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A unified framework for parabolic equations with mixed boundary conditions and diffusion on interfaces

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Abstract

In this paper we consider scalar parabolic equations in a general non-smooth setting with emphasis on mixed interface and boundary conditions. In particular, we allow for dynamics and diffusion on a Lipschitz interface and on the boundary, where diffusion coefficients are only assumed to be bounded, measurable and positive semidefinite. In the bulk, we additionally take into account diffusion coefficients which may degenerate towards a Lipschitz surface. For this problem class, we introduce a unified functional analytic framework based on sesquilinear forms and show maximal regularity for the corresponding abstract Cauchy problem.

1. Introduction

This paper presents a unified framework for a general class of linear inhomogeneous mixed initial-boundary value problems of the form

$$\zeta\partial_t u - \operatorname{div}(\mu_\Omega \nabla u) = f_{\Omega\setminus\Sigma} \qquad \text{in } J \times (\Omega \setminus \Sigma), \tag{1.1}$$

$$u = 0 \qquad \text{on } J \times \Gamma_D, \tag{1.2}$$

$$\nu \cdot \mu_\Omega \nabla u = 0 \qquad \text{on } J \times \Gamma_N, \tag{1.3}$$

$$\zeta\partial_t u - \operatorname{div}_{\Gamma_\mathrm{d}}(\mu_{\Gamma_\mathrm{d}} \nabla_{\Gamma_\mathrm{d}} u) + \nu \cdot \mu_\Omega \nabla u = f_{\Gamma_\mathrm{d}} \qquad \text{on } J \times \Gamma_\mathrm{d}, \tag{1.4}$$

$$\zeta\partial_t u - \operatorname{div}_\Sigma(\mu_\Sigma \nabla_\Sigma u) + [\nu_\Sigma \cdot \mu_\Omega \nabla u] = f_\Sigma \qquad \text{on } J \times \Sigma, \tag{1.5}$$

$$u(0) = u_0 \qquad \text{in } \Omega \times \Gamma_\mathrm{d} \times \Sigma. \tag{1.6}$$

Here $J = (0, T)$ is a time interval and $\Omega \subset \mathbb{R}^d$ is a bounded domain with boundary $\partial\Omega$ and outer unit normal vector field ν. The boundary is disjointly decomposed into a Dirichlet part Γ_D, a Neumann part Γ_N and a dynamic part Γ_d, i.e.,

$$\partial\Omega = \Gamma_D \,\dot\cup\, \Gamma_N \,\dot\cup\, \Gamma_\mathrm{d}.$$

Moreover, $\Sigma \subset \Omega$ is a $(d-1)$-dimensional hypersurface with unit normal vector field ν_Σ, on which a further dynamic condition is imposed, and $[\nu_\Sigma \cdot \mu_\Omega \nabla u]$ denotes the jump of $\nu_\Sigma \cdot \mu_\Omega \nabla u$ over Σ. The surface gradients on Γ_d and on Σ are denoted by $\nabla_{\Gamma_\mathrm{d}}$ and ∇_Σ. Accordingly, we write $\operatorname{div}_{\Gamma_\mathrm{d}}$ and $\operatorname{div}_\Sigma$ for the surface divergences, such that $\Delta_{\Gamma_\mathrm{d}} = \operatorname{div}_{\Gamma_\mathrm{d}} \nabla_{\Gamma_\mathrm{d}}$ and $\Delta_\Sigma = \operatorname{div}_\Sigma \nabla_\Sigma$ are the Laplace-Beltrami operators which model the tangential flux on the dynamic surfaces. The diffusion coefficients μ_Ω, μ_{Γ_d} and μ_Σ are matrix-valued, and the relaxation coefficent ζ is positive and uniformly bounded away from zero. The source terms $f_{\Omega\setminus\Sigma}$, f_{Γ_d} and f_Σ as well as the initial data u_0 are assumed to be given. Initial data has to be prescribed at $\Omega \setminus \Sigma$, Γ_{d} and Σ due to the corresponding dynamic equations on these sets.

Well-posedness and qualitative properties of parabolic problems with dynamic boundary conditions are well-studied, see [4, 5, 11, 14, 21, 41], as well as [2, 9, 15, 16, 23, 22, 32, 38, 40] for more recent developments. Here, mostly the case of a smooth boundary and nondegenerate diffusion coefficients is considered. Nonlinear degenerate bulk diffusion is investigated in [2, 15, 23], and the case of mixed boundary conditions on a Lipschitz boundary, nonsmooth diffusion coefficients and dynamics on interfaces is considered in [9]. Mixed Dirichlet-Wentzell boundary conditions with a smooth Wentzell boundary are treated in [41].

The present paper extends the results of [9] in two directions: we consider surface diffusion on Lipschitz boundaries and interfaces with diffusion coefficients which may degenerate arbitrarily, and further allow the bulk diffusion coefficients to degenerate moderately towards a Lipschitz hypersurface. In addition, mixed boundary conditions, nonsmooth diffusion and relaxation coefficients are still taken into account.

We present a unified setting based on recent abstract results for sesquilinear forms from [3], which handles all these nonsmooth scenarios and their combinations at once. It yields maximal parabolic L^p-regularity for the corresponding Cauchy problem, which in particular implies that solutions are governed by an analytic C_0-semigroup (see [7, 36]). We can even show that the underlying elliptic operator admits a bounded holomorphic functional calculus. The setting further provides sufficient conditions for fractional power domains of the corresponding elliptic operator to embed into spaces of bounded functions, such that local-in-time well-posed for semilinear versions of (1.1)–(1.6), i.e., where the right-hand side $(f_{\Omega\setminus\Sigma}, f_{\Gamma_{\mathrm{d}}}, f_\Sigma)$ depends nonlinearly on the solution itself, can be deduced.

The limits of our approach seem to be inhomogeneous Dirichlet and Neumann boundary conditions, as well as boundary parts and interfaces evolving in time.

