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# Damage processes in thermoviscoelastic materials with damage-dependent thermal expansion coefficients


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**Abstract**

In this paper we prove existence of global in time weak solutions for a highly nonlinear PDE system arising in the context of damage phenomena in thermoviscoelastic materials. The main novelty of the present contribution with respect to the ones already present in the literature consists in the possibility of taking into account a damage-dependent thermal expansion coefficient. This term implies the presence of nonlinear couplings in the PDE system, which make the analysis more challenging.

# 1 Introduction

We consider the PDE system, in $\Omega \times (0,T)$, where $\Omega \subseteq \mathbb{R}^d$ (with $d \in \{1,2,3\}$) is a bounded and sufficiently regular domain and $T$ denotes a final time,

$$\mathsf{c}(\theta)\theta_t - \operatorname{div}(\mathsf{K}(\theta)\nabla\theta) + \rho(\chi)\theta\operatorname{div}(u_t) + \theta\chi_t + \rho'(\chi)\theta\operatorname{div}(u)\chi_t = g, \tag{1a}$$

$$u_{tt} - \operatorname{div}(b(\chi)\mathbf{D}\varepsilon(u)) - \operatorname{div}(a(\chi)\mathbf{C}\varepsilon(u_t)) + \operatorname{div}(\rho(\chi)\theta\mathbf{1}) = l, \tag{1b}$$

$$\chi_t + \xi + \varphi - \Delta_p\chi + \gamma(\chi) + \frac{b'(\chi)}{2}\varepsilon(u) : \mathbf{D}\varepsilon(u) - \theta - \rho'(\chi)\theta\operatorname{div}(u) = 0 \tag{1c}$$

with subgradients $\xi \in \partial I_{[0,\infty)}(\chi)$ and $\varphi \in \partial I_{(-\infty,0]}(\chi_t)$. The initial-boundary conditions are

$$\theta(0) = \theta^0, \qquad u(0) = u^0, \qquad u_t(0) = v^0, \qquad \chi(0) = \chi^0 \qquad \text{in } \Omega, \tag{2a}$$

$$\mathsf{K}(\theta)\nabla\theta\cdot\nu = 0, \qquad u = 0, \qquad \nabla\chi\cdot\nu = 0 \qquad \text{on } \partial\Omega\times(0,T). \tag{2b}$$

The state variables and unknowns of the problem are the absolute temperature $\theta$, whose evolution is ruled by the internal energy balance (1a), the vector of small displacements $u$, satisfying the momentum balance (1b), and the damage parameter $\chi$, representing the local proportion of damage: $\chi = 1$ means that the material is completely safe, while $\chi = 0$ means it is completely damaged. Indeed, the two contraints $\chi \in [0,+\infty)$, $\chi_t \le 0$ together with the assumption $\chi^0 \in [0,1]$ implies that $\chi \in [0,1]$ during all the evolution, as it results from its physical meaning.

The main novelty of this contribution consists in the possibility of taking into account the dependence on the damage of the thermal expansion coefficient $\rho \in C^1([0,1])$. This provokes the presence of two new nonlinear terms in (1a) coupling it nonlinearly with both the momentum balance (1b) and the damage evolution (1c). Especially the coupling term $\rho'(\chi)\theta\operatorname{div}(u)\chi_t$ in (1a) complicates the analysis and requires elaborate estimation techniques to gain the desired a priori estimates. Moreover, a dependence on $\chi$ in the $u$-equation (1b) appears explicitly as well as a further dependence on $u$ and $\theta$ in the $\chi$-equation (1c). The other two coeffiecients c and K appearing in equation (1a) represent respectively the heat capacity and the heat conductivity of the system and will have to satisfy proper growth condition (cf. Remark 2.1), while the function $g$ denotes a given heat source.

In the momentum balance (1b) $\varepsilon(u) := (u_{i,j} + u_{j,i})/2$ denotes the linearized symmetric strain tensor, while the functions $b,\ a \in C^1([0,1])$ demarcate the damage dependence of the elasticity and viscosity modula, respectively. In the present contribution we will restrict the the case of incomplete damage, i.e. to the case where $a(x), b(x) \ge$

$\eta > 0$ (cf. [15] for the complete damage model in case $\rho = 0$). The function $l$ on the right hand side in (1b) represents a given external force.

Finally, in the inclusion (1c), the selections $\xi$ and $\varphi$ of the two maximal monotone operators, acting on $\chi$ and $\chi_t$ respectively, are introduced in order to give the constraints on the damage parameter ($\chi \in [0,1]$ as soon as $\chi^0 \in [0,1]$) and on the irreversibility of the damage process ($\chi_t \le 0$). The $p$-Laplacian operator $\Delta_p\chi$ accounts for the nonlocal interactions between particles, but the restriction of the exponent $p > d$ is mainly due to analytical reasons. It is introduced, in particular, in order to obtain sufficient regularity on $\chi$ needed in (1b) to obtain an enhanced estimate on $\epsilon(u)$, which appears at power $2$ in (1c) and so it has to be estimated in a better space than $L^2(\Omega \times (0,T))$. In addition to that, the enhanced regularity of $\chi$ enables the usage of approximation techniques in order to treat the doubly nonlinear inclusion (1c) in a weak formulation. Moreover, the function $\gamma$ is assumed to be smooth but possibly non monotone.

In the remaining part of the Introduction we will briefly explain the derivation of (1) referring to [15] for more details.

The system (1a)-(1c) can be derived from fundamental balance laws in continuum mechanics supplemented with constitutive relations used to describe thermoviscoelastic solids. In this approach, we make use of the free energy $\mathcal{F}$ given by [6, Sec. 4.5, pp. 42-43]

$$\begin{aligned}\mathcal{F}(\theta, \varepsilon(u), \chi, \nabla\chi) = &\int_\Omega \left(\frac{1}{p}|\nabla\chi|^p + \widehat{\gamma}(\chi) + \frac{b(\chi)}{2}\varepsilon(u) : \mathbf{D}\varepsilon(u)\right)\,\mathrm{d}x \\ &+ \int_\Omega \Big(f(\theta) - \theta\chi - \rho(\chi)\theta\,\mathrm{div}(u) + I_{[0,\infty)}(\chi)\Big)\,\mathrm{d}x\end{aligned}$$

and the dissipation potential defined by

$$\mathcal{P}_{\theta,\chi}(\nabla\theta, \chi_t, \varepsilon(u_t)) = \int_\Omega \left(\frac{\mathsf{K}(\theta)}{2}|\nabla\theta|^2 + \frac{1}{2}|\chi_t|^2 + \frac{a(\chi)}{2}\varepsilon(u_t) : \mathbf{C}\varepsilon(u_t) + I_{(-\infty,0]}(\chi_t)\right)\mathrm{d}x.$$

For notational convenience, we write $\mathcal{P}$ instead of $\mathcal{P}_{\theta,\chi}$. Let us point out that the gradient of $\chi$ accounts for the influence of damage at a material point, undamaged in its neighborhood. In this sense the term $\frac{1}{p}|\nabla\chi|^p$ models nonlocality of the damage process and effects like possible hardening or softening (cf. also [3] for further comments on this topic). Gradient regularizations of $p$-Laplacian type are often adopted in the mathematical papers on damage (see for example [1, 2, 9, 13]), and in the modeling literature as well (cf., e.g., [7, 5]).

Equation (1a) is obtained from the internal energy balance which reads as

$$e_t + \mathrm{div} q = g + \sigma : \varepsilon(u_t) + B\chi_t + H \cdot \nabla\chi_t,$$

where $e$ denotes the internal energy, $q$ the heat flux, $g$ the heat source, $\sigma$ the stress tensor, $\varepsilon(u_t)$ the linearized strain rate tensor, $H$ and $B$ the so-called microscopic forces (cf. . The quantities above are given by the following constitutive relations

$$\begin{aligned}&\sigma = \frac{\partial\mathcal{F}}{\partial\varepsilon(u)} + \frac{\partial\mathcal{P}}{\partial\varepsilon(u_t)}, && B \in \frac{\partial\mathcal{F}}{\partial\chi} + \frac{\partial\mathcal{P}}{\partial\chi_t}, && H = \frac{\partial\mathcal{F}}{\partial\nabla\chi} + \frac{\partial\mathcal{P}}{\partial\nabla\chi_t}, \\ &e = \mathcal{F} - \theta\frac{\partial\mathcal{F}}{\partial\theta}, && q = -\frac{\partial\mathcal{P}}{\partial\nabla\theta}.\end{aligned}$$

Note that, for analytical reasons, we have neglected the quadratic contributions $a(\chi)\varepsilon(u_t) : \mathbf{C}\varepsilon(u_t) + |\chi_t|^2$ on the right hand side of (1a), using the so-called *small perturbation assumption* (cf. [8]). In fact, to our knowledge only few results are available on diffuse interface models in thermoviscoelasticity (i.e. also accounting for the evolution of the displacement variables, besides the temperature and the order parameter): among others, we quote [14, 15] where the small perturbation assumption is adopted in case of constant $\rho$ and [17] where a PDE system coupling the momentum balance equation, the temperature equation (with quadratic nonlinearities) and a *rate-independent* flow rule for an internal dissipative variable $\chi$ (such as the damage parameter) has been analyzed. Finally, a temperature-dependent, *full* model for (rate-dependent) damage has been addressed in [1] as well, but only with local-in-time existence results.

Moreover, we make use of the assumption

$$\mathsf{c}(\theta) = -\theta f''(\theta),$$

where $f$ is a concave function. Eventually, the equation for the balance of forces (1b) can be written as

$$u_{tt} - \operatorname{div}\sigma = l$$

with external volume forces $l$ and the evolution of the damage processes as described in equation (1c) is derived from a balance equation of the microscopic forces, i.e.

$$B - \operatorname{div}H = 0.$$

To handle non-constant heat capacities $\mathsf{c}$, we perform an enthalpy transformation of system (1a)-(1c). To this end, we introduce the primitive $\widehat{\mathsf{c}}$ of $\mathsf{c}$ as $\widehat{\mathsf{c}}(r) := \int_0^r \mathsf{c}(\theta)\,\mathrm{d}\theta$. The enthalpy transformation of system (1) yields

$$w_t - \operatorname{div}(K(w)\nabla w) + \Theta(w)\chi_t + \rho(\chi)\Theta(w)\operatorname{div}(u_t) + \rho'(\chi)\Theta(w)\operatorname{div}(u)\chi_t = g, \tag{3a}$$

$$u_{tt} - \operatorname{div}(b(\chi)\mathbf{D}\varepsilon(u)) - \operatorname{div}(a(\chi)\mathbf{C}\varepsilon(u_t)) + \operatorname{div}(\rho(\chi)\Theta(w)\mathbf{1}) = l, \tag{3b}$$

$$\chi_t + \xi + \varphi - \Delta_p\chi + \gamma(\chi) + \frac{b'(\chi)}{2}\varepsilon(u) : \mathbf{D}\varepsilon(u) - \Theta(w) - \rho'(\chi)\Theta(w)\operatorname{div}(u) = 0. \tag{3c}$$

with $\xi \in \partial I_{[0,\infty)}(\chi)$ and $\varphi \in \partial I_{(-\infty,0]}(\chi_t)$ and the transformed quantities

$$w := \widehat{\mathsf{c}}(\theta), \qquad \Theta(w) := \widehat{\mathsf{c}}^{-1}(w), \qquad K(w) := \frac{\mathsf{K}(\Theta(w))}{\mathsf{c}(\Theta(w))}.$$

As already mentioned in the Introduction, the main difficulty here, with respect to the previous works in the literature, consists in the presence of the nonlinearities due to the fact that the temperature expansion term depends on $\chi$. Indeed, following [17, 15], here we will combine the conditions on $\mathsf{K}$ with conditions on the heat capacity coefficient $\mathsf{c}$ to handle the nonlinearities $\rho(\chi)\theta\operatorname{div}(u_t)$, $\theta\chi_t$, $\rho'(\chi)\theta\operatorname{div}(u)\chi_t$ in (1a) by means of a so-called Boccardo-Gallouët type estimate on $\theta$.

As for the triply nonlinear inclusion (1c), we will use a notion of solution derived in [9] (dealing with Cahn-Hilliard systems coupled with elasticity and damage processes; see also [10, 11, 12]). The authors have devised a weak formulation consisting of a *one-sided* variational inequality (i.e. with test functions having a fixed sign), and of an *energy inequality*, see Definition 3.1 later. Finally, let us notice that uniqueness of solutions remains and open problem even in the isothermal case. The main problem is, in general, the doubly nonlinear character of (1c) (cf. also [4] for examples of non-uniqueness in general doubly nonlinear equations).

The paper is organized as follows. In Section 2, we list all assumptions which are used throughout this paper and introduce some notation. Subsequently, a suitable notion of weak solutions for system (1) as well as the main result, existence of weak solutions (see Theorem 3.4), are stated in Section 3. In the main part, the proof of the existence theorem is firstly performed for a truncated system in Section 4 and finally for the limit system in Section 5.

## 2 Notation and assumptions

Let $d \in \{1,2,3\}$ denote the space dimension. For the analysis of the transformed system (3a)-(3c), the central hypotheses are stated below.

**Assumptions**

(A1) $\Omega \subseteq \mathbb{R}^d$ is a bounded $C^2$-domain.

(A2) The function $\Theta : \mathbb{R} \to \mathbb{R}$ is assumed to be Lipschitz continuous with $\Theta(w) \geq 0$ and $\Theta'(w) \geq 0$ for a.e. $w \geq 0$ and should satisfy the growth condition

$$\Theta(w) \leq c_0(w^{1/\sigma} + 1)$$

for all $w \geq 0$ and for constants $\sigma \geq 3$ and $c_0 > 0$. Moreover, we assume $\Theta(w) = 0$ for all $w \leq 0$.

(A3) The heat conductivity function $K : \mathbb{R} \to \mathbb{R}$ is assumed to be continuous and should satisfy the estimate

$$c_1(w^{2q} + 1) \leq K(w) \leq c_2(w^{2q_0} + 1)$$

for all $w \geq 0$ and for constants $c_1, c_2, q, q_0 > 0$ satisfying

$$1/\sigma \leq 2q - 1, \qquad q \leq q_0 < q + \frac{1}{2}.$$

(A4) The damage-dependent potential function $\widehat{\gamma}$ is assumed to satisfy $\widehat{\gamma} \in C^1([0,1])$.

(A5) The coefficient functions $a \in C^1([0,1])$ and $b \in C^2([0,1])$ should satisfy the estimate $a(x), b(x) \geq \eta$ for all $x \in [0,1]$ and a constant $\eta > 0$.

