Weierstraß-Institut
für Angewandte Analysis und Stochastik
Leibniz-Institut im Forschungsverbund Berlin e. V.

On nonlocal Cahn-Hilliard-Navier-Stokes systems in two dimensions

Sergio Frigeri[1], Ciprian G. Gal [2], Maurizio Grasselli[3]

submitted: February 20, 2014

[1] Weierstrass Institute Mohrenstr. 39 10117 Berlin Germany E-Mail: SergioPietro.Frigeri@wias-berlin.de	[2] Department od Mathematics Florida International University Miami, FL 33199 USA E-Mail: cgal@fiu.edu

[3] Dipartimento di Matematica
Politecnico di Milano
20133 Milano
Italy
E-Mail: maurizio.grasselli@polimi.it

No. 1923

Berlin 2014

Abstract

We consider a diffuse interface model which describes the motion of an incompressible isothermal mixture of two immiscible fluids. This model consists of the Navier-Stokes equations coupled with a convective nonlocal Cahn-Hilliard equation. Several results were already proven by two of the present authors. However, in the two-dimensional case, the uniqueness of weak solutions was still open. Here we establish such a result even in the case of degenerate mobility and singular potential. Moreover, we show the strong-weak uniqueness in the case of viscosity depending on the order parameter, provided that the mobility is constant and the potential is regular. In the case of constant viscosity, on account of the uniqueness results we can deduce the connectedness of the global attractor whose existence was obtained in a previous paper. The uniqueness technique can be adapted to show the validity of a smoothing property for the difference of two trajectories which is crucial to establish the existence of an exponential attractor.

1 Introduction

In a series of recent papers (see [8, 11, 12, 13, 14]) the following nonlinear evolution system has been analyzed

$$\varphi_t + u \cdot \nabla \varphi = \mathsf{div}(m(\varphi)\nabla\mu), \tag{1.1}$$

$$\mu = a\varphi - J * \varphi + F'(\varphi), \tag{1.2}$$

$$u_t - 2\mathsf{div}(\nu(\varphi)Du) + (u \cdot \nabla)u + \nabla\pi = \mu\nabla\varphi + h(t), \tag{1.3}$$

$$\mathsf{div}(u) = 0, \tag{1.4}$$

on a bounded domain $\Omega \subset \mathbb{R}^d$, $d = 2, 3$, for $t > 0$. This system describes the evolution of an isothermal mixture of two incompressible and immiscible fluids through the (relative) concentration φ of one species and the (averaged) velocity field u. Here m denotes the mobility, μ is the so-called chemical potential, J is a spatial-dependent interaction kernel and $J * \varphi$ stands for spatial convolution over Ω, a is defined as follows $a(x) = \int_\Omega J(x - y)dy$, F is a double-well potential, ν is the viscosity and h is an external force acting on the mixture. The density is supposed to be constant and equal to one (i.e., matched densities).

Such a system is the nonlocal version of the well-known Cahn-Hilliard-Navier-Stokes system which has been the subject of a number of papers (cf., e.g., [1, 2, 6, 7, 15, 16, 17, 27, 29] and references therein, see also the review [26] for modeling and numerical simulation issues). We recall that the nonlocal term seems physically more appropriate than its approximation, i.e., when in place of $a\varphi - J * \varphi$ there is $-\Delta\varphi$. For this issue, we refer the reader to the basic papers [20, 21, 22] (see also [4, 18, 19, 24, 25]). However, from the mathematical viewpoint, the present

system is more challenging since the regularity of φ is lower and so the Korteweg force $\mu \nabla \varphi$ acting on the fluid can be less regular than the convective term $(u \cdot \nabla)u$, even in dimension two (cf. [8, (3.4) and (3.7)]). Therefore, it is not straightforward to extend some of the results which hold for the Navier-Stokes equations as well as for the standard Cahn-Hilliard-Navier-Stokes system. This is particularly meaningful in dimension two. In fact, in dimension three, the only known results are comparable with the standard ones for the Navier-Stokes equations, namely, the existence of a global weak solution under various assumptions on m and F and a generalized notion of attractor (cf. [8, 11, 12, 14]).

In dimension two, under reasonable assumptions on F which ensure a suitable regularity of φ, it is possible to prove that there exists a weak solution which satisfies the energy identity. Therefore, such a solution is strongly continuous in time (see [8]). In addition, taking advantage of the energy identity, it is also possible to prove the existence of a the global attractor for the corresponding semiflow (cf. [11, 12, 14]). More recently, in [13], assuming constant (in φ) ν and m and taking a regular potential F, it has been shown the existence of a (unique) strong solution and that any weak solution which satisfies the energy identity regularizes in finite time. This entails some smoothness for the global attractor. Also, the convergence of any weak solution to a single equilibrium was established through the Łojasiewicz-Simon inequality approach. However, uniqueness of weak solutions was still an open issue in [8, 11, 12, 14].

The main goal of this paper is to prove the uniqueness of weak solutions when ν is constant; while, when ν is non constant, we are able to show weak-strong uniqueness. Uniqueness entails the connectedness of the global attractor. In addition, modifying the uniqueness argument we can also show the validity of a suitable smoothing property of the difference of two trajectories (see [9, 10]). This is the basic step to establish the existence of an exponential attractor. The fractal dimension of the global attractor is thus finite.

As in the previous contributions we take the following boundary and initial conditions

$$\frac{\partial \mu}{\partial n} = 0, \quad u = 0 \quad \text{on } \partial\Omega \times (0, T), \tag{1.5}$$

$$u(0) = u_0, \quad \varphi(0) = \varphi_0 \quad \text{in } \Omega. \tag{1.6}$$

The plan of the paper is the following. In the next section we recall the basic assumptions and the related existence of a weak solution. Section 3 is devoted to the uniqueness of weak solutions for constant viscosity. The strong-weak uniqueness is shown in Section 4. The final Section 5 is concerned with the connectedness of the global attractor and the existence of an exponential attractor.

2 Functional setup and preliminary results

Let us introduce the classical Hilbert spaces for the Navier-Stokes equations with no-slip boundary condition (see, e.g., [28])

$$G_{div} := \overline{\{u \in C_0^\infty(\Omega)^d : \text{div}(u) = 0\}}^{L^2(\Omega)^d},$$

2

and
$$V_{div} := \{u \in H_0^1(\Omega)^d : \text{div}(u) = 0\}.$$

We set $H := L^2(\Omega)$, $V := H^1(\Omega)$, and denote by $\|\cdot\|$ and $(\cdot,\cdot)$ the norm and the scalar product, respectively, on both H and G_{div}. H will also be used for L^2 spaces of vector or matrix valued functions. The notation $\langle\cdot,\cdot\rangle$ will stand for the duality pairing between a Banach space X and its dual X'. V_{div} is endowed with the scalar product

$$(u,v)_{V_{div}} = (\nabla u, \nabla v) = 2(Du, Dv), \qquad \forall u, v \in V_{div},$$

where D is the symmetric gradient, defined by $Du := (\nabla u + (\nabla u)^{tr})/2$.

The trilinear form b which appears in the weak formulation of the Navier-Stokes equations is defined as usual

$$b(u,v,w) = \int_\Omega (u \cdot \nabla)v \cdot w, \qquad \forall u, v, w \in V_{div},$$

and the associated bilinear operator $\mathcal{B}$ from $V_{div} \times V_{div}$ into V'_{div} is defined by $\langle\mathcal{B}(u,v), w\rangle := b(u,v,w)$, for all $u, v, w \in V_{div}$. We recall that we have $b(u,w,v) = -b(u,v,w)$, for all $u, v, w \in V_{div}$, and that the following estimate holds in dimension two

$$|b(u,v,w)| \le c\|u\|^{1/2}\|\nabla u\|^{1/2}\|\nabla v\|\|w\|^{1/2}\|\nabla w\|^{1/2}, \quad \forall u, v, w \in V_{div}.$$

In particular we have the following standard estimate in 2D which holds for all $u \in V_{div}$

$$\|\mathcal{B}(u,u)\|_{V'_{div}} \le c\|u\|\|\nabla u\|. \tag{2.1}$$

For every $f \in V'$ we denote by $\overline{f}$ the average of f over Ω, i.e., $\overline{f} := |\Omega|^{-1}\langle f, 1\rangle$. Here $|\Omega|$ is the Lebesgue measure of Ω. We assume that $\partial\Omega$ is smooth enough.

We also need to introduce the Hilbert spaces

$$V_0 := \{v \in V : \overline{v} = 0\}, \qquad V'_0 := \{f \in V' : \overline{f} = 0\},$$

and the operator $A_N : V \to V'$, $A_N \in \mathcal{L}(V, V')$, defined by

$$\langle A_N u, v\rangle := \int_\Omega \nabla u \cdot \nabla v \qquad \forall u, v \in V.$$

We recall that A_N maps V onto V'_0 and the restriction B_N of A_N to V_0 maps V_0 onto V'_0 isomorphically. Further, we denote by $B_N^{-1} : V'_0 \to V_0$ the inverse map. As is well known, for every $f \in V'_0$, $B_N^{-1}f$ is the unique solution with zero mean value of the Neumann problem

$$\begin{cases} -\Delta u = f, & \text{in } \Omega, \\ \frac{\partial u}{\partial n} = 0, & \text{on } \partial\Omega. \end{cases}$$

In addition, we have

$$\langle A_N u, B_N^{-1} f\rangle = \langle f, u\rangle, \qquad \forall u \in V, \quad \forall f \in V'_0,$$

$$\langle f, B_N^{-1} g\rangle = \langle g, B_N^{-1} f\rangle = \int_\Omega \nabla(B_N^{-1}f) \cdot \nabla(B_N^{-1}g), \qquad \forall f, g \in V'_0.$$

Furthermore, B_N can be also viewed as an unbounded linear operator on H with domain $D(B_N) = \{v \in H^2(\Omega) : \partial_n v = 0 \text{ on } \partial\Omega\}$.

If X is a Banach space and $\tau \in \mathbb{R}$, we shall denote by $L^p_{tb}(\tau, \infty; X)$, $1 \leq p < \infty$, the space of functions $f \in L^p_{loc}([\tau, \infty); X)$ that are translation bounded in $L^p_{loc}([\tau, \infty); X)$, that is,

$$\|f\|^p_{L^p_{tb}(\tau,\infty;X)} := \sup_{t \geq \tau} \int_t^{t+1} \|f(s)\|^p_X ds < \infty.$$

We now recall the result on existence of weak solutions and on the validity of the energy identity and of a dissipative estimate in dimension two for the nonlocal Cahn-Hilliard-Navier-Stokes system in the case of constant mobility, nonconstant viscosity and regular potential. This is the main case we shall deal with in this paper.

Let us list the assumptions (see [8]).

(H1) $J \in W^{1,1}(\mathbb{R}^d)$, $\quad J(x) = J(-x)$, $\quad a \geq 0$ $\quad$ a.e. in Ω.

(H2) The mobility $m(s) = 1$ for all $s \in \mathbb{R}$, the viscosity ν is locally Lipschitz on $\mathbb{R}$ and there exist $\nu_1, \nu_2 > 0$ such that
$$\nu_1 \leq \nu(s) \leq \nu_2, \qquad \forall s \in \mathbb{R}.$$

(H3) $F \in C^{2,1}_{loc}(\mathbb{R})$ and there exists $c_0 > 0$ such that
$$F''(s) + a(x) \geq c_0, \qquad \forall s \in \mathbb{R}, \quad \text{a.e. } x \in \Omega.$$

(H4) $F \in C^2(\mathbb{R})$ and there exist $c_1 > 0$, $c_2 > 0$ and $q > 0$ such that
$$F''(s) + a(x) \geq c_1|s|^{2q} - c_2, \qquad \forall s \in \mathbb{R}, \quad \text{a.e. } x \in \Omega.$$

(H5) There exist $c_3 > 0$, $c_4 \geq 0$ and $r \in (1, 2]$ such that
$$|F'(s)|^r \leq c_3|F(s)| + c_4, \qquad \forall s \in \mathbb{R}.$$

Remark 1. Assumption $J \in W^{1,1}(\mathbb{R}^d)$ can be weakened. Indeed, it can be replaced by $J \in W^{1,1}(B_\delta)$, where $B_\delta := \{z \in \mathbb{R}^d : |z| < \delta\}$ with $\delta := \text{diam}(\Omega)$, or also by (see, e.g., [4])
$$\sup_{x \in \Omega} \int_\Omega \big(|J(x - y)| + |\nabla J(x - y)|\big)dy < \infty.$$

Remark 2. Since F is bounded from below, it is easy to see that (H5) implies that F has polynomial growth of order p', where $p' \in [2, \infty)$ is the conjugate index to p. Namely, there exist $c_5 > 0$ and $c_6 \geq 0$ such that
$$|F(s)| \leq c_5|s|^{p'} + c_6, \qquad \forall s \in \mathbb{R}.$$

Observe that assumption (H5) is fulfilled by a potential of arbitrary polynomial growth. For example, (H3)–(H5) are satisfied for the case of the well-known double-well potential $F(s) = (s^2 - 1)^2$.

4

The following result follows from [8, Theorem 1, Corollaries 1 and 2].

Theorem 1. *Let* $h \in L^2_{loc}([0, \infty); V'_{div})$, $u_0 \in G_{div}$, $\varphi_0 \in H$ *such that* $F(\varphi_0) \in L^1(\Omega)$ *and suppose that (H1)-(H5) are satisfied. Then, for every given* $T > 0$, *there exists a weak solution* $[u, \varphi]$ *to* (1.1)–(1.6) *such that*

$$u \in L^\infty(0, T; G_{div}) \cap L^2(0, T; V_{div}), \quad \varphi \in L^\infty(0, T; L^{2+2q}(\Omega)) \cap L^2(0, T; V), \quad (2.2)$$

$$u_t \in L^{4/3}(0, T; V'_{div}), \quad \varphi_t \in L^{4/3}(0, T; V'), \qquad d = 3, \quad (2.3)$$

$$u_t \in L^2(0, T; V'_{div}), \quad d = 2, \quad (2.4)$$

$$\varphi_t \in L^2(0, T; V'), \quad d = 2 \quad \textit{or} \quad d = 3 \textit{ and } q \geq 1/2, \quad (2.5)$$

and satisfying the energy inequality

$$\mathcal{E}(u(t), \varphi(t)) + \int_0^t \left(2\|\sqrt{\nu(\varphi)}Du\|^2 + \|\nabla\mu\|^2 \right) d\tau \leq \mathcal{E}(u_0, \varphi_0) + \int_0^t \langle h(\tau), u \rangle d\tau,$$

for every $t > 0$, *where we have set*

$$\mathcal{E}(u(t), \varphi(t)) = \frac{1}{2}\|u(t)\|^2 + \frac{1}{4}\int_\Omega \int_\Omega J(x - y)(\varphi(x, t) - \varphi(y, t))^2 dxdy + \int_\Omega F(\varphi(t)).$$

If $d = 2$, *then any weak solution satisfies the energy identity*

$$\frac{d}{dt}\mathcal{E}(u, \varphi) + 2\|\sqrt{\nu(\varphi)}Du\|^2 + \|\nabla\mu\|^2 = \langle h(t), u \rangle, \quad (2.6)$$

In particular we have $u \in C([0, \infty); G_{div})$, $\varphi \in C([0, \infty); H)$ *and* $\int_\Omega F(\varphi) \in C([0, \infty))$. *Furthermore, if* $d = 2$ *and* $h \in L^2_{tb}(0, \infty; V'_{div})$, *then any weak solution satisfies also the dissipative estimate*

$$\mathcal{E}(u(t), \varphi(t)) \leq \mathcal{E}(u_0, \varphi_0)e^{-kt} + F(m_0)|\Omega| + K, \qquad \forall t \geq 0, \quad (2.7)$$

where $m_0 = (\varphi_0, 1)$ *and* k, K *are two positive constants which are independent of the initial data, with* K *depending on* Ω, ν, J, F *and* $\|h\|_{L^2_{tb}(0,\infty;V'_{div})}$.

In all the following sections it will be $d = 2$.

3   Uniqueness of weak solutions (constant viscosity)

In this section we prove that the weak solution of the nonlocal Cahn-Hilliard-Navier-Stokes system with constant viscosity $\nu = 1$ is unique and provide a continuous dependence estimate. In Subsection 3.1 we shall first address the case of constant mobility ($m = 1$) and regular potential F. Nevertheless, we shall see in Subsection 3.2 and Subsection 3.3 that the arguments used for this case can also be applied to the cases of singular potential and constant or degenerate mobility (see [12] or [14] for the existence).

5

3.1 Regular potential and constant mobility

The main result is the following.

