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Fluctuations near the limit shape of random permutations under a conservative measure

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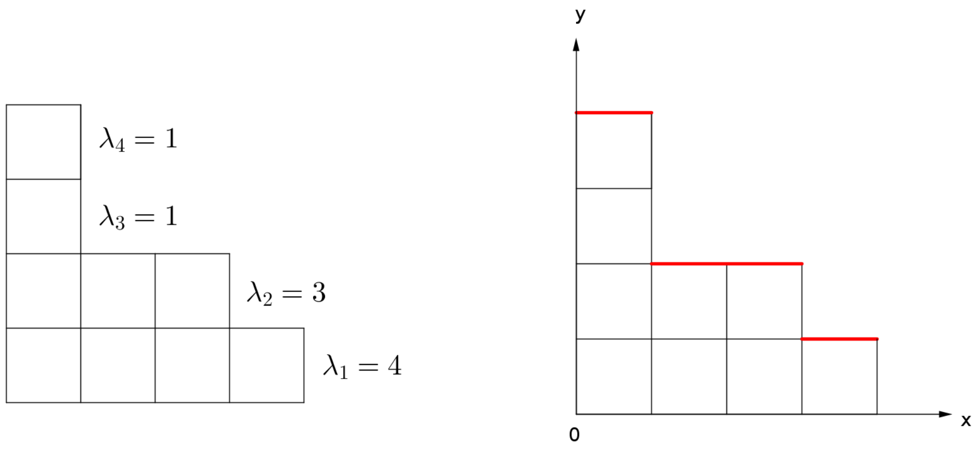
ABSTRACT. In this work we are considering the behavior of the limit shape of Young diagrams associated to random permutations on the set $\{1, \ldots, n\}$ under a particular class of multiplicative measures. Our method is based on generating functions and complex analysis (saddle point method). We show that fluctuations near a point behave like a normal random variable and that the joint fluctuations at different points of the limiting shape have an unexpected dependence structure. We will also compare our approach with the so-called randomization of the cycle counts of permutations and we will study the convergence of the limit shape to a continuous stochastic process.

CONTENTS

1. Introduction

The aim of this paper is to study the limit shape of a random permutation under the generalised Ewens measure with polynomial growing cycle weights and the fluctuations at each point of the limit shape. The study of such objects has a long history, which started with the papers of Temperley [24] and Vershik [25]. Later on Young diagrams have been approached under a different direction, as in the independent works of [26] and [20], which first derived the limit shape when the underpinned measure on partitions is the so-called *Plancherel measure*. We will not handle this approach here, even though it presents remarkable connections with random matrix theory and random polymers, among others (see for example [10]).

We first specify what we define as the limit shape of a permutation. We denote by $\mathfrak{S}_n$ the set of permutations on n elements and write each permutation $\sigma \in \mathfrak{S}_n$ as $\sigma = \sigma_1 \cdots \sigma_\ell$ with σ_j disjoint cycles of length λ_j. Disjoint cycles commute and we thus can assume $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_\ell$. This assigns to each permutation $\sigma \in \mathfrak{S}_n$ in a unique way a partition of n and this partition $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_\ell)$ is called the *cycle type* of σ. We will indicate that λ is such a partition with the notation $\lambda \vdash n$. We define the size $|\lambda| := \sum_i \lambda_i$ (so obviously if $\lambda \vdash n$ then $|\lambda| = n$). λ features a nice geometric visualisation by its Young diagram Υ_λ. This is a left- and bottom-justified diagram of ℓ rows with the $j-$th row consisting of λ_j squares, see Figure 1(a). It is clear that the area of Υ_λ is n if $\lambda \vdash n$. After introducing a coordinate system as



(a) The Young diagram (b) The shape function $w_n(\cdot)$

FIGURE 1. Illustration of the Young diagram and the shape of $\sigma = (3578)(129)(4)(6) \in \mathfrak{S}_9$

in Figure 1(b), we see that the upper boundary of a Young diagram Υ_λ is a piecewise constant and right continuous function $w_n : \mathbb{R}^+ \to \mathbb{N}^+$ with

$$w_n(x) := \sum_{j=1}^{n} \mathbb{1}_{\{\lambda_j \geq x\}} \tag{1.1}$$

with the convention that $C_0 := 0$. The cycle type of a permutation becomes a random partition if we endow the space $\mathfrak{S}_n$ with a probability measure $\mathbb{P}_n$. What we are then interested in studying is the now random shape $w_n(\cdot)$ as $n \to +\infty$, and more specifically to determine its limit shape. The limit shape with respect to a sequence of probability measures $\mathbb{P}_n$ on $\mathfrak{S}_n$ (and sequences of positive real numbers A_n and B_n with $A_n \cdot B_n = n$) is understood as a function $w_\infty : \mathbb{R}^+ \to \mathbb{R}^+$ such that for each $\epsilon, \delta > 0$

$$\lim_{n\to+\infty} \mathbb{P}_n \left[\left\{ \sigma \in \mathfrak{S}_n : \sup_{x\geq\delta} |A_n^{-1} w_n(B_n x) - w_\infty(x)| \leq \epsilon \right\} \right] = 1. \tag{1.2}$$

The assumption $A_n \cdot B_n = n$ ensures that the area under the rescaled Young diagram is 1. One of the most frequent choices is $A_n = B_n = n^{1/2}$, but we will see that it's useful to adjust the choice of A_n and B_n to the measures $\mathbb{P}_n$. Equation (1.2) can be viewed as a law of large numbers for the process $w_n(\cdot)$. The next natural question is then whether fluctuations satisfy a central limit theorem, namely whether

$$A_n w_n(B_n x) - w_\infty(x)$$

converges (after centering and normalization) in distribution to a Gaussian process on the space of càdlàg functions, for example. Of course the role of the probability distribution with which we equip the set of partitions will be crucial to this end.

In this paper, we work with the following measure on $\mathfrak{S}_n$:

$$\mathbb{P}_n[\sigma] = \frac{1}{h_n n!} \prod_{j=1}^{\ell} \vartheta_{\lambda_j}. \tag{1.3}$$

where $(\lambda_1, \ldots, \lambda_\ell)$ is the cycle type of σ, $(\vartheta_m)_{m\geq1}$ is a sequence of non-negative weights and h_n is a normalization constant (h_0 is defined to be 1). From time to time we will also use $\vartheta_0 := 0$ introduced as convention.

This measure has recently appeared in mathematical physics for a model of the quantum gas in statistical mechanics and has a possible connection with the Bose-Einstein condensation (see e.g. [6] and [12]). Classical cases of this measure are the uniform measure ($\vartheta_m \equiv 1$) and the Ewens measure ($\vartheta_m \equiv \vartheta$). The uniform measure is well studied and has a long history (see e.g. the first chapter of [3] for a detailed account with references). The Ewens measure originally appeared in population genetics, see [14], but has also various practical applications through its connection with Kingman's coalescent process, see [18]. The measure in (1.3) also has some similarities to multiplicative measure for partitions, see for instance [8]. It is clear that we have to make some assumptions on the sequence $(\vartheta_m)_{m\geq1}$ to be able study the behaviour as $n \to +\infty$. We use in this paper polynomial growing cycle weights ϑ_m as considered in the recent work Ercolani and Ueltschi [12] and of Maples, Nikeghbali and Zeindler [22]. More precisely, we use the weights

$$\vartheta_m = \frac{m^\alpha}{\Gamma(\alpha+1)} + O\left(m^\beta\right) \tag{1.4}$$

with some $\alpha > 0$ and $0 \leq \beta < \alpha/2$. We would like to point out that the requirement $0 \leq \beta < \alpha/2$ and the normalisation constant $\Gamma(\alpha+1)$ are not essential and it only simplifies the notation and the computations. Our argumentation indeed works also for $\vartheta_m \sim$ const $\cdot\, m^\alpha$. Note that the limit shape and the fluctuations at points of the limit shape with the weights (1.4) have already been studied by Erlihson and Granovsky in [13] in the context of Gibbs distributions on integer partitions. However, the approach in this paper is slightly different and allows to simplify the computations and to get at the same time the behaviour of the cumulants and to give some large deviation estimates for the fluctuations at the limit shape. We may mention two popular methods in the literature to study the asymptotic behaviour of the function $w_n(x)$ under such assumptions. The first one is is complex analytic and uses the saddle-point method as described in Section 4. This method was used in [12] and [22] and an introduction can be found for instance in [15, Section VIII]. The second one is stochastic and based on randomisation and was used in [8] and [13]. We present in Section 3 an argumentation similar to [8] and give at the begin of Section 3 an idea how to deduce the behaviour for $n \to \infty$ from the randomised setting. It is typically expected that the 'randomised' $w_n(x)$ has the same asymptotic behaviour as the 'unrandomized' $w_n(x)$. We will see here that this is not the case. More precisely, we show that the 'randomized' and the 'unrandomized' $w_n(x)$ have different limit shapes and different behaviours of the fluctuations around the limit shape.

Notation. We denote with $\mathbb{R}^+ := \mathbb{R} \setminus (-\infty, 0)$ and with $\mathbb{N}^+ := \mathbb{N} \setminus \{0\}$. With bold fonts we will always indicate vectors and $\langle \cdot, \cdot \rangle$ will be the Euclidean scalar product on $\mathbb{R}^\ell$. $M_\ell(\mathbb{R})$ is the set of $\ell \times \ell$ matrices with real coefficients. $\mathcal{L}(X)$ stands for the law of a random variable X and thus $\xrightarrow{\mathcal{L}}$ is the symbol representing convergence in distribution.

2. Preliminaries

We introduce in this section the notation of the cycle counts and the notation of generating functions.

2.1. **Cycle counts.** The notation $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_\ell)$ is very useful for the illustration of λ via its Young diagram, but in the computations it is better to work with the *cycle counts* C_k. These are defined as

$$C_k(\sigma) = C_k := \#\left\{j \geq 1; \lambda_j = k\right\} \tag{2.1}$$

for $k \geq 1$ and $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_\ell)$ the cycle type of $\sigma \in \mathfrak{S}_n$. We obviously have for $k \geq 1$

$$C_k \geq 0 \ \text{ and } \ \sum_{k=1}^{n} kC_k = n. \tag{2.2}$$

It is also clear that the cycle type of permutation (or a partition) is uniquely determined by the vector $(C_1, C_2, \dots)$. The function $w_n(x)$ and the measure $\mathbb{P}_n[\,\cdot\,]$ in (1.1) and (1.3) can now be written as

$$w_n(x) = \sum_{k \geq x} C_k \text{ and } \mathbb{P}_n[\sigma] = \frac{1}{h_n n!} \prod_{k=1}^{n} \vartheta_k^{C_k}. \tag{2.3}$$

Our aim is to study the behaviour of $w_n(x)$ as $n \to \infty$. It is thus natural to consider the asymptotic behaviour of C_k with respect to the measure $\mathbb{P}_n[\,\cdot\,]$.

Lemma 2.1 ([12], Corollary 2.3)**.** *Under the condition $h_{n-1}/h_n \to 1$ the random variables C_k converge for each $k \in \mathbb{N}$ in distribution to a Poisson distributed random variable Y_k with $\mathbb{E}[Y_k] = \frac{\vartheta_k}{k}$. More generally for all $b \in \mathbb{N}$ the following limit in distribution holds:*

$$\lim_{n \to +\infty} (C_1, C_2 \dots, C_b) = (Y_1, Y_2 \dots, Y_b)$$

with Y_k independent Poisson random variables with mean $\mathbb{E}[Y_k] = \frac{\vartheta_k}{k}$.

One might expect at this point that $w_n(x)$ is close to $\sum_{k=x}^{n} Y_k$. Unfortunately we will see in Section 4 that the asymptotic behaviour of $w_n(x)$ is more complicate.

