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On a fractional harmonic replacement

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ABSTRACT. Given $s \in (0, 1)$, we consider the problem of minimizing the Gagliardo seminorm in H^s with prescribed condition outside the ball and under the further constraint of attaining zero value in a given set K .

We investigate how the energy changes in dependence of such set. In particular, under mild regularity conditions, we show that adding a set A to K increases the energy of at most the measure of A (this may be seen as a perturbation result for small sets A).

Also, we point out a monotonicity feature of the energy with respect to the prescribed sets and the boundary conditions.

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1. INTRODUCTION

This paper deals with an harmonic replacement of nonlocal type with a prescribed zero set. We obtain energy monotonicity results with respect to the data and the zero set, and some perturbative estimates with respect to the variation of the zero set.

We fix $s \in (0, 1)$, the unit ball $B_1 \subseteq \mathbb{R}^n$ and a function $\varphi : \mathbb{R}^n \rightarrow \mathbb{R}$ such that $\varphi \in L^\infty(\mathbb{R}^n)$ and

$$(1.1) \quad \iint_{\mathbb{R}^{2n} \setminus (B_1^c)^2} \frac{|\varphi(x) - \varphi(y)|^2}{|x - y|^{n+2s}} dx dy < +\infty,$$

where B_1^c , as usual, denotes the complementary set of B_1 .

Given (measurable) functions v and $w : \mathbb{R}^n \rightarrow \mathbb{R}$ we use the notation

$$(1.2) \quad \langle v, w \rangle := \iint_{\mathbb{R}^{2n} \setminus (B_1^c)^2} \frac{(v(x) - v(y))(w(x) - w(y))}{|x - y|^{n+2s}} dx dy$$

and $\nu(v) := \sqrt{\langle v, v \rangle} = \sqrt{\iint_{\mathbb{R}^{2n} \setminus (B_1^c)^2} \frac{|v(x) - v(y)|^2}{|x - y|^{n+2s}} dx dy}.$

This functional setting is naturally compatible with the fractional Laplace operator

$$(-\Delta)^s v(x) = \int_{\mathbb{R}^n} \frac{2v(x) - v(x+y) - v(x-y)}{|y|^{n+2s}} dy.$$

We consider the space

$$(1.3) \quad \mathcal{D}^\varphi := \left\{ v : \mathbb{R}^n \rightarrow \mathbb{R} \text{ s.t. } v - \varphi \in L^2(\mathbb{R}^n), v = \varphi \text{ a.e. in } B_1^c \text{ and } \nu(v) < +\infty \right\}.$$

Also, given a measurable set $K \subseteq B_1$, we set

$$(1.4) \quad \mathcal{D}_K^\varphi := \left\{ v \in \mathcal{D}^\varphi \text{ s.t. } v = 0 \text{ a.e. in } K \right\}.$$

Roughly speaking, φ will play the role of a boundary prescription outside B_1 , in a sense compatible with the functional structure introduced in (1.2). We will deal with fractional energy minimizers among functions that vanish in the set K according to the following definition.

Definition 1.1. Assume that there exists $\bar{v} \in \mathcal{D}_K^\varphi$ with $\nu(\bar{v}) < +\infty$.

Then we denote by φ_K^φ (or simply φ_K when there is no ambiguity) the function that attains the minimal energy value

$$(1.5) \quad \min_{v \in \mathcal{D}_K^\varphi} \nu(v).$$

Using the direct method in the calculus of variations, in the subsequent Lemma 2.1 we will show that the minimum in (1.5) indeed exists, so Definition 1.1 is well posed. Notice that when $K := \emptyset$ then φ_\emptyset is simply the s -harmonic function in B_1 that minimizes the fractional Gagliardo seminorm ν with prescribed datum φ outside B_1 . When $K \neq \emptyset$, we are “replacing” such φ_\emptyset with a new φ_K that has the additional prescription to vanish in K , by paying the less possible amount of energy. For this reason, we call φ_K the “ s -harmonic replacement” outside K .

The s -harmonic replacement enjoys a useful monotonicity property.

Theorem 1.2. Let $A_1, A_2 \in B_1$.

Given $i \in \{1, 2\}$, we define $\varphi_i := \varphi_{K_i}^{\varphi_{i,0}}$ and $\tilde{\varphi}_i := \varphi_{K_i \cup A_i}^{\varphi_{i,0}}$ (that is: φ_i is the s -harmonic extension vanishing in K_i with datum $\varphi_{i,0}$, and $\tilde{\varphi}_i$ is the s -harmonic extension vanishing in $K_i \cup A_i$, with the same datum).

Assume that

$$0 \leq \varphi_{1,0} \leq \varphi_{2,0}, \quad K_1 \supseteq K_2 \quad \text{and} \quad A_1 \subseteq A_2.$$

Then

$$\nu^2(\tilde{\varphi}_1) - \nu^2(\varphi_1) \leq \nu^2(\tilde{\varphi}_2) - \nu^2(\varphi_2).$$

Next purpose of our paper is to estimate the energy difference of the s -harmonic replacements of K and $K \cup A$, for a given set A (which can be seen as a “perturbation”) in terms of the Lebesgue measure of A . The results that we provide are the following:

Theorem 1.3. Let $\varphi \geq 0$ and $\rho \in [1/4, 3/4]$ and let $A := B_\rho \setminus K$. Then

$$\nu^2(\varphi_{K \cup A}) - \nu^2(\varphi_K) \leq C |A| \|\varphi_K\|_{L^\infty(\mathbb{R}^n)}^2,$$

for some $C > 0$ depending on n and s .

Theorem 1.4. Let $\varphi \geq 0$, $K \supseteq B_{1/2}$ and $A \subseteq B_{3/4} \setminus B_{1/2}$. Suppose that A is closed and satisfies the following density property: there exists $c > 0$ such that for every $x \in \partial A$ and every $r \in (0, 2)$, we have that

$$(1.6) \quad |A \cap B_r(x)| \geq c |B_r|.$$

Then

$$\nu^2(\varphi_{K \cup A}) - \nu^2(\varphi_K) \leq C |A| \|\varphi_K\|_{L^\infty(\mathbb{R}^n)}^2,$$

for some $C > 0$ depending on c , n and s .

We observe that sets with Lipschitz boundary obviously satisfy the density property in (1.6). Also, we notice that the geometry of the perturbing set A in Theorem 1.4 is different from the one in Theorem 1.3: namely, in Theorem 1.3 the set A may be thought as “exiting” from K in the interior of B_1 , while in Theorem 1.4 the set A “stretches out” from K towards the boundary. Possible pictures for the geometries of the sets involved in Theorems 1.3 and 1.4 are depicted in Figures A and B respectively. In both the figures the set K is painted in black and A is the dark gray region (of course, Theorems 1.3 and 1.4 are interesting when A is a “small” perturbation, but for obvious aesthetic reasons the sets A drawn in the figures are “not so small”).

