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Boundary element collocation methods using splines with multiple knots

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Abstract. We extend the theory of boundary element collocation methods by allowing reduced inter-element smoothness (or in other words, by allowing trial functions that are splines with multiple knots). Our convergence analysis is based on a recurrence relation for the Fourier coefficients of the numerical solution, and so is restricted to uniform grids on smooth, closed curves. Superconvergence is possible with special choices of the collocation points. Numerical experiments with a model problem confirm the convergence rates predicted by our theory.

1. Introduction

In this paper, we prove asymptotic error estimates for boundary element collocation methods in which the trial functions are splines with multiple knots, i.e., splines having less than the maximum possible smoothness. Such trial functions are often preferred to splines with simple knots (i.e., to *smoothest* splines), because enforcing high inter-element differentiability complicates the assembly of the stiffness matrix. We treat the case of *double* knots in some detail, and confirm our analysis with numerical experiments using continuous quadratic splines and Hermite cubic splines. Our results generalise those obtained by Prößdorf and Schmidt [15], Arnold and Wendland [1], Saranen [19] and Schmidt [20] for splines with simple knots. There is also a connection with the work of Szyska [22] on collocation methods for singular integral equations using splines with multiple knots.

Consider a boundary integral equation over a smooth, closed curve in the plane. Via a suitable parametric representation of the curve, we can recast this problem as a 1-periodic integral equation over the real line, or equivalently, as an integral equation over the additive group $\mathbf{T} = \mathbb{R}/\mathbb{Z}$, say

$$Lu = f. \tag{1.1}$$

Technically, we shall assume that L is a scalar, elliptic pseudodifferential operator on $\mathbf{T}$, having order $\beta \in \mathbb{R}$. It follows that

$$L : H^s \to H^{s-\beta} \tag{1.2}$$

is a Fredholm operator for all $s \in \mathbb{R}$, where $H^s = H^s(\mathbf{T})$ is the usual periodic Sobolev space of order s. We further assume that L has index zero (as is the case if, for instance, L is *strongly* elliptic) and that the homogeneous equation $Lu = 0$ has only the trivial solution $u = 0$. Thus, the linear operator (1.2) has a bounded inverse.

Let r, M and N be positive integers, and suppose that

$$1 \leq M \leq r \quad \text{and} \quad h = \frac{1}{N}. \tag{1.3}$$

We define a knot sequence

$$t_k = nh \quad \text{for } nM \leq k < (n+1)M, \tag{1.4}$$

so that each integer multiple of h is a knot with multiplicity M. Observe that $(n+N)h = nh$ and $t_{k+MN} = t_k + 1$, so we can think of n as an element of the cyclic group $\mathbb{Z}_N = \mathbb{Z}/N\mathbb{Z}$ and of k as an element of $\mathbb{Z}_{MN}$, provided the knots are viewed as points of $\mathbf{T}$. Let

$$S_h^{r,M} = S_h^{r,M}(\mathbf{T})$$

denote the space of 1-periodic splines of order r with knot sequence $(t_k)_{k\in\mathbb{Z}_{MN}}$. In other words, $S_h^{r,r}$ is the set of all 1-periodic piecewise polynomials of degree at most $r-1$ with breakpoints nh for $n \in \mathbb{Z}_N$, and

$$S_h^{r,M} = S_h^{r,r} \cap C^{r-M-1} \quad \text{for } 1 \le M < r,$$

where $C^k = C^k(\mathbf{T})$ is the space of 1-periodic, k times continuously differentiable functions.

Since $\dim S_h^{r,M} = MN$, we select M parameters

$$0 \le \epsilon_1 < \epsilon_2 < \cdots < \epsilon_M < 1, \tag{1.5}$$

and define MN collocation points,

$$x_{n,j} = (n+\epsilon_j)h \quad \text{for } n \in \mathbb{Z}_N \text{ and } 1 \le j \le M. \tag{1.6}$$

Our numerical method is then as follows: find $u_h \in S_h^{r,M}$ satisfying

$$(Lu_h)(x_{n,j}) = f(x_{n,j}) \quad \text{for } n \in \mathbb{Z}_N \text{ and } 1 \le j \le M. \tag{1.7}$$

By expressing u_h in terms of a basis for the trial space $S_h^{r,M}$, one obtains a square linear system of order MN.

Our aim is to study the convergence of u_h to u as $h \to 0$ (holding r and M fixed). In Section 2, after setting out some notation we state our main result as Theorem 2.2. The stability of the method—and thus in particular the existence and uniqueness of u_h for h sufficiently small—is determined by the behaviour of a certain $M \times M$ matrix-valued function

$$D : \mathbf{T} \times [-\tfrac{1}{2}, \tfrac{1}{2}] \to \mathbf{C}^{M\times M}. \tag{1.8}$$

The function D depends on the order r of the splines, on the knot multiplicity M, on the collocation parameters $\epsilon_1, \ldots, \epsilon_M$, and on the principal symbol of L. If the inverse matrix $D(x,y)^{-1}$ exists and is uniformly bounded (in some matrix norm) for $(x,y) \in \mathbf{T} \times [-\frac{1}{2}, \frac{1}{2}]$, then the collocation method (1.7) is stable, and u_h satisfies quasi-optimal error estimates for a range of Sobolev norms. In Theorem 2.3, we give a criterion for *superconvergence*, i.e., for faster convergence rates in lower-order Sobolev norms. When $M = 1$, the function D is scalar-valued, and our results reduce to those of [15], [1], [19] and [20].

The method of analysis in the papers just cited relies on the fact that a 1-periodic function $f : \mathbf{T} \to \mathbf{C}$ belongs to the space $S_h^{r,1}$ of smoothest splines of order r if and only if the Fourier coefficients of f satisfy the recurrence relation

$$(1.9) \qquad (l + kh)^r \hat{f}(k + lN) = (kh)^r \hat{f}(k) \quad \text{for } l \in \mathbb{Z} \text{ and } -N/2 \le k < N/2,$$

a result that goes back to Quade and Collatz [18]; see the historical remarks in the introduction to Gautschi [7]. In Section 3, we generalise (1.9) by allowing any knot multiplicity M with $1 \le M \le r$. For instance, in the case of double knots it turns out that $f \in S_h^{r,2}$ if and only if

$$(1.10) \qquad \begin{bmatrix} (2l - 1 + kh)^r \hat{f}(k - N + 2lN) \\ (2l + kh)^r \hat{f}(k + 2lN) \end{bmatrix} = \begin{bmatrix} 1 - 2l & 2l \\ -2l & 1 + 2l \end{bmatrix} \begin{bmatrix} (kh - 1)^r \hat{f}(k - N) \\ (kh)^r \hat{f}(k) \end{bmatrix}$$

for $l \in \mathbb{Z}$ and $-N/2 \le k < N/2$. The recurrence relation for the general case is given in Theorem 3.2, and shows that all Fourier coefficients of a spline $f \in S_h^{r,M}$ can be obtained from the MN consecutive coefficients

$$\hat{f}(k + pN) \quad \text{for } -N/2 \le k < N/2 \text{ and } -M/2 \le p < M/2,$$

reflecting the fact that $\dim S_h^{r,M} = MN$.

The heart of the paper is Section 4, where the error estimates of Theorems 2.2 and 2.3 are proved for the special case when L is *translation-invariant*, or in other words, when L has a convolution kernel and hence the principal symbol has constant coefficients. The general case then follows by a standard localization argument (i.e., by *freezing coefficients*).

In Section 5, we investigate the stability and superconvergence criteria when the splines have double knots ($M = 2$). To make the analysis tractable, we restrict our attention to cases in which the principal symbol and the interpolation points possess natural symmetries.

Finally, Section 6 presents the results of some numerical experiments that confirm our theoretical analysis for a standard model problem: the first-kind integral equation with logarithmic kernel (Symm's equation).

2. Statement of the Main Result

We denote the complex Fourier coefficients of a 1-periodic distribution $f : \mathbf{T} \to \mathbf{C}$ by

$$\hat{f}(m) = \int_{\mathbf{T}} e^{-i2\pi mx} f(x)\, dx \quad \text{for } m \in \mathbb{Z},$$

so that

$$f(x) = \sum_{m=-\infty}^{\infty} \hat{f}(m) e^{i2\pi mx} \quad \text{for } x \in \mathbf{T}.$$

Let J^s denote the 1-periodic Bessel potential of order $s \in \mathbb{R}$, i.e., let

$$(J^s f)(x) = \sum_{m=-\infty}^{\infty} \langle m \rangle^s \hat{f}(m) e^{i2\pi m x} \quad \text{where} \quad \langle m \rangle = \begin{cases} 1 & \text{if } m = 0, \\ |m| & \text{if } m \neq 0, \end{cases}$$

then the norm $\|\cdot\|_s$ in the Sobolev space $H^s = H^s(\mathbf{T})$ is given by

$$\|f\|_s^2 = \|J^s f\|_{L_2(\mathbf{T})}^2 = \sum_{m=-\infty}^{\infty} \langle m \rangle^{2s} |\hat{f}(m)|^2.$$

Our assumptions on L, set out in the Introduction, mean that

$$L = L_0 + L_1, \tag{2.1}$$

where L_0 has a homogeneous symbol of order β, and where the order of the pseudodifferential operator L_1 is strictly less than β. Thus,

$$(L_0 u)(x) = \sum_{m=-\infty}^{\infty} \sigma_0(x, m) \hat{u}(m) e^{i2\pi m x} \quad \text{for } x \in \mathbf{T},$$

where the symbol σ_0 satisfies

$$\sigma_0(x, t\xi) = t^\beta \sigma_0(x, \xi) \quad \text{for } x \in \mathbf{T},\ t > 0 \text{ and } 0 \neq \xi \in \mathbb{R}. \tag{2.2}$$

It follows that σ_0 has the form

$$\sigma_0(x, \xi) = a_+(x)|\xi|^\beta + a_-(x)\,\text{sign}(\xi)|\xi|^\beta \quad \text{for } x \in \mathbf{T} \text{ and } 0 \neq \xi \in \mathbb{R}, \tag{2.3}$$

where the coefficients a_+ and a_- are functions in $C^\infty(\mathbf{T})$. Changing $\sigma_0(x, 0)$ only perturbs L_0 by a smoothing operator, so we shall assume without loss of generality that the splitting (2.1) has been chosen in such a way that

$$\sigma_0(x, 0) = 1 \quad \text{for } x \in \mathbf{T}.$$

By hypothesis, L is elliptic, i.e., $\sigma_0(x, \xi) \neq 0$ for $|\xi| = 1$, so

$$a_+(x) + a_-(x) \neq 0 \quad \text{and} \quad a_+(x) - a_-(x) \neq 0 \quad \text{for all } x \in \mathbf{T}.$$

A well known consequence of ellipticity is that $L_0 : H^s \to H^{s-\beta}$ is a Fredholm operator with index $-\kappa$, where κ is the winding number of the closed curve traced out by the complex-valued function $(a_+ + a_-)/(a_+ - a_-)$, i.e.,

$$\kappa = \frac{1}{2\pi} \left[\arg \frac{a_+(x) + a_-(x)}{a_+(x) - a_-(x)} \right]_{x=0}^{1}.$$

Since $L_1 : H^s \to H^{s-\beta}$ is a compact operator, it follows that $L : H^s \to H^{s-\beta}$ is Fredholm with index $-\kappa$, and by hypothesis this index is zero, i.e., we assume

$$\kappa = 0.$$

The remaining assumption that $L : H^s \to H^{s-\beta}$ has a trivial null space then guarantees the existence of a bounded inverse $L^{-1} : H^{s-\beta} \to H^s$. Furthermore, via a canonical factorization of σ_0 one can explicitly construct a bounded inverse for the principal part $L_0 : H^s \to H^{s-\beta}$.