2.2. **Generating functions.** The (ordinary) generating function of a sequence $(a_k)_{k \geq 0}$ of complex numbers is defined as the formal power series

$$g(z) := \sum_{j=0}^{\infty} a_k z^k. \tag{2.4}$$

As usual, we define the *extraction symbol* $[z^k]\, g(z) := a_k$, that is, as the coefficient of z^k in the power series expansion (2.4) of $g(z)$.

A generating function that plays an important role in this paper is

$$g_\Theta(t) := \sum_{m \geq 1} \frac{\vartheta_m}{m} t^m. \tag{2.5}$$

As mentioned in the introduction, we will use $\vartheta_m = \frac{m^\alpha}{\Gamma(\alpha+1)} + O\left(m^\beta\right)$. We stress that generating functions of the type $(1-t)^{-\alpha}$ fall also in this category, and for them we will recover the limiting shape as previously done in [13]. We will see in particular this case in Section 4.
The reason why generating functions are useful is that it is often possible to write down a generating function without knowing a_n explicitly. In this case one can try to use tools from analysis to extract information about a_n, for large n, from the generating function. It should be noted that there are several variants in the definition of generating functions. However, we will use only the ordinary generating function and thus call it 'just' generating function without risk of confusion.

The following well-known identity is a special case of the general *Pólya's Enumeration Theorem* [23, 16, p. 17] and is the main tool in this paper to obtain generating functions.

Lemma 2.2. *Let $(a_m)_{m\in\mathbb{N}}$ be a sequence of complex numbers. We then have as formal power series in t*

$$\sum_{n\in\mathbb{N}}\frac{t^n}{n!}\sum_{\sigma\in\mathfrak{S}_n}\prod_{j=1}^{n}a_j^{C_j}=\sum_{n\in\mathbb{N}}t^n\sum_{\lambda\vdash n}\frac{1}{z_\lambda}\prod_{k=1}^{\infty}a_k^{C_k}=\exp\left(\sum_{m\geq 1}\frac{a_m}{m}t^m\right)$$

where $z_\lambda:=\prod_{k=1}^{n}k^{C_k}C_k!$. If one series converges absolutely, so do the others.

We omit the proof of this lemma, but details can be found for instance in [21, p. 5].

2.3. **Approximation of sums.** We require for our argumentation the asymptotic behaviour of the generating function $g_\Theta(t)$ as t tends to the radius of convergence, which is 1 in our case.

Lemma 2.3. *Let $(v_n)_{n\in\mathbb{N}}$ a sequence of postive numbers with $v_n\downarrow 0$ as $n\to+\infty$. We have for all $\delta\in\mathbb{R}\setminus\{-1,-2,-3,\dots\}$*

$$\sum_{k=1}^{\infty}k^\delta e^{-kv_n}=\Gamma(\delta+1)v_n^{-\delta-1}+\zeta(-\delta)+O(v_n). \tag{2.6}$$

$\zeta(\cdot)$ indicates the Riemann Zeta function.

This lemma can proven with Euler Maclaurin summation formula or with the Mellin transformation. The computations with Euler Maclaurin summation are straightforward and the details of the proof with the Mellin transformation can be found for instance in [15, Chapter VI.8]. We thus omit it.

We require also the behaviour of partial sum $\sum_{k=x}^{\infty}\frac{\theta_m}{m}t^m$ as $x\to\infty$ and as $t\to 1$. We have

Lemma 2.4 (Approximation of sums)**.** *Let v_n, z_n be given with $z_n\to+\infty$ and $z_nv_n=a_0+a_1n^{-\beta}$ for $\beta>0, a_0>0$ and $a_0,a_1\in\mathbb{R}$. We then have for all $\delta\in\mathbb{R}$ and all $q\in\mathbb{N}$*

$$\begin{aligned}\sum_{k=\lfloor z_n\rfloor}^{\infty}k^\delta e^{-kv_n}=&\left(\frac{z_n}{a_0}\right)^{\delta+1}\left(\sum_{k=0}^{q}\frac{\Gamma(\delta+k+1,a_0)}{k!}\left(-\frac{a_1}{a_0}n^{-\beta}\right)^k+O\left(n^{-(q+1)\beta}\right)\right)\\&-B_1(z_n-\lfloor z_n\rfloor)z_n^\delta e^{-a_0}+\int_{z_n}^{+\infty}B_1(y-\lfloor y\rfloor)(\delta-v_ny)y^{\delta-1}e^{-v_ny}\,\mathrm{d}y.\end{aligned}$$

with $\Gamma(a,x):=\int_x^{+\infty}s^{a-1}e^{-s}\mathrm{d}s$ the incomplete Gamma function and $B_1(x):=x-\frac{1}{2}$ the first Bernoulli polynomial.

Proof. The proof of this lemma is based on the Euler Maclaurin summation formula, see [2] or [1, Theorem 3.1]. We use the here the following version: let $f:\mathbb{R}^+\to\mathbb{R}$ have a continuous derivative and suppose that f and f' are integrable. Then

$$\sum_{k\geq\lfloor c\rfloor}f(k)=\int_c^{+\infty}f(x)\,\mathrm{d}x-B_1(c-\lfloor c\rfloor)f(c)+\int_c^{+\infty}B_1(x-\lfloor x\rfloor)f'(x)\,\mathrm{d}x. \tag{2.7}$$

We substitute $f(x) := x^{\delta} e^{-xv_n}$, $c := z_n$ and notice that f and all its derivatives tend to zero exponentially fast as $x \longrightarrow +\infty$. Now by the change of variables $x := \frac{z_n}{a_0} y$

$$\int_{z_n}^{+\infty} e^{-v_n x} x^{\delta} \mathrm{d}x = \left(\frac{z_n}{a_0}\right)^{\delta+1} \int_{a_0}^{+\infty} y^{\delta} e^{-y} e^{-\frac{a_1}{a_0} n^{-\beta} y} \mathrm{d}y =$$

$$= \left(\frac{z_n}{a_0}\right)^{\delta+1} \left(\sum_{k=0}^{q} \frac{\Gamma(\delta+k+1, a_0)}{k!} \left(-\frac{a_1}{a_0} n^{-\beta}\right)^k + O\left(n^{-(q+1)\beta}\right)\right) \tag{2.8}$$

where we have swapped integral and series expansion of the exponential by Fubini's theorem. □

Remark. One can obtain more error terms in the expansion in Lemma 2.4 by using more terms in the Euler Maclaurin summation formula. We have stated in Appendix A a version of the Euler Maclaurin summation formula with non-integer boundaries, which is more suitable for this than the usual one.

3. Randomization

We introduce in this section a probability measure $\mathbb{P}_t\left[\,\cdot\,\right]$ on $\dot\bigcup_{n\geq 1} \mathfrak{S}_n$, where $\dot\bigcup$ denotes the disjoint union, dependent on a parameter $t > 0$ with $\mathbb{P}_t\left[\,\cdot\,|\mathfrak{S}_n\right] = \mathbb{P}_n\left[\,\cdot\,\right]$ and consider the asymptotic behaviour of $w_n(x)$ with respect to $\mathbb{P}_t\left[\,\cdot\,\right]$ as $t \longrightarrow 1$.

3.1. **Grand canonical ensemble.** Computations on $\mathfrak{S}_n$ can turn out to be difficult and many formulas can not be used to study the behaviour as $n \longrightarrow \infty$. A possible solution to this problem is to adopt to a suitable randomization. This has been successfully introduced by [16] and used also by [8] as a tool to investigate combinatorial structures, and later applied in many contexts. The main idea of randomization is to define a one-parameter family of probability measures on $\dot\bigcup_{n\geq 1} \mathfrak{S}_n$ for which cycle counts turn out to be independent. Then one is able to study their behavior more easily, and ultimately the parameter is tuned in such a way that randomized functionals are distributed as in the non-randomized context. Let us see how to apply this in our work. We define

$$G_{\Theta}(t) = \exp\big(g_{\Theta}(t)\big) \tag{3.1}$$

with $g_{\Theta}(t)$ as in (2.5). If $G_{\Theta}(t)$ is finite for some $t > 0$, then for each $\sigma \in \mathfrak{S}_n$ let us define the probability measure

$$\mathbb{P}_t\left[\sigma\right] := \frac{1}{G_{\Theta}(t)} \frac{t^n}{n!} \prod_{k=1}^{n} \vartheta_k^{C_k}. \tag{3.2}$$

Lemma 2.2 shows that $\mathbb{P}_t$ is indeed a probability measure on $\dot\bigcup_{n\geq 1} \mathfrak{S}_n$. The induced distribution on cycle counts C_k can easily be determined.

Lemma 3.1. *Under $\mathbb{P}_t\left[\,\cdot\,\right]$ the C_k's are independent and Poisson distributed with*

$$\mathbb{E}_t\left[C_k\right] = \frac{\vartheta_k}{k} t^k.$$

Proof. From Pólya's enumeration theorem (Lemma 2.2) we obtain

$$\begin{aligned}
\mathbb{E}_t\left[e^{-sC_k}\right] &= \sum_{n\geq 0}\sum_{\sigma\in\mathfrak{S}_n} e^{-sC_k}\mathbb{P}_t\left[\sigma\right] = \frac{1}{G_\Theta(t)}\sum_{n\geq 0}\sum_{\sigma\in\mathfrak{S}_n}\frac{t^n}{n!}(\vartheta_k e^{-s})^{C_k}\prod_{\substack{j\leq n\\ j\neq k}}(\vartheta_j)^{C_j} \\
&= \frac{1}{G_\Theta(t)}\exp\left(\sum_{j=0}^{+\infty}\frac{\vartheta_j}{j}t^j\right)\exp\left(\left(e^{-s}-1\right)\frac{\vartheta_k}{k}t^k\right) \\
&= \exp\left(\left(e^{-s}-1\right)\frac{\vartheta_k}{k}t^k\right).
\end{aligned}$$

Analogously one proves the pairwise independence of cycle counts. □

Obviously the following conditioning relation holds:

$$\mathbb{P}_t\left[\,\cdot\,|\,\mathfrak{S}_n\right] = \mathbb{P}_n\left[\,\cdot\,\right].$$

A proof of this fact is easy and can be found for instance in [17, Equation (1)]. We note that $w_n(x)$ is $\mathbb{P}_t$-a.s. finite, since $\mathbb{E}_t\left[w_n(x)\right] < +\infty$. Now since the conditioning relation holds for all t with $G_\Theta(t) < +\infty$, one can try to look for t satisfying "$\mathbb{P}_n\left[\,\cdot\,\right] \approx \mathbb{P}_t\left[\,\cdot\,\right]$", which heuristically means that we choose a parameter for which permutations on $\mathfrak{S}_n$ weigh as most of the mass of the measure $\mathbb{P}_t$. We have on $\mathfrak{S}_n$

$$n = \sum_{j=1}^{\ell}\lambda_j = \sum_{k=1}^{n} kC_k.$$

A natural choice for t is thus the solution of

$$n = \mathbb{E}_t\left[\sum_{k=1}^{\infty} kC_k\right] = \sum_{k=1}^{\infty}\vartheta_k t^k. \tag{3.3}$$

which is guaranteed to exist if the series on the right-hand side is divergent at the radius of convergence (we will see this holds true for our particular choice of weights). We write $t = e^{-v_n}$ and use Lemma 2.3 in our case $\vartheta_k = \frac{k^\alpha + O\left(k^\beta\right)}{\Gamma(\alpha+1)}$ to obtain

$$n \overset{!}{=} (v_n)^{-\alpha-1} + O\big((v_n)^{-\beta-1}\big) \Longrightarrow v_n = (n^*)^{-1} + O\big((n^*)^{\beta-\alpha-1}\big) \tag{3.4}$$

with $n^* := n^{\frac{1}{1+\alpha}}$. We will fix this choice for the rest of the section.