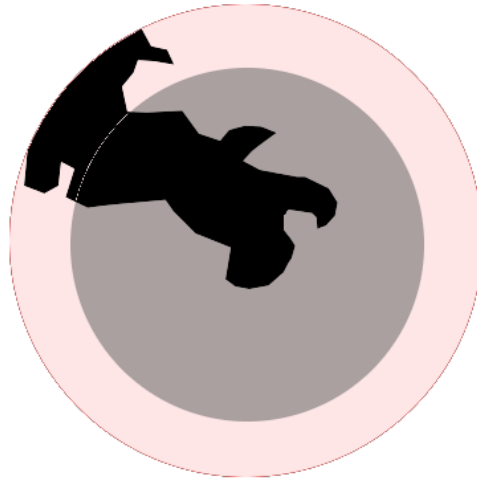


Figure A: The geometry involved in Theorem 1.3.

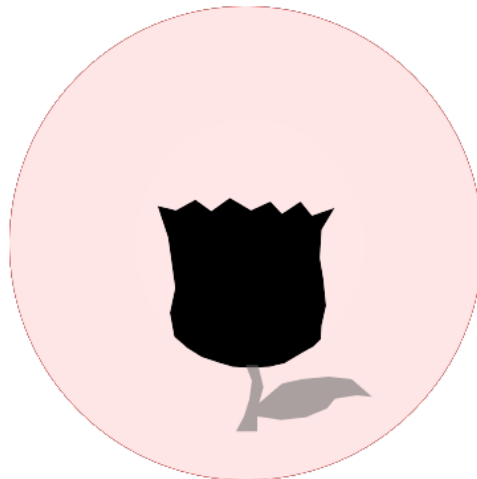


Figure B: The geometry involved in Theorem 1.4.

In the local case of the harmonic replacement (i.e. the classical minimization problem of the Dirichlet energy) the results presented in this paper were obtained in [2]. Theorems 1.3 and 1.4 may be seen as perturbative statements, namely they estimate the change of energy in terms of the (possibly

small) set A . It is worth pointing out that the estimates obtained are simply in terms of the Lebesgue measure of A and only require very mild regularity assumptions on the set (in fact, only the density assumption (1.6), and no high derivative of the boundary of A comes into play).

We also observe that once Theorems 1.3 and 1.4 are proved for minimizers in a ball (say, B_1), then they hold true for minimizers in any open set Ω : this follows from the fact that one can suppose $\Omega \supset B_1$ (up to scaling) and obtain from the set inclusions that

$$\begin{aligned} & \iint_{\mathbb{R}^{2n} \setminus (\Omega^c)^2} \frac{|\varphi(x) - \varphi(y)|^2}{|x - y|^{n+2s}} dx dy - \iint_{\mathbb{R}^{2n} \setminus (B_1^c)^2} \frac{|\varphi(x) - \varphi(y)|^2}{|x - y|^{n+2s}} dx dy \\ &= \iint_{(\Omega \setminus B_1)^2} \frac{|\varphi(x) - \varphi(y)|^2}{|x - y|^{n+2s}} dx dy + 2 \iint_{\Omega^c \times (\Omega \setminus B_1)} \frac{|\varphi(x) - \varphi(y)|^2}{|x - y|^{n+2s}} dx dy \end{aligned}$$

and the latter two integral terms do not depend on the values of φ in B_1 : accordingly, a minimizer in a domain Ω which contains B_1 is also a minimizer in B_1 .

Also, the values $1/4$, $1/2$ and $3/4$ in Theorems 1.3 and 1.4 do not play any role, in the sense that they can be replaced by some r_1 , r_2 and r_3 respectively, with $0 < r_1 < r_2 < r_3 < 1$ (but in this case the constants would depend on r_1 , r_2 and r_3).

As for the applications of our result, we notice that, in the local setting, the Dirichlet integral may be interpreted in terms of the classical heat equation as a sort of thermal energy: in this sense, the Dirichlet integral of the harmonic replacement with boundary data $\varphi \geq 0$ that vanishes in K represents the insulating energy of a room whose walls are fixed at temperature φ and having a “fridge” at the set K where the temperature is zero. In this framework, we may consider the fractional harmonic replacement as a nonlocal modification of this problem, in which the classical heat equation is replaced by a nonlocal one, which is generated by a non-Gaussian diffusive process, see e.g. [9]. Also, harmonic replacements play an important role in the Perron method and in free boundary problems, see e.g. [5] and [1].

The paper is organized as follows. In Section 2 we show some properties of the fractional harmonic replacement. In Section 3 we deal with the monotonicity property given in Theorem 1.2. Finally, Sections 4 and 5 are devoted to the proofs of Theorems 1.3 and 1.4

2. PRELIMINARIES ON THE s -HARMONIC REPLACEMENT

Lemma 2.1. *The minimum in (1.5) is attained by a unique minimizer.*

Proof. First we prove the existence of the minimum. For this, let $v_j \in \mathcal{D}_K^\varphi$ be a minimizing sequence. In particular, from (1.1), we may suppose that $\nu(v_j) \leq \nu(\bar{v})$, which is finite. Set $w_j := v_j - \varphi$. Then w_j vanishes outside B_1 , thus

$$\begin{aligned} & \sqrt{\iint_{\mathbb{R}^{2n}} \frac{|w_j(x) - w_j(y)|^2}{|x - y|^{n+2s}} dx dy} = \sqrt{\iint_{\mathbb{R}^{2n} \setminus (B_1^c)^2} \frac{|w_j(x) - w_j(y)|^2}{|x - y|^{n+2s}} dx dy} \\ &= \nu(v_j - \varphi) \leq \nu(v_j) + \nu(\varphi) \leq \nu(\bar{v}) + \nu(\varphi), \end{aligned}$$

which is finite, thanks to (1.1). Therefore (see, e.g. Theorem 7.1 in [3]) we obtain, up to subsequence, that w_j converges in $L^2(B_1)$ and a.e. in \mathbb{R}^n to some w . Accordingly, v_j converges in $L^2(B_1)$ and a.e.

in \mathbb{R}^n to $v := w + \varphi$. In particular, $v \in L^2(B_1)$, $v = 0$ a.e. in K and $v = \varphi$ a.e. in B_1^c . Moreover, by Fatou Lemma,

$$\nu(v) \leq \liminf_{j \rightarrow +\infty} \nu(v_j),$$

which says that $\nu(v)$ is finite. Thus $v \in \mathcal{D}_K^\varphi$ and attains the desired minimum.

