\tilde{\psi}_j$ are equal to zero outside $\tilde{\Omega}_j$ (recall (7.43)), and therefore we can use formula $(1.5)$ in [30] and we get

$$\int_{\mathbb{R}^n}\left((-\Delta)^s\eta+\eta-pw^{p-1}\eta\right)\tilde{\psi}_j\,dx+\int_{\mathbb{R}^n}\tilde{g}_j\,\eta\,dx=\sum_{i=1}^n c_i^j\int_{\mathbb{R}^n}\tilde{Z}_i\,\eta\,dx. \tag{7.48}$$

Now, we have that

$$\|\tilde{g}_j\|_{\star,0}=\|g_j\|_{\star,\xi_j}\searrow 0 \text{ as } j\to+\infty.$$

Moreover,

$$\left|\int_{\mathbb{R}^n}\tilde{g}_j\,\eta\,dx\right|\leqslant\|\tilde{g}_j\|_{\star,0}\int_{\mathbb{R}^n}\rho_0\,\eta\leqslant C\|\tilde{g}_j\|_{\star,0},$$

since $2\mu>n$, which implies that

$$\int_{\mathbb{R}^n}\tilde{g}_j\,\eta\,dx\to 0 \text{ as } j\to+\infty. \tag{7.49}$$

Also,

$$\left|\sum_{i=1}^n c_i^j\int_{\mathbb{R}^n}\tilde{Z}_i\,\eta\,dx\right|\leqslant C\sum_{i=1}^n|c_i^j|,$$

and so, thanks to (7.38), we obtain that

$$\sum_{i=1}^n c_i^j\int_{\mathbb{R}^n}\tilde{Z}_i\,\eta\,dx\to 0 \text{ as } j\to+\infty. \tag{7.50}$$

Finally, we fix $r>0$ and we estimate

(7.51)

$$\begin{aligned}
&\left|\int_{\mathbb{R}^n}\left((-\Delta)^s\eta+\eta-pw^{p-1}\eta\right)\tilde{\psi}_j\,dx-\int_{\mathbb{R}^n}\left((-\Delta)^s\eta+\eta-pw^{p-1}\eta\right)\bar{\psi}\,dx\right|\\
&\quad\leqslant\int_{\mathbb{R}^n}|(-\Delta)^s\eta+\eta-pw^{p-1}\eta|\,|\tilde{\psi}_j-\bar{\psi}|\,dx\\
&\quad=\int_{B_r}|(-\Delta)^s\eta+\eta-pw^{p-1}\eta|\,|\tilde{\psi}_j-\bar{\psi}|\,dx+\int_{\mathbb{R}^n\setminus B_r}|(-\Delta)^s\eta+\eta-pw^{p-1}\eta|\,|\tilde{\psi}_j-\bar{\psi}|\,dx.
\end{aligned}$$

We define the function

$$\tilde{\eta}:=(-\Delta)^s\eta+\eta-pw^{p-1}\eta$$

and we notice that it satisfies the following decay

$$|\tilde{\eta}(x)|\leqslant\frac{C_1}{(1+|x|)^{n+2s}}. \tag{7.52}$$

Hence,

$$\begin{aligned}
\int_{B_r}|(-\Delta)^s\eta+\eta-pw^{p-1}\eta|\,|\tilde{\psi}_j-\bar{\psi}|\,dx &\leqslant C_1\|\tilde{\psi}_j-\bar{\psi}\|_{L^\infty(B_r)}\int_{B_r}\frac{1}{(1+|x|)^{n+2s}}\\
&\leqslant C_2\|\tilde{\psi}_j-\bar{\psi}\|_{L^\infty(B_r)},
\end{aligned}$$

which implies that

$$\int_{B_r}|(-\Delta)^s\eta+\eta-pw^{p-1}\eta|\,|\tilde{\psi}_j-\bar{\psi}|\,dx\searrow 0 \text{ as } j\to+\infty, \tag{7.53}$$

due to the uniform convergence of $\tilde{\psi}_j$ to $\bar{\psi}$ on compact sets. On the other hand, from (7.44), (7.46) and (7.52) we have that

$$\begin{aligned}\int_{\mathbb{R}^n\setminus B_r} |(-\Delta)^s\eta+\eta-pw^{p-1}\eta|\,|\tilde{\psi}_j-\bar{\psi}|\,dx &\leqslant 2\,C_1\int_{\mathbb{R}^n\setminus B_r}\frac{1}{(1+|x|)^{n+2s}}\,dx\\ &\leqslant 2\,C_1\int_{\mathbb{R}^n\setminus B_r}\frac{1}{|x|^{n+2s}}\,dx\\ &\leqslant C_3 r^{-2s}.\end{aligned}$$

Hence, sending $r\longrightarrow+\infty$, we obtain that

$$\int_{\mathbb{R}^n\setminus B_r} |(-\Delta)^s\eta+\eta-pw^{p-1}\eta|\,|\tilde{\psi}_j-\bar{\psi}|\,dx\searrow 0. \tag{7.54}$$

Putting together (7.51), (7.53) and (7.54), we obtain that

$$\int_{\mathbb{R}^n}\left((-\Delta)^s\eta+\eta-pw^{p-1}\eta\right)\tilde{\psi}_j\,dx\to\int_{\mathbb{R}^n}\left((-\Delta)^s\eta+\eta-pw^{p-1}\eta\right)\bar{\psi}\,dx \text{ as } j\to+\infty.$$

This, (7.49), (7.50) and (7.48) imply that

$$\int_{\mathbb{R}^n}\left((-\Delta)^s\eta+\eta-pw^{p-1}\eta\right)\bar{\psi}\,dx=0$$

for any $\eta\in C_0^\infty(\mathbb{R}^n)$. This means that $\bar{\psi}$ is a weak solution to (7.47), and so a strong solution, thanks to [33].

Hence, recalling the nondegeneracy result in [21], we have that

$$\bar{\psi}=\sum_{i=1}^n\beta_i\frac{\partial w}{\partial x_i}, \tag{7.55}$$

for some coefficients $\beta_i\in\mathbb{R}$.

On the other hand, the orthogonality condition in (7.34) passes to the limit, that is

$$\int_{\mathbb{R}^n}\bar{\psi}\,\tilde{Z}_i\,dx=0,\ \text{ for any } i=1,\dots,n. \tag{7.56}$$

Indeed, we fix $r>0$ and we compute

$$\int_{\mathbb{R}^n}(\bar{\psi}-\tilde{\psi}_j)\tilde{Z}_i\,dx=\int_{B_r}(\bar{\psi}-\tilde{\psi}_j)\tilde{Z}_i\,dx+\int_{\mathbb{R}^n\setminus B_r}(\bar{\psi}-\tilde{\psi}_j)\tilde{Z}_i\,dx. \tag{7.57}$$

Concerning the first term on the right-hand side, we use the uniform convergence of $\tilde{\psi}_j$ to $\bar{\psi}$ on compact sets together with the fact that $\tilde{Z}_i$ is bounded to obtain that

$$\int_{B_r}(\bar{\psi}-\tilde{\psi}_j)\tilde{Z}_i\,dx\to 0 \text{ as } j\to+\infty.$$

As for the second term, we use (7.44), (7.46) and Lemma 5.2 and we get

$$\int_{\mathbb{R}^n\setminus B_r}(\bar{\psi}-\tilde{\psi}_j)\tilde{Z}_i\,dx\leqslant C\int_{\mathbb{R}^n\setminus B_r}\frac{1}{|x|^{n+2s}}\,dx\leqslant\bar{C}\,r^{-2s},$$

which tends to zero as $r\longrightarrow+\infty$. Using the above two formulas into (7.57) we obtain that

$$0=\int_{\mathbb{R}^n}\tilde{\psi}_j\tilde{Z}_i\,dx\to\int_{\mathbb{R}^n}\bar{\psi}\tilde{Z}_i\,dx,$$

which implies (7.56).

Therefore, recalling also (5.3), (7.55) and (7.56) imply that $\bar{\psi} \equiv 0$, thus contradicting (7.45). This proves (7.37).

Now, from Corollary 6.3 (notice that we can take $R$ sufficiently big in order to apply the corollary) we have that

$$\begin{aligned} \|\psi_j\|_{\star,\xi_j} &\leqslant C\left(\|\psi_j\|_{L^\infty(B_R(\xi_j))} + \left\|g_j + \sum_{i=1}^n c_i^j\, Z_i^j\right\|_{\star,\xi_j}\right) \\ &\leqslant C\left(\|\psi_j\|_{L^\infty(B_R(\xi_j))} + \|g_j\|_{\star,\xi_j} + \left\|\sum_{i=1}^n |c_i^j|\, Z_i^j\right\|_{\star,\xi_j}\right) \\ &= C\left(\|\psi_j\|_{L^\infty(B_R(\xi_j))} + \|g_j\|_{\star,\xi_j} + \left\|\sum_{i=1}^n |c_i^j|\, \rho_{\xi_j}^{-1}\, Z_i^j\right\|_{L^\infty(\mathbb{R}^n)}\right) \\ &\leqslant C\left(\|\psi_j\|_{L^\infty(B_R(\xi_j))} + \|g_j\|_{\star,\xi_j} + \sum_{i=1}^n |c_i^j|\right) \end{aligned}$$

up to renaming $C$, where we have used the decay of $Z_i^j$ (see Lemma 5.2) and the fact that $\mu < n+2s$. Therefore, (7.36), (7.37) and (7.38) imply that

$$\|\psi_j\|_{\star,\xi_j} \to 0 \text{ as } j \to +\infty,$$

which contradicts (7.35) and concludes the proof. □

Now we consider an auxiliary problem: we look for a solution $\psi \in \Psi$ of

$$(-\Delta)^s\psi + \psi + g = \sum_{i=1}^n c_i\, Z_i \text{ in } \Omega_\varepsilon, \tag{7.58}$$

and we prove the following:

**Proposition 7.4.** *Let $g \in L^2(\mathbb{R}^n)$ with $\|g\|_{\star,\xi} < +\infty$. Then, there exists a unique $\psi \in \Psi$ solution to (7.58).*

*Furthermore, there exists a constant $C > 0$ such that*

$$\|\psi\|_{\star,\xi} \leqslant C\|g\|_{\star,\xi}. \tag{7.59}$$

*Proof.* We first prove the existence of a solution to (7.58).

First of all, we notice that formula $(1.5)$ in [30] implies that, for any $\psi, \varphi \in \Psi$,

$$\int_{\mathbb{R}^n} (-\Delta)^s\psi\, \varphi\, dx = \int_{\mathbb{R}^n} (-\Delta)^{s/2}\psi\, (-\Delta)^{s/2}\varphi\, dx = \int_{\mathbb{R}^n} \psi\, (-\Delta)^s\varphi\, dx.$$

Now, given $g \in L^2(\mathbb{R}^n)$, we look for a solution $\psi \in \Psi$ of the problem

$$\int_{\mathbb{R}^n} (-\Delta)^{s/2}\psi\, (-\Delta)^{s/2}\varphi\, dx + \int_{\mathbb{R}^n} \psi\, \varphi\, dx + \int_{\mathbb{R}^n} g\, \varphi\, dx = 0, \tag{7.60}$$

for any $\varphi \in \Psi$. Subsequently we will show that $\psi$ is a solution to the original problem (7.58).

We observe that

$$\langle \psi, \varphi \rangle := \int_{\mathbb{R}^n} (-\Delta)^{s/2}\psi\, (-\Delta)^{s/2}\varphi\, dx + \int_{\mathbb{R}^n} \psi\, \varphi\, dx$$

defines an inner product in $\Psi$, and that

$$F(\varphi) := -\int_{\mathbb{R}^n} g\,\varphi\,dx$$

is a linear and continuous functional on $\Psi$. Hence, from Riesz's Theorem we have that there exists a unique function $\psi \in \Psi$ which solves (7.60).

We claim that

$$\psi \text{ is a strong solution to (7.58).} \tag{7.61}$$

For this, we take a radial cutoff $\tau \in C_0^\infty(\Omega_\varepsilon)$ of the form $\tau(x) = \tau_o(|x-\xi|)$, for some smooth and compactly supported real function, and we use Lemma 5.5. So, for any $\phi \in H^s(\mathbb{R}^n)$ such that $\phi = 0$ outside $\Omega_\varepsilon$, we define

$$\tilde{\phi} := \phi - \sum_{i=1}^{n} \lambda_i(\phi)\,\tilde{Z}_i,$$

where

$$\lambda_i(\phi) := \tilde{\alpha}^{-1} \int_{\mathbb{R}^n} \phi\,Z_i\,dx, \tag{7.62}$$

and $\tilde{Z}_i$ and $\tilde{\alpha}$ are as in Lemma 5.5. We remark that $\tilde{Z}_i$ vanishes outside $\Omega_\varepsilon$, hence so does $\tilde{\phi}$. Furthermore, for any $j \in \{1, \dots, n\}$,

$$\begin{aligned}
\int_{\Omega_\varepsilon} \tilde{\phi}\,Z_j\,dx &= \int_{\mathbb{R}^n} \tilde{\phi}\,Z_j\,dx \\
&= \int_{\mathbb{R}^n} \phi\,Z_j\,dx - \sum_{i=1}^{n} \lambda_i(\phi) \int_{\mathbb{R}^n} \tilde{Z}_i\,Z_j\,dx \\
&= \int_{\mathbb{R}^n} \phi\,Z_j\,dx - \sum_{i=1}^{n} \lambda_i(\phi)\,\tilde{\alpha}\,\delta_{ij} \\
&= \int_{\mathbb{R}^n} \phi\,Z_j\,dx - \lambda_j(\phi)\,\tilde{\alpha} \\
&= 0,
\end{aligned}$$

thanks to Lemma 5.5 and (7.62). This shows that $\tilde{\phi} \in \Psi$.