Theorem 2. *Let $d = 2$ and suppose that assumptions (H1)–(H5) are satisfied with $\nu = 1$. Take $h \in L^2_{loc}([0, \infty); V'_{div})$, $u_0 \in G_{div}$ and $\varphi_0 \in H$ such that $F(\varphi_0) \in L^1(\Omega)$... Then, the weak solution $[u, \varphi]$ corresponding to $[u_0, \varphi_0]$ and given by Theorem 1 is unique. Furthermore, if we consider two weak solutions $z_i := [u_i, \varphi_i]$ corresponding to two initial data $z_{0i} := [u_{0i}, \varphi_{0i}]$, with $h_i \in L^2_{loc}([0, \infty); V'_{div})$, $u_{0i} \in G_{div}$ and $\varphi_{0i} \in H$ such that $F(\varphi_{0i}) \in L^1(\Omega)$ and $|\overline{\varphi}_{0i}| \leq \eta$ for some positive constant η, $i = 1, 2$, then the following continuous dependence estimate holds*

$$\|u_2(t) - u_1(t)\|^2 + \|\varphi_2(t) - \varphi_1(t)\|^2_{V'}$$
$$+ \int_0^t \left(\frac{c_0}{2} \|\varphi_2(\tau) - \varphi_1(\tau)\|^2 + \frac{\nu}{4} \|\nabla(u_2(\tau) - u_1(\tau))\|^2 \right) d\tau$$
$$\leq \left(\|u_2(0) - u_1(0)\|^2 + \|\varphi_2(0) - \varphi_1(0)\|^2_{V'} \right) \Lambda_0(t) + |\overline{\varphi}| C_\eta \left(\mathcal{E}(z_{02}), \mathcal{E}(z_{01}) \right) \Lambda_1(t)$$
$$+ \|h_2 - h_1\|^2_{L^2(0,T;V'_{div})} \Lambda_2(t), \tag{3.1}$$

for all $t \in [0, T]$, where Λ_0, Λ_1 and Λ_2 are continuous functions which depend on the norms of the two solutions and C_η is a positive constant which depends on η and on the energies $\mathcal{E}(z_{02}), \mathcal{E}(z_{01})$.

Proof. Let us start by rewriting the Korteweg force by making explicit the dependence on φ. Indeed, we have

$$\mu \nabla \varphi = \left(a\varphi - J * \varphi + F'(\varphi) \right) \nabla \varphi = \nabla \left(F(\varphi) + a\frac{\varphi^2}{2} \right) - \nabla a \frac{\varphi^2}{2} - (J * \varphi) \nabla \varphi.$$

Hence we can write the Navier-Stokes equation with an extra-pressure $\widetilde{\pi} := \pi - F(\varphi) + a\frac{\varphi^2}{2}$ as follows

$$u_t - \nu \Delta u + (u \cdot \nabla)u + \nabla \widetilde{\pi} - h = -\nabla a \frac{\varphi^2}{2} - (J * \varphi) \nabla \varphi =: K(\varphi).$$

Let us now consider two weak solutions $[u_i, \varphi_i]$ corresponding to two initial data $[u_{0i}, \varphi_{0i}]$, with $u_{0i} \in G_{div}$ and $\varphi_{i0} \in H$ and $F(\varphi_{0i}) \in L^1(\Omega)$, $i = 1, 2$. Set $u := u_2 - u_1$ and $\varphi := \varphi_2 - \varphi_1$. Then, the difference $[u, \varphi]$ satisfies the system

$$\varphi_t = \Delta \widetilde{\mu} - u \cdot \nabla \varphi_1 + u_2 \cdot \nabla \varphi, \tag{3.2}$$
$$\widetilde{\mu} = a\varphi - J * \varphi + F'(\varphi_2) - F'(\varphi_1), \tag{3.3}$$
$$u_t - \nu \Delta u + (u_2 \cdot \nabla)u_2 - (u_1 \cdot \nabla)u_1 + \nabla \widetilde{\pi}$$
$$= -\varphi(\varphi_1 + \varphi_2)\frac{\nabla a}{2} - (J * \varphi)\nabla \varphi_2 - (J * \varphi_1)\nabla \varphi + h, \tag{3.4}$$

where $\widetilde{\pi} := \widetilde{\pi}_2 - \widetilde{\pi}_1$ and $h := h_2 - h_1$. We multiply (3.4) by u in G_{div}. After standard calculations, the following terms (cf. (3.4))

$$I_1 = -\frac{1}{2} \left(\varphi(\varphi_1 + \varphi_2) \nabla a, u \right), \quad I_2 = -\left((J * \varphi) \nabla \varphi_2, u \right), \quad I_3 = -\left((J * \varphi_1) \nabla \varphi, u \right),$$

6

can be estimated in this way

$$I_1 \leq \left|\left(\varphi(\varphi_1 + \varphi_2)\nabla a, u\right)\right| \leq \|\varphi\|\|\varphi_1 + \varphi_2\|_{L^4}\|\nabla a\|_{L^\infty}\|u\|_{L^4}$$

$$\leq c\|\varphi\|\|\varphi_1 + \varphi_2\|_{L^4}\|\nabla a\|_{L^\infty}\|u\|^{1/2}\|\nabla u\|^{1/2}$$

$$\leq \frac{c_0}{10}\|\varphi\|^2 + c\|\varphi_1 + \varphi_2\|_{L^4}^2\|\nabla a\|_{L^\infty}^2\|u\|\|\nabla u\|$$

$$\leq \frac{c_0}{10}\|\varphi\|^2 + \frac{\nu}{6}\|\nabla u\|^2 + c\|\varphi_1 + \varphi_2\|_{L^4}^4\|\nabla a\|_{L^\infty}^4\|u\|^2, \tag{3.5}$$

$$I_2 \leq \left|\left(\varphi_1, (\nabla J * \varphi)u\right)\right| \leq \|\varphi_1\|_{L^4}\|\nabla J * \varphi\|\|u\|_{L^4}$$

$$\leq c\|\varphi_1\|_{L^4}\|\nabla J\|_{L^1}\|\varphi\|\|u\|^{1/2}\|\nabla u\|^{1/2}$$

$$\leq \frac{c_0}{10}\|\varphi\|^2 + c\|\nabla J\|_{L^1}^2\|\varphi_1\|_{L^4}^2\|u\|\|\nabla u\|$$

$$\leq \frac{c_0}{10}\|\varphi\|^2 + \frac{\nu}{6}\|\nabla u\|^2 + c\|\nabla J\|_{L^1}^4\|\varphi_1\|_{L^4}^4\|u\|^2, \tag{3.6}$$

$$I_3 \leq \left|\left((\nabla J * \varphi_2)\varphi, u\right)\right| \leq \|\nabla J * \varphi_2\|_{L^4}\|\varphi\|\|u\|_{L^4}$$

$$\leq c\|\nabla J\|_{L^1}\|\varphi_2\|_{L^4}\|\varphi\|\|u\|^{1/2}\|\nabla u\|^{1/2}$$

$$\leq \frac{c_0}{10}\|\varphi\|^2 + c\|\nabla J\|_{L^1}^2\|\varphi_2\|_{L^4}^2\|u\|\|\nabla u\|$$

$$\leq \frac{c_0}{10}\|\varphi\|^2 + \frac{\nu}{6}\|\nabla u\|^2 + c\|\nabla J\|_{L^1}^4\|\varphi_2\|_{L^4}^4\|u\|^2. \tag{3.7}$$

Taking such estimates into account, it is easy see that from (3.4) we are led to the following differential inequality

$$\frac{1}{2}\frac{d}{dt}\|u\|^2 + \frac{\nu}{4}\|\nabla u\|^2 \leq \frac{3}{10}c_0\|\varphi\|^2 + \alpha\|u\|^2 + \frac{1}{\nu}\|h\|_{V_{div}'}^2, \tag{3.8}$$

where the function α is given by

$$\alpha := c\|\nabla J\|_{L^1}^4\left(\|\varphi_1\|_{L^4}^4 + \|\varphi_2\|_{L^4}^4\right) + c\|\nabla u_2\|^2. \tag{3.9}$$

Since $\varphi_1, \varphi_2 \in L^\infty(0, T; L^2) \cap L^2(0, T, H^1)$
and $L^\infty(0, T; L^2) \cap L^2(0, T, H^1) \hookrightarrow L^4(0, T; L^4(\Omega))$, thanks to the Gagliardo-Nirenberg inequality, then we have $\alpha \in L^1(0, T)$.

Let us now multiply (3.2) by $B_N^{-1}(\varphi - \overline{\varphi})$ (notice that we have $\overline{\varphi} = \overline{\varphi}_{01} - \overline{\varphi}_{02}$). We get

$$\frac{1}{2}\frac{d}{dt}\|B_N^{-1/2}(\varphi - \overline{\varphi})\|^2 + (a\varphi + F'(\varphi_1) - F'(\varphi_2), \varphi) = (J * \varphi, \varphi) + |\Omega|\overline{\varphi}\overline{\widetilde{\mu}} + I_4 + I_5, \tag{3.10}$$

where

$$I_4 = -\left(u \cdot \nabla\varphi_2, B_N^{-1}(\varphi - \overline{\varphi})\right), \ I_5 = \left(u_1 \cdot \nabla\varphi, B_N^{-1}(\varphi - \overline{\varphi})\right).$$

By using assumption (H3), we find

$$\frac{1}{2}\frac{d}{dt}\|B_N^{-1/2}(\varphi - \overline{\varphi})\|^2 + c_0\|\varphi\|^2 \leq |(J * \varphi, \varphi)| + |\Omega|\overline{\varphi}\overline{\widetilde{\mu}} + I_4 + I_5. \tag{3.11}$$

7

The first term on the right hand side of (3.11) can be controlled as follows

$$\big|(J * \varphi, \varphi - \overline{\varphi})\big| + \big|(J * \varphi, \overline{\varphi})\big| \leq \frac{c_0}{10}\|\varphi\|^2 + c\|B_N^{-1/2}(\varphi - \overline{\varphi})\|^2 + \frac{c_0}{4}\|\varphi\|^2 + c\overline{\varphi}^2,$$

(3.12)

while the terms I_4 and I_5 can be estimated as

$$I_4 \leq \big|\big(u \cdot \nabla B_N^{-1}(\varphi - \overline{\varphi}), \varphi_2\big)\big| \leq \|u\|_{L^4}\|\nabla B_N^{-1}(\varphi - \overline{\varphi})\|\|\varphi_2\|_{L^4}$$
$$\leq \frac{\nu}{8}\|\nabla u\|^2 + c\|\varphi_2\|_{L^4}^2\|\nabla B_N^{-1}(\varphi - \overline{\varphi})\|^2,$$

(3.13)

and

$$I_5 \leq \big|(u_2 \cdot \nabla B_N^{-1}(\varphi - \overline{\varphi}), \varphi)\big| \leq \|\varphi\|\|u_2\|_{L^4}\|\nabla B_N^{-1}(\varphi - \overline{\varphi})\|_{L^4}$$
$$\leq \frac{c_0}{20}\|\varphi\|^2 + c\|u_2\|_{L^4}^2\|\nabla B_N^{-1}(\varphi - \overline{\varphi})\|_{L^4}^2$$
$$\leq \frac{c_0}{20}\|\varphi\|^2 + c\|u_2\|_{L^4}^2\|\nabla B_N^{-1}(\varphi - \overline{\varphi})\|\|\nabla B_N^{-1}(\varphi - \overline{\varphi})\|_{H^1}.$$

(3.14)

Observe that on $D(B_N)$ (recall that $\phi := B_N^{-1}(\varphi - \overline{\varphi}) \in D(B_N)$) the H^2-norm of ϕ is equivalent to the L^2-norm of $B_N\phi + \phi$. Thus we have

$$\|\nabla B_N^{-1}(\varphi - \overline{\varphi})\|_{H^1} \leq \|B_N^{-1}(\varphi - \overline{\varphi})\|_{H^2} \leq c\|(B_N + I)B_N^{-1}(\varphi - \overline{\varphi})\| \leq c\|\varphi - \overline{\varphi}\|.$$

Therefore, from (3.14) we get

$$I_5 \leq \frac{c_0}{10}\|\varphi\|^2 + c\|u_2\|_{L^4}^4\|B_N^{-1/2}(\varphi - \overline{\varphi})\|^2 + |\Omega|\overline{\varphi}^2.$$

(3.15)

Plugging estimates (3.5)–(3.7) and (3.12)–(3.15) into (3.8) and (3.11), we deduce the differential inequality

$$\frac{1}{2}\frac{d}{dt}\Big(\|u\|^2 + \|B_N^{-1/2}(\varphi - \overline{\varphi})\|^2\Big) + \frac{c_0}{4}\|\varphi\|^2 + \frac{\nu}{8}\|\nabla u\|^2$$
$$\leq \beta\Big(\|u\|^2 + \|B_N^{-1/2}(\varphi - \overline{\varphi})\|^2\Big) + c\overline{\varphi}^2 + |\Omega|\overline{\varphi}\overline{\widetilde{\mu}} + \frac{1}{\nu}\|h\|_{V'_{div}}^2,$$

(3.16)

where the function β is given by

$$\beta := \alpha + c(1 + \|\varphi_2\|_{L^4}^2 + \|u_1\|_{L^4}^4) \in L^1(0, T).$$

If we consider two weak solutions corresponding to the same initial data and to the same external force, then we have $\overline{\varphi} = 0$ and $h = 0$. Therefore, from (3.16) by using Gronwall's lemma we get $u = 0$ and $\varphi = 0$ on $[0, T]$ and this proves uniqueness.

If the two weak solutions correspond to different initial data and to different external forces, we have

$$|\Omega||\overline{\widetilde{\mu}}| \leq \int_\Omega \big(|F'(\varphi_2)| + |F'(\varphi_1)|\big) \leq \int_\Omega \big(|F(\varphi_2)| + |F(\varphi_1)|\big) + c$$
$$\leq C_\eta\big(\mathcal{E}(z_{02}), \mathcal{E}(z_{01})\big), \qquad \forall t \geq 0,$$

where we have used (H5) (which implies that $|F'(s)| \leq cF(s) + c$, for all $s \in \mathbb{R}$) and (2.7). Here, η is a constant such that $|\overline{\varphi}_{0i}| \leq \eta$, $i = 1, 2$. Therefore (3.16) can be rewritten as

$$\frac{d}{dt}\Big(\|u\|^2 + \|B_N^{-1/2}(\varphi - \overline{\varphi})\|^2\Big) + \frac{c_0}{2}\|\varphi\|^2 + \frac{\nu}{4}\|\nabla u\|^2$$
$$\leq \beta\Big(\|u\|^2 + \|B_N^{-1/2}(\varphi - \overline{\varphi})\|^2\Big) + |\overline{\varphi}|C_\eta\big(\mathcal{E}(z_{02}), \mathcal{E}(z_{01})\big) + \frac{2}{\nu}\|h\|_{V'_{div}}^2. \qquad (3.17)$$

By using Gronwall's lemma once more, from (3.17) we deduce

$$\|u(t)\|^2 + \|B_N^{-1/2}(\varphi(t) - \overline{\varphi})\|^2 \leq \big(\|u(0)\|^2 + \|B_N^{-1/2}(\varphi(0) - \overline{\varphi})\|^2\big)\Gamma_0(t)$$
$$+ |\overline{\varphi}|C_\eta\big(\mathcal{E}(z_{02}), \mathcal{E}(z_{01})\big)\Gamma_1(t) + \frac{2}{\nu}\Gamma_0(t)\|h\|_{L^2(0,T;V'_{div})}^2, \qquad (3.18)$$

where $\Gamma_0(t) := e^{\int_0^t \beta(s)ds}$ and $\Gamma_1(t) := \int_0^t e^{\int_s^t \beta(\tau)d\tau}ds$. By integrating (3.17) between 0 and t and using (3.18), we have

$$\|u(t)\|^2 + \|B_N^{-1/2}(\varphi(t) - \overline{\varphi})\|^2 + \int_0^t \Big(\frac{c_0}{2}\|\varphi\|^2 + \frac{\nu}{4}\|\nabla u\|^2\Big)d\tau$$
$$\leq \big(\|u(0)\|^2 + \|B_N^{-1/2}(\varphi(0) - \overline{\varphi})\|^2\big)\Gamma_2(t) + |\overline{\varphi}|C_\eta\big(\mathcal{E}(z_{02}), \mathcal{E}(z_{01})\big)\Gamma_3(t)$$
$$+ \frac{2}{\nu}\Gamma_0(t)\|h\|_{L^2(0,T;V'_{div})}^2, \qquad (3.19)$$

for all $t \in [0, T]$, where the continuous functions Γ_2 and Γ_3 are given by

$$\Gamma_2(t) := 1 + \int_0^t \beta(s)\Gamma_0(s)ds, \qquad \Gamma_3(t) := \int_0^t \beta(s)\Gamma_1(s)ds + T.$$

Finally, from (3.19) we deduce (3.1) by suitably defining the functions Λ_0, Λ_1 in terms of Γ_0, Γ_2 and Γ_3. $\qquad \square$

3.2 Singular potential and constant mobility

The proof of existence of a weak solution with initial data $u_0 \in G_{div}$ and $\varphi_0 \in L^\infty(\Omega)$ with $F(\varphi_0) \in L^1(\Omega)$ is given in [12], where also a nonconstant viscosity is considered. We recall that in this case the assumption $|\overline{\varphi}_0| < 1$ is needed in order to control the average of the chemical potential. For the assumptions on the singular potential F we refer the reader to [12]. We recall, in particular, the physically relevant case of the so-called logarithmic potential, that is,

$$F(s) = -\frac{\theta_c}{2}s^2 + \frac{\theta}{2}\big((1 + s)\log(1 + s) + (1 - s)\log(1 - s)\big), \qquad (3.20)$$

where $0 < \theta < \theta_c$, θ being the absolute temperature and θ_c a given critical temperature below which the phase separation takes place.