Let us give more details on the assumptions for the geometry and the coefficients. The boundary parts Γ_D, Γ_N and Γ_{d} are allowed to meet, and also the interface Σ may meet arbitrary parts of the boundary. No conditions on the Dirichlet part Γ_D are imposed, except near points where it meets the remainder of $\partial\Omega$. The diffusion coefficients μ_Ω, $\mu_{\Gamma_{\mathrm{d}}}$ and μ_Σ do not have to be symmetric and are only assumed to be measurable and bounded. To describe their degeneracies in a precise way, we assume pointwise estimates of the form

$$(\mu(x)v, v) \geq c_1\mu^*(x)|v|^2, \quad v \in \mathbb{R}^d, \qquad \|\mu(x)\|_{\mathcal{L}(\mathbb{R}^d)} \leq c_2\mu^*(x),$$

where μ stands for μ_Ω, $\mu_{\Gamma_{\mathrm{d}}}$ or μ_Σ, and μ^* is in each case a measurable, bounded and nonnegative function. We may allow for *arbitrary* support of $\mu^*_{\Gamma_{\mathrm{d}}}$ and μ^*_Σ.

It is well-known that the heat equation with dynamic boundary conditions is well-posed with or without surface diffusion. Our results show that, as one expects, linear surface diffusion only makes things better and improves the regularity on the boundary and the interface. It does not destroy the good properties (i.e., maximal regularity) of the corresponding Cauchy problem.

For the function μ_Ω^* we assume that

$$\mu_\Omega^*(x) = \mathrm{dist}(x, S)^\gamma, \qquad x \in \Omega,$$

where $S \subset \overline{\Omega}$ is an arbitrary $(d-k)$-dimensional Lipschitz submanifold of $\mathbb{R}^d$, $1 \le k \le d$, and the exponent is in the range $0 < \gamma < k$, which makes μ_Ω^* a Muckenhoupt weight. It is of particular interest when $S = \Sigma$, i.e., when diffusion degenerates towards and on Σ, but is possible along Σ. In this case we will have to assume that $\gamma < 1$.

We describe the setting in which (1.1)–(1.6) is realized. The basis of the approach is the sesquilinear form

$$\mathfrak{t}(u, v) = \int_\Omega (\mu_\Omega \nabla u, \overline{\nabla v})\, dx + \int_{\Gamma_\mathrm{d}} \big(\mu_{\Gamma_\mathrm{d}} \nabla_{\Gamma_\mathrm{d}} u, \overline{\nabla_{\Gamma_\mathrm{d}} v}\big)\, d\mathcal{H}_{d-1} + \int_\Sigma \big(\mu_\Sigma \nabla_\Sigma u, \overline{\nabla_\Sigma v}\big)\, d\mathcal{H}_{d-1},$$

where $\mathcal{H}_{d-1}$ denotes the Hausdorff measure. The surface gradients $\nabla_{\Gamma_\mathrm{d}}$ and ∇_Σ on the Lipschitz surfaces Γ_d and Σ are introduced in a simple, straightforward way in terms of local coordinates, such that the definitions coincide with the corresponding well-known objects in a smooth situation (see Section 2). To precisely obtain the boundary and interface regularity which is dictated by (1.1)–(1.6) we define the domain of $\mathfrak{t}$ as the closure of the set of smooth functions (vanishing on the Dirichlet boundary Γ_D) with respect to

$$\|u\|^2_{\mathrm{Dom}(\mathfrak{t})} = \|u\|^2_{W^{1,2}(\Omega,\mu_\Omega^*)} + \|\nabla_{\Gamma_\mathrm{d}} u\|^2_{L^2(\Gamma_\mathrm{d},\mu^*_{\Gamma_\mathrm{d}})} + \|\nabla_\Sigma u\|^2_{L^2(\Sigma,\mu^*_\Sigma)}.$$

Here $W^{1,2}(\Omega, \mu_\Omega^*)$ is a Sobolev space with weight μ_Ω^* in the gradient norm, and $L^2(\Gamma_\mathrm{d}, \mu^*_{\Gamma_\mathrm{d}})$ and $L^2(\Sigma, \mu^*_\Sigma)$ are Lebesgue spaces equipped with the weights $\mu^*_{\Gamma_\mathrm{d}}$ and μ^*_Σ.

Based on the results of [3], to the form $\mathfrak{t}$ we associate an operator A_2 on the Lebesgue space

$$\mathbb{L}^2 = L^2\big(\Omega \cup \Gamma_\mathrm{d} \cup \Sigma, (dx + d\mathcal{H}_{d-1})\big) = L^2(\Omega) \oplus L^2(\Gamma_\mathrm{d}) \oplus L^2(\Sigma),$$

which realizes the spatial derivatives in (1.1), (1.4) and (1.5) in a weak setting. The constitutive relation for $A_2 u$ is

$$\int_{\Omega \cup \Gamma_\mathrm{d} \cup \Sigma} A_2 u\, \overline{\psi} (dx + d\mathcal{H}_{d-1}) = \mathfrak{t}(u, \psi),$$

for all suitable test functions ψ. To see the formal connection of A_2 to the spatial derivatives in (1.1), (1.4) and (1.5), extend Σ to a surface $\widetilde{\Sigma}$ which decomposes Ω into two subdomains. Choosing test functions ψ which vanish on the points where the boundary parts meet and on the boundary of the interface, formal integration by parts in the integrals defining $\mathfrak{t}$ yields

$$\int_{\Omega\cup\Gamma_{\mathrm{d}}\cup\Sigma}(A_2u)\overline{\psi}\,(dx+d\mathcal{H}_{d-1}) = -\int_{\Omega}\operatorname{div}(\mu_\Omega\nabla u)\overline{\psi}\,dx + \int_{\Gamma_N}(\nu\cdot\mu_\Omega\nabla u)\overline{\psi}\,d\mathcal{H}_{d-1}$$
$$+\int_{\Gamma_{\mathrm{d}}}\big(\nu\cdot\mu_\Omega\nabla u - \operatorname{div}_{\Gamma_{\mathrm{d}}}(\mu_{\Gamma_{\mathrm{d}}}\nabla_{\Gamma_{\mathrm{d}}}u)\big)\overline{\psi}\,d\mathcal{H}_{d-1}$$
$$+\int_{\widetilde{\Sigma}}[\nu_{\widetilde{\Sigma}}\cdot\mu_\Omega\nabla u]\overline{\psi}\,d\mathcal{H}_{d-1} - \int_{\Sigma}\operatorname{div}_\Sigma(\mu_\Sigma\nabla_\Sigma u)\overline{\psi}\,d\mathcal{H}_{d-1}.$$

