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Trace formulas for singular perturbations

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Abstract

Trace formulas for pairs of self-adjoint, maximal dissipative and other types of resolvent comparable operators are obtained. In particular, the existence of a *complex-valued spectral shift function* for a resolvent comparable pair $\{H', H\}$ of maximal dissipative operators is proved. We also investigate the existence of a *real-valued spectral shift function*. Moreover, we treat in detail the case of additive trace class perturbations. Assuming that H and $H' = H + V$ are maximal dissipative and V is of trace class, we prove the existence of a *summable* complex-valued spectral shift function. We also obtain trace formulas for a pair $\{A, A^*\}$ *assuming only that A and A^* are resolvent comparable*. In this case the determinant of a characteristic function of A is involved in the trace formula.

In the case of singular perturbations we apply the technique of boundary triplets. It allows to express the spectral shift function of a pair of extensions in terms of abstract Weyl function and boundary operator.

We improve and generalize certain classical results of M.G. Krein for pairs of self-adjoint and dissipative operators, the results of A. Rybkin for such pairs, as well as the results of V. Adamyan, B. Pavlov, and M.Krein for pairs $\{A, A^*\}$ with a maximal dissipative operator A.

1 Introduction

An important tool in operator theory is the so-called trace formulas originally introduced by I.M. Lifshitz [28]. Let H_0 be a self-adjoint operators and let V be a finite dimensional self-adjoint operator. He has observed that for a wide class of functions $\Phi(\cdot) : \mathbb{R} \longrightarrow \mathbb{R}$ the relation

$$\mathrm{tr}\,(\Phi(H) - \Phi(H_0)) = \int_{\mathbb{R}} \Phi'(t)\xi(t)dt \tag{1.1}$$

takes place where $H = H_0 + V$. The function $\xi(\cdot) : \mathbb{R} \longrightarrow \mathbb{R}$ depends only on H_0 and V. The self-adjoint operators H_0 and H may have point or continuous spectrum. I.M. Lifshitz applied formula (1.1) to compute the free energy of a crystal in the case of presence of an impurity at some lattice point. Since that time formulas of type (1.1) are called trace formulas. The function $\xi(\cdot)$ is called the *spectral shift function (in short SSF)*.

Outgoing from [28], Krein [23] extended trace formula (1.1) to trace class perturbations $V = H - H_0 \in \mathfrak{S}_1(\mathfrak{H})$. Moreover, for the rigorous justification of the existence of the SSF of a pair $\{H, H_0\}$ he introduced the concept of perturbation determinant $\Delta_{H/H_0}(\cdot)$ and proved the inversion formula

$$\xi(t) = \frac{1}{\pi}\lim_{y\downarrow 0} \mathrm{Im}\,(\log(\Delta_{H'/H}(t + iy))) \quad \text{for a.e.} \quad t \in \mathbb{R}, \tag{1.2}$$

expressing $\xi(\cdot)$ by means of $\Delta_{H/H_0}(\cdot)$. Here the branch of the logarithm is fixed by the condition $\lim_{y\to+\infty}\log(\Delta_{H'/H}(t + iy)) = 0$. Such treatment has allowed him to show that there exists the unique SSF satisfying $\xi(\cdot) \in L^1(\mathbb{R}; dt)$. Krein [25] introduced the class $\mathcal{K}(\mathbb{R})$ of functions admitting the integral representation

$$\Phi(t) = i\int_{\mathbb{R}} \frac{e^{-ist} - 1}{s}\, dp(s), \quad t \in \mathbb{R}, \tag{1.3}$$

with $p(\cdot) : \mathbb{R} \longrightarrow \mathbb{C}$ being of bounded variation, and proved (1.1) for $\Phi \in \mathcal{K}(\mathbb{R})$. Clearly, the class $\mathcal{K}(\mathbb{R})$ consists of absolutely continuous functions $\Phi(\cdot)$ with the derivative $\Phi'(t) = \int_{\mathbb{R}} e^{-ist} \ dp(s)$ being the Fourier-Stieltjes transform. The widest known class of functions for which (1.1) holds was found by V. Peller [32, 33]. Namely, he proved (1.1) for functions $\Phi \in B^1_{\infty 1}(\mathbb{R})$, i.e. the Besov class.

In subsequent publications M. Krein [24, 25] extended (1.1) to a pair $\{H, H_0\}$ of self-adjoint *resolvent comparable operators*, i.e., operators satisfying

$$(H - z)^{-1} - (H_0 - z)^{-1} \in \mathfrak{S}_1(\mathfrak{H}), \quad z \in \rho(H) \cap \rho(H_0). \tag{1.4}$$

This extension has been motivated by applications to Schrödinger operators $H = H_0 + q$ (and other differential operators). Clearly, H is not a trace class perturbation of $H_0 = -\Delta$ while the pair $\{H, H_0\}$ satisfies (1.4) for certain classes of decaying potentials q.

For pairs satisfying (1.4) a spectral shift function $\xi(\cdot) = \overline{\xi(\cdot)}$ exists and belongs to $L^1(\mathbb{R}; \frac{dt}{1+t^2})$, which is determined up to an additive real constant.

A first attempt to generalize trace formulas to pairs of non-selfadjoint and non-unitary operators goes back to Langer [27]. Pairs $\{H, H^*\}$ with a *maximal dissipative* operator H were treated by different methods in [2] and [40]. Namely, using the functional model and assuming that the pair $\{H, H^*\}$ is resolvent comparable, Adamyan and Pavlov [2] proved the following trace formula

$$\mathrm{tr}\,(\Phi(H) - \Phi(H^*)) = \sum_{z_k \in \sigma_p(H)} m_k(\Phi(z_k) - \Phi(\overline{z}_k)) + \frac{i}{\pi} \int_{\mathbb{R}} \Phi'(t) d\mu(t) - ia\, \mathrm{res}_\infty(\Phi) \tag{1.5}$$

for functions Φ, which are holomorphic in a neighborhood of $\sigma(H) \cup \sigma(H^*)$. Here μ is a Borel measure on $\mathbb{R}$ satisfying $\int_{\mathbb{R}} \frac{d\mu}{1+t^2} < \infty$, $\{z_k\}_k \subset \mathbb{C}_+$ the set of eigenvalues of H, m_k the algebraic multiplicity of z_j, $a \geq 0$ and $\mathrm{res}_\infty(\Phi)$ is the residuum of Φ at infinity. Earlier L. Sakhnovich [40] using the triangular model of M.S. Livsic proved (1.5) for bounded H with $H - H^* \in \mathfrak{S}_1$ and rational Φ.

A. Rybkin [35, 36] considered a pair $\{H, H_0\}$ consisting of a maximal dissipative operator H and $H_0 = H_0^*$. Assuming (1.4) he proved the following trace formula

$$\mathrm{tr}\,(\Phi(H) - \Phi(H_0)) = \sum_{z_k \in \sigma(H)} m_k \Phi(z_k) + \int_{\mathbb{R}} \Phi'(t) \omega(t) dt. \tag{1.6}$$

Here $\Phi(\cdot)$ is holomorphic on a neighborhood of $\sigma(H) \cup \sigma(H_0)$ and decays sufficiently fast at infinity. The meaning of the integral in (1.6) was clarified by A. Rybkin in the subsequent publications [37, 38, 39].

M. Krein [26] extended formula (1.5) to pairs $\{H^*, H\}$, $H := H_0 + iG$, where H_0 and $G \in \mathfrak{S}_1(\mathfrak{H})$ are selfadjoint. Notice that H is *not necessarily dissipative* $H := H_0 + iG$. He proved (cf. [26, Theorem 8.4]) that the trace formula

$$\mathrm{tr}\,(\Phi(H) - \Phi(H^*)) = \sum_{z_k \in \sigma_p(H)} m_k \left(\Phi(z_k) - \Phi(\overline{z}_k)\right) + i \int_{\mathbb{R}} \Phi'(t) d\omega(t) \tag{1.7}$$

holds with $\omega(\cdot) = \overline{\omega(\cdot)}$ being a function of bounded variation on $\mathbb{R}$. Since now $\mathrm{res}_\infty(\Phi) = 0$, (1.7) coincides with (1.5) if $\omega(\cdot)$ is non-decreasing. However as distinguished from (1.5) in the case of additive perturbations $\omega(\cdot)$ is bounded.

Studying the accumulative case $H := H_0 - iG$, $G \geq 0$, Krein introduced the class $\mathcal{K}(\mathbb{R}_+)$ of functions holomorphic in $\mathbb{C}_-$ and admitting integral representations (1.3) with $p(\cdot) : \mathbb{R}_+ \longrightarrow \mathbb{C}$ of bounded variation and supported on $\mathbb{R}_+$. He proved [26, Theorem 9.2] that if $\Phi \in \mathcal{K}(\mathbb{R}_+)$, then $\Phi(H) - \Phi(H_0) \in \mathfrak{S}_1(\mathfrak{H})$, and instead of (1.1) the following trace formula holds

$$\mathrm{tr}\,(\Phi(H) - \Phi(H_0)) = -i \int_{\mathbb{R}} \Phi'(t)\; d\omega_K(t). \tag{1.8}$$

Here $\omega_K(\cdot) = \overline{\omega_K(\cdot)}$ is a bounded non-decreasing function.

Finally, pairs $\{H, H_0\}$ with $H_0 = H_0^*$ and $H := H_0 - iG$ where $G \geq 0$, and $G \log(G) \in \mathfrak{S}_1(\mathfrak{H})$, were studied in [1]. It is proved in [1] that under the assumption $G \log(G) \in \mathfrak{S}_1(\mathfrak{H})$ there exists a *real-valued* function $\xi(\cdot) \in L^1(\mathbb{R}, \frac{dt}{1+t^2})$ such that in place of (1.8) one has

$$\mathrm{tr}\,(\Phi(H) - \Phi(H_0)) = \int_{\mathbb{R}} \Phi'(t)\; \xi(t)\; dt \tag{1.9}$$

for Φ from a certain class of holomorphic in $\mathbb{C}_-$ functions, which is smaller than $\mathcal{K}(\mathbb{R}_+)$. Notice that $G \log(G) \in \mathfrak{S}_1(\mathfrak{H})$ is stronger than $G \in \mathfrak{S}_1(\mathfrak{H})$.

Our goal is to improve and generalize the above mentioned results as well as extend them to a broader classes of operators. To demonstrate our achievements we first summarize our main results on pairs of maximal accumulative operators.

Theorem 1.1. *Let $\widetilde{A}'$ and $\widetilde{A}$ be two maximal accumulative resolvent comparable operators in $\mathfrak{H}$ and $\rho(\widetilde{A}) \cap \mathbb{C}_- \neq \emptyset$. If $\Phi \in \mathcal{F}_+(\widetilde{A}, \widetilde{A}')$ (see D), then $\Phi(\widetilde{A}') - \Phi(\widetilde{A}) \in \mathfrak{S}_1(\mathfrak{H})$. Moreover, the following holds:*

(i) *There exists a complex-valued function $\omega(\cdot) \in L^1(\mathbb{R}; \frac{dt}{1+t^2})$ (a SSF of $\{\widetilde{A}', \widetilde{A}\}$) such that the following trace formula holds*

$$\mathrm{tr}\,(\Phi(\widetilde{A}') - \Phi(\widetilde{A})) = \frac{1}{\pi} \int_{\mathbb{R}} \Phi'(t)\omega(t)dt. \tag{1.10}$$

A complex-valued function $\widetilde{\omega} \in L^1(\mathbb{R}; \frac{dt}{1+t^2})$ is also a SSF of the pair $\{\widetilde{A}', \widetilde{A}\}$, i.e. (1.10) *holds with $\widetilde{\omega}$ in place of ω, if and only if $\widetilde{\omega}(\cdot) - \omega(\cdot) \in H^1_-(\mathbb{R}, \frac{dt}{1+t^2})$.*

(ii) *If in addition, the imaginary part of ω satisfies the Zygmund condition (cf.* (3.25)*), then there exists a* real-valued SSF $\xi(\cdot) \in L^1\left(\mathbb{R}; \frac{dt}{1+t^2}\right)$. *The latter happens if, in particular, $\omega(\cdot) \in L^2\left(\mathbb{R}; \frac{dt}{(1+t^2)^{\alpha/2}}\right)$ for some $\alpha \in [0, 2]$. Moreover, if $\alpha \in (0, 1)$, then there is real SSF $\xi(\cdot)$ satisfying $\xi(\cdot) \in L^2\left(\mathbb{R}; \frac{dt}{(1+t^2)^{\alpha/2}}\right)$.*

(iii) *If $\widetilde{A} = \widetilde{A}^*$ (resp. $\widetilde{A}' = \widetilde{A}'^*$), then there is a SSF $\omega(\cdot)$ of $\{\widetilde{A}', \widetilde{A}\}$ satisfying $\mathrm{Im}\,(\omega(t)) \leq 0$ (resp. $\mathrm{Im}\,(\omega(t)) \geq 0$) for a.e. $t \in \mathbb{R}$.*

In the case of additive perturbations, $\widetilde{A} = H$, $\widetilde{A}' = H' := H_0 - iG$, $G \in \mathfrak{S}_1$, Theorem 1.1 can be specified. Namely, in this case a *complex-valued SSF* $\omega(\cdot)$ can be chosen, which is summable, i.e. $\omega(\cdot) \in L^1(\mathbb{R}; dt)$ (Theorem 4.6). Moreover, in this case trace formula (1.10) is extended to the class $\mathcal{K}(\mathbb{R}_+)$ of holomorphic functions (Theorem 4.11). In particular, if $H = H^*$ Theorem 4.11 improves Krein's formula (1.8): the measure $d\omega_K$ becomes absolutely continuous.

We treat the problem in the framework of extension theory. Namely, we set

$$\begin{gathered} Af := \widetilde{A}'f = \widetilde{A}f, \quad f \in \mathrm{dom}\,(A), \\ \mathrm{dom}\,(A) := \left\{ \begin{array}{c} f \in \mathrm{dom}\,(\widetilde{A}') \cap \mathrm{dom}\,(\widetilde{A}'^*) \cap \mathrm{dom}\,(\widetilde{A}) \cap \mathrm{dom}\,(\widetilde{A}^*) : \\ \widetilde{A}'f = \widetilde{A}'^*f = \widetilde{A}f = \widetilde{A}^*f \end{array} \right\}. \end{gathered} \tag{1.11}$$

The operator A is closed symmetric but not necessarily densely defined. It might even happen that $\mathrm{dom}\,(A) = \{0\}$. However, in what follows we restrict ourselves to the case of a densely defined operator A. In this case the pair $\{\widetilde{A}', \widetilde{A}\}$ is called singular. We consider $\widetilde{A}$ and $\widetilde{A}'$ as proper extensions of A and apply the boundary triplet technique elaborated in [8]–[11] and especially in our recent papers [30, 31]. Namely, we systematically use the formula for perturbation determinants $\Delta_{\widetilde{A}'/\widetilde{A}}(\cdot)$ of singular perturbations, which expresses $\Delta_{\widetilde{A}'/\widetilde{A}}(\cdot)$ by means of the Weyl function and boundary operators (see Section 2 for the precise definitions). It allows us to obtain formula of type (3.10) for the SSF that complements the Krein inversion formula (1.2) in the selfadjoint case. For instance, if $n_{\pm}(A) = n < \infty$ and $\widetilde{A} = \widetilde{A}^*$, $\widetilde{A}' = (\widetilde{A}')^*$, a SSF of the pair $\{\widetilde{A}', \widetilde{A}\}$ admits the representation (3.13) (see below). Both formulas (3.10) and (3.13) are important in applications of SSF to boundary value problems for differential operators.

The paper is organized as follows. In Section 2 we give a short introduction into the boundary triplet approach to extension theory. In Section 3.1 we present a new proof of the trace formula for self-adjoint extensions of A and complement formula (1.2) for the singular case. In Section 3.2 we prove Theorem 1.1 stated above. Moreover, we investigate in detail *existence of a real-valued SSF* $\omega(\cdot)$.

Section 3.3 is devoted to pairs $\{\widetilde{A}', \widetilde{A}\}$ where $\widetilde{A}$ is maximal accumulative and $\widetilde{A}'$ is arbitrary. In this case we obtain the following trace formula (Theorem 3.22)

$$\mathrm{tr}\,(\Phi(\widetilde{A}') - \Phi(\widetilde{A})) = \sum_k m_k(\Phi(z_k) - \Phi(\overline{z}_k)) + \frac{1}{\pi}\int_{\mathbb{R}} \Phi'(t)d\nu(t) + i\alpha_+ \mathrm{res}_\infty(\Phi), \tag{1.12}$$

where $d\nu(t) := \omega(t)dy + id\mu_+(t)$, $\{z_k\}_k$ is the set of eigenvalues of $\widetilde{A}'$ in $\mathbb{C}_+$, m_k is the algebraic multiplicity of z_k, $\alpha_+ \geq 0$ and $\Phi \in \mathcal{F}_+(\widetilde{A}, \widetilde{A}')$. This formula generalizes (1.8).

In Section 3.4 we consider pairs $\{\widetilde{A}, \widetilde{A}^*\}$ where $\widetilde{A}$ is an arbitrary proper extension of A with non-empty resolvent set $\rho(\widetilde{A})$. We show in Theorem 3.24 that the perturbation determinant $\Delta^{\Pi}_{\widetilde{A}/\widetilde{A}^*}(\cdot)$ coincides with one of the characteristic functions of $\widetilde{A}$ in the sense of [9] and prove the following trace formula

$$\mathrm{tr}\,(\Phi(\widetilde{A}) - \Phi(\widetilde{A}^*)) = \sum_n m_n(\Phi(z_n) - \Phi(\overline{z}_n)) + \frac{i}{\pi}\int_{\mathbb{R}} \Phi'(t)d\mu(t) + i\alpha\, \mathrm{res}_\infty(\Phi) \tag{1.13}$$

for $\Phi \in \mathcal{F}_+(\widetilde{A}^*, \widetilde{A})$. Here $\{z_n\}_n := \sigma(\widetilde{A}) \cap (\mathbb{C} \setminus \mathbb{R})$ and m_n denotes the algebraic multiplicity of z_n.

If $\widetilde{A}$ is maximal dissipative, this formula is just formula (1.5) by Adamyan and Pavlov [2]. On the other hand, formula (1.13) extends Krein's formula (1.7) to the case of singular perturbations of not necessarily maximal dissipative operator $\widetilde{A}$.

Furthermore, in Section 3.5 we consider the case $\{\widetilde{A}', \widetilde{A}\}$ where $\widetilde{A}$ is maximal dissipative and $\widetilde{A}'$ is arbitrary.

Finally, in Section 4 we specify the previous results for the case of additive perturbations. Namely, in Section 4.1 we consider pairs $\{H', H\}$ with *maximal accumulative (in particular, self-adjoint) operators* H and $H' := H + V$ where $V \in \mathfrak{S}_1(\mathfrak{H})$. In particular, we prove Theorems 4.6 and 4.11 described above.

Besides, in Section 4.2, we consider pairs $\{H', H\}$ with maximal accumulative H and arbitrary $V \in \mathfrak{S}_1(\mathfrak{H})$, hence *not necessarily accumulative* H'. In Theorem 4.15 we prove the following trace formula

$$\mathrm{tr}(\Phi(H') - \Phi(H)) = \sum_k m_k(\Phi(z_k) - \Phi(\overline{z}_k)) + \frac{1}{\pi}\int_{\mathbb{R}} \Phi'(t)d\nu(t) \qquad (1.14)$$

for $\Phi \in \mathcal{F}_+(H, H')$. Here $d\nu(t) := id\mu_+(t) + \omega(t)dt$, $\{z_k\}_k$ is the set of eigenvalues of H' lying in $\mathbb{C}_+$ and $\{m_k\}_k$ is the set of the corresponding algebraic multiplicities. Formula (1.14) clarifies and complements (1.5) and (1.7). To make the paper self-contained a few appendices are added.

The main results of the paper have been published as a preprint [30].

Notation. By $\mathfrak{H}$ and $\mathcal{H}$ we denote separable Hilbert spaces. Linear operators in $\mathfrak{H}$ or $\mathcal{H}$ are always denoted by capital Latin letters. Denote by $\mathcal{C}(\mathcal{H})$ the set of all closed linear (not necessarily densely defined) operators in $\mathcal{H}$. The set of bounded linear operators from $\mathfrak{H}_1$ to $\mathfrak{H}_2$ is denoted by $[\mathfrak{H}_1, \mathfrak{H}_2]$; $[\mathfrak{H}] := [\mathfrak{H}, \mathfrak{H}]$. The Schatten-v.Neumann ideals of compact operators on $\mathfrak{H}$ are denoted by $\mathfrak{S}_p(\mathfrak{H})$, $0 < p \le \infty$; in particular, $\mathfrak{S}_\infty(\mathfrak{H})$ is the ideal of compact operators in $\mathfrak{H}$.

By $\mathrm{dom}\,(A)$, $\mathrm{ran}\,(A)$, and $\sigma(A)$ we denote the domain, range and spectrum of an operator $A \in \mathcal{C}(\mathcal{H})$, respectively. The symbols $\sigma_p(A)$, $\sigma_c(A)$ stand for the point and continuous spectrum of A. Recall that $z \in \sigma_c(H)$ if $\ker(H - z) = \{0\}$ and $\mathrm{ran}\,(H - z) \neq \overline{\mathrm{ran}\,(H - z)} = \mathfrak{H}$.

Recall that an operator $T \in \mathcal{C}(\mathcal{H})$ is called dissipative if $\mathrm{Im}\,((Tf, f)) \ge 0$, $f \in \mathrm{dom}\,(T)$, and maximal dissipative (m-dissipative) if it does not admit closed dissipative extensions. $T \in \mathcal{C}(\mathcal{H})$ is called accumulative (resp. m-accumulative) if $-T$ is dissipative (resp. m-dissipative).

2 Preliminaries

2.1 Linear relations

A linear relation Θ in $\mathcal{H}$ is a closed linear subspace of $\mathcal{H} \oplus \mathcal{H}$. The set of all linear relations in $\mathcal{H}$ is denoted by $\widetilde{\mathcal{C}}(\mathcal{H})$. Identifying each operator $T \in \mathcal{C}(\mathcal{H})$ with its graph $\mathrm{gr}\,(T)$ we regard $\mathcal{C}(\mathcal{H})$ as a subset of $\widetilde{\mathcal{C}}(\mathcal{H})$.

The role of the set $\widetilde{\mathcal{C}}(\mathcal{H})$ in extension theory becomes clear from Proposition 2.3. However, its role in the operator theory is substantially motivated by the following circumstances: in contrast to $\mathcal{C}(\mathcal{H})$, the set $\widetilde{\mathcal{C}}(\mathcal{H})$ is closed with respect to taking inverse and adjoint relations Θ^{-1} and Θ^*. The latter are given by: $\Theta^{-1} = \{\{g, f\} : \{f, g\} \in \Theta\}$ and

$$\Theta^* = \left\{ \begin{pmatrix} k \\ k' \end{pmatrix} : (h', k) = (h, k') \text{ for all } \begin{pmatrix} h \\ h' \end{pmatrix} \in \Theta \right\}.$$

A linear relation Θ is called symmetric if $\Theta \subset \Theta^*$ and self-adjoint if $\Theta = \Theta^*$.

2.2 Boundary triplets and proper extensions

Let A be a densely defined closed symmetric operator in $\mathfrak{H}$ with equal deficiency indices $n_\pm(A) = \dim(\mathfrak{N}_{\pm i})$, $\mathfrak{N}_z := \ker(A^* - z)$, $z \in \mathbb{C}_\pm$.

Definition 2.1.

(i) A closed extension $\widetilde{A}$ of A is called a proper extension if $A \subseteq \widetilde{A} \subseteq A^*$.

(ii) Two proper extensions $\widetilde{A}'$, $\widetilde{A}$ are called disjoint if $\mathrm{dom}\,(\widetilde{A}') \cap \mathrm{dom}\,(\widetilde{A}) = \mathrm{dom}\,(A)$ and transversal if in addition $\mathrm{dom}\,(\widetilde{A}') + \mathrm{dom}\,(\widetilde{A}) = \mathrm{dom}\,(A^*)$.

Denote by $\widetilde{A} \in \mathrm{Ext}\,_A$ the set of proper extensions of A completed by non-proper extensions A and A^*. Any self-adjoint, m-dissipative or m-accumulative extension is proper.

Definition 2.2 ([19]). A triplet $\Pi = \{\mathcal{H}, \Gamma_0, \Gamma_1\}$, where $\mathcal{H}$ is an auxiliary Hilbert space and $\Gamma_0, \Gamma_1 : \mathrm{dom}\,(A^*) \to \mathcal{H}$ are linear mappings, is called a boundary triplet for A^* if the äbstract Green's identity"

$$(A^* f, g) - (f, A^* g) = (\Gamma_1 f, \Gamma_0 g)_{\mathcal{H}} - (\Gamma_0 f, \Gamma_1 g)_{\mathcal{H}}, \qquad f, g \in \mathrm{dom}\,(A^*), \tag{2.1}$$

holds and the mapping $\Gamma := (\Gamma_0, \Gamma_1)^\top : \mathrm{dom}\,(A^*) \to \mathcal{H} \oplus \mathcal{H}$ is surjective.

A boundary triplet $\Pi = \{\mathcal{H}, \Gamma_0, \Gamma_1\}$ for A^* always exists whenever $n_+(A) = n_-(A)$. Note also that $n_\pm(A) = \dim(\mathcal{H})$ and $\ker(\Gamma_0) \cap \ker(\Gamma_1) = \mathrm{dom}\,(A)$.

With any boundary triplet Π one associates two canonical self-adjoint extensions $A_j := A^* \upharpoonright \ker(\Gamma_j)$, $j \in \{0, 1\}$. Conversely, for any $A_0 = A_0^* \in \mathrm{Ext}\,_A$ there exists a boundary triplet $\Pi = \{\mathcal{H}, \Gamma_0, \Gamma_1\}$ for A^* such that $A_0 := A^* \upharpoonright \ker(\Gamma_0)$.

Using the concept of boundary triplets one can parametrize the set of proper extensions of A in the following way.

Proposition 2.3 ([8, 29]). *Let $\Pi = \{\mathcal{H}, \Gamma_0, \Gamma_1\}$ be a boundary triplet for A^*. Then the mapping*

$$\operatorname{Ext}_A \ni \widetilde{A} \to \Gamma \operatorname{dom}(\widetilde{A}) = \{\{\Gamma_0 f, \Gamma_1 f\} : \ f \in \operatorname{dom}(\widetilde{A})\} =: \Theta \in \widetilde{\mathcal{C}}(\mathcal{H}) \tag{2.2}$$

establishes a bijective correspondence between the sets Ext_A and $\widetilde{\mathcal{C}}(\mathcal{H})$. We write $\widetilde{A} = A_\Theta$ if $\widetilde{A}$ corresponds to Θ by (2.2). *Moreover, the following holds:*

(i) $A_\Theta^* = A_{\Theta^*}$*, in particular, $A_\Theta^* = A_\Theta$ if and only if $\Theta^* = \Theta$.*

(ii) A_Θ *is symmetric (resp. self-adjoint) if and only if Θ is symmetric (resp. self-adjoint).*

(iii) A_Θ *is m-dissipative (resp. m-accumulative) if and only if so is Θ.*

(iv) *The extensions A_Θ and A_0 are disjoint (resp. transversal) if and only if $\Theta = \operatorname{gr}(B)$ and $B \in \mathcal{C}(\mathcal{H})$ (resp. $B \in [\mathcal{H}]$). In this case* (2.2) *takes the form*

$$\widetilde{A} = A_B := A_{\operatorname{gr}(B)} = A^* \upharpoonright \ker(\Gamma_1 - B\Gamma_0). \tag{2.3}$$

The operator B is called the boundary operator *of $\widetilde{A}$ with respect to Π.*

The following concepts are important in the sequel.

Definition 2.4.

(i) An extension $\widetilde{A} \in \operatorname{Ext}_A$ is called almost solvable if there exists a self-adjoint extension $\widehat{A}$ of A such that $\widehat{A}$ and $\widetilde{A}$ are transversal, see Definition 2.1(ii).

(ii) The family $\{\widetilde{A}_j\}_{j=1}^N \subset \operatorname{Ext}_A$ is called jointly almost solvable if there exists a self-adjoint extension $\widehat{A} \in \operatorname{Ext}_A$ transversal to each $\widetilde{A}_j$, $j \in \{1, \ldots, N\}$.

The class of almost solvable extensions of A was introduced and investigated in [12] (see also [9, 10, 11]). In particular, it is shown in [12, 9] that $\widetilde{A} \in \operatorname{Ext}_A$ is almost solvable whenever there exist $z_1, z_2 \in \rho(\widetilde{A}) \cup \sigma_c(\widetilde{A})$ such that $\operatorname{Im}(z_1)\operatorname{Im}(z_2) < 0$.

Definition 2.5 ([31]). A boundary triplet $\Pi = \{\mathcal{H}, \Gamma_0, \Gamma_1\}$ for A^* *is called regular* for the family $\{\widetilde{A}_j\}_{j=1}^N \subset \operatorname{Ext}_A$ if there exist bounded operators $B_j \in [\mathcal{H}]$ such that $\widetilde{A}_j = A_{B_j} := A^* \upharpoonright \ker(\Gamma_1 - B_j\Gamma_0)$, $j \in \{1, \ldots, N\}$ (cf. (2.3)).

Theorem 2.6 ([31, Theorem 3.5]). *Let A be as above and let $\{\widetilde{A}_j\}_{j=1}^N \subset \operatorname{Ext}_A$ and $\widetilde{A} =: \widetilde{A}_{N+1} \in \operatorname{Ext}_A$. Assume also that $\bigcap_{j=1}^{N+1} \rho(\widetilde{A}_j) \neq \emptyset$ and*

$$(\widetilde{A} - z_1)^{-1} - (\widetilde{A}_j - z_1)^{-1} \in \mathfrak{S}_\infty(\mathfrak{H}), \quad z_1 \in \bigcap_{j=1}^{N+1} \rho(\widetilde{A}_j), \quad j \in \{1, \ldots, N\}. \tag{2.4}$$

If there is $z_2 \in \rho(\widetilde{A}) \cup \sigma_c(\widetilde{A})$ such that $\operatorname{Im}(z_1)\operatorname{Im}(z_2) < 0$, then the family $\{\widetilde{A}_j\}_{j=1}^{N+1}$ is jointly almost solvable and, hence, admits a regular boundary triplet. In particular, the last condition is satisfied whenever $\rho(\widetilde{A}) \cap \mathbb{C}_\pm \neq \emptyset$.

2.3 Weyl functions and spectra of proper extensions

It is well known that Weyl functions are an important tool in the direct and inverse spectral theory of singular Sturm-Liouville operators. In [7, 8, 12] the concept of Weyl function was generalized to the case of an arbitrary symmetric operator A with $n_+(A) = n_-(A) \le \infty$. Following [8] we briefly recall basic facts on Weyl functions and γ-fields associated with a boundary triplet Π.

Definition 2.7 ([7, 8]). Let $\Pi = \{\mathcal{H}, \Gamma_0, \Gamma_1\}$ be a boundary triplet for A^* and $A_0 = A^* \upharpoonright \ker(\Gamma_0)$. The operator valued functions $\gamma(\cdot) : \rho(A_0) \to [\mathcal{H}, \mathfrak{H}]$ and $M(\cdot) : \rho(A_0) \to [\mathcal{H}]$ defined by

$$\gamma(z) := \bigl(\Gamma_0 \upharpoonright \mathfrak{N}_z\bigr)^{-1} \quad \text{and} \quad M(z) := \Gamma_1 \gamma(z), \quad z \in \rho(A_0), \tag{2.5}$$

$\mathfrak{N}_z := \ker(A^* - z)$, are called the γ-field and Weyl function, respectively, corresponding to Π.

Clearly, the Weyl function can equivalently be defined by

$$M(z)\Gamma_0 f_z = \Gamma_1 f_z, \qquad f_z \in \mathfrak{N}_z, \quad z \in \rho(A_0).$$

The γ-field $\gamma(\cdot)$ and the Weyl function $M(\cdot)$ in (2.5) are well defined. Moreover, both $\gamma(\cdot)$ and $M(\cdot)$ are holomorphic on $\rho(A_0)$ and $M(\cdot)$ is a $[\mathcal{H}]$-valued Nevanlinna function, i.e. $M(\cdot)$ is a $[\mathcal{H}]$-valued holomorphic function on $\mathbb{C}\backslash\mathbb{R}$ satisfying

$$M(z) = M(\overline{z})^* \quad \text{and} \quad \frac{\mathrm{Im}\,(M(z))}{\mathrm{Im}\,(z)} \ge 0, \qquad z \in \mathbb{C}_+ \cup \mathbb{C}_-.$$

We note that $0 \in \rho(\mathrm{Im}\,(M(z)))$ for all $z \in \mathbb{C}\backslash\mathbb{R}$. Moreover, the following counterpart of the classical result from the Sturm-Liouville theory holds.

Proposition 2.8 ([8]). *Let A be a simple closed densely defined symmetric operator in $\mathfrak{H}$ and let $\Pi = \{\mathcal{H}, \Gamma_0, \Gamma_1\}$ be a boundary triplet for A^* and $M(\cdot)$ the corresponding Weyl function. Assume that $\Theta \in \widetilde{\mathcal{C}}(\mathcal{H})$ and $z \in \rho(A_0)$. Then the following holds:*

(i) $z \in \rho(A_\Theta)$ *if and only if* $0 \in \rho(\Theta - M(z))$.

(ii) $z \in \sigma_\tau(A_\Theta)$ *if and only if* $0 \in \sigma_\tau(\Theta - M(z))$, $\tau \in \{p, c\}$.

Moreover, $\dim(\ker(A_\Theta - z)) = \dim(\ker(\Theta - M(z)))$.

2.4 Krein-type formula for resolvents and comparability

Let $\Pi = \{\mathcal{H}, \Gamma_0, \Gamma_1\}$ be a boundary triplet for A^*, $M(\cdot)$ and $\gamma(\cdot)$ the corresponding Weyl function and γ-field, respectively. For any proper (not necessarily self-adjoint) extension $\widetilde{A}_\Theta \in \mathrm{Ext}_A$ with non-empty resolvent set $\rho(\widetilde{A}_\Theta)$ the following Krein-type formula holds (cf. [7, 8, 11, 12])

$$(A_\Theta - z)^{-1} - (A_0 - z)^{-1} = \gamma(z)(\Theta - M(z))^{-1}\gamma(\overline{z})^*, \quad z \in \rho(A_0) \cap \rho(A_\Theta). \tag{2.6}$$

The proof of the following result relies on formula (2.6).