As a consequence, we can use $\tilde{\phi}$ as a test function in (7.60) and conclude that

$$\begin{aligned}
&\int_{\mathbb{R}^n} (-\Delta)^{s/2}\psi\,(-\Delta)^{s/2}\left(\phi - \sum_{i=1}^{n} \lambda_i(\phi)\,\tilde{Z}_i\right)\,dx \\
&\qquad + \int_{\mathbb{R}^n} \psi\left(\phi - \sum_{i=1}^{n} \lambda_i(\phi)\,\tilde{Z}_i\right)\,dx + \int_{\mathbb{R}^n} g\left(\phi - \sum_{i=1}^{n} \lambda_i(\phi)\,\tilde{Z}_i\right)\,dx = 0,
\end{aligned}$$

that is

$$\begin{aligned}
&\int_{\mathbb{R}^n} (-\Delta)^{s/2}\psi\,(-\Delta)^{s/2}\phi\,dx + \int_{\mathbb{R}^n} \psi\,\phi\,dx + \int_{\mathbb{R}^n} g\,\phi\,dx \\
&\qquad = \int_{\mathbb{R}^n} (\psi+g)\sum_{i=1}^n \lambda_i(\phi)\,\tilde{Z}_i\,dx + \int_{\mathbb{R}^n} (-\Delta)^s\psi\,\sum_{i=1}^n \lambda_i(\phi)\,\tilde{Z}_i\,dx \\
&\qquad = \sum_{i=1}^n \lambda_i(\phi)\int_{\mathbb{R}^n} \left(\psi+g+(-\Delta)^s\psi\right)\,\tilde{Z}_i\,dx.
\end{aligned} \tag{7.63}$$

Now we define

$$b_i := \tilde{\alpha}^{-1}\int_{\mathbb{R}^n} \left(\psi+g+(-\Delta)^s\psi\right)\,\tilde{Z}_i\,dx,$$

we recall (7.62) and we write (7.63) as

$$\begin{aligned}
&\int_{\mathbb{R}^n} (-\Delta)^{s/2}\psi\,(-\Delta)^{s/2}\phi\,dx + \int_{\mathbb{R}^n} \psi\,\phi\,dx + \int_{\mathbb{R}^n} g\,\phi\,dx \\
&\qquad = \sum_{i=1}^n \lambda_i(\phi)\,\tilde{\alpha}\,b_i \\
&\qquad = \sum_{i=1}^n b_i\int_{\mathbb{R}^n} \phi\,Z_i\,dx.
\end{aligned}$$

Since $\phi$ is any test function, this means that $\psi$ is a solution of

$$(-\Delta)^s\psi + \psi + g = \sum_{i=1}^n b_i\,Z_i.$$

in a weak sense, and therefore in a strong sense, thanks to [33], thus proving (7.61).

Now, we prove the uniqueness of the solution to (7.58). For this, suppose by contradiction that there exist $\psi_1$ and $\psi_2$ in $\Psi$ that solve (7.58). We set

$$\tilde{\psi} := \psi_1 - \psi_2,$$

and we observe that $\tilde{\psi}\in\Psi$ and solves

$$(-\Delta)^s\tilde{\psi} + \tilde{\psi} = \sum_{i=1}^n a_i\,Z_i \text{ in } \Omega_\varepsilon, \tag{7.64}$$

for suitable coefficients $a_i\in\mathbb{R}$, $i=1,\dots,n$.

We multiply the equation in (7.64) by $\tilde{\psi}$ and we integrate over $\Omega_\varepsilon$, obtaining that

$$\int_{\Omega_\varepsilon} (-\Delta)^s\tilde{\psi}\,\tilde{\psi} + \tilde{\psi}^2\,dx = 0,$$

since $\tilde{\psi}\in\Psi$ (and so it is orthogonal to $Z_i$ in $L^2(\Omega_\varepsilon)$ for any $i=1,\dots,n$). Since $\tilde{\psi}=0$ outside $\Omega_\varepsilon$, we can apply formula $(1.5)$ in [30] and we obtain that

$$\int_{\mathbb{R}^n} \left|(-\Delta)^{s/2}\tilde{\psi}\right|^2 + \tilde{\psi}^2\,dx = 0,$$

that is

$$\|\tilde{\psi}\|_{H^s(\mathbb{R}^n)} = 0,$$

which implies that $\tilde{\psi}=0$. Thus $\psi_1=\psi_2$ and this concludes the proof of the uniqueness.

It remains to establish (7.59). Thanks to (7.58) and Corollary 6.4, we have that

(7.65) $$\|\psi\|_{\star,\xi} \leqslant C \left\| g - \sum_{i=1}^{n} c_i\, Z_i \right\|_{\star,\xi} \leqslant C \left( \|g\|_{\star,\xi} + \left\| \sum_{i=1}^{n} c_i\, Z_i \right\|_{\star,\xi} \right).$$

First, we observe that for any $i = 1, \ldots, n$

$$\|Z_i\|_{\star,\xi} = \sup_{\mathbb{R}^n} |\rho_\xi^{-1}\, Z_i| \leqslant C_1,$$

due to Lemma 5.2 and the fact that $\mu < n + 2s$ (recall also (6.2)). Hence,

(7.66) $$\left\| \sum_{i=1}^{n} c_i\, Z_i \right\|_{\star,\xi} \leqslant \sum_{i=1}^{n} |c_i|\, \|Z_i\|_{\star,\xi} \leqslant C_1 \sum_{i=1}^{n} |c_i|.$$

Now, we claim that

(7.67) $$\left\| \sum_{i=1}^{n} c_i\, Z_i \right\|_{\star,\xi} \leqslant C_2 \left( \|\psi\|_{L^2(\mathbb{R}^n)} + \|g\|_{L^2(\mathbb{R}^n)} \right).$$

Indeed, we recall Lemma 5.5, we multiply equation (7.58) by $\tilde{Z}_j$, for $j \in \{1, \ldots, n\}$, and we integrate over $\mathbb{R}^n$, obtaining that

(7.68) $$\int_{\mathbb{R}^n} (-\Delta)^s \psi\, \tilde{Z}_j + \psi\, \tilde{Z}_j + g\, \tilde{Z}_j\, dx = \tilde{\alpha} c_j,$$

where $\tilde{Z}_j$ and $\tilde{\alpha}$ are as in Lemma 5.5. Thanks to formula $(1.5)$ in [30], we have that

$$\left| \int_{\mathbb{R}^n} (-\Delta)^s \psi\, \tilde{Z}_j\, dx \right| = \left| \int_{\mathbb{R}^n} \psi\, (-\Delta)^s \tilde{Z}_j\, dx \right| \leqslant \|(-\Delta)^s \tilde{Z}_j\|_{L^2(\mathbb{R}^n)} \|\psi\|_{L^2(\mathbb{R}^n)},$$

where we have used Hölder inequality. Therefore, this and (7.68) give that

$$\tilde{\alpha} |c_j| \leqslant \|(-\Delta)^s \tilde{Z}_j\|_{L^2(\mathbb{R}^n)} \|\psi\|_{L^2(\mathbb{R}^n)} + \|\tilde{Z}_j\|_{L^2(\mathbb{R}^n)} \|\psi\|_{L^2(\mathbb{R}^n)} + \|\tilde{Z}_j\|_{L^2(\mathbb{R}^n)} \|g\|_{L^2(\mathbb{R}^n)},$$

which, together with (7.66), implies (7.67), since both $\|(-\Delta)^s \tilde{Z}_j\|_{L^2(\mathbb{R}^n)}$ and $\|\tilde{Z}_j\|_{L^2(\mathbb{R}^n)}$ are bounded (recall Lemma 5.4).

Now, we observe that

(7.69) $$\|\psi\|_{L^2(\mathbb{R}^n)} \leqslant \|g\|_{L^2(\mathbb{R}^n)}.$$

Indeed, we multiply equation (7.58) by $\psi$ and we integrate over $\Omega_\varepsilon$: we obtain

$$\int_{\Omega_\varepsilon} (-\Delta)^s \psi\, \psi + \psi^2 + g\, \psi\, dx = 0,$$

since $\psi \in \Psi$. We notice that the first term in the above formula is quadratic, and so, using Hölder inequality, we have that

$$\int_{\Omega_\varepsilon} \psi^2\, dx \leqslant \int_{\Omega_\varepsilon} (-g) \psi\, dx \leqslant \|g\|_{L^2(\mathbb{R}^n)} \|\psi\|_{L^2(\mathbb{R}^n)},$$

which implies (7.69).

Therefore, from (7.67) and (7.69), we deduce that

$$\left\| \sum_{i=1}^{n} c_i\, Z_i \right\|_{\star,\xi} \leqslant 2\, C_3\, \|g\|_{L^2(\mathbb{R}^n)}.$$

Moreover,

$$\|g\|_{L^2(\mathbb{R}^n)} \leqslant \|g\|_{\star,\xi} \left( \int_{\mathbb{R}^n} \rho_\xi^2 \, dx \right)^{1/2} = \|g\|_{\star,\xi} \left( \int_{\mathbb{R}^n} \frac{1}{(1+|x-\xi|)^{2\mu}} \, dx \right)^{1/2} \leqslant C\|g\|_{\star,\xi},$$

since $2\mu > n$. The above two formulas give that

$$\left\| \sum_{i=1}^{n} c_i \, Z_i \right\|_{\star,\xi} \leqslant C_4 \, \|g\|_{\star,\xi}.$$

This and (7.65) shows (7.59) and conclude the proof. $\square$

Now, for any $g \in L^2(\mathbb{R}^n)$ with $\|g\|_{\star,\xi} < +\infty$, we denote by $\mathcal{A}[g]$ the unique solution to (7.58). We notice that Proposition 7.4 implies that the operator $\mathcal{A}$ is well defined and that

$$\|\mathcal{A}[g]\|_{\star,\xi} \leqslant C\|g\|_{\star,\xi}.$$

We also remark that $\mathcal{A}$ is a linear operator.

We consider the Banach space

(7.70) $$Y_\star := \{\psi : \mathbb{R}^n \to \mathbb{R} \text{ s.t. } \|\psi\|_{\star,\xi} < +\infty\}$$

endowed with the norm $\|\cdot\|_{\star,\xi}$.

With this notation we can prove the main theorem of the linear theory, i.e. Theorem 7.1.

*Proof of Theorem 7.1.* We notice that solving (7.7) is equivalent to find a function $\psi \in \Psi$ such that

(7.71) $$\psi - \mathcal{A}[-pw_\xi^{p-1}\psi] = \mathcal{A}[g].$$

For this, we set

(7.72) $$\mathcal{B}[\psi] := \mathcal{A}[-pw_\xi^{p-1}\psi].$$

Recalling the definition of $Y_\star$ given in (7.70), we observe that

(7.73) $$\text{if } \psi \in Y_\star \text{ then } \mathcal{B}[\psi] \in Y_\star.$$

Indeed, from Proposition 7.4 we deduce that $\mathcal{B}[\psi] \in \Psi$ solves (7.58) with $g := -pw_\xi^{p-1}\psi$, and so

$$\|\mathcal{B}[\psi]\|_{\star,\xi} \leqslant C\| - pw_\xi^{p-1}\psi\|_{\star,\xi} \leqslant \tilde{C}\|\psi\|_{\star,\xi},$$

for some $\tilde{C} > 0$ (recall that $w_\xi$ is bounded thanks to (1.4)), which proves (7.73).

We claim that

(7.74) $\mathcal{B}$ defines a compact operator in $Y_\star$ with respect to the norm $\|\cdot\|_{\star,\xi}$.

Indeed, let $(\psi_j)_j$ a bounded sequence in $Y_\star$ with respect to the norm $\|\cdot\|_{\star,\xi}$. Then, thanks to Lemma 6.1, the fact that $w_\xi^{p-1}$ and $Z_i^j$ are bounded and $w_\xi^{p-1}\rho_\xi$ and $Z_i^j$ belong to $L^2(\mathbb{R}^n)$ and

Lemma 7.2, we have that

$$
\begin{aligned}
&\sup_{x\neq y}\frac{|\mathcal{B}[\psi_j](x)-\mathcal{B}[\psi_j](y)|}{|x-y|^s}\\
&\leqslant C_1\left(\left\|-pw_\xi^{p-1}\psi_j+\sum_{i=1}^n c_i^j\, Z_i^j\right\|_{L^\infty(\mathbb{R}^n)}+\left\|-pw_\xi^{p-1}\psi_j+\sum_{i=1}^n c_i^j\, Z_i^j\right\|_{L^2(\mathbb{R}^n)}\right)\\
&\leqslant C_2\left(\|\psi_j\|_{L^\infty(\mathbb{R}^n)}+\sum_{i=1}^n|c_i^j|\|Z_i^j\|_{L^\infty(\mathbb{R}^n)}+\|\psi_j\|_{\star,\xi}\,\|w_\xi^{p-1}\rho_\xi\|_{L^2(\mathbb{R}^n)}+\sum_{i=1}^n|c_i^j|\|Z_i^j\|_{L^2(\mathbb{R}^n)}\right)\\
&\leqslant C_3\left(\|\psi_j\|_{\star,\xi}+\sum_{i=1}^n|c_i^j|\right)\\
&\leqslant C_4,
\end{aligned}
$$

for suitable positive constants $C_1, C_2, C_3$ and $C_4$. This gives the equicontinuity of the sequence $\mathcal{B}[\psi_j]$, and so it converges to a function $\bar b$ uniformly on compact sets. Hence, for any $R>0$, we have

$$
\|\mathcal{B}[\psi_j]-\bar b\|_{L^\infty(B_R(\xi))}\to 0 \text{ as } j\to+\infty. \tag{7.75}
$$

On the other hand, for any $x\in\mathbb{R}^n\setminus B_R(\xi)$, we have the following estimate

$$
\begin{aligned}
|w_\xi^{p-1}(x)\psi_j(x)| &\leqslant \|\psi_j\|_{\star,\xi}|w_\xi^{p-1}(x)\rho_\xi(x)|\\
&\leqslant C_5\|\psi_j\|_{\star,\xi}\left|\frac{1}{(1+|x-\xi|)^{(n+2s)(p-1)}}\rho_\xi(x)\right|\\
&\leqslant C_5\|\psi_j\|_{\star,\xi}\rho_\xi^{1+\frac{(n+2s)(p-1)}{\mu}}(x),
\end{aligned}
$$

for some $C_5>0$, where we have used the decay of $w_\xi$ in (1.4) and the expression of $\rho_\xi$ given in (6.2). This implies that

$$
\sup_{x\in\mathbb{R}^n\setminus B_R(\xi)}|\rho_\xi^{-1}w_\xi^{p-1}\psi_j|\leqslant C_5\|\psi_j\|_{\star,\xi}\sup_{x\in\mathbb{R}^n\setminus B_R(\xi)}\rho_\xi^\sigma(x),
$$

where

$$
\sigma:=\frac{(n+2s)(p-1)}{\mu}>0. \tag{7.76}
$$

Hence, since $\psi_j$ is a uniformly bounded sequence with respect to the norm $\|\cdot\|_{\star,\xi}$, we obtain that

$$
\sup_{x\in\mathbb{R}^n\setminus B_R(\xi)}|\rho_\xi^{-1}\mathcal{B}[\psi_j]|\leqslant C_6\sup_{x\in\mathbb{R}^n\setminus B_R(\xi)}\rho_\xi^\sigma(x). \tag{7.77}
$$

It follows that

$$
\sup_{x\in\mathbb{R}^n\setminus B_R(\xi)}|\rho_\xi^{-1}\bar b|\leqslant C_6\sup_{x\in\mathbb{R}^n\setminus B_R(\xi)}\rho_\xi^\sigma(x). \tag{7.78}
$$