Now we show that the minimizer is unique. Suppose that u and v are minimizers in \mathcal{D}_K^φ , i.e.

$$\min_{\mathcal{D}_K^\varphi} \nu = \nu(u) = \nu(v).$$

Let $w(x) := (u(x) + v(x))/2$. Notice that $w \in \mathcal{D}_K^\varphi$, hence

$$(2.1) \quad \nu^2(w) \geq \min_{\mathcal{D}_K^\varphi} \nu^2 = \frac{1}{2}(\nu^2(u) + \nu^2(v)).$$

We denote $\delta u(x, y) := u(x) - u(y)$. For any $r \in \mathbb{R}^n$, let also $f(r) := |r|^2$. By convexity

$$(2.2) \quad f(\delta w(x, y)) = f\left(\frac{\delta u(x, y) + \delta v(x, y)}{2}\right) \leq \frac{1}{2}\left(f(\delta u(x, y)) + f(\delta v(x, y))\right)$$

and

$$(2.3) \quad \text{strict inequality in (2.2) holds whenever } \delta u(x, y) \neq \delta v(x, y).$$

Let $\mathcal{Z} := \{(x, y) \in \mathbb{R}^{2n} \text{ s.t. a strict inequality holds in (2.2)}\}$. We claim that

$$(2.4) \quad \mathcal{Z} \text{ is of measure zero.}$$

Indeed, assume by contradiction that \mathcal{Z} has positive measure. By dividing by $|x - y|^{n+2s}$ and integrating (2.2), and recalling (2.1), we see that

$$\begin{aligned} 0 &\leq \nu^2(w) - \frac{1}{2}(\nu^2(u) + \nu^2(v)) \\ &= \iint_{\mathbb{R}^{2n} \setminus (B_1^c)^2} \frac{f(\delta w(x, y)) - \frac{1}{2}\left(f(\delta u(x, y)) + f(\delta v(x, y))\right)}{|x - y|^{n+2s}} dx dy \\ &= \iint_{\mathbb{R}^{2n}} \frac{f(\delta w(x, y)) - \frac{1}{2}\left(f(\delta u(x, y)) + f(\delta v(x, y))\right)}{|x - y|^{n+2s}} dx dy \\ &= \iint_{\mathcal{Z}} \frac{f(\delta w(x, y)) - \frac{1}{2}\left(f(\delta u(x, y)) + f(\delta v(x, y))\right)}{|x - y|^{n+2s}} dx dy \\ &< 0. \end{aligned}$$

This contradiction establishes (2.4).

By construction, we have that equality holds in (2.2) for every $(x, y) \in \mathbb{R}^{2n} \setminus \mathcal{Z}$, and therefore, by (2.3),

$$(2.5) \quad u(x) - u(y) = \delta u(x, y) = \delta v(x, y) = v(x) - v(y) \text{ for every } (x, y) \in \mathbb{R}^{2n} \setminus \mathcal{Z}.$$

Now we observe that

$$(2.6) \quad \begin{aligned} &\text{there exist } \bar{y} \in \mathbb{R}^n \text{ and } \mathcal{V} \subset \mathbb{R}^n \text{ such that} \\ &|\mathcal{V}| = 0, \text{ and} \\ &(x, \bar{y}) \in \mathbb{R}^{2n} \setminus \mathcal{Z} \text{ for any } x \in \mathbb{R}^n \setminus \mathcal{V}. \end{aligned}$$

The proof of (2.6) relies on Fubini's theorem, we give the details for completeness. For any $y \in \mathbb{R}^n$, let

$$b(y) := \int_{\mathbb{R}^n} \chi_{\mathcal{Z}}(x, y) dx.$$

Then b is a nonnegative and measurable function, and

$$\int_{\mathbb{R}^n} b(y) dy = \iint_{\mathbb{R}^{2n}} \chi_{\mathcal{Z}}(x, y) dx dy = |\mathcal{Z}| = 0,$$

due to (2.4). Accordingly $b(y) = 0$ for a.e. $y \in \mathbb{R}^n$. In particular, we can fix $\bar{y} \in \mathbb{R}^n$ such that $b(\bar{y}) = 0$, that is

$$\int_{\mathbb{R}^n} \chi_{\mathcal{Z}}(x, \bar{y}) dx = 0.$$

As a consequence $\chi_{\mathcal{Z}}(x, \bar{y}) = 0$ for a.e. $x \in \mathbb{R}^n$ (say, for every $x \in \mathbb{R}^n \setminus \mathcal{V}$, for a suitable $\mathcal{V} \subset \mathbb{R}^n$ of zero measure). This concludes the proof of (2.6).

Using (2.5) and (2.6) we deduce that $u(x) - u(\bar{y}) = v(x) - v(\bar{y})$ for every $x \in \mathbb{R}^n \setminus \mathcal{V}$, that is, setting $c := u(\bar{y}) - v(\bar{y})$, we have that $u(x) = v(x) + c$ for a.e. $x \in \mathbb{R}^n$.

By taking $x \in B_1^c$ such that $u(x) = v(x) = \varphi(x)$, we obtain that $c = 0$, and therefore $u = v$ a.e. in \mathbb{R}^n . This completes the uniqueness result and ends the proof of Lemma 2.1. \square

Lemma 2.2. *For any $\psi \in H^s(\mathbb{R}^n)$ with $\psi = 0$ a.e. in $B_1^c \cup K$ we have*

$$(2.7) \quad \langle \varphi_K, \psi \rangle = 0$$

and

$$(2.8) \quad \nu^2(\varphi_K - \psi) - \nu^2(\varphi_K) = \nu^2(\psi).$$

Proof. Given $\varepsilon \in (-1, 1)$, we observe that $\varphi_K + \varepsilon\psi \in \mathcal{D}_K^\varphi$, hence $\nu(\varphi_K + \varepsilon\psi) \geq \nu(\varphi_K)$, which gives (2.7). Then, (2.8) easily follows from (2.7), using that

$$\nu^2(\varphi_K - \psi) = \nu^2(\varphi_K) + \nu^2(\psi) - 2\langle \varphi_K, \psi \rangle. \quad \square$$

Lemma 2.3. *We have that $(-\Delta)^s \varphi_K(x) = 0$ for any x in the interior of $B_1 \setminus K$.*

Proof. Let x be in the interior of $B_1 \setminus K$. Then, there exists $\rho > 0$ such that $B_\rho(x) \subset B_1 \setminus K$. So, by Lemma 2.2, for any $\psi \in C_0^\infty(B_\rho(x))$, we have that $\langle \varphi_K, \psi \rangle = 0$. This says that φ_K is s -harmonic in $B_1 \setminus K$ (in the weak and so in the strong sense, see e.g. [8]). \square

Additional properties of the s -harmonic replacement hold true if the datum φ has a sign, according to the next results.

Lemma 2.4. *We have that $\varphi_K \leq \|\varphi\|_{L^\infty(\mathbb{R}^n)}$. Moreover, if $\varphi \geq 0$ then $\varphi_K \geq 0$.*

Proof. First we point out that, if $u^+ := \max\{u, 0\}$, then

$$(2.9) \quad |u^+(x) - u^+(y)| \leq |u(x) - u(y)|$$

and

$$(2.10) \quad |u^+(x) - u^+(y)|^2 \leq (u^+(x) - u^+(y))(u(x) - u(y)),$$

see e.g. (8.10) in [4] for a simple proof.

Now let $\psi_1 := (-\varphi_K)^+$ (if $\varphi \geq 0$, otherwise disregard the argument involving ψ_1) and $\psi_2 := (\varphi_K - \|\varphi\|_{L^\infty(\mathbb{R}^n)})^+$ (independently from the fact that $\varphi \geq 0$). We have that $\psi_1 = \psi_2 = 0$ in $B_1^c \cup K$. Also, by (2.9), $\nu(\psi_i) \leq \nu(\varphi_K) < +\infty$ and $\|\psi_i\|_{L^2(B_1)} \leq \|\varphi_K\|_{L^2(B_1)} + \|\varphi\|_{L^2(B_1)} < +\infty$, thus $\psi \in H^s(\mathbb{R}^n)$.