Next, we define the $M \times M$ matrices $X^r(y)$, $V(y)$, $\Sigma(x,y)$ and $\Phi_l(\epsilon)$ with pq-entries

$$\begin{aligned} X^r_{pq}(y) &= (p+y)^r \delta_{pq}, \\ V_{pq}(y) &= (p+y)^{q+\lfloor M/2 \rfloor}, \\ \Sigma_{pq}(x,y) &= \sigma_0(x, p+y)\delta_{pq}, \end{aligned} \tag{2.4}$$

and

$$\Phi_{l;pq}(\epsilon) = \frac{1}{M} \sum_{j=1}^{M} e^{i2\pi(q-p+Ml)\epsilon_j}. \tag{2.5}$$

In each case, the row index p and column index q range over

$$-M/2 \le p < M/2 \quad \text{and} \quad -M/2 \le q < M/2,$$

so for instance, when $M = 2$,

$$X^r(y) = \begin{bmatrix} (y-1)^r & 0 \\ 0 & y^r \end{bmatrix}, \qquad V(y) = \begin{bmatrix} 1 & y-1 \\ 1 & y \end{bmatrix}$$

and

$$\Sigma(x,y) = \begin{bmatrix} \sigma_0(x, y-1) & 0 \\ 0 & \sigma_0(x,y) \end{bmatrix}.$$

The matrix $D(x,y)$ mentioned in the Introduction is defined by

$$D(x,y) = \Phi_0(\epsilon) + Z(x,y)X^r(y)\Sigma(x,y)^{-1} \quad \text{for } x \in \mathbf{T} \text{ and } -\tfrac{1}{2} \le y \le \tfrac{1}{2}, \tag{2.6}$$

where

$$Z(x,y) = \sum_{l \ne 0} \Phi_l(\epsilon)\Sigma(x, Ml+y)X^{-r}(Ml+y)V(Ml+y)V(y)^{-1}. \tag{2.7}$$

We remark that the product $X^r(y)\Sigma(x,y)^{-1}$ is continuous if $\beta < r$, even though $\Sigma(x,y)^{-1}$ is generally discontinuous at $y = 0$. Note that if the principal symbol has constant coefficients, then we can write $\sigma_0(x,\xi) = \sigma_0(\xi)$, $\Sigma(x,y) = \Sigma(y)$, $Z(x,y) = Z(y)$ and $D(x,y) = D(y)$.

Let us introduce the following terminology.

Definition 2.1 *The collocation method* (1.7) *is* stable *if*

$$|D(x,y)^{-1}| \le C \quad \textit{for } x \in \mathbf{T} \textit{ and } -\tfrac{1}{2} \le y \le \tfrac{1}{2}.$$

Our main result can now be stated as follows.

Theorem 2.2 *Consider the collocation method* (1.7) *applied to the pseudodifferential equation* (1.1). *Assume*

$$\beta + M < r, \tag{2.8}$$

and let s and t be real numbers satisfying

$$s < r - M + \tfrac{1}{2}, \quad \beta + \tfrac{1}{2} < t, \quad s \le t \le r. \tag{2.9}$$

If the method is stable and if $u \in H^t$, then the collocation equations are uniquely solvable for all h sufficiently small, and moreover

$$\|u_h - u\|_s \le Ch^{t-s}\|u\|_t \quad \textit{provided } \beta \le s. \tag{2.10}$$

The hypothesis $s < r - M + \frac{1}{2}$ implies that $S_h^{r,M} \subset H^s$ (see Theorem 3.4), so the error estimate (2.10) makes sense. Via the Sobolev imbedding,

$$H^{\delta+1/2} \subset C^{0,\delta} \quad \text{for } 0 < \delta < 1,$$

we see that the assumption (2.8) ensures Lu_h is (Hölder) continuous, and likewise the inequality $\beta + \frac{1}{2} < t$ from (2.9) implies that the right hand side f is continuous if the exact solution $u \in H^t$. Hence, the pointwise values in the collocation equations (1.7) are well-defined. The assumption (2.8) also ensures that the sum in the definition (2.7) of $Z(x,y)$ is absolutely convergent; see (4.3). The remaining inequalities in (2.9) are explained by the approximation properties of the trial space:

$$\inf_{v \in S_h^{r,M}} \|u - v\|_s \le Ch^{t-s}\|u\|_t \quad \text{for } s \le t \le r \text{ and } s < r - M + \tfrac{1}{2};$$

cf. Theorem 3.4 iv). The highest rate of convergence given by the error estimate (2.10) occurs when $s = \beta$ and $t = r$, in which case we have

$$\|u_h - u\|_\beta \le Ch^{r-\beta}\|u\|_r, \tag{2.11}$$

provided, of course, that $u \in H^r$. For $M = 1$, Saranen [19] showed that sometimes a superconvergence effect occurs, allowing one to relax the restriction $\beta \le s$ in (2.10). We show for a general M that the following holds.

Theorem 2.3 *Assume that the principal symbol* (2.3) *of L has constant coefficients a_+ and a_-, that* (2.8) *holds, that the real numbers s and t satisfy* (2.9) *and that*

$$\frac{1}{2} < b \leq r - \beta.$$

If the collocation method is stable, if

$$\text{(2.12)} \qquad 00\text{-}entry\ of\ D(y)^{-1}Z(y) = O(|y|^b) \quad as\ y \to 0,$$

and if in the splitting (2.1),

$$\text{(2.13)} \qquad L_1 \text{ is a pseudodifferential operator of order } \beta - b,$$

then

$$\text{(2.14)} \qquad \|u_h - u\|_s \leq Ch^{t-s}\|u\|_{t+\beta-s} \quad \text{provided } \beta - b \leq s \leq \beta.$$

The highest rate of convergence given by (2.14) occurs when $s = \beta - b$ and $t = r$, in which case

$$\text{(2.15)} \qquad \|u_h - u\|_{\beta-b} \leq Ch^{r-\beta+b}\|u\|_{r+b},$$

giving an improvement of $O(h^b)$ over (2.11) at the cost of some additional regularity of u.

3. Splines with Multiple Knots

Our first task in this section is to characterize the spline functions from $S_h^{r,M}$ in terms of their Fourier coefficients. We begin with a simple observation.

Lemma 3.1 *A 1-periodic distribution f belongs to $S_h^{r,M}$ if and only if there exist trigonometric polynomials a_q for $0 \leq q \leq M-1$, each with unit period, such that*

$$\text{(3.1)} \qquad m^r \hat{f}(m) = \sum_{q=0}^{M-1} m^q a_q(mh) \quad \text{for all } m \in \mathbb{Z}.$$

Proof. Since the r-th derivative of a polynomial of degree $r-1$ is identically zero, one sees that if $f \in S_h^{r,M}$ then

$$\text{(3.2)} \qquad f^{(r)}(x) = \sum_{q=0}^{M-1} \sum_{n \in \mathbb{Z}_N} c_{nq}\delta^{(q)}(x - nh),$$

where δ is the 1-periodic Dirac delta function, and the coefficients c_{nq} are just the jumps in the derivatives of f at the breakpoints:

$$c_{nq} = f^{(r-q+1)}(nh+) - f^{(r-q+1)}(nh-).$$

Hence, the Fourier coefficients of f satisfy

$$(\mathrm{i}2\pi m)^r \hat{f}(m) = \sum_{q=0}^{M-1} \sum_{n\in\mathbb{Z}_N} c_{nq}(\mathrm{i}2\pi m)^q e^{-\mathrm{i}2\pi mnh},$$

so (3.1) holds with $a_q(\xi) = \sum_{n=0}^{N-1} c_{nq}(\mathrm{i}2\pi)^{q-r} e^{-\mathrm{i}2\pi n\xi}$. Conversely, if (3.1) holds, then $f^{(r)}$ has the form (3.2), and thus $f \in S_h^{r,M}$. □

It is convenient to define an index set

$$\Lambda_h = \{\, k \in \mathbb{Z} : -N/2 \le k < N/2 \,\},$$

and to let $\hat{f}(l,k)$ denote the M-dimensional column vector with components

$$\hat{f}_p(l,k) = \hat{f}(k + pN + MlN) \quad \text{for } -M/2 \le p < M/2,\ l \in \mathbb{Z} \text{ and } k \in \Lambda_h;$$

for instance,

$$\hat{f}(l,k) = \begin{bmatrix} \hat{f}(k-N+2lN) \\ \hat{f}(k+2lN) \end{bmatrix} \quad \text{when } M = 2.$$

Recall the definitions of $X^r(y)$ and $V(y)$ given in (2.4); the next theorem gives the desired Fourier characterization of $S_h^{r,M}$, and shows that $\hat{f}(0,k)$ determines $\hat{f}(l,k)$ for all $l \in \mathbb{Z}$ whenever $f \in S_h^{r,M}$. Notice that in the special case $M = 1$, we get back to the recurrence relation (1.9) of Quade and Collatz.

Theorem 3.2 *A 1-periodic distribution f belongs to $S_h^{r,M}$ if and only if*

$$X^r(Ml+kh)\hat{f}(l,k) = V(Ml+kh)V(kh)^{-1}X^r(kh)\hat{f}(0,k)$$

for all $l \in \mathbb{Z}$ and $k \in \Lambda_h$.

Proof. Putting $m = k + pN + MlN$ and noting that $a_q(mh) = a_q(kh)$, we see that (3.1) is equivalent to

$$(kh+p+Ml)^r \hat{f}(k+pN+MlN) = \sum_{q=0}^{M-1} (kh+p+Ml)^q h^{r-q} a_q(kh)$$

for $k \in \Lambda_h$, $-M/2 \le p < M/2$ and $l \in \mathbb{Z}$, and in turn this condition is equivalent to

$$X^r(Ml+kh)\hat{f}(l,k) = V(Ml+kh) \begin{bmatrix} h^r a_0(kh) \\ \vdots \\ h^{r-M+1} a_{M-1}(kh) \end{bmatrix}.$$

The Vandermonde matrix $V(y)$ is non-singular for all $y \in \mathbb{R}$ because

$$\det V(y) = \prod_{1\le p<q\le M} (q-p) \ne 0,$$

so if $f \in S_h^{r,M}$ then the combination

$$V(Ml+kh)^{-1}X^r(Ml+kh)\hat{f}(l,k) = \begin{bmatrix} h^r a_0(kh) \\ \vdots \\ h^{r-M+1}a_{M-1}(kh) \end{bmatrix} \tag{3.3}$$

is independent of l, giving

$$V(Ml+kh)^{-1}X^r(Ml+kh)\hat{f}(l,k) = V(kh)^{-1}X^r(kh)\hat{f}(0,k) \tag{3.4}$$

for $l \in \mathbb{Z}$ and $k \in \Lambda_h$. Conversely, if the Fourier coefficients of f satisfy (3.4) then it is possible to find trigonometric polynomials a_q for $0 \le q \le M-1$ such that (3.3) holds for $k \in \Lambda_h$ and $l = 0$, implying that (3.1) holds and therefore $f \in S_h^{r,M}$. □

Remark 3.3 It is not difficult to show using elementary facts about polynomial interpolation that the matrix product $V(x+y)V(y)^{-1}$ is independent of y, and has entries that are polynomials in x of degree $\le M-1$. Hence, we may define the $M \times M$ matrix

$$W_l = V(Ml+y)V(y)^{-1} \quad \text{for } l \in \mathbb{Z};$$

for instance, we see from (2.4) that

$$W_l = \begin{bmatrix} 1-2l & 2l \\ -2l & 1+2l \end{bmatrix} \quad \text{when } M = 2,$$

which explains (1.10) in the Introduction. Actually, for our purposes it suffices to know that

$$|W_l| \le C(1+|l|)^{M-1} \quad \text{for } l \in \mathbb{Z}, \tag{3.5}$$

and we shall make no use of the fact that W_l does not depend on y.