Proposition 2.9 ([8, Theorem 2]). *Let $\Pi = \{\mathcal{H}, \Gamma_0, \Gamma_1\}$ be a boundary triplet for A^*, $\Theta', \Theta \in \widetilde{\mathcal{C}}(\mathcal{H})$ and $\rho(A_{\Theta'}) \cap \rho(A_\Theta) \neq \emptyset$. If $\rho(\Theta') \cap \rho(\Theta) \neq \emptyset$, then for any Neumann-Schatten ideal $\mathfrak{S}_p$, $p \in (0, \infty]$, the following holds:*

(i) *The inclusion*

$$(A_{\Theta'} - z)^{-1} - (A_\Theta - z)^{-1} \in \mathfrak{S}_p(\mathfrak{H}), \quad z \in \rho(A_{\Theta'}) \cap \rho(A_\Theta), \tag{2.7}$$

is equivalent to the inclusion

$$\big(\Theta' - \zeta\big)^{-1} - \big(\Theta - \zeta\big)^{-1} \in \mathfrak{S}_p(\mathcal{H}), \quad \zeta \in \rho(\Theta') \cap \rho(\Theta). \tag{2.8}$$

In particular, $(A_\Theta - z)^{-1} - (A_0 - z)^{-1} \in \mathfrak{S}_p(\mathfrak{H})$ for $z \in \rho(A_\Theta) \cap \rho(A_0)$ if and only if $\big(\Theta - \zeta\big)^{-1} \in \mathfrak{S}_p(\mathcal{H})$ for $\zeta \in \rho(\Theta)$.

(ii) *If $B', B \in \mathcal{C}(\mathcal{H})$ and $\mathrm{dom}\,(B') = \mathrm{dom}\,(B)$, then the implication*

$$\overline{B' - B} \in \mathfrak{S}_p(\mathcal{H}) \Longrightarrow (A_{B'} - z)^{-1} - (A_B - z)^{-1} \in \mathfrak{S}_p(\mathfrak{H}), \quad z \in \rho(A_{\Theta'}) \cap \rho(A_\Theta), \tag{2.9}$$

holds. Moreover, if $B', B \in [\mathcal{H}]$, then (2.7) *is equivalent to $B' - B \in \mathfrak{S}_p(\mathcal{H})$.*

2.5 Perturbation determinants

Following [18] let us briefly recall some basic facts on infinite determinants.

Definition 2.10. *Let T be a trace class operator, i.e. $T \in \mathfrak{S}_1(\mathcal{H})$, and let $\{\lambda_j(T)\}_{j=1}^\infty$ be its eigenvalues counted with respect to their algebraic multiplicities. Then the determinant $\det(I + T)$ is defined by $\det(I + T) := \Pi_{j=1}^\infty\big(1 + \lambda_j(T)\big)$.*

The determinants have the following interesting properties.

Proposition 2.11 ([18, Section 4.1]). *Let $T_1 \in [\mathcal{H}_1, \mathcal{H}_2]$ and $T_2 \in [\mathcal{H}_2, \mathcal{H}_1]$.*

(i) *If $T_1T_2 \in \mathfrak{S}_1(\mathcal{H}_2)$ and $T_2T_1 \in \mathfrak{S}_1(\mathcal{H}_1)$, then*

$$\det{}_{\mathcal{H}_2}(I + T_1T_2) = \det{}_{\mathcal{H}_1}(I + T_2T_1). \tag{2.10}$$

(ii) *If $\mathcal{H} := \mathcal{H}_1 = \mathcal{H}_2$ and $T_1, T_2 \in \mathfrak{S}_1(\mathcal{H})$, then*

$$\det[(I + T_1)(I + T_2)] = \det(I + T_1) \cdot \det(I + T_2). \tag{2.11}$$

(iii) *If $T \in \mathfrak{S}_1(\mathcal{H})$, then $\det(I + T^*) = \overline{\det(I + T)}$.*

Following [30, 31] with the pair $\{\widetilde{A}', \widetilde{A}\}$ satisfying

$$(\widetilde{A}' - z)^{-1} - (\widetilde{A} - z)^{-1} \in \mathfrak{S}_1(\mathfrak{H}), \qquad z \in \rho(\widetilde{A}') \cap \rho(\widetilde{A}), \tag{2.12}$$

one associates a perturbation determinant as follows:

Definition 2.12 ([30, 31]). Let A be a densely defined closed symmetric operator in $\mathfrak{H}$, let $\Pi = \{\mathcal{H}, \Gamma_0, \Gamma_1\}$ be a boundary triplet for A^* and $M(\cdot)$ the corresponding Weyl function. We say that the ordered pair $\{\widetilde{A}', \widetilde{A}\}$ of proper extensions of A belongs to the class $\mathfrak{D}^\Pi$ if $\widetilde{A}'$ and $\widetilde{A}$ admit representations (2.3) with boundary operators B' and B, respectively, and the following conditions

(i) $\rho(A_0) \cap \rho(A_B) \neq \emptyset$,

(ii) $\mathrm{dom}\,(B') = \mathrm{dom}\,(B)$,

(iii) $(B' - B)(B - M(z))^{-1} \in \mathfrak{S}_1(\mathfrak{H})$ for $z \in \rho(A_0) \cap \rho(A_B)$.

are satisfied. If $\{\widetilde{A}', \widetilde{A}\} \in \mathfrak{D}^\Pi$, then the scalar-valued function

$$\Delta^\Pi_{\widetilde{A}'/\widetilde{A}}(z) := \det\Big(I_\mathcal{H} + \big(B' - B\big)\big(B - M(z)\big)^{-1}\Big), \quad z \in \rho(A_0) \cap \rho(A_B), \tag{2.13}$$

is called the perturbation determinant of the pair $\{\widetilde{A}', \widetilde{A}\}$ with respect to Π.

It is shown in [30, 31] that definition (2.13) allows one to express $\Delta^\Pi_{\widetilde{A}'/\widetilde{A}}(\cdot)$ as *a ratio of two ordinary determinants involving only boundary operators and the corresponding Weyl function.* For instance, if A has finite deficiency indices $n_\pm(A) = n < \infty$, then (2.13) becomes

$$\Delta^\Pi_{\widetilde{A}'/\widetilde{A}}(z) := \frac{\det(B' - M(z))}{\det(B - M(z))}, \qquad z \in \rho(\widetilde{A}') \cap \rho(\widetilde{A}). \tag{2.14}$$

By Proposition 2.9 (ii), condition (iii) implies (2.12) but not vice versa. However, if the boundary operators B' and B are bounded, then condition (ii) is obviously satisfied and, by Proposition 2.9(ii), condition (iii) is equivalent to $B' - B \in \mathfrak{S}_1(\mathcal{H})$. These circumstances motivate to introduce Definition 2.5 of a regular boundary triplet. Emphasize that by Theorem 2.6, *there always exists a regular boundary triplet* Π for a pair $\{\widetilde{A}', \widetilde{A}\}$ whenever $\rho(\widetilde{A}) \cap \rho(\widetilde{A}_1) \cap \mathbb{C}_\pm \neq \emptyset$.

The perturbation determinant $\Delta^\Pi_{\widetilde{A}',\widetilde{A}}(\cdot)$ posses the same properties as the classical perturbation determinant $\Delta_{H',H}(\cdot)$ (see [18, Sect. 4.3], [44, Sect. 8.1]). Let us summarize those properties of the perturbation determinant from [30, 31] which will be important in the following.

Proposition 2.13. *Let A be a densely defined closed symmetric operator and let $\widetilde{A}', \widetilde{A} \in \mathrm{Ext}\,_A$. Assume also that $\{\widetilde{A}', \widetilde{A}\} \in \mathfrak{D}^\Pi$ for a boundary triplet $\Pi = \{\mathcal{H}, \Gamma_0, \Gamma_1\}$ for A^*. Then the following holds:*

(i) *The perturbation determinant $\Delta^\Pi_{\widetilde{A}'/\widetilde{A}}(\cdot)$ admits a holomorphic continuation from $\rho(\widetilde{A}) \cap \rho(A_0)$ to the domain $\rho(\widetilde{A})$.*

(ii) *If $\Pi' = \{\mathcal{H}', \Gamma'_0, \Gamma'_1\}$ is another boundary triplet for A^* such that $\{\widetilde{A}', \widetilde{A}\} \in \mathfrak{D}^{\Pi'}$, then there is $c \in \mathbb{C}$ such that for any $z \in \rho(\widetilde{A})$ the following identity holds*

$$\Delta^\Pi_{\widetilde{A}'/\widetilde{A}}(z) = c\,\Delta^{\Pi'}_{\widetilde{A}'/\widetilde{A}}(z). \tag{2.15}$$

(iii) *The following identity holds*

$$\frac{1}{\Delta^{\Pi}_{\widetilde{A}',\widetilde{A}}(z)}\frac{d}{dz}\Delta^{\Pi}_{\widetilde{A}',\widetilde{A}}(z) = \mathrm{tr}\,\left((\widetilde{A}-z)^{-1} - (\widetilde{A}'-z)^{-1}\right), \quad z \in \rho(\widetilde{A}') \cap \rho(\widetilde{A}). \tag{2.16}$$

(iv) *Let* $\{\widetilde{A}', \widetilde{A}\} \in \mathfrak{D}^{\Pi}$ *and let* z_0 *be either a regular point or a normal eigenvalue of the operators* $\widetilde{A}'$ *and* $\widetilde{A}$ *of algebraic multiplicities* $m_{z_0}(\widetilde{A}')$ *and* $m_{z_0}(\widetilde{A})$*. Then* $\mathrm{ord}\,\left(\Delta^{\Pi}_{\widetilde{A}'/\widetilde{A}}(z_0)\right) = m_{z_0}(\widetilde{A}') - m_{z_0}(\widetilde{A})$*. In particular,* $\mathrm{ord}\,\left(\Delta^{\Pi}_{\widetilde{A}'/\widetilde{A}}(z_0)\right) = m_{z_0}(\widetilde{A}')$ *for any* $z_0 \in \rho(A_0) \cap \rho(\widetilde{A})$.

(v) *Let* $\widetilde{A}''$ *be a proper extension of* A *such that* $\rho(\widetilde{A}'') \cap \rho(\widetilde{A}') \cap \rho(\widetilde{A}) \neq \emptyset$*. If* $\{\widetilde{A}'', \widetilde{A}'\} \in \mathfrak{D}^{\Pi}$*, then* $\{\widetilde{A}'', \widetilde{A}\} \in \mathfrak{D}^{\Pi}$ *and the chain rule*

$$\Delta^{\Pi}_{\widetilde{A}''/\widetilde{A}'}(z)\Delta^{\Pi}_{\widetilde{A}'/\widetilde{A}}(z) = \Delta^{\Pi}_{\widetilde{A}''/\widetilde{A}}(z), \quad z \in \rho(\widetilde{A}') \cap \rho(\widetilde{A}), \tag{2.17}$$

holds.

(vi) *If* $\{\widetilde{A}'^*, \widetilde{A}^*\} \in \mathfrak{D}^{\Pi}$*, then* $\Delta^{\Pi}_{\widetilde{A}'^*,\widetilde{A}^*}(z) = \overline{\Delta^{\Pi}_{\widetilde{A}',\widetilde{A}}(\overline{z})}$*,* $z \in \rho(\widetilde{A}'^*)$.

For the proof the reader is referred to Theorem 4.2 and Proposition 5.1 of [31].

Note that a different approach to perturbation determinants for singular perturbations was proposed in [15, 17]. It is based on the use of positive-type operators and its applicability requires that one of the square root domains of H and H' contains the other instead of condition (ii) of Definition 2.12.

3 Trace formulas for singular perturbations

3.1 Pairs of self-adjoint resolvent comparable extensions

Krein [22] proved that for a pair $\{H' = H + V, H\}$ of self-adjoint operators with $V \in \mathfrak{S}_1(\mathfrak{H})$ there exists a unique *real-valued* function $\xi(\cdot) \in L^1(\mathbb{R})$ such that the following trace formula holds

$$\mathrm{tr}\,\left((H'-z)^{-1} - (H-z)^{-1}\right) = -\int_{\mathbb{R}} \frac{\xi(t)}{(t-z)^2}dt, \quad z \in \rho(H') \cap \rho(H). \tag{3.1}$$

Moreover, Krein proved the inversion formula (1.2) and

$$\int_{\mathbb{R}} |\xi(t)|dt \le \|V\|_{\mathfrak{S}_1} \quad \text{and} \quad \int_{\mathbb{R}} \xi(t)dt = \mathrm{tr}\,(V), \tag{3.2}$$

cf. [44, Theorem 8.2.1]. In addition, it was shown in [44, Theorem 8.2.1] that for $V \ge 0$ the SSF $\xi(\cdot)$ is non-negative for a.e. $t \in \mathbb{R}$. In particular, let V_1 and V_2 be self-adjoint trace class operators and let $H_j := H_0 + V_j$, $j = 1, 2$. Further, if $V_1 \le V_2$, then there are SSFs $\xi_j(\cdot)$ associated with the pairs $\{H_j, H_0\}$, $j = 1, 2$ such that $\xi_1(t) \le \xi_2(t)$ for a.e. $t \in \mathbb{R}$.

Formula (3.1) has been extended in [24] to a pair of self-adjoint operators $\{H', H\}$ which are only resolvent comparable, that is, $(H'-\zeta)^{-1}-(H-\zeta)^{-1} \in \mathfrak{S}_1(\mathfrak{H})$ for some $\zeta \in \rho(H')\cap\rho(H)$. In this case formula (3.1) remains valid while relations (3.2) are no longer true. Instead of $\xi(\cdot) \in L^1(\mathbb{R}; dt)$ one has only $\xi(\cdot) = \overline{\xi(\cdot)} \in L^1\big(\mathbb{R}, \frac{dt}{1+t^2}\big)$. In what follows any function $\xi(\cdot) = \overline{\xi(\cdot)} \in L^1\big(\mathbb{R}, \frac{dt}{1+t^2}\big)$ satisfying (3.1) *will be called a SSF for the ordered pair* $\{H', H\}$. Clearly, $\xi(\cdot)$ is not unique since alongside with $\xi(\cdot)$ any function $\xi(\cdot)+c,\ c\in\mathbb{R}$, satisfies (3.1) too. First we show that the converse is also true.

Lemma 3.1. *Let H' and H be self-adjoint operators which are resolvent comparable and let $\xi(\cdot) \in L^1(\mathbb{R}; \frac{dt}{1+t^2})$ be a SSF of $\{H', H\}$. The real valued function $\widetilde{\xi}(\cdot) \in L^1(\mathbb{R}; \frac{dt}{1+t^2})$ is also a SSF of the pair $\{H', H\}$ if and only if $\widetilde{\xi}(t)-\xi(t) = c$ for a.e. $t\in\mathbb{R}$ where c* is a real constant*.*

Proof. Let $\eta(t) := \widetilde{\xi}(t)-\xi(t)$, $t\in\mathbb{R}$. Then

$$\int_{-\infty}^{\infty} \frac{\eta(t)dt}{(t-z)^2} = 0, \qquad z \in \mathbb{C}_+ \cup \mathbb{C}_-, \tag{3.3}$$

and $\eta(\cdot) \in L^1(\mathbb{R}; \frac{dt}{1+t^2})$. We set

$$\mathcal{P}_\eta(z,\overline{z}) := \frac{1}{\pi}\int \frac{y\ \eta(t)dt}{|t-z|^2} = \frac{1}{2i\pi}\int\left(\frac{1}{t-z}-\frac{1}{t-\overline{z}}\right)\eta(t)dt, \tag{3.4}$$

where $z = x+iy \in \mathbb{C}_\pm$. Differentiating $\mathcal{P}_\eta(z,\overline{z})$ with respect to z and $\overline{z}$ and taking (3.3) into account we get

$$\frac{\partial}{\partial z}\mathcal{P}_\eta(z,\overline{z}) = \frac{\partial}{\partial \overline{z}}\mathcal{P}_\eta(z,\overline{z}) = 0.$$

Thus, $\mathcal{P}_\eta(z,\overline{z})$ is holomorphic and anti-holomorphic in $\mathbb{C}_+\cup\mathbb{C}_-$. Hence $\mathcal{P}_\eta(z,\overline{z}) = a = \mathrm{const.}$, $z\in\mathbb{C}_+$. Applying the Fatou theorem to (3.4) we get $\eta(t) = \mathcal{P}(t+i0, t-i0) = a = \overline{a} = \mathrm{const.}$ for a.e. $t\in\mathbb{R}$. $\square$

In this section we reprove Krein's result for the case of singular perturbations and complement it in certain directions. To this end we need the following technical lemma.

Lemma 3.2. *Let $\Pi = \{\mathcal{H}, \Gamma_0, \Gamma_1\}$ be a boundary triplet for A^*, $M(\cdot)$ the corresponding Weyl function and let B be a m-accumulative (in particular, self-adjoint operator). Then the following statements are true:*

(i) *If $V_+ \in \mathfrak{S}_1(\mathcal{H})$ and $V_+ \geq 0$, then there exist a constant $c_+ > 0$ and a non-negative function $\xi_+(\cdot) \in L^1(\mathbb{R}; \frac{dt}{1+t^2})$ such that the following representation*

$$\det\big(I + V_+(B-M(z))^{-1}\big) = c_+ \exp\left\{\frac{1}{\pi}\int_R\left(\frac{1}{t-z}-\frac{t}{1+t^2}\right)\xi_+(t)dt\right\}, \tag{3.5}$$

$z\in\mathbb{C}_\pm$*, holds.*

(ii) *If* $V = V^* \in \mathfrak{S}_1(\mathcal{H})$*, then there exist a constant* $c > 0$ *and a* real-valued function $\xi(\cdot) \in L^1(\mathbb{R}; \frac{dt}{1+t^2})$ *such that the representation*

$$\det\left(I + V(B - M(z))^{-1}\right) = c\,\exp\left\{\frac{1}{\pi}\int_R \left(\frac{1}{t-z} - \frac{t}{1+t^2}\right)\xi(t)dt\right\}, \tag{3.6}$$

$z \in \mathbb{C}_\pm$*, holds. Moreover, there exists* $n \in \mathbb{Z}$ *such that for a.e.* $t \in \mathbb{R}$

$$\xi(t) = \lim_{y\to+0} \operatorname{Im}\left(\log\left(\det\left(I + V(B - M(t+iy))^{-1}\right)\right)\right) + 2\pi n. \tag{3.7}$$

Proof. (i) We introduce the operator-valued Nevanlinna function

$$\Omega_+(z) := I + \sqrt{V_+}(B - M(z))^{-1}\sqrt{V_+}, \quad z \in \mathbb{C}_+.$$

Since $\Omega_+(z)$ is m-dissipative for $z \in \mathbb{C}_+$ and $0 \in \rho(\Omega_+(z))$, $z \in \mathbb{C}_+$, the operator-valued function $\log(\Omega_+(z))$ is well-defined by (A.2) for $z \in \mathbb{C}_+$. Since $\log(\Omega_+(z)) \in \mathfrak{S}_1(\mathcal{H})$ Theorem 2.8 of [16] guarantees the existence of a measurable function $\Xi_+(\cdot) : \mathbb{R} \longrightarrow \mathfrak{S}_1(\mathcal{H})$ such that $\Xi(t) \geq 0$ for a.e. $t \in \mathbb{R}$ and $\operatorname{tr}(\Xi(\cdot)) \in L^1(\mathbb{R}; \frac{dt}{1+t^2})$. Moreover, the following representation holds

$$\log(\Omega_+(z)) = \Omega_+ + \frac{1}{\pi}\int_{\mathbb{R}} \left(\frac{1}{t-z} - \frac{t}{1+t^2}\right)\Xi_+(t)dt, \quad z \in \mathbb{C}_+,$$

where the integral is taken in the weak sense and $\Omega_+ = \Omega_+^* \in \mathfrak{S}_1(\mathcal{H})$ and $\log(\Omega(z))$ is defined in accordance with A. Setting $\xi_+(t) := \operatorname{tr}(\Xi_+(t))$, $t \in \mathbb{R}$, we define a non-negative function $\xi_+(\cdot)$ satisfying $\xi_+(\cdot) \in L^1(\mathbb{R}; \frac{dt}{1+t^2})$ and such that

$$\operatorname{tr}(\log(\Omega_+(z))) = \operatorname{tr}(\Omega_+) + \frac{1}{\pi}\int_{\mathbb{R}} \left(\frac{1}{t-z} - \frac{t}{1+t^2}\right)\xi_+(t)dt, \quad z \in \mathbb{C}_+.$$

Taking into account (A.3) we verify (3.5) for $z \in \mathbb{C}_+$ with $c_+ := \exp\{\operatorname{tr}(\Omega_+)\} > 0$. By taking the adjoint and using the property $M(z)^* = M(\overline{z})$, $z \in \mathbb{C}_\pm$, we immediately verify (3.5) for $z \in \mathbb{C}_-$.

(ii) Using the decomposition $V = V_+ - V_-$, $V_\pm \geq 0$, we set $B_- := B - V_-$. It follows from the identity

$$\left(I + V(B - M(z))^{-1}\right)\left(I + V_-(B_- - M(z))^{-1}\right) = I + V_+(B_- - M(z))^{-1}$$

that

$$\det\left(I + V(B - M(z))^{-1}\right) = \frac{\det\left(I + V_+(B_- - M(z))^{-1}\right)}{\det\left(I + V_-(B_- - M(z))^{-1}\right)}, \qquad z \in \mathbb{C}_\pm. \tag{3.8}$$

Combining (3.8) with the representation (3.5) we arrive at (3.6).

Let $z_0 \in \mathbb{C}_+$ such that $\det\left(I + V(B - M(z_0))^{-1}\right)$ does not belong to the negative imaginary axis. In this case we have

$$\log\left(\det\left(I + V(B - M(z_0))^{-1}\right)\right) = \frac{1}{\pi}\int_{\mathbb{R}} \left(\frac{1}{t-z_0} - \frac{t}{1+t^2}\right)\xi(t)dt + 2\pi n i$$

for some $n \in \mathbb{Z}$. Since both sides are analytic we get by analytic continuation that the equality holds for any $z \in \mathbb{C}_+$. Hence we find

$$\mathrm{Im}\,\left(\log\left(\det\left(I + V(B - M(z))^{-1}\right)\right)\right) = \frac{1}{\pi}\int_{\mathbb{R}} \frac{y}{(t-x)^2 + y^2}\,\xi(t)dt + 2\pi n$$

where $z = x + iy$, $y > 0$. Applying Fatou's theorem we obtain (3.7). $\square$

Remark 3.3. Notice that (3.7) shows that the function $\xi(\cdot)$ is determined by the representation (3.6) uniquely up to modulo 2π.

Lemma 3.2 leads to the following representation theorem.

Theorem 3.4. *Let $\widetilde{A}', \widetilde{A} \in \mathrm{Ext}_A$ be resolvent comparable self-adjoint extensions of A. Then the following holds:*

(i) *There exists a boundary triplet $\Pi = \{\mathcal{H}, \Gamma_0, \Gamma_1\}$ for A^* which is regular for the pair $\{\widetilde{A}', \widetilde{A}\}$ and satisfies $\{\widetilde{A}', \widetilde{A}\} \in \mathfrak{D}^{\Pi}$.*

(ii) *If $\{\widetilde{A}', \widetilde{A}\} \in \mathfrak{D}^{\Pi}$, then there exists a* real-valued function $\xi(\cdot) \in L^1\big(\mathbb{R}; \frac{dt}{1+t^2}\big)$ *and a constant $c > 0$ such that the following representation holds*

$$\Delta^{\Pi}_{\widetilde{A}'/\widetilde{A}}(z) = c\,\exp\left\{\frac{1}{\pi}\int_{\mathbb{R}}\left(\frac{1}{t-z} - \frac{t}{1+t^2}\right)\xi(t)dt\right\}, \quad z \in \mathbb{C}_\pm. \tag{3.9}$$

Moreover, there exists $n \in \mathbb{Z}$ such that $\xi(\cdot)$ is given by

$$\xi(t) = \lim_{\varepsilon\to+0}\mathrm{Im}\left(\log(\Delta^{\Pi}_{\widetilde{A}'/\widetilde{A}}(t+i\varepsilon))\right) + 2n\pi \quad \textit{for a.e.} \quad t \in \mathbb{R}. \tag{3.10}$$

(iii) *The function $\xi(\cdot)$ given by* (3.10) *is a SSF for the pair $\{\widetilde{A}', \widetilde{A}\}$, that is, the following trace formula holds*

$$\mathrm{tr}\,\left((\widetilde{A}' - z)^{-1} - (\widetilde{A} - z)^{-1}\right) = -\frac{1}{\pi}\int_{-\infty}^{\infty}\frac{\xi(t)}{(t-z)^2}dt, \qquad z \in \mathbb{C}_\pm. \tag{3.11}$$

Any other real-valued function $\widetilde{\xi}(\cdot) \in L^2(\mathbb{R}; \frac{1}{1+t^2})$ is a SSF of the pair $\{\widetilde{A}', \widetilde{A}\}$ if and only if it differs from $\xi(\cdot)$ by a real constant.

Proof. (i) This statement follows immediately from Theorem 2.6.

(ii) At first, let us assume that $\widetilde{\Pi}$ is regular for $\{\widetilde{A}', \widetilde{A}\}$. Then $\widetilde{A}' = A_{B'}$ and $\widetilde{A} = A_B$, where the boundary operators B' and B are bounded and self-adjoint. By Lemma 3.2(ii), there exist a real constant $\widetilde{c} > 0$ and a real-valued function $\widetilde{\xi}(\cdot) \in L^1(\mathbb{R}; \frac{dt}{1+t^2})$ such that the following representation holds

$$\begin{aligned}\Delta^{\widetilde{\Pi}}_{\widetilde{A}'/\widetilde{A}}(z) &= \det\left(I + (B'-B)(B - M(z))^{-1}\right)\\ &= \widetilde{c}\,\exp\left\{\frac{1}{\pi}\int_R\left(\frac{1}{t-z} - \frac{t}{1+t^2}\right)\widetilde{\xi}(t)dt\right\},\end{aligned} \tag{3.12}$$

$z \in \mathbb{C}_\pm$. This proves (3.9) for a regular boundary triplet $\widetilde{\Pi}$. If Π is not regular, then by Proposition 2.13 (ii) there is a constant $a \in \mathbb{C}$ such that $\Delta^{\Pi}_{\widetilde{A}',\widetilde{A}}(z) = a\,\Delta^{\widetilde{\Pi}}_{\widetilde{A}',\widetilde{A}}(z)$, $z \in \mathbb{C}_\pm$. In general, is not clear whether a is real. However

$$\overline{\Delta^{\Pi}_{\widetilde{A}',\widetilde{A}}(z)} = \overline{a}\,\overline{\Delta^{\widetilde{\Pi}}_{\widetilde{A}',\widetilde{A}}(z)}, \quad z \in \mathbb{C}_\pm.$$

By Proposition 2.13 (vi) we get

$$\Delta^{\Pi}_{\widetilde{A}',\widetilde{A}}(\overline{z}) = \overline{a}\,\Delta^{\widetilde{\Pi}}_{\widetilde{A}',\widetilde{A}}(\overline{z}), \quad z \in \mathbb{C}_\pm,$$

which yields $a = \overline{a}$. Hence

$$\Delta^{\Pi}_{\widetilde{A}',\widetilde{A}}(z) = \widetilde{c}\,a\exp\left\{\frac{1}{\pi}\int_R \left(\frac{1}{t-z} - \frac{t}{1+t^2}\right)\widetilde{\xi}(t)dt\right\}, \quad z \in \mathbb{C}_\pm,$$

where $c := \widetilde{c}\,a \in \mathbb{R}$. If $c > 0$, then we set $\xi(t) := \widetilde{\xi}(t)$, $t \in \mathbb{R}$, which proves (3.9). If $c < 0$, then we replace c by $|c|$ and set $\xi(t) := \widetilde{\xi}(t) + \pi$, $t \in \mathbb{R}$, which verifies also (3.9). The representation (3.10) follows from (3.7).

(iii) Combining (2.16) with (3.9) we immediately prove the trace formula (3.11). By Lemma 3.1, any real-valued function $\widetilde{\xi}(\cdot)$ satisfying (3.11) differs from $\xi(\cdot)$ by a real constant. $\square$

Remark 3.5. Theorem 3.4 restores the celebrated Krein result of [24] for singular perturbations in a quite different way. Our contribution concerns formula (3.10) which complements Krein's inversion formula (1.2). Namely, together with (2.13) it expresses the SSF $\xi(\cdot)$ in terms of the basic objects of the extension theory: Weyl function $M(\cdot)$ and boundary operators B, B'.

The most simple expression one obtains if $n_\pm(A) = n < \infty$, and hence $B, B' \in \mathbb{C}^{n\times n}$. Indeed, combining (3.10) with (2.14) this yields that for a.e. $t \in \mathbb{R}$

$$\xi(t) = \lim_{\varepsilon\to+0} \operatorname{Im}\left[\log(\det(B - M(t+i\varepsilon))) - \log(\det(B' - M(t+i\varepsilon)))\right]. \tag{3.13}$$

This formula makes it possible to compute explicitly the SSF for a pair of boundary value problems for certain classes of ordinary differential operators as well as for Dirac type systems, etc. (see [30, 31]).

3.2 Pairs of accumulative extensions

By analogy with the self-adjoint case we introduce the following definition.

Definition 3.6. Let $\{H', H\}$ be a pair of m-accumulative operators in $\mathfrak{H}$ which are resolvent comparable. A complex-valued function $\omega(\cdot) \in L^1(\mathbb{R}; \frac{dt}{1+t^2})$ satisfying Krein's trace identity

$$\operatorname{tr}\left((H'-z)^{-1} - (H-z)^{-1}\right) = -\frac{1}{\pi}\int_{\mathbb{R}} \frac{\omega(t)}{(t-\zeta)^2}\,dt, \quad z \in \mathbb{C}_+, \tag{3.14}$$

is referred to as a SSF of the ordered pair $\{H', H\}$. By $\mathcal{S}\{H', H\}$ the set of all SSFs of $\{H', H\}$ is denoted. The subset of all $\omega(\cdot) \in \mathcal{S}\{H', H\}$ satisfying the stronger condition $\omega(\cdot) \in L^1(\mathbb{R}; dt)$ is denoted by $\mathcal{S}_1\{H', H\}$. The identity (3.14) is called the trace formula of the pair $\{H', H\}$.

A SSF is in not unique. To investigate the character of this non-uniqueness let us recall a concept of weighted Hardy space $H^p(\mathbb{C}_\pm; h\,dt)$, $p \in [1,\infty)$, with non-negative $h \in L^\infty(\mathbb{R}; dt)$, satisfying the condition

$$\int_{\mathbb{R}} \frac{\log(h(t))}{1+t^2} dt > -\infty. \tag{3.15}$$

The weighted Hardy space $H^p(\mathbb{C}_\pm; h\,dt)$ is the space of holomorphic functions in $\mathbb{C}_\pm$ satisfying

$$\|f\|^p_{H^p(\mathbb{C};h\,dt)} = \sup_{\pm y>0} \int_{\mathbb{R}} |f(t+iy)|^p\, h(t)\, dt < \infty. \tag{3.16}$$

Due to condition (3.15) there exists an outer function $w(\cdot) \in H^\infty(\mathbb{C}_\pm; dt)$ such that $|w(t)| = h(t)$ for a.e. $t \in \mathbb{R}$ (see [14, Theorem II.4.4]). Clearly, $f(\cdot) \in H^p(\mathbb{C}_\pm; h\,dt)$ if and only if $f\, w^{1/p} \in H^p(\mathbb{C}_\pm; dt)$. Moreover, by (3.16) the weighted Hardy space $H^p(\mathbb{C}_\pm; h\,dt)$ is isometrically identified with a subspace of the weighted L^p-space $L^2(\mathbb{R}; h\,dt)$ denoted by $H^p_\pm(\mathbb{R}; h\,dt)$.

It is well known that the weighted Hardy space $H^p(\mathbb{C}_\pm; \frac{1}{1+t^2}\,dt)$, $p \in [1,\infty)$, is isomorphic to the Hardy space $H^p(\mathbb{D})$, for definition see [14, Section II.1].

First we slightly clarify the Riesz Brothers' theorem [14, Theorem 2.3.8].

Lemma 3.7. *Let $\mu(\cdot)$ be a complex-valued Borel measure on $\mathbb{R}$ satisfying*

$$\int_{\mathbb{R}} \frac{|d\mu(t)|}{1+t^2} < \infty.$$

Then $\mu(\cdot)$ is absolutely continuous, $d\mu(t) = f(t)dt$, if and only if

$$\int_{\mathbb{R}} \frac{d\mu(t)}{(t-z)^2} = 0, \quad z \in \mathbb{C}_+. \tag{3.17}$$

Moreover, (3.17) *implies $d\mu(t) = f(t)dt$ with a density $f \in H^1_-\left(\mathbb{R}; \frac{dt}{1+t^2}\right)$.*

Proof. Letting $\nu(\delta) := \int_\delta \frac{1}{1+t^2}\, d\mu(t)$, $\delta \in \mathcal{B}(\mathbb{R})$, we introduce the R-function

$$g(z) := \frac{1}{\pi}\int_{\mathbb{R}} \left(\frac{1}{t-z} - \frac{t}{1+t^2}\right) d\mu(t) = \frac{1}{\pi}\int_{\mathbb{R}} \frac{1+tz}{t-z} d\nu(t), \quad z \in \mathbb{C}_+. \tag{3.18}$$

By (3.17), $g'(z) = 0$, $z \in \mathbb{C}_+$, so there exists a constant $c \in \mathbb{C}$ such that

$$g(z) = \frac{1}{\pi}\int_{\mathbb{R}} \left(\frac{1}{t-z} - \frac{t}{1+t^2}\right) d\mu(t) = \frac{1}{\pi}\int_{\mathbb{R}} \frac{1+tz}{t-z} d\nu(t) = c, \quad z \in \mathbb{C}_+. \tag{3.19}$$

Let $\xi : \mathbb{C}_+ \to \mathbb{D}$, $\xi = \frac{i-z}{i+z}$, be the conformal mapping of $\mathbb{C}_+$ onto the disc $\mathbb{D}$,

$$t = \tan(\theta/2),\ \theta \in [-\pi,\pi), \quad \text{and} \quad \varrho(\theta) := \nu(\tan(\theta/2)). \tag{3.20}$$

Noting that $\int_{-\pi}^{\pi} |d\varrho(\theta)| < \infty$ one transforms (3.19) into

$$c = \frac{i}{\pi}\int_{-\pi}^{\pi} \frac{1+\xi e^{-i\theta}}{1-\xi e^{-i\theta}}\, d\varrho(\theta) = -\frac{i}{\pi}\int_{-\pi}^{\pi} d\varrho(\theta) + \frac{2i}{\pi}\int_{-\pi}^{\pi} \frac{1}{1-\xi e^{-i\theta}}\, d\varrho(\theta), \tag{3.21}$$

$\xi \in \mathbb{D}$, which is equivalent to

$$c + \frac{i}{\pi}\int_{-\pi}^{\pi} d\varrho(\theta) \;\; = \;\; \frac{2i}{\pi}\int_{-\pi}^{\pi} \frac{1}{1-\xi e^{-i\theta}}\, d\varrho(\theta) = \frac{2i}{\pi}\sum_{n=0}^{\infty}\xi^n \int_{-\pi}^{\pi} e^{-in\theta}\, d\varrho(\theta).$$

The latter implies

$$\int_{-\pi}^{\pi} e^{in\theta}\, d\overline{\varrho(\theta)} = 0, \qquad n \in \mathbb{N}. \tag{3.22}$$

By the Riesz Brothers' theorem, cf. [21, Section II A], the measure $\overline{\varrho(\cdot)}$ is absolutely continuous, that is, $\overline{d\varrho(\theta)} = F(\theta)d\theta$ with a density $F(\cdot) \in H^1(\mathbb{D})$. Hence $d\varrho(\theta) = \overline{F(\theta)}d\theta$. Since $d\varrho(2\arctan(t)) = d\nu(t) = \frac{d\mu(t)}{1+t^2}$, the measure $\mu(\cdot)$ is absolutely continuous too. Moreover, one derives with account of (3.20)

$$\frac{1}{1+t^2}\frac{d\mu(t)}{dt} = \frac{d\nu(t)}{dt} = \frac{d\varrho(2\arctan(t))}{2d\arctan(t)}\frac{2d\arctan(t)}{dt} = \frac{1}{1+t^2}f(t)$$

where $f(\cdot) := 2\overline{F(2\arctan(\cdot))}$. Hence $d\mu(t) = f(t)dt$. Let $h(z) = F(\frac{i-z}{i+z})$, $z \in \mathbb{C}_+$. Since $\overline{f(t)} = 2h(t)$, $t \in \mathbb{R}$, the equivalence $F \in H^1(\mathbb{D}) \iff h \in H^1\left(\mathbb{C}_+; \frac{dt}{1+t^2}\right)$ yields $f \in H^1_-(\mathbb{R}; \frac{dt}{1+t^2})$. This proves the first part.