Varying the support of ψ suitably, we formally obtain that A_2 reflects the Dirichlet and Neumann conditions and has the three components

$$A_2u = \begin{pmatrix} -\operatorname{div}(\mu_\Omega\nabla u) \\ -\operatorname{div}_{\Gamma_{\mathrm{d}}}(\mu_{\Gamma_{\mathrm{d}}}\nabla_{\Gamma_{\mathrm{d}}}u) + \nu\cdot\mu_\Omega\nabla u \\ -\operatorname{div}_\Sigma(\mu_\Sigma\nabla_\Sigma u) + [\nu_\Sigma\cdot\mu_\Omega\nabla u] \end{pmatrix} \in \mathbb{L}^2.$$

Observe at this point that the jump of $\nu_{\widetilde{\Sigma}}\cdot\mu_\Omega\nabla u$ over $\widetilde{\Sigma}\setminus\Sigma$ is forced to vanish, since the measure in $\mathbb{L}^2$ is supported only on Σ.

In case when the bulk diffusion degenerates on Γ_{d} or Σ, the regularity of the trace of functions from $\mathrm{Dom}(\mathfrak{t})$ on these sets becomes worse. To obtain the trace regularity as it is sufficient to realize A_2 on $\mathbb{L}^2$, we essentially rely on the weighted Sobolev embedding

$$W^{1,2}(\mathbb{R}^d,\operatorname{dist}(\cdot,S)^\gamma)\subset W^{\theta,q}(\mathbb{R}^d), \qquad 1-\frac{d+\gamma}{2}\geq\theta-\frac{d}{q}, \qquad q\geq 2,$$

which seems to be new in this explicit form and is deduced from the very general embedding results in [19] (see Proposition 4.3). Here $W^{\theta,q}(\mathbb{R}^d)$ denotes the usual Slobodetskii space.

It turns out that $-A_2$ generates an analytic C_0-semigroup $T_2(\cdot)$ of contractions on $\mathbb{L}^2$, see Proposition 3.4. This already yields the solvability of our realization of (1.1)–(1.6) for source terms $(f_{\Omega\setminus\Sigma}, f_{\Gamma_{\mathrm{d}}}, f_\Sigma)$ from $L^2(J;\mathbb{L}^2)$ and initial data $u_0\in\mathbb{L}^2$. We emphasize that the components of the initial data must not be related.

To treat semilinear problems, the realization of (1.1)–(1.6) on an L^2-space is not sufficient, due to the lack of embeddings for the fractional power domains A_2 into bounded functions. To this end we extend A_2 consistently to the whole $\mathbb{L}^p$-scale, $p\in[1,\infty]$. This is achieved by showing that $T_2(\cdot)$ is $\mathbb{L}^\infty$-contractive (see Proposition 3.6), such that it extends to a contraction semigroup $T_p(\cdot)$ on $\mathbb{L}^p$ by interpolation and duality. For the $\mathbb{L}^\infty$-contractivity, in Lemma 3.5 we in particular have to overcome a technical difficulty concerning the nonlinear operator $u\mapsto u\wedge\mathbb{1}$ on the domain of $\mathfrak{t}$. The negative generator A_p of the analytic semigroup $T_p(\cdot)$ is then the desired consistent extension of A_2 to $\mathbb{L}^p$.

The analyticity of $T_p(\cdot)$ for $p \in (1,\infty)$ together with the contractivity of $T_p(\cdot)$ for $p \in [1,\infty]$ now allows to apply a deep result from harmonic analysis due to [8, 26, 30, 42] (see [31, Proposition 2.2]) to conclude that A_p admits a bounded holomorphic functional calculus and maximal Lebesgue regularity (see [7, 29, 36] for surveys on these topics). Hence the realization is as good as it can be, despite the variety of nonsmooth effects it takes into account. The precise formulation is given in the Theorems 3.8 and 4.7. Maximal regularity of the linearization is the key to treat semilinear and quasilinear parabolic problems [36], see also [9, Section 4] for a detailed discussion and references related to the present setting.

Employing again that A_p is given on a scalar L^p-space, the multiplication with the inverse relaxation coefficient ζ^{-1} does not change the described properties. This essentially follows from the abstract results of [9, Proposition 2.20].

Finally, embeddings of the type $\mathrm{Dom}(A_p^\theta) \subset \mathbb{L}^\infty$, for θ sufficiently close to 1 and $p > 2$ sufficiently large, are obtained in Section 5 from semigroup estimates and an integral formula for negative fractional powers of A_p. By means of this embedding we can quantify the impact of the degeneracy of the surface diffusion on the regularity of solutions. As mentioned before, the embedding is crucial to treat the corresponding semilinear problems.

This paper is organized as follows. In Section 2 we introduce tangent spaces and the surface gradient for Lipschitz hypersurfes in graph representation. To separate the technical difficulties, in Section 3 we consider the case of nondegenerate bulk diffusion, while in Section 4 we treat degenerate bulk diffusion. In Section 5 embeddings into spaces of bounded functions are investigated.

Notations. Generic positive constant are denoted by C or c. By $\mathcal{L}(\mathbb{R}^d)$ we designate the space of linear operators on $\mathbb{R}^d$, which we canonically identify with the set of $(d \times d)$-matrices. The euclidian scalar product of $v, w \in \mathbb{R}^d$ is denoted by $v \cdot w$ or (v, w). For $p \in [1,\infty]$, the usual complex-valued Lebesgue space is denoted by $L^p(\Omega)$.

2. The surface gradient on Lipschitz hypersurfaces

In this section we introduce tangent spaces and the surface gradient for a Lipschitz hypersurface $\mathcal{S}$ in graph representation in an elementary way. The idea is that Lipschitz coordinates are differentiable almost everywhere, which allows us to give definitions in coordinates analogous to the smooth case. Hence for smooth $\mathcal{S}$ we automatically recover the standard notions, see [1, Chapter VII] and [20, 25] for basic accounts. For Lipschitz surfaces we also refer to [12, 18, 34, 37].