It is easy to see that, assuming the viscosity ν constant and $d = 2$, the uniqueness argument can also be applied to the present case. Indeed, estimates (3.5)-(3.8) obviously still hold. Moreover, considering (3.10) we immediately see that (3.11) still follows from (3.10), since in the case of singular potential we have

$$F''(s) + a(x) \geq c_0, \qquad \forall s \in (-1, 1), \qquad c_0 > 0.$$

In particular, this assumption is ensured by [12, (A6)]. Therefore, uniqueness is given by

Theorem 3. *Let $u_0 \in G_{div}$, $\varphi_0 \in L^\infty(\Omega)$ such that $F(\varphi_0) \in L^1(\Omega)$ and $|\overline{\varphi}_0| < 1$. Suppose that assumptions (A1)–(A8) of [12] are satisfied with $\nu = 1$ and that $d = 2$. Then, the weak solution $[u, \varphi]$ corresponding to $[u_0, \varphi_0]$ which is given by [12, Theorem 1] is unique. Furthermore, if we consider two weak solutions $z_i := [u_i, \varphi_i]$ corresponding to two initial data $z_{0i} := [u_{0i}, \varphi_{0i}]$, with $u_{0i} \in G_{div}$ and $\varphi_{0i} \in L^\infty(\Omega)$ such that $F(\varphi_{0i}) \in L^1(\Omega)$ and $|\overline{\varphi}_{0i}| \leq \eta$ for some constant $\eta \in [0, 1)$, $i = 1, 2$, then estimate* (3.1) *holds.*

3.3   Singular potential and degenerate mobility

This physically relevant case was addressed in [14] to which we refer for all the assumptions on the degenerate mobility m and on the singular potential F as well as for the weak formulation. However, it is worth recalling that a typical situation is $m(s) = k_1(1 - s^2)$ and F given by (3.20).

We recall that in [14] the viscosity ν was assumed to be constant just to avoid technicalities, but the results therein also hold for a nonconstant viscosity satisfying (H2). In [14, Theorem 2] the existence of a weak solution has been established with initial data $u_0 \in G_{div}$ and $\varphi_0 \in L^\infty(\Omega)$ with $F(\varphi_0) \in L^1(\Omega)$ and $M(\varphi_0) \in L^1(\Omega)$, where $M \in C^2(-1, 1)$ is defined by $m(s)M''(s) = 1$ for all $s \in (-1, 1)$ and $M(0) = M'(0) = 0$.

Furthermore, in [14, Proposition 4] uniqueness of the weak solution was proven for the convective nonlocal Cahn-Hilliard equation with degenerate mobility and with a given velocity $u \in L^2_{loc}([0, \infty); V_{div} \cap L^\infty(\Omega)^d)$ $(d = 2, 3)$. By combining the proof of [14, Proposition 4] with the arguments of Theorem 2 we can now prove uniqueness of the weak solution for the nonlocal Cahn-Hilliard-Navier-Stokes system with singular potential and degenerate mobility.

Theorem 4. *Let all the assumptions of [14, Theorem 2 and Proposition 4] be satisfied and let $d = 2$ and $\nu = 1$. Then, the weak solution to system* (1.1)-(1.6) *(cf. [14, Definition 2]) is unique.*

Proof. Arguing as in the first part of the proof of Theorem 2 we can obtain (3.8) that we now write in the following form

$$\frac{1}{2}\frac{d}{dt}\|u\|^2 + \frac{\nu}{2}\|\nabla u\|^2 \leq \frac{1}{4}(1 - \rho)\alpha_0\|\varphi\|^2 + \alpha\|u\|^2, \tag{3.21}$$

where $\rho \in [0, 1)$ and $\alpha_0 > 0$ are some constants which appear in the assumptions on the singular potential (see [14, Theorem 3]).

Regarding the estimates for the difference of the nonlocal Cahn-Hilliard, let us first recall the approach in the proof of [14, Proposition 4].

Following [22], one can introduce

$$\Lambda_1(s) := \int_0^s m(\sigma)F_1''(\sigma)d\sigma, \qquad \Lambda_2(s) := \int_0^s m(\sigma)F_2''(\sigma)d\sigma, \qquad \Gamma(s) := \int_0^s m(\sigma)d\sigma,$$

for all $s \in [-1, 1]$, and see that the assumptions on m and on F imply that $\Lambda_1 \in C^1([-1, 1])$ and $0 < \alpha_0 \leq \Lambda_1'(s) \leq \alpha_1$ for some positive constant α_1. The weak formulation of the

convective nonlocal Cahn-Hilliard equation with degenerate mobility (cf. [14, Definition 2]) can then be rewritten as follows

$$\langle \varphi_t, \psi \rangle + \big(\nabla \Lambda(\cdot, \varphi), \nabla \psi \big) - \big(\Gamma(\varphi) \nabla a, \nabla \psi \big)$$
$$+ \big(m(\varphi)(\varphi \nabla a - \nabla J * \varphi), \nabla \psi \big) = \big(u\varphi, \nabla \psi \big), \tag{3.22}$$

for all $\psi \in V$, where

$$\Lambda(x, s) := \Lambda_1(s) + \Lambda_2(s) + a(x)\Gamma(s).$$

Consider now two weak solutions $[u_1, \varphi_1]$ and $[u_2, \varphi_2]$. Let us assume for simplicity that the two initial data are the same (the case of different initial data can be handled without difficulties and leads to a continuous dependence estimate). Take the difference between the two identities (3.22), set $\varphi := \varphi_1 - \varphi_2$, $u := u_1 - u_2$ and choose $\psi = B_N^{-1}\varphi$ as test function in the resulting identity (notice that $\overline{\varphi} = 0$). This yields

$$\frac{1}{2}\frac{d}{dt}\|B_N^{-1/2}\varphi\|^2 + \big(\Lambda(\varphi_2) - \Lambda(\varphi_1), \varphi \big) - \big((\Gamma(\varphi_2) - \Gamma(\varphi_1))\nabla a, \nabla B_N^{-1}\varphi \big)$$
$$+ \big((m(\varphi_2) - m(\varphi_1))(\varphi_2 \nabla a - \nabla J * \varphi_2) + m(\varphi_1)(\varphi \nabla a - \nabla J * \varphi), \nabla B_N^{-1}\varphi \big)$$
$$= \big(u\varphi_1, \nabla B_N^{-1}\varphi \big) + \big(u_2\varphi, \nabla B_N^{-1}\varphi \big). \tag{3.23}$$

All the terms in (3.23) can be estimated as in the proof of [14, Proposition 4], with the exception of the two terms on the right hand side. These terms have now to be controlled in this way

$$|\big(u\varphi_1, \nabla B_N^{-1}\varphi \big)| \leq \|u\|_{L^4}\|\varphi_1\|_{L^4}\|\nabla B_N^{-1}\varphi\| \leq \frac{\nu}{4}\|\nabla u\|^2 + c\|\varphi_1\|_{L^4}^2\|\nabla B_N^{-1}\varphi\|^2, \tag{3.24}$$

$$|\big(u_2\varphi, \nabla B_N^{-1}\varphi \big)| \leq \|u_2\|_{L^4}\|\varphi\|\|\nabla B_N^{-1}\varphi\|_{L^4} \leq \frac{1}{8}(1-\rho)\alpha_0\|\varphi\|^2 + c\|u_2\|_{L^4}^2\|\nabla B_N^{-1}\varphi\|_{L^4}^2$$

$$\leq \frac{1}{8}(1-\rho)\alpha_0\|\varphi\|^2 + c\|u_2\|_{L^4}^2\|\nabla B_N^{-1}\varphi\|\|\nabla B_N^{-1}\varphi\|_{H^1}$$

$$\leq \frac{1}{4}(1-\rho)\alpha_0\|\varphi\|^2 + c\|u_2\|_{L^4}^4\|B_N^{-1/2}\varphi\|^2. \tag{3.25}$$

Therefore, plugging (3.24), (3.25) into (3.23) and using the estimates for the other terms in (3.23) written in the proof of [14, Proposition 4], we deduce the following differential inequality

$$\frac{1}{2}\frac{d}{dt}\|B_N^{-1/2}\varphi\|^2 + \frac{3}{4}(1-\rho)\alpha_0\|\varphi\|^2 \leq \frac{\nu}{4}\|\nabla u\|^2 + \zeta\|B_N^{-1/2}\varphi\|^2, \tag{3.26}$$

where the function $\zeta \in L^1(0, T)$ is given by $\zeta := c(1 + \|\varphi_1\|_{L^4}^2 + \|u_2\|_{L^4}^4)$ and α is the same as in (3.9). Inequalities (3.26) and (3.21) finally give

$$\frac{d}{dt}\Big(\|u\|^2 + \|B_N^{-1/2}\varphi\|^2 \Big) + (1-\rho)\alpha_0\|\varphi\|^2 + \frac{\nu}{2}\|\nabla u\|^2 \leq \theta\Big(\|u\|^2 + \|B_N^{-1/2}\varphi\|^2 \Big),$$

where $\theta = 2(\alpha + \zeta) \in L^1(0, T)$. Uniqueness of the weak solution hence follows from this last differential inequality by applying the standard Gronwall's lemma. $\qquad\square$

Remark 3. *Also in the present case a continuous dependence estimate like* (3.1) *holds.*

4 Weak-strong uniqueness (nonconstant viscosity)

Here we consider system (1.1)-(1.5) in dimension two with constant mobility, regular potential and with a nonconstant viscosity $\nu = \nu(\varphi)$. In this case we are not able to prove the uniqueness of weak solutions, due to the poor regularity of φ which makes difficult to estimate the difference of the dissipation term in the Navier-Stokes equations. However, we can prove a weak-strong uniqueness result. This means that, given a strong solution $[u_1, \varphi_1]$ and a weak solution $[u_2, \varphi_2]$ both corresponding to the same initial datum $[u_0, \varphi_0] \in V_{div} \times H^2(\Omega)$, then these two solutions coincide.

Before proving such result, let us first show that a global strong solution exists. Indeed, we observe that, while the existence of a weak solution with nonconstant viscosity easily follows easily from the same result for the constant viscosity case (see [8]), this does not occur as far as strong solutions are concerned. The difficulty essentially lies in the fact that the classical results for the Navier-Stokes equations in two dimensions with constant viscosity (see, e.g., [28]) cannot be used as in [13] to exploit the improved regularity for the convective term in the nonlocal Cahn-Hilliard equation.

Before stating the main results of this section we recall the definition of admissible kernel (see [5, Definition 1]).

Definition 1. *A kernel $J \in W^{1,1}_{loc}(\mathbb{R}^2)$ is admissible if the following conditions are satisfied:*

(A1) $J \in C^3(\mathbb{R}^d \setminus \{0\})$;

(A2) *J is radially symmetric, $J(x) = \tilde{J}(|x|)$ and $\tilde{J}$ is non-increasing;*

(A3) *$\tilde{J}''(r)$ and $\tilde{J}'(r)/r$ are monotone on $(0, r_0)$ for some $r_0 > 0$;*

(A4) $|D^3 J(x)| \leq C_d |x|^{-d-1}$ *for some $C_d > 0$.*

We recall that the Newtonian and Bessel potentials are admissible for all $d \geq 2$. Moreover, we report the following (cf. [5, Lemma 2])

Lemma 1. *Let J be admissible and $v = \nabla J * \psi$. Then, for all $p \in (1, \infty)$, there exists $C_p > 0$ such that*

$$\|\nabla v\|_p \leq C_p \|\psi\|_{L^p}.$$

The following result on existence of a strong solution generalizes [13, Theorem 2].

Theorem 5. *Let (H1)–(H5) be satisfied with $d = 2$ and either $J \in W^{2,1}(B_\delta)$ or J admissible. Assume that $u_0 \in V_{div}$, $\varphi_0 \in V \cap L^\infty(\Omega)$ and that $h \in L^2_{loc}(\mathbb{R}^+; G_{div})$. Then, for every $T > 0$ there exists a solution to (1.1)–(1.6) such that*

$$u \in L^\infty(0, T; V_{div}) \cap L^2(0, T; H^2(\Omega)^2), \qquad u_t \in L^2(0, T; G_{div}), \tag{4.1}$$

$$\varphi \in L^\infty(0, T; V) \cap L^\infty(\Omega \times (0, T)), \qquad \mu \in L^\infty(0, T; V), \qquad \varphi_t \in L^2(0, T; H). \tag{4.2}$$

Furthermore, suppose in addition that $F \in C^3(\mathbb{R})$ and that $\varphi_0 \in H^2(\Omega)$. Then, system (1.1)-(1.6) *admits a strong solution on* $[0, T]$ *satisfying* (4.1), (4.2) *and*

$$\varphi \in L^\infty(0, T; H^2(\Omega)), \tag{4.3}$$

$$\varphi_t \in L^\infty(0, T; H) \cap L^2(0, T; V). \tag{4.4}$$

Remark 4. *The assumption (H2) in the statement of Theorem 5 (and subsequent Theorem 6) can be replaced by a more general one, i.e., it suffices to assume that ν is locally Lipschitz on $\mathbb{R}$ and there exists $\nu_1 > 0$ such that*

$$\nu(s) \geq \nu_1, \qquad \forall s \in \mathbb{R}. \tag{4.5}$$

Indeed, an upper bound for $\nu(\varphi)$ (and $\nu'(\varphi)$, respectively) in $L^\infty(\Omega \times (0, T))$ can be easily produced on account of the fact that $\|\varphi\|_{L^\infty(\Omega \times (0,T))} \leq C_R$, for any $R > 0$ such that $\|\varphi_0\|_{L^\infty} \leq R$.

Proof. We first need to establish the $L^\infty(V)$ regularity for μ and φ. The argument used here differs from the one devised in [13]... Indeed, we cannot take advantage of the regularity $u \in L^2(H^2)$ as it happens for the constant viscosity case.

We begin with the nonlocal Cahn-Hilliard equation (1.1). First we recall that φ is bounded (see [18, Lemma 2.10], cf. also [13, Theorem 2]). Then we observe that

$$\begin{aligned}
\|\mu\|_{H^2}^2 &\leq c\| - \Delta\mu + \mu\|^2 \leq c\big(\|\varphi_t\|^2 + \|u \cdot \nabla\varphi\|^2\big) + Q(R) \\
&\leq c\big(\|\varphi_t\|^2 + \|u\|\|\nabla u\|\|\nabla\varphi\|\|\varphi\|_{H^2}\big) + Q(R) \\
&\leq \delta\|\varphi\|_{H^2}^2 + c\|\varphi_t\|^2 + c_\delta\|u\|^2\|\nabla u\|^2\|\nabla\varphi\|^2 + Q(R).
\end{aligned} \tag{4.6}$$

Henceforth we shall denote by Q a continuous monotone increasing function of its argument, and $R > 0$ is such that $\|\varphi\|_{L^\infty(\Omega \times (0,T))} \leq R$.