Theorem 3.2 shows that for each 1-periodic distribution f there exists a unique spline $P_h f = P_h^{r,M} f \in S_h^{r,M}$ satisfying

$$\widehat{P_h f}(k+pN) = \hat{f}(k+pN) \quad \text{for } k \in \Lambda_h \text{ and } -M/2 \le p < M/2,$$

and in this way, we obtain a projection P_h onto $S_h^{r,M}$. An alternative definition of $P_h f$ is

$$\langle P_h f, \phi \rangle = \langle f, \phi \rangle \quad \text{for all } \phi \in S_h^{\infty,M},$$

where

$$S_h^{\infty,M} = \text{span}\{\, e^{i2\pi(k+pN)x} : k \in \Lambda_h \text{ and } -M/2 \le p < M/2 \,\} \tag{3.6}$$

is a space of trigonometric polynomials, and $\langle\cdot,\cdot\rangle$ is the inner product in $L_2(\mathbf{T})$. We also introduce a discrete seminorm $\|\cdot\|_{s,h}$ given by

$$\|f\|_{s,h}^2 = \|P_h f\|_s^2 = \sum_{-M/2\le p<M/2} \sum_{k\in\Lambda_h} \langle k+pN\rangle^{2s} |\hat{f}(k+pN)|^2; \tag{3.7}$$

notice that the restriction of $\|\cdot\|_{s,h}$ to $S_h^{r,M}$ is a norm.

The approximation property and inverse inequality are are well-known for finite-element spaces. In parts iii) and iv) of the next theorem, we give a new and rather simple proof of these results for $S_h^{r,M}$ using the Fourier characterization of Theorem 3.2. References [6], [5, §6.1] and [16, Theorems 2.6 and 2.11] all contain proofs of these results for the complete range of Sobolev indices in the case of smoothest splines ($M = 1$) on a quasi-uniform grid.

In part v) of the next theorem, we prove the local approximation property for $S_h^{r,M}$. Related results have been exploited by Nitsche and Schatz [11, Assumption A.2] in the study of local error estimates for finite element methods. We shall use the local approximation property to justify the extension of our error estimates from the special case when the principal symbol of L has constant coefficients to the general case of variable coefficients. This idea was introduced by Arnold and Wendland [1] and, independently, by Prößdorf [12], [13]. Our proof of v) below generalises the one used in [1, Lemma 3.1] for $M = 1$; a different approach, again only for smoothest splines, appears in [17, Theorem 2.13]. For the case $s = 0$ and $M \ge 1$, a different proof of v) is given by Hagen, Roch and Silbermann [8] (see Theorem 2.8 and Section 2.12.4).

Theorem 3.4 *If $s < r - M + \frac{1}{2}$, then $S_h^{r,M} \subset H^s$ and the following hold.*

i) The norms $\|\cdot\|_{s,h}$ and $\|\cdot\|_s$ are uniformly equivalent on $S_h^{r,M}$:

$$c\|f\|_s \le \|f\|_{s,h} \le \|f\|_s \quad \text{for } f \in S_h^{r,M}.$$

ii) The projections P_h are uniformly bounded in H^s:

$$\|P_h f\|_s \le C\|f\|_s \quad \text{for } f \in H^s.$$

iii) The projections P_h have the approximation property:

$$\|f - P_h f\|_s \le Ch^{t-s}\|f\|_t \quad \text{for } s \le t \le r.$$

iv) There is an inverse inequality:

$$\|f\|_t \le Ch^{s-t}\|f\|_s \quad \text{for } s \le t < r - M + \tfrac{1}{2} \text{ and } f \in S_h^{r,M}.$$

v) The projections P_h have the local approximation property: given a fixed $g \in C^r(\mathbf{T})$,

$$\|gf - P_h(gf)\|_s \le Ch^{\rho}\|f\|_s \quad \text{for } f \in S_h^{r,M},$$

where $\rho = \min(1, r - M + 1 - s) > \frac{1}{2}$.

Proof. It is convenient to define the $M \times M$ diagonal matrix

$$Y^s(l,k) = \mathrm{diag}[\langle k + pN + MlN\rangle^s]_{-M/2 \le p < M/2},$$

so that

$$\widehat{J^s f}(l,k) = Y^s(l,k)\hat{f}(l,k). \tag{3.8}$$

Suppose $f \in S_h^{r,M}$, and observe that by Theorem 3.2 and Remark 3.3,

$$\widehat{J^s f}(l,k) = Y^s(l,k)X^{-r}(Ml + kh)W_l X^r(kh)Y^{-s}(0,k)\widehat{J^s f}(0,k).$$

We see that for $l \neq 0$, $k \in \Lambda_h$ and $s \le r$,

$$\begin{aligned}|Y^s(l,k)X^{-r}(Ml+kh)| &= \max_{-M/2 \le p < M/2} \langle k + pN + MlN\rangle^s |Ml + kh + p|^{-r} \\ &\le h^{-s} \max_{(l-1/2)M \le y < (l+1/2)M} |y|^{s-r} \le Ch^{-s}|l|^{s-r},\end{aligned}$$

and

$$\begin{aligned}|X^r(kh)Y^{-s}(0,k)| &= \max_{-M/2 \le p < M/2} |kh + p|^r \langle k + pN\rangle^{-s} \\ &\le h^s \max_{|y| \le M/2} |y|^{r-s} \le Ch^s,\end{aligned}$$

so by (3.5),

$$|\widehat{J^s f}(l,k)| \le C|l|^{s-r+M-1}|\widehat{J^s f}(0,k)| \quad \text{for } 0 \neq l \in \mathbb{Z} \text{ and } k \in \Lambda_h. \tag{3.9}$$

Hence, using the assumption that $s < r - M + \frac{1}{2}$,

$$\begin{aligned}\|f\|_s^2 - \|f\|_{s,h}^2 &= \sum_{l \neq 0} \sum_{k \in \Lambda_h} |\widehat{J^s f}(l,k)|^2 \le C \sum_{k \in \Lambda_h} |\widehat{J^s f}(0,k)|^2 \sum_{l \neq 0} |l|^{2(s-r+M-1)} \\ &\le C \frac{1}{(r - M + \frac{1}{2}) - s} \|f\|_{s,h}^2,\end{aligned}$$

which shows that $S_h^{r,M} \subseteq H^s$ and $\|f\|_s \le C\|f\|_{s,h}$. Since the inequality $\|f\|_{s,h} \le \|f\|_s$ is trivial, we have proved part i).

By (3.7), part ii) follows at once from part i).

To prove the approximation property, let $f \in H^t$ with $s \le t \le r$, and put

$$\Lambda_h^M = \{\, k + pN : k \in \Lambda_h \text{ and } -M/2 \le p < M/2 \,\}.$$

We write

$$\|f - P_h f\|_s^2 = \sum_{m \notin \Lambda_h^M} |m|^{2s} |\hat{f}(m) - \widehat{P_h f}(m)|^2 \le 2(\mathrm{I} + \mathrm{II}),$$

where

$$\mathrm{I} = \sum_{m \notin \Lambda_h^M} |m|^{2s} |\hat{f}(m)|^2 \quad \text{and} \quad \mathrm{II} = \sum_{m \notin \Lambda_h^M} |m|^{2s} |\widehat{P_h f}(m)|^2.$$

Since $|m| \geq N/2$ for all $m \notin \Lambda_h^M$,

$$\mathrm{I} \leq \sum_{|m| \geq N/2} |m|^{2(s-t)} |m|^{2t} |\hat{f}(m)|^2 \leq (N/2)^{2(s-t)} \|f\|_t^2 \leq C h^{2(t-s)} \|f\|_t^2,$$

and the matrix norm estimates leading to (3.9) show that

$$\begin{aligned} |(J^s P_h f)\hat{}\,(l,k)| &= \left|Y^s(l,k) X^{-r}(Ml + kh) W_l X^r(kh) Y^{-t}(k,0) \widehat{J^t f}(0,k)\right| \\ &\leq C |l|^{s-r+M-1} h^{t-s} |\widehat{J^t f}(0,k)| \end{aligned}$$

so

$$\mathrm{II} \leq \sum_{l \neq 0} \sum_{k \in \Lambda_h} |\widehat{J^s f}(l,k)|^2 \leq C h^{2(t-s)} \|f\|_t.$$

The proof of part iii) is now complete.

If we use the discrete norm, then the inverse inequality

$$\|f\|_{t,h} \leq C h^{s-t} \|f\|_{s,h}$$

holds for any real numbers s and t satisfying $s \leq t$, and for any 1-periodic distribution f. Thus, part iv) follows at once from part i).

To prove part v), we denote the n-th subinterval by $I_n = \big(nh, (n+1)h\big)$ for $n \in \mathbb{Z}_N$, and define the periodic *piecewise* Sobolev space

$$H_h^{l,M} = \{\, f \in H^{l-M} : u^{(l-M)}|_{I_n} \in H^M(I_n) \text{ for each } n \in \mathbb{Z}_N \,\}$$

for integers $l \geq M$. If $f \in H_h^{l,l}$, then for $0 \leq j \leq l$ we let $f^{\langle j \rangle} \in L_2(\mathbf{T})$ denote the piecewise (not distributional) derivative of f. We also define

$$Df(x) = \frac{1}{2\pi \mathrm{i}} f'(x) + \hat{f}(0),$$

and note that $D : H^t \to H^{t-1}$ is an isometry for all $t \in \mathbb{R}$, and that

$$D^{r-M} : S_h^{r,M} \to S_h^{M,M}$$

is an isomorphism. The proof of the following lemma is deferred until later in this section.

Lemma 3.5 *If $s < \frac{1}{2}$, then*

$$\|f - P_h^{M,M} f\|_s \leq C h^{M-s} \|f^{\langle M \rangle}\|_0 \quad \textit{for } f \in H_h^{M,M}.$$

Define a projection $\widetilde{P}_h^{r,M}$ onto $S_h^{r,M}$ by

$$\widetilde{P}_h^{r,M} = D^{M-r} P_h^{M,M} D^{r-M},$$

and suppose that $f \in S_h^{r,M}$, $g \in C^r(\mathbf{T})$ and $s < r - M + \frac{1}{2}$. We observe that

$$\begin{aligned} \|gf - \widetilde{P}_h^{r,M}(gf)\|_s &= \|D^{M-r}(I - P_h^{M,M})D^{r-M}(gf)\|_s \\ &= \|(I - P_h^{M,M})D^{r-M}(gf)\|_{s-r+M} \end{aligned}$$

and $D^{r-M}(gf) \in H_h^{M,M}$, so Lemma 3.5 implies that

$$\|gf - \widetilde{P}_h^{r,M}(gf)\|_s \leq Ch^{r-s}\|[D^{r-M}(gf)]^{\langle M\rangle}\|_0.$$

Since $f^{\langle r\rangle} = 0$ and $f^{\langle r-M\rangle} = f^{(r-M)}$, the Leibniz formula and the Bramble-Hilbert lemma give

$$\begin{aligned} \|[D^{r-M}(gf)]^{\langle M\rangle}\|_0 &\leq C\sum_{j=0}^{r-1}\|f^{\langle j\rangle}\|_0 \\ &\leq C\|f\|_{r-M} + C\sum_{j=r-M+1}^{r-1} h^{r-M-j}\|f^{\langle r-M\rangle}\|_0 \\ &\leq Ch^{1-M}\|f\|_{r-M}, \end{aligned}$$

(with C depending on g) and hence by the inverse inequality iv),

$$\|gf - \widetilde{P}_h^{r,M}(gf)\|_s \leq Ch^{r-s+1-M}\|f\|_{r-M} \leq Ch^{\rho}\|f\|_s.$$

Finally, $I - P_h^{r,M} = (I - P_h^{r,M})(I - \widetilde{P}_h^{r,M})$ and the $P_h^{r,M}$ are uniformly bounded, so both projections have the local approximation property v). □

Proof of Lemma 3.5. First we construct a projection Q_h onto $S_h^{M,M}(0,1)$ satisfying

(3.10) $$\|f - Q_h f\|_{H^s(0,1)} \leq Ch^{M-s}\|f\|_{H^M(0,1)} \quad \text{for } s < \tfrac{1}{2} \text{ and } f \in H^M(0,1).$$

(Note that here f may be *non*-periodic.) Let $\mathbf{T}_2 = \mathbb{R}/2\mathbb{Z}$, so that the functions in $H^s(\mathbf{T}_2)$ are 2-periodic, and let $E : H^s(0,1) \to H^s(\mathbf{T}_2)$ be an extension operator, i.e., a bounded linear operator satisfying $Ef|_{(0,1)} = f$ for all $f \in H^s(0,1)$. For instance, we can construct E using a standard extension operator from $H^s(0,1)$ to $H^s(\mathbb{R})$, together with a suitable cutoff function so that Ef has compact support in $(-1/2, 3/2)$. Write $P_h = P_h^{M,M}$ for brevity, and then define a projection $P_{h,2}$ onto $S_h^{M,M}(\mathbf{T}_2)$ by $P_{h,2}f = B^{-1}P_h Bf|_{(0,1)}$, where $Bf(x) = f(2x)$. Our projection Q_h is then defined by

$$Q_h f = (P_{h,2}Ef)|_{(0,1)}.$$

The approximation property of P_h implies that

$$\|f - P_{h,2}f\|_{H^s(\mathbf{T}_2)} \le Ch^{M-s}\|f\|_{H^M(\mathbf{T}_2)} \quad \text{for } s < \tfrac{1}{2} \text{ and } f \in H^M(\mathbf{T}_2),$$

and therefore, because $f - Q_h f = (Ef - P_{h,2}Ef)|_{(0,1)}$,

$$\begin{aligned}\|f - Q_h f\|_{H^s(0,1)} &\le C\|Ef - P_{h,2}Ef\|_{H^s(\mathbf{T}_2)} \\ &\le Ch^{M-s}\|Ef\|_{H^M(\mathbf{T}_2)} \le Ch^{M-s}\|f\|_{H^M(0,1)},\end{aligned}$$

which proves (3.10).