We observe that

$$\begin{aligned}\sup_{x\in\mathbb{R}^n}\left|\rho_\xi^{-1}\left(\mathcal{B}[\psi_j]-\bar{b}\right)\right| &= \sup_{x\in\mathbb{R}^n}\left|\rho_\xi^{-1}\left(\mathcal{B}[\psi_j]-\bar{b}\right)\chi_{B_R(\xi)}+\rho_\xi^{-1}\left(\mathcal{B}[\psi_j]-\bar{b}\right)\chi_{\mathbb{R}^n\setminus B_R(\xi)}\right|\\ &\leqslant \sup_{x\in\mathbb{R}^n}\left(\left|\rho_\xi^{-1}\left(\mathcal{B}[\psi_j]-\bar{b}\right)\chi_{B_R(\xi)}\right|+\left|\rho_\xi^{-1}\left(\mathcal{B}[\psi_j]-\bar{b}\right)\chi_{\mathbb{R}^n\setminus B_R(\xi)}\right|\right)\\ &\leqslant \sup_{x\in\mathbb{R}^n}\left|\rho_\xi^{-1}\left(\mathcal{B}[\psi_j]-\bar{b}\right)\chi_{B_R(\xi)}\right|+\sup_{x\in\mathbb{R}^n}\left|\rho_\xi^{-1}\left(\mathcal{B}[\psi_j]-\bar{b}\right)\chi_{\mathbb{R}^n\setminus B_R(\xi)}\right|\\ &= \sup_{x\in B_R(\xi)}\left|\rho_\xi^{-1}\left(\mathcal{B}[\psi_j]-\bar{b}\right)\right|+\sup_{x\in\mathbb{R}^n\setminus B_R(\xi)}\left|\rho_\xi^{-1}\left(\mathcal{B}[\psi_j]-\bar{b}\right)\right|.\end{aligned}$$

Therefore, we obtain that

$$\begin{aligned}\|\mathcal{B}[\psi_j]-\bar{b}\|_{\star,\xi} &= \sup_{x\in\mathbb{R}^n}\left|\rho_\xi^{-1}\left(\mathcal{B}[\psi_j]-\bar{b}\right)\right|\\ &\leqslant \sup_{x\in B_R(\xi)}\left|\rho_\xi^{-1}\left(\mathcal{B}[\psi_j]-\bar{b}\right)\right|+\sup_{x\in\mathbb{R}^n\setminus B_R(\xi)}\left|\rho_\xi^{-1}\left(\mathcal{B}[\psi_j]-\bar{b}\right)\right|\\ &\leqslant \sup_{x\in B_R(\xi)}\left|\rho_\xi^{-1}\left(\mathcal{B}[\psi_j]-\bar{b}\right)\right|+C_7\sup_{x\in\mathbb{R}^n\setminus B_R(\xi)}\rho_\xi^{\sigma}(x),\end{aligned}\tag{7.79}$$

where we have also used (7.77) and (7.78). Concerning the first term in the right-hand side, we have

$$\begin{aligned}\sup_{x\in B_R(\xi)}\left|\rho_\xi^{-1}\left(\mathcal{B}[\psi_j]-\bar{b}\right)\right| &= \sup_{x\in B_R(\xi)}\left|(1+|x-\xi|)^{\mu}\left(\mathcal{B}[\psi_j]-\bar{b}\right)\right|\\ &\leqslant (1+R)^{\mu}\|\mathcal{B}[\psi_j]-\bar{b}\|_{L^\infty(B_R(\xi))}.\end{aligned}$$

Therefore, sending $j\to+\infty$ and recalling (7.75), we obtain that

$$\sup_{x\in B_R(\xi)}\left|\rho_\xi^{-1}\left(\mathcal{B}[\psi_j]-\bar{b}\right)\right|\to 0 \text{ as } j\to+\infty.\tag{7.80}$$

Now, we send $R\to+\infty$ and, recalling (7.76), we get

$$\sup_{x\in\mathbb{R}^n\setminus B_R(\xi)}\rho_\xi^{\sigma}(x)\to 0 \text{ as } R\to+\infty.\tag{7.81}$$

Putting together (7.79), (7.80) and (7.81), we obtain that

$$\|\mathcal{B}[\psi_j]-\bar{b}\|_{\star,\xi}\to 0 \text{ as } j\to+\infty,$$

and this shows (7.74).

From (7.59) in Proposition 7.4, we deduce that if $g=0$ then $\psi=\mathcal{A}[g]=0$ is the unique solution to (7.58), and so by Fredholm's alternative we obtain that, for any $g\in Y_\star$, there exists a unique $\psi$ that solves (7.71) (recall (7.72) and (7.74)). This gives existence and uniqueness of the solution to (7.7), while estimate (7.8) follows from Lemma 7.3. This concludes the proof of Theorem 7.1. □

In the next proposition we deal with the differentiability of the solution $\psi$ to (7.7) with respect to the parameter $\xi$ (we recall Theorem 7.1 for the existence and uniqueness of such solution).

For this, we denote by $\mathcal{T}_\xi$ the operator that associates to any $g\in L^2(\mathbb{R}^n)$ with $\|g\|_{\star,\xi}<+\infty$ the solution to (7.7), that is

$$\psi:=\mathcal{T}_\xi[g] \text{ is the unique solution to (7.7) in } Y_\star,\tag{7.82}$$

where $Y_\star$ is given in (7.70).

We notice that, thanks to Theorem 7.1, $\mathcal{T}_\xi$ is a linear and continuous operator from $Y_\star$ to $Y_\star$ endowed with the norm $\|\cdot\|_{\star,\xi}$, and we will write $\mathcal{T}_\xi\in\mathcal{L}(Y_\star)$.

**Proposition 7.5.** *The map $\xi \in \Omega_\varepsilon \mapsto \mathcal{T}_\xi$ is continuously differentiable. Moreover there exists a positive constant $C$ such that*

$$\left\| \frac{\partial \mathcal{T}_\xi[g]}{\partial \xi} \right\|_{\star,\xi} \leqslant C \left( \|g\|_{\star,\xi} + \left\| \frac{\partial g}{\partial \xi} \right\|_{\star,\xi} \right). \tag{7.83}$$

*Proof.* First, let us prove (7.83) assuming the differentiability of $\xi \mapsto \mathcal{T}_\xi$. Given $\xi \in \Omega_\varepsilon$, $|t| < 1$ with $t \neq 0$ and a function $f$, we denote by $\xi_j^t := \xi + te_j$, and by

$$D_j^t f := \frac{f(\xi_j^t) - f(\xi)}{t},$$

for any $j = 1, \dots, n$.

Also, we set

$$\varphi_j^t := D_j^t \psi \text{ and } d_{i,j}^t := D_j^t c_i. \tag{7.84}$$

Using the fact that $\psi$ is a solution to (7.7), we have that $\varphi_j^t$ solves

$$(-\Delta)^s \varphi_j^t + \varphi_j^t - p w_\xi^{p-1} \varphi_j^t = p(D_j^t w_\xi^{p-1})\psi - D_j^t g + \sum_{i=1}^n c_i\, D_j^t Z_i + \sum_{i=1}^n d_{i,j}^t\, Z_i \text{ in } \Omega_\varepsilon. \tag{7.85}$$

Moreover, we have that $\varphi_j^t \in H^s(\mathbb{R}^n)$ and $\varphi_j^t = 0$ outside $\Omega_\varepsilon$.

Now, for the fixed index $j$, for any $i \in \{1, \dots, n\}$ we define

$$\lambda_i(\varphi_j^t) := \tilde{\alpha}^{-1} \int_{\mathbb{R}^n} \varphi_j^t Z_i \, dx, \tag{7.86}$$

where $\tilde{\alpha}$ is defined in (5.21), and

$$\tilde{\varphi}_j^t := \varphi_j^t - \sum_{i=1}^n \lambda_i(\varphi_j^t)\, \tilde{Z}_i, \tag{7.87}$$

where $\tilde{Z}_i$ are the ones in Lemma 5.5. We remark that $\varphi_j^t$ and $\tilde{Z}_i$ vanish outside $\Omega_\varepsilon$ by construction. Hence $\tilde{\varphi}_j^t$ vanishes outside $\Omega_\varepsilon$ as well. Moreover,

$$\begin{aligned}
\int_{\mathbb{R}^n} \tilde{\varphi}_j^t\, Z_k \, dx &= \int_{\mathbb{R}^n} \varphi_j^t\, Z_k \, dx - \sum_{i=1}^n \lambda_i(\varphi_j^t) \int_{\mathbb{R}^n} \tilde{Z}_i\, Z_k \, dx \\
&= \int_{\mathbb{R}^n} \varphi_j^t\, Z_k \, dx - \sum_{i=1}^n \lambda_i(\varphi_j^t)\, \tilde{\alpha}\, \delta_{ik} \\
&= \int_{\mathbb{R}^n} \varphi_j^t\, Z_k \, dx - \lambda_k(\varphi_j^t)\, \tilde{\alpha} \\
&= 0,
\end{aligned}$$

thanks to Lemma 5.5 and (7.86). This yields that

$$\tilde{\varphi}_j^t \in \Psi. \tag{7.88}$$

By plugging (7.87) into (7.85), we obtain that

$$(-\Delta)^s \tilde{\varphi}_j^t + \tilde{\varphi}_j^t - p w_\xi^{p-1} \tilde{\varphi}_j^t = \tilde{g}_j + \sum_{i=1}^n d_{i,j}^t Z_i, \tag{7.89}$$

where

$$\begin{aligned}\tilde g_j \;:=\; & -(-\Delta)^s \sum_{i=1}^n \lambda_i(\tilde\varphi_j^t)\,\tilde Z_i - \sum_{i=1}^n \lambda_i(\tilde\varphi_j^t)\,\tilde Z_i \\ & + p w_\xi^{p-1} \sum_{i=1}^n \lambda_i(\tilde\varphi_j^t)\,\tilde Z_i \\ & + p(D_j^t w_\xi^{p-1})\psi - D_j^t g + \sum_{i=1}^n c_i\, D_j^t Z_i.\end{aligned} \tag{7.90}$$

From (7.89), (7.88) and Lemma 7.3, we obtain that

$$\|\tilde\varphi_j^t\|_{\star,\xi} \leqslant C\,\|\tilde g_j\|_{\star,\xi}. \tag{7.91}$$

Now we observe that

$$\left\|\sum_{i=1}^n \lambda_i(\tilde\varphi_j^t)\,\tilde Z_i\right\|_{\star,\xi} \leqslant C\,\|g\|_{\star,\xi}. \tag{7.92}$$

To prove this, we notice that the orthogonality condition $\psi \in \Psi$ implies that

$$\int_{\Omega_\varepsilon} \varphi_j^t\, Z_k\,dx = -\int_{\Omega_\varepsilon} \psi\, D_j^t Z_k\,dx,$$

for any $k \in \{1,\dots,n\}$. Hence, recalling (7.87), (7.88) and Lemma 5.5,

$$\begin{aligned}-\int_{\Omega_\varepsilon} \psi\, D_j^t Z_k\,dx &= \int_{\Omega_\varepsilon}\left(\tilde\varphi_j^t + \sum_{i=1}^n \lambda_i(\tilde\varphi_j^t)\,\tilde Z_i\right) Z_k\,dx \\ &= \sum_{i=1}^n \lambda_i(\tilde\varphi_j^t) \int_{\Omega_\varepsilon} \tilde Z_i\, Z_k\,dx \\ &= \sum_{i=1}^n \lambda_i(\tilde\varphi_j^t)\,\tilde\alpha\,\delta_{ik} \\ &= \lambda_k(\tilde\varphi_j^t)\,\tilde\alpha.\end{aligned}$$

Therefore

$$\begin{aligned}|\lambda_k(\tilde\varphi_j^t)| &= |\tilde\alpha^{-1}|\left|\int_{\Omega_\varepsilon} \psi\, D_j^t Z_k\,dx\right| \\ &\leqslant |\tilde\alpha^{-1}| \int_{\Omega_\varepsilon} \rho_\xi^{-1}|\psi|\,\rho_\xi\,|D_j^t Z_k|\,dx \\ &\leqslant |\tilde\alpha^{-1}|\,\|\psi\|_{\star,\xi} \int_{\mathbb{R}^n} \rho_\xi\,|D_j^t Z_k|\,dx \\ &\leqslant C\,\|\psi\|_{\star,\xi},\end{aligned}$$

thanks to Lemma 5.3. Using this and Lemma 5.2, and possibly renaming the constants, we obtain that

$$\begin{aligned}\left\|\sum_{i=1}^{n}\lambda_i(\tilde\varphi_j^t)\,\tilde Z_i\right\|_{\star,\xi} &\leqslant \sum_{i=1}^{n}|\lambda_i(\tilde\varphi_j^t)|\,\|\tilde Z_i\|_{\star,\xi}\\ &\leqslant C\sum_{i=1}^{n}|\lambda_i(\tilde\varphi_j^t)| \leqslant C\,\|\psi\|_{\star,\xi}\ldots\end{aligned}$$

On the other hand, by Lemma 7.3, we have that $\|\psi\|_{\star,\xi}\leqslant C\,\|g\|_{\star,\xi}$, so the above estimate implies (7.92), as desired.

Now we claim that

$$\left\|(-\Delta)^s\sum_{i=1}^{n}\lambda_i(\tilde\varphi_j^t)\,\tilde Z_i\right\|_{\star,\xi}\leqslant C\,\|g\|_{\star,\xi}. \tag{7.93}$$

Indeed, $\tilde Z_i$ is compactly supported in a neighborhood of $\xi$, hence $(-\Delta)^s\tilde Z_i$ decays like $|x-\xi|^{-n-2s}$ at infinity. Accordingly, $\|(-\Delta)^s\tilde Z_i\|_{\star,\xi}$ is finite, and then we obtain

$$\begin{aligned}\left\|(-\Delta)^s\sum_{i=1}^{n}\lambda_i(\tilde\varphi_j^t)\,\tilde Z_i\right\|_{\star,\xi} &= \left\|\sum_{i=1}^{n}\lambda_i(\tilde\varphi_j^t)\,(-\Delta)^s\tilde Z_i\right\|_{\star,\xi}\\ &\leqslant \sum_{i=1}^{n}|\lambda_i(\tilde\varphi_j^t)|\,\left\|(-\Delta)^s\tilde Z_i\right\|_{\star,\xi}\\ &\leqslant C\sum_{i=1}^{n}|\lambda_i(\tilde\varphi_j^t)|\\ &\leqslant C\,\|g\|_{\star,\xi},\end{aligned}$$

due to (7.92), and this establishes (7.93).

Now we claim that

$$|D_j^t w_\xi^{p-1}|\leqslant C, \tag{7.94}$$

with $C$ independent of $t$. Indeed

$$\begin{aligned}D_j^t w_\xi^{p-1}(x) &= \frac{1}{t}\Big(w^{p-1}(x-\xi-te_j)-w^{p-1}(x-\xi)\Big)\\ &= \frac{1}{t}\int_0^t\frac{d}{d\tau}w^{p-1}(x-\xi-\tau e_j)\,d\tau\\ &= \frac{p-1}{t}\int_0^t w^{p-2}(x-\xi-\tau e_j)\frac{d}{d\tau}w(x-\xi-\tau e_j)\,d\tau\\ &= -\frac{p-1}{t}\int_0^t w^{p-2}(x-\xi-\tau e_j)\nabla w(x-\xi-\tau e_j)\cdot e_j\,d\tau.\end{aligned}$$

Also, by formulas (IV.2) and (IV.6) of [6], we know that

(7.95) $\quad w(x)$ is bounded both from above and from below by a constant times $\dfrac{1}{1+|x|^{n+2s}}$.