We now control the H^2-norm of φ (or at least the L^2-norm of the second derivatives $\partial_{ij}^2\varphi := \frac{\partial^2\varphi}{\partial x_i \partial x_j}$) in terms of the H^2-norm of μ. To this aim apply the second derivative operator ∂_{ij}^2 to (1.2), multiply the resulting identity by $\partial_{ij}^2\varphi$ and integrate on Ω. We get

$$\begin{aligned}
\int_\Omega \partial_{ij}^2\mu\partial_{ij}^2\varphi &= \int_\Omega (a + F''(\varphi))(\partial_{ij}^2\varphi)^2 + \int_\Omega (\partial_i a\partial_j\varphi + \partial_j a\partial_i\varphi)\partial_{ij}^2\varphi \\
&+ \int_\Omega (\varphi\partial_{ij}^2 a - \partial_i(\partial_j J * \varphi)\partial_{ij}^2\varphi + \int_\Omega F'''(\varphi)\partial_i\varphi\partial_j\varphi\partial_{ij}^2\varphi, \qquad i, j = 1, 2.
\end{aligned}$$

From this identity, by means of (H2) we obtain

$$\begin{aligned}
c_0\|\partial_{ij}^2\varphi\|^2 &\leq c\|\partial_{ij}^2\mu\|^2 \\
&+ c\big(\|\nabla a\|_{L^\infty}^2 + Q(R)\big)\|\nabla\varphi\|^2 + Q(R)\|\partial_{ij}^2 a\|^2 + \|\partial_i(\partial_j J * \varphi)\|^2, \tag{4.7}
\end{aligned}$$

and an estimate like this still holds if $\|\partial_{ij}^2\varphi\|$ and $\|\partial_{ij}^2\mu\|$ are replaced by $\|\varphi\|_{H^2}$ and $\|\mu\|_{H^2}$, respectively. By combining (4.6) with (4.7) and choosing $\delta > 0$ small enough we get

$$\begin{aligned}
\|\partial_{ij}^2\varphi\|^2 &\leq c\|\varphi_t\|^2 + c\|u\|^2\|\nabla u\|^2\|\nabla\varphi\|^2 + Q(R) \\
&+ c\big(\|\nabla a\|_{L^\infty}^2 + Q(R)\big)\|\nabla\varphi\|^2 + Q(R)\|\partial_{ij}^2 a\|^2 + c\|\partial_i(\partial_j J * \varphi)\|^2. \tag{4.8}
\end{aligned}$$

13

We now test the nonlocal Cahn-Hilliard equation by $\mu_t = \big(a + F''(\varphi)\big)\varphi_t - J * \varphi_t$ in H to deduce

$$\int_\Omega \varphi_t \mu_t + \int_\Omega (u \cdot \nabla\varphi)\mu_t + \frac{1}{2}\frac{d}{dt}\|\nabla\mu\|^2$$
$$= \int_\Omega (a + F''(\varphi))\varphi_t^2 - (\varphi_t, J * \varphi_t) + \int_\Omega (u \cdot \nabla\varphi)\mu_t + \frac{1}{2}\frac{d}{dt}\|\nabla\mu\|^2 = 0. \qquad (4.9)$$

This identity was considered also in [13], but now we must avoid to use the H^2-norm of u to estimate the term coming from convection. This term is then estimated as follows

$$\Big|\int_\Omega (u \cdot \nabla\varphi)\mu_t\Big| \leq \|u \cdot \nabla\varphi\|\|\mu_t\| \leq Q_J(R)\|\varphi_t\|\|u \cdot \nabla\varphi\|$$
$$\leq \frac{c_0}{4}\|\varphi_t\|^2 + Q_{c_0,J}(R)\|u\|_{L^4}^2\|\nabla\varphi\|_{L^4}^2$$
$$\leq \frac{c_0}{4}\|\varphi_t\|^2 + Q_{c_0,J}(R)\|u\|\|\nabla u\|\|\nabla\varphi\|\|\varphi\|_{H^2}$$
$$\leq \frac{c_0}{4}\|\varphi_t\|^2 + Q_{c_0,J,\epsilon}(R)\big(\|u\|^2\|\nabla u\|^2\big)\|\nabla\varphi\|^2 + \epsilon\|\varphi\|_{H^2}, \qquad (4.10)$$

for $\epsilon > 0$. Furthermore, we have

$$|(\varphi_t, J * \varphi_t)| \leq \|\varphi_t\|_{V'}\|J * \varphi_t\|_V \leq \|\varphi_t\|_{V'}\|J\|_{W^{1,1}}\|\varphi_t\|$$
$$\leq \frac{c_0}{4}\|\varphi_t\|^2 + c\|J\|_{W^{1,1}}^2\|\varphi_t\|_{V'}^2. \qquad (4.11)$$

Inserting (4.10), (4.11) into (4.9), using (4.8) together with (H3) and choosing $\epsilon > 0$ small enough, we get the following differential inequality

$$\frac{d}{dt}\|\nabla\mu\|^2 + c_0\|\varphi_t\|^2 \leq Q_{c_0,J}(R)\big(\|u\|^2\|\nabla u\|^2\big)\|\nabla\varphi\|^2 + c\|u\|^2\|\nabla u\|^2\|\nabla\varphi\|^2 + Q(R)$$
$$+ c\big(\|\nabla a\|_{L^\infty}^2 + Q(R)\big)\|\nabla\varphi\|^2 + Q(R)\sum_{i,j=1}^2 \|\partial_{ij}^2 a\|^2$$
$$+ c\sum_{i,j=1}^2 \|\partial_i(\partial_j J * \varphi)\|^2 + c\|J\|_{W^{1,1}}^2\|\varphi_t\|_{V'}^2. \qquad (4.12)$$

Moreover, notice that we have

$$c_0\|\nabla\varphi\|^2 - Q(R) \leq \|\nabla\mu\|^2 \leq Q(R)\big(\|\nabla\varphi\|^2 + 1\big).$$

Therefore, from (4.12) by means of Gronwall's lemma (cf. also Lemma 1), using the initial condition $\varphi_0 \in V$ and the regularity properties of the weak solution given by the first of (2.2) and by (2.5), we deduce the following bounds

$$\varphi \in L^\infty(0, T; V), \qquad \varphi_t \in L^2(0, T; H), \qquad \mu \in L^\infty(0, T; V). \qquad (4.13)$$

Let us now test the Navier-Stokes equations by u_t in G_{div} to deduce the identity

$$\|u_t\|^2 + 2\int_\Omega \nu(\varphi)Du : Du_t + b(u, u, u_t) = (l, u_t), \qquad (4.14)$$

14

where the function l is given by

$$l := -\frac{\varphi^2}{2}\nabla a - (J * \varphi)\nabla\varphi + h.$$

Notice that, due to the assumption on the external force h and to the regularity property for the φ component of a weak solution we have $l \in L^2(0,T;G_{div})$. From (4.14) we obtain

$$\frac{1}{2}\|u_t\|^2 + \frac{d}{dt}\int_\Omega \nu(\varphi)|Du|^2 + b(u,u,u_t) \leq \frac{1}{2}\|l\|^2 + \int_\Omega |Du|^2\nu'(\varphi)\varphi_t. \tag{4.15}$$

Observe that

$$\left|\int_\Omega |Du|^2\nu'(\varphi)\varphi_t\right| \leq \|\nu'(\varphi)\|_{L^\infty}\|\varphi_t\|\|Du\|_{L^4}^2 \leq Q(R)\|\varphi_t\|\|Du\|\|u\|_{H^2}$$
$$\leq \delta\|u\|_{H^2}^2 + Q_\delta(R)\|Du\|^2\|\varphi_t\|^2. \tag{4.16}$$

Furthermore, we have

$$|b(u,u,u_t)| \leq \frac{1}{4}\|u_t\|^2 + \|u \cdot \nabla u\|^2 \leq \frac{1}{4}\|u_t\|^2 + 2\|u\|_{L^4}^2\|\nabla u\|_{L^4}^2$$

$$\leq \frac{1}{4}\|u_t\|^2 + c\|u\|\|\nabla u\|\|\nabla u\|\|u\|_{H^2}$$

$$\leq \frac{1}{4}\|u_t\|^2 + \delta\|u\|_{H^2}^2 + c_\delta\big(\|u\|^2\|\nabla u\|^2\big)\|\nabla u\|^2. \tag{4.17}$$

Plugging (4.16) and (4.17) into (4.15), we get

$$\frac{1}{4}\|u_t\|^2 + \frac{d}{dt}\int_\Omega \nu(\varphi)|Du|^2 \leq \frac{1}{2}\|l\|^2 + 2\delta\|u\|_{H^2}^2 + c_\delta\big(\|u\|^2\|\nabla u\|^2\big)\|Du\|^2$$
$$+ Q_\delta(R)\|Du\|^2\|\varphi_t\|^2, \tag{4.18}$$

where $\delta > 0$ will be fixed later. Using (H2) we can write the following estimate which holds for every $u \in H^2(\Omega)^2 \cap G_{div}$ and every $\varphi \in W^{1,p}(\Omega)$ with $2 < p < \infty$

$$\|u\|_{H^2} \leq c\big(\|P\Delta u\| + \|u\|\big) \leq c_{\nu_1}\big(\|P\text{div}(\nu(\varphi)Du)\| + \|\nabla\varphi \cdot \nabla u\| + \|u\|\big)$$
$$\leq c_{\nu_1}\big(\|P\text{div}(\nu(\varphi)Du)\| + \|\nabla\varphi\|_{L^p}\|\nabla u\|_{L^q} + \|u\|\big)$$
$$\leq c_{\nu_1}\big(\|P\text{div}(\nu(\varphi)Du)\| + \|\nabla\varphi\|_{L^p}\|\nabla u\|^{2/q}\|u\|_{H^2}^{1-2/q} + \|u\|\big)$$
$$\leq \frac{1}{2}\|u\|_{H^2} + c_{\nu_1}\big(\|P\text{div}(\nu(\varphi)Du)\| + \|\nabla\varphi\|_{L^p}^{p/(p-2)}\|\nabla u\| + \|u\|\big),$$

where $2 < q < \infty$ is such that $p^{-1} + q^{-1} = 1/2$ and $P : L^2(\Omega)^2 \to G_{div}$ is the Leray projector. Hence we find

$$\|u\|_{H^2} \leq c_{\nu_1}\big(\|P\text{div}(\nu(\varphi)Du)\| + \|\nabla\varphi\|_{L^p}^{p/(p-2)}\|\nabla u\| + \|u\|\big). \tag{4.19}$$

On the other hand, from (1.3) we have

$$P\text{div}(\nu(\varphi)Du) = Pu_t + P\big((u \cdot \nabla)u\big) + Pl.$$

15

Therefore we deduce

$$\|P\text{div}(\nu(\varphi)Du)\| \leq \|u_t\| + \|u\|_{L^4}\|\nabla u\|_{L^4} + \|l\|$$
$$\leq \|u_t\| + c\|u\|^{1/2}\|\nabla u\|^{1/2}\|\nabla u\|^{1/2}\|u\|_{H^2}^{1/2} + \|l\|$$
$$\leq \|u_t\| + \sigma\|u\|_{H^2} + c_\sigma\|u\|\|\nabla u\|^2 + \|l\|, \tag{4.20}$$

where $\sigma > 0$. Plugging (4.20) into (4.19) and choosing σ small enough (i.e., $c_{\nu_1}\sigma < 1$) we get

$$\|u\|_{H^2} \leq c\big(\|u_t\| + \|u\|\|\nabla u\|^2 + \|l\| + \|\nabla\varphi\|_{L^p}^{p/(p-2)}\|\nabla u\| + \|u\|\big). \tag{4.21}$$

We now control $\nabla\varphi$ in terms of $\nabla\mu$ in L^p. We then take the gradient of $\mu = a\varphi - J*\varphi + F'(\varphi)$, multiply it by $\nabla\varphi|\nabla\varphi|^{p-2}$ and integrate the resulting identity on Ω. We get

$$\int_\Omega \nabla\varphi|\nabla\varphi|^{p-2}\cdot\nabla\mu = \int_\Omega (a + F''(\varphi))|\nabla\varphi|^p + \int_\Omega (\varphi\nabla a - \nabla J*\varphi)\cdot\nabla\varphi|\nabla\varphi|^{p-2}.$$

So that, by (H3), we find

$$c_0\|\nabla\varphi\|_{L^p}^p \leq \|\nabla\varphi\|_{L^p}^{p-1}\|\nabla\mu\|_{L^p} + (\|\nabla a\|_{L^\infty} + \|\nabla J\|_{L^1})\|\varphi\|_{L^p}\|\nabla\varphi\|_{L^p}^{p-1}$$
$$\leq \frac{c_0}{2}\|\nabla\varphi\|_{L^p}^p + c\|\nabla\mu\|_{L^p}^p + Q(R)(\|\nabla a\|_{L^\infty} + \|\nabla J\|_{L^1})^p,$$

which yields

$$\|\nabla\varphi\|_{L^p} \leq c\|\nabla\mu\|_{L^p} + Q(R). \tag{4.22}$$

Furthermore, from the nonlocal Cahn-Hilliard equation (1.1) we have

$$\|\nabla\mu\|_{L^p} \leq c\|\nabla\mu\|^{2/p}\|\nabla\mu\|_{H^1}^{1-2/p}$$
$$\leq c\|\nabla\mu\|^{2/p}\|\mu\|_{H^2}^{1-2/p} \leq c\|\nabla\mu\|^{2/p}(\|\Delta\mu\|^{1-2/p} + \|\mu\|^{1-2/p})$$
$$\leq Q(R, \|\varphi_0\|_V, \|u_0\|)\big(\|\varphi_t\|^{1-2/p} + \|u\cdot\nabla\varphi\|^{1-2/p} + 1\big)$$
$$\leq Q(R, \|\varphi_0\|_V, \|u_0\|)\big(\|\varphi_t\|^{1-2/p} + \|u\|_{L^q}^{1-2/p}\|\nabla\varphi\|_{L^p}^{1-2/p} + 1\big), \tag{4.23}$$

where we have used the fact that the H^2-norm of μ is equivalent to the L^2- norm of $-\Delta\mu + \mu$ (cf. (1.5)) and we have taken into account the improved regularity for μ given by the third of (4.13). By combining (4.13) with (4.23) we therefore get

$$\|\nabla\varphi\|_{L^p} \leq Q(R, \|\varphi_0\|_V, \|u_0\|)\big(\|\varphi_t\|^{1-2/p} + \|u\|_{L^{2p/(p-2)}}^{(p-2)/2} + 1\big)$$
$$\leq Q(R, \|\varphi_0\|_V, \|u_0\|)\big(\|\varphi_t\|^{1-2/p} + \|u\|^{(p-2)^2/2p}\|\nabla u\|^{1-2/p} + 1\big), \tag{4.24}$$

and inserting this estimate into (4.21) we get

$$\|u\|_{H^2} \leq Q(R, \|\varphi_0\|_V, \|u_0\|)\big(\|u_t\| + \|u\|\|\nabla u\|^2 + \|l\| + \|\varphi_t\|\|\nabla u\|$$
$$+ \|u\|^{(p-2)/2}\|\nabla u\|^2 + \|\nabla u\| + \|u\|\big). \tag{4.25}$$

16

We can now insert (4.25) into (4.18), take $\delta > 0$ small enough and then write the following differential inequality

$$\frac{d}{dt} \int_\Omega \nu(\varphi)|Du|^2 + \frac{1}{8}\|u_t\|^2$$
$$\leq Q(R, \|\varphi_0\|_V, \|u_0\|)\Big(\|l\|^2 + \big((\|u\|^2 + \|u\|^{p-2})\|\nabla u\|^2\big)\|Du\|^2$$
$$+ \|\varphi_t\|^2\|Du\|^2 + \|\nabla u\|^2\Big). \tag{4.26}$$

From (4.26), on account of (H2) and of the improved regularity for φ_t given by the second of (4.13), by means of Gronwall's lemma (cf. also (4.25)), we obtain

$$u \in L^\infty(0,T;V_{div}) \cap L^2(0,T;H^2(\Omega)^2), \qquad u_t \in L^2(0,T;G_{div}). \tag{4.27}$$

With these regularity properties for u at disposal we can now argue exactly as in the second step of the proof of [13, Theorem 2] by differentiating (1.1) with respect to time, multiplying the resulting identity by μ_t in H and using the assumptions that $F \in C^3(\mathbb{R})$ and $\varphi_0 \in H^2(\Omega)$ to deduce

$$\varphi_t \in L^\infty(0,T;H) \cap L^2(0,T,V).$$

From this property, on account of (4.24), (4.22) and the first of (4.27), we get

$$\varphi \in L^\infty(0,T;W^{1,p}(\Omega)).$$

Finally, by means of a comparison argument in the nonlocal Cahn-Hilliard equation as in [13] we get also $\mu \in L^\infty(0,T;H^2(\Omega))$ and from this we deduce

$$\varphi \in L^\infty\big(0,T;H^2(\Omega)\big).$$

This ends the proof. $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\square$

We can now state the weak-strong uniqueness result for the nonconstant viscosity case.