Next, we show that

$$\|f - Q_h f\|_s \le Ch^{M-s}\|f^{(M)}\|_{L_2(0,1)} \quad \text{for } s < \tfrac{1}{2} \text{ and } f \in H^M(0,1). \tag{3.11}$$

By Taylor's theorem, $f = f_1 + f_2$ where

$$f_1(x) = \sum_{j=0}^{M-1} \frac{f^{(j)}(0)}{j!}x^j \quad \text{and} \quad f_2(x) = \frac{1}{(M-1)!}\int_0^x (x-t)^{M-1} f^{(M)}(t)\,dt.$$

The polynomial f_1 belongs to $S_h^{M,M}(0,1)$, so $f_1 - Q_h f_1 = 0$, implying that

$$\|f - Q_h f\|_{H^s(0,1)} = \|f_2 - Q_h f_2\|_{H^s(0,1)} \le Ch^{M-s}\|f_2\|_{H^M(0,1)},$$

and the function f_2 satisfies

$$f_2^{(j)}(x) = \frac{1}{(M-1-j)!}\int_0^x (x-t)^{M-1-j} f^{(M)}(t)\,dt \quad \text{for } 0 \le j \le M-1,$$

with $f_2^{(M)}(x) = f^{(M)}(x)$, so $\|f_2\|_{H^M(0,1)} \le C\|f^{(M)}\|_{L_2(0,1)}$ and (3.11) holds.

Suppose now that $f \in H_h^{M,M}(0,1)$, and choose $\psi \in S_h^{M,M}(0,1)$ such that $f + \psi \in H^M(0,1)$. (In general, $f + \psi \notin H^M(\mathbf{T})$, however, which is why we require the non-periodic projection Q_h.) Since $\psi - Q_h\psi = 0$ and $\psi^{\langle M\rangle} = 0$, we have

$$\begin{aligned}\|f - Q_h f\|_{H^s(0,1)} &= \|(f+\psi) - Q_h(f+\psi)\|_{H^s(0,1)} \\ &\le Ch^{M-s}\|(f+\psi)^{(M)}\|_{L_2(0,1)} = Ch^{M-s}\|f^{\langle M\rangle}\|_{L_2(0,1)},\end{aligned}$$

for $s < \frac{1}{2}$. Finally, because the non-periodic spaces $H_h^{M,M}(0,1)$ and $S_h^{M,M}(0,1)$ are really no different from their periodic versions $H_h^{M,M}(\mathbf{T})$ and $S_h^{M,M}(\mathbf{T})$, and because also $H^s(0,1)$ and $H^s(\mathbf{T})$ are really the same if $s < \frac{1}{2}$, the result follows at once after writing $I - P_h = (I - P_h)(I - Q_h)$ and noting that the P_h are bounded on H^s. □

4. Error Estimates

Our strategy for proving the convergence of the collocation method (1.7) is the same as in [1]. We begin by studying the special case when $L = L_0$ and the symbol (2.3) has constant coefficients a_+ and a_-. We shall see that the MN collocation equations for $u_h \in S_h^{r,M}$,

$$(L_0 u_h)(x_{n,j}) = (L_0 u)(x_{n,j}) \quad \text{for } n \in \mathbb{Z}_N \text{ and } 1 \le j \le M, \tag{4.1}$$

are equivalent to N uncoupled $M \times M$ linear systems for the MN Fourier coefficients $\hat{u}_h(k + pM)$, where $k \in \Lambda_h$ and $-M/2 \le p < M/2$.

The following lemma explains the origin of the matrix $\Phi_l(\epsilon)$ defined in (2.5).

Lemma 4.1 *If $g \in H^t$ for some $t > \frac{1}{2}$, then the MN scalar equations*

$$g(x_{n,j}) = 0 \quad \text{for } n \in \mathbb{Z}_N \text{ and } 1 \le j \le M,$$

are equivalent to the N vector equations

$$\sum_{l=-\infty}^{\infty} \Phi_l(\epsilon)\hat{g}(l,k) = 0 \quad \text{for } k \in \Lambda_h.$$

Proof. The hypothesis of the lemma ensures that the Fourier series of g is absolutely convergent, so a simple argument involving discrete Fourier transformation shows that $g(x_{n,j}) = 0$ for all $n \in \mathbb{Z}_N$ (and a fixed j) if and only if

$$\sum_{m=-\infty}^{\infty} \hat{g}(k + mN)e^{i2\pi m\epsilon_j} = 0 \text{ for all } k \in \Lambda_h;$$

see, e.g., [1, Lemma 2.1]. Writing $m = q + Ml$, we see that

$$\sum_{j=1}^{M} e^{-i2\pi p\epsilon_j}\left\{\sum_{m=-\infty}^{\infty} \hat{g}(k + mN)e^{i2\pi m\epsilon_j}\right\}$$
$$= \sum_{l=-\infty}^{\infty} \sum_{-M/2 \le q < M/2} \left\{\sum_{j=1}^{M} e^{i2\pi(q-p+Ml)\epsilon_j}\right\}\hat{g}(k + qN + MlN),$$

and the right hand side is just the p-th component of $M\sum_{l=-\infty}^{\infty}\Phi_l(\epsilon)\hat{g}(l,k)$. It is easy to check that the Vandermonde-like matrix $[e^{-i2\pi p\epsilon_j}]$ is non-singular because by (1.5) the numbers $e^{-i2\pi\epsilon_j}$ $(1 \le j \le M)$ are distinct. The result follows at once. $\square$

Recalling the homogeneity property (2.2) of the principal symbol σ_0, we find that for $l \in \mathbb{Z}$, $k \in \Lambda_h$ and any distribution $u : \mathbf{T} \to \mathbf{C}$,

$$\widehat{L_0 u}(l,k) = \begin{cases} \Sigma_h \hat{u}(0,0) & \text{if } l = k = 0, \\ h^{-\beta}\Sigma(Ml + kh)\hat{u}(l,k) & \text{otherwise,} \end{cases} \tag{4.2}$$

where $\Sigma_h = \text{diag}[\sigma_0(pN)]_{-M/2\le p<M/2}$ and $\Sigma(y) = \text{diag}[\sigma_0(p+y)]_{-M/2\le p<M/2}$; cf. (2.4). We put

$$\Lambda_h^* = \{\, k \in \Lambda_h : k \neq 0 \,\},$$

and recall the definition (2.6) of the matrix $D(y)$.

Lemma 4.2 *Assume $\beta + M < r$. If $u_h \in S_h^{r,M}$ and $u \in H^t$ for some $t > \beta + \frac{1}{2}$, then the collocation equations* (4.1) *are equivalent to*

$$D(0)\Sigma_h \hat{u}_h(0,0) = \sum_{l=-\infty}^{\infty} \Phi_l(\epsilon)\widehat{L_0 u}(l,0)$$

and

$$D(kh)\Sigma(kh)\hat{u}_h(0,k) = h^\beta \sum_{l=-\infty}^{\infty} \Phi_l(\epsilon)\widehat{L_0 u}(l,k) \quad \textit{for } k \in \Lambda_h^*.$$

Proof. Taking $g = L_0 u_h - L_0 u$ in Lemma 4.1, we see that (4.1) holds if and only if

$$\sum_{l=-\infty}^{\infty} \Phi_l(\epsilon)\widehat{L_0 u_h}(l,k) = \sum_{l=-\infty}^{\infty} \Phi_l(\epsilon)\widehat{L_0 u}(l,k) \quad \text{for } k \in \Lambda_h.$$

The result now follows using (4.2) and Theorem 3.2, but we have to check that the infinite sum from the definition (2.7) of $Z(y)$ converges absolutely. In fact, using matrix norm estimates from Remark 3.3 and the proof of Theorem 3.4

$$\begin{aligned} |Z(y)| &\le \sum_{l\neq 0} |\Phi_l(\epsilon)||\Sigma(Ml+y)||X^{-r}(Ml+y)||W_l| \\ &\le C\sum_{l\neq 0} |l|^{\beta-r+M-1} \le \frac{C}{r-(\beta+M)}, \end{aligned} \tag{4.3}$$

because $\beta + M < r$. □

It follows from Lemma 4.2 that

$$D(0)\Sigma_h(u_h - u)\hat{}(0,0) = \{\Phi_0(\epsilon) - D(0)\}\Sigma_h\hat{u}(0,0) + \sum_{l\neq 0} \Phi_l(\epsilon)\widehat{L_0 u}(l,0) \tag{4.4}$$

and

$$\begin{aligned} D(kh)\Sigma(kh)(u_h - u)\hat{}(0,k) = \{\Phi_0(\epsilon) - D(kh)\}\Sigma(kh)\hat{u}(0,k) \\ + h^\beta \sum_{l\neq 0} \Phi_l(\epsilon)\widehat{L_0 u}(l,k) \end{aligned} \tag{4.5}$$

for $k \in \Lambda_h^*$. We use these equations to estimate the errors in the Fourier coefficients of u_h. It is convenient to let

$$Y_h^s = \text{diag}[\langle pN\rangle^s]_{-M/2\le p<M/2} \quad \text{and} \quad Y^s(y) = \text{diag}[|y+p|^s]_{-M/2\le p<M/2},$$

so that the matrix $Y^s(l,k)$ from (3.8) is given by

$$Y^s(l,k) = \begin{cases} Y_h^s & \text{if } l = k = 0, \\ h^{-s}Y^s(Ml+kh) & \text{otherwise;} \end{cases}$$

cf. (4.2). In the next three lemmas, we estimate the terms on the right hand sides of (4.4) and (4.5).