3.2. **Limit shape and mod-convergence.** In order to derive our main results from the measure $\mathbb{P}_t$ we will use a tool developed by [19], the *mod-Poisson convergence*.

Definition 3.2. *A sequence of random variables $(Z_n)_{n\in\mathbb{N}}$ converges in the mod-Poisson sense with parameters $(\mu_n)_{n\in\mathbb{N}}$ if the following limit*

$$\lim_{n\to+\infty}\exp(\mu_n(1-e^{iu}))\mathbb{E}\left[e^{iuZ_n}\right] = \Phi(u)$$

exists for every $u \in \mathbb{R}$, and the convergence is locally uniform. The limiting function Φ is then continuous and $\Phi(0) = 1$.

This type of converge gives stronger results than a central limit theorem, indeed it implies a CLT (and other properties we will see below). Our goal will then be to prove the following

Proposition 3.3. *Let $x \geq 0$ be arbitrary and $x^* := xn^*$ with $n^* = n^{\frac{1}{1+\alpha}}$. Furthermore, let $t = e^{-v_n}$ with v_n as in* (3.4). *Then the random variables $(w_n(x^*))_{n\in\mathbb{N}}$ converge in the mod-Poisson sense with parameters $\mu_n = (n^*)^\alpha w^{\mathbf{r}}_\infty(x) + o\big((n^*)^{\alpha/2}\big)$, where*

$$w^{\mathbf{r}}_\infty(x) := \frac{\Gamma(\alpha,\, x)}{\Gamma(\alpha+1)}. \tag{3.5}$$

$\Gamma(\alpha,\, x)$ is the upper incomplete Gamma function.

Proof. We have

$$\mathbb{E}_t\left[e^{isw_n(x^*)}\right] = \mathbb{E}_t\left[e^{is\sum_{\ell=\lfloor x^*\rfloor}^{\infty} C_\ell}\right] = \exp\left(\left(e^{is}-1\right)\sum_{j=\lfloor x^*\rfloor}^{\infty}\frac{\vartheta_j}{j}t^j\right). \tag{3.6}$$

This is the characteristic function of Poisson distribution. We thus obviously have mod-Poisson convergence with limiting function $\Phi(t) \equiv 1$. It remains to compute the parameter μ_n. Applying Lemma 2.3 for $x = 0$ and Lemma 2.4 for $x > 0$ together with (3.4) gives

$$\sum_{j=\lfloor x^*\rfloor}^{+\infty}\frac{j^{\alpha-1}+O\left(j^{\beta-1}\right)}{\Gamma(\alpha+1)}t^j = (n^*)^\alpha\frac{\Gamma(\alpha,x)}{\Gamma(\alpha+1)} + O\big((n^*)^\beta\big). \tag{3.7}$$

Since $\beta < \alpha/2$ by assumption, we deduce that $\lambda_n := (n^*)^\alpha w^{\mathbf{r}}_\infty(x) + o\big((n^*)^{\alpha/2}\big)$. This completes the proof. ☐

This yields a number of interesting consequences. In first place we can prove a CLT and detect the limit shape accordingly.

Corollary 3.4 (CLT and limit shape for randomization)**.** *With the notation as above, we have as $n \to \infty$ with respect to $\mathbb{P}_t$*

$$\widetilde{w}^{\mathbf{r}}_n(x) := \frac{w_n(x^*) - (n^*)^\alpha w^{\mathbf{r}}_\infty(x)}{(n^*)^{\frac{\alpha}{2}}} \xrightarrow{\mathcal{L}} \mathcal{N}(0, (\sigma^{\mathbf{r}}_\infty(x))^2). \tag{3.8}$$

Furthermore the limit shape of $w_n(x)$ is given by $w^{\mathbf{r}}_\infty(x)$ (with scaling $A_n = (n^*)^\alpha$ and $B_n = n^*$, see (1.2)*). In particular, we can choose $\delta = 0$ in* (1.2)*.*

Proof. The CLT follows immediately from [19, Prop. 2.4], but also can be deduced easily from (3.6) by replacing s by $s(n^*)^{-\alpha/2}$. It is also straightforward to show that $w^{\mathbf{r}}_\infty(x)$ is the limit shape. For a given $\epsilon > 0$, we choose $0 = x_0 < x_1 < \cdots < x_\ell$ such that $w^{\mathbf{r}}_\infty(x_{j+1}) -$

$w^{\mathbf{r}}_\infty(x_j) < \epsilon/2$ for $1 \le j \le \ell - 1$ and $w^{\mathbf{r}}_\infty(x_\ell) < \epsilon/2$. It is now easy to see that for each $x \in \mathbb{R}^+$

$$|(n^*)^{-\alpha} w_n(x^*) - w^{\mathbf{r}}_\infty(x)| > \epsilon \Longrightarrow \exists j \text{ with } |(n^*)^{-\alpha} w_n(x^*_j) - w^{\mathbf{r}}_\infty(x_j)| > \epsilon/2.$$

Thus

(3.9)

$$\mathbb{P}_t\left[\sup_{x\ge 0} |(n^*)^{\alpha} w_n(x^*) - w^{\mathbf{r}}_\infty(x)| \ge \epsilon\right] \le \sum_{j=1}^{\ell} \mathbb{P}_t\left[|(n^*)^{\alpha} w_n(x^*_j) - w^{\mathbf{r}}_\infty(x_j)| \ge \epsilon/2\right]$$

It now follows from (3.8) that each summand in (3.9) tends to 0 as $n \to \infty$. This completes the proof. $\square$

Another by-product of mod-Poisson convergence of a sequence $(Z_n)_{n\in\mathbb{N}}$ is that one can approximate Z_n with a Poisson random variable with parameter μ_n, see [19, Prop. 2.5]. However in our situation this is trivial since $w_n(x^*)$ is already Poisson distributed.

As we are going to do in the next section, we are also interested in the behavior of increments and their joint behaviour.

Proposition 3.5. *For all $x, y \in \mathbb{R}$, $y > x$, set*

$$w_n(x,y) := w_n(x) - w_n(y) \text{ and } w^{\mathbf{r}}_\infty(x,y) := \frac{\Gamma(\alpha,x) - \Gamma(\alpha,y)}{\Gamma(\alpha+1)}.$$

Then

$$\widetilde{w}^{\mathbf{r}}_n(x,y) := \frac{w_n(x^*,y^*) - (n^*)^{\alpha} w^{\mathbf{r}}_\infty(x,y)}{(n^*)^{\frac{\alpha}{2}}\sqrt{w^{\mathbf{r}}_\infty(x,y)}} \xrightarrow{\mathcal{L}} \mathcal{N}(0,1) \tag{3.10}$$

as $n \to \infty$ with $x^* = xn^{\frac{1}{\alpha+1}}$ and with $y^* = yn^{\frac{1}{\alpha+1}}$.

Furthermore, $\widetilde{w}^{\mathbf{r}}_n(x)$ and $\widetilde{w}^{\mathbf{r}}_n(x,y)$ are asymptotically independent.

Remark. As we will see, the proof of independence relies on the independence of cycles coming from Lemma 3.1. Therefore it is easy to generalize the above result to more than two points.

Proof. The proof of (3.10) almost the same as the proof of (3.8) and we thus omit it. Since

$$w_n(x,y) = \sum_{k=x^*}^{y^*-1} C_k \text{ and } w_n(y) = \sum_{k=y^*}^{\infty} C_k$$

and all C_k are independent, we have that $\widetilde{w}^{\mathbf{r}}_n(x)$ and $\widetilde{w}^{\mathbf{r}}_n(x,y)$ are independent for each $n \in \mathbb{N}$. Thus $\widetilde{w}^{\mathbf{r}}_n(x)$ and $\widetilde{w}^{\mathbf{r}}_n(x,y)$ are also independent in the limit. $\square$

3.3. **Functional CLT.** The topic of this section is to prove a functional CLT for the profile $w_n(x)$ of the Young diagram. Similar results were obtained in a different framework by [17, 11] on the number of cycle counts not exceeding $n^{\lfloor x \rfloor}$, and by [5] for Young diagrams confined in a rectangular box. We show

Theorem 3.6. *The process $\widetilde{w}_n^{\mathbf{r}} : \mathbb{R}^+ \to \mathbb{R}$ (see (3.8)) converges weakly with respect to $\mathbb{P}_t$ as $n \to \infty$ to a continuous process $\widetilde{w}_\infty^{\mathbf{r}} : \mathbb{R}^+ \to \mathbb{R}$ with $\widetilde{w}_\infty^{\mathbf{r}}(x) \sim \mathcal{N}(0, \sigma_\infty^{\mathbf{r}}(x))$ and independent increments.*

The technique we will exploit is quite standardized (see [17]). We remark that, unlike in this paper where the Ewens measure is considered, we do not obtain here a Brownian process, as the variance of $\widetilde{w}_\infty^{\mathbf{r}}(t) - \widetilde{w}_\infty^{\mathbf{r}}(s)$ for $r \geq s$ is more complicated than in the case of the Wiener measure.

We know from Proposition 3.5 the finite dimensional marginals of the process. More specifically we have for $x_\ell \geq x_{\ell-1} \geq \cdots \geq x_1 \geq 0$ that

$$(n^*)^{-\alpha/2}\big(w_n(x_\ell^*),\, w_n(x_{\ell-1}^*) - w_n(x_\ell^*),\, \ldots,\, w_n(x_1^*) - w_n(x_2^*)\big) \sim \mathcal{N}\left(\mathbf{0}, \Sigma'\right) \tag{3.11}$$

where Σ' is a diagonal matrix with

$$\Sigma'_{11} = w_\infty^{\mathbf{r}}(x_\ell) \text{ and } \Sigma'_{jj} = w_\infty^{\mathbf{r}}(x_{\ell-j+1}, x_{\ell-j+2}) \text{ for } j \geq 2.$$

Now all we need to show to complete the proof of Theorem 3.6 is the tightness of the process $\widetilde{w}_n^{\mathbf{r}}$. In order to do so, we will proceed similarly to [17], namely we will show that

Lemma 3.7. *We have for $0 \leq x_1 < x \leq x_2 < K$ with K arbitrary*

$$\mathbb{E}_t\left[(\widetilde{w}_n^{\mathbf{r}}(x) - \widetilde{w}_n^{\mathbf{r}}(x_1))^2(\widetilde{w}_n^{\mathbf{r}}(x_2) - \widetilde{w}_n^{\mathbf{r}}(x))^2\right] = O\left((x_2 - x_1)^2\right) \tag{3.12}$$

with $x^* := xn^{\frac{1}{\alpha+1}}$, $x_1^* := x_1 n^{\frac{1}{\alpha+1}}$ and $x_2^* := x_2 n^{\frac{1}{\alpha+1}}$.

Lemma 3.7 together with [7, Theorem 15.6] implies that the process $\widetilde{w}_n^{\mathbf{r}}$ is tight. This and the marginals in (3.11) prove Theorem 3.6.