Therefore, by (2.7),

$$\langle \varphi_K, \psi_1 \rangle = \langle \varphi_K, \psi_2 \rangle = 0.$$

Furthermore, using (2.10) with $u := -\varphi_K$, we have

$$|\psi_1(x) - \psi_1(y)|^2 \leq -(\psi_1(x) - \psi_1(y))(\varphi_K(x) - \varphi_K(y))$$

and so

$$(2.11) \quad \nu^2(\psi_1) \leq -\langle \psi_1, \varphi_K \rangle = 0.$$

Similarly, using (2.10) with $u := \varphi_K - \|\varphi\|_{L^\infty(\mathbb{R}^n)}$, we have

$$|\psi_2(x) - \psi_2(y)|^2 \leq (\psi_2(x) - \psi_2(y))(\varphi_K(x) - \varphi_K(y)),$$

which gives

$$(2.12) \quad \nu^2(\psi_2) \leq \langle \psi_2, \varphi_K \rangle = 0.$$

By (2.11) and (2.12) we conclude that ψ_1 and ψ_2 vanish identically, which implies the desired result. \square

Lemma 2.5. *If $\varphi \geq 0$ then*

$$\min_{v \in \mathcal{D}_K^\varphi} \nu(v) = \min_{v \in \mathcal{D}_K^{0,\varphi}} \nu(v),$$

where¹

$$(2.13) \quad \mathcal{D}_K^{0,\varphi} := \left\{ v \in \mathcal{D}^\varphi \text{ s.t. } v \leq 0 \text{ a.e. in } K \right\}.$$

Proof. Since $\mathcal{D}_K^\varphi \subseteq \mathcal{D}_K^{0,\varphi}$, we have that

$$\min_{v \in \mathcal{D}_K^\varphi} \nu(v) \geq \min_{v \in \mathcal{D}_K^{0,\varphi}} \nu(v).$$

Viceversa, given $v \in \mathcal{D}_K^{0,\varphi}$, from (2.9) we have that $\nu(v^+) \leq \nu(v)$. Also, $v^+ = 0$ in K and $v^+ = \max\{\varphi, 0\} = \varphi$ in B_1^c , therefore $v^+ \in \mathcal{D}_K^\varphi$. It follows that

$$\min_{v \in \mathcal{D}_K^\varphi} \nu(v) \leq \min_{v \in \mathcal{D}_K^{0,\varphi}} \nu(v). \quad \square$$

Lemma 2.6. *If $\varphi \geq 0$ then, for any $\psi \in H^s(\mathbb{R}^n)$ with $\psi \geq 0$ a.e. in B_1 and $\psi = 0$ a.e. in B_1^c ,*

$$\langle \varphi_K, \psi \rangle \leq 0.$$

Proof. Given $\varepsilon > 0$ we set $\psi_\varepsilon := \varphi_K - \varepsilon\psi$. By construction $\psi_\varepsilon \in \mathcal{D}_K^{0,\varphi}$ and so $\nu(\psi_\varepsilon) \geq \nu(\varphi_K)$, thanks to Lemma 2.5, which gives the desired result. \square

¹Clearly, the difference between (1.4) and (2.13) is in the fact that functions in $\mathcal{D}_K^{0,\varphi}$ are allowed to take also negative values in K .

Now we consider $\varphi_{K \cup A}$, for some measurable set $A \in B_1$. For this, we introduce the set

$$(2.14) \quad \mathcal{A}_{K,A}^\varphi := \left\{ v \in H^s(\mathbb{R}^n) \text{ s.t. } v = 0 \text{ a.e. in } B_1^c \cup K \text{ and } v \geq \varphi_K \text{ a.e. in } A \right\}.$$

Notice that

$$(2.15) \quad \mathcal{A}_{K,A}^\varphi \subseteq \varphi_K - \mathcal{D}_{K \cup A}^{0,\varphi}$$

and

$$(2.16) \quad \varphi_K - \mathcal{D}_{K \cup A}^\varphi \subseteq \mathcal{A}_{K,A}^\varphi.$$

Lemma 2.7. *If $\varphi \geq 0$, we have that*

$$\nu^2(\varphi_{K \cup A}) - \nu^2(\varphi_K) = \inf_{\psi \in \mathcal{A}_{K,A}^\varphi} \nu^2(\psi).$$

Proof. By Lemma 2.5, (2.15) and (2.8), we have that

$$\begin{aligned} \nu^2(\varphi_{K \cup A}) - \nu^2(\varphi_K) &= \inf_{v \in \mathcal{D}_{K \cup A}^{0,\varphi}} \nu^2(v) - \nu^2(\varphi_K) \\ &= \inf_{\psi \in \varphi_K - \mathcal{D}_{K \cup A}^{0,\varphi}} \nu^2(\varphi_K - \psi) - \nu^2(\varphi_K) \\ &\leq \inf_{\psi \in \mathcal{A}_{K,A}^\varphi} \nu^2(\varphi_K - \psi) - \nu^2(\varphi_K) \\ &= \inf_{\psi \in \mathcal{A}_{K,A}^\varphi} \nu^2(\psi). \end{aligned}$$

Viceversa, using (2.8) and (2.16), we have that

$$\begin{aligned} \nu^2(\varphi_{K \cup A}) - \nu^2(\varphi_K) &= \inf_{v \in \mathcal{D}_{K \cup A}^\varphi} \nu^2(v) - \nu^2(\varphi_K) \\ &= \inf_{\psi \in \varphi_K - \mathcal{D}_{K \cup A}^\varphi} \nu^2(\varphi_K - \psi) - \nu^2(\varphi_K) \\ &\geq \inf_{\psi \in \mathcal{A}_{K,A}^\varphi} \nu^2(\psi). \end{aligned}$$

By combining the two inequalities, we obtain the desired result. \square

3. MONOTONICITY PROPERTY AND PROOF OF THEOREM 1.2

In the light of the lemmata discussed in Section 2, we can now prove the monotonicity property of s -harmonic replacements:

Proof of Theorem 1.2. We let \bar{v} minimize ν among all the functions v such that $v - \tilde{\varphi}_2 \in H^s(\mathbb{R}^n)$, with $v = \tilde{\varphi}_2$ a.e. in $K_1 \cup B_1^c$. Notice that $\tilde{\varphi}_2$ is an admissible competitor for this definition, and $\nu(\tilde{\varphi}_2) < +\infty$, hence the minimum defining \bar{v} is attained by direct methods (see Lemma 2.1).