Lemma 4.3 *If $u \in H^t$ for some $t > \beta + \frac{1}{2}$, then*

$$\left|\sum_{l\neq 0} \Phi_l(\epsilon)\widehat{L_0 u}(l,k)\right|^2 \leq Ch^{2(t-\beta)}\sum_{l\neq 0}\left|\widehat{J^t u}(l,k)\right|^2 \quad \text{for } k \in \Lambda_h.$$

Proof. For $l \neq 0$,

$$\begin{aligned}\left|\Phi_l(\epsilon)\widehat{L_0 u}(l,k)\right| \leq C\left|\widehat{L_0 u}(l,k)\right| &= Ch^{t-\beta}\left|\Sigma(Ml+kh)Y^{-t}(Ml+kh)\widehat{J^t u}(l,k)\right| \\ &\leq Ch^{t-\beta}|l|^{\beta-t}\left|\widehat{J^t u}(l,k)\right|.\end{aligned}$$

Since $\sum_{l\neq 0}|l|^{2(\beta-t)}$ converges for $t > \beta + \frac{1}{2}$, we obtain the desired estimate by applying the Cauchy-Schwarz inequality. □

Lemma 4.4 *If $\beta + M < r$, then*

$$\left|\{\Phi_0(\epsilon) - D(0)\}\Sigma_h\hat{u}(0,0)\right| \leq Ch^{t-\beta}\left|\widehat{J^t u}(0,0)\right| \quad \text{for } t \in \mathbb{R}.$$

Proof. The 00-entry of the matrix $X^r(0)$ vanishes, so

$$\begin{aligned}\left|\{\Phi_0(\epsilon) - D(0)\}\Sigma_h\hat{u}(0,0)\right|^2 &= \left|Z(0)X^r(0)\Sigma(0)^{-1}\Sigma_h\hat{u}(0,0)\right|^2 \\ &\leq |Z(0)|^2 \sum_{-M/2\leq p<M/2}^{*} \left|p^r\sigma_0(p)^{-1}\sigma_0(pN)\hat{u}(pN)\right|^2 \\ &\leq Ch^{-2\beta}\sum_{p\neq 0}|\hat{u}(pN)|^2 \\ &\leq Ch^{2(t-\beta)}\sum_{p\neq 0}|pN|^{2t}|\hat{u}(pN)|^2 \\ &\leq Ch^{2(t-\beta)}\left|\widehat{J^t u}(0,0)\right|^2,\end{aligned}$$

where, in the second step, the asterisk on the sum indicates that the term with $p = 0$ is omitted, and, in the third step, we used the estimate (4.3). □

Lemma 4.5 *If $\beta + M < r$ and $k \in \Lambda_h^*$, then*

$$\left|\{\Phi_0(\epsilon) - D(kh)\}\Sigma(kh)\hat{u}(0,k)\right| \leq Ch^t\left|\widehat{J^t u}(0,k)\right| \quad \text{for } t \leq r.$$

Proof. This time,

$$\begin{aligned}\left|\{\Phi_0(\epsilon) - D(kh)\}\Sigma(kh)\hat{u}(0,k)\right|^2 &= \left|Z(kh)X^{\tau}(kh)\hat{u}(0,k)\right|^2 \\ &\le C \sum_{-M/2\le p<M/2} |p+kh|^{2r}|\hat{u}(k+pN)|^2 \\ &= Ch^{2t} \sum_{-M/2\le p<M/2} |p+kh|^{2(r-t)}|k+pN|^{2t}|\hat{u}(k+pN)|^2 \\ &\le Ch^{2t}\left|\widehat{J^t u}(0,k)\right|^2,\end{aligned}$$

where, in the final step, $|0+kh|^{2(r-t)} \le 1$ because we assume that $t \le r$. □

Now comes the main step in the proof of Theorems 2.2 and 2.3; recall the definition (3.7) of the discrete seminorm $\|\cdot\|_{s,h}$.

Theorem 4.6 *Assume that the collocation method* (4.1) *is stable, and that*

$$\beta + M < r \quad \textit{and} \quad \beta + \tfrac{1}{2} < t \le r.$$

For all h,

$$\|u_h - u\|_{s,h} \le Ch^{t-s}\|u\|_t \quad \textit{for } \beta \le s \le t, \tag{4.6}$$

and if (2.12) *holds, then in addition*

$$\|u_h - u\|_{s,h} \le Ch^{t-s}\|u\|_{t+\beta-s} \quad \textit{for } \beta - b \le s \le \beta. \tag{4.7}$$

Proof. Since

$$|Y_h^s \Sigma_h^{-1}| = \max_{-M/2\le p<M/2} \frac{\langle pN\rangle^s}{|\sigma_0(pN)|} \le C\max(1, h^{\beta-s}),$$

we see from (4.4) and the estimates in Lemmas 4.3 and 4.4 that

$$\begin{aligned}\left|J^s(u_h-u)\hat{}\,(0,0)\right|^2 &= \left|Y_h^s\Sigma_h^{-1}D(0)^{-1}D(0)\Sigma_h(u_h-u)\hat{}\,(0,0)\right|^2 \\ &\le C\max\left(1, h^{2(\beta-s)}\right)h^{2(t-\beta)} \sum_{l=-\infty}^{\infty}\left|\widehat{J^t u}(l,0)\right|^2.\end{aligned} \tag{4.8}$$

Similarly for $k \in \Lambda_h^*$, since

$$|Y(kh)^s\Sigma(kh)^{-1}| = \max_{-M/2\le p<M/2} \frac{\langle p+kh\rangle^s}{|\sigma_0(p+kh)|} \le C\max\left(1, |kh|^{s-\beta}\right), \tag{4.9}$$

we see from (4.5) and the estimates in Lemmas 4.3 and 4.5 that

$$\begin{aligned}&\left|J^s(u_h-u)\hat{}\,(0,k)\right|^2 \\ &= h^{-2s}\left|Y^s(kh)\Sigma(kh)^{-1}D(kh)^{-1}D(kh)\Sigma(kh)(u_h-u)\hat{}\,(0,k)\}\right|^2 \\ &\le C\max\left(1, |kh|^{2(s-\beta)}\right)h^{2(t-s)} \sum_{l=-\infty}^{\infty}\left|\widehat{J^t u}(l,k)\right|^2.\end{aligned} \tag{4.10}$$

Hence, if $\beta \le s$ then

$$\|u_h - u\|_{s,h}^2 = \sum_{k\in\Lambda_h} \left|J^s(u_h-u)\hat{}(0,k)\right|^2 \le Ch^{2(t-s)} \sum_{k\in\Lambda_h}\sum_{l=-\infty}^{\infty} \left|\widehat{J^t u}(l,k)\right|^2,$$

which proves (4.6).

Now suppose that (2.12) holds, and that $\beta - b \le s \le \beta$. In Lemmas 4.3 and 4.4 there is no upper limit on the index t, so the estimate (4.8) is valid with t replaced by $t+\beta-s$:

$$\left|J^s(u_h-u)\hat{}(0,0)\right|^2 \le Ch^{2(t-s)} \sum_{l=-\infty}^{\infty} \left|(J^{t+\beta-s}u)\hat{}(l,0)\right|^2. \tag{4.11}$$

However, in Lemma 4.5 we must have $t \le r$, so a sharper estimate than (4.10) is needed. Let $k \in \Lambda_h^*$, and define the M-dimensional column vector

$$v = D(kh)^{-1}\{\Phi_0(\epsilon) - D(kh)\}\Sigma(kh)\hat{u}(0,k) = D(kh)^{-1}Z(kh)X^r(kh)\hat{u}(0,k),$$

so that by (4.5)

$$\begin{aligned} J^s(u_h-u)\hat{}(0,k) &= h^{-s}Y^s(kh)\Sigma(kh)^{-1}v \\ &\quad + h^{\beta-s}Y^s(kh)\Sigma(kh)^{-1}D(kh)^{-1}\sum_{l\ne 0}\Phi_l(\epsilon)\widehat{L_0 u}(l,k). \end{aligned} \tag{4.12}$$

We will use the splitting $v = v' + v''$, where the components of v' are given by

$$v'_p = \begin{cases} v_0 & \text{if } p = 0, \\ 0 & \text{if } p \ne 0. \end{cases}$$

It follows from the assumption (2.12) that

$$|v'|^2 = |v_0|^2 \le C|kh|^{2(b+r)}|\hat{u}(k)|^2 + C \sum_{-M/2\le q<M/2}^{*} |q+kh|^{2r}|\hat{u}(k+qN)|^2,$$

and since $|Y^s(kh)\Sigma(kh)^{-1}v'| \le C|kh|^{2(s-\beta)}|v'|$, we estimate

$$|kh|^{s-\beta}|kh|^{b+r} = |kh|^{r-t+s-\beta+b}|kh|^t \le Ch^t|k|^t$$

and, for $q \ne 0$,

$$|kh|^{s-\beta}|q+kh|^r = |q+kh|^{r-t-\beta+s}|k|^{s-\beta}h^t|k+qN|^{t+\beta-s} \le Ch^t|k+qN|^{t+\beta-s},$$

to conclude that

$$\left|Y^s(kh)\Sigma(kh)^{-1}v'\right|^2 \le Ch^{2t}\left|(J^{t+\beta-s}u)\hat{}(0,k)\right|^2.$$

Next, since $v_0'' = 0$,

$$\begin{aligned}\left|Y^s(kh)\Sigma(kh)^{-1}v''\right|^2 &\le C \sum_{-M/2\le p<M/2}^{*} |p+kh|^{2(s-\beta)}|v_p''|^2 \\ &\le C|v''|^2 \le C|X^r(kh)\hat{u}(0,k)|^2 \\ &= C \sum_{-M/2\le q<M/2} |q+kh|^{2r}|\hat{u}(k+qN)|^2 \\ &\le Ch^{2t}\left|\widehat{J^t u}(0,k)\right|^2,\end{aligned}$$

so

$$\left|Y^s(kh)\Sigma(kh)^{-1}v\right|^2 \le Ch^{2t}\left|(J^{t+\beta-s}u)\hat{}\,(0,k)\right|^2.$$

Inserting this estimate in (4.12), and using (4.9) and Lemma 4.3, we obtain

$$\begin{aligned}\left|J^s(u_h-u)\hat{}\,(0,k)\right|^2 &\le Ch^{2(t-s)}\left|(J^{t+\beta-s}u)\hat{}\,(0,k)\right|^2 \\ &\quad + Ch^{2(\beta-s)}\max\left(1,|kh|^{2(s-\beta)}\right)h^{2(t-s)}\sum_{l\ne0}\left|(J^{t+\beta-s}u)\hat{}\,(l,k)\right|^2.\end{aligned}$$

Summing over k, and using (4.11), we finally arrive at (4.7). □

Proof of Theorem 2.2. If $s < r - M + \frac{1}{2}$, then

$$(4.13)\qquad \|u_h-u\|_s \le \|P_h(u_h-u)\|_s + \|P_hu-u\|_s = \|u_h-u\|_{s,h} + \|u-P_hu\|_s.$$

Hence, the approximation property from Theorem 3.4 shows that Theorem 2.2 holds in the special case when $L = L_0$ and the principal symbol has constant coefficients. Roughly speaking, the error estimates for a general L then follow by freezing coefficients. The details for the case of smoothest splines can by found in [1, §3], and with only slight modifications the same proof goes through for any knot multiplicity $M \ge 1$, given our Theorem 4.6. Alternatively, the local principle in [13] can be applied; see also [17, Chapter 13]. In both approaches, the local approximation property from part v) of Theorem 3.4 is crucial.

We comment briefly on the second approach. For the theory in [13], it is necessary to reformulate the collocation method (1.7) as a projection method,

$$\Pi_h L u_h = \Pi_h f,$$

where Π_h is the interpolation projection for a suitable subspace of dimension MN, not necessarily $S_h^{r,M}$. (Indeed, we shall see in the Section 5 that the interpolation projection for $S_h^{r,M}$ might not exist, even though the collocation method is stable.) If we assume that

$$\det \Phi_0(\epsilon) \ne 0,$$

then a good choice for Π_h is the interpolation projection onto the trigonometric polynomial space $S_h^{\infty,M}$ defined in (3.6), because by Lemma 4.1,

$$\Phi_0(\epsilon)\widehat{\Pi_h u}(0,k) = \sum_{l=-\infty}^{\infty} \Phi_l(\epsilon)\hat{u}(l,k) \quad \text{for } k \in \Lambda_h.$$

These projections satisfy

$$\|\Pi_h v\|_s \leq C\|v\|_s \quad \text{for } s > 1/2 \text{ and } v \in H^s,$$

and if the collocation method is stable in the sense of Definition 2.1, then it follows from Theorem 4.6 that in the special case when $L = L_0$ and σ_0 has constant coefficients,

$$\|v\|_s \leq C\|\Pi_h L v\|_{s-\beta} \quad \text{for } \beta + \tfrac{1}{2} < s < r - M + \tfrac{1}{2} \text{ and } v \in S_h^{r,M}.$$

Applying the local principle of [13], we deduce that this estimate holds for a general L and for all h sufficiently small. Thus,

$$\|P_h u - u_h\|_s \leq C\|\Pi_h L(P_h u - u_h)\|_{s-\beta} = C\|\Pi_h L(P_h u - u)\|_{s-\beta} \leq C\|P_h u - u\|_s,$$

and the error estimate of Theorem 2.2 follows from the approximation property, except that the case $\beta \leq s \leq \beta + \frac{1}{2}$ is not covered. (However, the method of proof in [1] gives the result for the complete range of values of s.) □

Proof of Theorem 2.3. Assume now that the principal symbol of L has constant coefficients, and that (2.12) and (2.13) hold. In the case $L = L_0$, the superconvergence result (2.14) follows at once from the second part of Theorem 4.6 using (4.13). The general case $L = L_0 + L_1$ can then be handled by applying the following perturbation argument, similar to [19, Theorem 3.5].