Conversely, if $d\mu(t) = f(t)dt$ with $f(\cdot) \in H^1_-(\mathbb{R}; \frac{dt}{1+t2})$, then

$$\begin{aligned} g(z) &= \frac{1}{\pi}\int_{\mathbb{R}}\left(\frac{1}{t-z} - \frac{t}{1+t^2}\right)f(t)dt = \frac{1}{\pi}\int_{\mathbb{R}}\frac{1+tz}{(t-z)(1+t^2)}f(t)dt \\ &= \frac{1}{\pi}\lim_{R\to\infty}\int_{\partial(\mathbb{D}_R\cap\mathbb{C}_-)}\frac{1+\zeta z}{(\zeta-z)(1+\zeta^2)}f(\zeta)d\zeta = if(-i), \qquad z \in \mathbb{C}_+, \end{aligned} \tag{3.23}$$

where $\mathbb{D}_R := \{z \in \mathbb{C} : |z| < R\}$, so $g(z) = if(-i) = const$, $z \in \mathbb{C}_+$. Hence, $g'(z) = 0$, which proves (3.17). $\square$

A counterpart of Lemma 3.1 for a pair of m-accumulative operators $\{H', H\}$ reads now as follows.

Proposition 3.8. *Let $\{H', H\}$ be a pair of resolvent comparable m-accumulative operators, $\omega(\cdot),\ \widetilde{\omega}(\cdot) \in L^1(\mathbb{R}; \frac{dt}{1+t^2})$ and $\omega(\cdot) \in \mathcal{S}\{H', H\}$. Then $\widetilde{\omega}(\cdot) \in \mathcal{S}\{H', H\}$ if and only if $\widetilde{\omega}(\cdot) - \omega(\cdot) \in H^1_-(\mathbb{R}; \frac{dt}{1+t^2})$.*

Moreover, if there exists a real valued $\xi(\cdot) \in \mathcal{S}\{H', H\}$*, then any other* real valued $\widetilde{\xi}(\cdot) \in \mathcal{S}\{H', H\}$ *differs from* $\xi(\cdot)$ by a real constant: $\widetilde{\xi}(\cdot) - \xi(\cdot) = c \in \mathbb{R}$*.*

Proof. (i) Let $\omega(\cdot) \in \mathcal{S}\{H', H\}$. It follows from (3.14) that $\widetilde{\omega}(\cdot) \in \mathcal{S}\{H', H\}$ if and only if $\eta(\cdot) := \widetilde{\omega}(\cdot) - \omega(\cdot) \in L^1(\mathbb{R}; \frac{dt}{1+t^2})$ and satisfies

$$\int_{\mathbb{R}} \frac{\eta(t)}{(t-z)^2}dt = 0, \quad z \in \mathbb{C}_+. \tag{3.24}$$

By Lemma 3.7, the latter is amount to say that $\eta(\cdot) \in H^1(\mathbb{C}_-; \frac{1}{1+t^2}\, dt)$.

(ii) If $\widetilde{\xi}(\cdot)$ and $\xi(\cdot)$ are two real-valued SSFs, then $\eta(\cdot) := \widetilde{\xi}(\cdot) - \xi(\cdot) \in H^1(\mathbb{C}_-; \frac{dt}{1+t^2})$. Hence $\eta(\cdot) = \overline{\eta(\cdot)} \in H^1(\mathbb{C}_+; \frac{1}{1+t^2}\, dt)$. Thus, $\eta(\cdot) \in H^1(\mathbb{C}_+; \frac{1}{1+t^2}\, dt) \cap H^1(\mathbb{C}_-; \frac{dt}{1+t^2})$, i.e. $\eta(\cdot) \equiv const$. The converse is obvious. □

Next we indicate certain sufficient conditions ensuring the existence of a *real valued* SSF $\xi(\cdot) \in L^1(\mathbb{R}; \frac{dt}{1+t^2})$ for the pair $\{H', H\}$.

Proposition 3.9. *Let $\{H', H\}$ be a pair of resolvent comparable m-accumulative operators and let $\omega(\cdot) \in \mathcal{S}\{H, H'\}$. Then there exists a* real valued *SSF provided that*

$$\int_{\mathbb{R}} |\omega_I(t)| \log(1 + |\omega_I(t)|) \frac{dt}{1+t^2} < \infty, \qquad \omega_I(\cdot) := \mathrm{Im}\,(\omega(\cdot)). \tag{3.25}$$

If in addition $\omega_I(t) \geq 0$ for a.e. $t \in \mathbb{R}$ and the pair $\{H', H\}$ admits a real-valued SSF $\xi(\cdot) \in L^1(\mathbb{R}; \frac{dt}{1+t^2})$, then (3.25) *is satisfied.*

Proof. Let $\widetilde{\omega}_I$ denote the function harmonic conjugate to ω_I and let H be the Hilbert transform. According to the Zygmund theorem, see [21, Section V.C.3],

$$\int_{\mathbb{R}} \frac{|\widetilde{\omega}_I(t)|}{1+t^2}\, dt = \int_{\mathbb{R}} |(H\omega_I)(t)| \frac{dt}{1+t^2} \leq C \int_{\mathbb{R}} |\omega_I(t)| \log(1+|\omega_I(t)|) \frac{dt}{1+t^2} < \infty, \tag{3.26}$$

i.e. $\widetilde{\omega}_I = H\omega_I \in L^1\left(\mathbb{R}; \frac{dt}{1+|t|^2}\right)$. Hence $F(\cdot) := \widetilde{\omega}_I(\cdot) + i\omega_I(\cdot) \in L^1\left(\mathbb{R}; \frac{dt}{1+t^2}\right)$. Moreover, $F(\cdot)$ admits a holomorphic continuation in $\mathbb{C}_-$ given by

$$F := \mathcal{P}(\widetilde{\omega}_I + i\omega_I) \in H^1_-\left(\mathbb{R}; \frac{dt}{1+t^2}\right), \tag{3.27}$$

where $\mathcal{P}$ denotes the Poisson integral transform,

$$(\mathcal{P}f)(z) = \frac{1}{\pi} \int_{\mathbb{R}} \frac{y}{(t-x)^2 + y^2} f(t)dt, \qquad z = x + iy, \qquad y < 0. \tag{3.28}$$

It follows from (3.27) that

$$\xi(t) := \omega(t) + F(t) = \omega_R(t) + i\omega_I(t) + \big(\widetilde{\omega}_I(t) - i\omega_I(t)\big) = \omega_R(t) + \widetilde{\omega}_I(t) = \overline{\xi(t)}.$$

By Proposition 3.8, inclusion (3.27) implies $\xi(\cdot)(= \overline{\xi(\cdot)}) \in \mathcal{S}\{H', H\}$, as claimed.

(ii) Now let $\omega(\cdot),\ \xi(\cdot) = \overline{\xi(\cdot)} \in \mathcal{S}\{H', H\}$, and let $\omega_I(\cdot) \geq 0$. Then, $\eta_I(\cdot) = -\omega_I(\cdot) \leq 0$ and, by Proposition 3.8, $\eta(\cdot) := \xi(\cdot) - \omega(\cdot) \in H^1_-(\mathbb{R}; \frac{dt}{1+t^2})$. Clearly, $g(\cdot) := \overline{\eta(\cdot)} \in H^1_+(\mathbb{R}; \frac{dt}{1+t^2})$ and $g_I(t) = \omega_I(t) \geq 0$ for a.e. $t \in \mathbb{R}$.

Next we put $F(z) := -ig(i\frac{1-z}{1+z})$, $z \in \mathbb{D}$, and note that $F(\cdot) \in H^1(\mathbb{D})$ since $g(\cdot) \in H^1_+(\mathbb{R}; \frac{dt}{1+t^2})$. Obviously, $\mathrm{Re}\,(F(e^{i\theta})) = g_I(\tan(\theta/2)) = w_I(\tan(\theta/2)) \geq 0$. Hence the inclusion $F(\cdot) \in H^1(\mathbb{D})$ ensures a representation (cf. [21, Section IV])

$$F(z) = \frac{1}{2\pi} \int_{-\pi}^{\pi} \frac{e^{i\theta} + z}{e^{i\theta} - z} \omega_I(\tan(\theta/2)) d\theta + i\mathrm{Im}\,(F(0), \qquad z \in \mathbb{D}.$$

Since $\omega_I(\cdot) \geq 0$, the Riesz theorem (cf. [21, Section V C 4]) applies, so

$$\int_{-\pi}^{\pi} \omega_I(\tan(\theta/2)) \log(1 + \omega_I(\tan(\theta/2)))\, d\theta < \infty.$$

The latter is equivalent to (3.25). $\square$

Next we clarify and specify Proposition 3.9 assuming that a SSF $\omega(\cdot) \in L^2\left(\mathbb{R}; \frac{dt}{(1+t^2)^{\alpha/2}}\right)$ with some $\alpha \in (0, 2)$.

Proposition 3.10. *Let $\{H', H\}$ be a pair of resolvent comparable m-accumulative operators and let $\omega(\cdot)$ be a SSF of $\{H', H\}$. Assume also that $\omega(\cdot) \in L^2(\mathbb{R}; \frac{dt}{(1+t^2)^{\alpha/2}})$ with some $\alpha \in (0, 2)$. Then:*

(i) *There exists a* real-valued *SSF* $\xi(\cdot) \in \mathcal{S}\{H', H\}$.

(ii) *If* $\alpha \in (0, 1)$*, then the SSF* $\xi(\cdot) = \overline{\xi(\cdot)} \in L^2\left(\mathbb{R}; \frac{dt}{(1+t^2)^{\alpha/2}}\right)$.

Proof. (i) Let us verify condition (3.25). Noting that $\log(1 + |\omega(t)|) \leq |\omega(t)|$ for $t \in \mathbb{R}$ and applying the Cauchy-Schwartz inequality one derives

$$\begin{aligned} \int_{\mathbb{R}} |\omega_I(t)| \log(1 + |\omega_I(t)|) \frac{dt}{1+t^2} &\leq \int_{\mathbb{R}} |\omega(t)| \log(1 + |\omega(t)|) \frac{dt}{1+t^2} \\ &\leq \left(\int_{\mathbb{R}} |\omega(t)|^2 \frac{dt}{(1+t^2)^{\alpha/2}}\right)^{1/2} \left(\int_{\mathbb{R}} |\omega(t)|^2 \frac{dt}{(1+t^2)^{2-\frac{\alpha}{2}}}\right)^{1/2}. \end{aligned} \tag{3.29}$$

Since $\alpha \leq 2$, one has

$$\frac{1}{(1+t^2)^{2-\frac{\alpha}{2}}} = \frac{1}{(1+t^2)^{\alpha/2}} \frac{1}{(1+t^2)^{2-\alpha}} \leq \frac{1}{(1+t^2)^{\alpha/2}}, \quad t \in \mathbb{R}.$$

Combining this inequality with (3.29) yields

$$\int_{\mathbb{R}} |\omega_I(t)| \log(1 + |\omega_I(t)|) \frac{dt}{1+t^2} \leq \int_{\mathbb{R}} |\omega(t)|^2 \frac{dt}{(1+t^2)^{\alpha/2}} < \infty$$

and (3.25) is verified. Proposition 3.9 completes the proof.

(ii) Clearly, one has $w_R(\cdot), w_I(\cdot) \in L^2(\mathbb{R}; \frac{dt}{(1+t^2)^{\alpha/2}})$. Note that for $\alpha \in (0, 1)$ the weight $(1 + t^2)^{-\alpha/2}$ satisfies the Muckenhoupt (A_2)-condition. Therefore, by Hunt-Muckenhoupt-Weeden theorem (see [14, Theorem 6.6.2]) the Hilbert transform H boundedly maps $L^2(\mathbb{R}; \frac{dt}{(1+t^2)^{\alpha/2}})$ onto $L^2(\mathbb{R}; \frac{dt}{(1+t^2)^{\alpha/2}})$. Hence

$$\widetilde{\omega}_I = H\omega_I \in L^2(\mathbb{R}; \tfrac{dt}{(1+t^2)^{\alpha/2}}), \tag{3.30}$$

where $\widetilde{\omega}_I$ denotes the function harmonic conjugate to ω_I. Setting $F(\cdot) := \widetilde{\omega}_I(\cdot) - i\omega_I(\cdot)$ we obtain from (3.30) that

$$\begin{aligned} \xi(t) := \omega(t) + F(t) &= \omega_R(t) + i\omega_I(t) + \big(\widetilde{\omega}_I(t) - i\omega_I(t)\big) \\ &= \omega_R(t) + \widetilde{\omega}_I(t) \in L^2(\mathbb{R}; \tfrac{dt}{(1+t^2)^{\alpha/2}}). \end{aligned} \tag{3.31}$$

Since $\widetilde{\omega}_I$ and ω_R are real-valued, the function $\xi(\cdot)$ is real-valued too. Moreover, the function $F(\cdot) = \widetilde{\omega}_I(\cdot) - i\omega_I(\cdot) \in L^2(\mathbb{R}; \frac{dt}{(1+t^2)^{\alpha/2}})$ admits a holomorphic continuation in $\mathbb{C}_-$ given by

$$F := \mathcal{P}(\widetilde{\omega}_I - i\omega_I) \in H^2_-(\mathbb{R}; \tfrac{dt}{(1+t^2)^{\alpha/2}}) \subset H^1_-(\mathbb{R}; \tfrac{dt}{(1+t^2)^2}), \tag{3.32}$$

where $\mathcal{P}$ is the Poisson transform (3.28). According to Proposition 3.8, relations (3.31) and (3.32) together imply that $\xi(\cdot) = \overline{\xi(\cdot)} \in \mathcal{S}\{H', H\}$, as claimed. □

Remark 3.11. According to [1, Theorem 4.1] a pair $\{H', H\}$ of resolvent comparable m-accumulative operators admits a *real-valued* SSF $\xi(\cdot) \in L^1(\mathbb{R}; \frac{dt}{1+t^2})$ if in addition the following conditions are verified

$$\begin{aligned} &(H'^* + i)^{-1} - (H' - i)^{-1} + 2i(H'^* + i)^{-1}(H' - i)^{-1} \in \mathfrak{S}_1^0(\mathfrak{H}), \\ &(H^* + i)^{-1} - (H - i)^{-1} + 2i(H^* + i)^{-1}(H - i)^{-1} \in \mathfrak{S}_1^0(\mathfrak{H}). \end{aligned} \tag{3.33}$$

Here $\mathfrak{S}_1^0(\mathfrak{H})$ denotes the ideal of compact operators T satisfying

$$\sum_{k=1}^{\infty} s_k(T) \log\left(1 + \frac{1}{s_k(T)}\right) < \infty,$$

where $\{s_k(T)\}_{k\in\mathbb{N}}$ is a sequence of singular numbers of $T \in \mathfrak{S}_\infty(\mathfrak{H})$. Notice that $\mathfrak{S}_1^0(\mathfrak{H})$ is a strict subset of $\mathfrak{S}_1(\mathfrak{H})$.

Next we are going to show that any pair $\{\widetilde{A}', \widetilde{A}\}$ of resolvent comparable m-accumulative extensions of A admits a SSF (in general non-real). Our considerations are heavily relied on the following technical lemma.

Lemma 3.12. *Let $\Pi = \{\mathcal{H}, \Gamma_0, \Gamma_1\}$ be a boundary triplet for A^*, $M(\cdot)$ the corresponding Weyl function, and B a bounded accumulative operator in $\mathcal{H}$. Then:*

(i) *If $0 \le V_+ \le |B_I| = -B_I$, $B_I := \mathrm{Im}\,(B) \le 0$ and $V_+ \in \mathfrak{S}_1(\mathcal{H})$, then the function $w_+(z) := \det(I + iV_+(B - M(z))^{-1})$, $z \in \mathbb{C}_+$, is holomorphic and contractive. Moreover, $w_+(\cdot)$ is an* outer function*, i.e. it admits a representation*

$$w_+(z) = \varkappa_+ \exp\left\{\frac{i}{\pi}\int_{\mathbb{R}} \left(\frac{1}{t-z} - \frac{t}{1+t^2}\right)\eta_+(t)dt\right\}, \qquad z \in \mathbb{C}_+, \tag{3.34}$$

with $\eta_+(t) = -\ln(|(w_+(t+i0))|) \ge 0$ for a.e. $t \in \mathbb{R}$, and $\eta_+(\cdot) \in L^1(\mathbb{R}; \frac{dt}{1+t^2})$, and a unimodular $\varkappa_+ \in \mathbb{T}$.

(ii) *If $V \le |B_I| = -B_I$ and $V = V^* \in \mathfrak{S}_1(\mathcal{H})$, then the function $w(z) := \det(I + iV(B - M(z))^{-1})$, $z \in \mathbb{C}_+$, is an* outer function *in $\mathbb{C}_+$, i.e. admits a representation*

$$w(z) = \varkappa \exp\left\{\frac{i}{\pi}\int_{\mathbb{R}} \left(\frac{1}{t-z} - \frac{t}{1+t^2}\right)\eta(t)dt\right\}, \qquad z \in \mathbb{C}_+, \tag{3.35}$$

with a unimodular $\varkappa \in \mathbb{T}$ and a real-valued function $\eta(\cdot) \in L^1(\mathbb{R}; \frac{dt}{1+t^2})$ satisfying $\eta(t) = -\ln(|(w(t+i0))|)$ for a.e. $t \in \mathbb{R}$, *Proof.* The trace class operator $V_+ = V_+^*$ admits a spectral decomposition $V_+ = \sum_{k\in\mathbb{N}} \mu_k(\cdot,\psi_k)\psi_k$ where $\{\mu_k\}_{k\in\mathbb{N}} \in l^1(\mathbb{N})$ is the system of eigenvalues, $\mu_k \geq 0$, and $\{\psi_k\}_{k\in\mathbb{N}}$ is an orthonormal system of eigenvectors. We set

$$B_0 := B_R + i(B_I + V_+) \quad \text{and} \quad B_l := B_0 - i\sum_{k=1}^{l} \mu_k(\cdot,\psi_k)\psi_k, \quad l \in \mathbb{N}.$$

Clearly, $B_l := B_0 + i\sum_{k=l+1}^{\infty} \mu_k(\cdot,\psi_k)\psi_k$ and $\lim_{l\to\infty} \|B_l - B\|_{\mathfrak{S}_1} = 0$ where $\|\cdot\|_{\mathfrak{S}_1}$ denotes the trace norm.

By assumption, $B_I + V_+ \leq 0$, so the operator B_0 is m-accumulative. Further, let us introduce the operator-valued function

$$W_l(z) := I + i\mu_l P_l(B_l - M(z))^{-1}P_l, \quad P_l := (\cdot,\psi_l)\psi_l, \quad z \in \mathbb{C}_+, \quad l \in \mathbb{N}.$$

We set $w_l(z) := \det(W_l(z))$, $z \in \mathbb{C}_+$, $l \in \mathbb{N}$. Clearly,

$$w_l(z) = 1 + i\mu_l((B_l - M(z))^{-1}\psi_l,\psi_l), \quad z \in \mathbb{C}_+, \quad l \in \mathbb{N}. \tag{3.36}$$

Further, we set

$$\Delta_{B_{l-1}/B_l}(z) := \det\left(I + (B_{l-1} - B_l)(B_l - M(z))^{-1}\right), \quad z \in \mathbb{C}_+, \quad l \in \mathbb{N}$$

Since $B_{l-1} - B_l = i\mu_l(\cdot,\psi_l)\psi_l$, $l \in \mathbb{N}$, we get $\Delta_{B_{l-1}/B_l}(z) = w_l(z)$, $z \in \mathbb{C}_+$, $l \in \mathbb{N}$. In accordance with the chain rule (2.17)

$$\Delta_{B_0/B_l}(z) = \prod_{k=1}^{l} \Delta_{B_{k-1}/B_k}(z) = \prod_{k=1}^{l} w_k(z), \quad z \in \mathbb{C}_+, \quad l \in \mathbb{N}. \tag{3.37}$$

Since $B_0 - B = iV_+$ we have $\det(W_+(z)) = \Delta_{B_0/B}(z)$ where

$$W_+(z) := I + i\sqrt{V_+}(B - M(z))^{-1}\sqrt{V_+}, \quad z \in \mathbb{C}_+. \tag{3.38}$$

By $\lim_{l\to\infty} \|B_l - B\|_{\mathfrak{S}_1} = 0$ we get from (3.37) that

$$w_+(z) := \det(W_+(z)) = \Delta_{B_0/B}(z) = \lim_{l\to\infty} \Delta_{B_0/B_l}(z) = \lim_{l\to\infty} \prod_{k=1}^{l} w_k(z) \tag{3.39}$$

for $z \in \mathbb{C}_+$. Note, that together with $W_+(\cdot)$, the operator function $W_l(\cdot)$, $l \in \mathbb{N}$, is holomorphic and contractive in $\mathbb{C}_+$. Hence $w_l(z) = \det(W_l(z))$, $l \in \mathbb{N}$, is holomorphic and contractive in $\mathbb{C}_+$, thus $w_l(\cdot) \in H^\infty(\mathbb{C}_+; dt)$. Next we set

$$\theta_l(z) := \Delta_{B_l/B_{l-1}}(z) := 1 - i\mu_l\left((B_{l-1} - M(z))^{-1}\psi_l,\psi_l\right), \quad z \in \mathbb{C}_+, \quad l \in \mathbb{N}.$$

Notice that $\theta_l(z)$ is well defined since B_{l-1} is accumulative. Moreover, one has

$$\theta_l(z)w_l(z) = w_l(z)\theta_l(z) = 1, \quad z \in \mathbb{C}_+, \quad l \in \mathbb{N}. \tag{3.40}$$

Since B_{l-1} is accumulative, one has $\mathrm{Im}\left((B_{l-1} - M(z))^{-1}\right) > 0$. Hence

$$\mathrm{Re}\,(\theta_l(z)) = 1 + \mu_l \mathrm{Im}\,\left(\left((B_{l-1} - M(z))^{-1}\psi_l, \psi_l\right)\right) > 1, \quad z \in \mathbb{C}_+, \quad l \in \mathbb{N}.$$

Combining this inequality with (3.40) yields

$$\mathrm{Re}\,(w_l(z)) = \frac{1}{|\theta_l(z)|^2}\mathrm{Re}\,(\theta_l(z)) > \frac{1}{|\theta_l(z)|^2} > 0, \quad z \in \mathbb{C}_+, \quad l \in \mathbb{N}.$$

By [14, Corollary II.4.8], for each $l \in \mathbb{N}$ the function $w_l(z)$ is an outer function. According to (B.5) it admits the representation

$$w_l(z) = \varkappa_l \, \exp\left\{\frac{i}{\pi}\int_R \left(\frac{1}{t-z} - \frac{t}{1+t^2}\right)\eta_l(t)dt\right\}, \quad \varkappa_l \in \mathbb{T}, \quad z \in \mathbb{C}_+, \quad l \in \mathbb{N},$$

where $\eta_l(t) := -\ln(|w_l(t + i0)|)$, $t \in \mathbb{R}$. Hence

$$\prod_{k=1}^{l} w_l(z) = \prod_{k=1}^{l} \varkappa_k \, \exp\left\{\sum_{k=1}^{l}\frac{i}{\pi}\int_{\mathbb{R}} \left(\frac{1}{t-z} - \frac{t}{1+t^2}\right)\eta_k(t)dt\right\}, \tag{3.41}$$

$z \in \mathbb{C}_+$, $l \in \mathbb{N}$. Now (3.37) yields

$$0 \le \left|\Delta_{B_0/B_l}(z)\right| = \exp\left\{-\sum_{k=1}^{l}\frac{1}{\pi}\int_R \frac{y}{(t-x)^2+y^2}\eta_k(t)dt\right\}$$

where $z = x + iy$. Since $w_k(z)$, $z \in \mathbb{C}_+$, is contractive, we get $\eta_k(t) \ge 0$ for a.e. $t \in \mathbb{R}$. Combining Corollary C.2 with (3.36) we obtain

$$-\int_R \ln(|w_k(t+i0)|)\frac{1}{1+t^2}dt \le 2\pi|w_k(i) - 1| \le 2\pi\mu_k \frac{1}{\|\mathrm{Im}\,(M(i))\|}, \quad k \in \mathbb{N}.$$

Since $\{\mu_k\}_{k\in\mathbb{N}} \in l_1$, the Beppo Levi theorem yields

$$0 \le \eta_+(t) := \sum_{k\in\mathbb{N}} \eta_k(t) = -\sum_{k\in\mathbb{N}} \ln(|w_k(t+i0)|) \in L^1(\mathbb{R}; \tfrac{dt}{1+t^2}),$$

and

$$\frac{i}{\pi}\int_R \left(\frac{1}{t-z} - \frac{t}{1+t^2}\right)\eta_+(t)dt = \sum_{k=1}^{\infty}\frac{i}{\pi}\int_R \left(\frac{1}{t-z} - \frac{t}{1+t^2}\right)\eta_k(t)dt. \tag{3.42}$$

It follows from (3.41) and (3.42) that

$$w_+(z) = \lim_{l\to\infty}\prod_{k=1}^{l} w_k(z) = \left(\lim_{l\to\infty}\prod_{k=1}^{l}\varkappa_k\right)\exp\left\{\frac{i}{\pi}\int_R \left(\frac{1}{t-z} - \frac{t}{1+t^2}\right)\eta_+(t)dt\right\},$$

$z \in \mathbb{C}_+$, where $w_+(\cdot) = \det(W_+(\cdot))$ and $W_+(\cdot)$ is given by (3.38). Hence the limit $\varkappa_+ := \lim_{l\to\infty}\prod_{k=1}^{l}\varkappa_k \in \mathbb{T}$ exists and we arrive at the representation (3.34). Thus, $w_+(\cdot)$ is an outer function and $\eta_+(t) = -\ln(|\det(w_+(t+i0))|)$ for a.e. $t \in \mathbb{R}$, see B.

(ii) Let $V = V_+ - V_-$, $V_\pm \geq 0$. We set $B_- := B - iV_-$. Since $(B_-)_I = B_I - V_- \leq 0$, the operator B_- is accumulative. One easily checks that

$$\det(I + iV(B - M(z))^{-1}) = \frac{\det(I + iV_+(B_- - M(z))^{-1})}{\det(I + iV_-(B_- - M(z))^{-1})}, \quad z \in \mathbb{C}_+.$$

The assumption $V \leq -B_I$ yields $0 \leq V_+ \leq -B_I + V_- = -(B_-)_I$. Applying (i) we get the existence of a complex number $\varkappa_+ \in \mathbb{T}$ and a non-negative function $\eta_+(\cdot) \in L^1(\mathbb{R}; \frac{dt}{1+t^2})$ such that the representation

$$\det(I + iV_+(B_- - M(z))^{-1}) = \varkappa_+ \exp\left\{\frac{i}{\pi}\int_{\mathbb{R}}\left(\frac{1}{t-z} - \frac{t}{1+t^2}\right)\eta_+(t)dt\right\}, \quad z \in \mathbb{C}_+,$$

is valid. From $0 \leq V_- \leq -B_I + V_- = (B_-)_I$ and (i) we get the existence of a complex number $\varkappa_- \in \mathbb{T}$ and a non-negative function $\eta_-(\cdot) \in L^1(\mathbb{R}; \frac{dt}{1+t^2})$ such that the representation

$$\det(I + iV_-(B_- - M(z))^{-1}) = \varkappa_- \exp\left\{\frac{i}{\pi}\int_{\mathbb{R}}\left(\frac{1}{t-z} - \frac{t}{1+t^2}\right)\eta_-(t)dt\right\}, \quad z \in \mathbb{C}_-,$$

holds. Setting $\varkappa := \varkappa_+/\varkappa_- \in \mathbb{T}$ and $\eta(t) := \eta_+(t) - \eta_-(t)$, $t \in \mathbb{R}$, we arrive at the representation (3.35). $\square$

Now we are ready to prove the existence of a SSF for a pair of m-accumulative operators.

Theorem 3.13. *Let $\{\widetilde{A}', \widetilde{A}\}$ be a pair of resolvent comparable m-accumulative extensions of A such that $\rho(\widetilde{A}) \cap \mathbb{C}_- \neq \emptyset$. Then the following assertions are valid:*

(i) *There exists a boundary triplet $\Pi_r = \{\mathcal{H}, \Gamma_0, \Gamma_1\}$ for A^*, which is regular for the pair $\{\widetilde{A}', \widetilde{A}\}$.*

(ii) *For any (not necessarily regular) boundary triplet Π satisfying $\{\widetilde{A}', \widetilde{A}\} \in \mathfrak{D}^\Pi$, the perturbation determinant $\Delta^\Pi_{\widetilde{A}'/\widetilde{A}}(\cdot)$ admits the representation*

$$\Delta^\Pi_{\widetilde{A}'/\widetilde{A}}(z) = c\ \exp\left\{\frac{1}{\pi}\int_{\mathbb{R}}\left(\frac{1}{t-z} - \frac{t}{1+t^2}\right)\omega(t)dt\right\}, \qquad z \in \mathbb{C}_+. \tag{3.43}$$

with a complex-valued function $\omega(\cdot) \in L^1(\mathbb{R}; \frac{dt}{1+t^2})$ *and a constant $c \in \mathbb{C}$. The representation is not unique.*

(iii) *Any $\omega(\cdot)$ from representation* (3.43) *is a SSF for the pair $\{\widetilde{A}', \widetilde{A}\}$, i.e. $\omega(\cdot) \in \mathcal{S}\{\widetilde{A}', \widetilde{A}\}$. In particular, the trace formula holds*

$$\operatorname{tr}\left((\widetilde{A}' - z)^{-1} - (\widetilde{A} - z)^{-1}\right) = -\frac{1}{\pi}\int_{\mathbb{R}}\frac{\omega(t)}{(t-z)^2}dt, \quad z \in \mathbb{C}_+. \tag{3.44}$$

(iv) *If the operator $\widetilde{A}$ (resp. $\widetilde{A}'$) is self-adjoint, then there is $\omega(\cdot) \in \mathcal{S}\{\widetilde{A}', \widetilde{A}\}$ and satisfying $\operatorname{Im}(\omega(t)) \leq 0$ (resp. $\operatorname{Im}(\omega(t)) \geq 0$) for a.e. $t \in \mathbb{R}$.*

(v) *If* $\xi(\cdot) \in \mathcal{S}\{\widetilde{A}', \widetilde{A}\}$ *is* real-valued*, then for a.e.* $x \in \mathbb{R}$

$$\xi(x) = \mathrm{Im}\,(\log(\Delta^{\Pi}_{\widetilde{A}'/\widetilde{A}}(x+i0))) := \lim_{y\downarrow 0} \mathrm{Im}\,(\log(\Delta^{\Pi}_{\widetilde{A}'/\widetilde{A}}(x+iy))). \tag{3.45}$$

Proof. (i) Since $\rho(\widetilde{A}) \cap \rho(\widetilde{A}') \supset \mathbb{C}_+$, then by Theorem 2.6, there exists a boundary triplet $\Pi_r = \{\mathcal{H}, \Gamma_0, \Gamma_1\}$ for A^* which is regular for the pair $\{\widetilde{A}', \widetilde{A}\}$.

(ii) First, let $\Pi = \Pi_r = \{\mathcal{H}, \Gamma_0, \Gamma_1\}$ be regular for the pair $\{\widetilde{A}', \widetilde{A}\}$, i.e. $\widetilde{A}' = A_{B'}$ and $\widetilde{A} = A_B$ with bounded operators B' and B. Moreover, by Proposition 2.3, B' and B are accumulative because so are $\widetilde{A}'$ and $\widetilde{A}$. By Proposition 2.9(ii) the pair $\{\widetilde{A}', \widetilde{A}\}$ is resolvent comparable if and only if the condition $B' - B \in \mathfrak{S}_1(\mathcal{H})$ is satisfied. We set

$$B'' := B'_R + iB_I. \tag{3.46}$$

By Proposition 2.3 (iii), $\widetilde{A}'' := A_{B''} = A^* \upharpoonright \ker(\Gamma_1 - B''\Gamma_0)$ is also m-accumulative because so is B''. Since $B' - B'' = i(B'_I - B_I) \in \mathfrak{S}_1(\mathcal{H})$, $\{\widetilde{A}', \widetilde{A}''\} \in \mathfrak{D}^{\Pi}$ and the perturbation determinant $\Delta^{\Pi}_{\widetilde{A}'/\widetilde{A}''}(\cdot)$ is well defined,

$$\Delta^{\Pi}_{\widetilde{A}'/\widetilde{A}''}(z) = \det(I + (B' - B'')(B'' - M(z))^{-1}) = \det(I + i(B'_I - B_I)(B'' - M(z))^{-1}),$$

$z \in \mathbb{C}_+$. Since $B'_I - B_I \le -B_I = -B''_I$, Lemma 3.12(ii) guarantees a representation

$$\Delta^{\Pi}_{\widetilde{A}'/\widetilde{A}''}(z) = \varkappa \exp\left\{\frac{i}{\pi}\int_R \left(\frac{1}{t-z} - \frac{t}{1+t^2}\right)\eta(t)dt\right\}, \qquad z \in \mathbb{C}_+, \tag{3.47}$$

with *a real-valued* function $\eta(\cdot) \in L^2(\mathbb{R}; \frac{dt}{1+t^2})$ and $\varkappa \in \mathbb{T}$.