Also, for any $g \in H^s(\mathbb{R}^n)$, with $g = 0$ a.e. in $K_1 \cup B_1^c$, we have that $\bar{v} + \varepsilon g$ is an admissible competitor with respect to the minimizing property of \bar{v} , therefore (see Lemma 2.2)

$$(3.1) \quad \langle \bar{v}, g \rangle = 0$$

and therefore

$$(3.2) \quad \nu^2(\bar{v} - g) - \nu^2(g) = \nu^2(\bar{v}) - 2\langle \bar{v}, g \rangle = \nu^2(\bar{v}).$$

Now we let $h := (\varphi_1 - \bar{v})^+$. By construction, a.e. in B_1^c we have that $\varphi_1 - \bar{v} = \varphi_{1,0} - \varphi_{2,0} \leq 0$. Also, a.e. in K_1 it holds that $\varphi_1 - \bar{v} = 0 - \tilde{\varphi}_2 \leq 0$, thanks to Lemma 2.4, and therefore $h = (\varphi_1 - \bar{v})^+ = 0$ in $K_1 \cup B_1^c$. So we can apply (3.1) with $g := h$. We obtain that

$$\langle \bar{v}, h \rangle = 0.$$

From this and recalling (2.10) with $u := \varphi_1 - \bar{v}$, we obtain that

$$\begin{aligned} \nu^2(h) &\leq \iint_{\mathbb{R}^{2n} \setminus (B_1^c)^2} \frac{(h(x) - h(y))((\varphi_1 - \bar{v})(x) - (\varphi_1 - \bar{v})(y))}{|x - y|^{n+2s}} dx dy \\ &= \langle h, \varphi_1 \rangle - \langle h, \bar{v} \rangle \\ &= \langle h, \varphi_1 \rangle. \end{aligned}$$

But the latter term also vanishes, thanks to (2.7): therefore we conclude that $\nu^2(h) \leq 0$ and so h vanishes identically.

This says that $\varphi_1 \leq \bar{v}$. Accordingly, if $\eta := \bar{v} - \tilde{\varphi}_2$, we have that

$$(3.3) \quad \eta \geq \varphi_1 - \tilde{\varphi}_2.$$

Also, since $A_1 \subseteq A_2$, we have that $\tilde{\varphi}_2 = 0$ a.e. in A_1 and so (3.3) gives that $\eta \geq \varphi_1$ in A_1 .

Moreover, by definition of \bar{v} , we have that $\eta = 0$ a.e. in $K_1 \cup B_1^c$, and so, by (2.14),

$$\eta \in \mathcal{A}_{K_1, A_1}^{\varphi_{1,0}}.$$

As a consequence, by Lemma 2.7

$$(3.4) \quad \nu^2(\tilde{\varphi}_1) - \nu^2(\varphi_1) = \inf_{\psi \in \mathcal{A}_{K_1, A_1}^{\varphi_{1,0}}} \nu^2(\psi) \leq \nu^2(\eta).$$

On the other hand, $\bar{v} = \tilde{\varphi}_2 = \varphi_{2,0}$ a.e. in B_1^c . Also, $K_2 \subseteq K_1$, so a.e. in K_2 we have that $\bar{v} = \tilde{\varphi}_2 = 0$. Accordingly, \bar{v} is an admissible competitor for the minimizing property of φ_2 , and we obtain

$$(3.5) \quad \nu^2(\varphi_2) \leq \nu^2(\bar{v}).$$

Now we point out that, by the definitions of η and \bar{v} , we see that $\eta = 0$ a.e. in $K_1 \cup B_1^c$, so we may use (3.2) with $g := \eta$. We conclude that

$$\nu^2(\tilde{\varphi}_2) - \nu^2(\eta) = \nu^2(\bar{v} - \eta) - \nu^2(\eta) = \nu^2(\bar{v}).$$

This and (3.5) give that

$$\nu^2(\tilde{\varphi}_2) - \nu^2(\varphi_2) \geq \nu^2(\tilde{\varphi}_2) - \nu^2(\bar{v}) = \nu^2(\eta).$$

By comparing this with (3.4), we obtain

$$\nu^2(\tilde{\varphi}_2) - \nu^2(\varphi_2) \geq \nu^2(\tilde{\varphi}_1) - \nu^2(\varphi_1). \quad \square$$

4. RADIAL ANALYSIS AND PROOF OF THEOREM 1.3

This section is devoted to the proof of Theorem 1.3. First we prove Theorem 1.3 in the particular case in which φ is constant, $K := B_r$ and $A := B_\rho \setminus B_r$, for some $r < \rho$, namely we have the following result:

Lemma 4.1. *Let $\rho \in [1/4, 3/4]$, $r \in (0, \rho)$ and $c_o \geq 0$. Assume that $\varphi(x) = c_o$ for any $x \in \mathbb{R}^n$. Then*

$$(4.1) \quad \nu^2(\varphi_{B_\rho}) - \nu^2(\varphi_{B_r}) \leq C |B_\rho \setminus B_r| \|\varphi_{B_r}\|_{L^\infty(\mathbb{R}^n)}^2,$$

for some $C > 0$ depending on n and s .

Proof. If $c_o = 0$ then both φ_{B_ρ} and φ_{B_r} vanish identically and the result is obvious. Hence, possibly dividing by c_o , we suppose that

$$(4.2) \quad c_o = 1.$$

We set $\mu := \rho - r$. Notice that

$$(4.3) \quad \begin{aligned} |B_\rho \setminus B_r| &= \bar{C}(\rho^n - r^n) = \bar{C}(\rho - r)(\rho^{n-1} + \rho^{n-2}r + \dots + \rho r^{n-2} + r^{n-1}) \\ &\geq \bar{C}(\rho - r)\rho^{n-1} \geq \tilde{C}(\rho - r) = \tilde{C}\mu, \end{aligned}$$

for suitable constants $\bar{C}, \tilde{C} > 0$.

Also, we fix a function $\varphi_o \in C^\infty(\mathbb{R}^n)$ with $\varphi_o = 1 = c_o$ in B_1^c and $\varphi_o = 0$ in $B_{3/4}$, so we write $C_0 := \nu^2(\varphi_o)$. In particular, from (4.2) and Lemma 2.4, we obtain

$$(4.4) \quad \nu^2(\varphi_o) = C_0 c_o^2 = C_0 \|\varphi_{B_r}\|_{L^\infty(\mathbb{R}^n)}^2.$$

Also, by construction, $\varphi_o = 0$ in $B_\rho \supseteq B_r$, hence the minimizing properties of φ_{B_ρ} and φ_{B_r} , together with (4.4), give that

$$(4.5) \quad \text{both } \nu^2(\varphi_{B_\rho}) \text{ and } \nu^2(\varphi_{B_r}) \text{ are less than or equal to } C_0 \|\varphi_{B_r}\|_{L^\infty(\mathbb{R}^n)}^2.$$

Now, for any $t \geq 0$ we define

$$a(t) := \begin{cases} -\rho^{-1}\mu t & \text{if } t \in [0, \rho], \\ (1 - \rho)^{-1}\mu(t - 1) & \text{if } t \in (\rho, 1], \\ 0 & \text{if } t > 1. \end{cases}$$

For any $x \in \mathbb{R}^n \setminus \{0\}$ let also

$$\beta(x) := \frac{x}{|x|} a(|x|) \text{ and } \alpha(x) := x + \beta(x).$$

We observe that

$$|a(t)| + |a'(t)| \leq 2(\rho^{-1} + (1 - \rho)^{-1})\mu \leq 16\mu$$

and therefore

$$(4.6) \quad |D\alpha - I_n| = |D\beta| \leq C_1\mu,$$

where I_n is the identity matrix and $C_1 > 0$ a suitable constant. In particular

$$(4.7) \quad |\det D\alpha| \geq 1 - C_2\mu,$$

if μ is small enough.