2.1. **Lipschitz hypersurfaces.** Let $\mathcal{S} \subset \mathbb{R}^d$ be a *Lipschitz hypersurface in graph representation.* This means that for each $x \in \mathcal{S}$ there are *Lipschitz-graph coordinates* (g, U) and an open neighbourhood V of x in $\mathbb{R}^d$ such that $U \subset \mathbb{R}^{d-1}$ is open and $\mathrm{g} : U \to \mathcal{S} \cap V$ is bijective and of the form

$$\mathrm{g}(y) = Q \begin{pmatrix} y \\ h(y) \end{pmatrix} + x^*, \qquad y \in U,$$

where $Q \in \mathcal{L}(\mathbb{R}^d)$ is orthogonal, $x^* \in \mathbb{R}^d$ is a fixed vector and $h : U \to \mathbb{R}$ is Lipschitz continuous. For this and equivalent definitions we refer to [34, Section 2]. Employing that the topology of $\mathbb{R}^d$ has a countable basis, standard arguments show that there is an at most countable number of Lipschitz graph coordinates $(\mathrm{g}_\alpha, U_\alpha)$ such that $\mathcal{S} \subseteq \bigcup_\alpha \mathrm{g}_\alpha(U_\alpha)$, see the proof of [34, Theorem 2.15].

By Rademacher's theorem (see [12, Theorem 3.1.2]), Lipschitz coordinates g are almost everywhere differentiable on U in the classical sense and one has $g \in W^{1,\infty}(U, \mathbb{R}^d)$, where

$$\mathrm{g}'(y) = Q \begin{pmatrix} \mathrm{id}_{d-1} \\ h'(y) \end{pmatrix} \in \mathcal{L}(\mathbb{R}^{d-1}, \mathbb{R}^d)$$

at points $y \in U$ where g is differentiable. Observe that $\mathrm{g}'(y)$ is injective and has rank $d-1$. Hence the corresponding *metric tensor* $\mathrm{G} : U \to \mathcal{L}(\mathbb{R}^{d-1})$, defined by

$$\mathrm{G}(y) = \mathrm{g}'(y)^T \mathrm{g}'(y) = \big((\partial_i \mathrm{g}(y), \partial_j \mathrm{g}(y))\big)_{ij},$$

is for almost all $y \in U$ symmetric and positive definite. With the usual abuse of notation we write $\mathrm{G} = (\mathrm{g}_{ij})_{ij}$, and $\mathrm{G}^{-1} = (\mathrm{g}^{ij})_{ij}$ for the pointwise inverse of G.

We call Lipschitz-graph coordinates g *regular* for $x \in \mathcal{S}$ if g is differentiable at $y = \mathrm{g}^{-1}(x)$. If such regular coordinates exist, we call x regular. For instance, all points of the boundary of a cube are regular except the ones on edges.

Lemma 2.1. *Let $\mathcal{S}$ be a Lipschitz hypersurface in graph representation. Then $\mathcal{H}_{d-1}$-almost every point $x \in \mathcal{S}$ is regular.*

Proof. Let $N \subset \mathcal{S}$ be the set of points which are not regular. Take at most countable many coordinates $(\mathrm{g}_\alpha, U_\alpha)$ such that $\mathcal{S} \subseteq \bigcup_\alpha V_\alpha$ for $V_\alpha = \mathrm{g}_\alpha(U_\alpha)$. Then $\mathcal{H}_{d-1}(N) \leq \sum_\alpha \mathcal{H}_{d-1}(N \cap V_\alpha)$. Let further N_α be the set of points where g_α is not differentiable. Then $\mathcal{H}_{d-1}(N_\alpha) = 0$ by Rademacher's theorem. Using $N \cap V_\alpha \subseteq \mathrm{g}_\alpha(N_\alpha)$ and [12, Theorem 2.4.1/1], for each α we obtain

$$\mathcal{H}_{d-1}(N \cap V_\alpha) \leq \mathcal{H}_{d-1}(\mathrm{g}_\alpha(N_\alpha)) \leq \mathrm{Lip}(\mathrm{g}_\alpha)^{d-1} \mathcal{H}_{d-1}(N_\alpha) = 0,$$

where $\mathrm{Lip}(\mathrm{g}_\alpha)$ is the Lipschitz constant of g_α. This shows $\mathcal{H}_{d-1}(N) = 0$. $\square$

As another preparation we consider the properties of *transition maps*.

Lemma 2.2. *Let* $(\mathrm{g}_\alpha, U_\alpha)$ *and* $(\mathrm{g}_\beta, U_\beta)$ *be Lipschitz-graph coordinates for* $\mathcal{S}$ *which are both regular for* $x \in \mathcal{S}$*. Set* $y_\alpha = \mathrm{g}_\alpha^{-1}(x) \in U_\alpha$ *and* $y_\beta = \mathrm{g}_\beta^{-1}(x) \in U_\beta$*. Then the following assertions hold true.*

(a) *The transition map* $\mathrm{g}_\beta^{-1} \circ \mathrm{g}_\alpha$ *is differentiable at* y_α*. The derivative* $(\mathrm{g}_\beta^{-1} \circ \mathrm{g}_\alpha)'(y_\alpha) \in \mathcal{L}(\mathbb{R}^{d-1})$ *is invertible with inverse* $(\mathrm{g}_\alpha^{-1} \circ \mathrm{g}_\beta)'(y_\beta)$*.*
(b) *The derivatives* $\mathrm{g}_\alpha'(y_\alpha)$ *and* $\mathrm{g}_\beta'(y_\beta)$ *have the same images in* $\mathbb{R}^d$*. We have* $v = \mathrm{g}_\alpha'(y_\alpha)\xi_\alpha$ *for* $\xi_\alpha \in \mathbb{R}^{d-1}$ *if and only if* $v = \mathrm{g}_\beta'(y_\beta)\xi_\beta$ *for* $\xi_\beta = (\mathrm{g}_\beta^{-1} \circ \mathrm{g}_\alpha)'(y_\alpha)\xi_\alpha$*.*
(c) *For the metric tensors* G_α *and* G_β *corresponding to* g_α *and* g_β *we have*

$$\mathrm{G}_\alpha(y_\alpha) = (\mathrm{g}_\beta^{-1} \circ \mathrm{g}_\alpha)'(y_\alpha)^T \mathrm{G}_\beta(y_\beta)(\mathrm{g}_\beta^{-1} \circ \mathrm{g}_\alpha)'(y_\alpha).$$

Proof. We write $\Phi = \mathrm{g}_\beta^{-1} \circ \mathrm{g}_\alpha$ for the transition map. Observe that Φ is a homeomorphism on a neighbourhood of y_α with inverse $\Phi^{-1} = \mathrm{g}_\alpha^{-1} \circ \mathrm{g}_\beta$.