Proof of Lemma 3.7. We define

$$E^* := \mathbb{E}_t\left[(\widetilde{w}_n^{\mathbf{r}}(x^*) - \widetilde{w}_n^{\mathbf{r}}(x_1^*))^2(\widetilde{w}_n^{\mathbf{r}}(x_2^*) - \widetilde{w}_n^{\mathbf{r}}(x^*))^2\right]. \tag{3.13}$$

Centering with $\mathbb{E}_t\left[w_n(\cdot)\right]$ and the independence of the cycle counts leads us to

$$
\begin{aligned}
E^* &= \left(\sum_{k=x_1^*}^{x^*-1}(n^*)^{-\alpha}\frac{\theta_k}{k}t^k\right)\cdot\left(\sum_{k=x^*}^{x_2^*-1}(n^*)^{-\alpha}\frac{\theta_k}{k}t^k\right)\\
&\overset{Lem.\ 2.4}{\sim} \left(\frac{(n^*)^{-\alpha}}{\Gamma(\alpha+1)}\int_{x_1^*}^{x^*}t^{\alpha-1}e^{-t}\mathrm{d}t\right)\left(\frac{(n^*)^{-\alpha}}{\Gamma(\alpha+1)}\int_{x^*}^{x_2^*}t^{\alpha-1}e^{-t}\mathrm{d}t\right)\\
&= \left(\frac{\Gamma(\alpha,x_1)-\Gamma(\alpha,x)}{\Gamma(\alpha+1)}\right)\left(\frac{\Gamma(\alpha,x)-\Gamma(\alpha,x_2)}{\Gamma(\alpha+1)}\right)\\
&\sim O\left((x-x_1)(x_2-x)\right)=O\left((x_2-x_1)^2\right).
\end{aligned}
$$

Here we have used the fact that $\Gamma(\alpha,\cdot)$ is a Lipschitz function and the assumption that $x_1 < x \leq x_2 < K$. $\square$

4. Saddle point method

The aim of this section is to study the asymptotic behaviour of $w_n(x)$ with respect to $\mathbb{P}_n\left[\cdot\right]$ as $n \to \infty$ and to compare the results with the results in Section 3.

There are at least two approaches with which to tackle this problem: one is more probabilistic and was employed by [13] in their paper. The second one was first developed in [22] from the standard saddle point method.

The first method to study the asymptotic statistics of $w_n(x)$ with respect to $\mathbb{P}_n\left[\cdot\right]$ as $n \to \infty$ is the so called Khintchine method. We illustrate this method briefly with the normalisation constant h_n (see (1.3)). The first step is to write down a Khintchine's type representation for the desired quantity. For h_n this is given by

$$
h_n = t^{-n}\exp\left(\sum_{k=1}^{n}\frac{\vartheta_k}{k}t^k\right)\mathbb{P}_t\left[\sum_{k=1}^{n}kC_k = n\right] \tag{4.1}
$$

with $t > 0$ and $\mathbb{P}_t\left[\,\cdot\,\right]$ as in Section 3. The second step is to choose the free parameter t in such a way that $\mathbb{P}_t\left[\sum_{k=1}^n kC_k = n\right]$ gets large. Here one can choose t to be the solution of the equation $\sum_{k=1}^n \vartheta_k t^k = n$.

This argumentation is very close to the argumentation relying on complex analysis and generating functions. Indeed, it is easy to see that (4.1) is equivalent to

$$
h_n = \left[t^n\right]\left[\exp\left(g_\Theta(t)\right)\right] \tag{4.2}
$$

with $g_\Theta(t)$ as in (2.5). Furthermore, the choice of t is (almost) the solution of the saddle point equation $tg'_\Theta(t) = n$. We have of course to justify (4.2) (or (4.1)). But this follows immediately from the definition of h_n and Lemma 2.2.

We prefer at this point to work with the second approach. We begin by writing down the generating functions of the quantities we would like to study.

Lemma 4.1. *We have for $x \geq 0$ and $s \in \mathbb{R}$*

$$\mathbb{E}_n\left[\exp\big(-sw_n(x)\big)\right] = \frac{1}{h_n}[t^n]\left[\exp\left(g_\Theta(t) + (e^{-s}-1)\sum_{k=\lfloor x \rfloor}^{\infty}\frac{\vartheta_k}{k}t^k\right)\right]. \tag{4.3}$$

Remark. Although the expressions in Lemmas 4.1 and 4.2 hold in broader generality, starting from Subsection 4.1 we will calculate moment generating functions on the positive half-line, namely we can assume all parameters $s_1, \ldots, s_\ell$ etc to be non-negative, according to [9, Theorem 2.2].

Proof. It follows from the definitions of $\mathbb{P}_n\left[\,\cdot\,\right]$ and $w_n(x)$ (see (2.3)) that

$$\begin{aligned} h_n\mathbb{E}_n\left[\exp\big(-sw_n(x)\big)\right] &= \frac{1}{n!}\sum_{\sigma\in\mathfrak{S}_n}\exp\left(-s\sum_{k=\lfloor x \rfloor}^{n}C_k\right)\prod_{k=1}^{n}\vartheta_k^{C_k} \\ &= \frac{1}{n!}\sum_{\sigma\in\mathfrak{S}_n}\prod_{k=1}^{\lfloor x \rfloor-1}\vartheta_k^{C_k}\prod_{m=\lfloor x \rfloor}^{\infty}(\vartheta_k e^{-s})^{C_k} \end{aligned} \tag{4.4}$$

Applying now Lemma 2.2, we obtain

$$\sum_{n=0}^{\infty}\frac{t^n}{n!}h_n\mathbb{E}_n\left[\exp\big(-sw_n(x)\big)\right] = \exp\left(\sum_{k=1}^{\lfloor x \rfloor-1}\frac{\vartheta_k}{k}t^k + e^{-s}\sum_{k=\lfloor x \rfloor}^{\infty}\frac{\vartheta_k}{k}t^k\right) \tag{4.5}$$

$$= \exp\left(g_\Theta(t) + (e^{-s}-1)\sum_{k=\lfloor x \rfloor}^{\infty}\frac{\vartheta_k}{k}t^k\right) \tag{4.6}$$

Equation (4.3) now follows by taking $\left[t^n\right]$ on both sides. □

We are also interested in the joint behaviour at different points of the limit shape. The results in Section 3 suggest that the increments of $w_n(x_{j+1}) - w_n(x_j)$ are independent for $x_\ell \geq x_{\ell-1} \geq \cdots \geq x_1 \geq 0$. It is thus natural to consider

$$\mathbf{w}_n(\mathbf{x}) = \big(w_n(x_\ell^*),\, w_n(x_{\ell-1}^*) - w_n(x_\ell^*),\, \ldots,\, w_n(x_1^*) - w_n(x_2^*)\big). \tag{4.7}$$

We obtain

Lemma 4.2. *We have for $\boldsymbol{x} = (x_1, \ldots, x_\ell) \in \mathbb{R}^\ell$ with $x_\ell \geq x_{\ell-1} \geq \cdots \geq x_1 \geq 0$ and $\boldsymbol{s} = (s_1, \ldots, s_\ell) \in \mathbb{R}^\ell$*

(4.8)

$$\mathbb{E}_n\left[\exp\big(-\langle \boldsymbol{s}, \boldsymbol{w}_n(\boldsymbol{x})\rangle\big)\right] = \frac{1}{h_n}[t^n]\left[\exp\left(g_\Theta(t) + \sum_{j=1}^{\ell}(e^{-s_j}-1)\sum_{k=\lfloor x_j \rfloor}^{\lfloor x_{j+1}-1 \rfloor}\frac{\vartheta_k}{k}t^k\right)\right]$$

with the convention $x_{\ell+1} := +\infty$. The proof of this lemma is almost the same as for Lemma 4.1 and we thus omit it.

4.1. **Log-n-admissibility.** The approach with which we first addressed the study of the limit shape is derived from the saddle point method for approximating integrals in the complex plane. We want to introduce the definition of log-n-admissible function, generalizing the analogous concept introduced in [22]. We stress that here, in comparison to the definition of log- (or equivalently Hayman) admissibility used there, we consider a family of functions parametrized by n for which log-admissibility holds simultaneously. The definition is therefore a natural extension.

Definition 4.3. *Let $\left(g_n(t)\right)_{n\in\mathbb{N}}$ with $g_n(t) = \sum_{k=0}^{\infty} g_{k,n}t^k$ be given with radius of convergence $\rho > 0$ and $g_{k,n} \geq 0$. We say that $\left(g_n(t)\right)_{n\in\mathbb{N}}$ is* $\log$-n-admissible *if there exist functions $a_n, b_n : [0,\rho) \to \mathbb{R}^+$, $R_n : [0,\rho) \times (-\pi/2, \pi/2) \to \mathbb{R}^+$ and a sequence $(\delta_n)_{n\in\mathbb{N}}$ s. t.*

Saddle-point: *For each n there exists $r_n \in [0,\rho)$ with*

$$a_n(r_n) = n \tag{4.9}$$

Approximation: *For all $|\varphi| \leq \delta_n$ we have the expansion*

$$g_n(r_n e^{i\varphi}) = g_n(r_n) + i\varphi a_n(r_n) - \frac{\varphi^2}{2} b_n(r_n) + R_n(r_n,\varphi) \tag{4.10}$$

where $R_n(r_n,\varphi) = o(\varphi^3\delta_n^{-3})$.

Divergence: *$b_n(r_n) \to \infty$ and $\delta_n \to 0$ as $n \to \infty$.*

Width of convergence: *We have $\delta_n^2 b_n(r_n) - \log b_n(r_n) \to +\infty$ as $n \to +\infty$.*

Monotonicity: *For all $|\varphi| > \delta_n$, we have*

$$\mathfrak{Re}\left(g_n(r_n e^{i\varphi})\right) \leq \mathfrak{Re}\left(g(r_n e^{\pm i\delta_n})\right). \tag{4.11}$$

The approximation condition allows us to compute the functions a and b exactly. We have

$$a_n(r) = r g_n'(r), \tag{4.12}$$

$$b_n(r) = r g_n'(r) + r^2 g_n''(r) \tag{4.13}$$

Clearly a_n and b_n are strictly increasing real analytic functions in $[0,\rho)$. The error in the approximation can similarly be bounded, so that

$$R_n(r,\varphi) = \varphi^3 O\left(r g_n'(r) + 3r^2 g_n''(r) + r^3 g_n'''(r)\right)$$

Having proved Lemma 4.1 we are now able to write down in a more explicit way generating functions. What we are left with is trying to extract the coefficients of the expansion given therein. This is the content of

Theorem 4.4. *Let $\left(g_n(t)\right)_{n\in\mathbb{N}}$ be $\log$-n-admissible with associated functions a_n, b_n and constants r_n. Call*

$$G_n := [t^n]e^{g_n(t)}.$$

Then

1 G_n *has the asymptotic expansion*

$$G_n = \frac{1}{\sqrt{2\pi}}(r_n)^{-n} b_n(r_n)^{-1/2} e^{g_n(r_n)}(1+o(1)). \tag{4.14}$$

2 *Recall* h_n *defined in* (1.3)*. For the class of functions with weights as in* (1.4)*,*

$$h_n = \frac{1}{\sqrt{2\pi(\alpha+1)n^{\frac{\alpha+2}{1+\alpha}}}} e^{2n^{\frac{\alpha}{1+\alpha}}}(1+o(1))$$

respectively as $n \to +\infty$*.*

Remark. As it is explained in [15, Chapter VIII] it is possible to take into account more error terms in the expansion of g_n.