Thus, supposing without loss of generality that $t > 0$, and recalling Lemma 5.2, we have that

$$\begin{aligned} |D_j^t w_\xi^{p-1}(x)| &\leqslant \frac{p-1}{t}\int_0^t w^{p-2}(x-\xi-\tau e_j)|\nabla w(x-\xi-\tau e_j)|\,d\tau \\ &\leqslant \frac{C}{t}\int_0^t \big(1+|x-\xi-\tau e_j|\big)^{-(p-2)(n+2s)}\big(1+|x-\xi-\tau e_j|\big)^{-(n+2s)}\,d\tau \\ &= \frac{C}{t}\int_0^t \big(1+|x-\xi-\tau e_j|\big)^{-(p-1)(n+2s)}\,d\tau \\ &\leqslant \frac{C}{t}\int_0^t 1\,d\tau \\ &= C, \end{aligned}$$

and this proves (7.94).

From (7.94) and Lemma 7.3 we obtain that

$$\|(D_j^t w_\xi^{p-1})\psi\|_{\star,\xi} \leqslant C\,\|\psi\|_{\star,\xi} \leqslant C\,\|g\|_{\star,\xi}. \tag{7.96}$$

Now we use Lemmata 5.3, 7.2 and 7.3 to see that

$$\begin{aligned} \left\|\sum_{i=1}^n c_i D_j^t Z_i\right\|_{\star,\xi} &\leqslant \sum_{i=1}^n |c_i|\,\left\|D_j^t Z_i\right\|_{\star,\xi} \\ &\leqslant C\sum_{i=1}^n |c_i| \\ &= C\sum_{i=1}^n \left|\frac{1}{\alpha}\int_{\mathbb{R}^n} g\,Z_i\,dx + f_i\right| \\ &\leqslant C\left(\|g\|_{L^2(\mathbb{R}^n)} + \sum_{i=1}^n |f_i|\right) \\ &\leqslant C\,\big(\|\psi\|_{L^2(\mathbb{R}^n)} + \|g\|_{L^2(\mathbb{R}^n)}\big) \\ &\leqslant C\,\big(\|\psi\|_{\star,\xi} + \|g\|_{\star,\xi}\big) \\ &\leqslant C\,\|g\|_{\star,\xi}. \end{aligned} \tag{7.97}$$

By plugging (7.92), (7.93), (7.96) and (7.97) into (7.90) we obtain that

$$\|\tilde g_j\|_{\star,\xi} \leqslant C\,\left(\|g\|_{\star,\xi} + \|D_j^t g\|_{\star,\xi}\right).$$

Therefore, by (7.91),

$$\|\tilde\varphi_j^t\|_{\star,\xi} \leqslant C\,\left(\|g\|_{\star,\xi} + \|D_j^t g\|_{\star,\xi}\right).$$

This and (7.92) imply that

$$\begin{aligned} \|\varphi_j^t\|_{\star,\xi} &\leqslant \|\tilde\varphi_j^t\|_{\star,\xi} + \left\|\sum_{i=1}^n \lambda_i(\tilde\varphi_j^t)\,\tilde Z_i\right\|_{\star,\xi} \\ &\leqslant C\,\left(\|g\|_{\star,\xi} + \|D_j^t g\|_{\star,\xi}\right). \end{aligned}$$

Hence we send $t \searrow 0$ and we obtain

$$\left\|\frac{\partial\psi}{\partial\xi}\right\|_{\star,\xi} \leqslant C\left(\|g\|_{\star,\xi} + \left\|\frac{\partial g}{\partial\xi}\right\|_{\star,\xi}\right),$$

which implies that

$$\left\| \frac{\partial \mathcal{T}_\xi[g]}{\partial \xi} \right\|_{\star,\xi} \leqslant C \left( \|g\|_{\star,\xi} + \left\| \frac{\partial g}{\partial \xi} \right\|_{\star,\xi} \right).$$

Using the previous computation and the implicit function theorem, a standard argument shows that $\xi \mapsto \mathcal{T}_\xi$ is continuously differentiable (see e.g. Section $2.2.1$ in [2], and in particular Lemma $2.11$ there, or [10] below formula $(4.20)$). $\square$

7.2. **The nonlinear projected problem.** In this subsection we solve the nonlinear projected problem

$$\begin{cases} (-\Delta)^s\psi + \psi - p w_\xi^{p-1}\psi = E(\psi) + N(\psi) + \sum_{i=1}^n c_i\, Z_i & \text{in } \Omega_\varepsilon, \\ \psi = 0 & \text{in } \mathbb{R}^n \setminus \Omega_\varepsilon, \\ \int_{\Omega_\varepsilon} \psi\, Z_i\, dx = 0 & \text{for any } i = 1, \dots, n, \end{cases} \tag{7.98}$$

where $E(\psi)$ and $N(\psi)$ are given in (7.3).

**Theorem 7.6.** *If $\varepsilon > 0$ is sufficiently small, there exists a unique $\psi \in H^s(\mathbb{R}^n)$ solution to (7.98) for suitable real coefficients $c_i$, for $i = 1, \dots, n$, and such that there exists a positive constant $C$ such that*

$$\|\psi\|_{\star,\xi} \leqslant C\, \varepsilon^{n+2s}. \tag{7.99}$$

Before proving Theorem 7.6, we show some estimates for the error terms $E(\psi)$ and $N(\psi)$.

**Lemma 7.7.** *There exists a positive constant $C$ such that*

$$|\bar{u}_\xi - w_\xi| \leqslant C\, \varepsilon^{n+2s}. \tag{7.100}$$

*Proof.* To prove (7.100), we define $\eta_\xi := \bar{u}_\xi - w_\xi$, and we observe that $\eta_\xi$ satisfies

$$\begin{cases} (-\Delta)^s\eta_\xi + \eta_\xi = 0 & \text{in } \Omega_\varepsilon, \\ \eta_\xi = -w_\xi & \text{in } \mathbb{R}^n \setminus \Omega_\varepsilon, \end{cases} \tag{7.101}$$

due to (1.3) and (1.9).

We have

$$|\eta_\xi| = |w_\xi| \leqslant C\varepsilon^{n+2s} \text{ outside } \Omega_\varepsilon,$$

thanks to (1.4). Hence, this together with (7.101) and the maximum principle give

$$|\eta_\xi| \leqslant C\varepsilon^{n+2s} \text{ in } \mathbb{R}^n,$$

which implies the thesis (recall the definition of $\eta_\xi$). $\square$

Moreover, we can prove the following:

**Lemma 7.8.** *There exists a positive constant $C$ such that*

$$\left| \frac{\partial \bar{u}_\xi}{\partial \xi} - \frac{\partial w_\xi}{\partial \xi} \right| \leqslant C\, \varepsilon^{\nu_1}, \tag{7.102}$$

*with $\nu_1 := \min\{(n + 2s + 1),\, p(n + 2s)\}$.*

*Proof.* We set $\eta_\xi := \bar{u}_\xi - w_\xi$. From (1.3) and (1.9), we have that $\eta_\xi$ solves

$$(-\Delta)^s \eta_\xi + \eta_\xi = 0 \text{ in } \Omega_\varepsilon.$$

Therefore, the derivative of $\eta_\xi$ with respect to $\xi$ satisfies

$$(-\Delta)^s \frac{\partial \eta_\xi}{\partial \xi} + \frac{\partial \eta_\xi}{\partial \xi} = 0 \text{ in } \Omega_\varepsilon. \tag{7.103}$$

Moreover, since $\bar{u}_\xi = 0$ outside $\Omega_\varepsilon$, we have that

$$\eta_\xi = \bar{u}_\xi - w_\xi = -w_\xi \text{ in } \mathbb{R}^n \setminus \Omega_\varepsilon,$$

which implies

$$\frac{\partial \eta_\xi}{\partial \xi} = -\frac{\partial w_\xi}{\partial \xi} = \frac{\partial w_\xi}{\partial x} \text{ in } \mathbb{R}^n \setminus \Omega_\varepsilon.$$

Therefore, from Lemma 5.2 (recall also (5.3)), we have that

$$\left| \frac{\partial \eta_\xi}{\partial \xi} \right| \leqslant C\varepsilon^{\nu_1} \text{ outside } \Omega_\varepsilon.$$

From this, (7.103) and the maximum principle we deduce that

$$\left| \frac{\partial \eta_\xi}{\partial \xi} \right| \leqslant C\varepsilon^{\nu_1} \text{ in } \mathbb{R}^n,$$

which gives the desired estimate (recall the definition of $\eta_\xi$). $\square$

In the next lemma we estimate the $\star$-norm of the error term $E(\psi)$. For this, we recall the definition of the space $Y_\star$ given in (7.70).

**Lemma 7.9.** *Let $\psi \in Y_\star$ with $\|\psi\|_{\star,\xi} \leqslant 1$. Then, there exists a positive constant $\bar{C}$ such that*

$$\|E(\psi)\|_{\star,\xi} \leqslant \bar{C}\, \varepsilon^{n+2s}.$$

*Proof.* Using (7.100) and Lemma 2.1 in [15] with $a := w_\xi + \psi$ and $b := \bar{u}_\xi - w_\xi$, we obtain that

$$\begin{aligned} |E(\psi)| &= |(\bar{u}_\xi - w_\xi + w_\xi + \psi)^p - (w_\xi + \psi)^p| \\ &\leqslant C_1 (w_\xi + \psi)^{p-1} |\bar{u}_\xi - w_\xi| \\ &\leqslant C_2\, \varepsilon^{n+2s} (w_\xi + \psi)^{p-1}. \end{aligned}$$

Hence, since $\|w_\xi\|_{\star,\xi}$ and $\|\psi\|_{\star,\xi}$ are bounded, we have

$$\|E(\psi)\|_{\star,\xi} \leqslant C_3 \varepsilon^{n+2s},$$

which gives the desired result. $\square$

Now, we give a bound for the $\star$-norm of the error term $N(\psi)$.

**Lemma 7.10.** *Let $\psi \in Y_\star$. Then, there exists a positive constant $C$ such that*

$$\|N(\psi)\|_{\star,\xi} \leqslant C \left( \|\psi\|^2_{\star,\xi} + \|\psi\|^p_{\star,\xi} \right).$$

*Proof.* We take $\psi \in Y_\star$ and we estimate

$$\begin{aligned} |N(\psi)| &= |(w_\xi + \psi)^p - w_\xi^p - p w_\xi^{p-1} \psi| \\ &\leqslant C \left( |\psi|^2 + |\psi|^p \right), \end{aligned}$$

for some positive constant $C$ (see, for instance, Corollary 2.2 in [15], applied here with $a := w_\xi$ and $b := \psi$). Hence,

$$\begin{aligned} \rho_\xi^{-1}|N(\psi)| &\leqslant C\,\rho_\xi^{-1}\left(|\psi|^2+|\psi|^p\right) \\ &\leqslant C\left(\rho_\xi^{-2}|\psi|^2+\rho_\xi^{-p}|\psi|^p\right) \\ &\leqslant C\left(\|\psi\|^2_{\star,\xi}+\|\psi\|^p_{\star,\xi}\right), \end{aligned}$$

which implies the desired estimate. □

For further reference, we now recall an estimate of elementary nature:

**Lemma 7.11.** *Fixed $\kappa > 0$, there exists a constant $C_\kappa > 0$ such that, for any $a, b \in [0,\kappa]$ we have*

$$|a^{p-1} - b^{p-1}| \leqslant C_\kappa |a-b|^q, \tag{7.104}$$

*where*

$$q := \min\{1, p-1\}. \tag{7.105}$$

*Proof.* Fixed any $\alpha \in (0,1)$, for any $t > 0$ we define

$$h(t) := \frac{(t+1)^\alpha - 1}{t^\alpha}.$$

Using de L'Hospital Rule we see that

$$\lim_{t\searrow 0} h(t) = \lim_{t\searrow 0} \frac{t^{1-\alpha}}{(t+1)^{1-\alpha}} = 0,$$

hence we can extend $h$ to a continuous function on $[0,+\infty)$ with $h(0) := 0$. Moreover

$$\lim_{t\to+\infty} h(t) = 1,$$

hence there exists

$$M_0 := \sup_{t\in[0,+\infty)} h(t) < +\infty. \tag{7.106}$$

Now we prove (7.104). For this, we may and do assume that $a > b$. If $p \geqslant 2$, we have that

$$\begin{aligned} a^{p-1} - b^{p-1} = (p-1)\int_b^a \tau^{p-2}\,d\tau \leqslant (p-1)\,a^{p-2}\,(a-b) \\ \leqslant (p-1)\,\kappa^{p-2}\,(a-b), \end{aligned}$$

that is (7.104) in this case. On the other hand, if $p \in (1,2)$ we take $t := (a/b) - 1 > 0$ and $\alpha := p-1$, so

$$M_0 \geqslant h(t) = \frac{(a/b)^{p-1} - 1}{((a/b)-1)^{p-1}} = \frac{a^{p-1} - b^{p-1}}{(a-b)^{p-1}},$$

thanks to (7.106), and this establishes (7.104) also in this case. □

Now we are ready to complete the proof of Theorem 7.6.

*Proof of Theorem 7.6.* Recalling the definition of the operator $\mathcal{T}_\xi$ in (7.82), we can write

$$\psi = \mathcal{T}_\xi[E(\psi) + N(\psi)].$$

We will prove Theorem 7.6 by a contraction argument. To do this, we set

$$\mathcal{K}_\xi(\psi) := \mathcal{T}_\xi[E(\psi) + N(\psi)]. \tag{7.107}$$

Moreover, we take a constant $C_0 > 0$ and $\varepsilon > 0$ small (we will specify the choice of $C_0$ and $\varepsilon$ in (7.118)), and we define the set

$$B := \{\psi \in Y_\star \text{ s.t } \|\psi\|_{\star,\xi} \leqslant C_0\, \varepsilon^{n+2s}\},$$

where $Y_\star$ is introduced in (7.70).

We claim that

(7.108)

$\mathcal{K}_\xi$ as in (7.107) is a contraction mapping from $B$ into itself with respect to the norm $\|\cdot\|_{\star,\xi}$.

First we prove that

$$\text{if } \psi \in B \text{ then } \mathcal{K}_\xi(\psi) \in B. \tag{7.109}$$

Indeed, if $\psi \in B$, we have that

$$\|N(\psi)\|_{\star,\xi} \leqslant C_1 \left(\|\psi\|^2_{\star,\xi} + \|\psi\|^p_{\star,\xi}\right) \tag{7.110}$$

thanks to Lemma 7.10.