Further, it follows from (3.46) and the inclusion $B' - B \in \mathfrak{S}_1(\mathcal{H})$ that $B'' - B = B'_R - B_R \in \mathfrak{S}_1(\mathcal{H})$. Hence $\{\widetilde{A}'', \widetilde{A}\} \in \mathfrak{D}^{\Pi}$ and $\Delta^{\Pi}_{\widetilde{A}''/\widetilde{A}}(\cdot)$ is well defined. Moreover, applying Lemma 3.2(ii), we arrive at the representation

$$\begin{aligned}\Delta^{\Pi}_{\widetilde{A}''/\widetilde{A}}(z) &= \det(I + (B'_R - B_R)(B - M(z))^{-1}) \\ &= \widetilde{c}\exp\left\{\frac{1}{\pi}\int_R \left(\frac{1}{t-z} - \frac{t}{1+t^2}\right)\xi(t)dt\right\},\end{aligned} \tag{3.48}$$

$z \in \mathbb{C}_+$, with a *real-valued function* $\xi(\cdot) \in L^2(\mathbb{R}; \frac{1}{1+t^2}dt)$ and a constant $\widetilde{c} > 0$. Combining (3.47) with (3.48), and applying the chain rule (2.17) we arrive at representation (3.43) with $c := \widetilde{c}\varkappa$ and $\omega(t) := \xi(t) + i\eta(t)$, $t \in \mathbb{R}$.

Finally, if Π is not regular for the pair $\{\widetilde{A}', \widetilde{A}\}$, formula (3.43) is implied by combining (2.15) with (3.43) just obtained for a regular boundary triplet.

(iii) Formula (3.44) is immediate by combining (3.43) with (2.16).

(iv) Let for definiteness $\widetilde{A} = \widetilde{A}^*$. Then $B = B^*$ and $B'' = B'_R = (B'')^*$. Hence $\widetilde{A}'' = A_{B''} = (\widetilde{A}'')^*$ and $B'_I - B_I = B'_I \le 0$. Therefore, applying Lemma 3.12 (i) with $V_+ = |B'_I|$ and $B = B'$ we derive

$$\Delta^{\Pi}_{\widetilde{A}'/\widetilde{A}''}(z) = \left(\Delta^{\Pi}_{\widetilde{A}''/\widetilde{A}'}(z)\right)^{-1} = \overline{\varkappa}\exp\left\{-\frac{i}{\pi}\int_R \left(\frac{1}{t-z} - \frac{t}{1+t^2}\right)\eta_+(t)dt\right\}, \tag{3.49}$$

$z \in \mathbb{C}_+$, where $\eta_+(t) \geq 0$ for a.e. $t \in \mathbb{R}$. Combining this identity with (3.48), applying the chain rule (2.17), and setting $\omega(t) := \xi(t) - i\eta_+(t)$ we arrive at (3.43) with $\mathrm{Im}\,(\omega(t)) \leq 0$, $t \in \mathbb{R}$, and $\varkappa \in \mathbb{T}$.

The case of $\widetilde{A}' = (\widetilde{A}')^*$ is treated similarly or just can be reduced to the previous one.

(v) Substituting $\xi(\cdot)$ in (3.43) in place of $\omega(\cdot)$, and taking the logarithm and then the imaginary part of both sides we get

$$\mathrm{Im}\,(\log(\Delta^{\Pi}_{\widetilde{A}'/\widetilde{A}}(x+iy))) = \frac{1}{\pi}\int_{\mathbb{R}} \frac{y}{(t-x)^2+y^2}\xi(t)dt + c_I, \tag{3.50}$$

$c_I := \mathrm{Im}\,(c)$. Applying the Fatou theorem as $y \downarrow 0$ we arrive at (3.45). $\square$

Corollary 3.14. *Let the assumptions of Theorem 3.13 be satisfied. If $\Phi \in \mathcal{F}_+(\widetilde{A}, \widetilde{A}')$, then $\Phi(\widetilde{A}') - \Phi(\widetilde{A}) \in \mathfrak{S}_1(\mathfrak{H})$ and the following trace formula holds*

$$\mathrm{tr}\,(\Phi(\widetilde{A}') - \Phi(\widetilde{A})) = \frac{1}{\pi}\int_{\mathbb{R}} \Phi'(t)\omega(t)dt. \tag{3.51}$$

Proof. By Lemma D.1, the assumption $\Phi \in \mathcal{F}_+(\widetilde{A}, \widetilde{A}')$ implies $\Phi(\widetilde{A}') - \Phi(\widetilde{A}) \in \mathfrak{S}_1(\mathfrak{H})$. Since the SSF $\omega(t)$ from representation (3.44) satisfies $\int_{\mathbb{R}} \frac{|\omega(t)|}{1+t^2}dt < \infty$, one gets from (D.2) that $\int_{\mathbb{R}} |\Phi'(t)\omega(t)|dt < \infty$ which guarantees the absolute convergence of the integral $\int_{\mathbb{R}} \Phi'(t)\omega(t)dt$. Multiplying both sides of (3.44) by $\Phi(z)$, then integrating with respect to dz, and using the identity

$$\int_{\mathbb{R}} \Phi'(t)\omega(t)dt = -\frac{1}{2\pi i}\oint_{\Gamma} \Phi(z)\left(\int_{\mathbb{R}} \frac{\omega(t)}{(t-z)^2}dt\right)dz, \tag{3.52}$$

which is immediate from (D.2), we arrive at (3.80). $\square$

Remark 3.15. Theorem 3.13 generalizes Krein's Theorem 3.4 and coincides with it whenever both operators $\widetilde{A}$ and $\widetilde{A}'$ are self-adjoint. Indeed, if $\widetilde{A} = \widetilde{A}^*$ and $\widetilde{A}' = (\widetilde{A}')^*$, then, by Theorem 3.13 (iv), $\mathrm{Im}\,\omega(\cdot) = 0$ and $\omega(\cdot)$ is real. Therefore formulas (3.43) and (3.44) can be extended to $\mathbb{C}_-$ by symmetry and turn into formulas (3.9) and (3.11), respectively. However, for the reader's convenience and because of its simplicity we presented a direct proof of Theorem 3.4.

Theorem 3.16. *Let $\widetilde{A}, \widetilde{A}' \in \mathrm{Ext}_A$ be m-accumulative extensions. Let also $\Pi = \{\mathcal{H}, \Gamma_0, \Gamma_1\}$ be a boundary triplet for A^* such that $\{\widetilde{A}', \widetilde{A}\} \in \mathfrak{D}^{\Pi}$. Then:*

(i) *If for some $\lambda_0 \in \rho(\widetilde{A}') \cap \rho(\widetilde{A}) \cap \rho(A_0) \cap \mathbb{R}$ the inequality*

$$\mathrm{Re}\,((\widetilde{A}' - \lambda_0)^{-1}) \leq \mathrm{Re}\,((\widetilde{A} - \lambda_0)^{-1}) \tag{3.53}$$

holds, then there exists a complex-valued $\omega(\cdot) \in \mathcal{S}\{\widetilde{A}', \widetilde{A}\}$ satisfying $\mathrm{Re}\,(\omega(t)) \geq 0$ for a.e. $t \in \mathbb{R}$.

(ii) *If for some $\lambda_0 \in \rho(\widetilde{A}') \cap \rho(\widetilde{A}) \cap \rho(A_0) \cap \mathbb{R}$ the inequality*

$$\mathrm{Im}\,((\widetilde{A}' - \lambda_0)^{-1}) \leq \mathrm{Im}\,((\widetilde{A} - \lambda_0)^{-1}), \tag{3.54}$$

holds, then there exists a complex-valued SSF $\omega(\cdot)$ for the pair $\{\widetilde{A}', \widetilde{A}\}$ such that $\mathrm{Im}\,(\omega(t)) \geq 0$ for a.e. $t \in \mathbb{R}$.

(iii) *If both conditions* (3.53) *and* (3.54) *are satisfied, then there exists* $\omega(\cdot) \in \mathcal{S}\{\widetilde{A}', \widetilde{A}\}$ *such that* $\mathrm{Re}\,(\omega(t)) \geq 0$ *and* $\mathrm{Im}\,(\omega(t)) \geq 0$ *for a.e.* $t \in \mathbb{R}$.

Proof. (i) By Definition 2.12, the inclusion $\{\widetilde{A}', \widetilde{A}\} \in \mathfrak{D}^{\Pi}$ implies existence of $B' \in \mathcal{C}(\mathcal{H})$ and $B \in \mathcal{C}(\mathcal{H})$ such that $\widetilde{A}' = A^* \upharpoonright \ker(\Gamma_1 - B'\Gamma_0)$ and $\widetilde{A} = A^* \upharpoonright \ker(\Gamma_1 - B\Gamma_0)$. By Proposition 2.3 (iii), B' and B are m-accumulative because so are $\widetilde{A}'$ and $\widetilde{A}$. By Proposition 2.8, $\lambda_0 \in \rho(\widetilde{A}') \cap \rho(\widetilde{A}) \cap \rho(A_0)$ implies $0 \in \rho(B' - M(\lambda_0)) \cap \rho(B - M(\lambda_0))$. According to the Krein-type formula (2.6)

$$(\widetilde{A} - \lambda_0)^{-1} - (\widetilde{A}' - \lambda_0)^{-1} = \gamma(\lambda_0)\left((B - M(\lambda_0))^{-1} - (B' - M(\lambda_0))^{-1}\right)\gamma(\lambda_0)^*. \quad (3.55)$$

Setting $\widetilde{B}' := -(B' - M(\lambda_0))^{-1}$ and $\widetilde{B} := -(B - M(\lambda_0))^{-1}$ and taking real parts we get

$$\mathrm{Re}\,((\widetilde{A} - \lambda_0)^{-1}) - \mathrm{Re}\,((\widetilde{A}' - \lambda_0)^{-1}) = \gamma(\lambda_0)\left(\mathrm{Re}\,(\widetilde{B}') - \mathrm{Re}\,(\widetilde{B})\right)\gamma(\lambda_0)^*$$

In turn, this identity yields the equivalence

$$\mathrm{Re}\,((\widetilde{A}' - \lambda_0)^{-1} \leq \mathrm{Re}\,((\widetilde{A} - \lambda_0)^{-1} \Longleftrightarrow \mathrm{Re}\,(\widetilde{B}) \leq \mathrm{Re}\,(\widetilde{B}'). \quad (3.56)$$

Next we introduce a new boundary triplet $\widetilde{\Pi} := \{\mathcal{H}, \widetilde{\Gamma}_0, \widetilde{\Gamma}_1\}$ for A^* by setting

$$\widetilde{\Gamma}_1 := -\Gamma_0, \quad \widetilde{\Gamma}_0 := \Gamma_1 - M(\lambda_0)\Gamma_0. \quad (3.57)$$

Clearly, $\widetilde{A}' = A_{\widetilde{B}'} = A^* \upharpoonright \ker(\widetilde{\Gamma}_1 - \widetilde{B}'\widetilde{\Gamma}_0)$ and $\widetilde{A} = A_{\widetilde{B}} = A^* \upharpoonright \ker(\widetilde{\Gamma}_1 - \widetilde{B}\widetilde{\Gamma}_0)$. Hence the triplet $\widetilde{\Pi}$ is regular for the pair $\{\widetilde{A}', \widetilde{A}\}$. By Proposition 2.3 (iii), $\widetilde{B}'_I := \mathrm{Im}\,(\widetilde{B}') \leq 0$ and $\widetilde{B}_I := \mathrm{Im}\,(\widetilde{B}) \leq 0$, since $\widetilde{A}'$ and $\widetilde{A}$ are m-accumulative. Next we set $\widetilde{B}'_R := \mathrm{Re}\,(\widetilde{B}')$ and $\widetilde{B}_R := \mathrm{Re}\,(\widetilde{B})$ and

$$\widetilde{B}'' := \widetilde{B}'_R + i\widetilde{B}_I. \quad (3.58)$$

By Proposition 2.3 (iii), $\widetilde{A}'' := A_{\widetilde{B}''}$ is m-accumulative because $\widetilde{B}''_I = \widetilde{B}_I \leq 0$.

Clearly, $\{\widetilde{A}', \widetilde{A}\} \in \mathfrak{D}^{\widetilde{\Pi}}$. Let us check that $\{\widetilde{A}', \widetilde{A}''\} \in \mathfrak{D}^{\widetilde{\Pi}}$ and $\{\widetilde{A}'', \widetilde{A}\} \in \mathfrak{D}^{\widetilde{\Pi}}$. Indeed, due to the assumption $\{\widetilde{A}', \widetilde{A}\} \in \mathfrak{D}^{\Pi}$ the operators $\widetilde{A}'$ and $\widetilde{A}$ are resolvent comparable. Therefore, it follows from (3.55) that

$$\widetilde{B}' - \widetilde{B} = (B' - M(\lambda_0))^{-1} - (B - M(\lambda_0))^{-1} \in \mathfrak{S}_1(\mathcal{H}).$$

Hence $\widetilde{B}' - \widetilde{B}'' = \widetilde{B}'_I - \widetilde{B}_I \in \mathfrak{S}_1(\mathcal{H})$ and $\widetilde{B}'' - \widetilde{B} = \widetilde{B}'_R - \widetilde{B}_R \in \mathfrak{S}_1(\mathcal{H})$. So the perturbation determinants $\Delta^{\widetilde{\Pi}}_{\widetilde{A}''/\widetilde{A}'}(\cdot)$ and $\Delta^{\widetilde{\Pi}}_{\widetilde{A}''/\widetilde{A}}(\cdot)$ are well defined and

$$\begin{aligned} \Delta^{\widetilde{\Pi}}_{\widetilde{A}''/\widetilde{A}'}(z) &= \det(I + (\widetilde{B}'' - \widetilde{B}')(\widetilde{B}' - \widetilde{M}(z))^{-1}), \\ \Delta^{\widetilde{\Pi}}_{\widetilde{A}''/\widetilde{A}}(z) &= \det(I + (\widetilde{B}'' - \widetilde{B})(\widetilde{B} - \widetilde{M}(z))^{-1}), \end{aligned} \qquad z \in \mathbb{C}_+,$$

where $\widetilde{M}(\cdot)$ is the Weyl function corresponding to $\widetilde{\Pi}$.

Moreover, it follows from (3.53), (3.56) and (3.58), that $\widetilde{B}'' - \widetilde{B} = \widetilde{B}'_R - \widetilde{B}_R \geq 0$. Therefore, by Lemma 3.2(i), $\Delta^{\widetilde{\Pi}}_{\widetilde{A}''/\widetilde{A}}(\cdot)$ admits a representation

$$\Delta^{\widetilde{\Pi}}_{\widetilde{A}''/\widetilde{A}}(z) = \widetilde{c}'' \exp\left\{\frac{1}{\pi}\int_{\mathbb{R}}\left(\frac{1}{t-z} - \frac{1}{1+t^2}\right)\xi_+(t)dt\right\}, \quad z \in \mathbb{C}_+, \tag{3.59}$$

with a *non-negative function* $\xi_+(\cdot) \in L^1(\mathbb{R}; \frac{dt}{1+t^2})$ and a *positive* $\widetilde{c}'' > 0$. Since

$$\Delta^{\widetilde{\Pi}}_{\widetilde{A}'/\widetilde{A}''}(z) = \det(I + (\widetilde{B}' - \widetilde{B}'')(\widetilde{B}'' - \widetilde{M}(z))^{-1}), \qquad z \in \mathbb{C}_+, \tag{3.60}$$

and $-i(\widetilde{B}' - \widetilde{B}'') = B'_I - B_I \leq -B_I$, Lemma 3.12(ii) implies a representation

$$\Delta^{\widetilde{\Pi}}_{\widetilde{A}'/\widetilde{A}''}(z) = \widetilde{\varkappa} \exp\left\{\frac{i}{\pi}\int_{R}\left(\frac{1}{t-z} - \frac{t}{1+t^2}\right)\eta(t)dt\right\}, \qquad z \in \mathbb{C}_+, \tag{3.61}$$

with a *real-valued function* $\eta(\cdot) \in L^1(\mathbb{R}; \frac{dt}{1+t^2})$ and a uni-modular constant $\widetilde{\varkappa} \in \mathbb{T}$. Combining (3.59) with (3.61), using the chain rule (2.17), and setting $\widetilde{c}' := \widetilde{c}''\widetilde{\varkappa} \in \mathbb{C}$, we arrive at the following representation

$$\Delta^{\widetilde{\Pi}}_{\widetilde{A}'/\widetilde{A}}(z) = \widetilde{c}' \exp\left\{\frac{1}{\pi}\int_{\mathbb{R}}\left(\frac{1}{t-z} - \frac{t}{1+t^2}\right)\omega(t)dt\right\}, \quad z \in \mathbb{C}_+, \tag{3.62}$$

with $\omega(t) := \xi_+(t) + i\eta(t)$. Since $\mathrm{Re}\,(\omega(t)) = \mathrm{Re}\,(\xi_+(t)) \geq 0$, $t \in \mathbb{R}$, this representation proves the statement for the regular boundary triplet $\widetilde{\Pi}$.

Going over to the general case, note that $\{\widetilde{A}', \widetilde{A}\} \in \mathfrak{D}^{\Pi} \cap \mathfrak{D}^{\widetilde{\Pi}}$, so that property (2.15) implies the equality $\Delta^{\Pi}_{\widetilde{A}'/\widetilde{A}}(\cdot) = c'\,\Delta^{\widetilde{\Pi}}_{\widetilde{A}''/\widetilde{A}}(\cdot)$ with a constant $c' \in \mathbb{C}$. Combining this equality with (3.62) and setting $c := c'\widetilde{c}'$, we derive

$$\Delta^{\Pi}_{\widetilde{A}'/\widetilde{A}}(z) = c \exp\left\{\frac{1}{\pi}\int_{\mathbb{R}}\left(\frac{1}{t-z} - \frac{t}{1+t^2}\right)\omega(t)dt\right\}, \quad z \in \mathbb{C}_+. \tag{3.63}$$

(ii) Recall that $\widetilde{B}'_I \leq 0$ and $\widetilde{B}_I \leq 0$. Further, it follows from (3.55) that

$$\mathrm{Im}\,((\widetilde{A}' - \lambda_0)^{-1}) \leq \mathrm{Im}\,((\widetilde{A} - \lambda_0)^{-1}) \Longleftrightarrow \widetilde{B}_I \leq \widetilde{B}'_I. \tag{3.64}$$

Combining this equivalence with assumption (3.54) implies $\widetilde{B}'_I - \widetilde{B}_I \geq 0$. On the other hand, it follows from (3.58) that $\widetilde{B}' - \widetilde{B}'' = i(\widetilde{B}'_I - \widetilde{B}_I)$ and $0 \leq \widetilde{B}'_I - \widetilde{B}_I \leq |\widetilde{B}''_I| = |\widetilde{B}_I| = -\widetilde{B}_I$. Therefore, by Lemma 3.12(i), the perturbation determinant $\Delta^{\widetilde{\Pi}}_{\widetilde{A}'/\widetilde{A}''}(\cdot)$ given by (3.60) admits the representation (3.61) with *a non-negative function* $\eta(\cdot) = \eta_+(\cdot) \in L^1(\mathbb{R}; \frac{dt}{1+t^2})$.

Further, due to (3.58) $\widetilde{B}'' - \widetilde{B}' = (\widetilde{B}'' - \widetilde{B})^*$, so that, by Lemma 3.2 (ii), the perturbation determinant $\Delta^{\widetilde{\Pi}}_{\widetilde{A}''/\widetilde{A}}(\cdot)$ admits representation (3.59) with a *real-valued function* $\xi(\cdot) \in L^1(\mathbb{R}; \frac{dt}{1+t^2})$. Therefore the representation (3.62) holds with $\omega(t) := \xi(t) + i\eta_+(t)$, $t \in \mathbb{R}$, and $\mathrm{Im}\,(\omega(t)) = \eta_+(t) \geq 0$, $t \in \mathbb{R}$. Hence representation (3.63) for $\Delta^{\Pi}_{\widetilde{A}'/\widetilde{A}}(\cdot)$ proves the result.

(iii) By (i) and (ii) the representation (3.62) for $\Delta^{\widetilde{\Pi}}_{\widetilde{A}'/\widetilde{A}}(\cdot)$ holds with $\omega(t) := \xi_+(t) + i\eta_+(t)$, $t \in \mathbb{R}$, so that $\mathrm{Re}\,(\omega(t)) \geq 0$ and $\mathrm{Im}\,(\omega(t)) \geq 0$. $\square$

Corollary 3.17. *Assume in addition to the conditions of Theorem 3.16 that* $\widetilde{A}' = (\widetilde{A}')^*$ *and* $\widetilde{A} = \widetilde{A}^*$. *If there is* $\lambda_0 \in \rho(\widetilde{A}') \cap \rho(\widetilde{A}) \cap \rho(A_0) \cap \mathbb{R}$ *such that*

$$(\widetilde{A}' - \lambda_0)^{-1} \leq (\widetilde{A} - \lambda_0)^{-1}, \tag{3.65}$$

then there is a SSF $\xi(\cdot) \in \mathcal{S}\{\widetilde{A}', \widetilde{A}\}$ *such that* $\xi(t) \geq 0$ *for a.e.* $t \in \mathbb{R}$. *Moreover, representation* (3.9) *holds with* $c > 0$.

Proof. By Theorem 3.4, representation (3.9) holds with a real-valued function $\xi(\cdot)$. Combining this fact with Lemma 3.1 and Theorem 3.16(i), yields that $\xi(\cdot)$ can be chosen to be non-negative for a.e. $t \in \mathbb{R}$. If $c > 0$, the statement is proved. Otherwise we replace $\xi(\cdot)$ by $\xi(\cdot) + \pi$ (cf. the proof of Theorem 3.4). □

As a simple consequence of Theorem 3.16 we obtain the following result.

Proposition 3.18. *Let* $H = H^* \in \mathrm{Ext}_A$ *and let* $\widetilde{A}', \widetilde{A} \in \mathrm{Ext}_A$ *be* m*-accumulative extensions. Let also* $\Pi = \{\mathcal{H}, \Gamma_0, \Gamma_1\}$ *be a boundary triplet for* A^*, *such that* $\{\widetilde{A}', H\} \in \mathfrak{D}^\Pi$ *and* $\{\widetilde{A}, H\} \in \mathfrak{D}^\Pi$. *Then:*

(i) *If for some* $\lambda_0 \in \rho(\widetilde{A}') \cap \rho(\widetilde{A}) \cap \rho(A_0) \cap \mathbb{R}$ *the inequality* (3.53) *holds, then there exist* a complex-valued *SSFs* $\omega_{\widetilde{A}'}(\cdot) \in \mathcal{S}\{\widetilde{A}', H\}$ *and* $\omega_{\widetilde{A}}(\cdot) \in \mathcal{S}\{\widetilde{A}, H\}$ *such that* $\mathrm{Re}\,(\omega_{\widetilde{A}}(t)) \leq \mathrm{Re}\,(\omega_{\widetilde{A}'}(t))$ *for a.e.* $t \in \mathbb{R}$.

(ii) *If for some* $\lambda_0 \in \rho(\widetilde{A}') \cap \rho(\widetilde{A}) \cap \rho(A_0) \cap \mathbb{R}$ *inequality* (3.54) *holds, then there exist* complex-valued *SSFs* $\omega_{\widetilde{A}'}(\cdot) \in \mathcal{S}\{\widetilde{A}', H\}$ *and* $\omega_{\widetilde{A}}(\cdot) \in \mathcal{S}\{\widetilde{A}, H\}$ *such that* $\mathrm{Im}\,(\omega_{\widetilde{A}}(t)) \leq \mathrm{Im}\,(\omega_{\widetilde{A}'}(t)) \leq 0$ *for a.e.* $t \in \mathbb{R}$.

(iii) *If both conditions* (3.53) *and* (3.54) *are satisfied, then there exist SSFs* $\omega_{\widetilde{A}'}(\cdot) \in \mathcal{S}\{\widetilde{A}', \widetilde{H}\}$ *and* $\omega_{\widetilde{A}}(\cdot) \in \mathcal{S}\{\widetilde{A}', \widetilde{A}\}$ *such that* $\mathrm{Re}\,(\omega_{\widetilde{A}}(t)) \leq \mathrm{Re}\,(\omega_{\widetilde{A}'}(t))$ *and* $\mathrm{Im}\,(\omega_{\widetilde{A}}(t)) \leq \mathrm{Im}\,(\omega_{\widetilde{A}'}(t)) \leq 0$ *for a.e.* $t \in \mathbb{R}$.

Proof. (i) Since $\{\widetilde{A}', H\} \in \mathfrak{D}^\Pi$, Theorem 3.13 (ii) applies and gives the representation

$$\Delta^\Pi_{\widetilde{A}'/H}(z) = c_{\widetilde{A}'} \exp\left\{\frac{1}{\pi}\int_{\mathbb{R}} \left(\frac{1}{t-z} - \frac{t}{1+t^2}\right)\omega_{\widetilde{A}'}(t)dt\right\}, \quad z \in \mathbb{C}_+, \tag{3.66}$$

with a *complex-valued function* $\omega_{\widetilde{A}'}(\cdot) \in L^1(\mathbb{R}; \frac{dt}{1+t^2})$. Further, $\{\widetilde{A}', H\} \in \mathfrak{D}^\Pi$ and $\{\widetilde{A}, H\} \in \mathfrak{D}^\Pi$ imply $\{\widetilde{A}', \widetilde{A}\} \in \mathfrak{D}^\Pi$ and, by the chain rule (2.17),

$$\Delta^\Pi_{\widetilde{A}/H}(z) = \frac{\Delta^\Pi_{\widetilde{A}'/H}(z)}{\Delta^\Pi_{\widetilde{A}'/\widetilde{A}}(z)}, \quad z \in \mathbb{C}_+. \tag{3.67}$$

Inserting (3.63) and (3.66) into (3.67), and setting $c_{\widetilde{A}} := \frac{c_{\widetilde{A}'}}{c} \in \mathbb{C}$ we find

$$\Delta^\Pi_{\widetilde{A}/H}(z) = c_{\widetilde{A}} \exp\left\{\frac{1}{\pi}\int_{\mathbb{R}} \left(\frac{1}{t-z} - \frac{t}{1+t^2}\right)\omega_{\widetilde{A}}(t)dt\right\}, \quad z \in \mathbb{C}_+, \tag{3.68}$$

where $\omega_{\widetilde{A}}(t) := \omega_{\widetilde{A}'}(t) - \omega(t)$, $t \in \mathbb{R}$, with $\mathrm{Re}\,(\omega(t)) \geq 0$, cf. Theorem 3.16 (i). Hence $\mathrm{Re}\,(\omega_{\widetilde{A}}(t)) = \mathrm{Re}\,(\omega_{\widetilde{A}'}(t)) - \mathrm{Re}\,(\omega(t)) \leq \mathrm{Re}\,(\omega_{\widetilde{A}'}(t))$, $t \in \mathbb{R}$.

(ii) By Theorem 3.13 (iv), $\Delta^{\Pi}_{\widetilde{A}'/H}(\cdot)$ admits the representation (3.66) with *a complex valued function* $\omega_{\widetilde{A}'}(\cdot) \in L^1(\mathbb{R}; \frac{dt}{1+t^2})$ satisfying $\mathrm{Im}\,(\omega_{\widetilde{A}'}(t)) \leq 0$ for a.e. $t \in \mathbb{R}$. Inserting (3.66) and (3.63) into (3.67) we arrive at representation (3.68) for $\Delta^{\Pi}_{\widetilde{A}/H}(z)$ with $\omega_{\widetilde{A}}(t) := \omega_{\widetilde{A}'}(t) - \omega(t)$, $t \in \mathbb{R}$. Further, by Theorem 3.16(ii), $\mathrm{Im}\,(\omega(t)) \geq 0$, $t \in \mathbb{R}$, hence $\mathrm{Im}\,(\omega_{\widetilde{A}}(t)) \leq \mathrm{Im}\,(\omega_{\widetilde{A}'}(t)) \leq 0$ for a.e. $t \in \mathbb{R}$.

(iii) By (i), representation (3.68) for $\Delta^{\Pi}_{\widetilde{A}/H}(\cdot)$ holds with $\omega_{\widetilde{A}}(t) = \omega_{\widetilde{A}'}(t) - \omega(t)$ and $\mathrm{Re}\,(\omega_{21}(t)) \geq 0$. By (ii), $\omega(\cdot)$ also satisfies $\mathrm{Im}\,(\omega(t)) \geq 0$. Hence $\mathrm{Re}\,(\omega_{\widetilde{A}}(t)) \leq \mathrm{Re}\,(\omega_{\widetilde{A}'}(t))$ and $\mathrm{Im}\,(\omega_{\widetilde{A}}(t)) \leq \mathrm{Im}\,(\omega_{\widetilde{A}'}(t)) \leq 0$ for a.e. $t \in \mathbb{R}$. □

Corollary 3.19. *Let $\widetilde{A}', \widetilde{A}, H \in \mathrm{Ext}_A$ be self-adjoint extensions of A such that $\{\widetilde{A}', H\} \in \mathfrak{D}^{\Pi}$ and $\{\widetilde{A}, H\} \in \mathfrak{D}^{\Pi}$ for certain boundary triplet Π. If for some $\lambda_0 \in \rho(\widetilde{A}') \cap \rho(\widetilde{A}) \cap \rho(A_0) \cap \mathbb{R}$ the condition* (3.65) *is valid, then there exist real SSFs $\xi_{\widetilde{A}'}(\cdot)$ and $\xi_{\widetilde{A}}(\cdot)$ for the pairs $\{\widetilde{A}', H\}$ and $\{\widetilde{A}, H\}$, respectively, such that $\xi_{\widetilde{A}}(t) \leq \xi_{\widetilde{A}'}(t)$ for a.e. $t \in \mathbb{R}$. Moreover, the perturbation determinants $\Delta^{\Pi}_{\widetilde{A}'/H}(\cdot)$ and $\Delta^{\Pi}_{\widetilde{A}/H}(\cdot)$ admit representation* (3.9) *with $c_{\widetilde{A}'} > 0$ and $c_{\widetilde{A}} > 0$ as well as $\xi_{\widetilde{A}'}(\cdot)$ and $\xi_{\widetilde{A}}(\cdot)$ in place of c and $\xi(\cdot)$, respectively.*

Proof. Since $\widetilde{A}'$ and $\widetilde{A}$ are self-adjoint and $\lambda_0 \in \mathbb{R}$, condition (3.53) turns into the condition (3.65) and the result is implied by Theorem 3.16(i) due to the equalities $\xi_{\widetilde{A}'}(t) = \overline{\xi_{\widetilde{A}}(t)}$ and $\xi_{\widetilde{A}]}(t) = \overline{\xi_{\widetilde{A}}(t)}$. □

Remark 3.20. For other results on trace formulas for non-self-adjoint operators we refer to the papers of A. Rybkin [36, 35, 37, 38, 39].

3.3 Pairs of extensions with one m-accumulative operator

Here we consider trace formulas for pairs $\{\widetilde{A}', \widetilde{A}\}$ of proper extensions of a closed symmetric operator A assuming that $\widetilde{A}$ is m-accumulative extension. In the following we denote by $\{z_k\}_k$ be a sequence in $\mathbb{C}_+$ which might be finite or infinite.

Lemma 3.21. *Let $\Pi = \{\mathcal{H}, \Gamma_0, \Gamma_1\}$ be a boundary triplet for A^*, $M(\cdot)$ the corresponding Weyl function and let $B \in [\mathcal{H}]$ be an accumulative operator.*

(i) *If $0 \leq V_+ \leq 2|B_I| = -2B_I$, $V \in \mathfrak{S}_1(\mathcal{H})$, then the holomorphic function $w_+(z) := \det(I + iV_+(B - M(z))^{-1})$, $z \in \mathbb{C}_+$, is contractive. In particular, there exist a non-negative Borel measure $\mu_+(\cdot)$ satisfying $\int_{\mathbb{R}} \frac{1}{1+t^2} d\mu_+(t) < \infty$ and numbers $\varkappa_+ \in \mathbb{T}$ and $\alpha_+ \geq 0$ such that the multiplicative representation*

$$w_+(z) = \varkappa_+ \mathcal{B}(z, \mathfrak{Z}_+) \exp\left\{\frac{i}{\pi}\int_{\mathbb{R}} \left(\frac{1}{t-z} - \frac{t}{1+t^2}\right) d\mu_+(t)\right\} e^{i\alpha_+ z}, \quad z \in \mathbb{C}_+, \tag{3.69}$$

holds where $\mathfrak{Z}_+ := \{(\mathfrak{z}_k^+, \mathfrak{m}_k^+)\}_k$ is the Blaschke sequence consisting of the zeros of $w_+(\cdot)$ lying in $\mathbb{C}_+$ and their multiplicities.

(ii) *If $V \le 2|B_I| = -2B_I$ and $V = V^* \in \mathfrak{S}_1(\mathcal{H})$, then the function $w(z) := \det(I + iV(B - M(z))^{-1})$, $z \in \mathbb{C}_+$, belongs to the Smirnov class $\mathcal{N}(\mathbb{C}_+)$, see B. In particular, there exist a non-negative Borel measure $\mu_+(\cdot) \ge 0$ satisfying $\int_{\mathbb{R}} \frac{1}{1+t^2} d\mu_+(t) < \infty$, a non-negative function $\eta(\cdot) \in L^1(\mathbb{R}; \frac{dt}{1+t^2})$ as well as numbers $\varkappa \in \mathbb{T}$ and $\alpha \ge 0$ such that*

$$w(z) = \varkappa\, \mathcal{B}(z, \mathfrak{Z}) \exp\left\{\frac{i}{\pi}\int_{\mathbb{R}}\left(\frac{1}{t-z} - \frac{t}{1+t^2}\right) d\mu(t)\right\} e^{i\alpha z}, \quad z \in \mathbb{C}_+, \tag{3.70}$$

where $\mathfrak{Z} = \{(\mathfrak{z}_k, \mathfrak{m}_k)\}_k$ consists of the zeros $\mathfrak{z}_k$ of $w(\cdot)$ in $\mathbb{C}_+$ and their multiplicities $\mathfrak{m}_k$ and $\mu(\cdot) := \mu_+(\cdot) - \eta(\cdot)dt$.