Both L and L_0 are bounded and invertible operators from H^s onto $H^{s-\beta}$, so the operator $L_0^{-1}L = I + L_0^{-1}L_1$ has a bounded inverse on H^s for all $s \in \mathbb{R}$. Thus, on the one hand,

$$\|u_h - u\|_s \leq C\|u_h - u + w\|_s \quad \text{where } w = L_0^{-1}L_1(u_h - u), \tag{4.14}$$

but on the other hand, the collocation equations

$$(Lu_h)(x_{n,j}) = (Lu)(x_{n,j}) \quad \text{for } n \in \mathbb{Z}_N \text{ and } 1 \leq j \leq M,$$

are equivalent to

$$(L_0 u_h)(x_{n,j}) = L_0(u - w)(x_{n,j}) \quad \text{for } n \in \mathbb{Z}_N \text{ and } 1 \leq j \leq M.$$

Thus, if $\tilde{u}_h$, $w_h \in S_h^{r,M}$ satisfy the collocation equations,

$$L_0\tilde{u}_h(x_{n,j}) = L_0 u(x_{n,j}) \quad \text{and} \quad L_0 w_h(x_{n,j}) = L_0 w(x_{n,j})$$

for $n \in \mathbb{Z}_N$ and $1 \le j \le M$, then $u_h = \tilde{u}_h - w_h$. Now assume that s and t satisfy (2.9), and that $\beta - b \le s \le \beta$. The superconvergence result for the case $L = L_0$ gives

$$\|\tilde{u}_h - u\|_s \le Ch^{t-s}\|u\|_{t+\beta-s} \quad \text{for } \beta - b \le s \le \beta,$$

and since $u_h \in H^\beta$ and $L_0^{-1}L_1 : H^\beta \to H^{\beta+b}$ is bounded, we have $w \in H^{\beta+b}$. Applying the basic error estimate (2.10) yields

$$\begin{aligned}\|w_h - w\|_\beta &\le Ch^{\beta+b-\beta}\|w\|_{\beta+b} \le Ch^b\|u_h - u\|_\beta \\ &\le Ch^{b+t-\beta}\|u\|_t \le Ch^{t-s}\|u\|_{t+\beta-s},\end{aligned}$$

where, in the second step, we used the hypothesis $\beta + \frac{1}{2} < \beta + b \le r$. Finally, by (4.14),

$$\|u_h - u\|_s \le C\|\tilde{u}_h - w_h - u + w\|_s \le C\|\tilde{u}_h - u\|_s + C\|w_h - w\|_\beta,$$

implying the desired estimate (2.14). □

5. Methods using Splines with Double Knots

Throughout this section, we assume that $M = 2$, and that the principal symbol (2.3) has constant coefficients satisfying

$$a_+ = 0 \quad \text{or} \quad a_- = 0. \tag{5.1}$$

We say that such a symbol is *even* if $a_- = 0$, because in this case $\sigma_0(-\xi) = \sigma_0(+\xi)$. Similarly, σ_0 is *odd* if $a_+ = 0$, because $\sigma_0(-\xi) = -\sigma_0(+\xi)$. It is useful to define

$$\theta = \begin{cases} + & \text{if } \sigma_0 \text{ and } r \text{ have like parity,} \\ - & \text{if } \sigma_0 \text{ and } r \text{ have opposite parity,} \end{cases}$$

so that

$$(-1)^r\sigma_0(-1) = \theta\sigma_0(+1)$$

and the formula (2.6) can be written as

$$D(y) = \Phi_0(\epsilon) + \frac{1}{\sigma_0(1)}Z(y)\begin{bmatrix} \theta(1-y)^{r-\beta} & 0 \\ 0 & \phi_{r-\beta}(y) \end{bmatrix} \quad \text{for } -\tfrac{1}{2} \le y \le \tfrac{1}{2},$$

where

$$\begin{aligned} Z(y) = \sigma_0(1)\Bigg\{ &\sum_{l=1}^{\infty} \Phi_l(\epsilon)\begin{bmatrix} (2l+y-1)^{\beta-r} & (2l+y-1)^{\beta-r+1} \\ (2l+y)^{\beta-r} & (2l+y)^{\beta-r+1} \end{bmatrix} \\ &+ \theta\sum_{l=1}^{\infty} \Phi_{-l}(\epsilon)\begin{bmatrix} (2l-y+1)^{\beta-r} & -(2l-y+1)^{\beta-r+1} \\ (2l-y)^{\beta-r} & -(2l-y)^{\beta-r+1} \end{bmatrix}\Bigg\} \begin{bmatrix} y & 1-y \\ -1 & 1 \end{bmatrix} \end{aligned}$$

and

$$\phi_\alpha(y) = \begin{cases} \theta|y|^\alpha & \text{if } -\frac{1}{2} \le y < 0, \\ y^\alpha & \text{if } 0 \le y \le \frac{1}{2}. \end{cases}$$

We shall assume without loss of generality that $\sigma_0(1) = 1$, and will study two choices of the collocation points, each possessing a natural symmetry.

First consider collocation at breakpoints and midpoints, i.e.,

$$\epsilon_1 = 0 \quad \text{and} \quad \epsilon_2 = \frac{1}{2}. \tag{5.2}$$

A quick calculation shows that with this choice of collocation points the matrix (2.5) is particularly simple:

$$\Phi_l(0, \tfrac{1}{2}) = I \quad \text{for all } l \in \mathbb{Z},$$

where I is the 2×2 identity matrix. The stability and superconvergence properties of the method follow from the next lemma.

Lemma 5.1 *Assume* (5.1) *and* (5.2).

i) If σ and r have like parity, then $\det D(0) = 0$.

ii) If σ and r have opposite parity, then

$$\det D(y) > 0 \quad \textit{for } -\tfrac{1}{2} \le y \le \tfrac{1}{2},$$

and

$$00\text{-}\textit{entry of } D(y)^{-1} Z(y) = O(y) \quad \textit{as } y \to 0.$$

Proof. Let

$$g_\alpha^\pm(y) = \sum_{l=1}^{\infty} [(2l+y)^{-\alpha} \pm (2l-y)^{-\alpha}] \quad \text{for } \alpha > 1 \text{ and } -2 < y < 2,$$

so that, recalling our assumption (2.8),

$$Z(y) = \begin{bmatrix} g_\alpha^\theta(y-1) & g_{\alpha-1}^{-\theta}(y-1) \\ g_\alpha^\theta(y) & g_{\alpha-1}^{-\theta}(y) \end{bmatrix} \begin{bmatrix} y & 1-y \\ -1 & 1 \end{bmatrix} \quad \text{with } \alpha = r - \beta > 2.$$

It is easy to see that

$$g_\alpha^+(-y) = g_\alpha^+(y) \quad \text{and} \quad g_\alpha^-(-y) = -g_\alpha^-(y),$$

with $g_\alpha^-(0) = 0$ and $g_\alpha^-(1) = -1$. Part i) follows at once because if $\theta = +$ then

$$D(0) = I + \begin{bmatrix} g_\alpha^+(1) & 1 \\ g_\alpha^+(0) & 0 \end{bmatrix} \begin{bmatrix} 0 & 1 \\ -1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}.$$

To prove part ii), suppose $\theta = -$. We find that

$$\det D(y) = 1 + \phi_\alpha(y)[g^+_{\alpha-1}(y) + (1-y)g^-_\alpha(y)] \\ + (1-y)^\alpha[g^+_{\alpha-1}(1-y) + yg^-_\alpha(1-y)] \\ + \phi_\alpha(y)(1-y)^\alpha[g^+_{\alpha-1}(y)g^-_\alpha(1-y) + g^+_{\alpha-1}(1-y)g^-_\alpha(y)]$$

where, as before, $\alpha = r - \beta$. In showing that $\det D(y) > 0$, we shall treat separately the cases $0 \le y \le 1/2$ and $-1/2 \le y \le 0$.

Suppose $0 \le y \le 1/2$, and write $\det D(y) = f_1(y) + f_2(y)$ where

$$f_1(y) = \frac{1}{2} + y^\alpha g^+_{\alpha-1}(y) + (1-y)^\alpha y g^-_\alpha(1-y) + y^\alpha(1-y)^\alpha g^+_{\alpha-1}(y)g^-_\alpha(1-y).$$

Since $-1 = g^-_\alpha(1) \le g^-_\alpha(1-y) \le 0$, we have

$$f_1(y) \ge [\tfrac{1}{2} + y^\alpha g^+_{\alpha-1}(y)][1 + (1-y)^\alpha g^-_\alpha(1-y)] \ge 0,$$

and taking into account that $g^-_\alpha(y) \ge -1$, one easily sees that

$$f_2(y) \ge \tfrac{1}{2} - y^\alpha + (1-y^\alpha)(1-y)^\alpha g^+_{\alpha-1}(1-y) > 0,$$

because $\alpha > 1$.

Next, suppose $-1/2 \le y \le 0$. In this case, we have

$$\det D(y) = 1 + f_3(y) + f_4(y) + f_5(y),$$

where

$$f_3(y) = (1-y)^\alpha g^+_{\alpha-1}(1-y)[1 - |y|^\alpha g^-_\alpha(y)], \\ f_4(y) = -|y|^\alpha g^+_{\alpha-1}(y)[1 + (1-y)^\alpha g^-_\alpha(1-y)], \\ f_5(y) = (1-y)^\alpha y g^-_\alpha(1-y) - |y|^\alpha(1-y)g^-_\alpha(y).$$

Since g^-_α is monotonically decreasing,

$$-g^-_\alpha(y) = g^-_\alpha(-y) \ge g^-_\alpha(1/2) > -(2/3)^\alpha > -1,$$

so

$$f_3(y) > (1-y)^\alpha g^+_{\alpha-1}(1-y)[1 - |y|^\alpha] > 0, \\ 1 + f_5(y) \ge 1 - |y|^\alpha(1-y)g^-_\alpha(y) \ge 1 - |y|^\alpha(1-y)(2/3)^\alpha > 1/2,$$

and $f_4(y) \ge 0$ because $g^-_\alpha(1-y) \le g^-_\alpha(1) = -1$. Thus, $\det D(y) > \frac{1}{2}$.

Finally,

$$D(0)^{-1}Z(0) = \frac{1}{1 + g^+_{\alpha-1}(1)} \begin{bmatrix} -g^+_{\alpha-1}(1) & 1 + g^+_{\alpha-1}(1) \\ -g^+_{\alpha-1}(0) & 0 \end{bmatrix}$$

so by Taylor's theorem the 00-entry of $D(y)^{-1}Z(y)$ is $O(y)$. □

Remark 5.2 The simplest example of the instability predicted in part i) of Lemma 5.1 occurs when L is the identity operator and $r = 4$. The collocation equations for $u_h \in S_h^{4,2}$ then reduce to

$$u_h(x_{n,j}) = u(x_{n,j}) \quad \text{for } n \in \mathbb{Z}_N \text{ and } j = 1,\, 2, \tag{5.3}$$

so our result says that Hermite cubic interpolation at breakpoints and midpoints is unstable. In fact, it is easy to see that there is a Hermite cubic u_h, not identically zero, that satisfies $u_h(x_{n,j}) = 0$ for all n and j. Thus, relative to any basis for $S_h^{4,2}$ the linear system arising from the interpolation equations (5.3) is singular.