Now, thanks to (7.8), we have that

$$\|\mathcal{K}_\xi(\psi)\|_{\star,\xi} = \|\mathcal{T}_\xi[E(\psi)+N(\psi)]\|_{\star,\xi} \leqslant C\|E(\psi)+N(\psi)\|_{\star,\xi}.$$

This, Lemma 7.9 and (7.110) give that

$$\begin{aligned}
\|\mathcal{K}_\xi(\psi)\|_{\star,\xi} &\leqslant C\left(\|E(\psi)\|_{\star,\xi} + \|N(\psi)\|_{\star,\xi}\right)\\
&\leqslant C\left(\|E(\psi)\|_{\star,\xi} + C_1\left(\|\psi\|^2_{\star,\xi} + \|\psi\|^p_{\star,\xi}\right)\right)\\
&\leqslant C\left(\bar C\, \varepsilon^{n+2s} + C_1\, C_0^2\, \varepsilon^{2(n+2s)} + C_1\, C_0^p\, \varepsilon^{p(n+2s)}\right)\\
&= C_0\, \varepsilon^{n+2s}\left(\frac{C\,\bar C}{C_0} + C\, C_1\, C_0 \varepsilon^{n+2s} + C\, C_1\, C_0^{p-1} \varepsilon^{(p-1)(n+2s)}\right),
\end{aligned} \tag{7.111}$$

since $\psi \in B$. We assume

$$C_0 > 2C\bar C \tag{7.112}$$

and

$$\varepsilon < \varepsilon_1 := \begin{cases} \left(\frac{1}{2C\, C_1 (C_0 + C_0^{p-1})}\right)^{1/(n+2s)} & \text{if } p \geqslant 2,\\ \left(\frac{1}{2C\, C_1 (C_0 + C_0^{p-1})}\right)^{1/(p-1)(n+2s)} & \text{if } 1 < p < 2. \end{cases} \tag{7.113}$$

With this choice of $C_0$ and $\varepsilon$, (7.111) implies that

$$\|\mathcal{K}_\xi(\psi)\|_{\star,\xi} \leqslant C_0\, \varepsilon^{n+2s},$$

which proves (7.109).

Now, we take $\psi_1, \psi_2 \in B$. Then,

$$\begin{aligned}
|N(\psi_1) - N(\psi_2)| &= |(w_\xi + \psi_1)^p - (w_\xi + \psi_2)^p - p w_\xi^{p-1}(\psi_1 - \psi_2)|\\
&\leqslant C_2\left(|\psi_1| + |\psi_2| + |\psi_1|^{p-1} + |\psi_2|^{p-1}\right)|\psi_1 - \psi_2|.
\end{aligned}$$

This and the fact that $\psi_1, \psi_2 \in B$ give that

(7.114)
$$\begin{aligned}\|N(\psi_1) - N(\psi_2)\|_{\star,\xi} &\leqslant C_2 \left(\|\psi_1\|_{\star,\xi} + \|\psi_2\|_{\star,\xi} + \|\psi_1\|^{p-1}_{\star,\xi} + \|\psi_2\|^{p-1}_{\star,\xi}\right) \|\psi_1 - \psi_2\|_{\star,\xi} \\ &\leqslant C_2 \left(2C_0\, \varepsilon^{n+2s} + 2C_0^{p-1}\, \varepsilon^{(p-1)(n+2s)}\right) \|\psi_1 - \psi_2\|_{\star,\xi} \\ &\leqslant 2\, C_2 \left(C_0 + C_0^{p-1}\right) \varepsilon^{q(n+2s)} \|\psi_1 - \psi_2\|_{\star,\xi},\end{aligned}$$

where $q$ is defined in (7.105).

We claim that

(7.115)
$$|E(\psi_1) - E(\psi_2)| \leqslant C|\bar{u}_\xi - w_\xi|^q\, |\psi_1 - \psi_2|,$$

where $q$ is given in (7.105).

Fixed $x \in \Omega_\varepsilon$, given $\tau$ in a bounded subset of $\mathbb{R}$ we consider the function

$$e(\tau) := (\bar{u}_\xi(x) + \tau)^p - (w_\xi(x) + \tau)^p.$$

We have that

$$|e'(\tau)| = p\Big|(\bar{u}_\xi(x) + \tau)^{p-1} - (w_\xi(x) + \tau)^{p-1}\Big| \leqslant C|\bar{u}_\xi - w_\xi|^q,$$

where we used (7.104) with $a := \bar{u}_\xi(x) + \tau$ and $b := w_\xi(x) + \tau$. This gives that

(7.116)
$$|e(\tau_1) - e(\tau_2)| \leqslant C|\bar{u}_\xi - w_\xi|^q\, |\tau_1 - \tau_2|.$$

Now we take $\tau_1 := \psi_1(x)$ and $\tau_2 := \psi_2(x)$: we remark that $\tau_1$ and $\tau_2$ range in a bounded set by our definition of $B$ and that $e(\tau_i) = E(\psi_i)$. Thus (7.115) follows from (7.116)

Hence, from (7.115) and (7.100), we obtain that

$$\|E(\psi_1) - E(\psi_2)\|_{\star,\xi} \leqslant \tilde{C}\varepsilon^{q(n+2s)} \|\psi_1 - \psi_2\|_{\star,\xi}.$$

This, (7.114) and (7.8) give that

(7.117)
$$\begin{aligned}&\|\mathcal{K}_\xi(\psi_1) - \mathcal{K}_\xi(\psi_2)\|_{\star,\xi} \\ &\leqslant C \left(\|E(\psi_1) - E(\psi_2)\|_{\star,\xi} + \|N(\psi_1) - N(\psi_2)\|_{\star,\xi}\right) \\ &\leqslant C \left(2\, C_2 \left(C_0 + C_0^{p-1}\right) \varepsilon^{q(n+2s)} + \tilde{C}\varepsilon^{q(n+2s)}\right) \|\psi_1 - \psi_2\|_{\star,\xi}.\end{aligned}$$

Now, we denote by

$$\varepsilon_2 := \left(\frac{1}{C(2\, C_2(C_0 + C_0^{p-1}) + \tilde{C})}\right)^{1/q(n+2s)}.$$

Therefore, recalling also (7.112) and (7.113), we obtain that if

(7.118)
$$C_0 > 2C\bar{C} \text{ and } \varepsilon < \min\{\varepsilon_1, \varepsilon_2\}$$

then, from (7.117) we have that

$$\|\mathcal{K}_\xi(\psi_1) - \mathcal{K}_\xi(\psi_2)\|_{\star,\xi} < \|\psi_1 - \psi_2\|_{\star,\xi},$$

which concludes the proof of (7.108).

From (7.108), we obtain the existence of a unique solution to (7.98) which belongs to $B$. This shows (7.99) and concludes the proof of Theorem 7.6. $\square$

For any $\xi \in \Omega_\varepsilon$, we say that

(7.119)
$$\Psi(\xi) \text{ is the unique solution to (7.98).}$$

Arguing as the proof of Proposition $5.1$ in [10], one can also prove the following:

**Proposition 7.12.** *The map $\xi \mapsto \Psi(\xi)$ is of class $C^1$, and*

$$\left\| \frac{\partial \Psi(\xi)}{\partial \xi} \right\|_{\star,\xi} \leqslant C \left( \| E(\Psi(\xi)) \|_{\star,\xi} + \left\| \frac{\partial E(\Psi(\xi))}{\partial \xi} \right\|_{\star,\xi} \right),$$

*for some constant $C > 0$.*

7.3. **Derivative estimates.** Here we deal with the derivatives of the solution $\psi = \Psi(\xi)$ to (7.98) with respect to $\xi$. This will also imply derivative estimates for the error term $\xi \mapsto E(\Psi(\xi))$.

We first show the following

**Lemma 7.13.** *Let $\psi \in \Psi$ be a solution[4] to (7.98), with $\|\psi\|_{\star,\xi} \leqslant C\varepsilon^{n+2s}$. Then, there exist positive constants $C$ and $\gamma$ such that*

$$\left\| \frac{\partial E(\psi)}{\partial \xi} \right\|_{\star,\xi} \leqslant C \left( \varepsilon^{q(n+2s)} \left\| \frac{\partial \psi}{\partial \xi} \right\|_{\star,\xi} + \varepsilon^{\gamma} \right),$$

*where $q$ is defined in (7.105).*

*Proof.* First of all, we observe that, thanks to Proposition 7.5 (applied here with $g := -\left(E(\psi) + N(\psi)\right)$), the function $\frac{\partial \psi}{\partial \xi}$ is well defined.

We make the following computations: from (7.3) we have that

$$\begin{aligned}
\frac{\partial E(\psi)}{\partial \xi} &= p(\bar{u}_\xi + \psi)^{p-1} \left( \frac{\partial \bar{u}_\xi}{\partial \xi} + \frac{\partial \psi}{\partial \xi} \right) - p(w_\xi + \psi)^{p-1} \left( \frac{\partial w_\xi}{\partial \xi} + \frac{\partial \psi}{\partial \xi} \right) \\
&= p\frac{\partial \psi}{\partial \xi} \left[ (\bar{u}_\xi + \psi)^{p-1} - (w_\xi + \psi)^{p-1} \right] + p(\bar{u}_\xi + \psi)^{p-1} \frac{\partial \bar{u}_\xi}{\partial \xi} - p(w_\xi + \psi)^{p-1} \frac{\partial w_\xi}{\partial \xi} \\
&= p\frac{\partial \psi}{\partial \xi} \left[ (\bar{u}_\xi + \psi)^{p-1} - (w_\xi + \psi)^{p-1} \right] + p(\bar{u}_\xi + \psi)^{p-1} \left( \frac{\partial \bar{u}_\xi}{\partial \xi} - \frac{\partial w_\xi}{\partial \xi} \right) \\
&\quad + p\left[ (\bar{u}_\xi + \psi)^{p-1} - (w_\xi + \psi)^{p-1} \right] \frac{\partial w_\xi}{\partial \xi}.
\end{aligned}$$

Thus, recalling (7.104), (7.100) and (7.102), we infer that

$$\begin{aligned}
\left| \frac{\partial E(\psi)}{\partial \xi} \right| &\leqslant Cp \left| \frac{\partial \psi}{\partial \xi} \right| |\bar{u}_\xi - w_\xi|^q + p\left( |\bar{u}_\xi| + |\psi| \right)^{p-1} \left| \frac{\partial \bar{u}_\xi}{\partial \xi} - \frac{\partial w_\xi}{\partial \xi} \right| \\
&\quad + Cp\, |\bar{u}_\xi - w_\xi|^q \left| \frac{\partial w_\xi}{\partial \xi} \right| \\
&\leqslant C \left| \frac{\partial \psi}{\partial \xi} \right| \varepsilon^{q(n+2s)} + C\left( |\bar{u}_\xi| + |\psi| \right)^{p-1} \varepsilon^{\nu_1} \\
&\quad + C\varepsilon^{q(n+2s)} \left| \frac{\partial w_\xi}{\partial \xi} \right|
\end{aligned} \tag{7.120}$$

for some $C > 0$. Now, we claim that

$$\begin{gathered}
\sup_{x \in \mathbb{R}^n} (1 + |x - \xi|)^{\mu} \, |\bar{u}_\xi(x)|^{p-1} \varepsilon^{\nu_1} \leqslant C\varepsilon^{\gamma} \\
\text{and} \quad \sup_{x \in \mathbb{R}^n} (1 + |x - \xi|)^{\mu} \, |\psi(x)|^{p-1} \varepsilon^{\nu_1} \leqslant C\varepsilon^{\gamma},
\end{gathered} \tag{7.121}$$

[4]We remark that a solution that fulfils the assumptions of Lemma 7.13 is provided by Theorem 7.6, as long as $\varepsilon$ is sufficiently small.

for suitable $C > 0$ and $\gamma > 0$. Let us prove the first inequality in (7.121). For this, we use that $\bar u_\xi$ vanishes outside $\Omega_\varepsilon$, together with (7.100) and (1.4), to see that

$$
\begin{aligned}
&\sup_{x\in\mathbb{R}^n}(1+|x-\xi|)^\mu\,|\bar u_\xi(x)|^{p-1}\varepsilon^{\nu_1}\\
&= \sup_{x\in\Omega_\varepsilon}(1+|x-\xi|)^\mu\,|\bar u_\xi(x)|^{p-1}\varepsilon^{\nu_1}\\
&\leqslant \sup_{x\in\Omega_\varepsilon}(1+|x-\xi|)^\mu\,\varepsilon^{(p-1)(n+2s)}\varepsilon^{\nu_1}+\sup_{x\in\Omega_\varepsilon}(1+|x-\xi|)^\mu\,|w_\xi(x)|^{p-1}\varepsilon^{\nu_1}\\
&\leqslant C\,\varepsilon^{-\mu}\,\varepsilon^{(p-1)(n+2s)}\varepsilon^{\nu_1}+\sup_{x\in\Omega_\varepsilon}(1+|x-\xi|)^{\mu(p-1)}\,|w_\xi(x)|^{p-1}(1+|x-\xi|)^{\mu(2-p)}\varepsilon^{\nu_1}\\
&\leqslant C\,\varepsilon^{-\mu}\,\varepsilon^{(p-1)(n+2s)}\varepsilon^{\nu_1}+\|w_\xi\|_{\star,\xi}^{p-1}\varepsilon^{-\mu(2-p)_+}\varepsilon^{\nu_1}\\
&\leqslant C\,\varepsilon^{-\mu}\,\varepsilon^{(p-1)(n+2s)}\varepsilon^{\nu_1}+C\,\varepsilon^{-\mu(2-p)_+}\varepsilon^{\nu_1}.
\end{aligned}\tag{7.122}
$$

Now we observe that

$$
\begin{aligned}
&-\mu+(p-1)(n+2s)+\nu_1\\
&=\min\{-\mu+(p-1)(n+2s)+n+2s+1,-\mu+(p-1)(n+2s)+p(n+2s)\}\\
&>\min\{-(n+2s)+(p-1)(n+2s)+n+2s+1,-(n+2s)+(p-1)(n+2s)+p(n+2s)\}\\
&=\min\{(p-1)(n+2s)+1,(2p-2)(n+2s)\}>0.
\end{aligned}\tag{7.123}
$$

Moreover, if $p\geqslant 2$, then

$$-\mu(2-p)_++\nu_1=\nu_1>0,$$

while if $1<p<2$, then

$$
\begin{aligned}
&-\mu(2-p)_++\nu_1\\
&=\min\{-\mu(2-p)+n+2s+1,-\mu(2-p)+p(n+2s)\}\\
&>\min\{-(n+2s)(2-p)+n+2s+1,-(n+2s)(2-p)+p(n+2s)\}\\
&=\min\{(p-1)(n+2s)+1,(2p-2)(n+2s)\}>0.
\end{aligned}
$$

Using this and (7.123) into (7.122) we obtain the first formula in (7.121). Now, we focus on the second inequality: from the assumptions on $\psi$ we have

$$
\begin{aligned}
&\sup_{x\in\mathbb{R}^n}(1+|x-\xi|)^\mu\,|\psi(x)|^{p-1}\varepsilon^{\nu_1}\\
&= \sup_{x\in\Omega_\varepsilon}(1+|x-\xi|)^\mu\,|\psi(x)|^{p-1}\varepsilon^{\nu_1}\\
&= \sup_{x\in\Omega_\varepsilon}(1+|x-\xi|)^{\mu(p-1)}\,|\psi(x)|^{p-1}\,(1+|x-\xi|)^{\mu(2-p)}\,\varepsilon^{\nu_1}\\
&\leqslant \|\psi\|_{\star,\xi}^{p-1}\,\varepsilon^{-\mu(2-p)_+}\,\varepsilon^{\nu_1}\\
&\leqslant C\,\varepsilon^{(p-1)(n+2s)}\,\varepsilon^{-\mu(2-p)_+}\,\varepsilon^{\nu_1}.
\end{aligned}\tag{7.124}
$$

If $p\geqslant 2$ we get the second inequality in (7.121), as desired, hence we focus on the case $1<p<2$. For this, we notice that

$$
\begin{aligned}
&(p-1)(n+2s)-\mu(2-p)_++\nu_1\\
&=\min\{(p-1)(n+2s)-\mu(2-p)+n+2s+1,(p-1)(n+2s)-\mu(2-p)+p(n+2s)\}\\
&>\min\{(p-1)(n+2s)-(2-p)(n+2s)+n+2s+1,(p-1)(n+2s)-(2-p)(n+2s)+p(n+2s)\}\\
&=\min\{(2p-2)(n+2s)+1,(3p-3)(n+2s)\}>0,
\end{aligned}
$$

and this, together with (7.124), implies the second inequality in (7.121) also in this case. Hence the proof of (7.121) is finished.