Theorem 6. *Assume that (H1)–(H5) are satisfied and $d = 2$. Let $u_0 \in G_{div}$, $\varphi_0 \in L^\infty(\Omega)$ and let $[u_1, \varphi_1]$ be a strong solution satisfying (4.1)–(4.4) and $[u_2, \varphi_2]$ be a weak solution both corresponding to $[u_0, \varphi_0]$. The existence of a strong solution is ensured by Theorem 5 if, in addition, $u_0 \in V_{div}$, $\varphi_0 \in H^2(\Omega)$, $F \in C^3(\mathbb{R})$ and either $J \in W^{2,1}(B_\delta)$ or J admissible. Then $u_1 = u_2$ and $\varphi_1 = \varphi_2$.*

Proof. Taking the difference between the variational formulation of (1.1) and (1.2) written for each solution and setting $u := u_1 - u_2$, $\varphi := \varphi_1 - \varphi_2$, we get

$$\langle u_t, v\rangle + 2\big((\nu(\varphi_1) - \nu(\varphi_2))Du_1, Dv\big) + 2\big(\nu(\varphi_2)Du, Dv\big) + b(u_1, u_1, v) - b(u_2, u_2, v)$$
$$= -\frac{1}{2}\big(\varphi(\varphi_1 + \varphi_2)\nabla a, v\big) - \big((J*\varphi)\nabla\varphi_1, v\big) - \big((J*\varphi_2)\nabla\varphi, v\big), \tag{4.28}$$
$$\langle\varphi_t, \psi\rangle + (\nabla\mu, \nabla\psi) = -(u\cdot\nabla\varphi_1, \psi) + (u_2\cdot\nabla\varphi, \psi), \tag{4.29}$$

for all $v \in V_{div}$ and $\psi \in V$, where $\mu = \mu_1 - \mu_2 = a\varphi - J * \varphi + F'(\varphi_1) - F'(\varphi_2)$. Let us choose $v = u$ and $\psi = \varphi$ as test functions in (4.28) and (4.29), respectively, and adding the resulting identities. Notice that the contribution from the second term on the right hand side of (4.29) vanishes due to the incompressibility condition. Hence, we get

$$
\frac{1}{2}\frac{d}{dt}\big(\|u\|^2 + \|\varphi\|^2\big) + 2\big((\nu(\varphi_1) - \nu(\varphi_2))Du_1, Du\big) + 2\big(\nu(\varphi_2)Du, Du\big) + b(u, u_2, u)
$$
$$
+ (\nabla\mu, \nabla\varphi) = I_1 + I_2 + I_3 + I_4, \tag{4.30}
$$

where I_1, I_2, I_3 are given again by

$$
I_1 = -\frac{1}{2}\big(\varphi(\varphi_1 + \varphi_2)\nabla a, u\big), \quad I_2 = -\big((J * \varphi)\nabla\varphi_1, u\big), \quad I_3 = -\big((J * \varphi_2)\nabla\varphi, u\big),
$$

while I_4 is given by

$$
I_4 = -(u \cdot \nabla\varphi_1, \varphi).
$$

Let us first estimate the terms in (4.30) coming from the Navier-Stokes equations. Due to assumption (H2) we have

$$
2\big|\big((\nu(\varphi_1) - \nu(\varphi_2))Du_1, Du\big)\big| \leq C\|\varphi\|_{L^4}\|Du_1\|_{L^4}\|\nabla u\|
$$
$$
\leq C\|\varphi\|^{1/2}\|\varphi\|_V^{1/2}\|Du_1\|^{1/2}\|Du_1\|_{H^1}^{1/2}\|\nabla u\|
$$
$$
\leq \frac{\nu_1}{12}\|\nabla u\|^2 + C\|\nabla u_1\|\|u_1\|_{H^2}\|\varphi\|^2 + C\|\nabla u_1\|\|u_1\|_{H^2}\|\varphi\|\|\nabla\varphi\|
$$
$$
\leq \frac{\nu_1}{12}\|\nabla u\|^2 + \frac{c_0}{4}\|\nabla\varphi\|^2 + C(1 + \|\nabla u_1\|^2\|u_1\|_{H^2}^2)\|\varphi\|^2, \tag{4.31}
$$

$$
2\big(\nu(\varphi_2)Du, Du\big) \geq \nu_1\|\nabla u\|^2,
$$

where henceforth in this proof C will denote a constant which depends on $\|\varphi_0\|_{L^\infty}$, and on $\|u_0\|$. Indeed, recall that, since $\varphi_0 \in L^\infty(\Omega)$, then we have

$$
\|\varphi_i\|_{L^\infty(\Omega \times (0,T))} \leq C_i = C_i\big(\|\varphi_0\|_{L^\infty}, \|u_0\|\big), \qquad \text{for } i = 1, 2.
$$

The term in the trilinear form is standard

$$
|b(u, u_2, u)| \leq c\|u\|\|\nabla u\|\|\nabla u_2\| \leq \frac{\nu_1}{12}\|\nabla u\|^2 + c\|\nabla u_2\|^2\|u\|^2,
$$

while the terms I_1, I_2, I_3 can now be estimated more easily in this way

$$
I_1 \leq \|\varphi\|\|\varphi_1 + \varphi_2\|_{L^4}\|\nabla a\|_{L^\infty}\|u\|_{L^4}
$$
$$
\leq \frac{\nu_1}{12}\|\nabla u\|^2 + c\big(\|\varphi_1\|_{L^4}^2 + \|\varphi_2\|_{L^4}^2\big)\|\varphi\|^2,
$$
$$
I_2 \leq \|\varphi_1\|_{L^4}\|\nabla J\|_{L^1}\|\varphi\|\|u\|_{L^4}
$$
$$
\leq \frac{\nu_1}{12}\|\nabla u\|^2 + c\|\varphi_1\|_{L^4}^2\|\varphi\|^2,
$$
$$
I_3 \leq \frac{\nu_1}{12}\|\nabla u\|^2 + c\|\varphi_2\|_{L^4}^2\|\varphi\|^2.
$$

18

Regarding the terms coming from the nonlocal Cahn-Hilliard equation we have

$$(\nabla\mu, \nabla\varphi) = \big((a + F''(\varphi_2))\nabla\varphi, \nabla\varphi\big) + \big(\varphi\nabla a - \nabla J * \varphi, \nabla\varphi\big)$$
$$+ \big((F''(\varphi_1) - F''(\varphi_2))\nabla\varphi_1, \nabla\varphi\big),$$

and the last term on the right hand side of this identity can be estimated as

$$\big|\big((F''(\varphi_1) - F''(\varphi_2))\nabla\varphi_1, \nabla\varphi\big)\big| \le \|F''(\varphi_1) - F''(\varphi_2)\|_{L^4}\|\nabla\varphi_1\|_{L^4}\|\nabla\varphi\|$$
$$\le C\|\varphi\|_{L^4}\|\nabla\varphi_1\|_{L^4}\|\nabla\varphi\| \le C(\|\varphi\| + \|\varphi\|^{1/2}\|\nabla\varphi\|^{1/2})\|\nabla\varphi_1\|^{1/2}\|\nabla\varphi_1\|_V^{1/2}\|\nabla\varphi\|$$
$$\le \frac{c_0}{4}\|\nabla\varphi\|^2 + C(1 + \|\nabla\varphi_1\|^2\|\varphi_1\|_{H^2}^2)\|\varphi\|^2.$$

Hence, by means of assumption (H3), we get

$$(\nabla\mu, \nabla\varphi) \ge c_0\|\nabla\varphi\|^2 - 2\|\nabla J\|_{L^1}\|\varphi\|\|\nabla\varphi\| - \frac{c_0}{4}\|\nabla\varphi\|^2 - C(1 + \|\nabla\varphi_1\|^2\|\varphi_1\|_{H^2}^2)\|\varphi\|^2$$
$$\ge \frac{c_0}{2}\|\nabla\varphi\|^2 - C(1 + \|\nabla\varphi_1\|^2\|\varphi_1\|_{H^2}^2)\|\varphi\|^2.$$

Finally, the last term in (4.30) coming from the nonlocal Cahn-Hilliard equation can be controlled as follows

$$I_4 \le \|u\|_{L^4}\|\nabla\varphi_1\|_{L^4}\|\varphi\| \le \frac{\nu_1}{12}\|\nabla u\|^2 + c\|\varphi_1\|_{H^2}^2\|\varphi\|^2. \tag{4.32}$$

By plugging estimates from (4.31) to (4.32) into (4.30) we are led to the following differential inequality

$$\frac{1}{2}\frac{d}{dt}\big(\|u\|^2 + \|\varphi\|^2\big) + \frac{\nu_1}{2}\|\nabla u\|^2 + \frac{c_0}{4}\|\nabla\varphi\|^2 \le \gamma\big(\|u\|^2 + \|\varphi\|^2\big), \tag{4.33}$$

where the function γ is given by

$$\gamma = c\big(1 + \|\nabla u_1\|^2\|u_1\|_{H^2}^2 + \|\nabla u_2\|^2 + \|\varphi_1\|_{L^4}^2 + \|\varphi_2\|_{L^4}^2 + \|\varphi_1\|_{H^2}^2 + \|\nabla\varphi_1\|^2\|\varphi_1\|_{H^2}^2\big),$$

and due to the regularity properties of the strong solution $[u_1, \varphi_1]$ and of the weak solution $[u_2, \varphi_2]$ we have $\gamma \in L^1(0, T)$. Strong-weak uniqueness follows by applying Gronwall's lemma to (4.33). In addition, a continuous dependence estimate in $L^2 \times L^2$ can also be deduced by considering two solutions with different initial data. $\qquad\square$

5 Global and exponential attractors

In this section we prove two results concerning the asymptotic behavior of the dynamical system generated by (1.1)–(1.5) in dimension two.

The first result is related to the property of connectedness of the global attractor whose existence was established in [11] for nonconstant viscosity, constant mobility and regular potential (see Remark 5 below, however).

The second result is the existence of an exponential attractor. This will be proven in details when mobility and viscosity are constant and the potential is regular. This kind of result relies on a regularization argument devised in [13] and on an abstract theorem (see [10]) which generalizes a well known result on the existence of exponential attractors in Banach spaces (cf. [9]). A similar argument will be carried out in the nonconstant viscosity case albeit we will work with strong solutions.

Let us define the dynamical system in the autonomous case. Take $d = 2$ and $h \in V'_{div}$. Then, as a consequence of Theorem 2, we have that for every fixed $\eta \geq 0$ system (1.1)–(1.5) generates a semigroup $\{S_\eta(t)\}_{t \geq 0}$ of *closed* operators on the metric space $\mathcal{X}_\eta$ given by

$$\mathcal{X}_\eta := G_{div} \times \mathcal{Y}_\eta \tag{5.1}$$

where

$$\mathcal{Y}_\eta := \{\varphi \in H : F(\varphi) \in L^1(\Omega), \, |\overline{\varphi}| \leq \eta\}.$$

It is convenient to endow the space $\mathcal{X}_\eta$ with the following metric

$$\rho_{\mathcal{X}_\eta}(z_2, z_1) = \|u_2 - u_1\| + \|\varphi_2 - \varphi_1\| + \left| \int_\Omega F(\varphi_2) - \int_\Omega F(\varphi_1) \right|,$$

for all $z_i := [u_i, \varphi_i] \in \mathcal{X}_\eta$, $i = 1, 2$. Notice that this metric is slightly different from the one which is naturally associated to the energy $\mathcal{E}$ (the difference is in the exponent in the third term, see [11]).

A first noteworthy consequence of the uniqueness result for weak solutions is the following

Theorem 7. *Let assumptions (H1)–(H5) be satisfied with $\nu = 1$. Suppose $d = 2$ and that $h \in V'_{div}$. Then, the global attractor in $\mathcal{X}_\eta$ for the semigroup $S_\eta(t)$ is connected.*

Proof. The conclusion follows immediately by applying [3, Corollary 4.3]. Indeed, the space $\mathcal{X}_\eta$ is (arcwise) connected, thanks to the fact that F is a quadratic perturbation of a convex function. Moreover, we have the strong time continuity of each trajectory $z = [u, \varphi]$ from $[0, \infty)$ to the metric space $\mathcal{X}_\eta$ (see Theorem 1). Thus Kneser's property is satisfied thanks to uniqueness. $\square$

Remark 5. *Theorem 7 also holds in the case of constant (or degenerate) mobility and singular potential on account of Theorem 3 and [12, Proposition 4] (or Theorem 4 and [14, Proposition 3]). The argument is similar.*

The second result is the existence of an exponential attractor. We first recall its definition.

Definition 2. *A compact set $\mathcal{M}_\eta \subset \mathcal{X}_\eta$ is an exponential attractor for the dynamical system $(\mathcal{X}_\eta, S_\eta(t))$ if the following properties are satisfied*

   (i) positive invariance: $S_\eta(t)\mathcal{M}_\eta \subseteq \mathcal{M}_\eta$ for all $t \geq 0$;

   (ii) finite dimensionality: $dim_F(\mathcal{M}_\eta, \mathcal{X}_\eta) < \infty$;

(iii) exponential attraction: $\exists Q : \mathbb{R}^+ \to \mathbb{R}^+$ *increasing and* $\kappa > 0$ *such that, for all* $R > 0$ *and for all* $\mathcal{B} \subset \mathcal{X}_\eta$ *with* $\sup_{z \in \mathcal{B}} \rho_{\mathcal{X}_\eta}(z, 0) \leq R$ *there holds*

$$dist_{\mathcal{X}_\eta}(S_\eta(t)\mathcal{B}, \mathcal{M}_\eta) \leq Q(R)e^{-\kappa t}, \qquad \forall t \geq 0.$$

Theorem 8. *Let* $d = 2$. *Assume that (H1)–(H5) are satisfied with* $\nu = 1$. *Then the dynamical system* $(\mathcal{X}_\eta, S_\eta(t))$ *possesses an exponential attractor* $\mathcal{M}_\eta$ *which is bounded in* $V_{div} \times W^{1,p}(\Omega)$, $2 < p < \infty$.

The proof of Theorem 8 is based on four lemmas. These lemmas allow us to apply the abstract result in [10]. For their proof we shall need the following regularization result which is an easy consequence of [13, Theorem 2 and Proposition 1] and has an independent interest. In the statement and proof of this result we shall denote by $\Gamma_\tau = \Gamma_\tau\big(\mathcal{E}(z_0), \eta\big)$ a positive constant depending on a positive time τ, on the energy $\mathcal{E}(z_0)$ of the initial datum $z_0 := [u_0, \varphi_0]$ of a weak solution, and on η, where $\eta \geq 0$ is such that $|\overline{\varphi}_0| \leq \eta$ (Γ_τ may of course depend also on h, F, J, ν and Ω). The value of Γ_τ may change even on the same line.

Proposition 1. *Let* $d = 2$ *and* $h \in L^2_{tb}(0, \infty; G_{div})$. *Assume that (H1)–(H5) are satisfied with* $\nu = 1$, *and suppose* $F \in C^3(\mathbb{R})$. *Let* $u_0 \in G_{div}$, $\varphi_0 \in H$ *with* $F(\varphi_0) \in L^1(\Omega)$ *and let* $[u, \varphi]$ *be the weak solution on* $(0, \infty)$ *to system* (1.1)–(1.6) *corresponding to* $[u_0, \varphi_0]$. *Then, for every* $\tau > 0$ *there exists* $\Gamma_\tau > 0$ *such that we have*

$$u \in L^\infty(\tau, \infty; V_{div}) \cap L^2_{tb}\big(\tau, \infty; H^2(\Omega)^2\big), \qquad u_t \in L^2_{tb}\big(\tau, \infty; G_{div}\big), \qquad (5.2)$$

$$\varphi \in L^\infty\big(\tau, \infty; W^{1,p}(\Omega)\big), \qquad 2 < p < \infty, \qquad \varphi_t \in L^\infty(\tau, \infty; H) \cap L^2_{tb}(\tau, \infty; V),$$
$$(5.3)$$

with norms controlled by Γ_τ. *In addition, there exists a constant* $\Lambda = \Lambda(\eta) > 0$ *depending only on* η *(and on* F, J, ν *and* Ω) *such that for every initial data* $z_0 := [u_0, \varphi_0] \in G_{div} \times H$ *with* $F(\varphi_0) \in L^1(\Omega)$ *and* $|\overline{\varphi}_0| \leq \eta$ *there exists a time* $t^* = t^*\big(\mathcal{E}(z_0)\big) \geq 0$ *starting from which the weak solution corresponding to* z_0 *regularizes, that is,*

$$\|\nabla u(t)\| + \|\varphi(t)\|_{W^{1,p}(\Omega)} + \int_t^{t+1} \|u(s)\|^2_{H^2(\Omega)^2} ds \leq \Lambda(\eta), \qquad \forall t \geq t^*. \qquad (5.4)$$

Remark 6. Notice that, differently from [13, Theorem 2], in Proposition 1 we do not require any further regularity assumption on J in addition to (H1).