(A6) The 4th order stiffness tensors $\mathbf{C}, \mathbf{D} \in \mathcal{L}(\mathbb{R}^{d\times d}_{\mathrm{sym}}; \mathbb{R}^{d\times d}_{\mathrm{sym}})$ are assumed to be symmetric and positive definite, i.e.

$$\begin{aligned} &\mathbf{C}_{ijlk} = \mathbf{C}_{jilk} = \mathbf{C}_{lkij}, && e : \mathbf{C}e \geq c_3|e|^2 \text{ for all } e \in \mathbb{R}^{d\times d}_{\mathrm{sym}},\\ &\mathbf{D}_{ijlk} = \mathbf{D}_{jilk} = \mathbf{D}_{lkij}, && e : \mathbf{D}e \geq c_4|e|^2 \text{ for all } e \in \mathbb{R}^{d\times d}_{\mathrm{sym}} \end{aligned}$$

for constants $c_3, c_4 > 0$.

(A7) The thermal expansion coefficient $\rho$ depending on $\chi$ is assumed to fulfill $\rho \in C^1([0,1])$.

(A8) The constant $p$ (occurring in the $p$-Laplacian in (1c) and in (3c), respectively) should satisfy $p > d$.

**Remark 2.1** *The assumptions (A2) and (A3) can also be formulated in terms of the original heat conductivity function* K *and the heat capacity function* c *as follows.*

*(A2') The function* c *should be continuous and should satisfy the estimate*

$$\widetilde{c}_0(\theta^{\sigma-1} + 1) \leq \mathsf{c}(\theta)$$

*for all $\theta \geq 0$ and for constants $\sigma \geq 3$ and $c_5 > 0$.*

*(A3') The function* $\mathsf{K}$ *is assumed to be continuous and should satisfy the estimate*

$$c_1(\widehat{\mathsf{c}}(\theta)^{2q}+1)\mathsf{c}(\theta) \le \mathsf{K}(\theta) \le c_2(\widehat{\mathsf{c}}(\theta)^{2q_0}+1)\mathsf{c}(\theta)$$

*for all* $\theta \ge 0$ *and for constants* $c_1, c_2, q, q_0 > 0$ *satisfying* $1/\sigma \le 2q-1$ *and* $q \le q_0 < q+\frac{1}{2}$.

For later use, we define the following subspaces (with $p, s \ge 1$):

$$\begin{aligned}
W^{2,s}_\nu(\Omega) &:= \big\{\zeta \in W^{2,s}(\Omega) \,|\, \nabla\zeta\cdot\nu = 0 \text{ on } \partial\Omega\big\},\\
W^{1,p}_+(\Omega) &:= \big\{\zeta \in W^{1,p}(\Omega) \,|\, \zeta \ge 0 \text{ a.e. in } \Omega\big\},\\
W^{1,p}_-(\Omega) &:= \big\{\zeta \in W^{1,p}(\Omega) \,|\, \zeta \le 0 \text{ a.e. in } \Omega\big\}.
\end{aligned}$$

The primitive of an integrable function $f : \mathbb{R} \to \mathbb{R}$ vanishing at $0$ is denoted by $\widehat{f}$.

## 3 Notion of weak solutions and main result

To keep the presentation short, we assume $l = g = 0$ in (3a)-(3b). We introduce the following notion of weak solutions.

**Definition 3.1** *A weak solution corresponding to the initial data* $(u^0, v^0, w^0, \chi^0)$ *is a* $4$*-tuple* $(u, w, \chi, \xi)$ *such that*

$$\begin{aligned}
&u \in H^1(0,T;H^2_0(\Omega;\mathbb{R}^d)) \cap W^{1,\infty}(0,T;H^1_0(\Omega;\mathbb{R}^d)) \cap H^2(0,T;L^2(\Omega;\mathbb{R}^d))\\
&\quad \text{with } u(0) = u^0 \text{ a.e. in } \Omega,\ \partial_t u(0) = v^0 \text{ a.e. in } \Omega,\\
&w \in L^2(0,T;H^1(\Omega)) \cap L^{2(q+1)}(0,T;L^{6(q+1)}(\Omega)) \cap L^\infty(0,T;L^2(\Omega))\\
&\qquad \cap W^{1,r}(0,T;(W^{2,s}_\nu(\Omega))^*),\ w(0) = w^0\\
&\quad \text{with } w(0) = w^0 \text{ a.e. in } \Omega,\ w \ge 0 \text{ a.e. in } \Omega_T,\\
&\chi \in L^\infty(0,T;W^{1,p}(\Omega)) \cap H^1(0,T;L^2(\Omega))\\
&\quad \text{with } \chi(0) = \chi^0 \text{ a.e. in } \Omega,\ \chi \ge 0 \text{ a.e. in } \Omega_T,\ \partial_t\chi \le 0 \text{ a.e. in } \Omega_T,\\
&\xi \in L^1(0,T;L^1(\Omega))
\end{aligned}$$

*with* $r = (2q+2)/(2q_0+1)$ *and* $s := (6q+6)/(6q-2q_0+5)$*, and for a.e.* $t \in (0,T)$*:*

*(i) heat equation: for all* $\zeta \in W^{2,s}_\nu(\Omega)$

$$\begin{aligned}
&\langle \partial_t w, \zeta\rangle_{H^1} + \int_\Omega \Big(-\widehat{K}(w)\Delta\zeta + \Theta(w)\partial_t\chi\zeta\Big)\,\mathrm{d}x\\
&\quad + \int_\Omega \big(\rho(\chi)\Theta(w)\,\mathrm{div}\,(\partial_t u)\,\zeta + \rho'(\chi)\Theta(w)\,\mathrm{div}(u)\partial_t\chi\zeta\big)\,\mathrm{d}x = 0,
\end{aligned} \tag{4}$$

*(ii) balance of forces: for a.e.* $x \in \Omega$

$$\partial_{tt}u - \mathrm{div}\,(b(\chi)\boldsymbol{D}\varepsilon(u)) - \mathrm{div}\,(a(\chi)\boldsymbol{C}\varepsilon(\partial_t u)) + \mathrm{div}\,(\rho(\chi)\Theta(w)\mathbf{1}) = 0, \tag{5}$$

*(iii) one-sided variational inequality: for all* $\zeta \in W^{1,p}_-(\Omega)$

$$0 \le \int_\Omega \left( \partial_t \chi \zeta + |\nabla \chi|^{p-2} \nabla \chi \cdot \nabla \zeta + \gamma(\chi)\zeta + \frac{b'(\chi)}{2} \varepsilon(u) : \boldsymbol{D}\varepsilon(u)\zeta + \xi\zeta \right) \mathrm{d}x + \int_\Omega \left( -\Theta(w)\zeta - \rho'(\chi)\Theta(w) \operatorname{div}(u)\zeta \right) \mathrm{d}x, \tag{6}$$

*and* $\xi \in \partial I_{W^{1,p}_+(\Omega)}(z)$*, i.e. for all* $\zeta \in W^{1,p}_+(\Omega)$

$$\int_\Omega \xi(\zeta - z)\,\mathrm{d}x \le 0, \tag{7}$$

*(iv) partial energy inequality:*

$$\int_\Omega |\nabla \chi(t)|^p\,\mathrm{d}x - \int_\Omega |\nabla \chi^0|^p\,\mathrm{d}x + \int_0^t \int_\Omega \Big( \gamma(\chi) + \frac{b'(\chi)}{2} \varepsilon(u) : \boldsymbol{D}\varepsilon(u) \Big) \partial_t \chi \,\mathrm{d}x\,\mathrm{d}s + \int_0^t \int_\Omega \Big( - \Theta(w) - \rho'(\chi)\Theta(w) \operatorname{div}(u) + \partial_t \chi \Big) \partial_t \chi \,\mathrm{d}x\,\mathrm{d}s \le 0 \tag{8}$$

*are satisfied.*

**Remark 3.2** *Due to the assumption* $q \le q_0 < q + \frac{1}{2}$ *(see (A3)), it holds* $1 < r < 2$.

By assuming better regularity for $\chi$, it is seen from the one-sided variational inequality and the partial energy inequality that the desired differential inclusion (3c) holds in $W^{1,p}(\Omega)^*$.

**Lemma 3.3** *If a weak solution additionally fulfills* $\chi \in H^1(0,T;W^{1,p}(\Omega))$ *then*

$$\chi_t + \xi + \varphi - \Delta_p \chi + \gamma(\chi) + \frac{b'(\chi)}{2} \varepsilon(u) : \boldsymbol{D}\varepsilon(u) - \theta - \rho'(\chi)\theta \operatorname{div}(u) = 0 \text{ in } W^{1,p}(\Omega)^*$$

*with subgradients* $\xi \in \partial I_{W^{1,p}_+(\Omega)}(\chi)$ *and* $\varphi \in \partial I_{W^{1,p}_-(\Omega)}(\partial_t \chi)$*. On the left hand side the operator* $\Delta_p : W^{1,p}(\Omega) \to W^{1,p}(\Omega)^*$ *denotes the usual* $p$*-Laplacian with no-flux condition.*

**Proof.** By setting

$$\varphi := -\Big( \chi_t + \xi - \Delta_p \chi + \gamma(\chi) + \frac{b'(\chi)}{2} \varepsilon(u) : \mathbf{D}\varepsilon(u) - \theta - \rho'(\chi)\theta \operatorname{div}(u) \Big) \in W^{1,p}(\Omega)^*,$$

and using (due to the enhanced regularity $\chi \in H^1(0,T;W^{1,p}(\Omega))$)

$$\int_\Omega |\nabla \chi(t)|^p\,\mathrm{d}x - \int_\Omega |\nabla \chi^0|^p\,\mathrm{d}x = \int_0^t \int_\Omega |\nabla \chi|^{p-2} \nabla \chi \cdot \nabla \partial_t \chi \,\mathrm{d}x\,\mathrm{d}s,$$

property (iii) and property (iv) from Definition 3.1 can be rewritten as

$$\langle \varphi\,,\ \zeta \rangle_{W^{1,p}} \le 0 \text{ and } -\langle \varphi\,,\ \partial_t \chi \rangle_{W^{1,p}} \le 0$$

for all $\zeta \in W^{1,p}_-(\Omega)$ and a.e. $t \in (0,T)$. Here we have used the fact that $\langle \xi, \partial_t \chi \rangle = 0$. Adding these inequalities yields the inclusion $\varphi \in \partial I_{W^{1,p}_-(\Omega)}(\partial_t \chi)$. $\square$

**Theorem 3.4** *Let the Assumptions (A1)-(A8) be satisfied. Moreover, let the initial values $u^0 \in H_0^2(\Omega;\mathbb{R}^d)$, $v^0 \in H_0^1(\Omega;\mathbb{R}^d)$, $w^0 \in L^2(\Omega)$ and $\chi^0 \in W^{1,p}(\Omega)$ be given and assume that $w^0 \geq 0$ and $0 \leq \chi^0 \leq 1$. Then, there exists a weak solution $(u, w, \chi, \xi)$ in the sense of Definition 3.1.*

The proof is carried out in the following two sections. It is based on a time-discretization scheme and on an approximation argument involving a truncation of $K$ and $\Theta$ (cf. also [15]).

# 4 Existence of weak solutions for the truncated system

As a first step in the proof of Theorem 3.4, we prove existence of weak solutions to the truncated system of (3a)-(3c) where $K$ and $\Theta$ are substituted by $K_M$ and $\Theta_M$ for $M \geq 0$ defined by

$$\Theta_M(x) := \begin{cases} \Theta(M) & \text{if } x > M, \\ \Theta(x) & \text{if } -M \leq x \leq M, \\ \Theta(-M) & \text{if } x < -M, \end{cases} \qquad K_M(x) := \begin{cases} K(M) & \text{if } x > M, \\ K(x) & \text{if } -M \leq x \leq M, \\ K(-M) & \text{if } x < -M. \end{cases}$$

We remind that $\Theta(w) = 0$ for all $w \leq 0$ by Assumption (A3).

The truncation function $\mathcal{T}_M : \mathbb{R} \to \mathbb{R}$ at the height $M$ is given via

$$\mathcal{T}_M(x) = \begin{cases} M & \text{if } x > M, \\ x & \text{if } -M \leq x \leq M, \\ -M & \text{if } x < -M. \end{cases}$$

Note that the crucial properties $\Theta_M(w) = \Theta(\mathcal{T}_M(w))$ and $K_M(w) = K(\mathcal{T}_M(w))$ are satisfied.

## 4.1 Time-discrete system

In this subsection, we will prove existence of weak solutions for a time-discrete and truncated version of system (3a)-(3c) by using a semi-implicit Euler scheme. The scheme is carefully chosen such that we can derive an energy estimate (see Lemma 4.4 (i)).

To this end, we consider an equidistant partition $\{0, \tau, 2\tau, \ldots, \tau T_\tau\}$ of $[0, T]$ where $\tau > 0$ denotes the time-discretization fineness and $T_\tau := T/\tau$ specifies the number of discrete time points. We set $(u_\tau^0, w_\tau^0, \chi_\tau^0) := (u^0, w^0, \chi^0)$ and $u_\tau^{-1} := u^0 - \tau v^0$ and perform a recursive procedure.

In the following, we adopt the notation $D_{\tau,k}(w) = \tau^{-1}(w_\tau^k - w_\tau^{k-1})$ (as well as for $D_{\tau,k}(u)$ and $D_{\tau,k}(\chi)$). Let, furthermore, $v_\tau^k$ be defined as

$$v_\tau^k := \frac{u_\tau^k - u_\tau^{k-1}}{\tau}. \tag{9}$$

The existence of weak solutions for the time-discrete system is proven in the following.