Proof. (i) Consider holomorphic operator-valued function $W_+(\cdot)$ given by (3.38). Since $B_I \le 0$, $\mathrm{Im}\,(M(z)) > 0$ and $0 \in \rho(\mathrm{Im}\,(M(z)))$ for $z \in \mathbb{C}_+$, the operator $(B - M(z))^{-1}$ is well-defined and bounded for $z \in \mathbb{C}_+$. It is easily seen that

$$\begin{aligned} I - W_+(z)^* W_+(z) = i\sqrt{V_+}\big((B^* - M(z)^*)^{-1} - (B - M(z))^{-1}\big)\sqrt{V_+} \\ - \sqrt{V_+}(B^* - M(z)^*)^{-1}\, V_+\, (B - M(z))^{-1}\sqrt{V_+}. \end{aligned} \tag{3.71}$$

Using the identity

$$\big(B^* - M(z)^*\big)^{-1} - \big(B - M(z)\big)^{-1} = -2i(B^* - M(z)^*)^{-1} \cdot (|B_I| + M_I(z)) \cdot (B - M(z))^{-1},$$

we rewrite (3.71) as

$$\begin{aligned} I - W_+(z)^* W_+(z) =& \sqrt{V_+}(B^* - M^*(z))^{-1} \\ & \cdot (2|B_I| - V_+ + 2\mathrm{Im}\,(M(z))) \cdot (B - M(z))^{-1}\sqrt{V_+}. \end{aligned}$$

Since $V_+ \le 2|B_I|$ and $\mathrm{Im}\,(M(z)) \ge 0$ one gets $I - W_+(z)^* W_+(z) \ge 0$ for $z \in \mathbb{C}_+$, i.e. $W_+(\cdot)$ is contractive in $\mathbb{C}_+$. Hence $w_+(\cdot) = \det(W_+(\cdot))$ is contractive in $\mathbb{C}_+$ too, so that the factorization result (B.3) implies representation (3.69).

(ii) Let $V = V_+ - V_-$, $V_\pm \ge 0$. We set $B_- := B - iV_-$. Since $(B_-)_I = B_I - V_- \le 0$, the operator B_- is accumulative too. Using (2.11) one easily gets

$$\det(I + iV(B - M(z))^{-1}) = \frac{\det(I + iV_+(B_- - M(z))^{-1})}{\det(I + iV_-(B_- - M(z))^{-1})}, \quad z \in \mathbb{C}_+. \tag{3.72}$$

The assumption $V \le -2B_I$ yields $0 \le V_+ \le -2B_I + V_- \le -2B_I + 2V_- = -2(B_-)_I$. Therefore the statement (i) applies and gives that the determinant $\det(I + iV_+(B_- - M(z))^{-1})$ is a contractive analytic function in $\mathbb{C}_+$. Further, since $0 \le V_- \le -B_I + V_- = -(B_-)_I$, it follows from Lemma 3.12(i) that the denominator in (3.72) is an outer function. Lemma B.1 completes the proof. □

Theorem 3.22. *Let $\widetilde{A}', \widetilde{A} \in \mathrm{Ext}_A$ and let $\widetilde{A}$ be an m-accumulative extension such that $\rho(\widetilde{A}) \cap \mathbb{C}_- \ne \emptyset$ and $\{\widetilde{A}', \widetilde{A}\}$ is resolvent comparable. Further, let $\mathcal{Z} = \{(z_k, m_k)\}_k$ be the sequence consisting of eigenvalues $\{z_k\}_k$ in $\mathbb{C}_+$ and their algebraic multiplicities $\{m_k\}_k$. Then the following holds:*

(i) *There exists a boundary triplet* $\Pi_r = \{\mathcal{H}, \Gamma_0, \Gamma_1\}$ *for* A^**, which is regular for* $\{\widetilde{A}', \widetilde{A}\}$ *and such that* $\{\widetilde{A}', \widetilde{A}\} \in \mathfrak{D}^{\Pi_r}$*.*

(ii) *The sequence* $\mathcal{Z}$ *is a Blaschke sequence. For any (not necessarily regular) boundary triplet* Π *satisfying* $\{\widetilde{A}', \widetilde{A}\} \in \mathfrak{D}^{\Pi}$*, the perturbation determinant* $\Delta^{\Pi}_{\widetilde{A}'/\widetilde{A}}(\cdot)$ *admits a representation*

$$\Delta^{\Pi}_{\widetilde{A}'/\widetilde{A}}(z) = c\, \mathcal{B}(z, \mathcal{Z}) \exp\left\{\frac{1}{\pi}\int_{\mathbb{R}} \left(\frac{1}{t-z} - \frac{t}{1+t^2}\right) d\nu(t)\right\} e^{i\alpha z}, \quad z \in \mathbb{C}_+, \tag{3.73}$$

where $d\nu(\cdot) := \omega(\cdot)dt + id\mu(\cdot)$, $\mu(\cdot)$ *is a non-negative Borel measure satisfying* $\int \frac{1}{1+t^2} d\mu(t) < \infty$, $\omega(\cdot) \in L^1(\mathbb{R}; \frac{dt}{1+t^2})$ *is complex-valued,* $\alpha \geq 0$ *and* $c \in \mathbb{C}$*.*

(iii) *The following trace formula holds*

$$\begin{aligned} \operatorname{tr}\Big((\widetilde{A}' - z)^{-1} - (\widetilde{A} - z)^{-1}\Big) \\ = -2i \sum_k \frac{m_k \operatorname{Im}(z_k)}{(z - z_k)(z - \overline{z}_k)} - \frac{1}{\pi}\int_{\mathbb{R}} \frac{1}{(t-z)^2} d\nu(t) - i\alpha, \end{aligned} \tag{3.74}$$

$z \in \rho(\widetilde{A}') \cap \mathbb{C}_+$.

Proof. (i) Since $\widetilde{A}$ is m-accumulative, $\mathbb{C}_+ \subseteq \rho(\widetilde{A})$, and by the assumption of the theorem, $\rho(\widetilde{A}) \cap \mathbb{C}_\pm \neq \emptyset$. Therefore, by Theorem 2.6, there exists a boundary triplet $\Pi = \{\mathcal{H}, \Gamma_0, \Gamma_1\}$ for A^* which is regular for the pair $\{\widetilde{A}', \widetilde{A}\}$. By Definition 2.5, there exist bounded operators $B', B \in [\mathcal{H}]$, such that $\widetilde{A}' = A_{B'}$ and $\widetilde{A} = A_B$. Since $\{\widetilde{A}', \widetilde{A}\}$ are resolvent comparable one gets from Proposition 2.9 (ii) that $B' - B \in \mathfrak{S}_1(\mathcal{H})$. Hence $\{\widetilde{A}', \widetilde{A}\} \in \mathfrak{D}^{\Pi_r}$.

(ii) First, let $\Pi = \Pi_r$. By Proposition 2.3 (iii) the operator B is accumulative because $\widetilde{A}$ is m-accumulative. Hence $B'_R - B_R \in \mathfrak{S}_1(\mathcal{H})$ and $V := B'_I - B_I \in \mathfrak{S}_1(\mathcal{H})$. We set

$$C := B'_R + iB_I \tag{3.75}$$

and note that C is also accumulative, $C_I = B_I \leq 0$. Let $V := B'_I - B_I = V^*$ and let $V = V_+ - V_-$, $V_\pm \geq 0$, be its orthogonal decomposition. We set

$$D := C - i(V_+ + V_-) \tag{3.76}$$

and note that D is accumulative because so is C and $V_\pm \geq 0$. Since

$$B' - D = B'_R + iB'_I - B'_R - iB_I + i(V_+ + V_-) = 2iV_+ \in \mathfrak{S}_1(\mathcal{H}), \tag{3.77}$$

we get $\{\widetilde{A}', A_D\} \in \mathfrak{D}^{\Pi}$. Moreover combining (3.76) with the obvious inequality $B_I \leq 0 \leq V_-$ implies

$$2V_+ \leq -2B_I + 2V_+ + 2V_- = -2(B_I - V_+ - V_-) = -2D_I.$$

According to (3.77) and Definition 2.12 the perturbation determinant $\Delta^{\Pi}_{\widetilde{A}'/A_D}(\cdot)$ is well defined, $\Delta^{\Pi}_{\widetilde{A}'/A_D}(\cdot) = \det(I + 2iV_+(D - M(\cdot))^{-1})$. By Lemma 3.21 (i), $\Delta^{\Pi}_{\widetilde{A}'/A_D}(\cdot)$ is holomorphic and contractive in $\mathbb{C}_+$, hence admits a representation

$$\Delta^{\Pi}_{\widetilde{A}'/A_D}(z) = \varkappa\, \mathcal{B}(z, \mathfrak{Z}) \exp\left\{\frac{i}{\pi}\int_{\mathbb{R}} \left(\frac{1}{t-z} - \frac{t}{1+t^2}\right) d\mu(t)\right\} e^{i\alpha z}$$

μ is a *non-negative Borel measure* on $\mathbb{R}$ satisfying $\int_{\mathbb{R}} \frac{1}{1+t^2} d\mu(t) < \infty$, $\varkappa \in \mathbb{T}$, $\alpha \geq 0$, and $\mathcal{B}(\cdot, \mathfrak{Z})$ is the Blaschke product associated with the set $\mathfrak{Z} = \{(\mathfrak{z}_k, \mathfrak{m}_k)\}_k$ which consists of the zeros $\mathfrak{z}_k$ of $\Delta^{\Pi}_{\widetilde{A}'/A_D}(\cdot)$ in $\mathbb{C}_+$ and their multiplicities $\mathfrak{m}_k$. Notice that $\mathfrak{Z}$ is a Blaschke sequence. By Proposition 2.13 (iv) the set $\mathfrak{Z}$ coincides with $\mathcal{Z} = \{(z_k, m_k)\}_k$. Hence, $\mathcal{Z}$ is a Blaschke sequence too, which yields the representation

$$\Delta^{\Pi}_{\widetilde{A}'/A_D}(z) = \varkappa\, \mathcal{B}(z, \mathcal{Z}) \exp\left\{\frac{i}{\pi} \int_{\mathbb{R}} \left(\frac{1}{t-z} - \frac{t}{1+t^2}\right) d\mu(t)\right\} e^{i\alpha z} \tag{3.78}$$

The inclusions $\{\widetilde{A}', \widetilde{A}\} \in \mathfrak{D}^{\Pi}$ and $\{A_D, \widetilde{A}'\} \in \mathfrak{D}^{\Pi}$ imply $\{A_D, \widetilde{A}\} \in \mathfrak{D}^{\Pi}$. Moreover, by Proposition 2.3 (iii), A_D is m-accumulative because so is D. Therefore Theorem 3.13(ii) applies and yields the following representation

$$\Delta^{\Pi}_{A_D/\widetilde{A}}(z) = c_D\ \exp\left\{\frac{1}{\pi} \int_{\mathbb{R}} \left(\frac{1}{t-z} - \frac{t}{1+t^2}\right) \omega(t) dt\right\}, \quad z \in \mathbb{C}_+, \tag{3.79}$$

with a *complex-valued function* $\omega(\cdot) \in L^1(\mathbb{R}; \frac{dt}{1+t^2})$ and a constant $c_D \in \mathbb{C}$. Combining (3.78) with (3.79) and applying the chain rule (2.17), we arrive at representation (3.73) with $c := c_D \varkappa$ and $d\nu(t) = \omega(t)dt + id\mu(t)$.

The case of a non-regular boundary triplet Π satisfying the only condition $\{\widetilde{A}', \widetilde{A}\} \in \mathfrak{D}^{\Pi}$ is reduced to the previous one by applying the property (2.15).

(iii) Combining (3.73) with (2.16) formula (3.74) immediately follows. $\square$

Using the Riesz-Dunford functional calculus, we extend trace formula (3.74) to the case of analytic functions of the class $\mathcal{F}_+(\widetilde{A}, \widetilde{A}')$, cf. D.

Corollary 3.23. *Let the assumptions of Theorem 3.22 be satisfied. If* $\Phi \in \mathcal{F}_+(\widetilde{A}, \widetilde{A}')$*, then* $\Phi(\widetilde{A}') - \Phi(\widetilde{A}) \in \mathfrak{S}_1(\mathfrak{H})$ *and*

$$\operatorname{tr}(\Phi(\widetilde{A}') - \Phi(\widetilde{A})) = \sum_k m_k(\Phi(z_k) - \Phi(\overline{z}_k)) + \frac{1}{\pi} \int_{\mathbb{R}} \Phi'(t) d\nu(t) + i\alpha \operatorname{res}_{\infty}(\Phi). \tag{3.80}$$

Using the Dunford-Schwartz functional calculus, see D, the proof easily follows from Theorem 3.22 (iii).

3.4 Pairs of an extension and its adjoint

Next we consider perturbation determinants and trace formulas for pairs $\{\widetilde{A}, \widetilde{A}^*\}$ of proper extensions $\widetilde{A}, \widetilde{A}^* \in \operatorname{Ext}_A$ assuming that $\rho(\widetilde{A}) \cap \rho(\widetilde{A}^*) \neq \emptyset$.

It turns out that in this case the perturbation determinant coincides with one of the characteristic functions (CF) of the operator $\widetilde{A}$. For the precise definition of a characteristic function of an unbounded operator with non-empty resolvent set we refer to the papers [41] and [9, 10], the most relevant to our considerations. In what follows we need only a representation of CF by means of the Weyl function and boundary operator.

By Definition 2.4, for any almost solvable extension $\widetilde{A}$ of A there exists a boundary triplet $\Pi = \{\mathcal{H}, \Gamma_0, \Gamma_1\}$ for A^* which is regular for $\widetilde{A}$, i.e. $\widetilde{A} = A_B = A^* \upharpoonright \ker(\Gamma_1 - B\Gamma_0)$ and $B \in [\mathcal{H}]$. According to [12, 9, 10] one of the characteristic functions of the operator A_B admits a representation

$$W^{\Pi}_{\widetilde{A}}(z) = I + 2i|B_I|^{1/2}\big(B^* - M(z)\big)^{-1}|B_I|^{1/2}J, \quad z \in \rho(\widetilde{A}^*) \cap \rho(A_0), \tag{3.81}$$

where $B_I = J|B_I|$, $J = \operatorname{sign}(B_I)$, is the polar decomposition of $B_I := \operatorname{Im}(B)$. Formula (3.81) express the CF $W^{\Pi}_{\widetilde{A}}(\cdot)$ by means of the Weyl function $M(\cdot)$ and boundary operator B. In particular, it shows that $W^{\Pi}_{\widetilde{A}}(\cdot)$ takes values in $[\mathcal{H}]$ although for general $\widetilde{A}$ its values might be unbounded. Moreover, it is known (see [41, 9, 10]) (and can be extracted from (3.81)) that as in the case of bounded operators (cf. [6]) $W^{\Pi}_{\widetilde{A}}(\cdot)$ is J-contractive in $\mathbb{C}_+$ and J-expansive in $\mathbb{C}_-$. If $\widetilde{A} = A_B$ is m-dissipative, then, by Proposition 2.3 (iii), B is m-dissipative, $J = I$, and $W^{\Pi}_{\widetilde{A}}(\cdot)$ is contractive in $\mathbb{C}_+$.

Theorem 3.24. *Let $\widetilde{A} \in \operatorname{Ext}_A$ such that the pair $\{\widetilde{A}, \widetilde{A}^*\}$ is resolvent comparable. Further, let $\{z_k^+\}_k := \sigma_p(\widetilde{A}) \cap \mathbb{C}_+$ and $\{z_l^-\}_l = \sigma_p(\widetilde{A}) \cap \mathbb{C}_-$ and let m_k^+ and m_l^- be the algebraic multiplicities of the eigenvalues z_k^+ and z_l^-, respectively. Let $\mathcal{Z}_+ := \{(z_k^+, m_k^+)\}_k$ and $\mathcal{Z}_- := \{(z_l^-, m_l^-)\}_l$. Then the following holds:*

(i) *There exists a boundary triplet $\Pi_r = \{\mathcal{H}, \Gamma_0, \Gamma_1\}$ for A^*, which is regular for the pair $\{\widetilde{A}, \widetilde{A}^*\}$ and satisfies $\{\widetilde{A}, \widetilde{A}^*\} \in \mathfrak{D}^{\Pi}$. Moreover, for any regular boundary triplet Π for $\{\widetilde{A}, \widetilde{A}^*\}$ the perturbation determinant $\Delta^{\Pi}_{\widetilde{A}/\widetilde{A}^*}(\cdot)$ is given by*

$$\Delta^{\Pi}_{\widetilde{A}/\widetilde{A}^*}(z) = \det(W^{\Pi}_{\widetilde{A}}(z)), \qquad z \in \rho(\widetilde{A}^*) \cap \mathbb{C}_\pm, \tag{3.82}$$

where $W^{\Pi}_{\widetilde{A}}(\cdot)$ is the characteristic operator-valued function of $\widetilde{A}$ given by (3.81).

(ii) *The sequences $\mathcal{Z}_+$ and $\mathcal{Z}_-^* := \{(\overline{z}_l^-, m_l^-)\}_l$ are Blaschke sequences. For any (not necessarily regular) boundary triplet Π for $\{\widetilde{A}, \widetilde{A}^*\}$ such that $\{\widetilde{A}, \widetilde{A}^*\} \in \mathfrak{D}^{\Pi}$ the perturbation determinant $\Delta^{\Pi}_{\widetilde{A}/\widetilde{A}^*}(\cdot)$ admits a representation*

$$\Delta^{\Pi}_{\widetilde{A}/\widetilde{A}^*}(z) = \varkappa\, \frac{\mathcal{B}(z, \mathcal{Z}_+)}{\mathcal{B}(z, \mathcal{Z}_-^*)} \exp\left\{\frac{i}{\pi}\int_{\mathbb{R}} \left(\frac{1}{t-z} - \frac{t}{1+t^2}\right) d\mu(t)\right\} e^{i\alpha z}, \tag{3.83}$$

$z \in \rho(\widetilde{A}^*) \cap \mathbb{C}_+$, with a real-valued measure *μ on $\mathbb{R}$ satisfying $\int_{\mathbb{R}} \frac{1}{1+t^2}|d\mu(t)| < \infty$ and constants $\alpha \in \mathbb{R}$, $\varkappa \in \mathbb{T}$.*

(iii) *The following trace formula holds*

$$\begin{aligned} \operatorname{tr}\Big((\widetilde{A}^* - z)^{-1} & - (\widetilde{A} - z)^{-1}\Big) \\ & = 2i\sum_n \frac{m_n^+ \operatorname{Im}(z_n)}{(z - z_n)(z - \overline{z}_n)} + \frac{i}{\pi}\int_R \frac{1}{(t-z)^2} d\mu(t) + i\alpha, \end{aligned} \tag{3.84}$$

$z \in \rho(\widetilde{A}^*) \cap \rho(\widetilde{A}) \cap \mathbb{C}_+$, where $\{z_n\}_n = \sigma_p(\widetilde{A}) \cap (\mathbb{C} \setminus \mathbb{R})$ and m_n denotes the algebraic multiplicity of the eigenvalue z_n.

Proof. (i) Let $z_0 \in \rho(\widetilde{A}) \cap \rho(\widetilde{A}^*)$. Then $\overline{z}_0 \in \rho(\widetilde{A}) \cap \rho(\widetilde{A}^*)$ and, by [9] (cf. also Theorem 2.6), the extension $\widetilde{A}$ is almost solvable. Hence there is a regular boundary triplet Π_r for $\{\widetilde{A}, \widetilde{A}^*\}$. Let now Π be any regular boundary triplet for $\{\widetilde{A}, \widetilde{A}^*\}$. Then there is a bounded operator B such that $\widetilde{A} = A_B$ and $\widetilde{A}^* = A_{B^*}$. By Proposition 2.9(ii), the resolvent comparability of $\{\widetilde{A}, \widetilde{A}^*\}$ yields $B_I = (B - B^*)/2i = B_I^* \in \mathfrak{S}_1(\mathcal{H})$ which implies $\{\widetilde{A}, \widetilde{A}^*\} \in \mathfrak{D}^\Pi$.

Further, let $B_I := J|B_I|$ be the polar decomposition of B_I where $J = J^*$. Combining Definition (2.12) with the property (2.10) we get

$$\begin{aligned}\Delta^\Pi_{\widetilde{A}/\widetilde{A}^*}(z) &= \det\bigl(I + (B - B^*)(B^* - M(z))^{-1}\bigr)\\ &= \det(I + 2i\sqrt{|B_I|}(B^* - M(z))^{-1}\sqrt{|B_I|}J), \quad z \in \rho(\widetilde{A}^*) \cap \mathbb{C}_\pm.\end{aligned}$$

Combining this formula with (3.81) we arrive at (3.82).

(ii) First, let Π be a regular boundary triplet for $\{\widetilde{A}, \widetilde{A}^*\}$. Consider the spectral decomposition $B_I = B_I^+ - B_I^-$ of B_I, where $B_I^\pm$ are orthogonal, $B_I^\pm \ge 0$, and $B_I^\pm \in \mathfrak{S}_1(\mathcal{H})$. Let us consider the accumulative operator $C = B_R - i|B_I| = B_R - iB_I^+ - iB_I^-$ and the m-accumulative extension A_C. Clearly, $B - C = 2iB_I^+ \in \mathfrak{S}_1(\mathcal{H})$, so $\{\widetilde{A}, A_C\} \in \mathfrak{D}^\Pi$. Moreover, since $0 \le V_+ := 2B_I^+ \le -2\mathrm{Im}\,(B_1^*)$, Lemma 3.21(i) applies with C in place of B gives the representation

$$\begin{aligned}\Delta^\Pi_{\widetilde{A}/A_C}(z) &= \det(I + 2iB_I^+(C - M(z))^{-1})\\ &= \varkappa_+\, \mathcal{B}(z, \mathfrak{Z}_+) \exp\left\{\frac{i}{\pi}\int_{\mathbb{R}}\left(\frac{1}{t-z} - \frac{t}{1+t^2}\right) d\mu_+(t)\right\} e^{i\alpha_+ z},\end{aligned} \tag{3.85}$$

$z \in \mathbb{C}_+$. Here μ_+ is a *non-negative Borel measure* satisfying $\int_R \frac{1}{1+t^2}\, d\mu_+(t) < \infty$, $\varkappa_+ \in \mathbb{T}$, $\alpha_+ \ge 0$, and $\mathfrak{Z}_+ = \{(\mathfrak{z}_k^+, \mathfrak{m}_k^+)\}_k$ consists of the zeros $\mathfrak{z}_k^+$ of $\Delta^\Pi_{\widetilde{A}/A_C}(\cdot)$ in $\mathbb{C}_+$ and their multiplicities $\mathfrak{m}_k^+$. Obviously, $\mathfrak{Z}_+$ is a Blaschke sequence.

Next, consider the perturbation determinant $\Delta^\Pi_{\widetilde{A}^*/A_C}(\cdot)$. Again Lemma 3.21(i) yields the representation

$$\begin{aligned}\Delta^\Pi_{\widetilde{A}^*/A_C}(z) &= \det(I + 2iB_I^-(C - M(z))\\ &= \varkappa_-\, \mathcal{B}(z, \mathfrak{Z}_+^*) \exp\left\{\frac{i}{\pi}\int_{\mathbb{R}}\left(\frac{1}{t-z} - \frac{t}{1+t^2}\right) d\mu_-(t)\right\} e^{i\alpha_- z},\end{aligned} \tag{3.86}$$

$z \in \mathbb{C}_+$. Here μ_- is a *non-negative Borel measure* satisfying $\int_R \frac{1}{1+t^2} d\mu_-(t) < \infty$, $\varkappa_- \in \mathbb{T}$, $\alpha_- \ge 0$, and $\mathfrak{Z}_+^* = \{(\mathfrak{z}_l^*, \mathfrak{m}_l^*)\}_l$ consists of zeros $\mathfrak{z}_l^*$ of $\Delta^\Pi_{\widetilde{A}^*/A_C}(\cdot)$ in $\mathbb{C}_+$ and their multiplicities $\mathfrak{m}*_l$. Obviously, $\mathfrak{Z}_+^*$ is a Blaschke sequence. By Proposition 2.13 (iv) we get $\mathfrak{Z}_+^* = \mathcal{Z}_-^*$ which shows that $\mathcal{Z}_-^*$ is a Blaschke sequence and $\mathfrak{Z}_+^* = \mathcal{Z}_-^*$.

Combining (3.85) with (3.86) and applying the chain rule (cf. (2.17)) we get

$$\Delta^\Pi_{\widetilde{A}/\widetilde{A}^*}(z) = \frac{\Delta^\Pi_{\widetilde{A}/A_C}(z)}{\Delta^\Pi_{\widetilde{A}^*/A_C}(z)}, \qquad z \in \rho(\widetilde{A}^*) \cap \rho(\widetilde{A}) \cap \mathbb{C}_+, \tag{3.87}$$

and setting $c := \frac{\varkappa_+}{\varkappa_-}$, $\mu := \mu_+ - \mu_-$ and $\alpha := \alpha_+ - \alpha_-$ as well as $\mathfrak{Z}_+ = \mathcal{Z}_+$ and $\mathfrak{Z}_+^* = \mathcal{Z}_-^*$ we prove (3.83).

To verify (3.83) for arbitrary (not necessarily regular) boundary triplet Π satisfying the condition $\{\widetilde{A}, \widetilde{A}^*\} \in \mathfrak{D}^\Pi$ it remains to apply property (2.15).

(iii) Combining (3.85) with the property (2.16) we derive

$$\begin{aligned}\operatorname{tr}\Big((\widetilde{A}-z)^{-1} &-(A_C-z)^{-1}\Big)\\ &= -2i\sum_k \frac{m_k^+ \operatorname{Im}(z_k^+)}{(z-z_k^+)(z-\overline{z}_k^+)} - \frac{i}{\pi}\int_R \frac{1}{(t-z)^2}d\mu_+(t) - i\alpha_+,\end{aligned} \tag{3.88}$$

for $z \in \rho(\widetilde{A}) \cap \mathbb{C}_+$. Similarly combining (3.86) with the property (2.16) we get

$$\begin{aligned}\operatorname{tr}\Big((\widetilde{A}^*-z)^{-1} &-(A_C-z)^{-1}\Big)\\ &= -2i\sum_l \frac{m_l^- \operatorname{Im}(\overline{z}_l^-)}{(z-z_l^-)(z-\overline{z}_l^-)} - \frac{i}{\pi}\int_R \frac{1}{(t-z)^2}d\mu_-(t) - i\alpha_-\\ &= 2i\sum_l \frac{m_l^- \operatorname{Im}(z_l^-)}{(z-z_l^-)(z-\overline{z}_l^-)} - \frac{i}{\pi}\int_R \frac{1}{(t-z)^2}d\mu_-(t) - i\alpha_-\end{aligned} \tag{3.89}$$

for $z \in \rho(\widetilde{A}^*) \cap \mathbb{C}_+$. Subtracting (3.88) from (3.89) we arrive at (3.84). □

Corollary 3.25. *Let the assumptions of Theorem 3.24 be satisfied. Further, let* $\{z_n\}_n = \sigma_p(\widetilde{A}) \cap (\mathbb{C}\backslash\mathbb{R})$ *and let* m_n *be the algebraic multiplicity of the eigenvalue* z_n. *If* $\Phi \in \mathcal{F}_+(\widetilde{A}, \widetilde{A}^*)$, *then* $\Phi(\widetilde{A}) - \Phi(\widetilde{A}^*) \in \mathfrak{S}_1(\mathfrak{H})$ *and*

$$\operatorname{tr}(\Phi(\widetilde{A}) - \Phi(\widetilde{A}^*)) = \sum_n m_n(\Phi(z_n) - \Phi(\overline{z}_n)) + \frac{i}{\pi}\int_{\mathbb{R}} \Phi'(t)d\mu(t) + i\alpha \operatorname{res}_\infty(\Phi). \tag{3.90}$$

Using the Dunford-Schwartz functional calculus, see D, the proof easily follows from Theorem 3.24 (iii).

Remark 3.26. Corollary 3.25 generalizes the known result of Adamyan and Pavlov [2] (see formula (1.5)) and coincides with it if $\widetilde{A}$ is a m-dissipative operator satisfying $\rho(\widetilde{A}) \cap \mathbb{C}_+ \neq \emptyset$. The proof in [2] is based on a functional model of m-dissipative operator [42]. On the other hand, Corollary 3.25 extends also Krein's formula (1.7) established in [26] for additive perturbations.

Next we complete and simplify Theorem 3.24 assuming in addition, that the resolvent of an extension is compact.

Theorem 3.27. *Let the assumptions of Theorem 3.24 be satisfied. If in addition* $(\widetilde{A} - \zeta)^{-1} \in \mathfrak{S}_\infty(\mathfrak{H})$, $\zeta \in \rho(\widetilde{A})$, *then the following holds:*

(i) *If* $\{\widetilde{A}, \widetilde{A}^*\} \in \mathfrak{D}^\Pi$, *then the perturbation determinant* $\Delta^\Pi_{\widetilde{A}/\widetilde{A}^*}(\cdot)$ *is holomorphic in a neighborhood of the real line* $\mathbb{R}$ *and*

$$|\Delta^\Pi_{\widetilde{A}/\widetilde{A}^*}(x)| = 1 \qquad \textit{for} \qquad x \in \mathbb{R}. \tag{3.91}$$

(ii) *If* $\{\widetilde{A},\widetilde{A}^*\}\in\mathfrak{D}^\Pi$*, then representation* (3.83) *is simplified to*

$$\Delta^\Pi_{\widetilde{A}/\widetilde{A}^*}(z) = c\,\frac{\mathcal{B}(z,\mathcal{Z}_+)}{\mathcal{B}(z,\mathcal{Z}_-^*)}e^{i\alpha z}, \qquad \alpha\in\mathbb{R}; \qquad z\in\rho(\widetilde{A})\cap\mathbb{C}_+. \tag{3.92}$$

(iii) *The following trace formula holds*

$$\mathrm{tr}\,\Big((\widetilde{A}^*-z)^{-1}-(\widetilde{A}-z)^{-1}\Big) = 2i\sum_n \frac{m_n \mathrm{Im}\,(z_n)}{(z-z_n)(z-\overline{z}_n)} + i\alpha, \tag{3.93}$$

for $z\in\rho(\widetilde{A}^*)\cap\rho(\widetilde{A})$*. In particular, if* $a=\overline{a}\in\rho(\widetilde{A})$*, then*

$$\alpha/2 = \mathrm{tr}\,(\mathrm{Im}\,(\widetilde{A}^*-a)^{-1}) - \sum_n \mathrm{Im}\,\left(\frac{m_n}{a-z_n}\right), \tag{3.94}$$

where $\{z_n\}_n=\sigma_p(\widetilde{A})\cap(\mathbb{C}\setminus\mathbb{R})$ *and* m_n *denotes the algebraic multiplicity of* z_n*.*

Proof. (i) Let Π be a boundary triplet for A^* regular for the pair $\{\widetilde{A},\widetilde{A}^*\}$, so that $\widetilde{A}=A_B=A^*\upharpoonright\ker(\Gamma_1-B\Gamma_0)$ and $\widetilde{A}^*=A_{B^*}$ where $B\in[\mathcal{H}]$. Therefore, the real part $\widetilde{A}_R$ of $\widetilde{A}$ is well defined, $\widetilde{A}_R:=A_{B_R}$. Since $\{\widetilde{A},\widetilde{A}^*\}$ is resolvent comparable we get $B_I\in\mathfrak{S}_1(\mathcal{H})$ and the perturbation determinants $\Delta^\Pi_{\widetilde{A}_R/\widetilde{A}}(\cdot)$ and $\Delta^\Pi_{\widetilde{A}_R/\widetilde{A}^*}(\cdot)$ are well defined,

$$\Delta^\Pi_{\widetilde{A}_R/\widetilde{A}}(z) = \det\big(I+(B_R-B)(B-M(z))^{-1}\big) = \det(I-iB_I(B-M(z))^{-1}),$$

$z\in\rho(\widetilde{A})\cap\rho(A_0)$, and $\Delta^\Pi_{\widetilde{A}_R/\widetilde{A}^*}(z)=\det(I+iB_I(B^*-M(z))^{-1})$ for $z\in\rho(\widetilde{A}^*)\cap\rho(A_0)$. Moreover, by Proposition 2.13 (i), both $\Delta^\Pi_{\widetilde{A}_R/\widetilde{A}}(\cdot)$ and $\Delta^\Pi_{\widetilde{A}_R/\widetilde{A}^*}(\cdot)$ admit holomorphic continuation from $\rho(\widetilde{A})\cap\rho(\widetilde{A}^*)\cap\rho(A_0)$ to $\rho(\widetilde{A})$ and $\rho(\widetilde{A}^*)$, respectively. Since the resolvents of $\widetilde{A}$ and $\widetilde{A}^*$ are compact, the determinants $\Delta^\Pi_{\widetilde{A}_R/\widetilde{A}}(\cdot)$ and $\Delta^\Pi_{\widetilde{A}_R/\widetilde{A}^*}(\cdot)$ are meromorphic. According to Proposition 2.13 (vi) we get $\Delta^\Pi_{\widetilde{A}_R/\widetilde{A}^*}(z)=\overline{\Delta^\Pi_{\widetilde{A}_R/\widetilde{A}}(\overline{z})}$ for $z\in\rho(\widetilde{A}^*)$. In particular,

$$\Delta^\Pi_{\widetilde{A}_R/\widetilde{A}^*}(x) = \overline{\Delta^\Pi_{\widetilde{A}_R/\widetilde{A}}(x)}, \qquad x\in\rho(\widetilde{A})\cap\rho(\widetilde{A}^*)\cap\mathbb{R}=\rho(\widetilde{A}^*)\cap\mathbb{R}=\rho(\widetilde{A})\cap\mathbb{R}.$$

Using this identity and applying the chain rule this yields

$$\Delta^\Pi_{\widetilde{A}/\widetilde{A}^*}(x) = \frac{\overline{\Delta^\Pi_{\widetilde{A}_R/\widetilde{A}}(x)}}{\Delta^\Pi_{\widetilde{A}_R/\widetilde{A}}(x)}, \qquad x\in\rho(\widetilde{A})\cap\rho(\widetilde{A}^*)\cap\rho(\widetilde{A}_R)\cap\mathbb{R}. \tag{3.95}$$

Hence $|\Delta^\Pi_{\widetilde{A}/\widetilde{A}^*}(x)|=1$ for $x\in\rho(\widetilde{A})\cap\rho(\widetilde{A}_R)\cap\mathbb{R}$. Since $(\widetilde{A}_R-z)^{-1}\in\mathfrak{S}_\infty(\mathfrak{H})$, $z\in\rho(\widetilde{A})$, the operator $\widetilde{A}_R$ has discrete spectrum. Thus, $|\Delta^\Pi_{\widetilde{A}/\widetilde{A}^*}(x)|=1$ for x outside a discrete set $(\sigma(\widetilde{A}_R)\cup\sigma(\widetilde{A}))\cap\mathbb{R}$. Hence any possible real pole x_0 of the meromorphic function $\Delta^\Pi_{\widetilde{A}/\widetilde{A}^*}(\cdot)$ is removable and $|\Delta^\Pi_{\widetilde{A}/\widetilde{A}^*}(x)|=1$ for any $x\in\mathbb{R}$. So, $\Delta^\Pi_{\widetilde{A}/\widetilde{A}^*}(\cdot)$ is holomorphic in a neighborhood of $\mathbb{R}$.