Furthermore

$$(4.8) \quad \alpha(B_\rho) \subseteq B_r \text{ and } \alpha(B_1^c) \subseteq B_1^c.$$

To check this, first take $x \in B_1^c$. Then $\beta(x) = 0$ and thus $|\alpha(x)| = |x| \geq 1$, so $\alpha(x) \in B_1^c$. Now take $x \in B_\rho$. Then

$$\beta(x) = -\rho^{-1}\mu x = \frac{r - \rho}{\rho} x$$

and so

$$\alpha(x) = x + \frac{r - \rho}{\rho} x = \frac{r}{\rho} x$$

hence $|\alpha(x)| < r$ and so $\alpha(x) \in B_r$ in this case. This proves (4.8).

So, we set $\varphi^*(x) := \varphi_{B_r}(\alpha(x))$. From (4.8), we have that φ^* vanishes a.e. in B_ρ and $\varphi^* = \varphi$ a.e. in B_1^c .

Accordingly, the minimizing property of φ_{B_ρ} implies that

$$(4.9) \quad \nu^2(\varphi_{B_\rho}) \leq \nu^2(\varphi^*).$$

Now we observe that

$$|\alpha(x) - \alpha(y)| \leq |x - y| + |\beta(x) - \beta(y)| \leq (1 + C_3\mu)|x - y|,$$

thanks to (4.6), and so, if μ is small enough,

$$\begin{aligned} \nu^2(\varphi^*) &= \iint_{\mathbb{R}^{2n} \setminus (B_1^c)^2} \frac{|\varphi_{B_r}(\alpha(x)) - \varphi(\alpha(y))|^2}{|x - y|^{n+2s}} dx dy \\ &\leq (1 + C_3\mu)^{n+2s} \iint_{\mathbb{R}^{2n} \setminus (B_1^c)^2} \frac{|\varphi_{B_r}(\alpha(x)) - \varphi(\alpha(y))|^2}{|\alpha(x) - \alpha(y)|^{n+2s}} dx dy \\ &\leq (1 - C_2\mu)^{-2} (1 + C_3\mu)^{n+2s} \iint_{\mathbb{R}^{2n} \setminus (B_1^c)^2} \frac{|\varphi_{B_r}(x^*) - \varphi(y^*)|^2}{|x^* - y^*|^{n+2s}} dx^* dy^*, \end{aligned}$$

where we have used (4.7) and the change of variable $x^* := \alpha(x)$, $y^* := \alpha(y)$. Hence, if μ is sufficiently small,

$$(4.10) \quad \nu^2(\varphi^*) \leq (1 + C_4\mu) \nu^2(\varphi_{B_r}).$$

By using (4.5) into (4.10), we conclude that

$$\nu^2(\varphi^*) \leq \nu^2(\varphi_{B_r}) + C_0 C_4 \mu \|\varphi_{B_r}\|_{L^\infty(\mathbb{R}^n)}^2.$$

So, recalling (4.3), we obtain, for small μ ,

$$\nu^2(\varphi^*) \leq \nu^2(\varphi_{B_r}) + C_5 |B_\rho \setminus B_r| \|\varphi_{B_r}\|_{L^\infty(\mathbb{R}^n)}^2.$$

This and (4.9) complete the proof of (4.1) when μ is sufficiently small, say $\mu \in [0, c)$ for some suitable constant $c \in (0, 1)$.

On the other hand, when $\mu \geq c$, we can prove (4.1) directly from the competitor φ_o introduced above. More precisely, if $\mu \geq c$, we infer from (4.5) that

$$\begin{aligned} \nu^2(\varphi_{B_\rho}) - \nu^2(\varphi_{B_r}) &\leq \nu^2(\varphi_{B_\rho}) \\ &\leq C_0 \|\varphi_{B_r}\|_{L^\infty(\mathbb{R}^n)}^2 \\ &\leq c^{-1} C_0 \mu \|\varphi_{B_r}\|_{L^\infty(\mathbb{R}^n)}^2. \end{aligned}$$

This and (4.3) say that (4.1) holds true also when $\mu \geq c$ and this completes the proof of Lemma 4.1. \square

Lemma 4.1 may be generalized to sets that are not necessarily rotationally symmetric, thanks to a rearrangement argument. The details go as follows:

Corollary 4.2. *Let $\rho \in [1/4, 3/4]$ and $c_o \geq 0$. Assume that $\varphi(x) = c_o$ for any $x \in \mathbb{R}^n$. Let $K \subseteq B_\rho$ and $A := B_\rho \setminus K$. Then*

$$\nu^2(\varphi_{K \cup A}) - \nu^2(\varphi_K) \leq C |A| \|\varphi_K\|_{L^\infty(\mathbb{R}^n)}^2,$$

for some $C > 0$ depending on n and s .

Proof. We take $r \geq 0$ such that $|B_r| = |K|$. Let also $\psi := c_o - \varphi_K$. Then $\psi = 0$ in B_1^c and $\psi = c_o = \max_{\mathbb{R}^n} \psi$ in K , thanks to Lemma 2.4. Then, its spherical rearrangement ψ^* satisfies $\psi^* = 0$ in B_1^c and $\psi^* = c_o$ in B_r . Accordingly, $c_o - \psi^*$ is a competitor against φ_{B_r} and so

$$\nu^2(\varphi_{B_r}) \leq \nu^2(c_o - \psi^*) = \nu^2(\psi^*).$$

On the other hand, spherical rearrangements decrease the Gagliardo seminorm (see e.g. [6]), therefore

$$\nu^2(\psi^*) = \iint_{\mathbb{R}^{2n}} \frac{|\psi^*(x) - \psi^*(y)|^2}{|x - y|^{n+2s}} dx dy \leq \iint_{\mathbb{R}^{2n}} \frac{|\psi(x) - \psi(y)|^2}{|x - y|^{n+2s}} dx dy = \nu^2(\psi).$$

We obtain that

$$\nu^2(\varphi_{B_r}) \leq \nu^2(\psi) = \nu^2(\varphi_K).$$

Notice also that $K \cup A = B_\rho$, hence, using Lemmata 4.1 and 2.4, we obtain that

$$\begin{aligned} \nu^2(\varphi_{K \cup A}) - \nu^2(\varphi_K) &= \nu^2(\varphi_{B_\rho}) - \nu^2(\varphi_K) \\ &\leq \nu^2(\varphi_{B_\rho}) - \nu^2(\varphi_{B_r}) \\ &\leq C |B_\rho \setminus B_r| \|\varphi_{B_r}\|_{L^\infty(\mathbb{R}^n)}^2 \\ &= C |B_\rho \setminus B_r| \|\varphi_K\|_{L^\infty(\mathbb{R}^n)}^2 \\ &= C |B_\rho \setminus K| \|\varphi_K\|_{L^\infty(\mathbb{R}^n)}^2. \end{aligned} \quad \square$$

Now we are ready for the proof of Theorem 1.3:

Proof of Theorem 1.3. We set $\varphi^\sharp := \|\varphi\|_{L^\infty(\mathbb{R}^n)}$, $K^\sharp := K \cap B_\rho$ and $A^\sharp := B_\rho \setminus K^\sharp$. We are now under the assumptions of Corollary 4.2, which gives that

$$\nu^2(\varphi_{K^\sharp \cup A^\sharp}) - \nu^2(\varphi_{K^\sharp}) \leq C |A^\sharp| \|\varphi_{K^\sharp}\|_{L^\infty(\mathbb{R}^n)}^2,$$

By construction

$$A^\sharp = B_\rho \setminus K^\sharp = B_\rho \cap (K \cap B_\rho)^c = B_\rho \cap K^c = A.$$

Also, $\varphi \leq \varphi^\sharp$ and $K^\sharp \subseteq K$: therefore, by Theorem 1.2,

$$\nu^2(\varphi_{K \cup A}) - \nu^2(\varphi_K) \leq \nu^2(\varphi_{K^\sharp \cup A^\sharp}) - \nu^2(\varphi_{K^\sharp}).$$

By collecting the above estimates, and recalling Lemma 2.4, we complete the proof of Theorem 1.3. \square

5. INTEGRAL CALCULATIONS AND PROOF OF THEOREM 1.4

Here we prove Theorem 1.4 in the particular case in which $K := B_{1/2}$ and φ is constant (the general case then will follow from Theorem 1.2).

Lemma 5.1. *Let $c_o \geq 0$, $A \subseteq B_{3/4} \setminus B_{1/2}$. Assume that $\varphi(x) = c_o$ for any $x \in \mathbb{R}^n$ and that A is closed and satisfies the following density property: there exists $c > 0$ such that for every $x \in \partial A$ and every $r \in (0, 2)$, we have that*

$$(5.1) \quad |A \cap B_r(x)| \geq c|B_r|.$$

Then

$$\nu^2(\varphi_{B_{1/2} \cup A}) - \nu^2(\varphi_{B_{1/2}}) \leq C |A| \|\varphi_{B_{1/2}}\|_{L^\infty(\mathbb{R}^n)}^2.$$

Proof. Notice that if $\|\varphi_{B_{1/2}}\|_{L^\infty(\mathbb{R}^n)} = 0$, then both $\varphi_{B_{1/2}}$ and $\varphi_{B_{1/2} \cup A}$ vanish identically and so the result is obvious. Hence, without loss of generality, we assume that $\|\varphi_{B_{1/2}}\|_{L^\infty(\mathbb{R}^n)} = 1/4$.

We claim that

$$(5.2) \quad \varphi_{B_{1/2}} \in C^s(\mathbb{R}^n),$$

and $\|\varphi_{B_{1/2}}\|_{C^s(\mathbb{R}^n)}$ is bounded by a constant that depends only on n and s . For this we take $\eta \in C^\infty(\mathbb{R}^n)$ such that $\eta = 0$ in $B_{1/2}$ and $\eta = 1/4$ in B_1^c . We define $\tilde{\eta} := \varphi_{B_{1/2}} - \eta$. We observe that $|(-\Delta)^s \tilde{\eta}| = |(-\Delta)^s \eta| \leq C_1$ in $\Omega := B_1 \setminus B_{1/2}$ and $\tilde{\eta} = 0$ in Ω^c . Hence, we deduce from Proposition 1.1 of [7] that $\|\tilde{\eta}\|_{C^s(\mathbb{R}^n)} \leq C_o$, for a universal $C_o > 0$ and so $\|\varphi_{B_{1/2}}\|_{C^s(\mathbb{R}^n)} \leq \|\eta\|_{C^s(\mathbb{R}^n)} + C_o$, ending the proof of (5.2).

Now we define $d(x)$ to be the distance function from the closed set A (with the standard convention that $d = 0$ in A and $d > 0$ in A^c), and we set $v(x) := \min\{d(x), \varphi_{B_{1/2}}(x)\}$. We observe that if $x \in B_1^c$ then $d(x) \geq 1/4$, since A is contained in $B_{3/4}$, therefore

$$d(x) \geq \|\varphi_{B_{1/2}}\|_{L^\infty(\mathbb{R}^n)} \geq \varphi_{B_{1/2}}(x).$$

Accordingly, $v(x) = \varphi_{B_{1/2}}(x)$ in B_1^c and $v = 0$ in $B_{1/2} \cup A$, so it is an admissible competitor for $\varphi_{B_{1/2} \cup A}$, and we conclude that

$$(5.3) \quad \nu(\varphi_{B_{1/2} \cup A}) \leq \nu(v).$$

Moreover

$$(5.4) \quad v(x) \leq \varphi_{B_{1/2}}(x) \leq \frac{1}{4}.$$

Now we show that

$$(5.5) \quad \text{for any } x \in \{d < \varphi_{B_{1/2}}\} \text{ we have } |v(x) - v(y)|^2 - |\varphi_{B_{1/2}}(x) - \varphi_{B_{1/2}}(y)|^2 \leq |x - y|^2.$$

To prove (5.5) we distinguish three cases:

$$(5.6) \quad \text{either } y \in \{\varphi_{B_{1/2}} < d\} \text{ and } \varphi_{B_{1/2}}(y) < d(x),$$

$$(5.7) \quad \text{or } y \in \{\varphi_{B_{1/2}} < d\} \text{ and } \varphi_{B_{1/2}}(y) \geq d(x),$$

$$(5.8) \quad \text{or } y \in \{\varphi_{B_{1/2}} \geq d\}.$$

First we deal with the case in (5.6). For this, we notice that, in this circumstance, $\varphi_{B_{1/2}}(x) > d(x) > \varphi_{B_{1/2}}(y)$ and so

$$\begin{aligned} |v(x) - v(y)| &= |d(x) - \varphi_{B_{1/2}}(y)| = d(x) - \varphi_{B_{1/2}}(y) \\ &< \varphi_{B_{1/2}}(x) - \varphi_{B_{1/2}}(y) = |\varphi_{B_{1/2}}(x) - \varphi_{B_{1/2}}(y)|. \end{aligned}$$

Accordingly,

$$|v(x) - v(y)|^2 \leq |\varphi_{B_{1/2}}(x) - \varphi_{B_{1/2}}(y)|^2$$

and so (5.5) follows in this case.

Now we deal with the case in (5.7). Here we have that $d(y) > \varphi_{B_{1/2}}(y) \geq d(x)$, hence

$$|v(x) - v(y)| = |d(x) - \varphi_{B_{1/2}}(y)| = \varphi_{B_{1/2}}(y) - d(x) < d(y) - d(x) \leq |x - y|.$$

Consequently

$$|v(x) - v(y)|^2 - |\varphi_{B_{1/2}}(x) - \varphi_{B_{1/2}}(y)|^2 \leq |v(x) - v(y)|^2 \leq |x - y|^2,$$

and so (5.5) follows in this case.

Finally, we take care of the case in (5.8): we have

$$|v(x) - v(y)| = |d(x) - d(y)| \leq |x - y|$$

and then (5.5) follows.