**Lemma 4.1** *For every equidistant partition of $[0, T]$ with fineness $\tau > 0$, there exists a sequence $\{(u_\tau^k, w_\tau^k, \chi_\tau^k, \xi_\tau^k)\}_{k=1}^{T_\tau}$ in the space $H_0^2(\Omega;\mathbb{R}^d) \times H^1(\Omega) \times W^{1,p}(\Omega) \times (W^{1,p}(\Omega))^*$ such that for all $k \in \{1, \ldots, T_\tau\}$:*

*(i) for all $\zeta \in H^1(\Omega)$*

$$
\begin{aligned}
&\int_\Omega \Big( D_{\tau,k}(w)\zeta + K(w_\tau^{k-1})\nabla w_\tau^k \cdot \nabla\zeta + \Theta_M(w_\tau^{k-1}) D_{\tau,k}(\chi)\zeta \Big)\,\mathrm{d}x \\
&\quad + \int_\Omega \rho(\chi_\tau^{k-1})\Theta_M(w_\tau^k)\,\mathrm{div}\,(D_{\tau,k}(u))\,\zeta\,\mathrm{d}x \\
&\quad + \int_\Omega \rho'(\chi_\tau^{k-1})\Theta_M(w_\tau^k)\,\mathrm{div}(u_\tau^{k-1}) D_{\tau,k}(\chi)\zeta\,\mathrm{d}x = 0,
\end{aligned}
\tag{10}
$$

*(ii) for a.e. $x \in \Omega$*

$$
\begin{aligned}
&D^2_{\tau,k}(u) - \,\mathrm{div}\,\Big( b(\chi_\tau^k)\varepsilon(u_\tau^k)\Big) - \,\mathrm{div}\,\Big( a(\chi_\tau^k) D_{\tau,k}(\varepsilon(u_\tau^k))\Big) \\
&\qquad + \,\mathrm{div}\,\Big(\rho(\chi_\tau^{k-1})\Theta_M(w_\tau^k)\mathbf{1}\Big) = 0,
\end{aligned}
\tag{11}
$$

*(iii) for all $\zeta \in W^{1,p}(\Omega)$*

$$
\begin{aligned}
0 = &\int_\Omega \Big( D_{\tau,k}(\chi)\zeta + |\nabla\chi_\tau^k|^{p-2}\nabla\chi_\tau^k \cdot \nabla\zeta + \gamma(\chi_\tau^k)\zeta - \Theta_M(w_\tau^{k-1})\zeta \Big)\,\mathrm{d}x \\
&+ \int_\Omega \left( \frac{b_1'(\chi_\tau^k) + b_2'(\chi_\tau^{k-1})}{2} |\varepsilon(u_\tau^{k-1})|^2\zeta - \rho'(\chi_\tau^{k-1})\Theta_M(w_\tau^k)\,\mathrm{div}(u_\tau^{k-1})\zeta \right)\mathrm{d}x \\
&+ \langle \xi, \zeta\rangle_{W^{1,p}}
\end{aligned}
\tag{12}
$$

*with $\xi \in \partial I_{Z_\tau^{k-1}}(\chi_\tau^k)$, where $Z_\tau^{k-1}$ is given by*

$$
Z_\tau^{k-1} := \left\{ f \in W^{1,p}(\Omega) \,|\, 0 \le f \le \chi_\tau^{k-1} \right\}
$$

*and $b = b_1 + b_2$ denotes a convex-concave decomposition of $b$, e.g.*

$$
\begin{aligned}
b_1(r) &:= b(0) + \int_0^r \Big( b'(0) + \int_0^s \max\{b''(\mu), 0\}\,\mathrm{d}\mu \Big)\,\mathrm{d}s, \\
b_2(r) &:= \int_0^r \Big( \int_0^s \min\{b''(\mu), 0\}\,\mathrm{d}\mu \Big)\,\mathrm{d}s.
\end{aligned}
$$

**Proof.** We will trace back this PDE problem to the abstract inclusion problem

$$
\partial\Psi(u) + A(u) \ni f,
\tag{13}
$$

where $\Psi : X \to \mathbb{R} \cup \{+\infty\}$ is a convex, proper and lower semicontinuous functional possessing a convex and Gâteaux differentiable regularization $\Psi_\varepsilon : X \to \mathbb{R}$ such that $\Psi_\varepsilon$ is bounded and radially continuous and

$$
\begin{aligned}
&\limsup_{\varepsilon\downarrow 0} \Psi_\varepsilon(g) \le \Psi(g) \text{ for all } g \in X, \\
&\liminf_{\varepsilon\downarrow 0} \Psi_\varepsilon(g_\varepsilon) \ge \Psi(g) \text{ for all } g_\varepsilon \to g \text{ weakly in } X.
\end{aligned}
$$

and $A : X \to X^*$ is pseudomonotone.

To this end, we define the spaces

$$
X := H_0^1(\Omega; \mathbb{R}^d) \times H^1(\Omega) \times W^{1,p}(\Omega),
$$

$$Y := \left\{(u,w,\chi) \in H_0^1(\Omega;\mathbb{R}^d) \times H^1(\Omega) \times W^{1,p}(\Omega) \,|\, 0 \le \chi \le \chi_\tau^{k-1}\right\} \subseteq X$$

and the operators (we write $A = (A_1, A_2, A_3)$)

$$\begin{aligned}
&\Psi := I_Y \text{ where } \Psi_\varepsilon \text{ is chosen to be the Yosida approximation of } \Psi,\\
&A_1(u,w,\chi) = u - \tau^2 \operatorname{div}(b(\chi)\varepsilon(u)) - \tau \operatorname{div}\left(a(\chi)\varepsilon(u - u_\tau^{k-1})\right) + \operatorname{div}\left(\rho(\chi_\tau^{k-1})\Theta_M(w)\mathbf{1}\right),\\
&A_2(u,w,\chi) = w - \tau \operatorname{div}\left(K(w_\tau^{k-1})\nabla w\right) + \Theta_M(w_\tau^{k-1})\chi + \rho(\chi_\tau^{k-1})\Theta_M(w)\operatorname{div}\left(u - u_\tau^{k-1}\right)\\
&\qquad\qquad + \rho'(\chi_\tau^{k-1})\Theta_M(w)\operatorname{div}\left(u_\tau^{k-1}\right)(\chi - \chi_\tau^{k-1}),\\
&A_3(u,w,\chi) = \chi - \tau\Delta_p\chi + \frac{b_1'(\chi)}{2}|\varepsilon(u_\tau^{k-1})|^2 + \gamma(\chi) - \rho'(\chi_\tau^{k-1})\Theta_M(w)\operatorname{div}(u_\tau^{k-1})
\end{aligned}$$

and the element $f \in X^*$ given by

$$f := \begin{pmatrix} 2u_\tau^{k-1} - u_\tau^{k-2} \\ w_\tau^{k-1} + \Theta_M(w_\tau^{k-1})\chi_\tau^{k-1} \\ \chi_\tau^{k-1} - \frac{b_2'(\chi_\tau^{k-1})}{2}|\varepsilon(u_\tau^{k-1})|^2 + \Theta_M(w_\tau^{k-1}) \end{pmatrix}.$$

Note that $Y$ is a convex, nonempty and closed subspace of $X$ since $\chi_\tau^{k-1} \in C(\overline{\Omega})$.

Now, it can be checked that the operator $A$ is pseudomonotone and coercive. A Leray-Lions type theorem for non-potential inclusions (see [16, Theorem 5.15]) yields a solution to the problem (13) and, therefore, to (i)-(iii). By standard elliptic regularity results, we obtain $u_\tau^k \in H_0^2(\Omega;\mathbb{R}^d)$. □

For later use, we define for a sequence of functions $\{h_\tau^k\}_{0\le k\le T_\tau}$ the piecewise constant and linear interpolation on the time interval $(0,T)$ as

$$\overline{h}_\tau(t) := h_\tau^k, \qquad \underline{h}_\tau(t) := h_\tau^{k-1}, \qquad h_\tau := \frac{t-(k-1)\tau}{\tau}h_\tau^k + \frac{k\tau - t}{\tau}h_\tau^{k-1}$$

for $t \in ((k-1)\tau, k\tau]$. Given a $t \in [0,T]$, we denote by $\overline{t}_\tau$ and $\underline{t}_\tau$ the left- and right-continuous piecewise constant interpolation, i.e.

$$\begin{aligned}
&\overline{t}_\tau := \tau k \text{ for } \tau(k-1) < t \le \tau k,\\
&\underline{t}_\tau := \tau(k-1) \text{ for } \tau(k-1) \le t < \tau k.
\end{aligned}$$

In what follows, we take for every $\tau > 0$ a time-discrete weak solution in the sense of Lemma 4.1 and adopt the convention above.

**Remark 4.2** *The differential inclusion* (12) *is equivalent to the following variational inequality:*

$$\begin{aligned}
0 \ge &- \int_\Omega \left( |\nabla\overline{\chi}_\tau|^{p-2}\nabla\overline{\chi}_\tau \cdot \nabla(\zeta - \overline{\chi}_\tau) + \left(\partial_t\chi_\tau + \gamma(\overline{\chi}_\tau) + \frac{1}{2}b'(\overline{\chi}_\tau)|\varepsilon(\underline{u}_\tau)|^2\right)(\zeta - \overline{\chi}_\tau)\right) \mathrm{d}x\\
&- \int_\Omega \left(-\Theta_M(\underline{w}_\tau) - \rho'(\underline{\chi}_\tau)\Theta_M(\overline{w}_\tau)\operatorname{div}(\underline{u}_\tau)\right)(\zeta - \overline{\chi}_\tau)\,\mathrm{d}x
\end{aligned} \tag{14}$$

*holding for all* $\zeta \in W^{1,p}(\Omega)$ *with* $0 \le \zeta \le \underline{\chi}_\tau$.

### 4.2 A priori estimates

We are going to prove a priori estimates for the discrete system in Lemma 4.1. We will make use of the following implication.

**Lemma 4.3** *A time-discrete weak solution constructed in the previous subsection satisfies $\overline{w}_M \geq 0$ and $\underline{w}_M \geq 0$.*

**Proof.** We show this lemma by induction over $k \in \{0, \ldots, T_\tau\}$. Assume that $w_\tau^{k-1}$ fulfills $w_\tau^{k-1} \geq 0$. Testing equation (10) with $\zeta = -(w_\tau^k)^- := \min\{w_\tau^k, 0\}$ yields

$$\begin{aligned}
&\frac{1}{\tau}\int_\Omega \underbrace{-w_\tau^k (w_\tau^k)^-}_{=|(w_\tau^k)^-|^2} \,\mathrm{d}x + \frac{1}{\tau}\int_\Omega \underbrace{w_\tau^{k-1}(w_\tau^k)^-}_{\geq 0} \,\mathrm{d}x + \int_\Omega \underbrace{K_M(w_\tau^{k-1})\nabla w_\tau^k \cdot \nabla(-(w_\tau^k)^-)}_{\geq c_1 |\nabla(w_\tau^k)^-|^2 \text{ by (A3)}} \,\mathrm{d}x \\
&\quad + \int_\Omega \underbrace{\Theta_M(w_\tau^{k-1})}_{\geq 0} \underbrace{D_{\tau,k}(\chi)}_{\leq 0} \underbrace{(-(w_\tau^k)^-)}_{\leq 0} \,\mathrm{d}x + \int_\Omega \rho(\chi_\tau^{k-1}) \operatorname{div}\left(D_{\tau,k}(u)\right) \underbrace{\Theta_M(w_\tau^k)(-(w_\tau^k)^-)}_{=0} \,\mathrm{d}x \\
&\quad + \int_\Omega \rho'(\chi_\tau^{k-1}) \operatorname{div}(u_\tau^{k-1}) D_{\tau,k}(\chi) \underbrace{\Theta_M(w_\tau^k)(-(w_\tau^k)^-)}_{=0} = 0.
\end{aligned}$$

$\square$

**Lemma 4.4 (A priori estimates independent of $\tau$)** *The following a priori estimates hold with respect to $\tau > 0$:*

*(i) First a priori estimate:*

| | |
|---|---|
| $\{u_\tau\}$ | *in* $H^1(0,T;H^1(\Omega;\mathbb{R}^d)) \cap W^{1,\infty}(0,T;L^2(\Omega;\mathbb{R}^d))$, |
| $\{\overline{u}_\tau\}, \{\underline{u}_\tau\}$ | *in* $L^\infty(0,T;H^1(\Omega;\mathbb{R}^d))$, |
| $\{\overline{w}_\tau\}, \{\underline{w}_\tau\}$ | *in* $L^\infty(0,T;L^1(\Omega))$, |
| $\{\chi_\tau\}$ | *in* $L^\infty(0,T;W^{1,p}(\Omega)) \cap H^1(0,T;L^2(\Omega))$, |
| $\{\overline{\chi}_\tau\}, \{\underline{\chi}_\tau\}$ | *in* $L^\infty(0,T;W^{1,p}(\Omega))$, |

*(ii) Second a priori estimate:*

| | |
|---|---|
| $\{\nabla\Theta_M(\overline{w}_\tau)\}$ | *in* $L^2(0,T;L^2(\Omega))$, |

*(iii) Third a priori estimate:*

| | |
|---|---|
| $\{u_\tau\}$ | *in* $H^1(0,T;H^2(\Omega;\mathbb{R}^d)) \cap W^{1,\infty}(0,T;H^1(\Omega;\mathbb{R}^d))$, |
| $\{\overline{u}_\tau\}, \{\underline{u}_\tau\}$ | *in* $L^\infty(0,T;H^2(\Omega;\mathbb{R}^d))$, |
| $\{v_\tau\}$ | *in* $L^2(0,T;H^2(\Omega;\mathbb{R}^d)) \cap L^\infty(0,T;H^1(\Omega;\mathbb{R}^d))$ $\cap H^1(0,T;L^2(\Omega;\mathbb{R}^d))$, |

*(iv) Forth a priori estimate:*

| | |
|---|---|
| $\{\overline{w}_\tau\}, \{\underline{w}_\tau\}$ | *in* $L^2(0,T;H^1(\Omega)) \cap L^\infty(0,T;L^2(\Omega))$, |

*(v) Fifth a priori estimate:*

| | |
|---|---|
| $\{w_\tau\}$ | *in* $H^1(0,T;(H^1(\Omega))^*)$. |

**Proof of the first a priori estimate.** The first a priori estimate is based on adding equation (3a) tested by $1$ with equation (3b) tested by $\partial_t u$ and with equation (3c) tested by $\partial_t \chi$. Here, we will develop this estimate on a time-discrete level.