(ii) We set $C := B_R - i|B_I|$ and introduce the extension A_C which is m-accumulative because C is accumulative. Since $\{\widetilde{A}, \widetilde{A}^*\}$ is resolvent comparable we get from Proposition 2.9 (ii) that $B_I \in \mathfrak{S}_1(\mathcal{H})$, so that $B - C = 2iB_I^+ \in \mathfrak{S}_1(\mathcal{H})$, and the perturbation determinant $\Delta^{\Pi}_{\widetilde{A}/A_C}(\cdot)$ is well defined,

$$F_+(z) := \Delta^{\Pi}_{\widetilde{A}/A_C}(z) = \det(I + 2iB_I^+(C - M(z))^{-1}), \quad z \in \rho(\widetilde{A}) \cap \rho(A_0).$$

By Lemma 3.21(i) $F(\cdot)$ is holomorphic and contractive in $\mathbb{C}_+$. Hence it admits the multiplicative representation (3.85) (cf. the proof of Theorem 3.24(ii)).

Further, by Proposition 2.9 (ii), $(\widetilde{A}-\zeta)^{-1}-(A_C-\zeta)^{-1} \in \mathfrak{S}_1(\mathfrak{H})$, $\zeta \in \rho(\widetilde{A})\cap\rho(A_C)$, because $B_I^+ \in \mathfrak{S}_1(\mathcal{H})$. Combining this relation with the assumption $(\widetilde{A} - \zeta)^{-1} \in \mathfrak{S}_\infty(\mathfrak{H})$ yields $(A_C - \zeta)^{-1} \in \mathfrak{S}_\infty(\mathfrak{H})$ i.e., the spectrum of A_C is discrete, $\sigma(A_C) = \sigma_d(A_C)$. Hence, the perturbation determinant $\Delta^{\Pi}_{\widetilde{A}/\widetilde{A}'}(\cdot)$ is holomorphic in $\mathbb{C}_+$ and meromorphic in $\mathbb{C}$. In particular, $F_+(\cdot)$ admits a holomorphic continuation through $\mathbb{R} \setminus \sigma(A_C)$ where $\sigma(A_C) \cap \mathbb{R}$ is a discrete set. Clearly, the same is valid with respect to $F_-(\cdot) := \Delta^{\Pi}_{\widetilde{A}^*/A_C}(\cdot)$ which is also holomorphic and contractive in $\mathbb{C}_+$. In particular, the function $F_-(\cdot)$ admits the representation (3.86).

Let $I_{F_\pm}(\cdot)$ and $\mathcal{O}_{F_\pm}(\cdot)$ be the inner and outer factors, respectively, of the holomorphic contractive function $F_\pm(\cdot)$ in $\mathbb{C}_+$, cf. B. By [14, Theorem II.6.3], the inner and outer factors $I_{F_\pm}(\cdot)$ and $\mathcal{O}_{F_\pm}(\cdot)$ also admit holomorphic continuation through $\mathbb{R} \setminus \sigma(A_C)$. Applying the Fatou theorem to representations (3.85) and (3.86) one gets

$$\lim_{y\to+0} |\Delta^{\Pi}_{\widetilde{A}/A_C}(x + iy)| = e^{-\mu_+'(x)} \quad \text{and} \quad \lim_{y\to+0} |\Delta^{\Pi}_{\widetilde{A}^*/A_C}(x + iy)| = e^{-\mu_-'(x)}$$

for a.e. $x \in \mathbb{R}$. Combining these relations with representation (3.87) this yields

$$|\Delta^{\Pi}_{\widetilde{A}/\widetilde{A}^*}(x)| = e^{-(\mu_+'(x)-\mu_-'(x))} \qquad \text{for a.e.} \quad x \in \mathbb{R}.$$

Since according to (3.91) $|\Delta^{\Pi}_{\widetilde{A}/\widetilde{A}^*}(x)| = 1$ for $x \in \mathbb{R}$ we get $\mu_+'(x) = \mu_-'(x)$ for a.e. $x \in \mathbb{R}$, i.e. $\mu_+^{ac} = \mu_-^{ac}$. Hence $\mathcal{O}_{F_+}(z) = \mathcal{O}_{F_-}(z)$, $z \in \mathbb{C}_+$, and representation (3.83) is reduced to

$$\Delta^{\Pi}_{\widetilde{A}/\widetilde{A}^*}(z) = \frac{F_+(z)}{F_-(z)} = \frac{I_{F_+}(z)}{I_{F_-}(z)} = \frac{\varkappa_+}{\varkappa_-}\frac{\mathcal{B}(z, \mathfrak{Z}_+)}{\mathcal{B}(z, \mathfrak{Z}_+^*)}\frac{S_{F_+}(z)}{S_{F_-}(z)}e^{i(\alpha_+-\alpha_-)}, \tag{3.96}$$

$z \in \rho(\widetilde{A}^*) \cap \mathbb{C}_+$. Recall that $\mathfrak{Z}_+ = \mathcal{Z}_+$ and $\mathfrak{Z}_+^* = \mathcal{Z}_-^*$ which yields $\mathcal{B}(\cdot, \mathfrak{Z}_+) = \mathcal{B}(\cdot, \mathcal{Z}_+)$ and $\mathcal{B}(\cdot, \mathfrak{Z}_+^*) = \mathcal{B}(\cdot, \mathcal{Z}_-^*)$. Since the spectrum of $\widetilde{A}$ is discrete, $\mathcal{B}(\cdot, \mathcal{Z}_+)$ and $\mathcal{B}(\cdot, \mathcal{Z}_-^*)$ admit a holomorphic continuation in a neighborhood of the real line $\mathbb{R}$. Furthermore, according to the statement (i), $\Delta^{\Pi}_{\widetilde{A}/\widetilde{A}^*}(\cdot)$ possesses the same property. Therefore representation (3.96) implies that $\frac{S_{F_+}(\cdot)}{S_{F_-}(\cdot)}$ also admits a holomorphic continuation in a neighborhood of $\mathbb{R}$.

Since the Blaschke products $\mathcal{B}(\cdot, \mathcal{Z}_+)$ and $\mathcal{B}(\cdot, \mathcal{Z}_-^*)$ admit a holomorphic continuation through $\mathbb{R} \setminus \sigma_p(A_C)$, the singular factor $S_{F_\pm}(\cdot)$ (cf. (B.4)) possesses this property too. By [14, Theorem II.6.2] the singular part $\mu_\pm^s$ of the measure $\mu_\pm$ is supported on $\mathbb{R} \cap \sigma(A_C)$. Thus, the singular continuous part $\mu_\pm^{sc}$ of the measure $\mu_\pm$ is missing, i.e. $\mu_\pm^{sc} \equiv 0$, $\mu_\pm^s$ is atomic, supported on

$\sigma(A_C)$ and by (B.4),

$$S(z) := \frac{S_{F_+}(z)}{S_{F_-}(z)} = \exp\left\{\frac{i}{\pi}\sum_{t_k\in\sigma(\widetilde{A}')\cap\mathbb{R}}\left(\frac{1}{t_k - z} - \frac{t_k}{1+t_k^2}\right)(\mu_+^s(\{t_k\}) - \mu_-^s(\{t_k\}))\right\},$$

$z \in \mathbb{C}_+ \cup \mathbb{R} \setminus \sigma(A_C)$. It is easily seen that $S(\cdot)$ is continuous at t_k if and only if $\mu_+^s(\{t_k\}) = \mu_-^s(\{t_k\})$, $t_k \in \sigma(A_C)\cap\mathbb{R}$. Indeed, $S(t_k+i0) = 0$ if $\mu_+^s(\{t_k\}) > \mu_-^s(\{t_k\})$ and $S(t_k+i0) = \infty$ if $\mu_+^s(\{t_k\}) < \mu_-^s(\{t_k\})$. This contradicts the equality $|S(x)| = 1$ which holds for all $x \in \mathbb{R}$ whenever $S(\cdot)$ is continuous on $\mathbb{R}$. Thus, we arrive at representation (3.92) with $c := \frac{\varkappa_+}{\varkappa_-}$ and $\alpha := \alpha_+ - \alpha_-$.

To prove (3.92) for any (not necessarily regular) boundary triplet Π satisfying $\{\widetilde{A}^*, \widetilde{A}\} \in \mathfrak{D}^\Pi$ it remains to apply Proposition 2.13 (ii).

(iii) Formula (3.93) immediately follows from (3.84) with $\mu = 0$. Formula (3.94) follows from (3.93) setting $z = a \in \mathbb{R}$. □

Corollary 3.28. *Assume the conditions of Theorem 3.24 and let* $(\widetilde{A} - \zeta)^{-1} \in \mathfrak{S}_\infty(\mathfrak{H})$, $\zeta \in \rho(\widetilde{A})$. *Further, let* $\{z_n\}_n = \sigma_p(\widetilde{A}) \cap (\mathbb{C}\setminus\mathbb{R})$ *and let* m_n *be the algebraic multiplicity of the eigenvalue* z_n. *If* $\Phi \in \mathcal{F}_+(\widetilde{A}, \widetilde{A}^*)$, *then* $\Phi(\widetilde{A}^*) - \Phi(\widetilde{A}) \in \mathfrak{S}_1(\mathfrak{H})$ *and*

$$\mathrm{tr}\,(\Phi(\widetilde{A}) - \Phi(\widetilde{A}^*)) = \sum_n m_n(\Phi(z_n) - \Phi(\overline{z}_n)) + i\alpha\,\mathrm{res}_\infty(\Phi).$$

Proof. The result is immediate from Corollary 3.25 since under our assumptions $\mu \equiv 0$. □

Remark 3.29.

(i) If $\widetilde{A}$ is m-dissipative and has discrete spectrum, then due to (3.91) the perturbation determinant $\Delta_{\widetilde{A}/\widetilde{A}^*}(\cdot)$ is an inner function in $\mathbb{C}_+$.

In contrast to this fact, in the non-dissipative case the perturbation determinant $\Delta^\Pi_{\widetilde{A}/\widetilde{A}^*}(\cdot)$ admits the representation $\Delta^\Pi_{\widetilde{A}/\widetilde{A}^*}(z) = \frac{F_+(z)}{F_-(z)}$, $z \in \mathbb{C}_+$, where the numerator and the denominator might really have outer factors despite of the analyticity of both determinants on the real line and the necessary condition $|\Delta^\Pi_{\widetilde{A}/\widetilde{A}^*}(x)| = 1$ for $x \in \mathbb{R}$ (cf. (3.91)).

(ii) Notice that the non-dissipative operator $\widetilde{A}$ might have real eigenvalues even if it is completely non-selfadjoint. However these eigenvalues do not appear neither in representation (3.92) nor in the trace formula (3.93). This fact is not surprising since if $\lambda_0 = \overline{\lambda}_0 \in \sigma_p(\widetilde{A})$ is a normal eigenvalue, then $\lambda_0 \in \sigma_p(\widetilde{A}^*)$ and $\dim\ker(\widetilde{A} - \lambda_0) = \dim\ker(\widetilde{A}^* - \lambda_0)$ and these zeros cancel out in the representation (3.92). Due to formula (3.82) such eigenvalues do not appear in the determinant of the characteristic function $W^\Pi_{\widetilde{A}}(\cdot)$. In this connection we mention the paper [43] where it is shown that even singular factors cancel in a formula for the determinant of the the characteristic function.

Corollary 3.30. *Assume the conditions of Theorem 3.24 and let* $(\widetilde{A} - \zeta)^{-1} \in \mathfrak{S}_1(\mathfrak{H})$, $\zeta \in \rho(\widetilde{A})$. *If* $\widetilde{A}$ *is* m*-dissipative, then the root vector system of* $\widetilde{A}$ *is complete if and only if* $\alpha = 0$.

Proof. Since the spectrum of $\widetilde{A}$ is discrete there exists a real $a \in \rho(\widetilde{A}) \cap \rho(\widetilde{A}^*)$. Clearly, $R := (\widetilde{A}^* - a)^{-1}$ is a compact dissipative operator. Since $\{\widetilde{A}, \widetilde{A}^*\}$ is resolvent comparable one has $R_I := \mathrm{Im}\,(R)$ is a trace class operator. The result is now immediate by combining the trace formula (3.94) for R with the classical Livsic theorem (see [18, Theorem V.2.1]). □

3.5 Pairs of an extension with one m-dissipative operator

Here certain results of Sections 3.3 and 3.4 are applied to prove a counterpart of Theorem 3.22 with $\widetilde{A}$ to be m-dissipative instead of m-accumulative.

Theorem 3.31. *Let $\widetilde{A}', \widetilde{A} \in \mathrm{Ext}\,_A$ and let $\widetilde{A}$ be an m-dissipative extension such that $\rho(\widetilde{A}') \cap \rho(\widetilde{A}^*) \cap \rho(\widetilde{A}) \cap \mathbb{C}_+ \neq \emptyset$. Further, let the pairs $\{\widetilde{A}', \widetilde{A}\}$ and $\{\widetilde{A}', \widetilde{A}^*\}$ be resolvent comparable. Let $\{z'_k\}_k = \sigma(\widetilde{A}') \cap \mathbb{C}_+ = \sigma_p(\widetilde{A}') \cap \mathbb{C}_+$, $\{z_l\}_l = \sigma(\widetilde{A}) \cap \mathbb{C}_+ = \sigma_p(\widetilde{A}) \cap \mathbb{C}_+$ and let m'_k, m_l be the algebraic multiplicities of the eigenvalues z'_k and z_l, respectively. Let $\mathcal{Z}' := \{(z'_k, m'_k)\}_k$ and $\mathcal{Z} := \{(z_l, m_l)\}_l$. Then the following holds:*

(i) *There exists a boundary triplet $\Pi_r = \{\mathcal{H}, \Gamma_0, \Gamma_1\}$ for A^*, which is regular for $\{\widetilde{A}', \widetilde{A}^*, \widetilde{A}\}$ and such that $\{\widetilde{A}', \widetilde{A}\} \in \mathfrak{D}^{\Pi_r}$, $\{\widetilde{A}', \widetilde{A}^*\} \in \mathfrak{D}^{\Pi_r}$ and $\{\widetilde{A}, \widetilde{A}^*\} \in \mathfrak{D}^{\Pi_r}$.*

(ii) *The sequences $\mathcal{Z}'$ and $\mathcal{Z}$ are Blaschke sequences. For any (not necessarily regular) boundary triplet Π satisfying $\{\widetilde{A}', \widetilde{A}\} \in \mathfrak{D}^{\Pi}$, the perturbation determinant $\Delta^{\Pi}_{\widetilde{A}'/\widetilde{A}}(\cdot)$ admits a representation*

$$\Delta^{\Pi}_{\widetilde{A}'/\widetilde{A}}(z) = c\frac{\mathcal{B}(z,\mathcal{Z}')}{\mathcal{B}(z,\mathcal{Z})} \exp\left\{\frac{1}{\pi}\int_{\mathbb{R}} \left(\frac{1}{t-z} - \frac{t}{1+t^2}\right) d\mu(t)\right\} e^{i\alpha z}, \tag{3.97}$$

$z \in \rho(\widetilde{A}) \cap \mathbb{C}_+$, where μ is a complex-valued Borel measure on $\mathbb{R}$ satisfying $\int_{\mathbb{R}} \frac{1}{1+t^2}|d\mu(t)| < \infty$, $c \in \mathbb{C}$ and $\alpha \in \mathbb{R}$.

(iii) *The following trace formula holds*

$$\begin{aligned} \mathrm{tr}\,\left((\widetilde{A}'-z)^{-1} - (\widetilde{A}-z)^{-1}\right) &= -2i\sum_k \frac{m_k \mathrm{Im}\,(z'_k)}{(z-z'_k)(z-\overline{z}'_k)} \\ &+ 2i\sum_l \frac{m_l \mathrm{Im}\,(z_l)}{(z-z_l)(z-\overline{z}_l)} - \frac{1}{\pi}\int_{\mathbb{R}} \frac{1}{(t-z)^2} d\mu(t) - i\alpha, \end{aligned} \tag{3.98}$$

$z \in \rho(\widetilde{A}') \cap \rho(\widetilde{A}) \cap \mathbb{C}_+$.

Proof. (i) Let $z_1 \in \rho(\widetilde{A}') \cap \rho(\widetilde{A}^*) \cap \rho(\widetilde{A})$. Since the pairs $\{\widetilde{A}', \widetilde{A}\}$ and $\{\widetilde{A}', \widetilde{A}^*\}$ are resolvent comparable, the pair $\{\widetilde{A}, \widetilde{A}^*\}$ is resolvent comparable too. First, let $z_1 \in \mathbb{C}_+$. Then $z_2 := \overline{z_1} \in \rho(\widetilde{A})$ because $\widetilde{A}$ is m-dissipative. Hence the assumptions of Theorem 2.6 are met and there exists a boundary triplet Π_r for A^* which is regular for the system $\{\widetilde{A}', \widetilde{A}, \widetilde{A}^*\}$.

Next, if $z_1 \in \mathbb{C}_-$, then $z_2 := \overline{z_1} \in \rho(\widetilde{A}^*)$ since $\widetilde{A}^*$ is m-accumulative. Again, by Theorem 2.6, there exists a boundary triplet Π_r for A^* which is regular for the system $\{\widetilde{A}', \widetilde{A}, \widetilde{A}^*\}$. The latter means that $\widetilde{A}' = A_{B'}$, $\widetilde{A} = A_B$ and $\widetilde{A}^* = A_{B^*}$ with bounded operators B' and B. By Proposition 2.9 (ii), $B' - B \in \mathfrak{S}_1(\mathcal{H})$ and $B' - B^* \in \mathfrak{S}_1(\mathcal{H})$ which implies $\{\widetilde{A}', \widetilde{A}\} \in \mathfrak{D}^{\Pi_r}$, $\{\widetilde{A}', \widetilde{A}^*\} \in \mathfrak{D}^{\Pi_r}$, and $\{\widetilde{A}, \widetilde{A}^*\} \in \mathfrak{D}^{\Pi_r}$.

(ii) According to the chain rule (see (2.17))

$$\Delta^{\Pi_r}_{\widetilde{A}'/\widetilde{A}}(z) = \frac{\Delta^{\Pi_r}_{\widetilde{A}'/\widetilde{A}^*}(z)}{\Delta^{\Pi_r}_{\widetilde{A}/\widetilde{A}^*}(z)}, \quad z \in \rho(\widetilde{A}^*) \cap \rho(\widetilde{A}). \tag{3.99}$$

Since $\widetilde{A}^*$ is m-accumulative and $\rho(\widetilde{A}^*) \cap \mathbb{C}_- \neq \emptyset$, Theorem 3.22 (ii) ensures the following representation

$$\Delta^{\Pi_r}_{\widetilde{A}'/\widetilde{A}^*}(z) = c'\mathcal{B}(z, \mathcal{Z}') \exp\left\{\int_{\mathbb{R}} \left(\frac{1}{t-z} - \frac{t}{1+t^2}\right) d\mu'(t)\right\} e^{i\alpha' z}, \quad z \in \mathbb{C}_+ \tag{3.100}$$

where $\mu'(\cdot)$ is a complex-valued Borel measure satisfying $\int_{\mathbb{R}} \frac{1}{1+t^2}|d\mu'(t)| < \infty$, $\alpha' \geq 0$ and $c' \in \mathbb{C}$.

On the other hand, since the pair $\{\widetilde{A}, \widetilde{A}^*\}$ is resolvent comparable, Theorem 3.24 (ii) implies the representation

$$\Delta^{\Pi_r}_{\widetilde{A}/\widetilde{A}^*}(z) = \varkappa^*\mathcal{B}(z, \mathcal{Z}) \exp\left\{\frac{i}{\pi}\int_{\mathbb{R}} \left(\frac{1}{t-z} - \frac{t}{1+t^2}\right) d\mu^*(t)\right\} e^{i\alpha^* z}, \tag{3.101}$$

$z \in \mathbb{C}_+$, with constants $\alpha^* \in \mathbb{R}$, $\varkappa^* \in \mathbb{T}$ and a real-valued measure μ^* satisfying $\int_{\mathbb{R}} \frac{1}{1+t^2}|d\mu^*(t)| < \infty$. Note that now formula (3.83) is simplified since the operator $\widetilde{A}$ is m-dissipative, hence the factor $\mathcal{B}_-(\cdot, \mathcal{Z}^*_-) \equiv 1$.

Inserting (3.100) and (3.101) into (3.99) we arrive at representation (3.97) where $c := \frac{c'}{\varkappa^*} \in \mathbb{C}$, $\alpha := \alpha' - \alpha^* \in \mathbb{R}$, and $\mu := \mu' - i\mu^*$ is a complex-valued measure satisfying $\int_R \frac{1}{1+t^2}|d\mu(t)| < \infty$. If Π is not regular but $\{\widetilde{A}', \widetilde{A}\} \in \mathfrak{D}^{\Pi}$, Proposition 2.13 (ii) completes the proof.

(iii) According to Theorem 3.22 (iii), the following trace formula holds

$$\begin{aligned} &\operatorname{tr}\left((\widetilde{A}' - z)^{-1} - (\widetilde{A}^* - z)^{-1}\right) \\ &\qquad = -2i\sum_k \frac{m'_k \operatorname{Im}(z'_k)}{(z - z'_k)(z - \overline{z}'_k)} - \frac{1}{\pi}\int_{\mathbb{R}} \frac{1}{(t-z)^2} d\mu'(t) - i\alpha', \end{aligned} \tag{3.102}$$

$z \in \rho(\widetilde{A}')$. At the same time, by Theorem 3.24 (iii),

$$\begin{aligned} &\operatorname{tr}\left((\widetilde{A}^* - z)^{-1} - (\widetilde{A} - z)^{-1}\right) \\ &\qquad = 2i\sum_l \frac{m_l \operatorname{Im}(z_l)}{(z - z_l)(z - \overline{z}_l)} + \frac{i}{\pi}\int_{\mathbb{R}} \frac{1}{(t-z)^2} d\mu^*(t) + i\alpha^*, \end{aligned} \tag{3.103}$$

$z \in \rho(\widetilde{A}) \cap \mathbb{C}_+$. Taking a sum we arrive at (3.98). □

Corollary 3.32. *Let the assumptions of Theorem 3.31 be satisfied. If $\Phi \in \mathcal{F}_+(\widetilde{A}, \widetilde{A}')$, then $\Phi(\widetilde{A}') - \Phi(\widetilde{A}) \in \mathfrak{S}_1(\mathfrak{H})$ and*

$$\begin{aligned}
&\mathrm{tr}\,(\Phi(\widetilde{A}') - \Phi(\widetilde{A})) \\
&= \sum_k m'_k(\Phi(z'_k) - \Phi(\overline{z}'_k)) - \sum_l m_l(\Phi(z_l) - \Phi(\overline{z}_l)) + \frac{1}{\pi}\int_{\mathbb{R}} \Phi'(t)d\mu(t) + i\alpha\,\mathrm{res}_\infty(\Phi).
\end{aligned}$$

The proof is immediate by combining Corollaries 3.23 and 3.25.

4 Trace formulas for trace class perturbations

Here we consider the case of additive trace class perturbations of m-accumulative operators. We clarify and improve certain results of Section 3.

4.1 The pairs of m-accumulative operators

We start with two technical statements.

Lemma 4.1. *Assume that H is m-accumulative operators in $\mathfrak{H}$. If $V \in \mathfrak{S}_1(\mathfrak{H})$, then*

$$\lim_{y\to\infty} y^2 \mathrm{tr}\left((H' - iy)^{-1}V(H - iy)^{-1}\right), = -\mathrm{tr}\,(V). \tag{4.1}$$

where $H' := H + V$.

Proof. Let $Z(y) := y^2(H - iy)^{-1}(H' - iy)^{-1}$, $y \in \mathbb{R}_+$. Since H and H' are m-accumulative,

$$\text{s-}\lim_{y\to\infty} y(H - iy)^{-1} = iI \quad \text{and} \quad \text{s-}\lim_{y\to\infty} y(H' - iy)^{-1} = iI,$$

and hence $\text{s-lim}_{y\to\infty} Z(y) = -I$. Since $V \in \mathfrak{S}_1(\mathfrak{H})$, the latter implies $\mathfrak{S}_1$-convergence by [18, Theorem III.6.3], i.e. $\lim_{y\to\infty} \|Z(y)V + V\|_{\mathfrak{S}_1} = 0$. Hence

$$\lim_{y\to\infty} y^2 \mathrm{tr}\left((H' - iy)^{-1}V(H - iy)^{-1}\right) = \lim_{y\to\infty} \mathrm{tr}\left(Z(y)V\right) = -\mathrm{tr}\,(V)$$

as claimed. □

Corollary 4.2. *Let $V \in \mathfrak{S}_1(\mathfrak{H})$ and let H be a m-accumulative operator in $\mathfrak{H}$. Let also $V_I := \mathrm{Im}\,(V) = V_I^+ - V_I^-$ where $V_I^\pm \ge 0$. If $H' := H + V$, then $z = x + iy \in \rho(H')$ for $y > \|V_I^+\|$ and* (4.1) *holds.*

Proof. Note that the operator $H' - i\|V_I^+\|$ is accumulative. Using the representation $H' - iy = H' - i\|V_I^+\| - i(y - \|V_I^+\|)$ we find that $i(y - \|V_I^+\|) \in \rho(H' - i\|V_I^+\|)$ provided that $y - \|V_I^+\| > 0$. Since $i(y - \|V_I^+\|) \in \rho(H)$ it remains to apply Lemma 4.1. □

Next we present a counterpart of Lemma 3.2 for additive perturbations.

Lemma 4.3. *Let H be a m-accumulative operator.*

(i) *If $0 \le V_+ = V_+^* \in \mathfrak{S}_1(\mathfrak{H})$, then there exists a non-negative function $\xi_+(\cdot) \in L^1(\mathbb{R}; dt)$ such that the following representation holds*

$$\det\left(I + V_+(H-z)^{-1}\right) = \exp\left\{\frac{1}{\pi}\int_{\mathbb{R}} \frac{\xi_+(t)}{t-z}dt\right\}, \quad z \in \mathbb{C}_+, \tag{4.2}$$

and $\mathrm{tr}\,(V_+) = \frac{1}{\pi}\int_{\mathbb{R}} \xi_+(t)dt$.

(ii) *If $V = V^* \in \mathfrak{S}_1(\mathfrak{H})$, then there exists a real-valued function $\xi(\cdot) \in L^1(\mathbb{R}; dt)$ such that the representation* (4.2) *is valid with V and $\xi(\cdot)$ in place of V_+ and $\xi_+(\cdot)$, in particular,*

$$\mathrm{tr}\,(V) = \int_R \xi(t)dt \qquad \text{and} \qquad \int_R |\xi(t)|dt \le \|V\|_{\mathfrak{S}_1}. \tag{4.3}$$

Proof. (i) Let $V = V_+ \ge 0$. We mimic the proof of Lemma 3.2(i) replacing B and $M(z)$ by H and z, respectively. Doing so we find a non-negative function $\xi_+(\cdot)$ satisfying $\int \frac{1}{1+t^2}\,\xi_+(t)dt < \infty$ and a positive constant c_+ such that the representation

$$\det(I + V_+(H-z)^{-1}) = c_+ \exp\left\{\frac{1}{\pi}\int_{\mathbb{R}} \left(\frac{1}{t-z} - \frac{t}{1+t^2}\right)\xi_+(t)dt\right\}, \quad z \in \mathbb{C}_+, \tag{4.4}$$

holds. Clearly, $T(z) := \sqrt{V_+}(H-z)^{-1}\sqrt{V_+}$, $z \in \mathbb{C}_+$, is a family of m-dissipative operators and $\lim_{y\to\infty}\|T(x+iy)\| = 0$, $x \in \mathbb{R}$. Hence, $0 \in \rho(I + T(x+iy))$ for any $x \in \mathbb{R}$ and sufficiently large $y > 0$. Thus, for y large enough we can take logarithm of both sides in (4.4) using definition (A.1),

$$\log\det(I+V_+(H-z)^{-1}) = \log(c_+) + \frac{1}{\pi}\int_{\mathbb{R}}\left(\frac{1}{t-z} - \frac{t}{1+t^2}\right)\xi_+(t)dt, \quad z \in \mathbb{C}_+. \tag{4.5}$$

Hence, applying property (A.3) we get

$$\mathrm{Im}\left(\mathrm{tr}\left(\log(I+T(z))\right)\right) = \mathrm{Im}\left(\log\det(I+T(z))\right) = \frac{1}{\pi}\int_{\mathbb{R}} \frac{y}{(t-x)^2+y^2}\xi_+(t)dt, \tag{4.6}$$

for $z = x+iy$. Clearly, $s-\lim_{y\uparrow\infty}(H-x-iy)^{-1} = 0$ and $s-\lim_{y\uparrow\infty}(-iy)(H-x-iy)^{-1} = I$. Since $V_+ \in \mathfrak{S}_1(\mathfrak{H})$, [18, Theorem 3.6.3] ensures that

$$\lim_{y\uparrow\infty}\|T(x+iy)\|_{\mathfrak{S}_1} = 0 \quad \text{and} \quad \lim_{y\uparrow\infty}\|(-iy)T(x+iy) - V_+\|_{\mathfrak{S}_1} = 0. \tag{4.7}$$

Combining these relations with definition (A.2) we obtain

$$\begin{aligned}
&\lim_{y\uparrow\infty} y\log\big(I+T(x+iy)\big)\\
&\qquad= -\lim_{y\uparrow\infty}(-iy)T(x+iy)\lim_{y\uparrow\infty}\int_{\mathbb{R}_+}\big(I+T(x+iy)+i\lambda\big)^{-1}(1+i\lambda)^{-1}d\lambda\\
&\qquad= -V_+\int_{\mathbb{R}_+}(1+i\lambda)^{-2}d\lambda = iV_+
\end{aligned}$$

for any fixed $x \in \mathbb{R}$. Notice that according to (4.7) the convergence in the last formula takes place in the $\mathfrak{S}_1$-norm and, hence

$$\lim_{y\to\infty} y\mathrm{Im}\,(\mathrm{tr}\,(\log(I + T(x+iy)))) = \mathrm{tr}\,(V_+). \tag{4.8}$$

On the other hand, multiplying identity (4.6) by y and tending y to $+\infty$ we arrive at the equality (4.8) with $\frac{1}{\pi}\int_{\mathbb{R}} \xi_+(t)dt$ in place of $\mathrm{tr}\,(V_+)$. So, we get $\frac{1}{\pi}\int_{\mathbb{R}} \xi_+(t)dt = \mathrm{tr}\,(V_+)$. In particular, $\xi_+(\cdot) \in L^1(\mathbb{R}; dt)$. Taking the last inclusion into account we obtain from (4.5) the representation

$$\det(I + V_+(H-z)^{-1}) = c'_+ \exp\left\{\frac{1}{\pi}\int_{\mathbb{R}} \frac{\xi_+(t)}{t-z}dt\right\}, \quad z \in \mathbb{C}_+, \tag{4.9}$$

where

$$c'_+ := c_+ \exp\left\{-\frac{1}{\pi}\int_{\mathbb{R}} \frac{t}{1+t^2}\xi_+(t)dt\right\}.$$

Tending y to $+\infty$ in (4.9) and using $\lim_{y\to\infty} \det\,(I + V_+(H-iy)^{-1}) = 1$ we find $c'_+ = 1$. Thus (4.9) coincides with (4.2).

(ii) Setting $K := H - V_-$ where $V = V_+ - V_-$, $V_\pm \ge 0$, and using the chain rule for perturbation determinants we get

$$w(z) := \det(I + V(H-z)^{-1}) = \frac{\det(I + V_+(K-z)^{-1})}{\det(I + V_-(K-z)^{-1})}, \qquad z \in \mathbb{C}_+. \tag{4.10}$$

According to (i) the perturbation determinant $\det(I + V_\pm(K-z)^{-1})$ admits exponential representation (4.2) with a non-negative $\xi_\pm(\cdot) \in L^1(\mathbb{R}; dt)$. Setting $\xi(\cdot) := \xi_+(\cdot) - \xi_-(\cdot)$ and applying (4.10) we arrive at representation (4.2) for $w(\cdot)$ with $\xi(\cdot)$ in place of $\xi_+(\cdot)$. Further, relations $\frac{1}{\pi}\int_{\mathbb{R}} \xi_\pm(t) = \mathrm{tr}\,(V_\pm)$ imply $\frac{1}{\pi}\int_{\mathbb{R}} \xi(t)dt = \mathrm{tr}\,(V)$. Moreover,

$$\frac{1}{\pi}\int_{\mathbb{R}} |\xi(t)|dt \le \frac{1}{\pi}\int_{\mathbb{R}} \xi_+(t)dt + \frac{1}{\pi}\int_{\mathbb{R}} \xi_-(t)dt = \mathrm{tr}\,(V_+) + \mathrm{tr}\,(V_-) = \|V\|_{\mathfrak{S}_1},$$

which proves (4.3). □

Remark 4.4. Let us outline another approach to the proof of Lemma 4.3 Since H is a m-accumulative operator, it admits a self-adjoint dilation, i.e. an operator $K = K^*$ acting in a larger Hilbert space $\mathfrak{K} \supseteq \mathfrak{H}$ and satisfying

$$(H-z)^{-1} = P^{\mathfrak{K}}_{\mathfrak{H}}(K-z)^{-1} \upharpoonright \mathfrak{H}, \qquad z \in \mathbb{C}_+.$$

cf.[42]. Setting $V \upharpoonright \mathfrak{K} \ominus \mathfrak{H} = 0$ we identify V with its (trivial) continuation to $\mathfrak{K}$ and put $H' := H + V$ and $K' := K + V$. Clearly,

$$\Delta_{H'/H}(z) = \det(I_{\mathfrak{H}} + V(H-z)^{-1}) = \det(I_{\mathfrak{K}} + V(K-z)^{-1}) = \Delta_{K'/K}(z), \tag{4.11}$$

$z \in \mathbb{C}_+$. Combining (4.11) with [24, Theorem 1] (see also [23, 25] and [5]) implies the conclusion of Lemma 4.3(ii). In particular, relation (4.3) holds with $\xi(\cdot) = \overline{\xi(\cdot)} \in L^1(\mathbb{R}; dt)$. Moreover, if $V \ge 0$, then the same Krein's theorem ensures the conclusion of Lemma 4.3(i) with a non-negative $\xi_+(\cdot) \in L^1(\mathbb{R}; dt)$.