Proof of Theorem 4.4. The proof is exactly the same as in [22, Prop. 2.2] and we thus give only a quick sketch of it, referring the reader to this paper for more details. As in the well-known saddle point method, we want to evaluate the integral

$$\frac{1}{2\pi i}\oint_\gamma \exp\left(g_n(z)\right)\frac{\mathrm{d}z}{z^{n+1}}.$$

We choose as contour the circle $\gamma := r_n e^{i\varphi}$ with $\varphi \in [-\pi, \pi]$. On $\varphi \in [-\delta_n, \delta_n]$ after changing to polar coordinates we can expand the function g as

$$\int_{-\delta_n}^{\delta_n} \exp\left(g_n(r) + i\varphi a_n(r) - \frac{\varphi^2}{2}b_n(r) + o(\varphi^3\delta_n^{-3}) - in\varphi\right)\mathrm{d}\varphi$$

We now choose r_n such that $a(r_n) = r_n g_n'(r_n) = n$ in order to cancel the linear terms in n. This allows us to approximate the integral on the minor arc with a Gaussian. One shows that away from the saddle point (so for $|\varphi| > \delta_n$) the contribution is exponentially smaller than on the minor arc and thus it can be neglected. $\square$

We would like to emphasize also that it will be not always possible to solve the saddle point equation (4.9) exactly. However it is enough to find an r_n such that

$$a(r_n) - n = o\left(\sqrt{b(r_n)}\right) \tag{4.15}$$

holds.

4.2. **Limit shape for polynomial weights.** In this section we will derive the limit shape for Young diagrams for the class of measures given by the weights. We will not go into all the details to prove the log-n-admissibility for the most general case, but will try to give a precise overview of the main steps nonetheless. One important remark we have to make is that our parameter s will not be fixed, but will be scaled and hence dependent on n. This comes from the fact that for a fixed s (4.9) becomes a fixed point equation whose solution cannot be given constructively, but has only an implicit form. We were not able to use this information for our purposes, and hence preferred to exploit a less general, but more explicit parameter to calculate asymptotics.

4.2.1. *Limit shape.* The main goal of this subsection is to prove that the weights (1.4) induce a sequence of log-n-admissible functions of which we can recover the asymptotics of $g_n(r_n)$. This will give us the limit shape of the Young diagram according to Theorem 4.4. More specifically

Theorem 4.5. *For the scaling $n^* = n^{\frac{1}{\alpha+1}}$, $x^* = xn^*$ and $s^* := s(n^*)^{-\alpha/2}$, define the functions*

$$w^{\mathbf{s}}_{\infty}(x) := \frac{\Gamma(\alpha, x)}{\Gamma(\alpha+1)},$$

$$\sigma^2_{\infty}(x) := -\frac{\Gamma(\alpha+1,\, x)^2}{2\Gamma(\alpha+1)\Gamma(\alpha+2)} + \frac{\Gamma(\alpha, x)}{\Gamma(\alpha+1)}.$$

Then

$$\widetilde{w}^{\mathbf{s}}_n(x) := \frac{w_n(x^*) - (n^*)^{\alpha} w^{\mathbf{s}}_{\infty}(x)}{(n^*)^{\alpha/2}} \xrightarrow{\mathcal{L}} \mathcal{N}\left(0, \sigma^2_{\infty}(x)\right).$$

In particular $w^{\mathbf{s}}_{\infty}(x)$ is the desired limit shape.

Remark. We note that the limit shape matches the one obtained in [13, Thm. 4.8] and also the one obtained in the present paper in the randomized case (cf. the definition of $w^{\mathbf{r}}_{\infty}(x)$ of Prop. 3.3).

Theorem 4.6. *Define the cumulant generating function as*

$$\Lambda(s) := \log \mathbb{E}_n \left[e^{s\tilde{w}_n(x)} \right] = \sum_{m \geq 1} q_m \frac{t^m}{m!}.$$

We then have for $m \geq 2$

$$q_m = \kappa_m(1 + o(1)). \tag{4.16}$$

with

$$\begin{aligned} \kappa_m \quad = \quad & [s^m] \left(\frac{(n^*)^{\alpha}}{\alpha} \left(1 - s^* \frac{\Gamma(\alpha+1,\, x)}{\Gamma(\alpha+2)} \right)^{-\alpha} \right. \\ & \left. + \frac{\left(e^{-s^*} - 1\right)}{\Gamma(\alpha+1)} \sum_{k \geq 0} \frac{\Gamma(\alpha+k,\, x)}{k!} \left(\frac{\Gamma(\alpha+1,\, x)}{\Gamma(\alpha+2)} s^* \right)^k \right). \end{aligned} \tag{4.17}$$

We can also determine the behavior of the increments of the function $w_n(\cdot)$. We will consider first the more general case and then give the example of the two-increment case (refer to (4.7) with $\ell = 2$).

Theorem 4.7. 1 *For $\ell \geq 2$ and $x_\ell \geq x_{\ell-1} \geq \cdots \geq x_1 \geq 0$, let*

$$\widetilde{\mathbf{w}}_n^{\mathbf{s}}(\mathbf{x}) = \left(\widetilde{w}_n^{\mathbf{s}}(x_\ell),\, \widetilde{w}_n^{\mathbf{s}}(x_{\ell-1}) - \widetilde{w}_n^{\mathbf{s}}(x_\ell),\, \ldots,\, \widetilde{w}_n^{\mathbf{s}}(x_1) - \widetilde{w}_n^{\mathbf{s}}(x_2)\right).$$

Set $x_{\ell+1} = +\infty$. For $1 \leq j < i < \ell$ we have that

$$\begin{aligned}\widetilde{w}_\infty^{\mathbf{s}}(x_i,\, x_j) &:= \lim_{n\to+\infty} \operatorname{Cov}\left(\widetilde{w}_n^{\mathbf{s}}(x_j) - \widetilde{w}_n^{\mathbf{s}}(x_{j+1}),\, \widetilde{w}_n^{\mathbf{s}}(x_i) - \widetilde{w}_n^{\mathbf{s}}(x_{i+1})\right) \\ &= \frac{\left(\Gamma(\alpha+1,\, x_i) - \Gamma(\alpha+1,\, x_{i+1})\right)\left(\Gamma(\alpha+1,\, x_j) - \Gamma(\alpha+1,\, x_{j+1})\right)}{\Gamma(\alpha+1)\Gamma(\alpha+2)}.\end{aligned}$$

Remark. Let us comment briefly on Thm. 4.7. What we obtained in this result is most unexpected: cycle counts are asymptotically independent under very mild assumptions (see Lemma 2.1). The assumption of the lemma holds in our case as the growth of the parameters ϑ_n is algebraic. The fact that the increments depend on disjoint sets of cycles would have suggested the asymptotic independence of $w_n(y^*)$ from $w_n(x^*) - w_n(y^*)$. We are aware of the work of [4] handling this issue in the case of the Ewens sampling formula, in particular showing that partial sums of cycle counts need not converge to processes with independent increments. Our result extends this idea in the sense that it shows the explicit covariance matrix for a whole category of generating functions. It would be interesting to provide a heuristic explanation for this theorem.

4.2.2. *Log-n-admissibility.* In order to determine the limit shape we would like to prove the log-n-admissibility of the function explicited in (4.3). To be more precise, what we have to prove is

Lemma 4.8. *Let $s \geq 0$, and recall $n^* = n^{\frac{1}{\alpha+1}}$, $s^* = s(n^*)^{\alpha/2}$. The function*

$$g_\Theta(t) + (e^{-s^*} - 1)\sum_{k=\lfloor x^*\rfloor}^{\infty} \frac{k^{\alpha-1} + O\left(k^{\beta-1}\right)}{\Gamma(\alpha+1)} t^k$$

is log-n-admissible for all $x \geq 0$, with $g_\Theta(t)$ as in (2.5) *and*

$$r_n := e^{-v_n} \tag{4.18}$$

with

$$v_n := (n^*)^{-1}\left(1 - s(n^*)^{-\alpha/2}\frac{\Gamma(\alpha+1,\, x)}{\Gamma(\alpha+2)}\right).$$

Proof of Lemma 4.8. **Saddle-point and approximation:** We start first with the case $\beta = 0$. By doing so one obtains that

$$
\begin{aligned}
a(r_n) &= \sum_{k=1}^{+\infty} \frac{k^\alpha}{\Gamma(\alpha+1)} e^{-kv_n} + (e^{-s^*}-1) \sum_{k=\lfloor x^* \rfloor}^{+\infty} \frac{k^\alpha}{\Gamma(\alpha+1)} e^{-kv_n} \\
&= (v_n)^{-\alpha-1} + O(1) + (e^{-s^*}-1) \frac{\Gamma(\alpha+1, x^* v_n)}{\Gamma(\alpha+1)} \\
&\quad + (e^{-s^*}-1) O\left(v_n^{-\alpha}\right) \qquad (4.19) \\
&= n \left(1 + (\alpha+1) s (n^*)^{-\alpha/2} \frac{\Gamma(\alpha+1,\, x)}{\Gamma(\alpha+2)} + O\left((n^*)^{-\alpha}\right)\right) \\
&\quad + n \left(-\frac{s}{(n^*)^{\alpha/2}} + O\left(\frac{s^2}{(n^*)^\alpha}\right)\right) \left(\frac{\Gamma(\alpha+1, x)}{\Gamma(\alpha+1)} + O\left((n^*)^{-\alpha/2}\right)\right) \\
&\quad + O\left(1 + v_n^{-\alpha} (n^*)^{-\alpha/2}\right) \\
&= n + O(n^*). \qquad (4.20)
\end{aligned}
$$

We also have that

$$
b(r_n) = O\left(\sum_{k=1}^{+\infty} \frac{k^{\alpha+1}}{\Gamma(\alpha+1)} e^{-kv_n}\right) \sim (\alpha+1)(n^*)^{\alpha+2} + O(n). \tag{4.21}
$$

Therefore (4.15) holds true for all α. In the case where β is turned on, we obtain by performing similar steps that

$$
a(r_n) = n + O\left((n^*)^{\beta+1}\right).
$$

Then (4.15) is satisfied if

$$
\frac{\beta+1}{\alpha+1} < \frac{\alpha+2}{2(\alpha+1)} \iff \beta < \frac{\alpha}{2} \tag{4.22}
$$

which holds by assumption.

Divergence: By the above calculations we set $\delta_n := (n^*)^{-\xi}$ with $\frac{\alpha+3}{3} < \xi < \frac{\alpha+2}{2}$. This position holds also in the case $\beta > 0$.

Monotonicity: In the region $|\varphi| = o(1)$ we wish to show that

$$
g\left(r_n e^{i\varphi}\right) = g(r_n)(1 + o(1)). \tag{4.23}
$$

First remember that $g_n\left(r_n e^{i \pm \delta_n}\right) = O\left((n^*)^\alpha\right)$ by Lemma 2.3. Thus here we have:

1 if $\varphi = o(v_n)$, then by a change of variable $t \rightsquigarrow (v_n - i\varphi)t$

$$
\begin{aligned}
&\sum_{k \geq \lfloor x^* \rfloor} \frac{k^{\alpha-1}}{\Gamma(\alpha+1)} e^{-k(v_n - i\varphi)} \\
&\sim \frac{(v_n - i\varphi)^{-\alpha}}{\Gamma(\alpha+1)} \int_x^{+\infty} t^{\alpha-1} e^{-t} \mathrm{d}t = \frac{\Gamma(\alpha, x)}{\Gamma(\alpha+1)} (v_n - i\varphi)^{-\alpha}
\end{aligned}
$$

which is asymptotic to $(n^*)^\alpha$. Considering the factor $e^{-s^*} - 1$ we obtain that the summand is negligible with respect to $\mathfrak{Re}\left(g(r_n e^{\pm i\delta_n})\right)$.