Exploiting (7.121) and Lemma 5.2, we infer from (7.120) that

$$\left\| \frac{\partial E(\psi)}{\partial \xi} \right\|_{\star,\xi} \leqslant C\varepsilon^{q(n+2s)} \left\| \frac{\partial \psi}{\partial \xi} \right\|_{\star,\xi} + C\varepsilon^{\gamma}, \tag{7.125}$$

for suitable $C > 0$ and $\gamma > 0$, and this concludes the proof of Lemma 7.13, up to renaming the constants. □

**Lemma 7.14.** *Let $\psi \in \Psi$ be a solution to (7.98), with $\|\psi\|_{\star,\xi} \leqslant C\varepsilon^{n+2s}$. Then, there exists a positive constant $C$ such that*

$$\left\| \frac{\partial \psi}{\partial \xi} \right\|_{\star,\xi} \leqslant C.$$

*Proof.* We observe that, thanks to Proposition 7.5 (applied here with $g := -(E(\psi) + N(\psi))$),

$$\left\| \frac{\partial \psi}{\partial \xi} \right\|_{\star,\xi} \leqslant C \left( \|E(\psi)\|_{\star,\xi} + \|N(\psi)\|_{\star,\xi} + \left\| \frac{\partial E(\psi)}{\partial \xi} \right\|_{\star,\xi} + \left\| \frac{\partial N(\psi)}{\partial \xi} \right\|_{\star,\xi} \right).$$

Therefore, from Lemmata 7.9, 7.10 and 7.13, we obtain that

$$\left\| \frac{\partial \psi}{\partial \xi} \right\|_{\star,\xi} \leqslant C \left( 1 + \varepsilon^{q(n+2s)} \left\| \frac{\partial \psi}{\partial \xi} \right\|_{\star,\xi} + \left\| \frac{\partial N(\psi)}{\partial \xi} \right\|_{\star,\xi} \right). \tag{7.126}$$

Now we observe that, from (7.3),

$$\begin{aligned} \frac{\partial N(\psi)}{\partial \xi} &= p(w_\xi + \psi)^{p-1} \left( \frac{\partial w_\xi}{\partial \xi} + \frac{\partial \psi}{\partial \xi} \right) - p w_\xi^{p-1} \frac{\partial w_\xi}{\partial \xi} - p(p-1) w_\xi^{p-2} \frac{\partial w_\xi}{\partial \xi} \psi - p w_\xi^{p-1} \frac{\partial \psi}{\partial \xi} \\ &= p\Big[(w_\xi + \psi)^{p-1} - w_\xi^{p-1}\Big] \frac{\partial \psi}{\partial \xi} + p\Big[(w_\xi + \psi)^{p-1} - w_\xi^{p-1}\Big] \frac{\partial w_\xi}{\partial \xi} - p(p-1) w_\xi^{p-2} \frac{\partial w_\xi}{\partial \xi} \psi. \end{aligned}$$

As a consequence, using (7.104) once again,

$$\left| \frac{\partial N(\psi)}{\partial \xi} \right| \leqslant C\,|\psi|^q \left[ \left| \frac{\partial \psi}{\partial \xi} \right| + \left| \frac{\partial w_\xi}{\partial \xi} \right| \right] + C\, w_\xi^{p-2} \left| \frac{\partial w_\xi}{\partial \xi} \right| \, |\psi|. \tag{7.127}$$

Now we claim that

$$w_\xi^{p-2} \left| \frac{\partial w_\xi}{\partial \xi} \right| \leqslant C, \tag{7.128}$$

for some $C > 0$. When $p \geqslant 2$, (7.128) follows from (1.4) and Lemma 5.2, hence we focus on the case $p \in (1, 2)$. In this case, we take $\nu_1$ as in Lemma 5.2 and we notice that

$$\begin{aligned} \tilde{\nu} := \nu_1 - (2-p)(n+2s) &= \min\{n + 2s + 1 + (p-2)(n+2s),\ (2p-2)(n+2s)\} \\ &= \min\{(p-1)(n+2s) + 1,\ (2p-2)(n+2s)\} > 0. \end{aligned}$$

Then we use (7.95) and we obtain that

$$w_\xi^{p-2} \left| \frac{\partial w_\xi}{\partial \xi} \right| \leqslant C\,|x-\xi|^{(2-p)(n+2s)}\,|x-\xi|^{-\nu_1} = C\,|x-\xi|^{-\tilde{\nu}}.$$

Since $w_\xi$ is positive and smooth in the vicinity of $\xi$, this proves (7.128).

Now, using (7.128) into (7.127), we obtain that

$$\left|\frac{\partial N(\psi)}{\partial \xi}\right| \leqslant C\,|\psi|^q \left[\left|\frac{\partial \psi}{\partial \xi}\right| + \left|\frac{\partial w_\xi}{\partial \xi}\right|\right] + C\,|\psi|. \tag{7.129}$$

We claim that

$$\left\|\frac{\partial N(\psi)}{\partial \xi}\right\|_{\star,\xi} \leqslant C\,\|\psi\|^q_{\star,\xi} \left[\left\|\frac{\partial \psi}{\partial \xi}\right\|_{\star,\xi} + \left\|\frac{\partial w_\xi}{\partial \xi}\right\|_{\star,\xi}\right] + C\,\|\psi\|_{\star,\xi}. \tag{7.130}$$

Indeed, the claim plainly follows from (7.129) if $q = 1$ (that is $p \geqslant 2$), hence we focus on the case $q = p-1$ (that is $1 < p < 2$). In this case, we observe that

$$\begin{aligned} & (1+|x-\xi|)^\mu |\psi|^q \left[\left|\frac{\partial \psi}{\partial \xi}\right| + \left|\frac{\partial w_\xi}{\partial \xi}\right|\right] \\ = \; & (1+|x-\xi|)^{\mu q}|\psi|^q\,(1+|x-\xi|)^\mu \left[\left|\frac{\partial \psi}{\partial \xi}\right| + \left|\frac{\partial w_\xi}{\partial \xi}\right|\right]\,(1+|x-\xi|)^{-\mu q} \\ \leqslant \; & C\|\psi\|^q_{\star,\xi}\left[\left\|\frac{\partial \psi}{\partial \xi}\right\|_{\star,\xi} + \left\|\frac{\partial w_\xi}{\partial \xi}\right\|_{\star,\xi}\right], \end{aligned}$$

and this implies (7.130) also in this case.

Hence, using our assumptions on $\psi$, we deduce that

$$\left\|\frac{\partial N(\psi)}{\partial \xi}\right\|_{\star,\xi} \leqslant C\varepsilon^{q(n+2s)}\left[\left\|\frac{\partial \psi}{\partial \xi}\right\|_{\star,\xi} + 1\right] + C,$$

up to renaming constants. By inserting this into (7.126) we conclude that

$$\left\|\frac{\partial \psi}{\partial \xi}\right\|_{\star,\xi} \leqslant C + \frac{1}{2}\left\|\frac{\partial \psi}{\partial \xi}\right\|_{\star,\xi},$$

as long as $\varepsilon$ is sufficiently small. By reabsorbing one term into the left hand side, we obtain the desired result. ☐

**Lemma 7.15.** *Let $\psi \in \Psi$ be a solution to (7.98), with $\|\psi\|_{\star,\xi} \leqslant C\varepsilon^{n+2s}$. Then, there exist positive constants $\tilde{C}$ and $\gamma$ such that*

$$\left\|\frac{\partial E(\psi)}{\partial \xi}\right\|_{\star,\xi} \leqslant \tilde{C}\,\varepsilon^\gamma.$$

*Proof.* The proof easily follows from Lemmata 7.13 and 7.14, up to renaming the constants. ☐

7.4. **The variational reduction.** We are looking for solutions to (1.8) of the form (7.1), that is, recalling also (7.119),

$$u_\xi = \bar{u}_\xi + \Psi(\xi). \tag{7.131}$$

We observe that, thanks to (1.9) and (7.98), the function $u_\xi$ satisfies the equation

$$(-\Delta)^s u_\xi + u_\xi = u_\xi^p + \sum_{i=1}^n c_i\,Z_i \text{ in } \Omega_\varepsilon. \tag{7.132}$$

Notice that if $c_i = 0$ for any $i = 1, \dots, n$ then we will have a solution to (1.8). Hence, aim of this subsection is to find a suitable point $\xi \in \Omega_\varepsilon$ such that all the coefficients $c_i$, $i = 1, \dots, n$, in (7.132) vanish.

In order to do this, we define the functional $J_\varepsilon : \Omega_\varepsilon \to \mathbb{R}$ as

$$J_\varepsilon(\xi) := I_\varepsilon(\bar{u}_\xi + \Psi(\xi)) = I_\varepsilon(u_\xi) \text{ for any } \xi \in \Omega_\varepsilon, \tag{7.133}$$

where $I_\varepsilon$ is introduced in (1.10). We have the following characterization:

**Lemma 7.16.** *If $\varepsilon > 0$ is sufficiently small, the coefficients $c_i$, $i = 1, \dots, n$, in (7.132) are equal to zero if and only if $\xi$ satisfies the following condition*

$$\frac{\partial J_\varepsilon}{\partial \xi}(\xi) = 0.$$

*Proof.* We make a preliminary observation. We write $\xi = (\xi_1, \dots, \xi_n)$ and, for any $j = 1, \dots, n$, we take the derivative of $u_\xi$ with respect to $\xi_j$.

We observe that

$$\frac{\partial u_\xi}{\partial \xi_j} = \frac{\partial \bar{u}_\xi}{\partial \xi_j} + \frac{\partial \Psi(\xi)}{\partial \xi_j}. \tag{7.134}$$

Thanks to (7.102), we have that

$$\frac{\partial \bar{u}_\xi}{\partial \xi_j} = \frac{\partial w_\xi}{\partial \xi_j} + O(\varepsilon^{\nu_1}). \tag{7.135}$$

Moreover, from Proposition 7.12 and Lemmata 7.9 and 7.15, we obtain that

$$\frac{\partial \Psi(\xi)}{\partial \xi_j} = O(\varepsilon^\gamma), \tag{7.136}$$

where $\gamma > 0$. Hence, (7.134), (7.135) and (7.136) imply that

$$\frac{\partial u_\xi}{\partial \xi_j} = \frac{\partial w_\xi}{\partial \xi_j} + O(\varepsilon^\gamma),$$

which means, recalling (5.3) and using the fact that $\frac{\partial w_\xi}{\partial \xi_j} = -\frac{\partial w_\xi}{\partial x_j}$, that

$$\frac{\partial u_\xi}{\partial \xi_j} = -Z_j + O(\varepsilon^\gamma). \tag{7.137}$$

In particular,

$$\int_{\Omega_\varepsilon} Z_i \frac{\partial u_\xi}{\partial \xi_j}\, dx = \int_{\Omega_\varepsilon} Z_i \Big( -Z_j + O(\varepsilon^\gamma)\Big)\, dx = -\int_{\Omega_\varepsilon} Z_i\, Z_j\, dx + O(\varepsilon^\gamma) \tag{7.138}$$

and, from Lemma 5.2, we deduce that

$$\left| \frac{\partial u_\xi}{\partial \xi_j} \right| \leqslant C_1(|Z_j| + \varepsilon^\gamma) \leqslant C_2. \tag{7.139}$$

With this, we introduce the matrix $M \in \text{Mat}(n \times n)$ whose entries are given by

$$M_{ji} := \int_{\Omega_\varepsilon} Z_i \frac{\partial u_\xi}{\partial \xi_j}\, dx. \tag{7.140}$$

We claim that

$$\text{the matrix } M \text{ is invertible.} \tag{7.141}$$

To prove this, we use (7.138), Corollary 5.6 and the fact that $\alpha > 0$ (recall (5.17)): namely we compute

$$\begin{aligned} M_{ji} &= -\int_{\Omega_\varepsilon} Z_i\, Z_j\, dx + O(\varepsilon^\gamma) \\ &= -\alpha\delta_{ij} + O(\varepsilon^\gamma). \end{aligned}$$

This says that the matrix $-\alpha^{-1}M$ is a perturbation of the identity and therefore it is invertible for $\varepsilon$ sufficiently small, hence (7.141) readily follows.

Now, we multiply (7.132) by $\frac{\partial u_\xi}{\partial \xi}$, obtaining that

$$\left((-\Delta)^s u_\xi + u_\xi - u_\xi^p\right) \frac{\partial u_\xi}{\partial \xi} = \sum_{i=1}^n c_i\, Z_i \frac{\partial u_\xi}{\partial \xi} \text{ in } \Omega_\varepsilon,$$

and therefore

$$\left|\left((-\Delta)^s u_\xi + u_\xi - u_\xi^p\right) \frac{\partial u_\xi}{\partial \xi}\right| \leqslant \sum_{i=1}^n |c_i|\, |Z_i| \left|\frac{\partial u_\xi}{\partial \xi}\right|.$$

This, together with (7.139) and Lemma 5.2, implies that the function $\left((-\Delta)^s u_\xi + u_\xi - u_\xi^p\right) \frac{\partial u_\xi}{\partial \xi}$ is in $L^\infty(\Omega_\varepsilon)$, and so in $L^1(\Omega_\varepsilon)$ uniformly with respect to $\xi$.