(a) The form of g_β shows that $\Phi(y)$ is given by the first $d-1$ entries of $Q_\beta^T(\mathrm{g}_\alpha(y) - x_\beta^*)$. Hence Φ is differentiable at y_α. In the same way we obtain the differentiability of Φ^{-1} at y_β. Therefore $\Phi'(y_\alpha)$ is invertible with inverse as asserted.

(b) This follows from $\mathrm{g}_\alpha'(y_\alpha) = \mathrm{g}_\beta'(y_\beta)\Phi'(y_\alpha)$ and the invertibility of $\Phi'(y_\alpha)$.

(c) We can repeat the short argument from [25, Section 1.4]. For arbitrary $\xi_\alpha, \eta_\alpha \in \mathbb{R}^{d-1}$ we use (b) to obtain

$$\begin{aligned}(\mathrm{G}_\alpha(y_\alpha)\xi_\alpha, \eta_\alpha) &= \big(\mathrm{g}_\alpha'(y_\alpha)\xi_\alpha, \mathrm{g}_\alpha'(y_\alpha)\eta_\alpha\big)\\ &= \big(\mathrm{g}_\beta'(y_\beta)\Phi'(y_\alpha)\xi_\alpha, \mathrm{g}_\beta'(y_\beta)\Phi'(y_\alpha)\eta_\alpha\big)\\ &= (\Phi'(y_\alpha)^T \mathrm{G}_\beta(y_\beta)\Phi'(y_\alpha)\xi_\alpha, \eta_\alpha).\end{aligned}$$

This implies the asserted formula. □

2.2. **Tangent space and surface gradient.** Now we can introduce the following notions.

Definition 2.3. Let $\mathcal{S}$ be a Lipschitz hypersurface in graph representation.

(a) Let $x \in \mathcal{S}$ be regular with Lipschitz graph coordinates (g, U). The *tangent space* at x is

$$T_x\mathcal{S} = \big\{v \in \mathbb{R}^d \,:\, \text{there is } \xi \in \mathbb{R}^{d-1} \text{ with } v = \mathrm{g}'(\mathrm{g}^{-1}(x))\xi\big\}.$$

We further set $T_x\mathcal{S} = \{0\} \subset \mathbb{R}^d$ if x is not regular.
(b) A function $u \in L^1_{\mathrm{loc}}(\mathcal{S})$ is called *weakly differentiable*, if for all Lipschitz graph coordinates (g, U) for $\mathcal{S}$ the function $u \circ \mathrm{g}$ is weakly differentiable on $U \subset \mathbb{R}^{d-1}$.

(c) Let $u \in L^1_{\mathrm{loc}}(\mathcal{S})$ be weakly differentiable. Then for a regular point $x \in \mathcal{S}$ the *surface gradient* $\nabla_{\mathcal{S}} u(x) \in T_x\mathcal{S}$ is given by

$$\nabla_{\mathcal{S}} u(x) = \mathrm{g}'(y)\mathrm{G}^{-1}(y)\nabla(u \circ \mathrm{g})(y) = \sum_{i,j=1}^{d-1} \mathrm{g}^{ij}(y)\partial_j(u \circ \mathrm{g})(y)\partial_i \mathrm{g}(y),$$

where (g, U) are arbitrary regular Lipschitz graph coordinates for x and $y = \mathrm{g}^{-1}(x)$. Setting $\nabla_{\mathcal{S}} u(x) = 0$ if x is not regular, this defines the *surface gradient field* $\nabla_{\mathcal{S}} u$ on $\mathcal{S}$.

These notions coincide with the usual ones if $\mathcal{S}$ is smooth, see, e.g., [1, Remark VII.10.11] for the representation of the surface gradient in coordinates. As in the smooth case one shows that these notions are well-defined.

Lemma 2.4. *At a regular point $x \in \mathcal{S}$, the tangent space as well as the surface gradient of a weakly differentiable function are independent of the chosen regular graph coordinates.*

Proof. The assertion for the tangent space follows from Lemma 2.2(b). For the surface gradient we let g_α and g_β be regular for x, set $y_\alpha = \mathrm{g}_\alpha^{-1}(x)$, $y_\beta = \mathrm{g}_\beta^{-1}(x)$ and

$$v_\alpha = \mathrm{g}_\alpha'(y_\alpha)\mathrm{G}_\alpha^{-1}(y_\alpha)\nabla(u \circ \mathrm{g}_\alpha)(y_\alpha), \qquad v_\beta = \mathrm{g}_\beta'(y_\beta)\mathrm{G}_\beta^{-1}(y_\beta)\nabla(u \circ \mathrm{g}_\beta)(y_\beta).$$

As above we write $\Phi = \mathrm{g}_\beta^{-1} \circ \mathrm{g}_\alpha$ for the transition map. By Lemma 2.2(b) we have $v_\alpha = v_\beta$ if and only if

$$\mathrm{G}_\beta^{-1}(y_\beta)\nabla(u \circ \mathrm{g}_\beta)(y_\beta) = \Phi'(y_\alpha)\mathrm{G}_\alpha^{-1}(y_\alpha)\nabla(u \circ \mathrm{g}_\alpha)(y_\alpha).$$

But this is a consequence of the identities

$$\nabla(u \circ \mathrm{g}_\alpha)(y_\alpha) = \Phi'(y_\alpha)^T\nabla(u \circ \mathrm{g}_\beta)(y_\beta), \qquad \mathrm{G}_\beta^{-1}(y_\beta) = \Phi'(y_\alpha)\mathrm{G}_\alpha^{-1}(y_\alpha)\Phi'(y_\alpha)^T,$$

where the latter follows from Lemma 2.2(c). □

3. Non-degenerate bulk diffusion

In this section we consider (1.1)–(1.6) with a uniformly elliptic diffusion coefficient μ_Ω in the bulk. The case when μ_Ω degenerates towards a compact Lipschitz surface is investigated in the next section.