Note that trace formula (3.1) immediately follows from Lemma 4.3. Coming to the case that H is m-accumulative H we firstly prove an "additive" counterpart of Lemma 3.12.

Lemma 4.5. *Let H be a m-accumulative operator in $\mathfrak{H}$.*

(i) *Let $0 \le V_+ = V_+^* \in \mathfrak{S}_1(\mathfrak{H})$. Assume also that*

$$(V_+f, f) \le -\mathrm{Im}\,(Hf, f), \quad f \in \mathrm{dom}\,(H). \tag{4.12}$$

Then the function $w_+(z) := \det(I + iV_+(H-z)^{-1})$ admits a representation

$$w_+(z) = \exp\left\{\frac{i}{\pi}\int_{\mathbb{R}} \frac{1}{t-z}\eta_+(t)dt\right\}, \qquad z \in \mathbb{C}_+, \tag{4.13}$$

with a non-negative $\eta_+(\cdot) \in L^1(\mathbb{R}; dt)$. Moreover, the inversion formula $\eta_+(t) = -\ln(|w_+(t+i0)|)$ holds for a.e. $t \in \mathbb{R}$ where $w_+(t+i0) := \lim_{y\downarrow 0} w_+(t+iy)$.

Moreover, there exists a non-negative $\eta^+(\cdot) \in L^1(\mathbb{R}; dt)$ such that

$$w^+(z) := \det(I - iV_+(H-z)^{-1}) = \exp\left\{-\frac{i}{\pi}\int_{\mathbb{R}} \frac{1}{t-z}\eta^+(t)dt\right\}, \qquad z \in \mathbb{C}_+, \tag{4.14}$$

and the inversion formula $\eta^+(t) = \ln(|w^+(t+i0)|)$ holds for a.e. $t \in \mathbb{R}$.

(ii) *If $V = V^* \in \mathfrak{S}_1(\mathfrak{H})$ and the condition*

$$(Vf, f) \le -\mathrm{Im}\,(Hf, f), \qquad f \in \mathrm{dom}\,(H), \tag{4.15}$$

is satisfied, then there exists a real-valued *$\eta(\cdot) \in L^1(\mathbb{R}; dt)$ such that*

$$w(z) := \det(I + iV(H-z)^{-1}) = \exp\left\{\frac{i}{\pi}\int_{\mathbb{R}} \left(\frac{1}{t-z}\right)\eta(t)dt\right\}, \qquad z \in \mathbb{C}_+, \tag{4.16}$$

and the inversion formula $\eta(t) = -\ln\left(|w(t+i0)|\right)$ holds for a.e. $t \in \mathbb{R}$.

Proof. (i) Clearly, the proof of Lemma 3.12(i) remains true if one replaces B and $M(z)$ by H and z, respectively. So according to (3.34) $w_+(\cdot)$ admits a representation

$$w_+(z) = \varkappa_+ \exp\left\{\frac{i}{\pi}\int_{\mathbb{R}} \left(\frac{1}{t-z} - \frac{t}{1+t^2}\right)\eta_+(t)dt\right\}, \qquad z \in \mathbb{C}_+, \tag{4.17}$$

with a non-negative function $\eta_+(\cdot) \in L^1(\mathbb{R}; \frac{dt}{1+t^2})$. Moreover, by the Fatou theorem, $\eta_+(t) = -\ln(|\det(w_+(t+i0))|)$ for a.e. $t \in \mathbb{R}$. Hence

$$\exp\left\{\frac{1}{\pi}\int_{\mathbb{R}} \frac{y}{(t-x)^2+y^2}\eta_+(t)dt\right\} = \left|\frac{1}{w_+(z)}\right|, \qquad z = x+iy \in \mathbb{C}_+. \tag{4.18}$$

Setting $H' := H + iV_+$ one easily gets $\frac{1}{w_+(z)} = \det(I - iV_+(H'-z)^{-1})$. Applying the known estimate for the determinant (see [18, Section IV.1]) we derive

$$\frac{1}{|w_+(iy)|} \le \exp\left\{\left\|V_+(H'-iy)^{-1}\right\|_{\mathfrak{S}_1}\right\} \le \exp\left\{\frac{\|V_+\|_{\mathfrak{S}_1}}{y - \|V_+\|}\right\}, \quad y > \|V_+\|.$$

Combining this estimate with relation (4.18) yields

$$\frac{1}{\pi}\int_{\mathbb{R}}\frac{y^2}{t^2+y^2}\eta_+(t)dt \le \frac{y}{y-\|V_+\|}\|V_+\|_{\mathfrak{S}_1}, \qquad y > \|V_+\|.$$

The monotone convergence theorem applies as $y \to +\infty$ and gives

$$\|\eta_+\|_{L^1(\mathbb{R};dt)} = \int_{\mathbb{R}}\eta_+(t)\,dt \le \pi\|V_+\|_{\mathfrak{S}_1}.$$

Hence $\int_{\mathbb{R}}\frac{t}{1+t^2}\eta_+(t)dt$ is finite and setting $\varkappa_+' := \varkappa_+\exp\left\{-\frac{i}{\pi}\int_{\mathbb{R}}\frac{t}{1+t^2}\eta_+(t)dt\right\}$ one simplifies (4.17) as

$$w_+(z) = \varkappa_+'\exp\left\{\frac{i}{\pi}\int_{\mathbb{R}}\frac{1}{t-z}\eta_+(t)dt\right\}, \quad z\in\mathbb{C}_+. \tag{4.19}$$

To prove (4.13) it remains to note that $\lim_{y\to\infty} w_+(iy) = 0$, and hence $\varkappa_+' = 1$.

Further, since $H - iV_+$ is a m-accumulative operator, a representation similar to (4.19) holds

$$\det(I + iV_+(H - iV_+ - z)^{-1}) = \exp\left\{\frac{i}{\pi}\int_{\mathbb{R}}\frac{\eta^+(t)}{t-z)}dt\right\}, \quad z\in\mathbb{C}_+,$$

with a non-negative function $\eta^+(\cdot) \in L^1(\mathbb{R};dt)$. Noting that

$$w^+(z) = \det(I - iV_+(H-z)^{-1}) = \frac{1}{\det(I + iV_+(H - iV_+ - z)^{-1})}, \quad z\in\mathbb{C}_+.$$

we arrive at representation (4.14).

(ii) Let $V = V_+ - V_-$, $V_\pm \ge 0$, and let $H_- := H - iV_-$. Clearly, H_- is also a m-accumulative operator. Setting $w_\pm(z) := \det(I + iV_\pm(H_- - z)^{-1})$ and using the chain rule for determinants we get

$$w(z) = \frac{w_+(z)}{w_-(z)}, \qquad z\in\mathbb{C}_+. \tag{4.20}$$

Next, rewriting condition (4.15) in the form

$$(V_+f,f) \le -\mathrm{Im}\,(Hf,f) + (V_-f,f) = -\mathrm{Im}\,(H_-f,f), \quad f\in\mathrm{dom}\,(H_-),$$

and applying (i) we find that $w_+(\cdot)$ admits the representation (4.13) with $\eta_+(\cdot) \ge 0$. Similarly, since $(V_-f,f) \le -\mathrm{Im}\,(H_-f,f)$, $f\in\mathrm{dom}\,(H_-)$, we obtain by applying (i) that $w_-(\cdot)$ also admits a representation of the type (4.13) with $\eta_-(\cdot)\ge 0$ in place of $\eta_+(\cdot)$. Combining (4.20) with these representations for $w_\pm(\cdot)$ and setting $\eta(\cdot) := \eta_+(\cdot) - \eta_-(\cdot)$, we arrive at (4.16). □

A counterpart of Theorem 3.13 reads now as follows.

Theorem 4.6. *Let H be a m-accumulative operator, $V\in\mathfrak{S}_1(\mathfrak{H})$ and let*

$$\mathrm{Im}\,(Vf,f) \le -\mathrm{Im}\,(Hf,f), \quad f\in\mathrm{dom}\,(H). \tag{4.21}$$

Then the operator $H' = H + V$ is also m-accumulative and there exists a complex-valued function $\omega(\cdot)\in L^1(\mathbb{R};dt)$ *such that the following holds:*

(i) *The perturbation determinant $\Delta_{H'/H}(\cdot)$ admits the representation*

$$\Delta_{H'/H}(z) = \exp\left\{\frac{1}{\pi}\int_{\mathbb{R}} \frac{\omega(t)}{t-z}dt\right\}, \qquad z \in \mathbb{C}_+. \tag{4.22}$$

(ii) *The classical trace formula*

$$\operatorname{tr}\left((H'-z)^{-1} - (H-z)^{-1}\right) = -\frac{1}{\pi}\int_{\mathbb{R}} \frac{\omega(t)}{(t-z)^2}dt, \quad z \in \mathbb{C}_+, \tag{4.23}$$

holds and

$$\operatorname{tr}(V) = \frac{1}{\pi}\int_{\mathbb{R}} \omega(t)dt. \tag{4.24}$$

Proof. Since H is m-accumulative, then, by (4.21), $H' := H + V$ is m-accumulative too.

(i) Let $V = V_R + iV_I$ and $K := H + V_R$. Clearly, K is m-accumulative too. In accordance with (4.21),

$$(V_I f, f) \leq -\operatorname{Im}(Kf, f) = -\operatorname{Im}(Hf, f), \qquad f \in \operatorname{dom}(K) = \operatorname{dom}(H).$$

By Lemma 4.5 (ii), there exists a *real-valued function* $\eta(\cdot) \in L^1(\mathbb{R}; dt)$ such that the perturbation determinant $\Delta_{H'/K}(\cdot)$ admits the representation

$$\Delta_{H'/K}(z) = \det(I + iV_I(K-z)^{-1}) = \exp\left\{\frac{i}{\pi}\int_{\mathbb{R}} \frac{\eta(t)}{t-z}dt\right\}, \quad z \in \mathbb{C}_+. \tag{4.25}$$

Furthermore, by Lemma 4.3(ii), there exists a *real-valued function* $\xi(\cdot) \in L^1(\mathbb{R}; dt)$ such that the perturbation determinant $\Delta_{K/H}(\cdot)$ admits the representation

$$\Delta_{K/H}(z) = \det(I + V_R(H-z)^{-1}) = \exp\left\{\frac{1}{\pi}\int_{\mathbb{R}} \frac{\xi(t)}{t-z}dt\right\}, \quad z \in \mathbb{C}_+. \tag{4.26}$$

Combining representations (4.25), (4.26), and applying the chain rule $\Delta_{H'/H}(\cdot) = \Delta_{H'/K}(\cdot)\Delta_{K/H}(\cdot)$ we arrive at (4.22) with $\omega := \xi + i\eta \in L^1(\mathbb{R}; dt)$.

(ii) Taking logarithmic derivative from both sides of (4.22) we derive the trace formula (4.23). To prove (4.24) we rewrite (4.23) in the form

$$\operatorname{tr}\left((H'-z)^{-1}V(H-z)^{-1}\right) = \frac{1}{\pi}\int_{\mathbb{R}} \frac{\omega(t)}{(t-z)^2}dt, \qquad z \in \mathbb{C}_+.$$

Setting here $z = iy$, then multiplying both sides by y^2 and tending y to $+\infty$. Finally, applying Lemma 4.1 and using $\omega \in L^1(\mathbb{R}; dt)$ we arrive at (4.24). □

Corollary 4.7. *Assume the conditions of Theorem 4.6 and let $\pm V_I \geq 0$. Then a* summable real-valued SSF $\xi(\cdot)$ *of the pair $\{H', H\}$ exists if and only if $V = V^*$.*

Proof. Sufficiency is ensured by Lemma 4.3. Conversely, if $\omega(\cdot) \in L^1(\mathbb{R}; dt)$ and is real valued, then trace formula (4.24) implies $\operatorname{tr}(V_I) = 0$. If $V_I \geq 0$, then $V_I = 0$, i.e. $V = V^*$. The same holds if $V_I \leq 0$. □

However we show that even in the case of $V \neq V^*$ but under an additional assumption there exists a *non-summable real-valued SSF* $\xi(\cdot) \in \mathcal{S}\{H', H\}$ that belongs to $L^p(\mathbb{R}; dt)$ and even to certain weighted $L^p(\mathbb{R}; dt)$-spaces. First we describe the set of all SSFs (and even measures) arising in the trace formula (4.23). A counterpart of Proposition 3.8 reads as follows.

Proposition 4.8. *Let the assumptions of Theorem 4.6 be satisfied and let* $\omega(\cdot) \in L^1(\mathbb{R}; dt)$ *be the SSF for the pair* $\{H', H\}$. *Let also* $\mu(\cdot)$ *be a complex-valued finite Borel measure on* $\mathbb{R}$. *Then the trace identity*

$$\operatorname{tr}\left((H'-z)^{-1} - (H-z)^{-1}\right) = -\frac{1}{\pi}\int_{\mathbb{R}} \frac{d\mu(t)}{(t-z)^2}, \quad z \in \mathbb{C}_+, \tag{4.27}$$

holds if and only if $\mu(\cdot)$ *is absolutely continuous, i.e.* $d\mu(t) = \widetilde{\omega}(t)dt$ *with* $\widetilde{\omega}(\cdot) \in L^1(\mathbb{R}; dt)$, *and* $\eta := \widetilde{\omega} - \omega \in H^1(\mathbb{C}_-; dt)$.

Proof. If $\eta(\cdot) \in H^1(\mathbb{C}_-; dt)$, then

$$\int_{\mathbb{R}} \frac{\eta(t)}{t-z} dt = 0, \quad z \in \mathbb{C}_+. \tag{4.28}$$

Setting $\widetilde{\omega}(\cdot) := \omega(\cdot) + \eta(\cdot)$, one gets that (4.28) implies (4.22) with $\widetilde{\omega}(\cdot)$ in place of $\omega(\cdot)$. In turn, (4.22) yields (4.23) with $\widetilde{\omega}(\cdot)$ in place of $\omega(\cdot)$.

Conversely, assume that (4.23) also holds with a complex-valued finite Borel measure $d\mu$ in place of $\omega(\cdot)dt$. Since $\omega(\cdot) \in L^1(\mathbb{R}; dt)$, the measure $d\nu(\cdot) := d\mu(\cdot) - \omega(\cdot)dt$ is also finite. Therefore the function

$$g(z) := \int_{\mathbb{R}} \frac{d\nu(t)}{t-z}, \qquad z \in \mathbb{C}_+. \tag{4.29}$$

is well defined and holomorphic in $\mathbb{C}_+$. By the assumption (4.23), $g'(z) = 0,\ z \in \mathbb{C}_+$, and hence $g(z) \equiv c = \mathrm{const},\ z \in \mathbb{C}_+$. Since the measure $d\nu(\cdot)$ is finite, $g(iy) \to 0$ as $y \to \infty$, by the dominated convergence theorem. Thus, $g(z) \equiv 0,\ z \in \mathbb{C}_+$, and by Riesz's Brothers theorem ([14, Theorem II.3.8], the measure $d\nu$ is absolutely continuous, i.e. $d\nu(t) = \eta(t)dt$ with $\eta(\cdot) \in H^1(\mathbb{C}_-; dt)$. Hence $d\mu(t) = \widetilde{\omega}(t)dt$, where $\widetilde{\omega} := \omega + \eta \in L^1(\mathbb{R}; dt)$. □

Proposition 3.10 looks now as follows.

Proposition 4.9. *Assume the conditions of Theorem 4.6. Let also* $p \in (1, \infty)$ *and* $\alpha \in (-1, p-1)$. *If* $\omega(\cdot) \in L^1(\mathbb{R}; dt) \cap L^p(\mathbb{R}; (1+t^2)^{\alpha/2}dt)$, *then there exists a real-valued SSF* $\xi(\cdot) \in \mathcal{S}\{H', H\}$ *such that* $\xi(\cdot) \in L^p(\mathbb{R}; (1+t^2)^{\alpha/2}dt)$.

Proof. The proof relies on Proposition 4.8 and is similar to that of Proposition 3.10. It suffices to note that for $\alpha \in (-1, p-1)$ the weight $(1+t^2)^{\alpha/2}$ satisfies the Muckenhoupt condition (A_p) and, by Hunt-Muckenhoupt-Weeden theorem ([14, Theorem 6.6.2]) the Hilbert transform boundedly maps $L^p(\mathbb{R}; (1+t^2)^{\alpha/2}dt)$ onto $L^p(\mathbb{R}; (1+t^2)^{\alpha/2}dt)$. □

Remark 4.10. Let us compare Theorem 4.6 with Krein's results [26].

(i) Krein [26] considered a m-accumulative operator $H' := H - iV_+$, with $H = H^*$ and $V_+ = V_+^* \geq 0$. He proved [26, Theorem 9.1] that the perturbation determinant $\Delta_{H/H'}(\cdot)$ admits a representation

$$\Delta_{H/H'}(z) = \exp\left\{ i \int_{\mathbb{R}} \frac{d\,\tau(t)}{t-z} \right\}, \quad z \in \mathbb{C}_+. \tag{4.30}$$

with a non-decreasing function $\tau(\cdot) : \mathbb{R} \longrightarrow \mathbb{R}$. Our Lemma 4.5(i) improves Krein's result since according to (4.13) the measure $d\tau(\cdot)$ *is necessarily absolutely continuous*, $d\tau(t) = \eta_+(t)dt$ where $\eta_+(\cdot) \geq 0$ and $\eta_+(\cdot) \in L^1(\mathbb{R}; dt)$.

(ii) Theorem 4.6 generalizes [26, Theorem 9.1] in two directions. Firstly, it allows H to be m-accumulative in place of self-adjoint and, secondly, condition $\mathrm{Im}\,(V) \leq 0$ in [26] is relaxed to (4.21), i.e. to the condition of accumulativity of the perturbed operator $H' = H + V$ itself instead of the perturbation V.

The trace formula (4.30) can be extended to a class of holomorphic in $\mathbb{C}_-$ functions $\Phi(\cdot)$ admitting the following representation

$$\Phi(z) = \int_{[0,\infty)} \Psi(z,t)dp(t), \quad z \in \mathbb{C}_- \cup \mathbb{R}, \tag{4.31}$$

where $p(\cdot)$ is a complex-valued Borel measure on $[0,\infty)$ of finite variation, i.e

$$\int_{[0,\infty)} |dp((t)| < \infty,$$

and

$$\Psi(z,t) := \begin{cases} \frac{e^{-itz}-1}{-it}, & t > 0, \\ z, & t = 0. \end{cases}, \qquad z \in \mathbb{C}_- \cup \mathbb{R}. \tag{4.32}$$

It is well known that any m-accumulative (in particular self-adjoint) operator H in $\mathfrak{H}$ generates a strongly continuous semigroup of contractions e^{-itH}, $t \geq 0$. This fact allows one to define the operator $\Phi(T)$ by setting

$$\Phi(H)h = \int_{[0,\infty)} \Psi(H,t)hdp(t), \qquad h \in \mathrm{dom}\,(H). \tag{4.33}$$

In general, $\Phi(H)$ is unbounded but closable and $\mathrm{dom}\,(\Phi(H)) \supseteq \mathrm{dom}\,(H)$ holds. However, if $\mathrm{supp}\,(p) \subset (0,\infty)$, then $\Phi(H)$ is bounded.

In [26, Theorem 9.2] Krein has shown that for a pair $\{H', H\}$ with self-adjoint $H = H^*$ and m-accumulative $H' = H - iV_+$, $V_+ = V_+^* \geq 0$, the trace formula

$$\mathrm{tr}\,\big(\Phi(H') - \Phi(H)\big) = -i \int_{\mathbb{R}} \Phi'(t)d\tau(t) \tag{4.34}$$

holds. Here $\Phi(\cdot)$ is given by (4.31) and $\tau(\cdot)$ is a non-decreasing bounded function from representation (4.30) for the perturbation determinant $\Delta_{H/H'}(\cdot)$.

Our generalization of [26, Theorem 9.2] reads as follows.

Theorem 4.11. *Let the assumptions of Theorem 4.6 be satisfied and let* $\omega(\cdot) \in \mathcal{S}_1\{H', H\}$*. Let also* $\Phi(\cdot)$ *be a function on* $\mathbb{C}_+$ *of the form* (4.31)*. Then both operators* $\Phi(H')$ *and* $\Phi(H)$ *are well defined,* $\Phi(H') - \Phi(H) \in \mathfrak{S}_1(\mathfrak{H})$*, and the following trace formula holds*

$$\operatorname{tr}\left(\Phi(H') - \Phi(H)\right) = \frac{1}{\pi}\int_{\mathbb{R}} \Phi'(t)\omega(t)dt\,. \tag{4.35}$$

Proof. We set

$$H_\alpha = H(I + i\alpha H)^{-1} \qquad \text{and} \qquad H'_\alpha = H'(I + i\alpha H')^{-1}, \quad \alpha > 0. \tag{4.36}$$

One easily verifies that H'_α and H_α are bounded accumulative operators. Moreover, it is easily seen that

$$(H_\alpha - z)^{-1} = \frac{i\alpha}{1 - i\alpha z}I + \frac{1}{(1-i\alpha z)^2}\left(H - \frac{z}{1-i\alpha z}\right)^{-1}. \tag{4.37}$$

and similar identity holds for $(H'_\alpha - z)^{-1}$. Hence

$$\text{s-}\lim_{\alpha\to+0}(H'_\alpha - z)^{-1} = (H' - z)^{-1} \quad \text{and} \quad \text{s-}\lim_{\alpha\to+0}(H_\alpha - z)^{-1} = (H - z)^{-1} \tag{4.38}$$

for $z \in \mathbb{C}_+$. Combining (4.37) with a similar identity for $(H'_\alpha - z)^{-1}$ and applying the trace formula (4.23) to bounded accumulative operators H'_α and H_α yields

$$\begin{aligned} \operatorname{tr}\left((H'_\alpha - z)^{-1} - (H_\alpha - z)^{-1}\right) &= -\frac{1}{(1-i\alpha z)^2}\int_{\mathbb{R}} \frac{\omega(t)}{(t - \frac{z}{1-i\alpha z})^2}dt \\ = -\int_{\mathbb{R}} \frac{\omega(t)}{\big(t - z(1+i\alpha t)\big)^2}dt &= -\frac{1}{\pi}\int_{\mathbb{R}} \frac{\omega(t)}{(1+i\alpha t)^2}\frac{1}{\big(z - \frac{t}{1+i\alpha t}\big)^2}dt. \end{aligned} \tag{4.39}$$

Let Γ be a simple closed curve such that its interior contains $\sigma(H'_\alpha) \cup \sigma(H_\alpha)$. Since H_α and H'_α are bounded, the Riesz-Dunford functional calculus applies

$$\begin{aligned} e^{-isH'_\alpha} - e^{-isH_\alpha} &= -\frac{1}{2\pi i}\oint_\Gamma e^{-isz}\left((H'_\alpha - z)^{-1} - (H_\alpha - z)^{-1}\right)dz \\ &= \frac{1}{2\pi i}\oint_\Gamma e^{-isz}(H'_\alpha - z)^{-1}V_\alpha(H_\alpha - z)^{-1}dz, \end{aligned} \tag{4.40}$$

where

$$V_\alpha := H'_\alpha - H_\alpha = H'(I+i\alpha H')^{-1} - H(I+i\alpha H)^{-1} = (I+i\alpha H')^{-1}V(I+i\alpha H)^{-1}, \tag{4.41}$$

$\alpha > 0$. Since $V \in \mathfrak{S}_1(\mathfrak{H})$, this identity yields $V_\alpha \in \mathfrak{S}_1(\mathfrak{H})$. In turn, the latter implies that the integrand operator-valued function in (4.40) is continuous in $\mathfrak{S}_1$-norm which gives $e^{-isH'_\alpha} - e^{-isH_\alpha} \in \mathfrak{S}_1(\mathfrak{H})$. Moreover, one gets from (4.40)

$$\operatorname{tr}\left(e^{-isH'_\alpha} - e^{-isH_\alpha}\right) = -\frac{1}{2\pi i}\oint_\Gamma e^{-isz}\operatorname{tr}\left((H'_\alpha - z)^{-1} - (H_\alpha - z)^{-1}\right)dz.$$

In turn, combining this formula with (4.39) we derive

$$\begin{aligned}
&\operatorname{tr}\left(e^{-isH'_\alpha}-e^{-isH_\alpha}\right) \qquad (4.42)\\
&=\frac{1}{\pi}\int_{\mathbb{R}} dt\ \frac{\omega(t)}{(1+i\alpha t)^2}\frac{1}{2\pi i}\oint_\Gamma \frac{e^{-isz}}{\left(z-\frac{t}{1+i\alpha t}\right)^2}dz=\frac{-is}{\pi}\int_{\mathbb{R}} e^{-is\frac{t}{1+i\alpha t}}\frac{\omega(t)}{(1+i\alpha t)^2}\ dt.
\end{aligned}$$

Further, it follows from [20, formula (IX.2.22)] and (4.41) that for $s>0$

$$e^{-isH'_\alpha}-e^{-isH_\alpha}=-i\int_0^s e^{-i(s-x)H'_\alpha}(I+i\alpha H')^{-1}V(I+i\alpha H)^{-1}e^{-ixH_\alpha}dx, \qquad (4.43)$$

and

$$e^{-isH'}-e^{-isH}=-i\int_0^s e^{-i(s-x)H'}Ve^{-ixH}dx, \qquad s>0. \qquad (4.44)$$

Since $V\in\mathfrak{S}_1(\mathfrak{H})$ we find $e^{-isH'_\alpha}-e^{-isH_\alpha}\in\mathfrak{S}_1(\mathfrak{H})$ and $e^{-isH'}-e^{-isH}\in\mathfrak{S}_1(\mathfrak{H})$, $s>0$, see above. Moreover, representations (4.43) and (4.44) imply the following important estimates

$$\left\|e^{-isH'_\alpha}-e^{-isH_\alpha}\right\|_{\mathfrak{S}_1}\le s\|V_\alpha\|_{\mathfrak{S}_1} \text{ and } \left\|e^{-isH'}-e^{-isH}\right\|_{\mathfrak{S}_1}\le s\|V\|_{\mathfrak{S}_1}. \qquad (4.45)$$

Since $V\in\mathfrak{S}_1(\mathfrak{H})$, it follows from (4.38) and (4.41) that $\lim_{\alpha\to 0}\|V_\alpha-V\|_{\mathfrak{S}_1}=0$. Combining this relation with integral representations (4.43) and (4.44) we obtain

$$\lim_{\alpha\to+0}\operatorname{tr}\left(e^{-isH'_\alpha}-e^{-isH_\alpha}\right)=\operatorname{tr}\left(e^{-isH'}-e^{-isH}\right), \quad s>0. \qquad (4.46)$$

In turn, combining (4.46) with (4.42) and applying the dominated convergence theorem (with the majorant $|\omega|\in L^1(\mathbb{R};dt)$) yields

$$\begin{aligned}
&\operatorname{tr}\left(e^{-isH'}-e^{-isH}\right) \qquad (4.47)\\
&=\lim_{\alpha\to+0}\frac{-is}{\pi}\int_{\mathbb{R}} e^{-is\frac{t}{1+i\alpha t}}\frac{\omega(t)}{(1+i\alpha t)^2}dt=\frac{-is}{\pi}\int_{\mathbb{R}} e^{-ist}\omega(t)dt, \quad s>0.
\end{aligned}$$

On the other hand, since both H and H' are m-accumulative, (4.32) and (4.33) imply the representation

$$\Phi(H')-\Phi(H)=\int_{[0,\infty)}\frac{e^{-itH'}-e^{-itH}}{-it}dp(t). \qquad (4.48)$$

Since the measure p is finite, one readily derives from (4.48) and (4.45) that $\Phi(H')-\Phi(H)\in\mathfrak{S}_1(\mathfrak{H})$ and

$$\|\Phi(H')-\Phi(H)\|_{\mathfrak{S}_1}\le\|V\|_{\mathfrak{S}_1}\int_{[0,\infty)}|dp(t)|.$$

Combining (4.47) with (4.48) we finally obtain

$$\begin{aligned}
\operatorname{tr}\left(\Phi(H')-\Phi(H)\right)&=\int_{[0,\infty)}\frac{\operatorname{tr}\left(e^{-itH'}-e^{-itH}\right)}{-it}dp(t)\\
&=\frac{1}{\pi}\int_{[0,\infty)}dp(t)\int_{\mathbb{R}}e^{-itx}\omega(x)\ dx\ =\frac{1}{\pi}\int_{\mathbb{R}}dx\ \omega(x)\int_{[0,\infty)}e^{-itx}dp(t).
\end{aligned}$$

Noting that $\Phi'(x)=\int_{[0,\infty)}e^{-itx}dp(t)$, $x\in\mathbb{R}$, we arrive at (4.35). □

Remark 4.12. In the case of $H = H^*$ treated by Krein [26] formula (4.35) improves Krein's formula (4.34). Namely, according to representations (4.30) and (4.22), $d\tau = i\omega dt$, i.e. the Krein measure $d\tau$ is absolutely continuous.

Note that for a regular pair $\{H', H\}$ of self-adjoint operators the Lifshitz–Krein trace formula (4.35) was established by M. Krein earlier [23, 25]. In [32, 33] this formula was extended to a broad class of functions. Namely, it was shown in [32] and [33] that if f belongs to a certain Besov space, then for arbitrary (not necessarily bounded) self-adjoint operators H' and H with $\overline{H' - H} \in \mathfrak{S}_1$, the operator $f(H') - f(H)$ is also of trace class and formula (4.35) holds.

Note also that Alexandrov and Peller [3, 4, 34] found sharp conditions on f analytic in $\mathbb{C}_+$ for the implication $\overline{H' - H} \in \mathfrak{S}_1 \implies f(H') - f(H) \in \mathfrak{S}_1$ to hold, whenever H' and H are m-dissipative operators. This condition is given in terms of a certain Besov space of functions analytic in the upper half-plane.

4.2 Pairs $\{H, H'\}$ with one m-accumulative operator

Our next goal is to remove condition (4.21), i.e. to prove trace formulas for pairs $\{H, H'\}$ with a m-accumulative operator H. At first we prove an analog of Lemma 3.21. To this end we recall a simple statement on Blaschke products in $\mathbb{C}_+$ associated with the Blaschke sequence $\mathcal{Z} = \{(z_k, m_k)\}_k$ in $\mathbb{C}_+$ satisfying in addition

$$\sum_k m_k |\mathrm{Im}\,(z_k)| < \infty \tag{4.49}$$

Lemma 4.13 ([26, Lemma 8.1]). *Let $\mathcal{Z} := \{(z_k, m_k)\}_k$ as above. If the condition* (4.49) *is satisfied, then*

$$\lim_{y\uparrow\infty} y^2 \sum_k m_k \frac{\mathrm{Im}\,(z_k)}{(iy - z_k)(iy - \overline{z}_k)} = -\sum_k m_k \mathrm{Im}\,(z_k). \tag{4.50}$$

and the (regularized) Blaschke product

$$\widetilde{\mathcal{B}}(z, \mathcal{Z}) := \prod_k \left(\frac{z - z_k}{z - \overline{z}_k}\right)^{m_k}$$

converges uniformly on any compact subset $\mathcal{K} \subset \mathbb{C}$ satisfying $\mathrm{dist}\,(\mathcal{K}, \{z_k\}_k) > 0$. Moreover, the following relations hold

$$\lim_{y\uparrow\infty} \widetilde{\mathcal{B}}(z, \mathcal{Z}) = 1 \quad \text{and} \quad \lim_{y\uparrow\infty} y \ln|\widetilde{\mathcal{B}}(z, \mathcal{Z})| = -2\sum_k \mathrm{Im}\, z_k, \quad z = x + iy.$$

A counterpart of Lemma 3.21 for additive perturbations reads as follows.