2 If $\varphi \neq o(v_n)$, then

$$\sum_{k \geq \lfloor x^* \rfloor} \frac{k^{\alpha-1}}{\Gamma(\alpha+1)} e^{-k(v_n - i\varphi)}$$
$$\sim \frac{(v_n - i\varphi)^{-\alpha}}{\Gamma(\alpha+1)} \int_{x - ix\varphi n^* + o(1)}^{+\infty} t^{\alpha-1} e^{-t} \mathrm{d}t = \frac{\Gamma(\alpha, x - ix\varphi n^*)}{\Gamma(\alpha+1)} (v_n - i\varphi)^{-\alpha}$$

and afterwards use the fact that $\Gamma(\alpha, x+iy) = O\left(y^{\alpha-1}\right)$ for $|y|$ large. Hence the RHS of (4.24) becomes

$$O\left((n^*)^{\alpha-1}\right)(v_n - i\varphi)^{1-\alpha} = O\left((n^*)^{\alpha-1}\varphi^{-1}\right)$$

As $\varphi \neq o(v_n)$, we obtain that $O\left((n^*)^{\alpha-1}\varphi^{-1}\right) = O\left((n^*)^{\alpha}\right) o(1)$ which is enough to show (4.23).

3 To conclude we consider the case $|\varphi| > C$: the function $g_n\left(r_n e^{i\varphi}\right)$ is bounded there by a constant uniform in n, and then by bounding $g_n\left(r_n e^{i\varphi}\right)$ through its modulus we have

$$\mathfrak{Re}\left(g_n(r_n e^{i\varphi})\right) \leq \mathfrak{Re}\left(g(r_n e^{\pm i\delta_n})\right)\left(1 + O\left((n^*)^{-\alpha/2}\right)\right). \tag{4.24}$$

□

In order to show Thms. 4.5, 4.6 and 4.7 we need to prove first an auxiliary proposition.

Proposition 4.9. *For the scaling $n^* := n^{\frac{1}{\alpha+1}}$, $x^* := xn^*$ and $s^* := s(n^*)^{-\alpha/2}$ the equality*

$$\begin{aligned} &\left(e^{-s^*} - 1\right) \sum_{k \geq \lfloor x^* \rfloor} k^{\alpha-1} r_n^k \\ &= \left(-s(n^*)^{\alpha/2}\Gamma(\alpha, x) + \frac{s^2}{2}\Gamma(\alpha, x) - \frac{\Gamma(\alpha+1, x)^2}{\Gamma(\alpha+2)} s^2\right) + o(1) \end{aligned} \tag{4.25}$$

holds asymptotically as $n \to +\infty$.

Proof. We apply Lemma 2.4 with

$$\begin{aligned} &f(t) := t^{\alpha-1} e^{-tv_n}, \\ &z_n = xn^*,\ v_n = \frac{1}{n^*}\left(1 - s(n^*)^{-\alpha/2}\frac{\Gamma(\alpha+1, x)}{\Gamma(\alpha+2)}\right) \quad \text{and} \\ &z_n v_n = x - \frac{sx}{(n^*)^{\alpha/2}} \frac{\Gamma(\alpha+1, x)}{\Gamma(\alpha+2)}. \end{aligned}$$

The first term of the expansion is

$$\begin{aligned} &\left(e^{-s^*} - 1\right)(n^*)^{\alpha}\,\Gamma(\alpha, x) \\ &= -s(n^*)^{\alpha/2}\Gamma(\alpha, x) + \frac{s^2}{2}\Gamma(\alpha, x) + o(1) \end{aligned}$$

because $(s^*)^3(n^*)^\alpha = o(1)$. If $\beta > 0$ instead we obtain

$$\begin{aligned}&(e^{-s^*}-1)\,(n^*)^\alpha\,\Gamma(\alpha,x)\\&= -s(n^*)^{\alpha/2}\Gamma(\alpha,x)+\frac{s^2}{2}\Gamma(\alpha,x)+o(1)\end{aligned}$$

To calculate the expansion up to a $O(1)$ term it is sufficient to consider for $k=1$

$$\begin{aligned}&(e^{-s^*}-1)\,(n^*)^\alpha\left(\frac{\Gamma(\alpha+1,x)}{\Gamma(\alpha+2)}sn^{-\beta}\right)\\&= -\Gamma(\alpha+1,x)\frac{\Gamma(\alpha+1,x)}{\Gamma(\alpha+2)}s^2+o(1)\end{aligned}$$

This tells us that

$$\begin{aligned}&(e^{-s^*}-1)\sum_{k\geq\lfloor x^*\rfloor}k^{\alpha-1}r_n^k\\&= \left(-s(n^*)^{\alpha/2}\Gamma(\alpha,x)+\frac{s^2}{2}\Gamma(\alpha,x)-\frac{\Gamma(\alpha+1,x)^2}{\Gamma(\alpha+2)}s^2\right)+o(1)\end{aligned}\tag{4.26}$$

As for the remainder, we can find an a priori bound on the Bernoulli polynomials independent of n on $x \in [0,1]$. Furthermore,

$$(e^{-s^*}-1)f(\lfloor x^*\rfloor) = O\left(s(n^*)^{-\alpha/2}\right)(\lfloor xn^*\rfloor)^{\alpha-1}e^{-x+o(1)} = O\left(s(n^*)^{\frac{\alpha-2}{2}}\right),$$

which is small compared to the magnitude of the leading coefficient in s. Moreover

$$\int_{xn^*}^{+\infty} B_1(x'-\lfloor x'\rfloor)f'(x')\mathrm{d}x' \tag{4.27}$$

$$\begin{aligned}&\leq C\int_{xn^*}^{+\infty}|f'(x')|\mathrm{d}x' = C\int_{xn^*}^{+\infty}e^{-x'v_n}(-v_n(x')^{\alpha-1}\\&+(\alpha-1)(x')^{\alpha-2})\mathrm{d}x'.\end{aligned}\tag{4.28}$$

With the same substitution $x' := \frac{z_n}{a_0}y$ we can interchange limit and integral by the dominated convergence theorem to obtain

$$(4.28) = O\left(\left(\frac{z_n}{a_0}\right)^{\alpha/2}\Gamma(1+\alpha,a_0)\right).$$

Combining this with the first order expansion of $(e^{-s^*}-1)$ we obtain

$$(4.27) = O\left(s(n^*)^{\alpha/2-\alpha/2}\right) = s\,O(1).$$

□

Proof of Thms. 4.5 and 4.6. To determine the behavior of G_n we would like to use Lemma 4.1. By (4.3)

$$\mathbb{E}_n\left[\exp(-s^*w_n(x^*))\right] = \frac{1}{h_n}[t^n]\left[\exp\left(g_\Theta(t)+(e^{-s^*}-1)\sum_{k=\lfloor x^*\rfloor}^{+\infty}\frac{\vartheta_k}{k}t^k\right)\right].$$

We have shown that $g_n(t) = g_\Theta(t) + (e^{-s^*} - 1)\sum_{k=\lfloor x^* \rfloor}^{+\infty} \frac{\vartheta_k}{k} t^k$ is log-n-admissible. Therefore Thm. 4.4 tells us how G_n behaves, and we have more precisely to recover three terms. In first place we collect the terms for the asymptotic of $e^{g_n(r_n)}$: one is

$$\begin{aligned}
g_\Theta(r_n) &\sim \frac{\Gamma(\alpha)}{\Gamma(\alpha+1)} v_n^{-\alpha} = \frac{(n^*)^\alpha}{\alpha}\left(1 - s(n^*)^{-\alpha/2}\frac{\Gamma(\alpha+1,\, x)}{\Gamma(\alpha+2)}\right)^{-\alpha} \\
&= \frac{(n^*)^\alpha}{\alpha} + (n^*)^{\alpha/2}\,\frac{s\Gamma(\alpha+1,\, x)}{\Gamma(\alpha+2)} \\
&+ \frac{s^2\Gamma(\alpha+1,\, x)^2}{2\Gamma(\alpha+2)\Gamma(\alpha+1)} + O\left((n^*)^\beta\right)
\end{aligned}$$

given by Lemma 2.3. The other is $\sum_{k \geq \lfloor x^* \rfloor} \frac{k^{\alpha-1} + O\left(k^{\beta-1}\right)}{\Gamma(\alpha+1)} r_n^k$ which we can approximate through Prop. 4.9. Secondly we obviously have

$$-n\log(r_n) = (n^*)^\alpha - (n^*)^{\alpha/2} s \frac{\Gamma(\alpha+1,\, x)}{\Gamma(\alpha+2)}.$$

Thirdly the behavior of $b(r_n)$ was determined in (4.21). All in all

$$\begin{aligned}
e^{g(r_n) - n\log(r_n)} &= \exp\left((n^*)^\alpha\left(1 + \frac{1}{\alpha}\right) + s(n^*)^{\alpha/2}\frac{\Gamma(\alpha, x)}{\Gamma(\alpha+1)}\right. \\
&+ \frac{s^2}{2}\left(-\frac{\Gamma(\alpha+1,\, x)^2}{2\Gamma(\alpha+1)\Gamma(\alpha+2)} + \frac{\Gamma(\alpha, x)}{\Gamma(\alpha+1)}\right) \\
&\left. + s^3 O\left((n^*)^{-3\alpha/2}\left(1 + (n^*)^\alpha\right)\right)\right).
\end{aligned} \tag{4.29}$$

Theorem 4.4 yields the behavior of h_n, and the same theorem allows us to conclude plugging in (4.14) the expressions obtained in (4.21), (4.29) and h_n of (2) therein. It is also clear that $w_\infty^{\mathbf{s}}$ is the limit shape, in the same fashion the result followed in the proof of Corollary 3.4.

For cumulants what we have to do is considering the logarithm of the expansion $(r_n)^{-n} b(r_n)^{-1/2} e^{g_n(r_n) - n\log(r_n)}$. We claim that it suffices to consider simply the logarithm of the expression (4.29). In fact,

$$\begin{aligned}
\log(b(r_n)) &= \log\left(O\left((n^*)^{\alpha+2}\left(1 - s^*\frac{\Gamma(\alpha, x)}{\Gamma(\alpha+1)}\right)^{-\alpha-2}\right)\right) \\
&= C_1\log(n) + C_2\sum_{k\geq 0} s^k (n^*)^{\frac{k\alpha}{2}}
\end{aligned}$$

whilst each coefficient of s^k in $g_n(r_n) - n\log(r_n)$ is of order $(n^*)^{\frac{\alpha(2-k)}{2}}$ (compare (4.29)). This confirms that the main contribution stems from (4.29). $\square$

Proof of Thm. 4.7. For multiple increments repeating the proof of Thm. 4.5 tells us that for a vector $\mathbf{w}_n(\mathbf{x}^*)$ as in (4.7) with length $\ell > 2$ we can set

$$v_n := (n^*)^{-1}\left(1 - \frac{(n^*)^{-\alpha/2}}{\Gamma(\alpha+2)}\left(s_\ell\Gamma(\alpha+1,\, x_\ell)\right.\right.$$
$$\left.\left. + \sum_{k=1}^{\ell-1} s_{\ell-k}\left(\Gamma(\alpha+1,\, x_{\ell-1-k}) - \Gamma(\alpha+1,\, x_{\ell-k})\right)\right)\right).$$

We deduce from this that

$$g_\Theta(r_n) \sim \frac{v_n^{-\alpha}}{\alpha} = \frac{(n^*)^\alpha}{\alpha} - \frac{(n^*)^{\alpha/2}}{\Gamma(\alpha+2)}$$
$$\left(s_\ell\Gamma(\alpha+1,\, x_\ell) + \sum_{k=1}^{\ell-1}(s_{\ell-k-1}(\Gamma(\alpha+1,\, x_{\ell-k-1}) - \Gamma(\alpha+1,\, x_{\ell-k}))\right)$$
$$+\frac{1}{2\Gamma(\alpha+2)\Gamma(\alpha+1)}\left(s_\ell\Gamma(\alpha+1,\, x_\ell)\right.$$
$$\left. + \sum_{k=1}^{\ell-1}(s_{\ell-k-1}(\Gamma(\alpha+1,\, x_{\ell-k-1}) - \Gamma(\alpha+1,\, x_{\ell-k}))\right)^2$$
$$+o\left(1\right). \tag{4.30}$$