In the following, we make use of a convex-concave estimate for $b_1$ and $b_2$ given by

$$\begin{aligned} b(\chi_\tau^{k-1}) - b(\chi_\tau^k) &= \big(b_1(\chi_\tau^{k-1}) - b_1(\chi_\tau^k)\big) + \big(b_2(\chi_\tau^{k-1}) - b_2(\chi_\tau^k)\big) \\ &\geq b_1'(\chi_\tau^k)(\chi_\tau^{k-1} - \chi_\tau^k) + b_2'(\chi_\tau^{k-1})(\chi_\tau^{k-1} - \chi_\tau^k) \\ &= (b_1'(\chi_\tau^k) + b_2'(\chi_\tau^{k-1}))(\chi_\tau^{k-1} - \chi_\tau^k). \end{aligned}$$

Testing (11) with $\zeta = u_\tau^k - u_\tau^{k-1}$ and using the estimates

$$\begin{aligned} & b(\chi_\tau^k)\varepsilon(u_\tau^k) : \varepsilon(u_\tau^k - u_\tau^{k-1}) \\ &\quad \geq \frac{b(\chi_\tau^k)}{2}|\varepsilon(u_\tau^k)|^2 - \frac{b(\chi_\tau^{k-1})}{2}|\varepsilon(u_\tau^{k-1})|^2 + \frac{1}{2}(b(\chi_\tau^{k-1}) - b(\chi_\tau^k))|\varepsilon(u_\tau^{k-1})|^2 \\ &\quad \geq \frac{b(\chi_\tau^k)}{2}|\varepsilon(u_\tau^k)|^2 - \frac{b(\chi_\tau^{k-1})}{2}|\varepsilon(u_\tau^{k-1})|^2 + \frac{1}{2}(b_1'(\chi_\tau^k) + b_2'(\chi_\tau^{k-1}))(\chi_\tau^{k-1} - \chi_\tau^k)|\varepsilon(u_\tau^{k-1})|^2 \end{aligned}$$

and

$$D_{\tau,k}^2(u) \cdot (u_\tau^k - u_\tau^{k-1}) \geq \frac{1}{2}|D_{\tau,k}(u)|^2 - \frac{1}{2}|D_{\tau,k-1}(u)|^2,$$

yield

$$\begin{aligned} &\frac{1}{2}\|D_{\tau,k}(u)\|_{L^2}^2 - \frac{1}{2}\|D_{\tau,k-1}(u)\|_{L^2}^2 + \int_\Omega \frac{b(\chi_\tau^k)}{2}|\varepsilon(u_\tau^k)|^2 \,\mathrm{d}x - \int_\Omega \frac{b(\chi_\tau^{k-1})}{2}|\varepsilon(u_\tau^{k-1})|^2 \,\mathrm{d}x \\ &\qquad + \tau \int_\Omega a(\chi_\tau^k)|D_{\tau,k}(\varepsilon(u))|^2 + R_1 \leq 0 \end{aligned} \tag{15}$$

with the remainder term

$$R_1 := \int_\Omega \frac{b_1'(\chi_\tau^k) + b_2'(\chi_\tau^{k-1})}{2}|\varepsilon(u_\tau^{k-1})|^2(\chi_\tau^{k-1} - \chi_\tau^k) - \int_\Omega \rho(\chi_\tau^{k-1})\Theta_M(w_\tau^k)\,\mathrm{div}\left(u_\tau^k - u_\tau^{k-1}\right).$$

Testing (12) with $\chi_\tau^{k-1} - \chi_\tau^k$ and using the convexity estimate

$$|\nabla\chi_\tau^k|^{p-2}\nabla\chi_\tau^k \cdot \nabla(\chi_\tau^k - \chi_\tau^{k-1}) \geq \int_\Omega \frac{1}{p}|\nabla\chi_\tau^k|^p \,\mathrm{d}x - \int_\Omega \frac{1}{p}|\nabla\chi_\tau^{k-1}|^p \,\mathrm{d}x$$

yield

$$\tau \int_\Omega |D_{\tau,k}(\chi)|^2 \,\mathrm{d}x + \int_\Omega \frac{1}{p}|\nabla\chi_\tau^k|^p \,\mathrm{d}x - \int_\Omega \frac{1}{p}|\nabla\chi_\tau^{k-1}|^p \,\mathrm{d}x + R_2 \leq 0 \tag{16}$$

with the remainder term

$$\begin{aligned} R_2 := &\int_\Omega \gamma(\chi_\tau^k)(\chi_\tau^k - \chi_\tau^{k-1}) \,\mathrm{d}x + \int_\Omega \frac{b_!'(\chi_\tau^k) + b_2'(\chi_\tau^{k-1})}{2}|\varepsilon(u_\tau^{k-1})|^2(\chi_\tau^k - \chi_\tau^{k-1}) \,\mathrm{d}x \\ &- \int_\Omega \Theta_M(w_\tau^{k-1})(\chi_\tau^k - \chi_\tau^{k-1}) \,\mathrm{d}x - \int_\Omega \rho'(\chi_\tau^{k-1})\Theta_M(w_\tau^k)\,\mathrm{div}(u_\tau^{k-1})(\chi_\tau^k - \chi_\tau^{k-1}) \,\mathrm{d}x. \end{aligned}$$

Testing (10) with $\tau$ shows

$$\int_\Omega \left(w_\tau^k - w_\tau^{k-1}\right)\,\mathrm{d}x + R_3 \le 0 \tag{17}$$

with the remainder term

$$\begin{aligned}R_3 := {}&\int_\Omega \Theta_M(w_\tau^{k-1})(\chi_\tau^k - \chi_\tau^{k-1})\,\mathrm{d}x + \int_\Omega \rho(\chi_\tau^{k-1})\Theta_M(w_\tau^k)\,\mathrm{div}\left(u_\tau^k - u_\tau^{k-1}\right)\,\mathrm{d}x\\&+ \int_\Omega \rho'(\chi_\tau^{k-1})\Theta_M(w_\tau^k)\,\mathrm{div}\left(u_\tau^{k-1}\right)(\chi_\tau^k - \chi_\tau^{k-1})\,\mathrm{d}x.\end{aligned}$$

By adding (15)-(17), noticing the crucial property

$$R_1 + R_2 + R_3 = \int_\Omega \gamma(\chi_\tau^k)(\chi_\tau^k - \chi_\tau^{k-1})\,\mathrm{d}x$$

and summing over $k = 1, \ldots, \bar{t}_\tau$, we obtain

$$\begin{aligned}&\int_\Omega \overline{w}_\tau(t)\,\mathrm{d}x + \frac{1}{2}\|\partial_t u_\tau(t)\|^2_{L^2(\Omega)} + c\|\varepsilon(\overline{u}_\tau(t))\|^2_{L^2(\Omega;\mathbb{R}^{d\times d})} + \frac{1}{p}\|\nabla\overline{\chi}_\tau(t)\|^p_{L^p(\Omega)}\\&+ \int_0^{\bar{t}_\tau}\left(c\|\varepsilon(\partial_t u_\tau(s))\|^2_{L^2(\Omega;\mathbb{R}^{d\times d})} + \|\partial_t\chi_\tau(s)\|^2_{L^2(\Omega)}\right)\mathrm{d}s\\&\quad\le \int_\Omega w^0\,\mathrm{d}x + \frac{1}{2}\|v^0\|^2_{L^2(\Omega)} + \int_\Omega \frac{b(\chi^0)}{2}|\varepsilon(u^0)|^2\,\mathrm{d}x + \frac{1}{p}\|\nabla\chi^0\|^p_{L^p(\Omega)}\\&\qquad+ \int_0^{\bar{t}_\tau}\int_\Omega -\gamma(\overline{\chi}_\tau)\partial_t\chi_\tau\,\mathrm{d}x\,\mathrm{d}s\end{aligned} \tag{18}$$

The last term on the right hand side can be estimated from above as follows:

$$\int_0^{\bar{t}_\tau}\int_\Omega -\gamma(\overline{\chi}_\tau)\partial_t\chi_\tau\,\mathrm{d}x\,\mathrm{d}s \le C\int_0^{\bar{t}_\tau}\|\partial_t\chi(s)\|_{L^2(\Omega)}\,\mathrm{d}s.$$

and, therefore, absorbed by the left hand side. Hence the left hand side of (18) is bounded with respect to $\tau$ and $t$. $\square$

**Proof of the second a priori estimate.** By testing (10) with $\tau\Theta_M(w_\tau^k)$ and using the convexity estimate

$$\Theta_M(w_\tau^k)(w_\tau^k - w_\tau^{k-1}) \ge \widehat{\Theta}_M(w_\tau^k) - \widehat{\Theta}_M(w_\tau^{k-1}),$$

where $\widehat{\Theta}_M$ denotes the antiderivative of $\Theta_M$ with $\widehat{\Theta}_M(0) = 0$ (note that $\widehat{\Theta}_M$ is convex due to $\Theta'_M \ge 0$), we obtain

$$\begin{aligned}&\int_\Omega \widehat{\Theta}_M(w_\tau^k)\,\mathrm{d}x - \int_\Omega \widehat{\Theta}_M(w_\tau^{k-1})\,\mathrm{d}x + \tau\int_\Omega K_M(w_\tau^{k-1})\nabla w_\tau^k\cdot\nabla\left(\Theta_M(w_\tau^k)\right)\,\mathrm{d}x\\&\quad\le -\tau\int_\Omega \Theta_M(w_\tau^{k-1})D_{\tau,k}(\chi)\Theta_M(w_\tau^k)\,\mathrm{d}x - \tau\int_\Omega \rho(\chi_\tau^{k-1})\Theta_M(w_\tau^k)\,\mathrm{div}\,(D_{\tau,k}(u))\,\Theta_M(w_\tau^k)\,\mathrm{d}x\\&\qquad- \tau\int_\Omega \rho'(\chi_\tau^{k-1})\Theta_M(w_\tau^k)\,\mathrm{div}(u_\tau^{k-1})D_{\tau,k}(\chi)\Theta_M(w_\tau^k)\,\mathrm{d}x\\&\quad\le \tau\|\Theta_M(w_\tau^{k-1})\|_{L^\infty(\Omega)}\|D_{\tau,k}(\chi)\|_{L^2(\Omega)}\|\Theta_M(w_\tau^k)\|_{L^\infty(\Omega)}\\&\qquad+ \tau\|\rho(\chi_\tau^{k-1})\|_{L^\infty(\Omega)}\|\Theta_M(w_\tau^k)\|_{L^\infty(\Omega)}\|\,\mathrm{div}\,(D_{\tau,k}(u))\,\|_{L^2(\Omega)}\|\Theta_M(w_\tau^k)\|_{L^\infty(\Omega)}\end{aligned}$$

$$+\tau\|\rho'(\chi_\tau^{k-1})\|_{L^\infty(\Omega)}\|\Theta_M(w_\tau^k)\|_{L^\infty(\Omega)}\|\operatorname{div}(u_\tau^{k-1})\|_{L^2(\Omega)}\|D_{\tau,k}(\chi)\|_{L^2(\Omega)}\|\Theta_M(w_\tau^k)\|_{L^\infty(\Omega)}.$$

By summing over all discrete time points $k = 1, \dots, T_\tau$, we end up with the estimate

$$\begin{aligned}
&\int_\Omega \widehat{\Theta}_M(\overline{w}_\tau(T))\,\mathrm{d}x + \int_{\Omega_T} K_M(\underline{w}_\tau)\nabla\overline{w}_\tau \cdot \nabla\left(\Theta_M(\overline{w}_\tau)\right)\,\mathrm{d}x\,\mathrm{d}t \\
&\quad\le \int_\Omega \widehat{\Theta}_M(w^0)\,\mathrm{d}x + \|\Theta_M(\underline{w}_\tau)\|_{L^\infty(L^\infty)}\|\partial_t\chi_\tau\|_{L^2(L^2)}\|\Theta_M(\overline{w}_\tau)\|_{L^2(L^\infty)} \\
&\qquad + \|\rho(\underline{\chi}_\tau)\|_{L^\infty(L^\infty)}\|\Theta_M(\overline{w}_\tau)\|_{L^\infty(L^\infty)}\|\operatorname{div}\left(\partial_t u_\tau\right)\|_{L^2(L^2)}\|\Theta_M(\overline{w}_\tau)\|_{L^2(L^\infty)} \\
&\qquad + \|\rho'(\underline{\chi}_\tau)\|_{L^\infty(L^\infty)}\|\Theta_M(\overline{w}_\tau)\|_{L^\infty(L^\infty)}\|\operatorname{div}(\underline{u}_\tau)\|_{L^2(L^2)}\|\partial_t\chi_\tau\|_{L^2(L^2)}\|\Theta_M(\overline{w}_\tau)\|_{L^\infty(L^\infty)}.
\end{aligned}$$

Together with the first a priori estimate and the Assumptions (A2) and (A3), we obtain boundedness of

$$\begin{aligned}
c_1\int_{\Omega_T}|\nabla\overline{w}_\tau|^2\Theta_M'(\overline{w}_\tau)\,\mathrm{d}x\,\mathrm{d}t &\le \int_{\Omega_T} K_M(\underline{w}_\tau)|\nabla\overline{w}_\tau|^2\Theta_M'(\overline{w}_\tau)\,\mathrm{d}x\,\mathrm{d}t \\
&= \int_{\Omega_T} K_M(\underline{w}_\tau)\nabla\overline{w}_\tau\cdot\nabla\left(\Theta_M(\overline{w}_\tau)\right)\,\mathrm{d}x\,\mathrm{d}t.
\end{aligned}$$

Since $\Theta_M'(w_\tau)$ is also bounded in $L^\infty(0,T;L^\infty(\Omega))$ by the Lipschitz continuity of $\Theta$ (see Assumption (A2)), we obtain the claim as follows:

$$\begin{aligned}
\|\nabla\Theta_M(\overline{w}_\tau)\|_{L^2(L^2)}^2 &= \int_{\Omega_T}|\nabla\overline{w}_\tau|^2(\Theta_M'(\overline{w}_\tau))^2\,\mathrm{d}x\,\mathrm{d}t \\
&\le \|\Theta_M'(w_\tau)\|_{L^\infty(L^\infty)}\int_{\Omega_T}|\nabla\overline{w}_\tau|^2\Theta_M'(\overline{w}_\tau)\,\mathrm{d}x\,\mathrm{d}t.
\end{aligned}$$

□

**Proof of the third a priori estimate.** We test equation (20) with $-\tau\operatorname{div}((\varepsilon(D_{\tau,k}(u)))$ and sum over $k = 1, \dots, \bar{t}_\tau$ for a chosen $t \in [0,T]$. The corresponding calculations without the term $\int_0^{\bar{t}_\tau}\int_\Omega \operatorname{div}\left(\rho(\overline{\chi}_\tau)\Theta_M(\overline{w}_\tau)\mathbf{1}\right)\cdot \operatorname{div}(\varepsilon(\partial_t u_\tau))\,\mathrm{d}x\,\mathrm{d}s$ are carried out in [15, Proposition 3.10].