Lemma 4.14. *Let H be a m-accumulative operator in $\mathfrak{H}$.*

(i) *Assume that $0 \le V_+ = V_+^* \in \mathfrak{S}_1(\mathfrak{H})$ and*

$$(V_+ f, f) \le -2\mathrm{Im}\,(Hf, f), \qquad f \in \mathrm{dom}\,(H).$$

Further, and let $\mathcal{Z}_+ = \{(z_k^+, m_k^+)\}_k$ where $\{z_k^+\}_k$ are the eigenvalues of $H_+' := H + V$ in $\mathbb{C}_+$ with algebraic multiplicities $\{m_k^+\}_k$. Then $\mathcal{Z}_+$ satisfies condition (4.49) *and the the function $w_+(z) := \det\bigl(I + iV_+(H-z)^{-1}\bigr)$ admits the representation*

$$w_+(z) = \widetilde{\mathcal{B}}(z, \mathcal{Z}_+) \exp\left\{\frac{i}{\pi}\int_{\mathbb{R}} \frac{1}{t-z} d\mu_+(t)\right\}, \quad z \in \mathbb{C}_+, \tag{4.51}$$

where $\mu_+(\cdot)$ is a non-negative finite *Borel measure.*

(ii) *Assume that* $0 \le V = V^* \in \mathfrak{S}_1(\mathfrak{H})$ *and*

$$(Vf, f) \le -2\mathrm{Im}\,(Hf, f), \quad f \in \mathrm{dom}\,(H). \tag{4.52}$$

Further, let $\mathcal{Z} = \{(z_k, m_k)\}_k$ where $\{z_k\}_k$ are the eigenvalues of $H' := H + V$ win $\mathbb{C}_+$ with algebraic multiplicity m_k. Then $\mathcal{Z}$ is a Blaschke sequence satisfying (4.49) *and the function $w(z) = \det(I + V(H-z)^{-1})$, $z \in \mathbb{C}_+$, admits the representation* (4.51) *with $\mathcal{Z}_+$ and $\mu_+(\cdot)$ replaced by $\mathcal{Z}$ and $\mu(\cdot)$, respectively, where $\mu(\cdot)$ is a* real-valued measure *satisfying $\int_{\mathbb{R}} |d\mu(t)| < \infty$.*

Proof. (i) Following the proof of Lemma 3.21 with $M(z)$ replaced by z, we arrive at representation (3.69) where $\mathfrak{Z}_+$ consists of the zeros $\mathfrak{z}_k^+$ of $w_+(\cdot)$ in $\mathbb{C}_+$ and their multiplicities $\mathfrak{m}_k^+$. Obviously,

$$|w_+(z)| = |\mathcal{B}(z, \mathfrak{Z}_+)| \ \exp\left\{-\frac{1}{\pi}\int_{\mathbb{R}} \frac{y}{(t-x)^2 + y^2} d\mu_+(t)\right\}, \quad z = x + iy \in \mathbb{C}_+,$$

which implies

$$\exp\left\{\frac{1}{\pi}\int_{\mathbb{R}} \frac{y}{(t-x)^2 + y^2} d\mu_+(t)\right\} = |\mathcal{B}(z, \mathfrak{Z}_+)| \left|\frac{1}{w_+(z)}\right|, \quad z \in \mathbb{C}_+ \setminus \{z_k^+\}_k, \tag{4.53}$$

where $\sigma(H_+') \cap \mathbb{C}_+ = \bigcup_k \{z_k^+\}_k$. One easily gets

$$\frac{1}{w_+(z)} = \det\left(I - i\sqrt{V_+}(H_+' - z)^{-1}\sqrt{V_+}\right), \quad z \in \mathbb{C}_+ \setminus \sigma(H_+').$$

Combining this identity with (4.53) and noting that $|\widetilde{\mathcal{B}}(z)| \le 1$, $z \in \mathbb{C}_+$, yields

$$\exp\left\{\frac{1}{\pi}\int_{\mathbb{R}} \frac{y}{(t-x)^2 + y^2} d\mu_+(t)\right\} \le \left|\det\left(I - i\sqrt{V_+}(H_+' - z)^{-1}\sqrt{V_+}\right)\right|,$$

$z \in \mathbb{C}_+ \setminus \sigma(H_+')$. In turn, combing this inequality with a simple estimate for determinants (see [18, Section IV.1]) we arrive at the estimate

$$\exp\left\{\frac{1}{\pi}\int_{\mathbb{R}} \frac{y}{(t-x)^2 + y^2} d\mu_+(t)\right\} \le \exp\left\{\left\|V_+(H_+' - z)^{-1}\right\|_{\mathfrak{S}_1}\right\}, \quad z \in \mathbb{C}_+ \setminus \sigma(H_+').$$

The latter with account of the resolvent estimate of the m-accumulative operator H_+' implies

$$\frac{1}{\pi}\int_{\mathbb{R}} \frac{y}{(t-x)^2 + y^2} d\mu_+(t) \le \left\|V_+(H_+' - z)^{-1}\right\|_{\mathfrak{S}_1} \le \|V_+\|_{\mathfrak{S}_1} \frac{1}{y - \|V_+\|},$$

$y > \|V_+\|$. Multiplying both sides of this estimate by y and tending y to infinity we derive

$$\frac{1}{\pi}\int_{\mathbb{R}} d\mu_+(t) \le \|V_+\|_{\mathfrak{S}_1} = \mathrm{tr}\,(V_+).$$

According to the classical property of perturbation determinants (see [18, Section 4.3]), $\{z_k^+\}_k = \sigma_p(H'_+) \cap \mathbb{C}_+$ and $\{m_k^+\}_k$ is the set of corresponding algebraic multiplicities, i.e $\mathfrak{Z}_+ = \mathcal{Z}_+$. Since $\mathrm{Im}\,(Hf,f) \le 0,\ f \in \mathrm{dom}\,(H)$, we have

$$\mathrm{Im}\,(H'_+f,f) = \mathrm{Im}\,(Hf,f) + (V_+f,f) \le (V_+f,f), \quad f \in \mathrm{dom}\,(H), \tag{4.54}$$

Denoting by $\mathfrak{H}_p^+$ the (closed) invariant subspace of H'_+ spanned by the (finite-dimensional) root subspaces $\mathfrak{L}_{z_k^+} := \ker(H' - z_k^+)^{m_k^+}$ and choose a Schur orthonormal basis $\{f_j\}_j$ in $\mathfrak{H}_p^+$ such that the matrix of the operator $H'_+ \upharpoonright \mathfrak{H}_p^+$ is triangular in this basis. Taking into account (4.54) we get

$$\begin{aligned} 0 \le \sum_k m_k^+ \mathrm{Im}\,(z_k^+) &= \sum_k \mathrm{Im}\,(H'_+f_k, f_k) \\ &= \sum_k \mathrm{Im}\,\big(Hf_k, f_k\big) + \sum_k (V_+f_k, f_k) \le \mathrm{tr}\,(V_+) < \infty. \end{aligned}$$

Hence condition (4.49) is satisfied. By Lemma 4.13, the product $\widetilde{\mathcal{B}}(z, \mathcal{Z}_+) = \prod_k \left(\frac{z - z_k^+}{z - \overline{z_k^+}}\right)^{m_k^+}$ converges uniformly on any compact subsets $\mathcal{K} \subseteq \mathbb{C}_+$ satisfying of $\mathrm{dist}\,(\mathcal{K}, \{z_k^+\}_k) > 0$. It is easily seen that $\mathcal{B}(z, \mathfrak{Z}_+) = \mathcal{B}(z, \mathcal{Z}_+) = \varkappa\, \widetilde{\mathcal{B}}(z, \mathcal{Z}_+)$ where $|\varkappa| = 1$. Since the measure μ is finite, representation (3.69) is simplified and becomes

$$w_+(z) = \varkappa'\, \widetilde{\mathcal{B}}(z, \mathcal{Z}_+) \exp\left\{\frac{i}{\pi}\int_{\mathbb{R}} \frac{1}{t - z} d\mu_+(t)\right\}, \quad z \in \mathbb{C}_+, \tag{4.55}$$

where

$$\varkappa' = \varkappa_+\ \varkappa\ \exp\left\{i\left(\alpha_+ - \frac{1}{\pi}\int_{\mathbb{R}} \frac{t}{1+t^2} d\mu_+(t)\right)\right\}.$$

Noting that always $\lim_{y\uparrow\infty} w_+(iy) = 1$ and $\lim_{y\uparrow\infty} \widetilde{\mathcal{B}}(iy, \mathcal{Z}_+) = 1$, by Lemma 4.13, one obtains from (4.55) that $\varkappa' = 1$. Thus (4.55) turns into (4.51).

(ii) We set $K := H - iV_-$ where $V := V_+ - V_-$, $V_\pm \ge 0$. Clearly, K is m-accumulative. According to the chain rule for determinants

$$w(z) = \frac{\det(I + iV_+(K - z)^{-1})}{\det(I + iV_-(K - z)^{-1})} = \frac{w_+(z)}{w_-(z)}, \quad z \in \mathbb{C}_+. \tag{4.56}$$

Rewriting (4.52) in the form $(V_+f,f) - (V_-f,f) \le -2\mathrm{Im}\,(Hf,f),\ f \in \mathrm{dom}\,(H)$, one gets that for any $f \in \mathrm{dom}\,(K) = \mathrm{dom}\,(H)$

$$(V_+f,f) \le -2\mathrm{Im}\,(Hf,f) + (V_-f,f) \le -2\mathrm{Im}\,(Hf,f) + 2(V_-f,f) = -2\mathrm{Im}\,(Kf,f).$$

Therefore, by the statement (i), the perturbation determinant $w_+(z) := \det(I+iV_+(K-z)^{-1})$, admits the representation (4.51). On the other hand, since

$$(V_-f,f) \le -\mathrm{Im}\,(Hf,f) + (V_-f,f) = -\mathrm{Im}\,(Kf,f), \quad f \in \mathrm{dom}\,(K) = \mathrm{dom}\,(H),$$

Lemma 4.5(i) ensures the following representation

$$\det(I+iV_-(K-z)^{-1}) = \exp\left\{\frac{i}{\pi}\int_{\mathbb{R}}\frac{\eta_+(t)}{t-z}dt\right\}, \quad z \in \mathbb{C}_+. \tag{4.57}$$

Inserting (4.51) and (4.57) into (4.56) we arrive at the representation

$$w(z) = \widetilde{B}(z,\mathcal{Z}_+)\exp\left\{\frac{i}{\pi}\int_{\mathbb{R}}\frac{1}{t-z}d\mu(t)\right\}, \quad z \in \mathbb{C}_+.$$

Here $d\mu(t) = d\mu_+(t) - \eta_+ dt$ and $\mathcal{Z}_+ = \{(z_k^+, m_k^+)\}_k$ consists of the eigenvalues $\{z_k^+\}_k$ of $K_+' := K + iV_+ = H'$ and their multiplicities $\{m_k^+\}_k$. Hence $\mathcal{Z}_+ = \mathcal{Z}$. Since the operator H is m-accumulative, the function $w_-(\cdot) = \Delta_{H/K}(\cdot)$ has no zeros in $\mathbb{C}_+$. Combining this fact with representation (4.51) for $w_+(\cdot)$ we complete the proof (ii). □

Now we are ready to prove the trace formulas for a pair $\{H, H+V\}$ with m-accumulative operator H. A counterpart of Theorem 3.22 reads as follows.

Theorem 4.15. *Let H be a m-accumulative operator in $\mathfrak{H}$, $V \in \mathfrak{S}_1(\mathfrak{H})$ and let $H' := H+V$. Further, let $\mathcal{Z} = \{(z_k, m_k)\}_k$ be the eigenvalues of H' in $\mathbb{C}_+$ with multiplicities $\{m_k\}_k$. Then the following holds:*

(i) *$\mathcal{Z}$ is a Blaschke sequence satisfying condition* (4.49). *There exists a complex-valued Borel measure $d\nu(t) := id\mu_+(t) + \omega(t)dt$ on $\mathbb{R}$, where $d\mu_+(\cdot)$ is* a non-negative finite Borel measure *on $\mathbb{R}$ and $\omega(\cdot) \in L^1(\mathbb{R}; dt)$, such that the perturbation determinant $\Delta_{H'/H}(\cdot)$ admits the representation*

$$\Delta_{H'/H}(z) = \widetilde{\mathcal{B}}(z,\mathcal{Z})\exp\left\{\frac{1}{\pi}\int_{\mathbb{R}}\frac{1}{t-z}d\nu(t)\right\}, \quad z \in \mathbb{C}_+. \tag{4.58}$$

(ii) *The following trace formula holds*

$$\mathrm{tr}\,\left((H'-z)^{-1} - (H-z)^{-1}\right) = -\sum_k \frac{2i\, m_k \cdot \mathrm{Im}\,(z_k)}{(z-z_k)(z-\overline{z_k})} - \frac{1}{\pi}\int_{\mathbb{R}}\frac{d\nu(t)}{(t-z)^2}, \qquad z \in \mathbb{C}_+. \tag{4.59}$$

In particular,

$$\begin{aligned}\mathrm{tr}\,(V) &= 2i\sum_k m_k \mathrm{Im}\,(z_k) + \frac{1}{\pi}\int_{\mathbb{R}} d\nu(t)\\ &= 2i\sum_k\ m_k\mathrm{Im}\,(z_k) + \frac{i}{\pi}\int_{\mathbb{R}} d\mu_+(t) + \frac{1}{\pi}\int_{\mathbb{R}}\omega(t)dt.\end{aligned} \tag{4.60}$$

and

$$\mathrm{tr}\,(V_I) = 2\sum_k\ m_k\mathrm{Im}\,(z_k) + \frac{1}{\pi}\int_{\mathbb{R}} d\mu_+(t) + \frac{1}{\pi}\int_{\mathbb{R}}\omega_I(t)dt. \tag{4.61}$$

where $V_I := \mathrm{Im}\,(V)$ and $\omega_I(\cdot) := \mathrm{Im}\,(\omega(\cdot)) \le 0$, *Proof.* (i) Let $V_I = V_I^+ - V_I^-$ be the spectral decomposition of V_I, i.e. $V_I^\pm \geq 0$ and $V_I^\pm V_I^\mp = 0$. We set $K := H + \widetilde{V}$ and $\widetilde{V} := V_R - i|V_I|$, where $|V_I| = V_I^+ + V_I^-$. Clearly, the operator K is m-accumulative because so are H and $\widetilde{V}(\in [\mathfrak{H}])$.

Noting that $H' - K = 2iV_I^+$ we put

$$w_+(z) := \det(I + 2V_I^+(K - z)^{-1}) = \Delta_{H'/K}(z). \tag{4.62}$$

Clearly, $(2V_I^+ f, f) \leq -2\mathrm{Im}\,(Kf, f)$, $f \in \mathrm{dom}\,(K)$. Therefore Lemma 4.14(i) applies and leads to the representation (4.51) for $w_+(\cdot)$, where $d\mu_+(\cdot)$ is the *non-negative finite Borel measure* and $\mathcal{Z}_+ = \{(z_k^+, m_k^+)\}_k$ consists of the set of eigenvalues of $\{z_k^+\}_k$ of $K'_+ := K + 2iV_I^+ = H'$ and their algebraic multiplicities $\{m_k\}_k$. Hence $\mathcal{Z}_+ = \mathcal{Z}$.

Further, since H is accumulative, we have

$$\mathrm{Im}\,(\widetilde{V}f, f) = -(|V|f, f) \leq -\mathrm{Im}\,(Hf, f), \qquad f \in \mathrm{dom}\,(H).$$

Therefore, by Theorem 4.6(i), there exists *a complex-valued function* $\omega(\cdot) \in L^1(\mathbb{R}; dt)$ such that the following representation holds

$$\Delta_{K/H}(z) = \det(I + (V_R - i|V_I|)(H - z)^{-1}) = \exp\left\{\frac{1}{\pi}\int_{\mathbb{R}} \frac{\omega(t)}{t - z} dt\right\}, \quad z \in \mathbb{C}_+. \tag{4.63}$$

Setting $d\nu(t) := id\mu_+(t) + \omega(t)dt$ we define a complex-valued Borel measure on $\mathbb{R}$ satisfying $\int_{\mathbb{R}} |d\nu(t)| < \infty$. Finally, combining representation (4.51) for $w_+(\cdot) := \Delta_{H'/K}(\cdot)$ with representation (4.63) for $\Delta_{K/H}(\cdot)$ and applying the chain rule $\Delta_{H'/H}(\cdot) = \Delta_{H'/K}(\cdot)\Delta_{K/H}(\cdot)$, we arrive at (4.58).

(ii) Clearly, $\{z \in \mathbb{C} : \mathrm{Im}\,(z) > \|V_I^+\|\} \subset \rho(H')$. Taking the logarithmic derivative of both sides of (4.58) with $\mathrm{Im}\,(z) > \|V_I^+\|$ and applying (2.16) we obtain

$$\mathrm{tr}\,\left((H - z)^{-1} - (H' - z)^{-1}\right) = \sum_k m_k \left(\frac{1}{z - z_k} - \frac{1}{z - \overline{z}_k}\right) + \frac{1}{\pi}\int_{\mathbb{R}} \frac{d\nu(t)}{(t - z)^2},$$

which proves (4.59). Since V is bounded one rewrites this identity as

$$\mathrm{tr}\,\left((H' - z)^{-1} V (H - z)^{-1}\right) = \sum_k \frac{2i\, m_k \cdot \mathrm{Im}\,(z_k)}{(z - z_k)(z - \overline{z}_k)} + \frac{1}{\pi}\int_{\mathbb{R}} \frac{d\nu(t)}{(t - z)^2}.$$

Setting here $z = iy$, $y > \|V_I^+\|$, multiplying both sides by y^2, then passing to the limit as $y \uparrow \infty$ and applying Lemma 4.1, Lemma 4.13, and the dominated convergence theorem the relation, we arrive at the relation

$$\begin{aligned}
-\mathrm{tr}\,(V) &= -2i\sum_k\ m_k \mathrm{Im}\,(z_k) + \lim_{y\uparrow\infty} \frac{y^2}{\pi}\int_{\mathbb{R}} \frac{d\nu(t)}{(t - iy)^2} \\
&= -2i\sum_k\ m_k \mathrm{Im}\,(z_k) - \frac{1}{\pi}\int_{\mathbb{R}} d\nu(t) \\
&= -2i\sum_k\ m_k \mathrm{Im}\,(z_k) - \frac{i}{\pi}\int_{\mathbb{R}} d\mu_+(t) - \frac{1}{\pi}\int_{\mathbb{R}} \omega(t)dt.
\end{aligned}$$

which implies (4.60). In turn, the latter yields (4.61). □

Remark 4.16.

(i) Let H be a m-accumulative operator and $V \in \mathfrak{S}_1(\mathfrak{H})$. Further, let $H' = H + V$ Using Theorem 4.15 one easily proves results similar to those of Theorem 3.24 for the pair $\{H', H'^*\}$. To this end it is sufficient to use the formula

$$\Delta_{H'/H'^*}(z) = \frac{\Delta_{H'/K}(z)}{\Delta_{H'^*/K}(z)}, \qquad z \in \rho(K) \cap \rho(H'^*) \cap \mathbb{C}_+,$$

where $K := H + V_R - iV_-$, $V = V_R + iV_I^+ - iV_I^-$, $V_I^\pm \geq 0$, is m-accumulative.

(ii) If H is m-dissipative we get results similar to those of Theorem 3.31 assuming $V := H' - H \in \mathfrak{S}_1(\mathfrak{H})$ and $V_* := H' - H^* \in \mathfrak{S}_1(\mathfrak{H})$. The results follow from the formula

$$\Delta_{H'/H}(z) = \frac{\Delta_{H'/H^*}(z)}{\Delta_{H/H^*}(z)}, \qquad z \in \rho(H^*) \cap \rho(H) \cap \mathbb{C}_+,$$

Theorem 4.15 and (i).

Appendix

A Logarithm

We define the logarithm $\log(z)$ of a complex number $z \in \mathbb{C}$ by setting

$$\log(z) := -i \int_0^\infty \left((z + i\lambda)^{-1} - (1 + i\lambda)^{-1}\right) d\lambda, \quad z \in \mathbb{C}_+ \setminus -i\mathbb{R}_+, \tag{A.1}$$

with a cut along the negative imaginary semi-axis. One proves that $\log(e^z) = z, \quad e^z \in \mathbb{C}_+ \setminus -i\mathbb{R}_+$, which yields $e^{\log(z)} = z, \quad z \in \mathbb{C}_+ \setminus -i\mathbb{R}_+$.

Let $f(\cdot)$ and $g(\cdot)$ be holomorphic functions in a domain Ω satisfying $f(z) \neq 0$ and $f(z) = e^{g(z)}$. Then for a neighborhood $\mathcal{O}$ of a fixed point $z_0 \in \Omega$ such that $f(z_0)$ does not belong to the negative imaginary semi-axis one has $\log(f(z)) = g(z) + 2n\pi i,\ z \in \mathcal{O},\ n \in \mathbb{Z}$. By analytical continuation this equality can be extended to the whole Ω. Using definition (A.1) we find

$$\frac{d}{dz}\log(f(z)) = \frac{1}{f(z)}\frac{d}{dz}f(z), \quad z \in \Omega.$$

Let G be a bounded dissipative operator such that $0 \in \rho(G)$. Following [16] we define the logarithm of G by setting

$$\log(G) := -i \int_0^\infty \left((G + i\lambda)^{-1} - (1 + i\lambda)^{-1}\right) d\lambda \tag{A.2}$$

where the integral is understood in the operator norm. It is proved in [16] that $e^{\log(G)} = G$ and (see [16, Lemma 2.6]) $0 \leq \mathrm{Im}\,(\log(G)) \leq \pi I$. If $G \in \mathfrak{S}_1(\mathfrak{H})$ and is dissipative, then $\log(I + G) \in \mathfrak{S}_1(\mathfrak{H})$ and

$$\det(I + G) = e^{\mathrm{tr}\,(\log(I+G))}, \quad G \in \mathfrak{S}_1(\mathfrak{H}). \tag{A.3}$$

B Holomorphic functions in $\mathbb{C}_+$

Here we briefly recall factorizations theorems on functions from $H^\infty(\mathbb{C}_+;dt)$ and $\mathcal{N}(\mathbb{C}_+)$ following [21, Section VI C] and [14]. Let $F(\cdot) \in H^\infty(\mathbb{C}_+;dt)$ and let $\{z_k\}_{k\in\mathbb{N}}$ be the set of its zeros in $\mathbb{C}_+$, m_k the multiplicity of z_k, $k \in \mathbb{N}$. Then

$$\sum_k \frac{m_k \mathrm{Im}\,(z_k)}{1+|z_k|^2} < \infty. \tag{B.1}$$

In the following we call the sequence $\mathcal{Z} := \{(z_k, m_k)\}_k$ which takes into account the multiplicities, a Blaschke sequence. If a sequence $\{\alpha_k\}_{k\in\mathbb{N}} \subset \mathbb{R}$ is chosen such that $e^{i\alpha_k}(i-z_k)(i-\overline{z}_k)^{-1} \geq 0$, $k \in \mathbb{N}$, then with each Blaschke sequence $\mathcal{Z}$ one associates a Blaschke product $\mathcal{B}(z,\mathcal{Z})$ defined by

$$\mathcal{B}(z,\mathcal{Z}) := \prod_k (b_{z_k})^{m_k} := \prod_k \left(e^{i\alpha_k}\frac{z-z_k}{z-\overline{z}_k}\right)^{m_k}, \qquad z \in \mathbb{C}_+, \tag{B.2}$$

which converges uniformly on compact subsets of $\mathbb{C}_+$.

Moreover, $F_+(\cdot)$ admits (see [21, Section VI C]) the following representation

$$F(z) = \varkappa\, \mathcal{B}(z,\mathcal{Z}) \exp\left\{\frac{i}{\pi}\int_{\mathbb{R}}\left(\frac{1}{t-z} - \frac{t}{1+t^2}\right) d\mu(t)\right\} e^{i\alpha z}, \quad z \in \mathbb{C}_+, \tag{B.3}$$

where $\varkappa \in \mathbb{T}$, $\alpha \geq 0$ and $\mu(\cdot)$ is a non-decreasing function on $\mathbb{R}$ generating by a non-negative Borel measure and satisfying $\int_{\mathbb{R}} \frac{1}{1+t^2} d\mu_+(t) < \infty$. If $F(\cdot)$ has no zeros in $\mathbb{C}_+$, the Blaschke product $\mathcal{B}(\cdot,\mathcal{Z})$ in (B.3) is missing. Let $\mu = \mu^s + \mu^{ac}$ be the Lebesgue decomposition of μ, where μ^s and μ^{ac} are the singular and the absolutely continuous measures, respectively. Setting

$$\begin{aligned}
I_F(z) &:= \mathcal{B}(z,\mathcal{Z}) S_F(z) e^{i\alpha z}, \quad \alpha \geq 0,\\
S_F(z) &:= \exp\left\{\frac{i}{\pi}\int_R \left(\frac{1}{t-z} - \frac{t}{1+t^2}\right) d\mu^s(t)\right\},\\
\mathcal{O}_F(z) &:= \exp\left\{\frac{i}{\pi}\int_{\mathbb{R}} \left(\frac{1}{t-z} - \frac{t}{1+t^2}\right) d\mu^{ac}(t)\right\},
\end{aligned} \tag{B.4}$$

one gets the unique factorization $F(z) = \varkappa\, I_F(z)\mathcal{O}_F(z)$, $z \in \mathbb{C}_+$, where $\varkappa \in \mathbb{T}$ and $I_F(z)$ and $\mathcal{O}_F(z)$ are the inner and the outer factors, respectively. Note, that $|I_F(t+i0)| = 1$, $|\mathcal{O}_F(t+i0)| = |F(t+i0)|$ for a.e. $t \in \mathbb{R}$ and $d\mu^{ac}(t) = -\ln(|F(t+i0)|)dt$. Hence $\mathcal{O}_F(\cdot)$ admits the representation

$$\mathcal{O}_F(z) = \varkappa_+ \exp\left\{-\frac{i}{\pi}\int_R \left(\frac{1}{t-z} - \frac{t}{1+t^2}\right)\ln(|F(t+i0)|)\, dt\right\}, \quad z \in \mathbb{C}_+. \tag{B.5}$$

Clearly, $\mathcal{O}_F(i)$ is real, if and only if $\varkappa = 1$.

A holomorphic function U belongs to the Smirnov class $\mathcal{N}(\mathbb{C}_+)$ if it admits the representation $U = F/G$ where $F, G \in H^\infty(\mathbb{C}_+;dt)$ and G is an outer function. Any function $U \in \mathcal{N}(\mathbb{C}_+)$

admits the representation

$$
\begin{aligned}
U(z) =& \varkappa B(z,\mathcal{Z}) \exp\left\{\frac{i}{\pi}\int_{\mathbb{R}}\left(\frac{1}{t-i}-\frac{t}{1+t^2}\right)d\mu^s(t)\right\}\times \\
& \exp\left\{-\frac{i}{\pi}\int_{\mathbb{R}}\left(\frac{1}{t-i}-\frac{t}{1+t^2}\right)h(t)dt\right\}e^{i\alpha z}, \quad z\in\mathbb{C}_+,
\end{aligned}
\tag{B.6}
$$

where $\varkappa \in \mathbb{T}$, $\alpha \geq 0$, $\mu^s(\cdot)$ is a singular non-negative Borel measure, and $h \in L^1(\mathbb{R}; \frac{dt}{1+t^2})$. According to (B.6) functions $F, G \in H^\infty(\mathbb{C}_+; dt)$ can be chosen to be contractive. Indeed, let $\eta(t) := \max\{h(t), 0\} \geq 0$, and $k(\cdot) := \eta(\cdot) - h(\cdot) \geq 0$. Setting $d\mu(\cdot) := d\mu^s(\cdot) + k(\cdot)dt$ and

$$
\begin{aligned}
F(z) :=& \varkappa B(z,\mathcal{Z}) \exp\left\{\frac{i}{\pi}\int_{\mathbb{R}}\left(\frac{1}{t-i}-\frac{t}{1+t^2}\right)d\mu(t)\right\}e^{i\alpha z}, \quad z\in\mathbb{C}_+, \\
G(z) :=& \exp\left\{\frac{i}{\pi}\int_{\mathbb{R}}\left(\frac{1}{t-i}-\frac{t}{1+t^2}\right)\eta(t)dt\right\}, \quad z\in\mathbb{C}_+,
\end{aligned}
\tag{B.7}
$$

we arrive at representation $U = F/G$ with contractive analytic F and contractive outer G. Summing up we arrive at the following statement.

Lemma B.1. *Assume that $U \in \mathcal{N}(\mathbb{C}_+)$. Then there exists a non-negative Borel measure $\mu(\cdot)$ satisfying $\int_{\mathbb{R}} \frac{d\mu(t)}{1+t^2}dt < \infty$, a non-negative $\eta(\cdot) \in L^1(\mathbb{R}; \frac{dt}{1+t^2})$, and constants $\varkappa \in \mathbb{T}$, $\alpha \geq 0$, such that the representation*

$$
U(z) = \varkappa B(z,\mathcal{Z}) \exp\left\{\frac{i}{\pi}\int_{\mathbb{R}}\left(\frac{1}{t-i}-\frac{t}{1+t^2}\right)d\nu(t)\right\}
\tag{B.8}
$$

holds where $d\nu(\cdot) = d\mu(\cdot) - \eta(\cdot)dt$.

C On $H^1(\mathbb{D})$ functions

Let $\mathbb{D} := \{w \in \mathbb{C} : |w| < 1\}$. By $\ln(\cdot)$ we denote a branch of the logarithm such that $\ln(z) \in \mathbb{R}$ for $z \in \mathbb{R}_+$ and $\mathrm{Im}\,(\ln(z)) \in (-\pi/2, \pi/2)$ for $\mathrm{Re}\,(z) > 0$.

Lemma C.1. *Let $H(w)$ be a holomorphic function in $\mathbb{D}$ such that $\mathrm{Re}\,(H(w)) \geq 0$ for $w \in \mathbb{D}$. Let $G(w) := \ln(1 + H(w))$ for $w \in \mathbb{D}$. Then $G(w) \in H^1(\mathbb{D})$ and the following estimate holds*

$$
0 \leq \int_{-\pi}^{\pi} \mathrm{Re}\,(G(e^{i\theta}))d\theta \leq 2\pi\,|H(0)|.
\tag{C.1}
$$

Proof. Obviously we have $|\mathrm{Im}\,(G(w))| \leq \pi/2$, $w \in \mathbb{D}$. Furthermore, we have

$$
G(0) = \frac{1}{2\pi}\int_{-\pi}^{\pi} G(re^{i\theta})d\theta, \quad r \in (0,1),
$$

which yields $2\pi\mathrm{Re}\,(G(0)) = \int_{-\pi}^{\pi} \mathrm{Re}\,(G(re^{i\theta}))d\theta$.

Further, since $\mathrm{Re}\,(G(re^{i\theta})) \geq 0$, it follows $\|G\|_{H^1} \leq 2\pi \mathrm{Re}\,(G(0)) + \pi^2$, which yields $G \in H^1(\mathbb{D})$. In particular, we have

$$\|G_R\|_{L^1} = 2\pi G_R(0), \quad G_R(w) = \mathrm{Re}\,(G(w)), \quad w \in \mathbb{D}. \tag{C.2}$$

Using the estimate $\mathrm{Re}\,(G(0)) = \ln(|1 + H(0)|) \leq |H(0)|$ we arrive at (C.1). $\square$

The result can be carried over to upper half-plane.

Corollary C.2. *Let $h(\cdot)$ be a holomorphic function in $\mathbb{C}_+$ with non-negative real part. Let $g(z) := \ln(1 + h(z))$ for $z \in \mathbb{C}_+$. Then the following estimate*

$$\int_{\mathbb{R}} |g(x + i0)| \frac{dx}{1 + x^2} \leq 2\pi\, |h(i)| \tag{C.3}$$

is valid where $g(x + i0) := \lim_{y \downarrow 0} g(x + iy)$.

Proof. We set $H(w) := h\left(i\frac{1+w}{1-w}\right)$ and $G(w) := \ln(1 + H(w)) = g\left(i\frac{1+w}{1-w}\right)$. Since

$$\int_{-\pi}^{\pi} |G(e^{i\theta})| d\theta = \int_{\mathbb{R}} |g(x + i0)| \frac{dx}{1 + x^2}$$

and $h(i) = H(0)$, the result is implied by (C.1). $\square$

D Riesz-Dunford functional calculus

Let T be a densely defined closed operator in $\mathfrak{H}$. A function Φ is put in the class $\mathcal{F}_\pm(T)$ if there is a simple closed curve Γ in $\mathbb{C}_\pm$ which does not intersect the real axis and such that

(i) the exterior domain $\Omega^{\mathrm{ext}}_\Gamma$ with $\partial\Omega^{\mathrm{ext}}_\Gamma = \Gamma$ contains the spectrum of T;

(ii) there is a neighborhood $\mathcal{O}$ of the closed set $\overline{\Omega^{\mathrm{ext}}_\Gamma}$ such that Φ is holomorphic in $\mathcal{O}$ including the infinity.

If $\rho(T) \cap \mathbb{C}_+ \neq \emptyset$ (resp. $\rho(T) \cap \mathbb{C}_- \neq \emptyset$), then the class $\mathcal{F}_+(T) \neq \emptyset$ ($\mathcal{F}_-(T) \neq \emptyset$. In this case one defines $\Phi(T)$ by

$$\Phi(T) := \Phi(\infty)I + \frac{1}{2\pi i}\oint_\Gamma \Phi(z)(T - z)^{-1} dz \tag{D.1}$$

where the integral $\oint_\Gamma$ is taken in mathematical positive sense with respect to the open inner domain $\Omega^{\mathrm{in}}_\Gamma$, see [13, Section VII.9]. We note that

$$\Phi(\xi) = \Phi(\infty) - \frac{1}{2\pi i}\oint_\Gamma \frac{\Phi(z)}{z - \xi} dz, \quad \xi \in \Omega^{\mathrm{ext}}_\Gamma, \tag{D.2}$$

Since $\oint_\Gamma |\Phi(z)| \; |dz| < \infty$, the integral $\oint_\Gamma \Phi(z)dz$ is well-defined. Hence the residuum $\mathrm{res}_\infty(\Phi)$, $\Phi \in \mathcal{F}(T)$, is well defined by

$$\mathrm{res}_\infty(\Phi) := -\frac{1}{2\pi i} \oint_\Gamma \Phi(z)dz \tag{D.3}$$

Since the curve Γ does not intersect the real axis, and $\oint_\Gamma |\Phi(z)| \; |dz| < \infty$, we get from (D.2) that $\sup_{t\in\mathbb{R}}(1+t^2)|\Phi'(t)| < \infty$ for $\Phi \in \mathcal{F}(T)$.

Let T and T' be two densely defined closed operators. We set $\mathcal{F}_\pm(T,T') := \mathcal{F}_\pm(T)\cap\mathcal{F}_\pm(T')$, that is, there is a simple closed curve Γ such that $\Omega_\Gamma^{\mathrm{ext}}$ contains the spectra of both T and T'. If $\rho(T)\cap\rho(T') \neq \emptyset$, then $\mathcal{F}(T,T') \neq \emptyset$.

Lemma D.1. *Let T and T' be two densely defined closed operators in $\mathfrak{H}$ such that $\rho(T) \cap \rho(T') \neq \emptyset$. If the condition $(T'-\xi)^{-1} - (T-\xi)^{-1} \in \mathfrak{S}_1(\mathfrak{H})$ for some $\xi \in \rho(T)\cap\rho(T')\cap\mathbb{C}_+$ (resp. $\xi \in \rho(T)\cap\rho(T')\cap\mathbb{C}_-$), then $\Phi(T') - \Phi(T) \in \mathfrak{S}_1(\mathfrak{H})$ for $\Phi \in \mathcal{F}_+(T,T')$ (resp. $\Phi \in \mathcal{F}_-(T,T')$).*

Proof. By definition (D.1), $\Phi(T') - \Phi(T) = \frac{1}{2\pi i}\oint_\Gamma \Phi(z)((T'-z)^{-1} - (T-z)^{-1})dz$. Hence the following estimate holds

$$\|\Phi(T') - \Phi(T)\|_{\mathfrak{S}_1} \le \sup_{z\in\Gamma} \|(T'-z)^{-1} - (T-z)^{-1}\|_{\mathfrak{S}_1} \frac{1}{2\pi} \oint_\Gamma |\Phi(z)||dz| < \infty,$$

which completes the proof. □

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