Since the coefficients of the form $\left(e^{-s_j^*} - 1\right)\sum_{k=x_j^*}^{x_{j+1}^*-1}\frac{\vartheta_k}{k}r_n^k$ do not give a contribution to covariances, the mixed terms will stem from the expansion of the square in (4.30). In particular we see that the coefficient of s_is_j, for $1 \le j < i < \ell$, is

$$\frac{\left(\Gamma(\alpha+1,\, x_i) - \Gamma(\alpha+1,\, x_{i+1})\right)\left(\Gamma(\alpha+1,\, x_j) - \Gamma(\alpha+1,\, x_{j+1})\right)}{2\Gamma(\alpha+1)\Gamma(\alpha+2)}.$$

□

4.3. **Functional CLT for $w_n(\cdot)$.** As in the randomized setting, a functional CLT can be obtained here too. Unlike the previous case though we do not have the independence of cycle counts, hence we will have to show the tightness of the fluctuations as in Sec. 3.3 in two steps (cf. [17]). The result we aim at is, precisely as before,

Theorem 4.10. *The process $\widetilde{w}_n^{\mathbf{s}} : \mathbb{R}^+ \to \mathbb{R}$ (see Thm. 4.5) converges weakly with respect to $\mathbb{P}_n$ as $n \to \infty$ to a continuous process $\widetilde{w}_\infty^{\mathbf{s}} : \mathbb{R}^+ \to \mathbb{R}$ with $\widetilde{w}_\infty^{\mathbf{s}}(x) \sim \mathcal{N}(0,\, (\sigma_\infty^{\mathbf{s}}(x))^2)$ and whose increments are* not *independent. The covariance structure is given in Thm. 4.7.*

Proof. We will proceed as in the proof of Thm. 3.6. Having shown already the behavior of the increments in Thm. 4.7 what we have to tackle now is their tightness. The proof's goal is again, analogoulsy as Lemma 3.7. However the evaluation of the LHS of (3.12) is more difficult this time; one possible approach is present in [11] and is based on Pólya's enumeration lemma and the calculation of factorial moments of cycle counts. We prefer rather to follow again [17]. We will proceed in two main steps.

i) We define for $0 < t < 1$ the measure $\mathbb{P}_t$ as in Section 3. By repeating the proof of [17, Lemma 2.1] we see that

$$\mathbb{P}_t\left[\sum kC_k = n\right] = t^n h_n e^{g_\Theta(t)}.$$

Mimicking Hansen's strategy one can also prove that for arbitrary functions $\Psi : \mathfrak{S} \to \mathbb{C}$, where $\mathfrak{S} := \cup_n \mathfrak{S}_n$ and $\Psi_n : \mathfrak{S}_n \to \mathbb{C}$ s. t. $\Psi_n = \Psi(C_1, \ldots, C_n, 0, 0, \ldots)$

$$\mathbb{E}_t\left[\Psi\right] e^{g_\Theta(t)} = \sum_{n\geq 1} t^n h_n \mathbb{E}\left[\Psi_n\right] + \Psi(0, 0, 0, \ldots). \tag{4.31}$$

ii) As a formal power series identity (4.31) holds for $|t| < 1$, thus we decide to set, for x_1, x_2 as in the assumptions,

$$\begin{aligned}
&\Psi(k_1, k_2, \ldots) := \\
&= \left(n^{-\gamma} \sum_{i=x_1^*+1}^{x^*} \left(k_i - \frac{\vartheta_i}{i} r_n^i\right)\right)^2 \left(n^{-\gamma} \sum_{j=x^*+1}^{x_2^*} \left(k_j - \frac{\vartheta_j}{j} r_n^j\right)\right)^2 \\
&= \left(n^{-\gamma} \sum_{i=x_1^*+1}^{x^*} k_i - \frac{\vartheta_i}{i} t^i + n^{-\gamma} \sum_{i=x_1^*+1}^{x^*} \frac{\vartheta_i}{i}\left(t^i - e^{-iv_n}\right)\right)^2 \\
&\left(n^{-\gamma} \sum_{j=x^*+1}^{x_2^*} k_j - \frac{\vartheta_j}{j} t^j + n^{-\gamma} \sum_{j=x^*+1}^{x_2^*} \frac{\vartheta_j}{j}\left(t^j - e^{-jv_n}\right)\right)^2
\end{aligned}$$

for $\gamma > 0$ to be tuned appropriately later. We now calculate, using the independence of cycle counts under the randomized measure and the fact that $\mathrm{Var}_{\mathbb{P}_t}[C_i] = \mathbb{E}_t\left[C_i\right] =$

$\frac{\vartheta_i}{i} t^i$,

$$\begin{aligned}
\mathbb{E}_t\left[\Psi\right] &= n^{-4\gamma}\left(\sum_{i=x_1^*+1}^{x^*}\frac{\vartheta_i}{i}t^i\right)\left(\sum_{j=x^*+1}^{x_2^*}\frac{\vartheta_j}{j}t^j\right)\\
&+n^{-4\gamma}\left(\sum_{i=x_1^*+1}^{x^*}\frac{\vartheta_i}{i}(t^i-e^{-iv_n})\right)^2\left(\sum_{j=x^*+1}^{x_2^*}\frac{\vartheta_j}{j}(t^j-e^{-jv_n})\right)^2\\
&+2n^{-4\gamma}\left(\sum_{i=x_1^*+1}^{x^*}\frac{\vartheta_i}{i}t^i\right)\left(\sum_{i=x_1^*+1}^{x^*}\frac{\vartheta_i}{i}\left(t^i-e^{-iv_n}\right)\right)\\
&\cdot\left(\sum_{j=x^*+1}^{x_2^*}\frac{\vartheta_j}{j}(t^j-e^{-jv_n})\right)^2\\
&+\ldots\\
&+2n^{-4\gamma}\left(\sum_{i=x_1^*+1}^{x^*}\frac{\vartheta_i}{i}t^i\right)\left(\sum_{j=x^*+1}^{x_2^*}\frac{\vartheta_j}{j}t^j\right)\left(\sum_{i=x_1^*+1}^{x^*}\frac{\vartheta_i}{i}\left(t^i-e^{-iv_n}\right)\right)\\
&\cdot\left(\sum_{j=x^*+1}^{x_2^*}\frac{\vartheta_j}{j}(t^j-e^{-jv_n})\right)\\
&=: G_\Theta^{(1)}(t,n)+G_\Theta^{(2)}(t,n)+\ldots+G_\Theta^{(9)}(t,n).
\end{aligned}$$

Let us define $g_a^b(z) := \sum_{j=a}^b \frac{\vartheta_j}{j} z^j$. From (4.31) we obtain

$$\mathbb{E}\left[\Psi_n\right] = \frac{1}{h_n}[t^n]\left(e^{g_\Theta(t)}G_\Theta^{(1)}(t,n)\right)+\ldots+\frac{1}{h_n}[t^n]\left(e^{g_\Theta(t)}G_\Theta^{(9)}(t,n)\right).$$

We therefore obtain several terms and we will analyze them one by one.

0.1 $\frac{1}{h_n}[t^n]\left(e^{g_\Theta(t)}G_\Theta^{(1)}(t,n)\right)$. One has

$$\begin{aligned}
&\frac{n^{-4\gamma}}{h_n}[t^n]\left(e^{g_\Theta(t)}g_{x_1^*+1}^{x^*}(t)g_{x^*+1}^{x_2^*}(t)\right)\\
&= \frac{n^{-4\gamma}}{h_n}[t^n]\left(e^{g_\Theta(t)+\log\left(g_{x_1^*+1}^{x^*}(t)\right)+\log\left(g_{x^*+1}^{x_2^*}(t)\right)}\right)
\end{aligned}\tag{4.32}$$

We want to apply the saddle-point method to the sequence of functions $g_n(t) := e^{g_\Theta(t)+\log\left(g_{x_1^*+1}^{x^*}(t)\right)+\log\left(g_{x^*+1}^{x_2^*}(t)\right)}$ to extract coefficients. Our first target is to show the log-n-admissibility. We consider again the radius $r_n := e^{-v_n}$ with $v_n :=$

$(n^*)^{-1}$. In this case as in (4.20)

$$
\begin{aligned}
a(r_n) &= \sum_{k=1}^{+\infty} \frac{k^\alpha}{\Gamma(\alpha+1)} e^{-kv_n} + \frac{\sum_{x_1^*+1}^{x^*} \frac{k^\alpha}{\Gamma(\alpha+1)} e^{-kv_n}}{g_{x_1^*+1}^{x^*}(r_n)} + \frac{\sum_{x^*+1}^{x_2^*} \frac{k^\alpha}{\Gamma(\alpha+1)} e^{-kv_n}}{g_{x^*+1}^{x_2^*}(r_n)} = \\
&= (v_n)^{-\alpha-1} + O(1) + \frac{\sum_{x_1^*+1}^{x^*} \frac{k^\alpha}{\Gamma(\alpha+1)} e^{-kv_n}}{\sum_{x_1^*+1}^{x^*} \frac{k^{\alpha-1}}{\Gamma(\alpha+1)} e^{-kv_n}} + \frac{\sum_{x^*+1}^{x_2^*} \frac{k^\alpha}{\Gamma(\alpha+1)} e^{-kv_n}}{\sum_{x^*+1}^{x_2^*} \frac{k^{\alpha-1}}{\Gamma(\alpha+1)} e^{-kv_n}} \\
&= n + \frac{(v_n)^{-\alpha-2} C_{\alpha+1,\, x,\, x_1}}{v_n^{-\alpha-1} C_{\alpha,\, x,\, x_1}} + \frac{(v_n)^{-\alpha-2} C_{\alpha+1,\, x,\, x_2}}{v_n^{-\alpha-1} C_{\alpha,\, x,\, x_2}} \\
&= n + O\left(v_n^{-1}\right).
\end{aligned}
$$

where $C_{\alpha+1,\, x,\, x_1}$, $C_{\alpha,\, x,\, x_1}$, $C_{\alpha+1,\, x,\, x_2}$ and $C_{\alpha,\, x,\, x_2}$ are constants independent of n. Very little changes also in the computions for $b(r_n)$ which lead to $b(r_n) = O\left((n^*)^{\alpha+2}\right)$, yielding the saddle point equation (4.15). As far as monotonicity is concerned, heuristically one can prove it using the fact that the order of $\log\left(g_{x_1^*+1}^{x^*}(t)\right)$ is smaller that that of the leading term $g_\Theta(t)$ (as one can already notice for example in the computations for $a(r_n)$ and $b(r_n)$ above). Since calculations are straightforward we omit them. Then by Thm. 4.4 one has that (recall that $h_n = [t^n] e^{g_\Theta(t)}$)

$$
\begin{aligned}
n^{-4\gamma} \left| \frac{1}{h_n} [t^n] e^{g_n(t)} \right| &= \left| g_{x_1^*+1}^{x^*}(r_n) g_{x^*+1}^{x_2^*}(r_n)(1+o(1)) \right| = \\
&\leq C n^{-4\gamma} \left| (v_n)^{-\alpha} \left(\Gamma(\alpha,\, x) - \Gamma(\alpha,\, x_1)\right) (v_n)^{-\alpha} \left(\Gamma(\alpha,\, x_2) - \Gamma(\alpha,\, x)\right) \right| \\
&\leq C n^{-4\gamma} (v_n)^{-2\alpha} \left| (x - x_1)(x_2 - x) \right| \\
&= O\left((x-x_1)(x_2-x)\right) = O\left((x_2 - x_1)^2\right)
\end{aligned}
$$

provided that $n^{-4\gamma}(n^*)^{2\alpha} = O(1)$ iff $\gamma := \frac{\alpha}{2(\alpha+1)}$. We highlight that in this case n^γ is precisely the variance of the process (cf. Thm. 4.5). Here we have also used the fact that the incomplete Gamma function is continuous on a compact $[0,\, K]$ for some K large.