Hence, we estimate the remaining term by

$$\begin{aligned}
&\left|\int_0^{\bar{t}_\tau}\int_\Omega \operatorname{div}\bigl(\rho(\overline{\chi}_\tau)\Theta_M(\overline{w}_\tau)\mathbf{1}\bigr)\cdot\operatorname{div}(\varepsilon(\partial_t u_\tau))\,\mathrm{d}x\,\mathrm{d}s\right| \\
&\quad\le \int_0^{\bar{t}_\tau}\int_\Omega\left|\Theta_M(\overline{w}_\tau)\rho'(\overline{\chi}_\tau)\nabla\overline{\chi}_\tau\cdot\operatorname{div}\left(\varepsilon\left(\partial_t u_\tau\right)\right)\right|\,\mathrm{d}x\,\mathrm{d}s \\
&\qquad + \int_0^{\bar{t}_\tau}\int_\Omega\left|\rho(\overline{\chi}_\tau)\nabla\bigl(\Theta_M(\overline{w}_\tau)\bigr)\cdot\operatorname{div}\left(\varepsilon\left(\partial_t u_\tau\right)\right)\right|\,\mathrm{d}x\,\mathrm{d}s \\
&\quad\le C\|\Theta_M(\overline{w}_\tau)\|_{L^\infty(L^\infty)}\|\rho'(\overline{\chi}_\tau)\|_{L^\infty(L^\infty)}\|\nabla\overline{\chi}_\tau\|_{L^2(L^2)}\|\partial_t u_\tau\|_{L^2(H^2)} \\
&\qquad + C\|\rho(\overline{\chi}_\tau)\|_{L^\infty(L^\infty)}\|\nabla\bigl(\Theta_M(\overline{w}_\tau)\bigr)\|_{L^2(L^2)}\|\partial_t u_\tau\|_{L^2(H^2)}.
\end{aligned}$$

By using the first and the second a priori estimates and the calculations in [15, Proposition 3.10], we obtain eventually for small $\delta > 0$:

$$\frac{1}{2}\|\varepsilon(\partial_t u_\tau(t))\|_{L^2(\Omega;\mathbb{R}^{d\times d})}^2 + \delta\|\partial_t u_\tau\|_{L^2(0,\bar{t}_\tau;H^2(\Omega;\mathbb{R}^d))}^2$$

$$\leq \frac{1}{2}\|\varepsilon(v^0)\|^2_{L^2(\Omega;\mathbb{R}^{d\times d})} + C\int_0^{\bar{t}_\tau} \|\partial_t u_\tau\|^2_{L^2(0,\bar{s}_\tau;H^2(\Omega;\mathbb{R}^d))}\,\mathrm{d}s + C\,\|\partial_t u_\tau\|_{L^2(0,\bar{t}_\tau;H^2(\Omega;\mathbb{R}^d))}\,.$$

Gronwall's lemma leads to the claim. ☐

**Proof of the fourth a priori estimate.** Testing (10) with $\tau w_\tau^k$ and using standard convexity estimates as well as Assumption (A3) yield

$$\begin{aligned}
&\frac{1}{2}\|w_\tau^k\|^2_{L^2(\Omega)} - \frac{1}{2}\|w_\tau^{k-1}\|^2_{L^2(\Omega)} + c_1\tau\|\nabla w_\tau^k\|^2_{L^2(\Omega;\mathbb{R}^d)}\\
&\quad\leq \tau\|\Theta_M(w_\tau^{k-1})w_\tau^k D_{\tau,k}(\chi)\|_{L^1(\Omega)} + C\tau\|\rho(\chi_\tau^{k-1})\Theta_M(w_\tau^k)\|_{L^\infty(\Omega)}\|\operatorname{div}\big(D_{\tau,k}(u)\big)w_\tau^k\|_{L^1(\Omega)}\\
&\qquad + C\tau\|\rho'(\chi_\tau^{k-1})\Theta_M(w_\tau^k)\|_{L^\infty(\Omega)}\|D_{\tau,k}(\chi)\operatorname{div}(u_\tau^{k-1})w_\tau^k\|_{L^1(\Omega)} \leq 0.
\end{aligned}$$

Summing over the discrete time points $k = 1,\ldots,\bar{t}_\tau$, using the continuous embedding $H^1(\Omega)\hookrightarrow L^6(\Omega)$ and standard estimates, we receive

$$\begin{aligned}
&\frac{1}{2}\|\overline{w}_\tau(t)\|^2_{L^2(\Omega)} + c_1\|\nabla\overline{w}_\tau\|^2_{L^2(0,\bar{t}_\tau,L^2(\Omega;\mathbb{R}^d))}\\
&\quad\leq \frac{1}{2}\|w^0\|^2_{L^2(\Omega)} + C\|\Theta_M(\underline{w}_\tau)\|_{L^\infty(L^\infty)}\left(\int_0^{\bar{t}_\tau}\|\overline{w}_\tau(s)\|^2_{L^2(\Omega)}\,\mathrm{d}s + \|\partial_t\chi_\tau\|^2_{L^2(L^2)}\right)\\
&\qquad + C\|\rho(\underline{\chi}_\tau)\|_{L^\infty(L^\infty)}\|\Theta_M(\overline{w}_\tau)\|_{L^\infty(L^\infty)}\left(\int_0^{\bar{t}_\tau}\|\overline{w}_\tau(s)\|^2_{L^2(\Omega)}\,\mathrm{d}s + \|\operatorname{div}\big(\partial_t u_\tau\big)\|^2_{L^2(L^2)}\right)\\
&\qquad + \|\rho'(\underline{\chi}_\tau)\|_{L^\infty(L^\infty)}\|\Theta_M(\overline{w}_\tau)\|_{L^\infty(L^\infty)}\times\\
&\qquad\quad\times\left(\delta\int_0^{\bar{t}_\tau}\|\overline{w}_\tau(s)\|^2_{H^1(\Omega)}\,\mathrm{d}s + C_\delta\|\partial_t\chi_\tau\|^2_{L^2(L^2)}\|\operatorname{div}(\underline{u}_\tau)\|^2_{L^\infty(L^3)}\right).
\end{aligned}$$

Chosing $\delta > 0$ sufficiently small, applying the first and the third a priori estimates, we obtain by Gronwall's inequality boundedness of the left hand side and, therefore, the claim. ☐

**Proof of the fifth a priori estimate.** A comparison argument in equation (10) shows the assertion. ☐

### 4.3 The passage $\tau \downarrow 0$

By utilizing Lemma 4.4 and by noticing $\partial_t u_\tau = \overline{v}_\tau$ (see (9)), we obtain by standard compactness and Aubin-Lions type theorems (see [18]) the following convergence properties.

**Corollary 4.5** *We obtain functions $(u, w, \chi)$ which are in the spaces*

$$\begin{aligned}
&u \in H^1(0,T;H_0^2(\Omega;\mathbb{R}^d)) \cap W^{1,\infty}(0,T;H_0^1(\Omega;\mathbb{R}^d)) \cap H^2(0,T;L^2(\Omega;\mathbb{R}^d))\\
&\quad \textit{with } u(0) = u^0 \textit{ a.e. in } \Omega,\ \partial_t u(0) = v^0 \textit{ a.e. in } \Omega,\\
&w \in L^2(0,T;H^1(\Omega)) \cap L^\infty(0,T;L^2(\Omega)) \cap H^1(0,T;(H^1(\Omega))^*)\\
&\quad \textit{with } w(0) = w^0 \textit{ a.e. in } \Omega,\ w \geq 0 \textit{ a.e. in } \Omega_T,\\
&\chi \in L^\infty(0,T;W^{1,p}(\Omega)) \cap H^1(0,T;L^2(\Omega))
\end{aligned}$$

*with* $\chi(0) = \chi^0$ *a.e. in* $\Omega,\ \chi \geq 0$ *a.e. in* $\Omega_T,\ \partial_t\chi \leq 0$ *a.e. in* $\Omega_T$

*such that (along a subsequence) for all* $\varepsilon \in (0,1]$, $\mu \geq 1$*:*

*(i)* $u_\tau \to u$ *weakly-star in* $H^1(0,T;H^2(\Omega;\mathbb{R}^d)) \cap W^{1,\infty}(0,T;H^1(\Omega;\mathbb{R}^d))$,
$\overline{u}_\tau, \underline{u}_\tau \to u$ *weakly-star in* $L^\infty(0,T;H^2(\Omega;\mathbb{R}^d))$,
$u_\tau \to u$ *strongly in* $H^1(0,T;H^{2-\varepsilon}(\Omega;\mathbb{R}^d))$,
$\overline{u}_\tau, \underline{u}_\tau \to u$ *strongly in* $L^\infty(0,T;H^{2-\varepsilon}(\Omega;\mathbb{R}^d))$,
$u_\tau, \overline{u}_\tau, \underline{u}_\tau \to u$ *a.e. in* $\Omega_T$,

*(ii)* $v_\tau \to \partial_t u$ *weakly-star in* $H^1(0,T;L^2(\Omega;\mathbb{R}^d))$,

*(iii)* $w_\tau \to w$ *weakly-star in* $L^2(0,T;H^1(\Omega)) \cap L^\infty(0,T;L^2(\Omega)) \cap H^1(0,T;(H^1(\Omega))^*)$,
$\overline{w}_\tau, \underline{w}_\tau \to w$ *weakly-star in* $L^2(0,T;H^1(\Omega)) \cap L^\infty(0,T;L^2(\Omega))$,
$\overline{w}_\tau, \underline{w}_\tau \to w$ *strongly in* $L^2(0,T;H^{1-\varepsilon}(\Omega)) \cap L^\mu(0,T;L^2(\Omega))$,
$w_\tau, \overline{w}_\tau, \underline{w}_\tau \to w$ *a.e. in* $\Omega_T$,

*(iv)* $\chi_\tau \to \chi$ *weakly-star in* $L^\infty(0,T;W^{1,p}(\Omega)) \cap H^1(0,T;L^2(\Omega))$,
$\overline{\chi}_\tau, \underline{\chi}_\tau \to \chi$ *weakly-star in* $L^\infty(0,T;W^{1,p}(\Omega))$,
$\overline{\chi}_\tau, \underline{\chi}_\tau \to \chi$ *strongly in* $L^\mu(0,T;W^{1-\varepsilon,p}(\Omega))$,
$\overline{\chi}_\tau, \underline{\chi}_\tau \to \chi$ *uniformly on* $\overline{\Omega_T}$.

**Lemma 4.6** *It even holds (along a subsequence as* $\tau \downarrow 0$*)*

$$\overline{\chi}_\tau \to \chi \text{ strongly in } L^p(0,T;W^{1,p}(\Omega)).$$

**Proof.** Applying an approximations result from [9, Lemma 5.2], we obtain a sequence $\{\zeta_\tau\}$ in the space $L^p(0,T;W^{1,p}_+(\Omega))$ such that $\zeta_\tau \to \chi$ in $L^p(0,T;W^{1,p}(\Omega))$ as $\tau \downarrow 0$ and

$$0 \leq \zeta_\tau(t) \leq \underline{\chi}_\tau(t) \text{ a.e. in } \Omega_T.$$

The claim can now be shown by using a uniform monotonicity estimate of the $L^p$-norm

$$\begin{aligned}
\|\overline{\chi}_\tau - \chi\|^p_{L^p(\Omega_T)} &\leq C\int_{\Omega_T} \big(|\nabla\overline{\chi}_\tau|^{p-2}\nabla\overline{\chi}_\tau - |\nabla\chi|^{p-2}\nabla\chi\big)\cdot\nabla(\overline{\chi}_\tau - \chi)\,\mathrm{d}x\,\mathrm{d}t\\
&= C\int_{\Omega_T} \big(|\nabla\overline{\chi}_\tau|^{p-2}\nabla\overline{\chi}_\tau - |\nabla\chi|^{p-2}\nabla\chi\big)\cdot\nabla(\overline{\chi}_\tau - \zeta_\tau)\,\mathrm{d}x\,\mathrm{d}t\\
&\quad + C\int_{\Omega_T} \big(|\nabla\overline{\chi}_\tau|^{p-2}\nabla\overline{\chi}_\tau - |\nabla\chi|^{p-2}\nabla\chi\big)\cdot\nabla(\zeta_\tau - \chi)\,\mathrm{d}x\,\mathrm{d}t,
\end{aligned}$$

by applying Corollary 4.5 and by testing the variational inequality (14) with $\zeta_\tau$, it can be shown that $lim\ sup$ of the right hand side is $\leq 0$. ☐

The passage to the limit $\tau \downarrow 0$ in the time-discrete system in Lemma 4.1 can now be performed as follows.

- **Heat equation.** Integrating equation (10) over the time $[0, T]$, Corollary 4.5 allows to pass to the limit $\tau \downarrow 0$ by taking into account the uniform boundedness of $K_M(\underline{w}_\tau)$, $\Theta_M(\underline{w}_\tau)$ and $\Theta_M(\overline{w}_\tau)$ in $L^\infty(\Omega)$. Then, by switching to an a.e. $t$ formulation in the limit, we obtain for every $\zeta \in H^1(\Omega)$ and a.e. $t \in (0, T)$:

$$\langle \partial_t w, \zeta\rangle_{H^1} + \int_\Omega \left(K_M(w)\nabla w \cdot \nabla\zeta + \Theta_M(w)\partial_t\chi\zeta\right)\,\mathrm{d}x \\ + \int_\Omega \Big(\rho(\chi)\Theta_M(w)\,\mathrm{div}\,(\partial_t u)\,\zeta + \rho'(\chi)\Theta_M(w)\,\mathrm{div}(u)\partial_t\chi\zeta\Big)\,\mathrm{d}x = 0. \tag{19}$$

- **Balance of forces.** To obtain the equation for the balance of forces, we integrate equation (11) over $\Omega_T$ and use Corollary 4.5 to pass to the limit $\tau \downarrow 0$. In the limit we have the necessary regularity properties to switch to an a.e. formulation in $\Omega_T$, i.e. it holds

$$\partial_{tt}u - \,\mathrm{div}\,(b(\chi)\varepsilon(u)) - \,\mathrm{div}\,(a(\chi)\varepsilon(\partial_t u)) + \,\mathrm{div}\,(\rho(\chi)\Theta_M(w)\mathbf{1}) = 0 \tag{20}$$

a.e. in $\Omega_T$.