Proof. Recalling the proof of [18, Lemma 2.10] and the dissipative estimate (2.7), observe first that, if $z_0 \in \mathcal{X}_\eta$, then for every $\tau > 0$ there exists $\Gamma_\tau = \Gamma_\tau\big(\mathcal{E}(z_0), \eta\big)$ such that

$$\|\varphi(t)\|_{L^\infty(\Omega)} \leq \Gamma_\tau, \qquad \forall t \geq \tau. \qquad (5.5)$$

This implies that $\|\mu(t)\|_{L^\infty(\Omega)} \leq \Gamma_\tau$ for all $t \geq \tau$, and hence that the Korteweg term $\mu\nabla\varphi \in L^2(\tau, T; L^2(\Omega)^2)$.

We can now repeat exactly the same argument in the proof of [13, Theorem 2], by writing the same estimates which now hold starting from a positive time, say for $t \geq \tau/2 > 0$. We recall that these estimates are obtained by multiplying the nonlocal Cahn-Hilliard by μ_t in H and then

by differentiating the nonlocal Cahn-Hilliard with respect to time and multiplying the resulting identity bu μ_t. By doing so we are led to a differential inequality of the following form

$$\frac{d}{ds} \log\Big(1 + \int_\Omega \big(a + F''(\varphi)\big)\varphi_t^2\Big) \leq \Gamma_\tau\big(\sigma(s) + \|\varphi_t\|^2\big), \qquad \forall s \geq \tau/2, \qquad (5.6)$$

where $\sigma = \Gamma_\tau\big(1 + \|u\|_{H^2}^2 + \|u_t\|^2\big)$ and we have $\sigma \in L^1(\tau/2, T)$, for all $T > \tau/2$. At this point we argue a bit differently from the proof of [13, Theorem 2]. Indeed, here we want to avoid the L^2-norm of φ_t in $\tau/2$ which would require the initial condition $\varphi(\tau/2) \in H^2$ and in addition would force us to make some further regularity assumptions on the kernel J (like, e.g., $J \in W^{2,1}$). Therefore, we multiply (5.6) by $(s - \tau/2)$ and integrate with respect to s between $\tau/2$ and $t \in (\tau/2, T)$. We get

$$\Big(t - \frac{\tau}{2}\Big)\log\Big(1 + \int_\Omega \big(a + F''(\varphi)\big)\varphi_t^2\Big) \leq \int_{\tau/2}^T \log\Big(1 + \int_\Omega \big(a + F''(\varphi)\big)\varphi_t^2\Big)ds$$

$$+ \Gamma_\tau\Big(T - \frac{\tau}{2}\Big)\big(\|\sigma\|_{L^1(\tau/2,T)} + \|\varphi_t\|_{L^2(\tau/2,T;H)}^2\big)$$

$$\leq \Gamma_\tau\|\varphi_t\|_{L^2(\tau/2,T;H)}^2 + \Gamma_\tau\Big(T - \frac{\tau}{2}\Big)\big(\|\sigma\|_{L^1(\tau/2,T)} + \|\varphi_t\|_{L^2(\tau/2,T;H)}^2\big), \qquad \forall t \in (\tau/2, T).$$

From this inequality, on account of the fact that we have $\|\varphi_t\|_{L^2(\tau/2,T;H)} \leq \Gamma_\tau$ (this was shown in the first step of the proof of [13, Theorem 2], before (5.6)) we deduce that

$$\varphi_t \in L^\infty(\tau, T; H). \qquad (5.7)$$

This bound, together with the following estimate (cf. proof of [13, Theorem 2])

$$\|\nabla\mu\|_{L^p} \leq \Gamma_\tau\big(1 + \|\varphi_t\|^{1-2/p}\big), \qquad 2 < p < \infty,$$

yield

$$\varphi \in L^\infty\big(\tau, T; W^{1,p}(\Omega)\big). \qquad (5.8)$$

Finally, arguing as in the proof of [13, Proposition 1] by applying the uniform Gronwall's lemma, and taking (5.7), (5.8) (together with the bounds for u on (τ, T)) into account, we get (5.2), (5.3) and (5.4), respectively. $\qquad\square$

For the statements and proofs of the following lemmas we shall denote by $C_\tau = C_\tau\big(\mathcal{E}(z_{0i}), \eta\big)$ a positive constant depending on a positive time τ, on the energies $\mathcal{E}(z_{01})$, $\mathcal{E}(z_{02})$ of the initial data $z_{01}, z_{02} \in \mathcal{X}_\eta$ of two weak solutions, and on η, where $\eta > 0$ is such that $|\overline{\varphi}_{01}|, |\overline{\varphi}_{02}| \leq \eta$ (of course, C_τ will generally depend also on h, F, J, ν and Ω). The value of C_τ may change even within the same line. Furthermore, we shall always set $u := u_2 - u_1$, $\varphi := \varphi_2 - \varphi_1$.

Lemma 2. *Let $d = 2$. Assume that (H1)–(H5) are satisfied with $\nu = 1$ and that $F \in C^3(\mathbb{R})$. Let $u_{0i} \in G_{div}$, $\varphi_{0i} \in H$ with $F(\varphi_{0i}) \in L^1(\Omega)$ and $[u_i, \varphi_i]$ be the corresponding weak solutions, $i = 1, 2$. Then, for every $\tau > 0$ there exists $C_\tau > 0$ such that we have*

$$\|u_2(t) - u_1(t)\|^2 + \|\varphi_2(t) - \varphi_1(t)\|^2$$

$$+ \int_\tau^t \Big(\frac{\nu}{4}\|\nabla(u_2(s) - u_1(s))\|^2 + \frac{c_0}{4}\|\nabla(\varphi_2(s) - \varphi_1(s))\|^2\Big)ds$$

$$\leq e^{C_\tau t}\big(\|u_2(\tau) - u_1(\tau)\|^2 + \|\varphi_2(\tau) - \varphi_1(\tau)\|^2\big), \qquad \forall t \geq \tau. \qquad (5.9)$$

Proof. Let us multiply (3.2) by φ in $L^2(\Omega)$. We get

$$\frac{1}{2}\frac{d}{dt}\|\varphi\|^2 = -(u \cdot \nabla\varphi_2, \varphi) - (\nabla\widetilde{\mu}, \nabla\varphi) \tag{5.10}$$

Taking the gradient of $\widetilde{\mu}$, on account of (3.3) we have

$$
\begin{aligned}
(\nabla\widetilde{\mu}, \nabla\varphi) &= \int_\Omega \big(a + F''(\varphi_1)\big)|\nabla\varphi|^2 + (\varphi\nabla a - \nabla J * \varphi, \nabla\varphi) \\
&\quad + \big((F''(\varphi_2) - F''(\varphi_1))\nabla\varphi_2, \nabla\varphi\big) \geq c_0\|\nabla\varphi\|^2 - c\|\varphi\|\|\nabla\varphi\| \\
&\quad - \|F''(\varphi_2) - F''(\varphi_1)\|_{L^4}\|\nabla\varphi_2\|_{L^4}\|\nabla\varphi\| \\
&\geq \frac{c_0}{2}\|\nabla\varphi\|^2 - c\|\varphi\|^2 - C_\tau\|\varphi\|_{L^4}\|\nabla\varphi_2\|_{L^4}\|\nabla\varphi\| \\
&\geq \frac{c_0}{2}\|\nabla\varphi\|^2 - c\|\varphi\|^2 - C_\tau\big(\|\varphi\| + \|\varphi\|^{1/2}\|\nabla\varphi\|^{1/2}\big)\|\nabla\varphi_2\|_{L^4}\|\nabla\varphi\| \\
&\geq \frac{c_0}{4}\|\nabla\varphi\|^2 - C_\tau\big(1 + \|\nabla\varphi_2\|_{L^4}^2 + \|\nabla\varphi_2\|_{L^4}^4\big)\|\varphi\|^2.
\end{aligned}
$$

Observe that

$$(\nabla\widetilde{\mu}, \nabla\varphi) \geq \frac{c_0}{4}\|\nabla\varphi\|^2 - C_\tau\big(1 + \|\nabla\varphi_2\|_{L^4}^4\big)\|\varphi\|^2. \tag{5.11}$$

Furthermore, we have

$$|(u \cdot \nabla\varphi_2, \varphi)| \leq \|u\|_{L^4}\|\nabla\varphi_2\|_{L^4}\|\varphi\| \leq \frac{\nu}{4}\|\nabla u\|^2 + c\|\nabla\varphi_2\|_{L^4}^2\|\varphi\|^2. \tag{5.12}$$

Therefore, plugging (5.11) and (5.12) into (5.10), we get

$$\frac{1}{2}\frac{d}{dt}\|\varphi\|^2 + \frac{c_0}{4}\|\nabla\varphi\|^2 \leq C_\tau\big(1 + \|\nabla\varphi_2\|_{L^4}^4\big)\|\varphi\|^2 + \frac{\nu}{4}\|\nabla u\|^2.$$

Adding this last differential inequality to (3.8), we obtain

$$\frac{1}{2}\frac{d}{dt}\big(\|u\|^2 + \|\varphi\|^2\big) + \frac{\nu}{4}\|\nabla u\|^2 + \frac{c_0}{4}\|\nabla\varphi\|^2 \leq \gamma(t)\big(\|u\|^2 + \|\varphi\|^2\big), \tag{5.13}$$

where

$$\gamma(t) := \alpha(t) + C_\tau\big(1 + \|\nabla\varphi_2\|_{L^4}^4\big).$$

Then, thanks to Proposition 1, for every $\tau > 0$ there exists $C_\tau > 0$ (always depending on τ, η and on the energies $\mathcal{E}(z_{01})$, $\mathcal{E}(z_{02})$) such that the following bounds for the solutions $z_i = [u_i, \varphi_i]$ corresponding to $[u_{0i}, \varphi_{0i}]$ hold

$$\|u_i\|_{L^\infty(\tau,\infty;V_{div})} + \|\varphi_i\|_{L^\infty(\tau,\infty;W^{1,p}(\Omega))} \leq C_\tau, \tag{5.14}$$

$$\|u_{i,t}\|_{L^2_{tb}(\tau,\infty;G_{div})} + \|\varphi_{i,t}\|_{L^\infty(\tau,\infty;H)} \leq C_\tau, \tag{5.15}$$

Thus we have $\gamma(t) \leq C_\tau$, for all $t \geq \tau$ and by applying the standard Gronwall's lemma to (5.13) written for $t \geq \tau$ we get

$$\|u(t)\|^2 + \|\varphi(t)\|^2 \leq \big(\|u(\tau)\|^2 + \|\varphi(\tau)\|^2\big)e^{C_\tau t}, \qquad \forall t \geq \tau. \tag{5.16}$$

By integrating (5.13) between τ and t and using (5.16) we get (5.9). $\qquad\square$

Lemma 3. *Let the assumptions of Lemma 2 be satisfied. Let* $u_{0i} \in G_{div}$, $\varphi_{0i} \in H$ *with* $F(\varphi_{0i}) \in L^1(\Omega)$ *and* $[u_i, \varphi_i]$ *be the corresponding weak solutions,* $i = 1, 2$. *Then, for every* $\tau > 0$ *there exists* $C_\tau > 0$ *such that we have*

$$\|u_2(t) - u_1(t)\|^2 + \|\varphi_2(t) - \varphi_1(t)\|^2 + \left| \int_\Omega F(\varphi_2(t)) - \int_\Omega F(\varphi_1(t)) \right|^2$$

$$\leq C_\tau \big(\|u_2(\tau) - u_1(\tau)\|^2 + \|\varphi_2(\tau) - \varphi_1(\tau)\|^2\big) e^{-kt}$$

$$+ C_\tau \int_\tau^t \big(\|u_2(s) - u_1(s)\|^2 + \|\varphi_2(s) - \varphi_1(s)\|^2\big) ds, \qquad \forall t \geq \tau. \tag{5.17}$$

Proof. By using Poincaré's inequality for u and the Poincaré-Wirtinger's inequality for φ, i.e.,

$$\lambda_1 \|u\|^2 \leq \|\nabla u\|^2, \qquad \|\varphi - \overline{\varphi}\|^2 \leq c_\Omega \|\nabla\varphi\|^2, \tag{5.18}$$

from (5.13) we have

$$\frac{d}{dt}\big(\|u\|^2 + \|\varphi\|^2\big) + \frac{\nu\lambda_1}{2}\|u\|^2 + \frac{c_0}{2c_\Omega}\|\varphi\|^2 \leq 2\gamma(t)\big(\|u\|^2 + \|\varphi\|^2\big) + \frac{c_0|\Omega|}{2c_\Omega}\overline{\varphi}^2,$$

which yields

$$\frac{d}{dt}\big(\|u\|^2 + \|\varphi\|^2\big) + k\big(\|u\|^2 + \|\varphi\|^2\big) \leq C_\tau\big(\|u\|^2 + \|\varphi\|^2\big), \tag{5.19}$$

where $k := \min(\lambda_1\nu, c_0/c_\Omega)/2$ and C_τ is a positive constant such that $2\gamma(t) + c_0/2c_\Omega \leq C_\tau$ for all $t \geq \tau$. By using Gronwall's lemma we immediately see from (5.19) that $\|u\|^2 + \|\varphi\|^2$ is controlled by the right hand side of (5.17). Furthermore, we also have

$$\left| \int_\Omega F(\varphi_2(t)) - \int_\Omega F(\varphi_1(t)) \right| \leq C_\tau \|\varphi(t)\|, \qquad \forall t \geq \tau.$$

Hence, the proof of (5.17) is complete. $\qquad\square$

Lemma 4. *Let the assumptions of Lemma 2 be satisfied. Let* $u_{0i} \in G_{div}$, $\varphi_{0i} \in H$ *with* $F(\varphi_{0i}) \in L^1(\Omega)$ *and* $[u_i, \varphi_i]$ *be the corresponding weak solutions,* $i = 1, 2$. *Then, for every* $\tau > 0$ *there exists* $C_\tau > 0$ *such that*

$$\|u_{2,t} - u_{1,t}\|^2_{L^2(\tau,t;V'_{div})} + \|\varphi_{2,t} - \varphi_{1,t}\|^2_{L^2(\tau,t;D(B_N)')}$$

$$\leq C_\tau e^{C_\tau t}\big(\|u_2(\tau) - u_1(\tau)\|^2 + \|\varphi_2(\tau) - \varphi_1(\tau)\|^2\big), \qquad \forall t \geq \tau. \tag{5.20}$$

Proof. Consider the variational formulation of (3.2) and (3.3), namely,

$$\langle \varphi_t, \psi \rangle = -(\nabla\widetilde{\mu}, \nabla\psi) - (u \cdot \nabla\varphi_1, \psi) - (u_2 \cdot \nabla\varphi, \psi), \qquad \forall \psi \in V, \tag{5.21}$$

and take $\psi \in D(B_N)$. Then, for every $\tau > 0$ we see that there exists $C_\tau > 0$ such that

$$|(\nabla\widetilde{\mu}, \nabla\psi)| = |(\widetilde{\mu}, B_N\psi)| \leq \|\widetilde{\mu}\|\|\psi\|_{D(B_N)} \leq C_\tau\|\varphi\|\|\psi\|_{D(B_N)}, \qquad \forall t \geq \tau. \tag{5.22}$$

Moreover, we have

$$|(u \cdot \nabla\varphi_1, \psi)| = |(u \cdot \nabla\psi, \varphi_1) \leq c\|\nabla u\|\|\varphi_1\|\|\psi\|_{D(B_N)} \leq C\|\nabla u\|\|\psi\|_{D(B_N)},$$

where in this case it is enough to use the dissipative estimate (2.7) and therefore the constant C does not depend on τ but depends on h, $\mathcal{E}(z_{01})$ and η only. Concerning the last term on the right hand side of (5.21) we have

$$|(u_2 \cdot \nabla\varphi, \psi)| = |(u_2 \cdot \nabla\psi, \varphi)| \le c\|\nabla u_2\|\|\varphi\|\|\psi\|_{D(B_N)} \le C_\tau\|\varphi\|\|\psi\|_{D(B_N)}, \qquad \forall t \ge \tau. \tag{5.23}$$