Having completed the proof of (5.5), we use it together with (5.4) to deduce that, for any $x \in \{d < \varphi_{B_{1/2}}\}$, we have

$$|v(x) - v(y)|^2 - |\varphi_{B_{1/2}}(x) - \varphi_{B_{1/2}}(y)|^2 \leq m(x - y),$$

where $m(x) := \min\{1, |x - y|^2\}$. As a consequence, for any $x \in \{d < \varphi_{B_{1/2}}\}$,

$$\begin{aligned} \int_{\mathbb{R}^n} \frac{|v(x) - v(y)|^2 - |\varphi_{B_{1/2}}(x) - \varphi_{B_{1/2}}(y)|^2}{|x - y|^{n+2s}} dy &\leq \int_{\mathbb{R}^n} \frac{m(x - y)}{|x - y|^{n+2s}} dy \\ &= \int_{\mathbb{R}^n} \frac{m(y)}{|y|^{n+2s}} dy = C, \end{aligned}$$

for a suitable $C > 0$.

Therefore

$$(5.9) \quad \int_{\{d < \varphi_{B_{1/2}}\}} \int_{\mathbb{R}^n} \frac{|v(x) - v(y)|^2 - |\varphi_{B_{1/2}}(x) - \varphi_{B_{1/2}}(y)|^2}{|x - y|^{n+2s}} dx dy \leq C |\{d < \varphi_{B_{1/2}}\}|.$$

Now we observe that if $x, y \in \{d \geq \varphi_{B_{1/2}}\}$

$$v(x) - v(y) = \varphi_{B_{1/2}}(x) - \varphi_{B_{1/2}}(y).$$

Using this and the fact that v and $\varphi_{B_{1/2}}$ coincide outside B_1 , we conclude that

$$\begin{aligned} & \nu^2(v) - \nu^2(\varphi_{B_{1/2}}) \\ &= \iint_{\mathbb{R}^{2n}} \frac{|v(x) - v(y)|^2 - |\varphi_{B_{1/2}}(x) - \varphi_{B_{1/2}}(y)|^2}{|x - y|^{n+2s}} dx dy \\ &= \int_{\{d < \varphi_{B_{1/2}}\}} \int_{\{d < \varphi_{B_{1/2}}\}} \frac{|v(x) - v(y)|^2 - |\varphi_{B_{1/2}}(x) - \varphi_{B_{1/2}}(y)|^2}{|x - y|^{n+2s}} dx dy \\ &\quad + 2 \int_{\{d < \varphi_{B_{1/2}}\}} \int_{\{d \geq \varphi_{B_{1/2}}\}} \frac{|v(x) - v(y)|^2 - |\varphi_{B_{1/2}}(x) - \varphi_{B_{1/2}}(y)|^2}{|x - y|^{n+2s}} dx dy. \end{aligned}$$

In particular

$$\nu^2(v) - \nu^2(\varphi_{B_{1/2}}) \leq 3 \int_{\{d < \varphi_{B_{1/2}}\}} \int_{\mathbb{R}^n} \frac{|v(x) - v(y)|^2 - |\varphi_{B_{1/2}}(x) - \varphi_{B_{1/2}}(y)|^2}{|x - y|^{n+2s}} dx dy.$$

Thus, recalling (5.3) and (5.9), we obtain

$$(5.10) \quad \nu^2(\varphi_{B_{1/2} \cup A}) - \nu^2(\varphi_{B_{1/2}}) \leq \nu^2(v) - \nu^2(\varphi_{B_{1/2}}) \leq 3C |\{d < \varphi_{B_{1/2}}\}|.$$

Now we show that

$$(5.11) \quad |\{d < \varphi_{B_{1/2}}\}| \leq C' |A|$$

for some $C' > 0$. The proof of (5.11) follows from the density property in (5.1) and a covering argument. Indeed, first of all we observe that

$$(5.12) \quad \{d < \varphi_{B_{1/2}}\} \subseteq \overline{B_{1/2}}^c.$$

To prove it, notice that if $x \in \{d < \varphi_{B_{1/2}}\}$ we have that

$$0 \leq d(x) < \varphi_{B_{1/2}}(x),$$

so (5.12) follows from the fact that $\varphi_{B_{1/2}}$ vanishes in $B_{1/2}$. Thanks to (5.12), for any $x \in \{d < \varphi_{B_{1/2}}\} \setminus A$ we can denote by $y(x)$ its projection onto A and by $z(x)$ its projection onto $\overline{B_{1/2}}$, hence

$$|x - y(x)| = d(x) < \varphi_{B_{1/2}}(x) = \varphi_{B_{1/2}}(x) - \varphi_{B_{1/2}}(z(x)) \leq C_* |x - z(x)|^s,$$

where C_* is a positive constant, whose existence is warranted by (5.2). That is, for any $x \in \{d < \varphi_{B_{1/2}}\} \setminus A$,

$$|x - y(x)| \leq \frac{\delta(x)}{6},$$

where

$$\delta(x) := 6C_* |x - z(x)|^s.$$

We write such estimate as

$$S := \{d < \varphi_{B_{1/2}}\} \setminus A \subseteq \bigcup_{x \in S} B_{\delta(x)/5}(y(x)).$$

Hence, by the Vitali covering theorem, we have that there exists a subcollection of disjoint balls such that

$$S \subseteq \bigcup_{i \in \mathbb{N}} B_{\delta(x_i)}(y(x_i))$$

and so

$$(5.13) \quad |S| \leq \sum_{i \in \mathbb{N}} |B_{\delta(x_i)}(y(x_i))|.$$

Using (5.1), we have that

$$|A \cap B_{\delta(x_i)}(y(x_i))| \geq c |B_{\delta(x_i)}(y(x_i))|.$$

So we can fix $N \in \mathbb{N}$, sum up this estimate and use that the balls are disjoint: we obtain that

$$|A| \geq \left| A \cap \left(\bigcup_{i=1}^N B_{\delta(x_i)}(y(x_i)) \right) \right| = \sum_{i=1}^N |A \cap B_{\delta(x_i)}(y(x_i))| \geq c \sum_{i=1}^N |B_{\delta(x_i)}(y(x_i))|.$$

Now we send $N \rightarrow +\infty$ and we recall (5.13), to establish that

$$|A| \geq c |S|.$$

Since $\{d < \varphi_{B_{1/2}}\} \subseteq S \cup A$, this implies (5.11).

Then, the claim of Theorem 1.4 follows from (5.10) and (5.11). \square

With the above results, we can now complete the proof of Theorem 1.4:

Proof of Theorem 1.4. We set $\varphi^\sharp := \|\varphi\|_{L^\infty(\mathbb{R}^n)}$ and $K^\sharp := B_{1/2}$. We are now under the assumptions of Lemma 5.1 which gives that

$$\nu^2(\varphi_{K^\sharp \cup A}) - \nu^2(\varphi_{K^\sharp}) \leq C |A| \|\varphi\|_{L^\infty(\mathbb{R}^n)}^2.$$

Notice also that $K^\sharp \subseteq K$ and $\varphi^\sharp \geq \varphi$, thus Theorem 1.2 implies that

$$\nu^2(\varphi_{K \cup A}) - \nu^2(\varphi_K) \leq \nu^2(\varphi_{K^\sharp \cup A}) - \nu^2(\varphi_{K^\sharp}),$$

so the claim of Theorem 1.4 readily follows. \square

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