0.2 $\frac{1}{h_n}[t^n]\left(e^{g_\Theta(t)} G_\Theta^{(j)}(t,\, n)\right)$, $2 \leq j \leq 9$. We want to show that all these terms are $O\left((x_2-x_1)^2\right)$ as well. We take for example $G_\Theta^{(3)}(t,\, n) := \left(\sum_{j=x_1^*+1}^{x^*} \frac{\vartheta_j}{i} (t^j - e^{-jv_n})\right)^2 \left(\sum_{j=x^*+1}^{x_2^*} \frac{\vartheta_j}{j} t^j\right)$. We define the auxiliary function $h_a^b(t) := \sum_{j=a}^{b} \frac{\vartheta_j}{j}(t^j - e^{-jv_n})$. We wish to apply again the saddle point method. In fact we decompose h as

$$
h_{x_1^*+1}^{x^*}(t) = g_{x_1^*+1}^{x^*}(t) - \sum_{x_1^*+1}^{x^*} \frac{\vartheta_j}{j} e^{-jv_n}.
$$

We now have

$$G_{\Theta}^{(3)}(t,n) = \left(g_{x_1^*+1}^{x^*}(t)\right)^2 g_{x^*+1}^{x_2^*}(t) - 2g_{x_1^*+1}^{x^*}(t) g_{x^*+1}^{x_2^*}(t) \left(\sum_{x_1^*+1}^{x^*} \frac{\vartheta_j}{j} e^{-jv_n}\right)$$
$$+ \left(\sum_{x_1^*+1}^{x^*} \frac{\vartheta_j}{j} e^{-jv_n}\right)^2 g_{x^*+1}^{x_2^*}(t).$$

It is clear then that in the first-order asymptotics $G_{\Theta}^{(3)}$ (as well as all other terms involving $t^j - r_n^j$) will not give any contribution, because $G_{\Theta}^{(3)}(r_n, n) = 0$. We ask then ourselves if admissibility holds true for each one of these terms, but this is fairly easy because of the previous computations. Indeed we can start for example with the middle one. We have already shown in (a) that

$$n^{-4\gamma}\frac{1}{h_n}[t^n]\left(e^{g_\Theta(t)} g_{x_1^*+1}^{x^*}(t) g_{x^*+1}^{x_2^*}(t)\right)$$

is log-n-admissible and the term $\left(\sum_{x_1^*+1}^{x^*} \frac{\vartheta_j}{j} e^{-jv_n}\right)$ is a constant independent of t. Both the other two summands are log-n-admissible with $r_n = e^{-v_n} := e^{-1/n^*}$: calculations can be performed in the same fashion as (a) and since they are direct we skip them.

□

4.4. **Large deviations estimates.** We are able to prove large deviations estimates for $w_n(\cdot)$ thanks to our method as well. In fact, knowing the behavior of the Laplace transform enables us to compute the asymptotics of the Young diagram in the limit. More precisely, let σ_n be the limit variance as in Thm. 4.5. Define the normalized moment generating function and its logarithm as

$$M(s) := \mathbb{E}\left[\exp\left(s\frac{(w_n(x) - (n^*)^\alpha w_\infty^{\mathbf{s}}(x))}{\sigma_n}\right)\right],$$
$$\Lambda(s) := \log M(s).$$

The strategy we adopt was first exploited in [22, Theorem 4.1], and relies on the fact that

Proposition 4.11. *There exist functions $\xi(n) = O\left((n^*)^\alpha\right)$, $\sigma(n) = O\left((n^*)^{\alpha/2}\right)$ such that for all $s = O\left(\sigma(n)\right)$ we obtain*

$$\Lambda(s) = \frac{s^2}{2} + O\left(\xi(n)\sigma(n)^{-3}\right)s^3.$$

It follows than that

$$\Lambda'(s) = O\left(\xi(n)\sigma(n)^{-3}\right)s^2,$$
$$\Lambda''(s) = O\left(\xi(n)\sigma(n)^{-3}\right)s.$$

From this we derive

Proposition 4.12. *For all* $a = O(\sigma_n)$ *let* $\delta := O\left(\xi(n)\sigma(n)^{-3}\right)$. *Then we have*

$$\mathbb{P}\left[\left|\frac{(w_n(x) - (n^*)^\alpha w_\infty^{\mathbf{s}}(x)}{\sigma_n} - a\right| < \epsilon\right] = \left(1 - \epsilon^{-2}(1+\delta)\right)\exp\left(-a^2/2 + O\left(\delta + \epsilon a\right)\right).$$

The error terms are absolute.

Proof. The proof can be performed analogously as [22], as we know that (4.29) holds. □

At this juncture we would like to apply our method to a simple but illustrative case.

4.5. **An example: the case** $g_\Theta(t) = (1-t)^{-1}$**.** We would like to begin by the easiest case, in other words to derive the limit shape for one point. We remark that here all our computations were performed using the function $g_\Theta(t) = t(1-t)^{-1}$. This does not affect the computations of the limit shape as it will "only" make a constant appear, which will be later simplified in all calculations.

Proposition 4.13. *For all* $x \in \mathbb{R}^+$

$$\frac{w_n(x\sqrt{n}) - \sqrt{n}e^{-x}}{n^{1/4}} \xrightarrow{\mathcal{L}} \mathcal{N}\left(0, e^{-x}\left(1 - \frac{1}{2}e^{-x}(x+1)^2\right)\right)$$

In particular, the limit shape is $w_\infty^{\mathbf{s}}(x) := e^{-x}$ *(cf. Thm. 4.5 plugging in* $\alpha = 1$*).*

We now pass to the joint behavior of $(w_n(x_1), \ldots, w_n(x_\ell))$ which can be recovered from

Proposition 4.14. *Let* $\ell \in \mathbb{N}^+$. *For all* $x_1, \ldots, x_\ell \in \mathbb{R}^+$, *set* $x_k^* := x_k\, n^{1/2}$; *then we have*

$$\left(\frac{w_n\left(x_k^*\right) - n^{1/2}e^{-x_k}}{n^{1/4}}\right)_{k=1,\ldots,\ell} \xrightarrow{\mathcal{L}} \mathcal{N}\left(\mathbf{0}, \Sigma\right)$$

with $\Sigma \in M_\ell(\mathbb{R})$ *defined through*

$$\begin{aligned} \Sigma_{k,k} &= -\tfrac{1}{2}\, e^{-x_k}\left(e^{-x_k}{x_k}^2 + 2\, e^{-x_k}x_k + e^{-x_k} - 1\right) \\ \Sigma_{k,j} &= -\tfrac{1}{2}\, e^{-x_k}\left(e^{-x_j}x_kx_j + e^{-x_j}x_k + e^{-x_j}x_j + e^{-x_j} - 1\right), \; j \neq k. \end{aligned} \tag{4.33}$$

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APPENDIX A. EULER MACLAURIN FORMULA WITH NON INTEGER BOUNDARIES

We prove in this section a slight extension of Euler Maclaurin formula, which allows to deal also with non-integer summation limits.

Theorem A.1. *Let $f : \mathbb{R} \to \mathbb{R}$ be a smooth function, $B_k(x)$ be the Bernoulli polynomials and $c < d$ with $c, d \in \mathbb{R}$. We then have for $p \in \mathbb{N}$*

$$\sum_{\lfloor c \rfloor \le k < d} f(k) = \int_c^d f(x)\,\mathrm{d}x - B_1(d - \lfloor d \rfloor) f(d) - B_1(c - \lfloor c \rfloor) f(c) + \sum_{k=1}^{p} (-1)^{k+1} \frac{B_{k+1}(d - \lfloor d \rfloor) f^{(k)}(d) - B_{k+1}(c - \lfloor c \rfloor) f^{(k)}(c)}{k!} + \frac{(-1)^{p+1}}{(p+1)!} \int_c^d B_{p+1}(x - \lfloor x \rfloor) f^{(p+1)}(x)\,\mathrm{d}x \tag{A.1}$$

Proof. The proof of this theorem follows the same lines as the proof of the Euler-Maclaurin summation formula with integer summation limits, see for instance [1, Theorem 3.1]. We give it here though for completeness. Our proof considers only the case $d \notin \mathbb{Z}$. The argumentation for $d \in \mathbb{Z}$ is completely similar. One possible definition of the Bernoulli polynomials is by induction:

$$B_0(y) \equiv 1, \tag{A.2}$$

$$B_k'(y) = k B_{k-1}(y) \text{ and } \int_0^1 B_k(y)\,\mathrm{d}y = 1 \text{ for } k \ge 1. \tag{A.3}$$

In particular, we have $B_1(y) = y - \frac{1}{2}$. We now have for $m \in \mathbb{Z}$

$$\begin{aligned}\int_m^{m+1} f(y)\,\mathrm{d}y &= \int_m^{m+1} B_0(y - m) f(y)\,\mathrm{d}y \\ &= [B_1(y-m) f(y)]|_{y=m}^{m+1} - \int_m^{m+1} B_1(y-m) f'(y)\,\mathrm{d}y \\ &= \frac{1}{2} f(m) + \frac{1}{2} f(m+1) - \int_m^{m+1} B_1(y - \lfloor y \rfloor) f'(y)\,\mathrm{d}y.\end{aligned}$$

since $B_1(0) = -\frac{1}{2}$ and $B_1(1) = \frac{1}{2}$. We obtain

$$\sum_{k=\lfloor c \rfloor}^{\lfloor d \rfloor} f(k) = \int_{\lfloor c \rfloor}^{\lfloor d \rfloor} f(x)\,\mathrm{d}x + \frac{1}{2} f(\lfloor c \rfloor) + \frac{1}{2} f(\lfloor d \rfloor) + \int_{\lfloor c \rfloor}^{\lfloor d \rfloor} B_1(y - \lfloor y \rfloor) f'(y)\,\mathrm{d}y.$$

Furthermore, we use

$$\int_{\lfloor d \rfloor}^{d} f(y)\,\mathrm{d}y = \frac{1}{2} f(\lfloor d \rfloor) + B_1(d - \lfloor d \rfloor) f(d) - \int_{\lfloor d \rfloor}^{d} B_1(y - \lfloor y \rfloor) f'(y)\,\mathrm{d}y.$$

and get

$$\sum_{k=\lfloor c \rfloor}^{\lfloor d \rfloor} f(k) = \int_{\lfloor c \rfloor}^{d} f(x)\,\mathrm{d}x + \frac{1}{2} f(\lfloor c \rfloor) - B_1(d - \lfloor d \rfloor) f(d) + \int_{\lfloor c \rfloor}^{d} B_1(y - \lfloor y \rfloor) f'(y)\,\mathrm{d}y.$$

The argumentation for replacing $\lfloor c \rfloor$ by c is similar. One gets

$$\sum_{\lfloor c \rfloor \le k < d} f(k) = \int_c^d f(x)\,\mathrm{d}x - B_1(c - \lfloor c \rfloor) f(c) - B_1(d - \lfloor d \rfloor) f(d) + \int_c^d B_1(y - \lfloor y \rfloor) f'(y)\,\mathrm{d}y.$$

The theorem now follows by successive partial integration of $\int_c^d B_1(y - \lfloor y \rfloor) f'(y)\, \mathrm{d}y$. $\square$

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