This allows us to compute the derivative of $J_\varepsilon$ with respect to $\xi_j$ as follows:

$$\begin{aligned} \frac{\partial J_\varepsilon}{\partial \xi_j}(\xi) &= \frac{\partial}{\partial \xi_j} I_\varepsilon(u_\xi) \\ &= \frac{\partial}{\partial \xi_j}\left(\int_{\Omega_\varepsilon} \frac{1}{2}(-\Delta)^s u_\xi\, u_\xi + \frac{1}{2}u_\xi^2 - \frac{1}{p+1}u_\xi^{p+1}\, dx\right) \\ &= \int_{\Omega_\varepsilon} \frac{1}{2}(-\Delta)^s \frac{\partial u_\xi}{\partial \xi_j}\, u_\xi + \frac{1}{2}(-\Delta)^s u_\xi \frac{\partial u_\xi}{\partial \xi_j} + \frac{\partial u_\xi}{\partial \xi_j}\, u_\xi - u_\xi^p \frac{\partial u_\xi}{\partial \xi_j}\, dx \\ &= \int_{\Omega_\varepsilon} \left((-\Delta)^s u_\xi + u_\xi - u_\xi^p\right) \frac{\partial u_\xi}{\partial \xi_j}\, dx \\ &= \sum_{i=1}^n c_i \int_{\Omega_\varepsilon} Z_i \frac{\partial u_\xi}{\partial \xi_j}\, dx, \end{aligned} \tag{7.142}$$

where we have used (7.132) in the last step. Thus, recalling (7.140), we can write

$$\frac{\partial J_\varepsilon}{\partial \xi_j}(\xi) = \sum_{i=1}^n c_i M_{ji},$$

for any $j \in \{1, \dots, n\}$, that is the vector $\frac{\partial J_\varepsilon}{\partial \xi}(\xi) := \left(\frac{\partial J_\varepsilon}{\partial \xi_1}(\xi), \dots, \frac{\partial J_\varepsilon}{\partial \xi_n}(\xi)\right)$ is equal to the product between the matrix $M$ and the vector $c := (c_1, \dots, c_n)$. From (7.141) we obtain that $\frac{\partial J_\varepsilon}{\partial \xi}(\xi)$ is equal to zero if and only if $c$ is equal to zero, as desired. $\square$

Thanks to Lemma 7.16, the problem of finding a solution to (1.8) reduces to the one of finding critical points of the functional defined in (7.133).To this end, we obtain an expansion of $J_\varepsilon$.

**Theorem 7.17.** *We have the following expansion of the functional $J_\varepsilon$:*

$$J_\varepsilon(\xi) = I_\varepsilon(\bar{u}_\xi) + o(\varepsilon^{n+4s}).$$

*Proof.* We know that

$$J_\varepsilon(\xi) = I_\varepsilon(\bar u_\xi + \Psi(\xi)).$$

Hence, we can Taylor expand in the vicinity of $\bar u_\xi$, thus obtaining

$$\begin{aligned} J_\varepsilon(\xi) &= I_\varepsilon(\bar u_\xi) + I'_\varepsilon(\bar u_\xi)[\Psi(\xi)] + I''(\bar u_\xi)[\Psi(\xi),\Psi(\xi)] + O(|\Psi(\xi)|^3) \\ &= I_\varepsilon(\bar u_\xi) + \int_{\Omega_\varepsilon} (-\Delta)^s \bar u_\xi\, \Psi(\xi) + \bar u_\xi\, \Psi(\xi) - \bar u_\xi^p\, \Psi(\xi)\, dx \\ &\qquad + \int_{\Omega_\varepsilon} (-\Delta)^s \Psi(\xi)\, \Psi(\xi) + \Psi^2(\xi) - p\bar u_\xi^{p-1}\Psi^2(\xi)\, dx + O(|\Psi(\xi)|^3) \\ &= I_\varepsilon(\bar u_\xi) + \int_{\Omega_\varepsilon} \left((-\Delta)^s u_\xi + u_\xi - u_\xi^p\right) \Psi(\xi)\, dx \\ &\qquad - \int_{\Omega_\varepsilon} \left((-\Delta)^s (u_\xi - \bar u_\xi) + u_\xi - \bar u_\xi - u_\xi^p + \bar u_\xi^p\right) \Psi(\xi)\, dx \\ &\qquad + \int_{\Omega_\varepsilon} (-\Delta)^s \Psi(\xi)\, \Psi(\xi) + \Psi^2(\xi) - p\bar u_\xi^{p-1}\Psi^2(\xi)\, dx + O(|\Psi(\xi)|^3). \end{aligned}$$

Therefore, using (7.131), we have that

$$\begin{aligned} J_\varepsilon(\xi) = {}& I_\varepsilon(\bar u_\xi) + \int_{\Omega_\varepsilon} \left((-\Delta)^s u_\xi + u_\xi - u_\xi^p\right) \Psi(\xi)\, dx \\ &+ \int_{\Omega_\varepsilon} \left(u_\xi^p - \bar u_\xi^p - p\bar u_\xi^{p-1}\Psi(\xi)\right) \Psi(\xi)\, dx + O(|\Psi(\xi)|^3). \end{aligned} \tag{7.143}$$

We notice that

$$\int_{\Omega_\varepsilon} \left((-\Delta)^s u_\xi + u_\xi - u_\xi^p\right) \Psi(\xi)\, dx = 0,$$

thanks to (7.132) and the fact that $\Psi(\xi)$ is orthogonal in $L^2(\Omega_\varepsilon)$ to any function in the space $\mathcal{Z}$. Hence, (7.143) becomes

$$J_\varepsilon(\xi) = I_\varepsilon(\bar u_\xi) + \int_{\Omega_\varepsilon} \left(u_\xi^p - \bar u_\xi^p - p\bar u_\xi^{p-1}\Psi(\xi)\right) \Psi(\xi)\, dx + O(|\Psi(\xi)|^3). \tag{7.144}$$

Now, we observe that

$$|u_\xi^p - \bar u_\xi^p - p\bar u_\xi^{p-1}\Psi(\xi)| \leqslant |u_\xi^p - \bar u_\xi^p| + p\,|\bar u_\xi^{p-1}\Psi(\xi)| \leqslant C\,|\bar u_\xi^{p-1}\Psi(\xi)|,$$

for a positive constant $C$, and so, using also (7.100), we have

$$\begin{aligned} &\left| \int_{\Omega_\varepsilon} \left(u_\xi^p - \bar u_\xi^p - p\bar u_\xi^{p-1}\Psi(\xi)\right) \Psi(\xi)\, dx \right| \\ &\leqslant C \int_{\Omega_\varepsilon} |\bar u_\xi|^{p-1} |\Psi(\xi)|^2\, dx \\ &\leqslant C \|\Psi(\xi)\|^2_{\star,\xi} \int_{\Omega_\varepsilon} |\bar u_\xi|^{p-1} \rho_\xi^2\, dx \\ &\leqslant C \|\Psi(\xi)\|^2_{\star,\xi} \int_{\Omega_\varepsilon} |w_\xi + O(\varepsilon^{n+2s})|^{p-1} \rho_\xi^2\, dx \\ &\leqslant C \|\Psi(\xi)\|^2_{\star,\xi} \int_{\Omega_\varepsilon} |w_\xi|^{p-1} \rho_\xi^2\, dx + C\,\varepsilon^{(p-1)(n+2s)} \|\Psi(\xi)\|^2_{\star,\xi} \int_{\Omega_\varepsilon} \rho_\xi^2\, dx. \end{aligned} \tag{7.145}$$

Recalling the definition of $\rho_\xi$ in (6.2) and the fact that $\mu > n/2$, we have that

$$\int_{\Omega_\varepsilon} \rho_\xi^2 \, dx \leqslant C_1, \tag{7.146}$$

for a suitable constant $C_1 > 0$. Moreover, thanks to (7.99) (recall also (7.119)), we obtain

$$\varepsilon^{(p-1)(n+2s)} \|\Psi(\xi)\|_{\star,\xi}^2 \leqslant C_2\, \varepsilon^{(p-1)(n+2s)}\, \varepsilon^{2(n+2s)} = C_2\, \varepsilon^{(p+1)(n+2s)},$$

which, together with (7.146), says that

$$C\, \varepsilon^{(p-1)(n+2s)} \|\Psi(\xi)\|_{\star,\xi}^2 \int_{\Omega_\varepsilon} \rho_\xi^2 \, dx = o(\varepsilon^{n+4s}). \tag{7.147}$$

Also, using (1.4) we have that

$$\int_{\Omega_\varepsilon} |w_\xi|^{p-1} \rho_\xi^2 \, dx \leqslant C_3 \int_{\Omega_\varepsilon} \frac{1}{(1+|x-\xi|)^{(p-1)(n+2s)}} \frac{1}{(1+|x-\xi|)^{2\mu}} \, dx \leqslant C_4,$$

and so, using also (7.99) we have that

$$C \|\Psi(\xi)\|_{\star,\xi}^2 \int_{\Omega_\varepsilon} |w_\xi|^{p-1} \rho_\xi^2 \, dx = o(\varepsilon^{n+4s}).$$

This, (7.144), (7.145) and (7.147) give the desired claim in Theorem 7.17. □

## 8. Proof of Theorem 1.1

In this section we complete the proof of Theorem 1.1. For this, we notice that, thanks to Theorems 4.1 and 7.17, we have that, for any $\xi \in \Omega_\varepsilon$ with $\operatorname{dist}(\xi, \partial\Omega_\varepsilon) \geqslant \delta/\varepsilon$ (for some $\delta \in (0,1)$),

$$J_\varepsilon(\xi) = I(w) + \frac{1}{2}\mathcal{H}_\varepsilon(\xi) + o(\varepsilon^{n+4s}), \tag{8.1}$$

where $J_\varepsilon$ and $I$ are defined in (7.133) and (4.1) respectively (see also (7.119)), where $\mathcal{H}_\varepsilon$ is given by (1.17).

Also, we recall the definition of the set $\Omega_{\varepsilon,\delta}$ given in (2.29), and we claim that $J_\varepsilon$ has an interior minimum, namely

$$\text{there exists } \bar\xi \in \Omega_{\varepsilon,\delta} \text{ such that } J_\varepsilon(\bar\xi) = \min_{\xi \in \overline{\Omega}_{\varepsilon,\delta}} J_\varepsilon(\xi). \tag{8.2}$$

For this, we observe that $J_\varepsilon$ is a continuous functional, and therefore

$$J_\varepsilon \text{ admits a minimizer } \bar\xi \in \overline{\Omega}_{\varepsilon,\delta}. \tag{8.3}$$

We have that

$$\bar\xi \in \Omega_{\varepsilon,\delta}. \tag{8.4}$$

Indeed, suppose by contradiction that $\bar\xi \in \partial\Omega_{\varepsilon,\delta}$. Then, from (8.1), we have that

$$\begin{aligned} J_\varepsilon(\bar\xi) &= I(w) + \frac{1}{2}\mathcal{H}_\varepsilon(\bar\xi) + o(\varepsilon^{n+4s}) \\ &\geqslant I(w) + \frac{1}{2}\min_{\partial\Omega_{\varepsilon,\delta}} \mathcal{H}_\varepsilon + o(\varepsilon^{n+4s}). \end{aligned} \tag{8.5}$$

On the other hand, by Proposition 2.8, we know that $\mathcal{H}_\varepsilon$ has a strict interior minimum: more precisely, there exists $\xi_o \in \Omega_{\varepsilon,\delta}$ such that

$$\mathcal{H}_\varepsilon(\xi_o) = \min_{\Omega_{\varepsilon,\delta}} \mathcal{H}_\varepsilon \leqslant c_1 \varepsilon^{n+4s} \tag{8.6}$$

and

$$\min_{\partial\Omega_{\varepsilon,\delta}} \mathcal{H}_\varepsilon \geqslant c_2 \left(\frac{\varepsilon}{\delta}\right)^{n+4s}, \tag{8.7}$$

for suitable $c_1, c_2 > 0$. Also, the minimality of $\bar{\xi}$ and (8.1) say that

$$\begin{aligned} J_\varepsilon(\bar{\xi}) &= \min_{\xi\in\overline{\Omega}_{\varepsilon,\delta}} J_\varepsilon(\xi) \\ &\leqslant J_\varepsilon(\xi_o) \\ &= I(w) + \frac{1}{2}\mathcal{H}_\varepsilon(\xi_o) + o(\varepsilon^{n+4s}). \end{aligned}$$

By comparing this with (8.5) and using (8.6) and (8.7) we obtain

$$\begin{aligned} \frac{c_2\varepsilon^{n+4s}}{2\,\delta^{n+4s}} + o(\varepsilon^{n+4s}) &\leqslant \frac{1}{2}\min_{\partial\Omega_{\varepsilon,\delta}} \mathcal{H}_\varepsilon + o(\varepsilon^{n+4s}) \\ &\leqslant J_\varepsilon(\bar{\xi}) - I(w) \\ &\leqslant \frac{1}{2}\mathcal{H}_\varepsilon(\xi_o) + o(\varepsilon^{n+4s}) \\ &\leqslant \frac{c_1\varepsilon^{n+4s}}{2} + o(\varepsilon^{n+4s}). \end{aligned}$$

So, a division by $\varepsilon^{n+4s}$ and a limit argument give that

$$\frac{c_2}{2\,\delta^{n+4s}} \leqslant \frac{c_1}{2}.$$

This is a contradiction when $\delta$ is sufficiently small, thus (8.4) is proved. Hence (8.2) follows from (8.3) and (8.4).

From (8.2), since $\Omega_{\varepsilon,\delta}$ is open, we conclude that

$$\frac{\partial J_\varepsilon}{\partial \xi}(\bar{\xi}) = 0.$$

Therefore, from Lemma 7.16 we obtain the existence of a solution to (1.1) that satisfies (1.5) for $\varepsilon$ sufficiently small, and this concludes the proof of Theorem 1.1.

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## Appendix A. Some physical motivation

Equation (1.1) is a particular case of the fractional Schrödinger equation

$$i\hbar\partial_t\psi = \hbar^{2s}(-\Delta)^s\psi + V\,\psi, \tag{A.1}$$

in case the wave function $\psi$ is a standing wave (i.e. $\psi(x,t) = U(x)e^{it/\hbar}$) and the potential $V$ is a suitable power of the density function (i.e. $V = V(|\psi|) = -|\psi|^{p-1}$). As usual, $\hbar$ is the Planck’s constant (then we write $\varepsilon := \hbar$ in (1.1)) and $\psi = \psi(x,t)$ is the quantum mechanical probability amplitude for a given particle (for simplicity, of unit mass) to have position $x$ at time $t$ (the corresponding probability density is $|\psi|^2$).

In this setting our Theorem 1.1 describes the confinement of a particle inside a given domain $\Omega$: for small values of $\hbar$ the wave function concentrates to a material particle well inside the domain.

Equation (A.1) is now quite popular (say, for instance, popular enough to have its own page on Wikipedia, see [36]) and it is based on the classical Schrödinger equation (corresponding to the case $s = 1$) in which the Brownian motion of the quantum paths is replaced by a Lévy flight. We refer to [25–27] for a throughout physical discussion and detailed motivation of equation (A.1) (see in particular formula (18) in [25]), but we present here some very sketchy heuristics about it.