Plugging (5.22)–(5.23) into (5.21), we get

$$\|\varphi_t\|_{D(B_N)'} \le C_\tau\big(\|\varphi\| + \|\nabla u\|\big), \qquad \forall t \ge \tau. \tag{5.24}$$

Therefore, taking also (5.9) into account, we have

$$\|\varphi_t\|_{L^2(\tau,t;D(B_N)')} \le C_\tau e^{C_\tau t}\big(\|u(\tau)\| + \|\varphi(\tau)\|\big), \qquad \forall t \ge \tau. \tag{5.25}$$

In order to obtain an estimate for $u_{2,t} - u_{1,t}$ let us consider the difference of the Navier-Stokes equations written for two weak solutions in the variational formulation, i.e.,

$$\langle u_t, v \rangle = -\nu(\nabla u, \nabla v) - b(u_2, u_2, v) + b(u_1, u_1, v)$$
$$- \frac{1}{2}\big(\nabla a\varphi(\varphi_1 + \varphi_2), v\big) - \big((J * \varphi)\nabla\varphi_2, v\big) - \big((J * \varphi_2)\nabla\varphi, v\big), \qquad \forall v \in V_{div}. \tag{5.26}$$

Thanks to (5.14) the last three terms on the right hand side can be easily estimated as follows

$$\frac{1}{2}\big|\big(\nabla a\varphi(\varphi_1 + \varphi_2), v\big)\big| \le c\|\nabla a\|_{L^\infty}\|\varphi\|\|\varphi_1 + \varphi_2\|_{L^\infty}\|v\| \le C_\tau\|\varphi\|\|v\|_{V_{div}},$$
$$\big|\big((J * \varphi)\nabla\varphi_2, v\big)\big| = \big|\big((\nabla J * \varphi)\varphi_2, v\big)\big| \le c\|\nabla J\|_{L^1}\|\varphi\|\|\varphi_2\|_{L^\infty}\|v\| \le C_\tau\|\varphi\|\|v\|_{V_{div}},$$
$$\big|\big((J * \varphi_2)\nabla\varphi, v\big)\big| = \big|\big((\nabla J * \varphi_2)\varphi, v\big)\big| \le c\|\nabla J\|_{L^1}\|\varphi_2\|_{L^\infty}\|\varphi\|\|v\| \le C_\tau\|\varphi\|\|v\|_{V_{div}},$$

for all $t \ge \tau$. Furthermore, the trilinear form can be controlled by using (2.1), that is,

$$|b(u_2, u_2, v) - b(u_1, u_1, v)| = |b(u_2, u, v) + b(u, u_1, v)|$$
$$\le c\big(\|\nabla u_1\| + \|\nabla u_2\|\big)\|\nabla u\|\|\nabla v\| \le C_\tau\|\nabla u\|\|\nabla v\|, \qquad \forall t \ge \tau.$$

Combining the last four estimates with (5.26) we obtain

$$\|u_t\|_{V_{div}'} \le C_\tau\big(\|\nabla u\| + \|\varphi\|\big), \qquad \forall t \ge \tau,$$

Thus, recalling (5.9), we deduce

$$\|u_t\|_{L^2(\tau,t;V_{div}')} \le C_\tau e^{C_\tau t}\big(\|u(\tau)\| + \|\varphi(\tau)\|\big), \qquad \forall t \ge \tau. \tag{5.27}$$

Finally, (5.25) and (5.27) yield (5.20). □

Lemma 5. *Let the assumptions of Lemma 2 be satisfied. Let $u_{0i} \in G_{div}$, $\varphi_{0i} \in H$ with $F(\varphi_{0i}) \in L^1(\Omega)$ $i = 1, 2$. Then, for every $\tau > 0$ and every $T > 0$ there exists $C_{\tau,T} > 0$ depending also on T such that*

$$\rho_{\mathcal{X}_\eta}(S_\eta(t_2)z_{02}, S_\eta(t_1)z_{01}) \le C_{\tau,T}\big(\rho_{\mathcal{X}_\eta}(S_\eta(\tau)z_{02}, S_\eta(\tau)z_{01}) + |t_2 - t_1|^{1/2}\big), \tag{5.28}$$

for all $t_1, t_2 \in [\tau, \tau + T]$, where $z_{0i} := [u_{0i}, \varphi_{0i}]$, $i = 1, 2$.

25

Proof. Setting $S_\eta(t)z_{0i} := [u_i(t), \varphi_i(t)]$, $i = 1, 2$, we have

$$\rho_{\mathcal{X}_\eta}(S_\eta(t_2)z_{01}, S_\eta(t_1)z_{01})$$

$$= \|u_1(t_2) - u_1(t_1)\| + \|\varphi_1(t_2) - \varphi_1(t_1)\| + \left| \int_\Omega F(\varphi_1(t_2)) - \int_\Omega F(\varphi_1(t_1)) \right|$$

$$\leq \|u_{1,t}\|_{L^2(t_1,t_2;G_{div})}|t_2 - t_1|^{1/2} + \|\varphi_{1,t}\|_{L^\infty(\tau,\infty;H)}|t_2 - t_1| + C_\tau\|\varphi_{1,t}\|_{L^\infty(\tau,\infty;H)}|t_2 - t_1|$$

$$\leq C_{\tau,T}|t_2 - t_1|^{1/2}, \qquad \forall t_1, t_2 \in [\tau, \tau + T], \tag{5.29}$$

where we have used (5.15). Furthermore we have

$$\rho_{\mathcal{X}_\eta}(S_\eta(t_2)z_{02}, S_\eta(t_2)z_{01})$$

$$= \|u_2(t_2) - u_1(t_2)\| + \|\varphi_2(t_2) - \varphi_1(t_2)\| + \left| \int_\Omega F(\varphi_2(t_2)) - \int_\Omega F(\varphi_1(t_2)) \right|$$

$$\leq C_\tau e^{C_\tau(\tau+T)}\big(\|u_2(\tau) - u_1(\tau)\| + \|\varphi_2(\tau) - \varphi_1(\tau)\|\big) \leq C_{\tau,T}\rho_{\mathcal{X}_\eta}(S_\eta(\tau)z_{02}, S_\eta(\tau)z_{01}).$$

$$\tag{5.30}$$

From (5.29) and (5.30) we get (5.28). □

We now recall the following abstract result on the existence of exponential attractors [10, Proposition 3.1]. This result, together with the lemmas above, will be used to prove Theorem 8.

Proposition 2. *Let $\mathcal{H}$ be a metric space (with metric $\rho_\mathcal{H}$) and let $\mathcal{V}, \mathcal{V}_1$ be two Banach spaces such that the embedding $\mathcal{V}_1 \hookrightarrow\hookrightarrow \mathcal{V}$ is compact. Let $\mathbb{B}$ be a bounded subset of $\mathcal{H}$ and let $\mathcal{S} : \mathbb{B} \to \mathbb{B}$ be a map such that*

$$\rho_\mathcal{H}\big(\mathcal{S}w_{02}, \mathcal{S}w_{01}\big) \leq \gamma\rho_\mathcal{H}(w_{02}, w_{01}) + K\|\mathcal{T}w_{02} - \mathcal{T}w_{01}\|_\mathcal{V}, \qquad \forall w_{01}, w_{02} \in \mathbb{B}, \quad (5.31)$$

where $\gamma < 1/2$, $K \geq 0$ and $\mathcal{T} : \mathbb{B} \to \mathcal{V}_1$ is a globally Lipschitz continuous map, i.e.,

$$\|\mathcal{T}w_{02} - \mathcal{T}w_{01}\|_{\mathcal{V}_1} \leq L\rho_\mathcal{H}(w_{02}, w_{01}), \qquad \forall w_{01}, w_{02} \in \mathbb{B}, \tag{5.32}$$

for some $L \geq 0$. Then, there exists a (discrete) exponential attractor $\mathcal{M}_d \subset \mathbb{B}$ for the (time discrete) semigroup $\{\mathcal{S}^n\}_{n=0,1,2,\dots}$ on $\mathbb{B}$ (with the topology of $\mathcal{H}$ induced on $\mathbb{B}$).

Proof of Theorem 8. Let $\mathcal{B}_0$ be a bounded absorbing set in $\mathcal{X}_\eta$. The existence of such a bounded absorbing set has been proven in [11]. Indeed, it is immediate to check that the argument of [11, Proposition 4] still applies with our choice for the metric $\rho_{\mathcal{X}_\eta}$... Let $t_0 = t_0(\mathcal{B}_0) \geq 0$ be a time such that $S_\eta(t)\mathcal{B}_0 \subset \mathcal{B}_0$ for all $t \geq t_0$. Due to (5.4) we can fix $t^* = t^*(\mathcal{B}_0) \geq t_0$ such that $S_\eta(t)\mathcal{B}_0 \subset B_{\mathcal{Z}_\eta^p}(0, \Lambda(\eta))$ for all $t \geq t^*$, where $B_{\mathcal{Z}_\eta^p}(0, \Lambda(\eta))$ is the closed ball in $\mathcal{Z}_\eta^p$ with radius $\Lambda(\eta)$ and $\Lambda(\eta)$ a positive constant which depends only on η. The (complete) metric space $\mathcal{Z}_\eta^p$ is given by

$$\mathcal{Z}_\eta^p := V_{div} \times \{\varphi \in W^{1,p}(\Omega) : |\overline{\varphi}| \leq \eta\},$$

endowed with the metric

$$d_{\mathcal{Z}_\eta^p}(z_2, z_1) = \|\nabla u_2 - \nabla u_1\| + \|\varphi_2 - \varphi_1\|_{W^{1,p}(\Omega)}, \quad \forall z_i := [u_i, \varphi_i] \in \mathcal{Z}_\eta^p, \qquad i = 1, 2.$$

26

Note that the terms in the integrals of $F(\varphi_1)$, $F(\varphi_2)$ are omitted in the metric since, for $p > 2$, we have the embedding $W^{1,p}(\Omega) \hookrightarrow C(\overline{\Omega})$.

Let us now set

$$\mathcal{B}_1 := \bigcup_{t \geq t^*} S_\eta(t)\mathcal{B}_0.$$

Then, $\mathcal{B}_1$ is bounded in $\mathcal{Z}_\eta^p$ and positively invariant for $S_\eta(t)$. It is easy to see that it is also absorbing in $\mathcal{X}_\eta$. Indeed, if B is a bounded subset of $\mathcal{X}_\eta$ and $t_0 = t_0(B)$ is such that $S_\eta(t_0)B \subset \mathcal{B}_0$, then we have $S_\eta(t)B \subset \cup_{\tau \geq t^*} S_\eta(\tau + t_0)B \subset \cup_{\tau \geq t^*} S_\eta(\tau)\mathcal{B}_0 =: \mathcal{B}_1$, for all $t \geq t_0 + t^*$. Furthermore, we set

$$\mathbb{B} := S_\eta(1)\mathcal{B}_1.$$

Then, $\mathbb{B} \subset B_{\mathcal{Z}_\eta^p}(0, \Lambda(\eta))$ is positively invariant and still absorbing in $\mathcal{X}_\eta$.

By choosing $\tau = 1$ in Lemma 3, then (5.17) can be written as follows

$$\rho_{\mathcal{X}_\eta}\big(S_\eta(t)z_{02}, S_\eta(t)z_{01}\big) \leq C_1 e^{-kt/2}\rho_{\mathcal{X}_\eta}\big(S_\eta(1)z_{02}, S_\eta(1)z_{01}\big)$$
$$+ C_1 \|S_\eta(\cdot)z_{02} - S_\eta(\cdot)z_{01}\|_{L^2(1,t;G_{div}\times H)}, \qquad \forall t \geq 1, \qquad \forall z_{01}, z_{02} \in \mathcal{X}_\eta, \qquad (5.33)$$

where $C_1 > 0$ depends only on $\mathcal{E}(z_{01})$, $\mathcal{E}(z_{02})$ and η. From (5.33) we therefore get

$$\rho_{\mathcal{X}_\eta}\big(S_\eta(t-1)w_{02}, S_\eta(t-1)w_{01}\big) \leq C_1 e^{-kt/2}\rho_{\mathcal{X}_\eta}\big(w_{02}, w_{01}\big)$$
$$+ C_1 \|S_\eta(\cdot)w_{02} - S_\eta(\cdot)w_{01}\|_{L^2(0,t-1;G_{div}\times H)}, \qquad \forall t > 1, \qquad \forall w_{01}, w_{02} \in \mathbb{B}. \qquad (5.34)$$

Observe that, since $w_{0i} = S(1)z_{0i}$, with $z_{0i} \in \mathcal{B}_1$, $i = 1, 2$, and $\mathcal{B}_1$ is bounded in $\mathcal{Z}_\eta^p$, then C_1 does not depend on w_{01}, w_{02}.

Choosing $\tau = 1$ also in Lemma 2 and in Lemma 4, and combining (5.9) with (5.20) we can write

$$\|S_\eta(\cdot)z_{02} - S_\eta(\cdot)z_{01}\|_{L^2(1,t;V_{div}\times V)}^2 + \|\partial_t S_\eta(\cdot)z_{02} - \partial_t S_\eta(\cdot)z_{01}\|_{L^2(1,t;V'_{div}\times D(B_N)')}^2$$
$$\leq C_1 e^{C_1 t}\rho_{\mathcal{X}_\eta}^2(S_\eta(1)z_{02}, S_\eta(1)z_{01}), \qquad \forall t \geq 1, \qquad \forall z_{01}, z_{02} \in \mathcal{X}_\eta.$$

Thus we find

$$\|S_\eta(\cdot)w_{02} - S_\eta(\cdot)w_{01}\|_{L^2(0,t-1;V_{div}\times V)}^2 + \|\partial_t S_\eta(\cdot)w_{02} - \partial_t S_\eta(\cdot)w_{01}\|_{L^2(0,t-1;V'_{div}\times D(B_N)')}^2$$
$$\leq C_1 e^{C_1 t}\rho_{\mathcal{X}_\eta}^2(w_{02}, w_{01}), \qquad \forall t \geq 1, \qquad \forall w_{01}, w_{02} \in \mathbb{B}, \qquad (5.35)$$

where, as pointed out above, the constant C_1 does not depend on w_{01}, w_{02}.

Let us now introduce the following spaces

$$\mathcal{H} := \mathcal{X}_\eta := G_{div} \times \mathcal{Y}_\eta,$$
$$\mathcal{V}_1 := L^2(0, T; V_{div} \times V) \cap H^1(0, T; V'_{div} \times D(B_N)'),$$
$$\mathcal{V} := L^2(0, T; G_{div} \times H),$$

with $T > 0$ fixed such that $C_1 e^{-k(T+1)/2} < 1/2$, where C_1 and k are the same constants that appear in the first term on the right hand side of (5.34). Notice that, due to the Aubin-Lions lemma, $\mathcal{V}_1$ is compactly embedded into $\mathcal{V}$.

Then, take $\mathcal{S} := S_\eta(T)$ and define a map $\mathcal{T} : \mathbb{B} \to \mathcal{V}_1$ in the following way: for every $w_0 \in \mathbb{B}$ we set $\mathcal{T}w_0 := w := S_\eta(\cdot)w_0$, i.e., $w \in \mathcal{V}_1$ is the (strong) solution corresponding to the initial datum w_0.

It is now easy to see that choosing the spaces $\mathcal{H}, \mathcal{V}, \mathcal{V}_1$, the set $\mathbb{B}$, and the maps $\mathcal{S}, \mathcal{T}$ as above, then the conditions of Proposition 2 are satisfied. Indeed, (5.31) and (5.32) follow from (5.17) and (5.35), respectively, both written for $t = T + 1$...

Therefore, Proposition 2 entails the existence of a (discrete) exponential attractor $\mathcal{M}_\eta^d \subset \mathbb{B}$ for the (time discrete) semigroup $\{\mathcal{S}^n\}_{n=0,1,2,...}$ on $\mathbb{B}$ (with the topology of $\mathcal{H}$ induced on $\mathbb{B}$). Since $\mathbb{B}$ is absorbing in $\mathcal{H}$, then the basin of attraction of $\mathcal{M}_\eta^d$ is the whole phase space $\mathcal{H}$.