- **One-sided variational inequality for the damage process.** The limit passage for equation (14) can be accomplished by an approximation argument developed in [9]. Note that this approach strongly relies on $p > d$ (see (A8)). We sketch the argument.
  - Initially, the main idea has been to consider time-depending test-functions $\Psi \in L^\infty(0, T; W^{1,p}_-(\Omega))$ which satisfies for a.e. $t \in (0, T)$ the constraint

    $$\{x \in \overline{\Omega} \,|\, \Psi(x, t) = 0\} \supseteq \{x \in \overline{\Omega} \,|\, \chi(x, t) = 0\}.$$

    Here, we make use of the embedding $W^{1,p}(\Omega) \hookrightarrow C(\overline{\Omega})$.
  - As shown in [9, Lemma 5.2], we obtain an approximation sequence $\{\Psi_\tau\} \subseteq L^p(0, T; W^{1,p}_-(\Omega))$ and constants $\nu = \nu(\tau, t) > 0$ (independent of $x$) such that $\Psi_\tau \to \Psi$ in $L^p(0, T; W^{1,p}(\Omega))$ as $\tau \downarrow 0$ and $0 \le -\nu\Psi_\tau(t) \le \overline{\chi}_\tau(t)$ in $\Omega$ for a.e. $t \in (0, T)$. Multiplying this inequality by -1, adding $\overline{\chi}_\tau(t)$ and using the monotonicity condition $\overline{\chi}_\tau \le \underline{\chi}_\tau$, we obtain

    $$0 \le \nu\Psi_\tau(t) + \overline{\chi}_\tau(t) \le \underline{\chi}_\tau(t) \text{ in } \Omega. \tag{21}$$

  - Because of (21), we are allowed to test (14) with $\nu_\tau(t)\Psi_\tau(t) + \overline{\chi}_\tau(t)$. Dividing the resulting inequality by $\nu$ (which is positive and independent of $x$), integrating in time over $[0, T]$, passing to the limit and switching back to an a.e. $t$ formulation, we obtain for a.e. $t \in (0, T)$

    $$0 \le \int_\Omega \left(\partial_t\chi\zeta + |\nabla\chi|^{p-2}\nabla\chi \cdot \nabla\zeta + \gamma(\chi)\zeta + \frac{b'(\chi)}{2}|\varepsilon(u)|^2\zeta\right)\,\mathrm{d}x \\ + \int_\Omega \left(-\Theta_M(w)\zeta - \rho'(\chi)\Theta_M(w)\,\mathrm{div}(u)\zeta\right)\,\mathrm{d}x,$$

    for all $\zeta \in W^{1,p}_-(\Omega)$ with $\{\zeta = 0\} \supseteq \{\chi(t) = 0\}$.
  - It is shown in [9, Lemma 5.3] that, in this case, we obtain

    $$0 \le \int_\Omega \left(\partial_t\chi\zeta + |\nabla\chi|^{p-2}\nabla\chi \cdot \nabla\zeta + \gamma(\chi)\zeta + \frac{b'(\chi)}{2}|\varepsilon(u)|^2\zeta\right)\,\mathrm{d}x \\ + \int_\Omega \left(-\Theta_M(w)\zeta - \rho'(\chi)\Theta_M(w)\,\mathrm{div}(u)\zeta + \xi\zeta\right)\,\mathrm{d}x \tag{22}$$

for all $\zeta \in W^{1,p}_-(\Omega)$ and for a.e. $t \in (0,T)$, where $\xi \in L^2(0,T;L^2(\Omega))$ is given by

$$\xi := -\mathbf{1}_{\{\chi=0\}}\Big(\gamma(\chi) + \frac{b'(\chi)}{2}|\varepsilon(u)|^2 - \Theta_M(w) - \rho'(\chi)\Theta_M(w)\,\mathrm{div}(u)\Big)^+, \tag{23}$$

with $(\cdot)^+ := \max\{\cdot,0\}$. Note that $\partial_t\chi$ does not appear in the bracket. In particular, $\xi$ fulfills

$$\int_\Omega \xi(\zeta - z)\,\mathrm{d}x \le 0 \tag{24}$$

for all $\zeta \in W^{1,p}_+(\Omega)$ and a.e. $t \in (0,T)$.

- **Partial energy inequality.** Test the variational inequality (14) with $\chi^k_\tau - \chi^{k-1}_\tau$, applying the convexity argument

$$\int_\Omega |\nabla\chi^k_\tau|^{p-2}\nabla\chi^k_\tau\cdot\nabla(\chi^k_\tau - \chi^{k-1}_\tau)\,\mathrm{d}x \ge \int_\Omega |\nabla\chi^k_\tau|^p\,\mathrm{d}x - \int_\Omega |\nabla\chi^{k-1}_\tau|^p\,\mathrm{d}x,$$

summing over the discrete time points $k = 1,\ldots,\bar{t}_\tau$, we end up with

$$\begin{aligned}&\int_\Omega |\nabla\overline{\chi}_\tau(t)|^p\,\mathrm{d}x - \int_\Omega |\nabla\chi^0|^p\,\mathrm{d}x + \int_0^{\bar{t}_\tau}\int_\Omega\Big(\gamma(\overline{\chi}_\tau) + \frac{b'(\overline{\chi}_\tau)}{2}|\varepsilon(\underline{u}_\tau)|^2\Big)\partial_t\chi_\tau\,\mathrm{d}x\,\mathrm{d}t\\
&\quad+\int_0^{\bar{t}_\tau}\int_\Omega\Big(-\Theta_M(\underline{w}_\tau) - \rho'(\underline{\chi}_\tau)\Theta_M(\overline{w}_\tau)\,\mathrm{div}(\underline{u}_\tau) + \partial_t\chi_\tau\Big)\partial_t\chi_\tau\,\mathrm{d}x\,\mathrm{d}t \le 0\end{aligned}$$

for a.e. $t \in (0,T)$. Passing to the limit $\tau \downarrow 0$ by using Corollary 4.5, weakly lower-semicontinuity arguments and the estimate $t \le \bar{t}_\tau$ for the quadratic term in $\partial_t\chi$, we get for a.e. $t \in (0,T)$ the desired partial energy inequality

$$\begin{aligned}&\int_\Omega |\nabla\chi(t)|^p\,\mathrm{d}x - \int_\Omega |\nabla\chi^0|^p\,\mathrm{d}x + \int_0^t\int_\Omega\Big(\gamma(\chi) + \frac{b'(\chi)}{2}|\varepsilon(u)|^2\Big)\partial_t\chi\,\mathrm{d}x\,\mathrm{d}t\\
&\quad+\int_0^t\int_\Omega\Big(-\Theta_M(w) - \rho'(\chi)\Theta_M(w)\,\mathrm{div}(u) + \partial_t\chi\Big)\partial_t\chi\,\mathrm{d}x\,\mathrm{d}t \le 0.\end{aligned} \tag{25}$$

In conclusion, we have proven existence of weak solutions to the truncated system given by (19), (20), (22), (24) and (25).

# 5 Existence of weak solutions for the limit system

In this section, we will perform the limit analysis for weak solutions of the truncated system as $M \uparrow \infty$. We consider for each $M \in \mathbb{N}$ a weak solution $(u_M, w_M, \chi_M, \xi_M)$ as proven in the previous section.

## 5.1 A priori estimates

The boundedness properties for $(u_M, w_M, \chi_M, \xi_M)$ with respect to $M$ are based on six different types of a priori estimates.

**Lemma 5.1 (A priori estimates independent of $M$)** *The following boundedness properties with respect to $M$ are satisfied:*

*(i) First a priori estimate:*

$\{u_M\}$ *in* $H^1(0,T;H^1(\Omega;\mathbb{R}^d))\cap W^{1,\infty}(0,T;L^2(\Omega;\mathbb{R}^d))$,

$\{w_M\}$ *in* $L^\infty(0,T;L^1(\Omega))$,

$\{\chi_M\}$ *in* $L^\infty(0,T;W^{1,p}(\Omega))\cap H^1(0,T;L^2(\Omega))$,

*(ii) Second a priori estimate:*

$\{\mathcal{T}_M(w_M)\}$ *in* $L^2(0,T;H^1(\Omega))$,

*(iii) Third a priori estimate:*

$\{u_M\}$ *in* $H^1(0,T;H^2(\Omega;\mathbb{R}^d))\cap W^{1,\infty}(0,T;H^1(\Omega;\mathbb{R}^d))$
$\cap\, H^2(0,T;L^2(\Omega;\mathbb{R}^d))$,

*(iv) Forth a priori estimate:*

$\{\mathcal{T}_M(w_M)\}$ *in* $L^\infty(0,T;L^2(\Omega))\cap L^{2(q+1)}(0,T;L^{6(q+1)}(\Omega))$,

*(v) Fifth a priori estimate:*

$\{w_M\}$ *in* $L^\infty(0,T;L^2(\Omega))\cap L^2(0,T;H^1(\Omega))$,

*(vi) Sixth a priori estimate:*

$\{w_M\}$ *in* $W^{1,r}(0,T;(W^{2,s}_\nu(\Omega))^*)$

*with the constants* $r:=(2q+2)/(2q_0+1)$ *and* $s:=(6q+6)/(6q-2q_0+5)$.

**Proof of the first a priori estimate.** The first a priori estimate in Lemma 4.4 which is based on the energy estimate (18) is also independent of $M$. Lower semi-continuity arguments show the energy estimate also for weak solutions $(u_M,w_M,\chi_M)$ of the time-continuous, truncated system. □

**Proof of the second a priori estimate.** We deduce the desired estimate by testing (19) with

$$\zeta=-(\mathcal{T}_M(w_M)+1)^{-\alpha}\in H^1(\Omega), \tag{26}$$

where $\alpha$ is a fixed real number satisfying $1/\sigma\le\alpha\le 2q-1$ (recap Assumption (A3)). We remind that $\mathcal{T}_M(w_M)\ge 0$ a.e. in $\Omega_T$. Integration in time reveals

$$\begin{aligned}&\int_0^T\big\langle\partial_t w_M,-(\mathcal{T}_M(w_M)+1)^{-\alpha}\big\rangle_{H^1}\,\mathrm{d}t+\int_{\Omega_T}\frac{K_M(w_M)}{(\mathcal{T}_M(w_M)+1)^{\alpha+1}}\nabla w_M\cdot\nabla\mathcal{T}_M(w_M)\,\mathrm{d}x\,\mathrm{d}t\\&+\int_{\Omega_T}\Big(\partial_t\chi_M+\rho(\chi_M)\,\mathrm{div}\,(\partial_t u_M)+\rho'(\chi_M)\,\mathrm{div}(u_M)\partial_t\chi_M\Big)\frac{-\Theta_M(w_M)}{(\mathcal{T}_M(w_M)+1)^{\alpha}}\,\mathrm{d}x\,\mathrm{d}t=0.\end{aligned}\tag{27}$$

The integral terms on the left hand side are transformed/estimates in the following calculations.

- Let $\psi_M$ denote the primitive of the function $\zeta$ given in (26) with $\psi_M(0)=0$. The use of a generalized chain-rule yields

$$\int_0^T\big\langle\partial_t w_M,-(\mathcal{T}_M(w_M)+1)^{-\alpha}\big\rangle\,\mathrm{d}t=\int_\Omega\psi_M(w_M(T))\,\mathrm{d}x-\int_\Omega\psi_M(w^0)\,\mathrm{d}x.$$

- By utilizing the identities $\nabla w_M \cdot \nabla \mathcal{T}_M(w_M) = |\nabla \mathcal{T}_M(w_M)|^2$ and $K_M(w_M) = K(\mathcal{T}_M(w_M))$, the growth assumption for $K$ (see Assumption (A3)) and the estimate $\alpha \le 2q-1$, we obtain

$$\begin{aligned}
&\int_{\Omega_T} \frac{K_M(w_M)}{(\mathcal{T}_M(w_M)+1)^{\alpha+1}} \nabla w_M \cdot \nabla \mathcal{T}_M(w_M)\,\mathrm{d}x\,\mathrm{d}t \\
&\quad = \int_{\Omega_T} \frac{K(\mathcal{T}_M(w_M))}{(\mathcal{T}_M(w_M)+1)^{\alpha+1}} |\nabla \mathcal{T}_M(w_M)|^2\,\mathrm{d}x\,\mathrm{d}t \\
&\quad \ge c_1 \int_{\Omega_T} \frac{(\mathcal{T}_M(w_M)^{2q}+1)}{(\mathcal{T}_M(w_M)+1)^{\alpha+1}} |\nabla \mathcal{T}_M(w_M)|^2\,\mathrm{d}x\,\mathrm{d}t \\
&\quad \ge \widetilde{c}_1 \|\nabla \mathcal{T}_M(w_M)\|^2_{L^2(\Omega_T;\mathbb{R}^d)}.
\end{aligned}$$

- The identity $\Theta_M(w_M) = \Theta(\mathcal{T}_M(w_M))$, the growth assumption for $\Theta$ (see Assumption (A2)) and the estimate $1/\sigma \le \alpha$ imply boundedness of

$$\left|\frac{\Theta_M(w_M)}{\mathcal{T}_M(w_M)+1}\right| = \frac{\Theta(\mathcal{T}_M(w_M))}{\mathcal{T}_M(w_M)+1} \le c_0 \frac{(\mathcal{T}_M(w_M)^{1/\sigma}+1)}{(\mathcal{T}_M(w_M)+1)^{\alpha}} \le C.$$

Putting the pieces together, (27) results in

$$\begin{aligned}
&\int_\Omega \psi_M(w_M(T))\,\mathrm{d}x - \int_\Omega \psi_M(w^0)\,\mathrm{d}x + \widetilde{c}_1 \|\nabla \mathcal{T}_M(w_M)\|^2_{L^2(\Omega_T;\mathbb{R}^d)} \\
&\quad \le C \left\| \partial_t \chi_M + \rho(\chi_M)\,\mathrm{div}\,(\partial_t u_M) + \rho'(\chi_M)\,\mathrm{div}(u_M)\partial_t \chi_M \right\|_{L^1(\Omega_T)}.
\end{aligned}$$

The right hand side estimates as

$$\begin{aligned}
\text{r.h.s.} \le C\big(&\|\partial_t \chi_M\|_{L^1(\Omega_T)} + \|\rho(\chi_M)\|_{L^\infty(\Omega_T)} \|\,\mathrm{div}\,(\partial_t u_M)\,\|_{L^1(\Omega_T)} \\
&+ \|\rho'(\chi_M)\|_{L^\infty(\Omega_T)} \|\,\mathrm{div}(u_M)\|_{L^2(\Omega_T)} \|\partial_t \chi_M\|_{L^2(\Omega_T)}\big)
\end{aligned}$$

and is bounded by the first a priori estimate.