The idea is that the evolution of the wave function $\psi(x,t)$ from its initial state $\psi_0(x) := \psi(x,0)$ is run by a quantum mechanics kernel (or amplitude) $K$ which produces the forthcoming values of the wave function by integration with the initial state, i.e.

$$\psi(x,t) = \int_{\mathbb{R}^n} dy\ K(x,y,t)\,\psi_0(y). \tag{A.2}$$

The main assumption is that such amplitude $K(x,y,t)$ is modulated by an action functional $S_t$ via the contributions of all the possible paths $\gamma$ that join $x$ to $y$ in time $t$, that is

$$K(x,y,t) = \int_{\mathcal{F}(x,y,t)} d\gamma\ e^{-iS_t(\gamma)/\hbar}. \tag{A.3}$$

The above integral denotes the Feynman path integral over "all possible histories of the system", that is over "all possible" continuous paths $\gamma : [0,t] \longrightarrow \mathbb{R}^n$ with $\gamma(0) = y$ and $\gamma(t) = x$, see [18]. We remark that such integral is indeed a functional integral, that is the domain of integration $\mathcal{F}(x,y,t)$ is not a region of a finite dimensional space, but a space of functions. The mathematical treatment of Feynman path integrals is by no means trivial: as a matter of fact, the convergence must rely on the highly oscillatory behavior of the system which produces the necessary cancellations. In some cases, a rigorous justification can be provided by the theory of Wiener spaces, but a complete treatment of this topic is far beyond the scopes of this appendix (see e.g. [4, 5] and [1, 23]).

The next structural ansatz we take is that the action functional $S_t$ is the superposition of a (complex) diffusive operator $H_0$ and a potential term $V$.

Though the diffusion and the potential operate "simultaneously", with some approximation we may suppose that, at each tiny time step, they operate just one at the time, interchanging their action[5] at a very high frequency. Namely, we discretize a path $\gamma$ into $N$ adjacent paths of time range $t/N$, say $\gamma_1, \dots, \gamma_N : [0, t/N] \longrightarrow \mathbb{R}^n$, with $\gamma_1(0) = y$ and $\gamma_N(t/N) = x$, and we suppose that, along each $\gamma_j$ the action reduces to the subsequent nonoverlapping superpositions of diffusion and

[5]In a sense, this is the quantum mechanics version of the Lie–Trotter product formula

$$e^{A+B} = \lim_{N\to+\infty} \left(e^{A/N} e^{B/N}\right)^N$$

for $A, B \in \mathrm{Mat}\,(n \times n)$. The procedure of disentangling mixed exponentials is indeed crucial in quantum mechanics computations, see e.g. [19]. In our computation, a more rigorous approximation scheme lies in explicitly writing $S_t(\gamma)$ as an integral from $0$ to $t$ of the Lagrangian along the path $\gamma$, then one splits the integral in $N$ time steps of size $t/N$ by supposing that in each of these time steps the Lagrangian is, approximatively, constant. One may also suppose that the Lagrangian involved in the action is a classical one, i.e. it is the sum of a kinetic term and the potential $V$. Then the effect of taking the integral over all the possible paths averages out the kinetic part reducing it to a diffusive operator. Since here we are not aiming at a rigorous justification of all these delicate procedures (such as infinite dimensional integrals, limit exchanges and so on), for simplicity we are just taking $H_0$ to be a diffusive operator from the beginning. In this spirit, it is also convenient to suppose that the potential is an operator, that is we identify $V$ with the operation of multiplying a function by $V$.

potential terms, according to the formula

(A.4) $$e^{-iS_t(\gamma)/\hbar} = \lim_{N\to+\infty} \left(e^{-itH_0/(\hbar N)} e^{-itV/(\hbar N)}\right)^N.$$

Once more, we do not indulge into a rigorous mathematical discussion of such a limit and we just plug (A.3) and (A.4) into (A.2). We obtain

(A.5) $$\begin{aligned} \psi(x,t) &= \int_{\mathbb{R}^n} dy \int_{\mathcal{F}(x,y,t)} d\gamma\, e^{-iS_t(\gamma)/\hbar}\, \psi_0(y) \\ &= \lim_{N\to+\infty} \int_{\mathbb{R}^n} dy \int_{\mathcal{F}(x,y,t)} d\gamma\, \left(e^{-itH_0/(\hbar N)} e^{-itV/(\hbar N)}\right)^N \psi_0(y). \end{aligned}$$

Therefore, if we formally[6] apply the time derivative to (A.5) we obtain that

(A.6)
$$\begin{aligned} & i\hbar\partial_t\psi(x,t) \\ &= \lim_{N\to+\infty} \int_{\mathbb{R}^n} dy \int_{\mathcal{F}(x,y,t)} d\gamma\, N \left(\frac{H_0}{N} e^{-itH_0/(\hbar N)} e^{-itV/(\hbar N)} + \frac{V}{N} e^{-itH_0/(\hbar N)} e^{-itV/(\hbar N)}\right) \\ &\quad \cdot \left(e^{-itH_0/(\hbar N)} e^{-itV/(\hbar N)}\right)^{N-1} \psi_0(y) \\ &= \lim_{N\to+\infty} \int_{\mathbb{R}^n} dy \int_{\mathcal{F}(x,y,t)} d\gamma\, \left(H_0 e^{-itH_0/(\hbar N)} e^{-itV/(\hbar N)} + V e^{-itH_0/(\hbar N)} e^{-itV/(\hbar N)}\right) \\ &\quad \cdot \left(e^{-itH_0/(\hbar N)} e^{-itV/(\hbar N)}\right)^{N-1} \psi_0(y) \\ &= (H_0 + V) \int_{\mathbb{R}^n} dy \int_{\mathcal{F}(x,y,t)} d\gamma\, e^{-iS_t(\gamma)/\hbar}\, \psi_0(y) \\ &= (H_0 + V)\,\psi \end{aligned}$$

by (A.2), (A.3) and (A.4). The classical Schrödinger equation follows by taking $H_0 := -\hbar^2\Delta$, that is the Gaussian diffusive process, while (A.1) follows by taking $H_0 := \hbar^{2s}(-\Delta)^s$, that is the $2s$-stable diffusive process with polynomial tail.

[6]The disentangling procedure allows to take derivative of the exponentials of the operators “as they were commuting ones”. Namely, by the Zassenhaus formula,

$$e^{t(A+B)} = e^{tA} e^{tB} e^{O(t^2)} = e^{tA} e^{tB}(1 + O(t^2))$$

that in our case gives

$$e^{-itH_0/(\hbar N)} e^{-itV/(\hbar N)} = e^{-it(H_0+V)/(\hbar N)}(1 + O(t^2/N^2))$$

and so

$$\left(e^{-itH_0/(\hbar N)} e^{-itV/(\hbar N)}\right)^N = e^{-it(H_0+V)/\hbar}(1 + O(t^2/N^2))$$

Hence

$$\begin{aligned} & \lim_{N\to+\infty} \partial_t \left(e^{-itH_0/(\hbar N)} e^{-itV/(\hbar N)}\right)^N \\ = & \lim_{N\to+\infty} -\frac{i(H_0+V)}{\hbar} e^{-it(H_0+V)/\hbar}(1 + O(t^2/N^2)) + O(t/N^2) \\ = & -\frac{i(H_0+V)}{\hbar}. \end{aligned}$$

Moreover, we point out that a couple of additional approximations are likely to be hidden in the computation in (A.6). Namely, first of all, we do not differentiate the functional domain of the Feynman integral. This is consistent with the ansatz that the set of the paths joining two points at a macroscopic scale in time $t$ “does not vary much” for small variations of $t$. Furthermore, we replace the action of $H_0$ and $V$ along the infinitesimal paths with their effective action after averaging, so that we take $(H_0 + V)$ outside the integral.

Having given a brief justification of (A.1), we also recall that the fractional Schrödinger case presents interesting differences with respect to the classical one. For instance, the energy of a particle of unit mass is proportional to $|p|^{2s}$ (instead of $|p|^2$, see e.g. formula (12) in [25]). Also the space/time scaling of the process gives that the fractal dimension of the Lévy paths is $2s$ (differently from the classical Brownian case in which it is $2$), see pages 300–301 of [25].

Now, for completeness, we discuss a nonlocal notion of canonical quantization, together with the associated Heisenberg Uncertainty Principle (see e.g. pages 17–28 of [22] for the classical canonical quantization and related issues).

For this, we introduce the canonical operators for $k \in \{1, \dots, n\}$

$$P_k := -i\hbar^s \partial_k (-\Delta)^{(s-1)/2} \text{ and } Q_k := x_k. \tag{A.7}$$

Notice that $Q_k$ is the classical position operator, namely the multiplication by the $k$th space coordinate. On the other hand, $P_k$ is a fractional momentum operator, that reduces[7] to the classical momentum $-i\hbar\partial_k$ when $s = 1$. In this setting, our goal is to check that the commutator

$$[Q, P] := \sum_{k=1}^{n} [Q_k, P_k]$$

does not vanish. For this, we suppose $0 < \sigma < n/2$ and use the Riesz potential representation of the inverse of the fractional Laplacian of order $\sigma$, that is

$$(-\Delta)^{-\sigma}\psi(x) = c(n, s) \int_{\mathbb{R}^n} \frac{\psi(x-y)}{|y|^{n-2\sigma}}\, dy = c(n, s) \int_{\mathbb{R}^n} \frac{\psi(y)}{|x-y|^{n-2\sigma}}\, dy, \tag{A.8}$$

for a suitable $c(n, s) > 0$, see [24].

In our case we use (A.8) with $\sigma := (1-s)/2 \in (0, 1/2) \subseteq (0, n/2)$. Then

$$P_k\psi(x) = -c(n, s)\, i\hbar^s \partial_k \int_{\mathbb{R}^n} \frac{\psi(y)}{|x-y|^{n+s-1}}\, dy = c(n, s)\, i\, \hbar^s\, (n+s-1) \int_{\mathbb{R}^n} \frac{(x_k - y_k)\, \psi(y)}{|x-y|^{n+s+1}}\, dy$$

and so

$$P_k Q_k \psi(x) = P_k(x_k \psi(x)) = c(n, s)\, i\, \hbar^s\, (n+s-1) \int_{\mathbb{R}^n} \frac{(x_k - y_k)\, y_k\, \psi(y)}{|x-y|^{n+s+1}}\, dy.$$

This gives that

$$\begin{aligned}
& Q_k P_k \psi - P_k Q_k \psi \\
= \;& c(n, s)\, i\, \hbar^s\, (n+s-1) \left[ \int_{\mathbb{R}^n} \frac{x_k\, (x_k - y_k)\, \psi(y)}{|x-y|^{n+s+1}}\, dy - \int_{\mathbb{R}^n} \frac{(x_k - y_k)\, y_k\, \psi(y)}{|x-y|^{n+s+1}}\, dy \right] \\
= \;& c(n, s)\, i\, \hbar^s\, (n+s-1) \int_{\mathbb{R}^n} \frac{(x_k - y_k)^2\, \psi(y)}{|x-y|^{n+s+1}}\, dy,
\end{aligned}$$

[7] Of course, from the point of view of physical dimensions, the fractional momentum is not a momentum, since it has physical dimension $[\mathtt{Planckconstant}]^s/[\mathtt{length}]^s$, while the classical momentum has physical dimension $[\mathtt{Planckconstant}]/[\mathtt{length}]$. Namely, the physical dimension of the fractional momentum is a fractional power of the physical dimension of the classical momentum. Clearly, the same phenomenon occurs for the physical dimension of the fractional Laplace operators in terms of the usual Laplacian.

and so, by summing up[8] and recalling (A.8), we conclude that

$$[Q,P]\psi = c(n,s)\, i\,\hbar^s\,(n+s-1)\int_{\mathbb{R}^n}\frac{\psi(y)}{|x-y|^{n+s-1}}\,dy = i\,(n+s-1)\,\hbar^s(-\Delta)^{(s-1)/2}\psi.$$

Notice that, as $s \longrightarrow 1$, this formula reduces to the the classical Heisenberg Uncertainty Principle.

We also point out that a similar computation shows that, differently from the local quantum momentum, the $k$th fractional quantum momentum does not commute with the $m$th spatial coordinates even when $k \neq m$, namely $[Q_m, P_k]\psi(x)$ is, up to normalizing constants,

$$i\hbar^s \int_{\mathbb{R}^n} \frac{(x_m - y_m)(x_k - y_k)\,\psi(y)}{|x-y|^{n+s+1}}\,dy.$$

This Heisenberg Uncertainty Principle is also compatible with equation (A.1), in the sense that the diffusive operator $H_0$ is exactly the one obtained by the canonical quantization in (A.7): indeed

$$\begin{aligned}
\sum_{k=1}^n P_k^2 &= \sum_{k=1}^n \Big(-i\hbar^s\partial_k(-\Delta)^{(s-1)/2}\Big)\Big(-i\hbar^s\partial_k(-\Delta)^{(s-1)/2}\Big)\\
&= -\hbar^{2s}\sum_{k=1}^n \partial_k^2(-\Delta)^{s-1}\\
&= -\hbar^{2s}\Delta\,(-\Delta)^{s-1}\\
&= \hbar^{2s}(-\Delta)^s\\
&= H_0.
\end{aligned}$$

Moreover, we mention that the fractional Laplace operator also arises naturally in the high energy Hamiltonians of relativistic theories. For further motivation of the fractional Laplacian in modern physics see e.g. [7] and references therein.

[8] Alternatively, one can also perform the commutator calculation in Fourier space and then reduce to the original variable by an inverse Fourier transform. This computation can be done easily by using the facts that the Fourier transform sends products into convolutions and that (up to constants)

$$\begin{aligned}
(\hat{x}_k * g)(\xi) &= \mathcal{F}\big(x_k\mathcal{F}^{-1}g(x)\big)(\xi)\\
&= \int_{\mathbb{R}^n} dx \int_{\mathbb{R}^n} dy\; e^{ix\cdot(y-\xi)}x_k g(y)\\
&= i^{-1}\int_{\mathbb{R}^n} dx \int_{\mathbb{R}^n} dy\; \partial_{y_k} e^{ix\cdot(y-\xi)} g(y)\\
&= i\int_{\mathbb{R}^n} dx \int_{\mathbb{R}^n} dy\; e^{ix\cdot(y-\xi)}\partial_k g(y)\\
&= i\int_{\mathbb{R}^n} dx\; e^{-ix\cdot\xi}\mathcal{F}^{-1}(\partial_k g)(x)\\
&= i\mathcal{F}\big(\mathcal{F}^{-1}(\partial_k g)\big)(\xi)\\
&= i\partial_k g(\xi).
\end{aligned}$$

Then we leave to the reader the computation of $\mathcal{F}([Q,P]\psi)(\xi)$.

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