3.1. **Assumptions on the geometry and the coefficients.** Throughout this section we impose the following.

Assumption 3.1. (a) $\Omega \subset \mathbb{R}^d$ is a bounded domain, $d \geq 2$.

(b) The closures $\overline{\Gamma_N}$, $\overline{\Gamma_{\mathrm{d}}}$ and $\overline{\Sigma}$ of Γ_N, Γ_{d} and Σ are contained in a Lipschitz hypersurface in graph representation, respectively.

(c) Γ_{d} and Σ are endowed with the $(d-1)$-dimensional Hausdorff measure $\mathcal{H}_{d-1}$.

(d) The coefficient $\mu_\Omega : \Omega \to \mathcal{L}(\mathbb{R}^d)$ is measurable, bounded and there is a constant $\mu_\Omega^* > 0$ such that

$$\big(\mu_\Omega(x)v, v\big) \geq \mu_\Omega^* |v|^2, \qquad x \in \Omega, \qquad v \in \mathbb{R}^d.$$

(e) Let $\mathcal{S}$ be either Γ_{d} or Σ. Then $\mu_{\mathcal{S}} : \mathcal{S} \to \mathcal{L}(\mathbb{R}^d)$ is measurable, and there are a measurable, bounded, nonnegative function $\mu_{\mathcal{S}}^* : \mathcal{S} \to \mathbb{R}$ and constants $c_1, c_2 > 0$ such that

$$\big(\mu_{\mathcal{S}}(x)v, v\big) \geq c_1 \mu_{\mathcal{S}}^*(x)|v|^2, \qquad \|\mu_{\mathcal{S}}(x)\|_{\mathcal{L}(\mathbb{R}^d)} \leq c_2 \mu_{\mathcal{S}}^*(x), \qquad x \in \mathcal{S}, \quad v \in T_x\mathcal{S}.$$

(f) The relaxation coefficient $\zeta : \Omega \cup \Gamma_{\mathrm{d}} \cup \Sigma \to \mathbb{R}$ is measurable, bounded and there is a constant $c > 0$ such that $\zeta(x) \geq c$ for all $x \in \Omega \cup \Gamma_{\mathrm{d}} \cup \Sigma$.

We emphasize that for the Dirichlet part Γ_D, there are only assumptions in a neighbourhood of points where Γ_D meets Γ_N or Γ_{d}. In particular, in the pure Dirichlet case $\Gamma_D = \partial\Omega$ there are no assumptions on the boundary. Moreover, it is not excluded that one or more of the sets Γ_D, Γ_N, Γ_{d} or Σ are empty.

The functions $\mu_{\Gamma_{\mathrm{d}}}^*$ and μ_Σ^* describe where diffusion takes place on Γ_{d} and Σ, and where diffusion degenerates. There are no restrictions on the support of these functions. An example we have in mind is $\mu_{\mathcal{S}}^*(x) = \operatorname{dist}(x, M)^\gamma$ for a subset $M \subset \mathcal{S}$ and $\gamma > 0$, which indicates that diffusion degenerates towards M and is impossible along and across M.

The above assumptions cover a large class of nonsmooth scenarios. However, our realization of (1.1)–(1.6) developed below also works under more general conditions. For instance, the interface Σ must only be a Lipschitz hypersurface in graph representation in a neighbourhood of the support of μ_Σ^*. Away from the support, as in [9] it suffices that Σ is a $(d-1)$-set (see [24, Section VII.1.1]). To avoid too many technical difficulties we do not take these issues into account.

3.2. **The realization on $\mathbb{L}^2$.** We construct the operator A_2 which yields a realization of the elliptic part of (1.1)–(1.6) on a suitable L^2-space. The assumptions of this section cover the ones of [9], such that the extension and trace results obtained there are available.

For $p \in (1, \infty)$ we denote by $W^{1,p}(\Omega)$ the usual complex-valued Sobolev space over Ω. We further define $W^{1,p}_D(\Omega)$ as the closure in $W^{1,p}(\Omega)$ of

$$C_D^\infty(\Omega) = \big\{ u|_\Omega : u \in C_c^\infty(\mathbb{R}^d),\ (\operatorname{supp} u) \cap \Gamma_D = \emptyset \big\}.$$

Roughly speaking, elements of $W^{1,p}_D(\Omega)$ vanish on the Dirichlet part Γ_D of $\partial\Omega$.

Let $\mathrm{tr}_{\Gamma_\mathrm{d}}$ and tr_Σ be the trace operators for Γ_d and Σ. Then [9, Proposition 2.8] implies the continuity of

$$\mathrm{tr}_{\Gamma_\mathrm{d}} : W^{1,2}_D(\Omega) \to L^2(\Gamma_\mathrm{d}), \qquad \mathrm{tr}_\Sigma : W^{1,2}_D(\Omega) \to L^2(\Sigma). \tag{3.1}$$

We shall also write $u_{\Gamma_\mathrm{d}} = \mathrm{tr}_{\Gamma_\mathrm{d}} u$ and $u_\Sigma = \mathrm{tr}_\Sigma u$ for the traces, and often write u for u_{Γ_d} and u_Σ with abuse of notation if it is clear from the context that traces are meant.

Definition 3.2. (a) On $C^\infty_D(\Omega)$ we introduce the scalar product $(\cdot,\cdot)_{\mathrm{Dom}(\mathfrak{t})}$ by

$$(u,v)_{\mathrm{Dom}(\mathfrak{t})} = (u,v)_{W^{1,2}(\Omega)} + \int_{\Gamma_\mathrm{d}} \big(\nabla_{\Gamma_\mathrm{d}} u, \overline{\nabla_{\Gamma_\mathrm{d}} v}\big)\, \mu^*_\Gamma \, d\mathcal{H}_{d-1} + \int_\Sigma \big(\nabla_\Sigma u, \overline{\nabla_\Sigma v}\big)\, \mu^*_\Sigma \, d\mathcal{H}_{d-1},$$

where $(\cdot,\cdot)_{W^{1,2}(\Omega)}$ is the usual scalar product on $W^{1,2}(\Omega)$. The corresponding Hilbert norm is denoted by $\|\cdot\|_{\mathrm{Dom}(\mathfrak{t})}$.