In order to prove the existence of the exponential attractor $\mathcal{M}_\eta$ for $(\mathcal{X}_\eta, S_\eta(t))$ with continuous time we observe first that (5.28) written with $\tau = 1$ (the time T is chosen as above) yields

$$\rho_{\mathcal{X}_\eta}(S_\eta(t_2 - 1)w_{02}, S_\eta(t_1 - 1)w_{01}) \le C_{1,T}\big(\rho_{\mathcal{X}_\eta}(w_{02}, w_{01}) + |t_2 - t_1|^{1/2}\big),$$

for all $w_{01}, w_{02} \in \mathbb{B}$ and for all $t_1, t_2 \in [1, 1 + T]$. Hence

$$\rho_{\mathcal{X}_\eta}(S_\eta(t'')w_{02}, S_\eta(t')w_{01}) \le C_{1,T}\big(\rho_{\mathcal{X}_\eta}(w_{02}, w_{01}) + |t'' - t'|^{1/2}\big),$$

for all $w_{01}, w_{02} \in \mathbb{B}$ and for all $t'', t' \in [0, T]$. Therefore, the map $[t, z] \mapsto S_\eta(t)z$ is uniformly Hölder continuous (with exponent $1/2$) on $[0, T] \times \mathbb{B}$, where $\mathbb{B}$ is endowed with the $\mathcal{H}-$metric. Therefore, the exponential attractor $\mathcal{M}_\eta$ for the continuous time case can be obtained by the classical expression

$$\mathcal{M}_\eta = \bigcup_{t \in [0,T]} S_\eta(t)\mathcal{M}_\eta^d,$$

and this concludes the proof of the theorem. $\qquad\qquad\square$

We conclude by proving a the existence of exponential attractors when the viscosity ν is not constant and satisfies the assumption (4.5) in Remark 4. In view of Theorems 5 and 6 we can define a dynamical system for the strong solutions. Indeed, taking $d = 2$ and $h \in G_{div}$, we have that for every fixed $\eta \ge 0$ system (1.1)–(1.5) generates a semigroup $\{Z_\eta(t)\}_{t \ge 0}$ of *closed* operators on the metric space $\mathcal{K}_\eta$ given by

$$\mathcal{K}_\eta := V_{div} \times \{\varphi \in H^2(\Omega) : |\overline{\varphi}| \le \eta\},$$

endowed with the (weaker) metric

$$\varrho(z_2, z_1) = \|u_2 - u_1\| + \|\varphi_2 - \varphi_1\|, \quad \forall z_i := [u_i, \varphi_i] \in \mathcal{K}_\eta, \; i = 1, 2.$$

We are now ready to state and prove the following.

Theorem 9. *Assume (H1), (H3)-(H5) and (4.5). Consider either $J \in W^{2,1}(B_\delta)$ or J admissible. The dynamical system $(\mathcal{K}_\eta, Z_\eta(t))$ possesses an exponential attractor $\mathcal{E}_\eta$ which is bounded in $V_{div} \times H^2(\Omega)$ such that the following properties are satisfied:*

■ *positive invariance: $Z_\eta(t)\mathcal{E}_\eta \subseteq \mathcal{E}_\eta$ for all $t \ge 0$;*

- *finite dimensionality:* $dim_F(\mathcal{E}_\eta, G_{div} \times H) < \infty$;

- *exponential attraction:* $\exists Q : \mathbb{R}^+ \to \mathbb{R}^+$ *increasing and* $\kappa > 0$ *such that, for all* $R > 0$ *and for all* $\mathcal{B} \subset \mathcal{K}_\eta$ *with* $\sup_{z \in \mathcal{B}} \rho(z, 0) \leq R$ *there holds*

$$dist_{\mathcal{K}_\eta}(Z_\eta(t)\mathcal{B}, \mathcal{E}_\eta) \leq Q(R)e^{-\kappa t}, \qquad \forall t \geq 0.$$

Proof. Step 1. We will briefly show that a dissipative estimate like (5.4) still holds for the strong solution of (1.1)–(1.5) under the assumptions of the theorem. More precisely, the following estimate holds

$$\|\nabla u(t)\| + \|\varphi(t)\|_{H^2(\Omega)} + \int_t^{t+1} \|u(s)\|^2_{H^2(\Omega)^2} ds \leq \Lambda(\eta), \qquad \forall t \geq t_*, \qquad (5.36)$$

for some positive constant Λ independent of the initial data and time, and some time $t_* > 0$ which depends only $\mathcal{E}(z_0)$. In order to get this estimate, first we recall estimate (2.7) by Theorem 1 which also holds for nonconstant viscosity. The proof of (5.36) follows immediately from the proof of Theorem 5. Indeed, we observe preliminarily that (5.5) and (5.7) already hold uniformly with respect to time and initial data in the nonconstant case, i.e., there exists a time $t_\# > 0$, depending only on $\mathcal{E}(z_0)$, such that

$$\varphi \in L^\infty(t_\#, \infty; L^\infty(\Omega) \cap V) \cap W^{1,2}(t_\#, \infty; H). \qquad (5.37)$$

In particular, this regularity allows us to obtain $\mu \in L^\infty(t_\#, \infty; L^\infty(\Omega) \cap V)$ and $l \in L^2(t_\#, \infty; L^2(\Omega)^2)$ uniformly. This can be done by arguing exactly in the same fashion as in the derivation of estimates (4.7)-(4.13), with the exception that the constant $R > 0$ is such that $\operatorname{ess\,sup}_{t \in (t_\#, \infty)} \|\varphi(t)\|_{L^\infty} \leq R$. Then, we can employ the same procedure as in (4.16)-(4.26) (with a function $Q = Q(R) > 0$ which is now independent of the initial data, by (5.37)) to deduce by virtue of the uniform Gronwall lemma (see [28, Chapter III, Lemma 1.1]) that

$$u \in L^\infty(t_*, \infty; V_{div}) \cap L^2(t_*, \infty; H^2(\Omega)^2), \quad u_t \in L^2(t_*, \infty; G_{div}),$$

for some $t_* \geq 1$ depending only on $t_\#$. Finally, arguing exactly as in the proof of Theorem 5 we deduce $\varphi \in L^\infty(t_*, \infty; H^2(\Omega))$ uniformly with respect to time and the data. Note that estimate (5.36) entails the existence of a bounded absorbing set $\mathcal{B}_2 \subset \mathcal{K}_\eta$ for the semigroup $Z_\eta(t)$.

Step 2. As in the proof of Theorem 8, it will be sufficient to construct the exponential attractor for the restriction of $Z_\eta(t)$ on this set $\mathcal{B}_2$. Thus, it suffices to verify the validity of Lemmas 3 and 4 for the difference $u = u_1 - u_2$, $\varphi = \varphi_1 - \varphi_2$, where (u_i, φ_i) is a (given) strong solution and $i = 1, 2$. The first one is an immediate consequence of estimate (4.33) (see the proof of Theorem 6) and the application of Poincaré-type inequalities (5.18) (see the proof of Lemma 3). Indeed, in the nonconstant case we have

$$\|u(t)\|^2 + \|\varphi(t)\|^2$$
$$\leq C(\|u(\tau))\|^2 + \|\varphi(\tau)\|^2)e^{-kt} + C \int_\tau^t (\|u(s)\|^2 + \|\varphi(s)\|^2)ds, \qquad \forall t \geq \tau, \qquad (5.38)$$

29

for some constant $C = C_\tau > 0$, where $(u_i(\tau), \varphi_i(\tau)) \in \mathcal{B}_2$ for each $i = 1, 2$. For the second one, we observe that in order to estimate $u_t := u_{2,t} - u_{1,t}$, we have

$$
\begin{aligned}
\langle u_t, v \rangle = {} &-(\nu(\varphi_2)\nabla u, \nabla v) - ((\nu(\varphi_1) - \nu(\varphi_2))\nabla u_1, \nabla v) \\
&- b(u_2, u_2, v) + b(u_1, u_1, v) \\
&- \frac{1}{2}(\nabla a\varphi(\varphi_1 + \varphi_2), v) - ((J * \varphi)\nabla\varphi_2, v) - ((J * \varphi_2)\nabla\varphi, v),
\end{aligned}
\tag{5.39}
$$

for all $v \in W := H^{2+\varepsilon}(\Omega)^2 \cap V_{div}$ and some $\varepsilon > 0$ (such that the embedding $H^{2+\varepsilon} \subset W^{1,\infty}$ holds). While all the terms on the right-hand side of (5.39), with the exception of the first two, can be word by word estimated exactly as in the proof of Lemma 4, we notice that assumption (4.5) and the essential L^∞-bound on φ yield

$$
|(\nu(\varphi_2)\nabla u, \nabla v)| \leq C \|\nabla u\| \|\nabla v\|,
$$
$$
|((\nu(\varphi_1) - \nu(\varphi_2))\nabla u_1, \nabla v)| \leq C \|\varphi\| \|\nabla u_1\| \|v\|_{H^{2+\varepsilon}}.
$$

Thus, we easily get

$$
\|u_t\|_{W'} \leq C(\|\nabla u\| + \|\varphi\|), \quad \forall t \geq \tau,
$$

which together with (4.33) and (5.24) yields the following estimate

$$
\|u_t(t)\|^2_{L^2(\tau,t;W')} + \|\varphi_t\|^2_{L^2(\tau,t;D(B_N)')} \leq Ce^{Ct}(\|u(\tau)\|^2 + \|\varphi(\tau)\|^2), \quad \forall t \geq \tau.
\tag{5.40}
$$

Estimates (5.38) and (5.40) convey that a certain smoothing property holds for the difference of any two strong solutions associated with any two given initial data in $\mathcal{B}_2$.

Step 3. It is now not difficult to finish the proof of the theorem, using the abstract scheme of Proposition 2 by arguing in a similar fashion as in the proof of Theorem 8. The differences are quite minor and so we leave them to the interested reader. $\qquad\square$

Remark 7. *On account of [13, Proofs of Proposition 1 and Lemma 3] and* (4.21)*, using uniform Gronwall's lemma (see [28, Chapter III, Lemma 1.1]), it is possible to show that any weak solution becomes a strong solution in finite time. We remind that this property is based on the validity of the energy identity* (2.6)*. Indeed, estimate* (5.36) *ensures that, given a weak trajectory z starting from $z_0 \in \mathcal{X}_\eta$ (cf.* (5.1)*), there exists a time $t^* = t^*(z_0) \geq 0$ such that $z(t) \in \mathcal{B}_1(\Lambda(\eta))$ for all $t \geq t^*$, where $\mathcal{B}_1(\Lambda(\eta))$ is the closed ball in the space $V_{div} \times H^2(\Omega)$ with radius $\Lambda(\eta)$ and constraint $|\overline{\varphi}| \leq \eta$. Let us briefly mention some consequences of this property. First, the global attractor of the generalized semiflow on $\mathcal{X}_\eta$ generated by the problem with nonconstant viscosity (see [12]) is bounded in $V_{div} \times H^2(\Omega)$. Therefore we can show the validity of a smoothing property (cf.* (5.38) *and* (5.40)*) on the global attractor and deduce that it has finite fractal dimension. Moreover, the regularizing effect also allows to prove the precompactness of (weak) trajectories (see [13, Lemma 3]). This is an essential ingredient to establish the convergence of a weak solution to a single equilibrium which can be done along the lines of [13, Section 5].*

References

[1] H. Abels, *On a diffusive interface model for two-phase flows of viscous, incompressible fluids with matched densities*, Arch. Ration. Mech. Anal. **194** (2009), 463-506.

[2] H. Abels, *Longtime behavior of solutions of a Navier-Stokes/Cahn-Hilliard system*, Proceedings of the Conference "Nonlocal and Abstract Parabolic Equations and their Applications", Bedlewo, Banach Center Publ. **86** (2009), 9-19.

[3] J.M. Ball, *Continuity properties and global attractors of generalized semiflows and the Navier-Stokes equation*, J. Nonlinear Sci. **7** (1997), 475-502 (Erratum, J. Nonlinear Sci. **8** (1998), 233).

[4] P.W. Bates, J. Han, *The Neumann boundary problem for a nonlocal Cahn-Hilliard equation*, J. Differential Equations **212** (2005), 235-277.

[5] J. Bedrossian, N. Rodríguez, A. Bertozzi, *Local and global well-posedness for an aggregation equation and Patlak-Keller-Segel models with degenerate diffusion*, Nonlinearity **24** (2011), 1683-1714.

[6] F. Boyer, *Mathematical study of multi-phase flow under shear through order parameter formulation*, Asymptot. Anal. **20** (1999), 175-212.

[7] C. Cao, C.G. Gal, *Global solutions for the 2D NS-CH model for a two-phase flow of viscous, incompressible fluids with mixed partial viscosity and mobility*, Nonlinearity **25** (2012), 3211-3234.

[8] P. Colli, S. Frigeri, M. Grasselli, *Global existence of weak solutions to a nonlocal Cahn-Hilliard-Navier-Stokes system*, J. Math. Anal. Appl. **386** (2012), 428-444.

[9] M. Efendiev, A. Miranville, S. Zelik, *Exponential attractors for a nonlinear reaction-diffusion system in $\mathbb{R}^3$*, C. R. Acad. Sci. Paris, Ser. I **330** (2000), 713-718.

[10] M. Efendiev, S. Zelik, *Finite-dimensional attractors and exponential attractors for degenerate doubly nonlinear equations*, Math. Methods Appl. Sci. **32** (2009), 1638-1668.

[11] S. Frigeri, M. Grasselli, *Global and trajectories attractors for a nonlocal Cahn-Hilliard-Navier-Stokes system*, J. Dynam. Differential Equations **24** (2012), 827-856.

[12] S. Frigeri, M. Grasselli, *Nonlocal Cahn-Hilliard-Navier-Stokes systems with singular potentials*, Dyn. Partial Differ. Equ. **9** (2012), 273-304.

[13] S. Frigeri, M. Grasselli, P. Krejčí, *Strong solutions for two-dimensional nonlocal Cahn-Hilliard-Navier-Stokes systems*, J. Differential Equations **255** (2013), 2597-2614.

[14] S. Frigeri, M. Grasselli, E. Rocca, *A diffuse interface model for two-phase incompressible flows with nonlocal interactions and nonconstant mobility*, arXiv 1303.6446 (2013).

[15] C.G. Gal, M. Grasselli, *Asymptotic behavior of a Cahn-Hilliard-Navier-Stokes system in 2D*, Ann. Inst. H. Poincaré Anal. Non Linéaire **27** (2010), 401-436.

[16] C.G. Gal, M. Grasselli, *Trajectory attractors for binary fluid mixtures in 3D*, Chinese Ann. Math. Ser. B **31** (2010), 655-678.

[17] C.G. Gal, M. Grasselli, *Instability of two-phase flows: a lower bound on the dimension of the global attractor of the Cahn-Hilliard-Navier-Stokes system,* Phys. D **240** (2011), 629-635.

[18] C.G. Gal, M. Grasselli, *Longtime behavior of nonlocal Cahn-Hilliard equations*, Discrete Contin. Dyn. Syst. Ser. A **34** (2014), 145-179.

[19] H. Gajewski, K. Zacharias, *On a nonlocal phase separation model*, J. Math. Anal. Appl. **286** (2003), 11-31.

[20] G. Giacomin, J.L. Lebowitz, *Exact macroscopic description of phase segregation in model alloys with long range interactions*, Phys. Rev. Lett. **76** (1996), 1094-1097.

[21] G. Giacomin, J.L. Lebowitz, *Phase segregation dynamics in particle systems with long range interactions. I. Macroscopic limits*, J. Statist. Phys. **87** (1997), 37-61.

[22] G. Giacomin, J.L. Lebowitz, *Phase segregation dynamics in particle systems with long range interactions. II. Phase motion*, SIAM J. Appl. Math. **58** (1998), 1707-1729.

[23] C. Liu, J. Shen, *A phase field model for the mixture of two incompressible fluids and its approximation by a Fourier spectral method*, Phys. D **179** (2003), 211-228.

[24] S.-O. Londen, H. Petzeltová, *Convergence of solutions of a non-local phase-field system*, Discrete Contin. Dyn. Syst. Ser. S **4** (2011), 653-670.

[25] S.-O. Londen, H. Petzeltová, *Regularity and separation from potential barriers for a non-local phase-field system*, J. Math. Anal. Appl. **379** (2011), 724-735.

[26] J.S. Kim, *Phase-field models for multi-component fluid flows*, Commun. Comput. Phys. **12** (2012), 613-661.

[27] V.N. Starovoitov, *The dynamics of a two-component fluid in the presence of capillary forces*, Math. Notes **62** (1997), 244-254.

[28] R. Temam, *Navier-Stokes equations and nonlinear functional analysis*, Second edition, CBMS-NSF Reg. Conf. Ser. Appl. Math., **66**, SIAM, Philadelphia, PA, 1995.

[29] L. Zhao, H. Wu, H. Huang, *Convergence to equilibrium for a phase-field model for the mixture of two viscous incompressible fluids*, Commun. Math. Sci. **7** (2009), 939-962.