If we take $r = 3$, however, then part ii) applies, showing that interpolation at midpoints and breakpoints using continuous, piecewise-quadratic splines is superconvergent: more precisely, the interpolant $u_h \in S_h^{3,2}$ satisfies

$$\|u_h - u\|_{-1} \leq ch^4 \|u\|_4.$$

Remark 5.3 In part ii) of Lemma 5.1, the inequality $\det D > 0$ on $[-1/2, 1/2]$ suffices to guarantee stability, but is not sharp. We observed in numerical investigations that $\det D$ decreases monotonically on the interval $[-1/2, 1/2]$, having the minimum value $\det D(1/2) = [1 + 2^{1-\alpha} g_{\alpha-1}^{+}(1/2)][1 + 2^{-\alpha} g_{\alpha}^{-}(1/2)] > 1$.

Next, we consider collocation at points given by

$$\epsilon_1 = \epsilon \quad \text{and} \quad \epsilon_2 = 1 - \epsilon \quad \text{with } 0 < \epsilon < \tfrac{1}{2}. \tag{5.4}$$

In this case,

$$\Phi_l(\epsilon, 1-\epsilon) = \begin{bmatrix} \cos 2\pi(2l)\epsilon & \cos 2\pi(2l+1)\epsilon \\ \cos 2\pi(2l-1)\epsilon & \cos 2\pi(2l)\epsilon \end{bmatrix},$$

and we shall see in a moment that the stability properties are the other way around from in Lemma 5.1. Moreover, superconvergence is possible if ϵ is a zero of the function $G_{r-\beta}$, where

$$G_\alpha(\epsilon) = 2 \sum_{m=1}^{\infty} \frac{1}{m^\alpha} \cos 2\pi m\epsilon \quad \text{for } \alpha > 0.$$

The zeros of the function G_α have appeared elsewhere in connection with the *qualocation* method [3], [21] and related fully-discrete schemes [10].

Table 5.1. The unique zero $\epsilon_\alpha^\star$ of G_α in the interval $(0, 1/2)$.

α	$\epsilon_\alpha^\star$
1	1/6
2	0 · 21132 48654 051871
3	0 · 23082 96502 521382
4	0 · 24033 51888 203859
5	0 · 24511 88417 393386
6	0 · 24754 07162 436733
∞	1/4

Various properties of G_α and some other trigonometric series were proved in [2]. In particular, G_α is strictly decreasing on the interval $(0,1/2)$, where it has a unique zero that we shall denote by $\epsilon_\alpha^\star$. When α is a positive integer, it is possible to evaluate G_α efficiently and accurately using a technique described in [9]. By applying any standard rootfinding algorithm, one can then compute $\epsilon_\alpha^\star$; see Table 5.1.

Lemma 5.4 *Assume* (5.1) *and* (5.4).

i) If σ and r have like parity, and if $\epsilon = \epsilon_{r-\beta}^\star$, then

$$00\text{-entry of } D(y)^{-1}Z(y) = O(y^2) \quad \text{as } y \to 0.$$

ii) If σ and r have opposite parity, then $\det D(0) = 0$.

Proof. Let

$$g_\alpha^\pm(y) = \sum_{l=1}^\infty [(2l+y)^{-\alpha} \pm (2l-y)^{-\alpha}] \cos 2\pi l(2\epsilon)$$

and

$$h_\alpha^\pm(y) = \sum_{l=1}^\infty [(2l+y)^{-\alpha} \pm (2l-y)^{-\alpha}] \sin 2\pi l(2\epsilon);$$

notice that the functions g_α^+ and h_α^+ are even, whereas g_α^- and h_α^- are odd. Since

$$\Phi_l(\epsilon, 1-\epsilon) = \cos 2\pi l(2\epsilon) \begin{bmatrix} 1 & \cos 2\pi\epsilon \\ \cos 2\pi\epsilon & 1 \end{bmatrix} + \sin 2\pi l(2\epsilon) \begin{bmatrix} 0 & -\sin 2\pi\epsilon \\ \sin 2\pi\epsilon & 0 \end{bmatrix},$$

we find that

$$\begin{aligned} Z(y) = \Bigg\{ &\begin{bmatrix} 1 & \cos 2\pi\epsilon \\ \cos 2\pi\epsilon & 1 \end{bmatrix} \begin{bmatrix} g_\alpha^\theta(y-1) & g_{\alpha-1}^{-\theta}(y-1) \\ g_\alpha^\theta(y) & g_{\alpha-1}^{-\theta}(y) \end{bmatrix} \\ &+ \begin{bmatrix} 0 & -\sin 2\pi\epsilon \\ \sin 2\pi\epsilon & 0 \end{bmatrix} \begin{bmatrix} h_\alpha^{-\theta}(y-1) & h_{\alpha-1}^{\theta}(y-1) \\ h_\alpha^{-\theta}(y) & h_{\alpha-1}^{\theta}(y) \end{bmatrix} \Bigg\} \begin{bmatrix} y & 1-y \\ -1 & 1 \end{bmatrix}. \end{aligned}$$

To prove part i), suppose that $\theta = +$. After some calculation, one finds that

$$D(y) = \frac{1}{\sin 2\pi\epsilon} \Phi_0(\epsilon, 1-\epsilon) \begin{bmatrix} a(y) & b(y) \\ c(y) & d(y) \end{bmatrix},$$

where

$$\begin{aligned} a(y) &= (1-y)^\alpha \sum_{m=1}^\infty \left[\frac{m}{(m+y)^\alpha} + \frac{m}{(m-y)^\alpha} \right] \sin 2\pi m\epsilon, \\ b(y) &= -|y|^\alpha \sum_{m=1}^\infty \left[\frac{m+1}{(m+y)^\alpha} + \frac{m-1}{(m-y)^\alpha} \right] \sin 2\pi m\epsilon, \\ c(y) &= -(1-y)^\alpha \sum_{m=1}^\infty \left[\frac{m-1}{|m+y-1|^\alpha} + \frac{m+1}{(m-y+1)^\alpha} \right] \sin 2\pi m\epsilon, \\ d(y) &= |y|^\alpha \sum_{m=1}^\infty \left[\frac{m}{|m+y-1|^\alpha} + \frac{m}{(m-y+1)^\alpha} \right] \sin 2\pi m\epsilon. \end{aligned} \tag{5.5}$$

Here, $d(0) = \sin 2\pi\epsilon$, and since $\det \Phi_0(\epsilon, 1-\epsilon) = (\sin 2\pi\epsilon)^2$, we see that

$$\det D(0) = H_{\alpha-1}(\epsilon) \sin 2\pi\epsilon,$$

where $H_\alpha(\epsilon) = 2\sum_{m=1}^{\infty} m^{-\alpha} \sin 2\pi m\epsilon$. It is known from [2] that $H_\alpha(\epsilon) > 0$ for $0 < \epsilon < \frac{1}{2}$ and for any $\alpha > 0$. Thus, $\det D(0) > 0$ and so $D(0)^{-1}$ exists, and some further calculation reveals that the 00-entry of $D(0)^{-1}Z(0) = G_\alpha(\epsilon)$. To complete the proof of part i), we will show that the 00-entry of $D(y)^{-1}Z(y)$ is an even function of y.

In fact, simple calculations show that the entries of the matrix

$$Z(y) = \begin{bmatrix} a_1(y) & b_1(y) \\ c_1(y) & d_1(y) \end{bmatrix}$$

are given by

$$a_1(y) = \sum_{m=1}^{\infty} \left[\frac{m+1}{(m-y+1)^\alpha} - \frac{m-1}{|m+y-1|^\alpha} \right] \cos 2\pi m\epsilon,$$

$$b_1(y) = \sum_{m=1}^{\infty} \left[\frac{m}{|m+y-1|^\alpha} - \frac{m}{(m-y+1)^\alpha} \right] \cos 2\pi m\epsilon - \frac{1}{|y|^\alpha} \cos 2\pi\epsilon,$$

$$c_1(y) = \sum_{m=1}^{\infty} \left[\frac{m}{(m-y)^\alpha} - \frac{m}{(m+y)^\alpha} \right] \cos 2\pi m\epsilon - \frac{1}{(1-y)^\alpha} \cos 2\pi\epsilon,$$

$$d_1(y) = \sum_{m=1}^{\infty} \left[\frac{m+1}{(m+y)^\alpha} - \frac{m-1}{(m-y)^\alpha} \right] \cos 2\pi m\epsilon.$$

Putting

$$\lambda_{1,\alpha}(y) = \sum_{m=1}^{\infty} \left[\frac{1}{(m+y)^\alpha} + \frac{1}{(m-y)^\alpha} \right] \cos 2\pi m\epsilon,$$

$$\lambda_{2,\alpha}(y) = \sum_{m=1}^{\infty} \left[\frac{1}{(m+y)^\alpha} - \frac{1}{(m-y)^\alpha} \right] \sin 2\pi m\epsilon,$$

$$\lambda_{3,\alpha}(y) = \sum_{m=1}^{\infty} \left[\frac{m}{(m-y)^\alpha} - \frac{m}{(m+y)^\alpha} \right] \cos 2\pi m\epsilon,$$

one finds that the 00-entry of $D(y)^{-1}Z(y)$ is

$$\frac{1}{|y|^\alpha} \left[1 - \frac{a(y) \sin 2\pi\epsilon}{a(y)d(y) - b(y)c(y)} \right]$$
$$= \frac{1}{|y|^\alpha} \left[1 - \frac{(1-y)^{-\alpha} a(y)}{(1-y)^{-\alpha} a(y)[1 + |y|^\alpha \lambda_{1,\alpha}(y)] + |y|^\alpha \lambda_{3,\alpha}(y) \lambda_{2,\alpha}(y)} \right],$$

which is an even function of y.

Finally, we find that if $\theta = -$, then

$$D(0) = \begin{bmatrix} 1 + g^+_{\alpha-1}(1) + g^+_{\alpha-1}(0)\cos 2\pi\epsilon & \cos 2\pi\epsilon \\ \cos 2\pi\epsilon + g^+_{\alpha-1}(1)\cos 2\pi\epsilon + g^+_{\alpha-1}(0) - h^-_{\alpha-1}(1)\sin 2\pi\epsilon & 1 \end{bmatrix}$$

and

$$\det D(0) = \sin 2\pi\epsilon\{[1 + g^+_{\alpha-1}(1)]\sin 2\pi\epsilon + h^-_{\alpha-1}(1)\cos 2\pi\epsilon\} = 0,$$

which proves part ii). □

Remark 5.5 We conjecture that if, as in part i), σ and r have like parity, then for any choice of $\epsilon \in (0, 1/2)$,

$$\det D(y) > 0 \quad \text{for } -1/2 \le y \le 1/2.$$

Computer plots of $\det D$ for a wide range of values of $\alpha > 2$ and $\epsilon \in (0, 1/2)$ indicated that this function is always monotonically decreasing on the interval $[-1/2, 1/2]$, and one can easily see that

$$\det D(1/2) > 0 \quad \text{for any choice of } \epsilon \in (0, 1/2). \tag{5.6}$$

Indeed, if we define

$$H^\pm_\alpha(\epsilon, y) = \sum_{m=1}^{\infty} \left\{ \frac{1}{(m+y)^\alpha} \mp \frac{1}{(m-y)^\alpha} \right\} \sin 2\pi m\epsilon,$$

then

$$a(1/2) + c(1/2) = b(1/2) + d(1/2) = -2^{-\alpha} H^+_\alpha(\epsilon, 1/2)$$

and

$$d(1/2) - c(1/2) = 2^{1-\alpha} H^-_{\alpha-1}(\epsilon, 1/2),$$

so

$$\begin{aligned} \det D(1/2) &= [a(1/2) + c(1/2)][d(1/2) - c(1/2)] \\ &= -2^{1-2\alpha} H^+_\alpha(\epsilon, 1/2) H^-_{\alpha-1}(\epsilon, 1/2). \end{aligned}$$

We know from [2] that $H^+_\alpha(\epsilon, 1/2) < 0$ and $H^-_{\alpha-1}(\epsilon, 1/2) > 0$ for $0 < \epsilon < 1/2$, so (5.6) holds. In the particular case $\epsilon = 1/4$, there is a complete proof of our conjecture, based on the fact that in (5.5) we have only simple, alternating series that can easily be estimated from above and below.