(b) The Hilbert space $\mathrm{Dom}(\mathfrak{t})$ is defined by

$$\mathrm{Dom}(\mathfrak{t}) = \text{closure of } C^\infty_D(\Omega) \text{ with respect to } \|\cdot\|_{\mathrm{Dom}(\mathfrak{t})}.$$

(c) For $p \in [1,\infty]$ we set $\mathbb{L}^p = L^p\big(\Omega \cup \Gamma_\mathrm{d} \cup \Sigma, (dx + \mathcal{H}_{d-1})\big)$.

(d) The map $\mathfrak{J} : \mathrm{Dom}(\mathfrak{t}) \to \mathbb{L}^2$ is given by $\mathfrak{J}(u) = (u, u_{\Gamma_\mathrm{d}}, u_\Sigma)$.

The regularity of elements of $\mathrm{Dom}(\mathfrak{t})$ on Γ_d and Σ is determined by the supports of $\mu^*_{\Gamma_\mathrm{d}}$ and μ^*_Σ. It thus fits precisely to the regularity which is expected from the dynamical equations (1.4) and (1.5). We always have $\mathrm{Dom}(\mathfrak{t}) \subseteq W^{1,2}_D(\Omega)$ and

$$\big\{u \in W^{1,2}_D(\Omega) : u_{\Gamma_\mathrm{d}} \in W^{1,2}(\Gamma_\mathrm{d}), u_\Sigma \in W^{1,2}(\Sigma)\big\} \subseteq \mathrm{Dom}(\mathfrak{t}),$$

with equalities if $\mu^*_{\Gamma_\mathrm{d}}$ and μ^*_Σ vanish resp. are bounded away from zero. For $\mathcal{S} \in \{\Gamma_\mathrm{d}, \Sigma\}$ we will also write

$$\|f\|^2_{L^2(\mathcal{S},\mu^*_\mathcal{S})} = \int_\mathcal{S} |f|^2 \, \mu^*_\mathcal{S} \, d\mathcal{H}_{d-1},$$

such that the Hilbert norm may be expressed as

$$\|u\|^2_{\mathrm{Dom}(\mathfrak{t})} = \|u\|^2_{W^{1,2}(\Omega)} + \|\nabla_{\Gamma_\mathrm{d}} u\|^2_{L^2(\Gamma_\mathrm{d},\mu^*_{\Gamma_\mathrm{d}})} + \|\nabla_\Sigma u\|^2_{L^2(\Sigma,\mu^*_\Sigma)}. \tag{3.2}$$

In view of $\mathrm{Dom}(\mathfrak{t}) \subseteq W^{1,2}_D(\Omega)$ and the continuity of the traces (3.1), the map $\mathfrak{J}$ is indeed well-defined. The space $\mathbb{L}^p$ can be identified as

$$\mathbb{L}^p = L^p(\Omega) \oplus L^p(\Gamma_\mathrm{d}) \oplus L^p(\Sigma).$$

In general there is no relation between these components of an element of $\mathbb{L}^p$.

The operator A_2 will be derived from the sesquilinear form

$$\mathfrak{t}(u,v) = \int_\Omega (\mu_\Omega \nabla u, \overline{\nabla v})\, dx + \int_{\Gamma_\mathrm{d}} \big(\mu_{\Gamma_\mathrm{d}} \nabla_{\Gamma_\mathrm{d}} u, \overline{\nabla_{\Gamma_\mathrm{d}} v}\big)\, d\mathcal{H}_{d-1} + \int_\Sigma \big(\mu_\Sigma \nabla_\Sigma u, \overline{\nabla_\Sigma v}\big)\, d\mathcal{H}_{d-1},$$

which is originally defined for $u, v \in C_D^\infty(\Omega)$.

Lemma 3.3. *The form* $\mathfrak{t}$ *extends continuously to a sesquilinear form on* $\mathrm{Dom}(\mathfrak{t})$*. It is* $\mathfrak{J}$*-elliptic, i.e., there is* $c > 0$ *such that*

$$\operatorname{Re}\mathfrak{t}(u,u) + \|\mathfrak{J}u\|_{\mathbb{L}^2}^2 \geq c\|u\|_{\mathrm{Dom}(\mathfrak{t})}^2, \qquad u \in \mathrm{Dom}(\mathfrak{t}).$$

Moreover, the map $\mathfrak{J} : \mathrm{Dom}(\mathfrak{t}) \to \mathbb{L}^2$ *has dense range and is continuous and compact.*

Proof. The continuity and the compactness of $\mathfrak{J}$ follow from $\mathrm{Dom}(\mathfrak{t}) \subseteq W_D^{1,2}(\Omega)$ and [9, Lemma 2.10]. The proof in [9] also shows that $\mathfrak{J}C_D^\infty(\Omega)$ is dense in $\mathbb{L}^2$, hence $\mathfrak{J}\,\mathrm{Dom}(\mathfrak{t})$ is dense since $C_D^\infty(\Omega) \subset \mathrm{Dom}(\mathfrak{t})$.