It remains to show boundedness of $\int_\Omega \psi_M(w_M(T))\,\mathrm{d}x$. Since

$$|\psi_M'(x)| = |(\mathcal{T}_M(x)+1)^{-\alpha}| \in [-1,0],$$

we obtain the growth condition $|\psi_M(x)| \le |x|$. Hence

$$\left|\int_\Omega \psi_M(w_M(T))\,\mathrm{d}x\right| \le \int_\Omega w_M(T)\,\mathrm{d}x.$$

Eventually, we obtain boundedness of $\|\nabla \mathcal{T}_M(w_M)\|_{L^2(\Omega_T;\mathbb{R}^d)}$. The claim follows together with the boundedness of $\|\mathcal{T}_M(w_M)\|_{L^\infty(0,T;L^1(\Omega))}$ by the first a priori estimate. □

**Proof of the third a priori estimate.** We test (20) with $\zeta = -\,\mathrm{div}(\varepsilon(u_t))$ and adapt a calculation performed in [15, Sixth a priori estimate]. Additionally, we need to estimate the following integral term:

$$\begin{aligned}
&\left|\int_{\Omega_t} \mathrm{div}\,(\rho(\chi_M)\Theta_M(w_M)\mathbf{1}) \cdot \mathrm{div}(\varepsilon(\partial_t u_M))\,\mathrm{d}x\,\mathrm{d}s\right| \\
&\quad \le \int_{\Omega_t} |(\rho'(\chi_M)\nabla \chi_M \Theta_M(w_M)) \cdot \mathrm{div}(\varepsilon(\partial_t u_M))|\,\mathrm{d}x\,\mathrm{d}s \\
&\qquad + \int_{\Omega_t} |\rho(\chi_M)\Theta'(\mathcal{T}_M(w_M))\nabla\big(\mathcal{T}_M(w_M)\big) \cdot \mathrm{div}(\varepsilon(\partial_t u_M))|\,\mathrm{d}x\,\mathrm{d}s \\
&\quad \le C\|\rho'(\chi_M)\|_{L^\infty(L^\infty)} \|\nabla \chi_M\|_{L^\infty(L^p)} \|\Theta(\mathcal{T}_M(w_M))\|_{L^2(L^{2p/(p-2)})} \|\partial_t u_M\|_{L^2(H^2)} \\
&\qquad + C\|\rho(\chi_M)\|_{L^\infty(L^\infty)} \|\Theta_M'(w_M)\|_{L^\infty(L^\infty)} \|\nabla\big(\mathcal{T}_M(w_M)\big)\|_{L^2(L^2)} \|\partial_t u_M\|_{L^2(H^2)}.
\end{aligned}$$

By using the Lipschitz continuity of $\Theta$ (see Assumption (A2)) and the first as well as the second a priori estimates, it only remains to show boundedness of the term $\|\Theta(\mathcal{T}_M(w_M))\|_{L^2(L^{2p/(p-2)})}$. Indeed, by using the growth assumption in (A2),

$$\|\Theta(\mathcal{T}_M(w_M))\|_{L^2(L^{2p/(p-2)})} \leq c_0 \left( \|\mathcal{T}_M(w_M)\|^{1/\sigma}_{L^{2/\sigma}(L^{2p/(\sigma(p-2))})} + 1 \right). \tag{28}$$

In the case $d = 3$, we have $p > 3$ and, in particular, $2p/(\sigma(p-2)) \leq 6$ since $\sigma \geq 3$ by Assumption (A2). Consequently, by using the second a priori estimate, the right hand side of (28) is bounded.

In the cases $d \in \{1, 2\}$, boundedness of the right hand side of (28) follows immediately from the second a priori estimate and $\sigma \geq 3$. □

**Proof of the fourth a priori estimate.** Testing (19) with $\mathcal{T}_M(w_M)$, integration in time over $[0, t]$ and using the generalized chain-rule yield

$$\begin{aligned}
&\int_\Omega \widehat{\mathcal{T}}_M(w_M(t))\,\mathrm{d}x - \int_\Omega \widehat{\mathcal{T}}_M(w^0)\,\mathrm{d}x + \int_{\Omega_t} K(\mathcal{T}_M(w_M))|\nabla \mathcal{T}_M(w_M)|^2\,\mathrm{d}x\,\mathrm{d}s \\
&+ \int_{\Omega_t} \big(\partial_t \chi_M + \rho(\chi_M)\,\mathrm{div}\,(\partial_t u_M) + \rho'(\chi_M)\,\mathrm{div}(u_M)\partial_t\chi_M\big)\,\Theta_M(w_M)\mathcal{T}_M(w_M)\,\mathrm{d}x\,\mathrm{d}s = 0,
\end{aligned}$$

where $\widehat{\mathcal{T}}_M$ denotes the primitive of $\mathcal{T}_M$ vanishing at $0$. By using Assumption (A3), the estimates (cf. [15, Remark 2.10])

$$c\int_0^t \|\mathcal{T}_M(w_M)\|^{2(q+1)}_{L^{6(q+1)}(\Omega)}\,\mathrm{d}s \leq \int_{\Omega_t} (\mathcal{T}_M(w_M)^{2q} + 1)|\nabla \mathcal{T}_M(w_M)|^2\,\mathrm{d}x\,\mathrm{d}s,$$

and

$$\frac{1}{2}|\mathcal{T}_M(w_M)|^2 \leq \widehat{\mathcal{T}}_M(w_M),$$

we obtain by using Hölder's inequality in space and time

$$\begin{aligned}
&\int_\Omega \frac{1}{2}|\mathcal{T}_M(w_M)|^2\,\mathrm{d}x - \int_\Omega \widehat{\mathcal{T}}_M(w^0)\,\mathrm{d}x + \widetilde{c}\|\mathcal{T}_M(w_M)\|^{2(q+1)}_{L^{2(q+1)}(0,t;L^{6(q+1)}(\Omega))} \\
&\quad\leq \|\partial_t\chi_M\|_{L^2(L^2)}\|\Theta_M(w_M)\mathcal{T}_M(w_M)\|_{L^2(0,t;L^2(\Omega))} \\
&\qquad + \|\rho(\chi_M)\|_{L^\infty(L^\infty)}\|\,\mathrm{div}(\partial_t u_M)\|_{L^2(L^2)}\|\Theta_M(w_M)\mathcal{T}_M(w_M)\|_{L^2(0,t;L^2(\Omega))} \\
&\qquad + \|\rho'(\chi_M)\|_{L^\infty(L^\infty)}\|\partial_t\chi_M\|_{L^2(L^2)}\|\,\mathrm{div}(u_M)\|_{L^\infty(L^6)}\|\Theta_M(w_M)\mathcal{T}_M(w_M)\|_{L^2(0,t;L^3(\Omega))} \\
&\quad\leq C\|\Theta_M(w_M)\mathcal{T}_M(w_M)\|_{L^2(0,t;L^3(\Omega))}.
\end{aligned}$$

Notice the following implications:

$$\begin{cases}
\text{if } 0 \leq q \leq 1 & \text{then } 1/\sigma \leq q \text{ (since } 2q-1 \leq q \text{ and } 1/\sigma \leq 2q-1 \text{ by (A3))}, \\
\text{if } q > 1 & \text{then } 1/\sigma \leq q \text{ (since } \sigma \geq 3 \text{ by (A2))}.
\end{cases}$$

Therefore, in both cases $1/\sigma \leq q$ and we can estimate the right hand side above as follows by using Assumption (A2):

$$\begin{aligned}
\|\Theta_M(w_M)\mathcal{T}_M(w_M)\|_{L^2(0,t;L^3(\Omega))} &\leq C\big(\||T_M(w_M)|^{1/\sigma+1}\|_{L^2(0,t;L^3(\Omega))} + 1\big) \\
&\leq C\big(\|\mathcal{T}_M(w_M)\|^{q+1}_{L^{2(q+1)}(0,t;L^{3(q+1)}(\Omega))} + 1\big) \\
&\leq C\big(\|\mathcal{T}_M(w_M)\|^{q+1}_{L^{2(q+1)}(0,t;L^{6(q+1)}(\Omega))} + 1\big).
\end{aligned}$$

Thus the r.h.s. can be absorbed by the l.h.s. and we obtain the assertion. $\square$

**Proof of the fifth a priori estimate.** We test equation (19) with $w_M$, integrate over the time interval $[0,t]$ and obtain

$$\frac{1}{2}\int_\Omega |w_M(t)|^2\,\mathrm{d}x - \frac{1}{2}\int_\Omega |w_M(0)|^2\,\mathrm{d}x + \int_{\Omega_t} K_M(w_M)|\nabla w_M|^2\,\mathrm{d}x\,\mathrm{d}t$$
$$+ \int_{\Omega_t} \big(\partial_t\chi_M + \rho(\chi_M)\,\mathrm{div}\,(\partial_t u_M) + \rho'(\chi_M)\,\mathrm{div}(u_M)\partial_t\chi_M\big)\,\Theta_M(w_M)w_M\,\mathrm{d}x\,\mathrm{d}t = 0.$$

We introduce the sublevel and the strict superlevel set of $w_M(t)$ at height $M$ as

$$l_M^-(t) := \{x\in\Omega \,|\, w_M(x,t)\le M\}, \tag{29a}$$
$$l_M^+(t) := \{x\in\Omega \,|\, w_M(x,t) > M\} \tag{29b}$$

and receive by utilizing Hölder's inequality as in the fourth a priori estimate

$$\begin{aligned}
&\frac{1}{2}\int_\Omega |w_M(t)|^2\,\mathrm{d}x - \frac{1}{2}\int_\Omega |w_M(0)|^2\,\mathrm{d}x + c\int_{\Omega_t} |\nabla w_M|^2\,\mathrm{d}x\,\mathrm{d}t + c\int_0^t \|w_M\|_{L^{6(q+1)}(l_M^-(t))}^{2(q+1)}\,\mathrm{d}s\\
&\le C\|\Theta_M(w_M)w_M\|_{L^2(0,t;L^3(\Omega))}\\
&\le C\left(\int_0^t \|\Theta_M(w_M)w_M\|_{L^3(l_M^-(t))}^2\,\mathrm{d}s\right)^{1/2} + C\left(\int_0^t \|\Theta_M(w_M)w_M\|_{L^3(l_M^+(t))}^2\,\mathrm{d}s\right)^{1/2}.
\end{aligned} \tag{30}$$

We treat the last two terms on the right hand side as follows.

- By using the definition $l_M^-(t)$, the growth assumption for $\Theta$ in (A2) and the estimate $1/\sigma\le q$ (see the proof of the fourth a priori estimate), we obtain

$$\begin{aligned}
\int_0^t \|\Theta_M(w_M)w_M\|_{L^3(l_M^-(t))}^2\,\mathrm{d}s &= \int_0^t \|\Theta(w_M)w_M\|_{L^3(l_M^-(t))}^2\,\mathrm{d}s\\
&\le C\left(\int_0^t \|w_M\|_{L^{6(q+1)}(l_M^-(t))}^{2(q+1)}\,\mathrm{d}s + 1\right).
\end{aligned}$$

- Hölder's inequality and the embedding $H^1(\Omega)\hookrightarrow L^6(\Omega)$ yield

$$\begin{aligned}
\int_0^t \|\Theta_M(w_M)w_M\|_{L^3(l_M^-(t))}^2\,\mathrm{d}s &\le \operatorname*{ess\,sup}_{t\in(0,T)} \|\Theta_M(w_M(t))\|_{L^6(l_M^+(t))}^2 \int_0^t \|w_M\|_{L^6(l_M^+(t))}^2\,\mathrm{d}s\\
&\le \operatorname*{ess\,sup}_{t\in(0,T)} \|\Theta_M(w_M(t))\|_{L^6(l_M^+(t))}^2 \|w_M\|_{L^2(0,t;H^1(\Omega))}^2.
\end{aligned}$$

  By the fourth a priori estimate, we have the boundedness of

$$M^2 \operatorname*{ess\,sup}_{t\in(0,T)} |l_M^+(t)| = \operatorname*{ess\,sup}_{t\in(0,T)} \int_{l_M^+(t)} M^2\,\mathrm{d}x = \|\mathcal{T}_M(w_M)\|_{L^\infty(0,T;L^2(\Omega))}^2\,\mathrm{d}x \le C.$$

  This implies by using the growth condition for $\Theta$ in (A2):

$$\operatorname*{ess\,sup}_{t\in(0,T)} \|\Theta_M(w_M(t))\|_{L^6(l_M^+(t))}^2 = \Theta(M)^2 \operatorname*{ess\,sup}_{t\in(0,T)} |l_M^+(t)|^{1/3}$$

$$\le c_0(M^{2/\sigma}+1)\operatorname*{ess\,sup}_{t\in(0,T)}|l_M^+(t)|^{1/3}$$
$$\le c_0(M^{2/\sigma}+1)\frac{C}{M^{2/3}}.$$

Since $\sigma \ge 3$, we obtain boundedness of $\operatorname{ess\,sup}_{t\in(0,T)}\|\Theta_M(w_M(t))\|^2_{L^6(l_M^+(t))}$ and hence

$$\int_0^t \|\Theta(w_M)w_M\|^2_{L^3(l_M^+(t))}\,\mathrm{d}s \le C\|w_M\|^2_{L^2(0,t;H^1(\Omega))}.$$

Eventually, estimate (30) yields to

$$\frac{1}{2}\int_\Omega |w_M(t)|^2\,\mathrm{d}x - \frac{1}{2}\int_\Omega |w_M(0)|^2\,\mathrm{d}x + c\|\nabla w_M\|^2_{L^2(0,t;L^2(\Omega;\mathbb{R}^d))} + c\int_0^t \|w_M\|^{2(q+1)}_{L^{6(q+1)}(l_M^-(t))}\,\mathrm{d}s$$
$$\le C\left(\int_0^t \|w_M\|^{2(q+1)}_{L^{6(q+1)}(l_M^-(t))}\,\mathrm{d}s + 1\right)^{1/2} + C\|w_M\|_{L^2(0,t;H^1(\Omega))}$$

and thus the claim. $\square$

To tackle the sixth a priori estimate, we will make use of the primitive $\widehat{K}_M$ of $K_M$ vanishing at $0$ and use the property

$$\widehat{K}_M(x) = \begin{cases} \widehat{K}(x) & \text{if } 0\le x\le M, \\ \widehat{K}(M)+x-M & \text{if } x>M. \end{cases} \tag{31}$$