Remark 5.6 As in the Remark 5.2, the simplest examples arise by choosing $r = 3$ or 4, and taking L to be the identity operator, so that u_h just has to satisfy the interpolation equations (5.3). This time, with the interpolation points given by (5.4), the unstable case is $r = 3$, and indeed it is easy to construct a continuous, piecewise-quadratic function $u_h \in S_h^{3,2}$ that satisfies $u_h(x_{n,j}) = 0$ for all n and j, and yet is not identically zero. The stable case is now $r = 4$ (see Remark 5.5), and if $\epsilon = \epsilon_4^\star$ then we have superconvergence with $b = 2$, i.e., the Hermite cubic interpolant $u_h \in S_h^{4,2}$ satisfies

$$\|u_h - u\|_{-2} \le ch^6 \|u\|_6.$$

6. Numerical Experiments

Consider the Dirichlet problem for the Laplace equation,

$$\begin{aligned} \nabla^2 V &= 0 \quad \text{on } \Omega, \\ V &= F \quad \text{on } \Gamma, \end{aligned} \tag{6.1}$$

where Ω is a bounded domain in $\mathbb{R}^2$ with boundary Γ. We assume that the curve Γ is smooth, and take a regular parametric representation $\gamma : \mathbf{T} \to \Gamma$. The solution of (6.1) can be represented as a single layer potential,

$$V(X) = \int_{\mathbf{T}} u(y) \log \frac{\omega}{|X - \gamma(y)|}\, dy \quad \text{for } X \in \Omega,$$

where ω is a parameter chosen so that

$$\omega > \text{diameter of } \Gamma. \tag{6.2}$$

In order to satisfy the boundary condition in (6.1), the density function u must be a solution of the integral equation $Lu = f$, where

$$(Lu)(x) = \int_{\mathbf{T}} u(y) \log \frac{\omega}{|\gamma(x) - \gamma(y)|}\, dy \quad \text{and} \quad f(x) = F[\gamma(x)] \quad \text{for } x \in \mathbf{T}.$$

It is well known that the principal symbol of L is

$$\sigma_0(\xi) = \frac{1}{2|\xi|} \quad \text{for } \xi \neq 0,$$

so L is strongly elliptic and of order $\beta = -1$. Moreover, in the decomposition (2.1), the Schwartz kernel of L_1 is C^∞ on $\mathbf{T} \times \mathbf{T}$, i.e., L_1 is a smoothing operator. The condition (6.2) ensures that the homogeneous equation $Lu = 0$ has only the trivial solution $u = 0$, so L satisfies all of the hypotheses required for Theorems 2.2 and 2.3.

Let $V_h(X)$ be the single layer potential of the collocation solution $u_h \in S_h^{r,M}$, then it follows from (2.11) and (2.15) that

$$V_h(X) = V(X) + O(h^{r+1+b}),$$

uniformly for X in any compact subset of Ω, where $b = 0$ if there is no superconvergence. (We assume that F, and hence u, is smooth.)

For our numerical experiments, we used the domain shown in Figure 6.1, whose boundary is parameterised by

$$\gamma(t) = \big(\cos 2\pi t, (1 - \tfrac{1}{2}\sin 2\pi t)\sin 2\pi t\big), \tag{6.3}$$

and chose as our boundary data

$$F(X) = \operatorname{Re}\sin[(X_1 - 0{\cdot}33) + \mathrm{i}(x_2 - 0{\cdot}22)] = \sin(X_1 - 0{\cdot}33)\cosh(x_2 - 0{\cdot}22).$$

In this way, F is harmonic on $\mathbb{R}^2$, and so $V(X) = F(X)$ for $X \in \Omega$. We chose $\omega = 3$ so as to satisfy (6.2).

Table 6.1 shows some numerical results for continuous, piecewise-quadratic splines ($r = 3$, $M = 2$), collocating at breakpoints and midpoints ($\epsilon_1 = 0$ and $\epsilon_2 = 1/2$). We give the error in $V_h(X)$ at the point $X = (0{\cdot}6, -0{\cdot}2)$ for $N = 2^J$ where $J = 2$, 4, ..., 9. According to Lemma 5.1 ii), the method is stable and superconvergent with $b = 1$, so we expect the error to be $O(h^5)$. Our numerical results agree with this theoretical prediction, bearing in mind that for $N = 512$ the roundoff error dominates the discretisation error; the computations were performed on a workstation in double precision, i.e., with a unit roundoff of around 10^{-16}.

As well as tabulating the error and empirical convergence rate for the potential, we give (estimates for) the ℓ_∞ condition number of the stiffness matrix and the CPU time for the overall computation. The diagonal and near-diagonal entries of the stiffness matrix have to be evaluated carefully because they involve singular or near-singular integrands. In such cases, we used the splitting

$$\log \frac{\omega}{|\gamma(x) - \gamma(y)|} = \log \frac{1}{|x + q - y|} + \text{smooth function},$$

with the integer q chosen so that $|x + q - y| = \min_{l \in \mathbb{Z}} |x + l - y|$. The integrals of the form

$$\int \log \frac{1}{|x + q - y|} \times (\text{polynomial in } y)\, dy,$$

were evaluated analytically, leaving only integrals of smooth functions to be handled with quadratures. For the computations in Table 6.1, we used 3 Gauss points per interval, so that the quadrature error integrating a smooth function on Γ would be $O(h^6)$.

Tables 6.2–6.4 give numerical results for Hermite cubic splines ($r = 4$ and $M = 2$) using two symmetrically-located collocation points in each interval, i.e., with $\epsilon_1 = \epsilon$ and $\epsilon_2 = 1 - \epsilon$ where $0 < \epsilon < 1/2$. As above, the error in the potential $V_h(X)$ was calculated at the point $X = (0{\cdot}6, -0{\cdot}2)$. From Lemma 5.4 and Table 5.1, if

$$\epsilon = \epsilon_5^* = 0 \cdot 24511\ 88417\ 393386,$$

then we expect the method to be superconvergent with $b = 2$, i.e., we expect $O(h^7)$ convergence. For any other value of ϵ, the error ought to be $O(h^5)$.

Table 6.2 shows our results using the special value $\epsilon = \epsilon_5^*$ and 4 Gauss points per interval, but unfortunately the roundoff error seems to dominate the discretisation error before the convergence rate has a chance to stabilise. We therefore repeated the calculation using a black box, adaptive quadrature routine for all integrations. The results are shown in Table 6.3, and the $O(h^7)$ convergence is reasonably clear. (The very small error for $N = 256$ is presumably a fluke, given the size of the condition number.) Notice that for $N = 64$, using adaptive quadrature increases the overall execution time by a factor of about 16 but has virtually no effect on the accuracy of the potential. Finally, in Table 6.4 we give the results using $\epsilon = 1/4$ with adaptive quadrature; as expected, the error increases to $O(h^5)$.

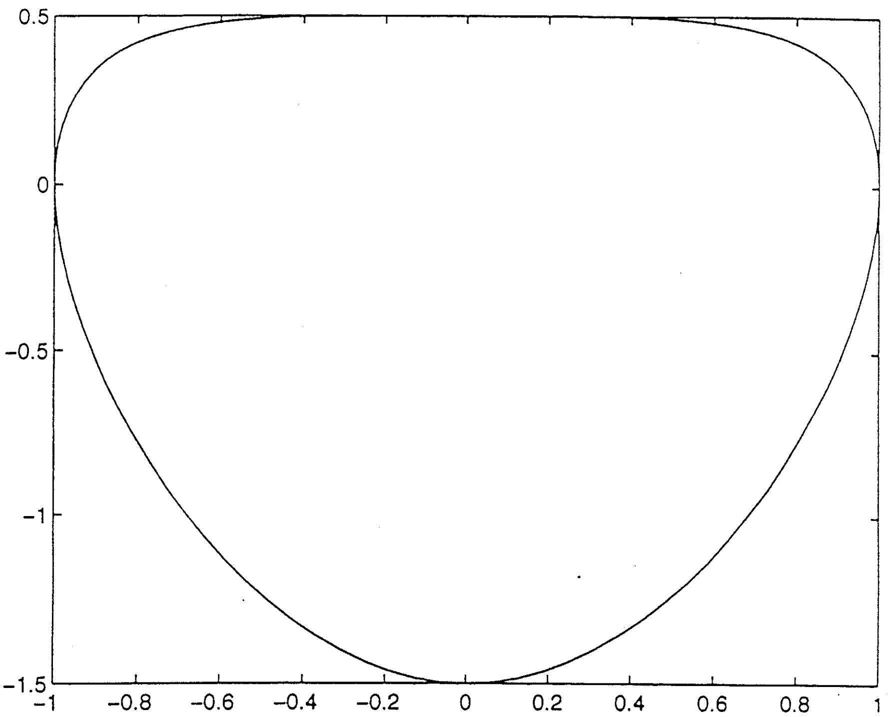

Figure 6.1. The domain with boundary parameterised (6.3)

Table 6.1. Numerical results for continuous, piecewise-quadratic splines, collocating at breakpoints and midpoints, using 3 Gauss points per interval.

N	error	convergence rate	condition number	CPU time seconds
4	2.211e–02		1.70e+01	
8	5.417e–04	5.35	3.07e+01	0.04
16	6.182e–06	6.45	6.15e+01	0.12
32	5.463e–07	3.50	1.23e+02	0.49
64	1.675e–08	5.03	2.47e+02	2.14
128	5.310e–10	4.98	4.94e+02	10.50
256	1.679e–11	4.98	9.89e+02	65.81
512	1.839e–12	3.19	1.98e+03	483.65

Table 6.2. Numerical results for Hermite cubic splines with collocation parameter $\epsilon = \epsilon_5^\star$, using 4 Gauss points per interval.

N	error	convergence rate	condition number	CPU time seconds
4	−2.325e−03		1.07e+02	
8	1.110e−04	4.39	2.09e+20	0.05
16	5.771e−07	7.59	4.13e+02	0.20
32	2.160e−09	8.06	8.18e+02	0.79
64	2.764e−11	6.29	1.63e+03	3.36
128	2.871e−13	6.59	3.24e+03	15.53
256	2.267e−12	−2.98	6.47e+03	86.21
512	−2.659e−12	−0.23	1.29e+04	564.22

Table 6.3. Numerical results for Hermite cubic splines with collocation parameter $\epsilon = \epsilon_5^\star$, using adaptive quadrature.

N	error	convergence rate	condition number	CPU time seconds
4	−3.865e−03		1.07e+02	
8	1.746e−04	4.47	2.09e+02	1.79
16	−4.494e−10	18.57	4.13e+02	5.00
32	3.767e−09	−3.07	8.18e+02	15.73
64	2.762e−11	7.09	1.63e+03	54.56
128	2.120e−13	7.03	3.24e+03	202.69
256	1.332e−15	7.31	6.47e+03	799.28

Table 6.4. Numerical results for Hermite cubic splines with collocation parameter $\epsilon = 1/4$, using adaptive quadrature.

N	error	convergence rate	condition number	CPU time seconds
4	−3.418e−03		1.07e+02	
8	1.705e−04	4.32	2.10e+02	1.78
16	−7.972e−07	7.74	4.14e+02	4.96
32	−3.038e−08	4.71	8.21e+02	15.57
64	−1.121e−09	4.76	1.63e+03	54.02
128	−3.628e−11	4.95	3.25e+03	201.02
256	−1.143e−12	4.99	6.49e+03	797.88

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