Note that the identity $\widehat{K}_M(x) = \widehat{K}(\mathcal{T}_M(x))$ is not fulfilled while $K_M(x) = K(\mathcal{T}_M(x))$ is true. By exploiting growth assumption (A3), we obtain the crucial estimate

$$\left|\widehat{K}_M(x)\right| \le \left.\begin{cases} C(x^{2q_0+1}+1) & \text{if } 0\le x\le M, \\ C(M^{2q_0+1}+1)+x-M & \text{if } x>M \end{cases}\right\} \le C(\mathcal{T}_M(x)^{2q_0+1}+1)+x. \tag{32}$$

**Proof of the sixth a priori estimate.** We will use a comparison argument in (19).

In what follows let $r := \frac{2q+2}{2q_0+1}$ and $s := \frac{6q+6}{6q-2q_0+5}$ as in Definition 3.1. Applying integration by parts in (19), we receive for all $\zeta \in W^{2,s}_\nu(\Omega)$:

$$\langle \partial_t w_M, \zeta\rangle = \int_\Omega \Big(\widehat{K}_M(w_M)\Delta\zeta - \left(\Theta_M(w_M)\partial_t\chi_M + \rho(\chi_M)\Theta_M(w_M)\operatorname{div}(\partial_t u_M)\right)\zeta\Big)\,\mathrm{d}x$$
$$- \int_\Omega \rho'(\chi_M)\Theta_M(w_M)\operatorname{div}(u_M)\partial_t\chi_M\zeta\,\mathrm{d}x. \tag{33}$$

Let $s^{**} := \frac{6q+6}{2q-2q_0+1} > 0$ denote the constant resulting from the continuous embedding $W^{2,s}(\Omega)\hookrightarrow L^{s^{**}}(\Omega)$. Due to the crucial identities

$$\frac{1}{\frac{6q+6}{2q_0+1}} + \frac{1}{\frac{6q+6}{6q-2q_0+5}} = 1,\ \frac{1}{\frac{6q+6}{2q_0+1}} + \frac{1}{6} + \frac{1}{2} + \frac{1}{\frac{6q+6}{2q-2q_0+1}} = 1 \text{ and } \frac{1}{\frac{6q+6}{q+2q_0+2}} + \frac{1}{2} + \frac{1}{\frac{6q+6}{2q-2q_0+1}} = 1, \tag{34}$$

Hölder's inequality reveals

$$\langle \partial_t w_M, \zeta\rangle \le \|\widehat{K}_M(w_M)\|_{L^{\frac{6q+6}{2q_0+1}}}\|\Delta\zeta\|_{L^s} + \|\Theta_M(w_M)\|_{L^{\frac{6q+6}{q+2q_0+2}}}\|\partial_t\chi_M\|_{L^2}\|\zeta\|_{L^{s^{**}}}$$
$$+ \|\rho(\chi_M)\|_{L^\infty}\|\Theta_M(w_M)\|_{L^{\frac{6q+6}{q+2q_0+2}}}\|\operatorname{div}(\partial_t u_M)\|_{L^2}\|\zeta\|_{L^{s^{**}}}$$
$$+ \|\rho'(\chi_M)\|_{L^\infty}\|\Theta_M(w_M)\|_{L^{\frac{6q+6}{2q_0+1}}}\|\operatorname{div}(u_M)\|_{L^6}\|\partial_t\chi_M\|_{L^2}\|\zeta\|_{L^{s^{**}}}.$$

By using boundedness of $\chi_M$ in $L^\infty(0,T;L^\infty(\Omega))$ and $\frac{6q+6}{q+2q_0+2} \leq \frac{6q+6}{2q_0+1}$, we obtain

$$\|\partial_t w_M\|_{(W^{2,s}_\nu)^*} \leq C\|\Theta_M(w_M)\|_{L^{\frac{6q+6}{2q_0+1}}}\Big(\|\partial_t\chi_M\|_{L^2} + \|\operatorname{div}(\partial_t u_M)\|_{L^2} \\ + \|\operatorname{div}(u_M)\|_{L^6}\|\partial_t\chi_M\|_{L^2}\Big) + \|\widehat{K}_M(w_M)\|_{L^{\frac{6q+6}{2q_0+1}}}.$$

Calculating $L^r$-norm in time and using Hölder's inequality show

$$\begin{aligned}
&\|\partial_t w_M\|_{L^r\left((W^{2,s}_\nu)^*\right)} \\
&\quad\leq C\|\Theta_M(w_M)\|_{L^{\frac{2r}{2-r}}\left(L^{\frac{6q+6}{2q_0+1}}\right)}\Big(\|\partial_t\chi_M\|_{L^2(L^2)} + \|\operatorname{div}(\partial_t u_M)\|_{L^2(L^2)} \\
&\qquad + \|\operatorname{div}(u_M)\|_{L^\infty(L^6)}\|\partial_t\chi_M\|_{L^2(L^2)}\Big) + \|\widehat{K}_M(w_M)\|_{L^r\left(L^{\frac{6q+6}{2q_0+1}}\right)}.
\end{aligned}$$

Keeping the first and the third a priori estimates in mind, it still remains to show

$$\{\widehat{K}_M(w_M)\} \text{ bounded in } L^r\big(0,T;L^{\frac{6q+6}{2q_0+1}}(\Omega)\big), \tag{35a}$$

$$\{\Theta_M(w_M)\} \text{ bounded in } L^{\frac{2r}{2-r}}\big(0,T;L^{\frac{6q+6}{2q_0+1}}(\Omega)\big). \tag{35b}$$

- Estimate (32) leads to

  $$\|\widehat{K}_M(w_M)\|_{L^r\left(L^{\frac{6q+6}{2q_0+1}}\right)} \leq C(\|\mathcal{T}_M(w_M)\|^{2q_0+1}_{L^{r(2q_0+1)}(L^{6q+6})} + 1) + \|w_M\|_{L^r\left(L^{\frac{6q+6}{2q_0+1}}\right)}. \tag{36}$$

  Since, by definition, $r(2q_0+1) = 2(q+1)$, we infer boundedness of

  $$\{\mathcal{T}_M(w_M)\} \text{ in } L^{r(2q_0+1)}(0,T;L^{6q+6}(\Omega))$$

  by the fourth a priori estimate and boundedness of

  $$\{w_M\} \text{ in } L^r\big(0,T;L^{\frac{6q+6}{2q_0+1}}(\Omega)\big)$$

  by the fifth a priori estimate and by $r \in (1,2)$ and $\frac{6q+6}{2q_0+1} \leq 6$ using (A3). Finally, we obtain (35a).
- By Assumption (A2), we obtain

  $$\|\Theta_M(w_M)\|_{L^{\frac{2r}{2-r}}\left(0,T;L^{\frac{6q+6}{2q_0+1}}(\Omega)\right)} \leq C(\|w_M\|^{1/\sigma}_{L^{\frac{2r}{(2-r)\sigma}}\left(0,T;L^{\frac{6q+6}{(2q_0+1)\sigma}}(\Omega)\right)} + 1).$$

  Because of $\frac{6q+6}{(2q_0+1)\sigma} \leq 2$ (since $\sigma \geq 3$ by (A2) and $q \leq q_0$ by (A3)), we obtain (35b) by the fifth a priori estimate.

$\square$

### 5.2 The passage $M \uparrow \infty$

The a priori estimates from Lemma 5.1 give rise to the subsequent convergence properties for $\{u_M\}$, $\{w_M\}$ and $\{\chi_M\}$ along subsequences by Aubin-Lions type compactness results (cf. [18]) and by adapting Lemma 4.6 to this case.

**Corollary 5.2** *There exists limit functions $(u, w, \chi)$ defined in spaces given in Definition 3.1 such that the following convergence properties are satisfied for all $\mu \geq 1$, $s > 3$ and all $\varepsilon \in (0,1]$ (as $M \uparrow \infty$ for a subsequence):*

$$\begin{aligned}
&\text{(i) } u_M \to u && \text{weakly-star in } H^1(0,T;H^2(\Omega;\mathbb{R}^d)) \cap W^{1,\infty}(0,T;H^1(\Omega;\mathbb{R}^d)) \\
& && \qquad \cap H^2(0,T;L^2(\Omega;\mathbb{R}^d)), \\
&u_M \to u && \text{strongly in } H^1(0,T;H^{2-\varepsilon}(\Omega;\mathbb{R}^d)), \\
&u_M \to u && \text{a.e. in } \Omega_T, \\
&\text{(ii) } w_M \to w && \text{weakly-star in } L^2(0,T;H^1(\Omega)) \cap L^\infty(0,T;L^2(\Omega)) \\
& && \qquad \cap W^{1,r}(0,T;(W^{2,s}_\nu(\Omega))^*), \\
&w_M \to w && \text{strongly in } L^2(0,T;H^{1-\varepsilon}(\Omega)) \cap L^\mu(0,T;L^2(\Omega)), \\
&w_M \to w && \text{a.e. in } \Omega_T, \\
&\text{(iii) } \chi_M \to \chi && \text{weakly-star in } L^\infty(0,T;W^{1,p}(\Omega)) \cap H^1(0,T;L^2(\Omega)), \\
&\chi_M \to \chi && \text{strongly in } L^\mu(0,T;W^{1,p}(\Omega)), \\
&\chi_M \to \chi && \text{uniformly on } \overline{\Omega_T}.
\end{aligned}$$

Corollary 5.2 can be used to prove convergence of $\widehat{K}_M(w_M)$, $\Theta_M(w_M)$ and $\xi_M$ as $M \uparrow \infty$ in suitable spaces. More precisely, we obtain the following result.

**Corollary 5.3** *There exists an element $\xi \in L^2(0,T;L^2(\Omega))$ such that for all $1 \leq \mu < 6$ (as $M \uparrow \infty$ for a subsequence):*

$$\begin{aligned}
&\text{(i) } \widehat{K}_M(w_M) \to \widehat{K}(w) && \text{weakly in } L^{\frac{2q+2}{2q_0+1}}\big(0,T;L^{\frac{6q+6}{2q_0+1}}(\Omega)\big), \\
&\text{(ii) } \Theta_M(w_M) \to \Theta(w) && \text{strongly in } L^{2\sigma}(0,T;L^{\mu\sigma}(\Omega)), \\
&\text{(iii) } \xi_M \to \xi && \text{weakly in } L^2(0,T;L^2(\Omega)).
\end{aligned}$$

**Proof.**

(i) We obtain the estimate

$$\begin{aligned}
&\|\widehat{K}_M(w_M)\|_{L^{\frac{2q+2}{2q_0+1}}\left(L^{\frac{6q+6}{2q_0+1}}\right)} \\
&\quad \leq C(\|\mathcal{T}_M(w_M)\|^{2q_0+1}_{L^{2q+2}(L^{6q+6})} + 1) + \|w_M\|_{L^{\frac{2q+2}{2q_0+1}}\left(L^{\frac{6q+6}{2q_0+1}}\right)}.
\end{aligned}$$

due to (32). The first summand on the right hand side is bounded by the fourth a priori estimate while the second is bounded by the fifth a priori estimate.

This enables us to choose a subsequence (we omit the subindex) such that

$$\widehat{K}_M(w_M) \to \eta \text{ weakly in } L^{\frac{2q+2}{2q_0+1}}\big(0,T;L^{\frac{6q+6}{2q_0+1}}(\Omega)\big) \tag{37}$$

for an element $\eta \in L^{\frac{2q+2}{2q_0+1}}(0,T;L^{\frac{6q+6}{2q_0+1}}(\Omega))$.

Furthermore, noticing that $w_M \to w$ a.e. in $\Omega_T$ as $M \uparrow \infty$, we conclude

$$\widehat{K}_M(w_M) \to \widehat{K}(w) \text{ a.e. in } \Omega_T. \tag{38}$$

From (37) and (38) we conclude (i).

(ii) This item follows from the fact that $w_M \to w$ converge strongly in $L^2(0,T;L^\mu(\Omega))$ for all $1 \leq \mu < 6$ and from the growth condition for $\Theta$ in (A2).

(iii) By referring to the construction of $\xi_M$ in (23), we choose a weakly-star cluster point for the sequence $\{\mathbf{1}_{\{\chi_M=0\}}\}$, i.e.

$$\pi_M := \mathbf{1}_{\{\chi_M=0\}} \to \pi \text{ weakly-star in } L^\infty(0,T;L^\infty(\Omega))$$

as $M \uparrow \infty$ for a subsequence. By the already known convergence properties, we deduce that the sequence of functions

$$\eta_M := \Big(\gamma(\chi_M) + \frac{b'(\chi_M)}{2}|\varepsilon(u_M)|^2 - \Theta_M(w_M) - \rho'(\chi_M)\Theta_M(w_M)\,\mathrm{div}(u_M)\Big)^+$$

converges strongly to the corresponding limit function $\eta$ in $L^2(0,T;L^2(\Omega))$. This proves

$$\xi_M = -\pi_M\eta_M \to -\pi\eta =: \xi \text{ weakly in } L^2(0,T;L^2(\Omega))$$

as desired.

□

**Proof of Theorem 3.4.** The limit passage of the truncated system given by (19), (20), (22), (24) and (25) as $M \uparrow \infty$ can now be perform with Corollary 5.2 and Corollary 5.3.

- **Heat equation.** Integrating (19) in time and applying integration by parts show

$$\begin{aligned}&\int_0^T \langle \partial_t w_M, \Psi\rangle\,\mathrm{d}t\\ &\quad - \int_{\Omega_T} \Big(\widehat{K}_M(w_M)\Delta\Psi - \left(\Theta_M(w_M)\partial_t\chi_M + \rho(\chi_M)\Theta_M(w_M)\,\mathrm{div}\,(\partial_t u_M)\right)\Psi\Big)\,\mathrm{d}x\,\mathrm{d}t\\ &\quad + \int_{\Omega_T} \rho'(\chi_M)\Theta_M(w_M)\,\mathrm{div}(u_M)\partial_t\chi_M\Psi\,\mathrm{d}x\,\mathrm{d}t = 0,\end{aligned}$$

  for all test-functions $\Psi \in C([0,T];W^{2,s}_\nu(\Omega))$. Taking (34) into account, passing $M \uparrow \infty$ by employing the convergence results in Corollary 5.2 and Corollary 5.3 and switching back to an a.e. in time formulation, we end up with (4).

- **Balance of momentum equation** and **one-sided variational inequality.** Translating (20), (22) and (24) to a weak formulation involving test-functions in time and space, we can pass $M \uparrow \infty$. Translating the results back to an a.e. in time formulation, we obtain (5), (6) and (7).

- **Partial energy inequality.** The inequality (8) is gained from (25) by using lower semi-continuity arguments in the transition $M \uparrow \infty$. □

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