It is clear that $\mathfrak{t} : C_D^\infty(\Omega) \times C_D^\infty(\Omega) \to \mathbb{C}$ is sesquilinear. Given $u, v \in C_D^\infty(\Omega)$ we use the assumption $\|\mu_{\mathcal{S}}(x)\|_{\mathcal{L}(\mathbb{R}^d)} \leq c_2\mu_{\mathcal{S}}^*(x)$ for $\mathcal{S} \in \{\Gamma_\mathrm{d}, \Sigma\}$, Hölder's inequality and (3.2) to estimate

$$\begin{aligned}
|\mathfrak{t}(u,v)| &\leq \|\mu_\Omega\|_\infty \|\nabla u\|_{L^2(\Omega)} \|\nabla v\|_{L^2(\Omega)} \\
&\quad + c_2\|\nabla_{\Gamma_\mathrm{d}} u\|_{L^2(\Gamma_\mathrm{d},\mu_{\Gamma_\mathrm{d}}^*)} \|\nabla_{\Gamma_\mathrm{d}} v\|_{L^2(\Gamma_\mathrm{d},\mu_{\Gamma_\mathrm{d}}^*)} + c_2\|\nabla_\Sigma u\|_{L^2(\Sigma,\mu_\Sigma^*)} \|\nabla_\Sigma v\|_{L^2(\Sigma,\mu_\Sigma^*)} \\
&\leq C\|u\|_{\mathrm{Dom}(\mathfrak{t})}\|v\|_{\mathrm{Dom}(\mathfrak{t})}.
\end{aligned}$$

Hence $\mathfrak{t}$ extends continuously to a sesquilinear form on $\mathrm{Dom}(\mathfrak{t})$. To show its $\mathfrak{J}$-ellipticity, for $u \in C_D^\infty(\Omega)$ we use the assumption $(\mu_{\mathcal{S}}v, v) \geq c_1\mu_{\mathcal{S}}^*|v|^2$ for $\mathcal{S} \in \{\Gamma_\mathrm{d}, \Sigma\}$ to get

$$\begin{aligned}
\operatorname{Re}\mathfrak{t}(u,u) + \|\mathfrak{J}u\|_{\mathbb{L}^2}^2 &\geq \mu_\Omega^*\|\nabla u\|_{L^2(\Omega)}^2 + c_1\|\nabla_{\Gamma_\mathrm{d}} u\|_{L^2(\Gamma_\mathrm{d},\mu_{\Gamma_\mathrm{d}}^*)}^2 + c_1\|\nabla_\Sigma u\|_{L^2(\Sigma,\mu_\Sigma^*)}^2 + \|u\|_{L^2(\Omega)}^2 \\
&\geq c\|u\|_{\mathrm{Dom}(\mathfrak{t})}^2.
\end{aligned}$$

This inequality carries over to all $u \in \mathrm{Dom}(\mathfrak{t})$ by density and the continuity of $\mathfrak{J}$. □

Now the operator A_2 can be derived from $\mathfrak{t}$ as follows.

Proposition 3.4. *There is a closed, densely defined operator* A_2 *on* $\mathbb{L}^2$ *associated to the form* $\mathfrak{t}$*: for* $\varphi, \psi \in \mathbb{L}^2$ *we have* $\varphi \in \mathrm{Dom}(A_2)$ *and* $A_2\varphi = \psi$ *if and only if there is* $u \in \mathrm{Dom}(\mathfrak{t})$ *such that* $\varphi = \mathfrak{J}u$ *and*

$$(\psi, \mathfrak{J}v)_{\mathbb{L}^2} = \mathfrak{t}(u,v) \qquad \text{for all } v \in \mathrm{Dom}(\mathfrak{t}).$$

The operator $-A_2$ *generates an analytic* C_0*-semigroup*

$$T_2(\cdot) = (T_2(t))_{t\geq 0}$$

of contractions on $\mathbb{L}^2$*. Furthermore,* A_2 *has compact resolvent.*

Proof. All assertions except the contraction property are a consequence of Lemma 3.3 and the general results of [3, Theorem 2.1, Lemma 2.7]. For the contractivity we observe that for $\varphi \in \mathrm{Dom}(A_2)$ with $\varphi = \mathfrak{J}u$ for $u \in \mathrm{Dom}(\mathfrak{t})$ we have $\mathrm{Re}\,(A_2\varphi, \varphi) = \mathrm{Re}\,\mathfrak{t}(u,u) \geq 0$. Hence the vertex of A_2 is zero and the contractivity of the semigroup follows from [27, Theorem IX.1.24]. □

3.3. **Properties of A_2 and extension to $\mathbb{L}^p$.** The key to the extension of A_2 to all $\mathbb{L}^p$-spaces is the $\mathbb{L}^\infty$-contractivity of the semigroup $T_2(\cdot)$. For the contractivity we will employ that $-A_2$ is associated to the form $\mathfrak{t}$. In this situation powerful invariance criteria for closed convex sets are available.

We need the following technical result. For a real-valued function u we define $u \wedge \mathbb{1}$ by $(u \wedge \mathbb{1})(x) = \min(u(x), 1)$.

Lemma 3.5. *Let $u \in \mathrm{Dom}(\mathfrak{t})$ be real-valued. Then $u \wedge \mathbb{1} \in \mathrm{Dom}(\mathfrak{t})$, and there is a sequence $(u_n)_{n\geq 0} \subset C_D^\infty(\Omega)$ such that $u_n \to u$ and $u_n \wedge \mathbb{1} \to u \wedge \mathbb{1}$ in $\mathrm{Dom}(\mathfrak{t})$ as $n \to \infty$.*

Proof. Step 1. Take real-valued $u_n \in C_D^\infty(\Omega)$ such that $u_n \to u$ in $\mathrm{Dom}(\mathfrak{t})$ as $n \to \infty$. Recall from [28, Theorem II.A.1] that for the euclidian gradient we have the formula

$$\nabla(v \wedge \mathbb{1})(x) = \begin{cases} \nabla v(x), & v(x) < 1, \\ 0, & v(x) \geq 1. \end{cases}$$

Therefore, using the formula for the surface gradient in coordinates from Definition 2.3(c), we obtain that $u_n \wedge \mathbb{1} \in \mathrm{Dom}(\mathfrak{t})$ with

$$\|u_n \wedge \mathbb{1}\|_{\mathrm{Dom}(\mathfrak{t})} \leq \|u_n\|_{\mathrm{Dom}(\mathfrak{t})}.$$

Hence $u_n \wedge \mathbb{1}$ is bounded and a subsequence converges weakly in $\mathrm{Dom}(\mathfrak{t})$ to some $v \in \mathrm{Dom}(\mathfrak{t})$. Since $u_n \wedge \mathbb{1} \to u \wedge \mathbb{1}$ in $L^2(\Omega)$, the uniqueness of weak limits gives $v = u \wedge \mathbb{1} \in \mathrm{Dom}(\mathfrak{t})$.

For the second assertion we are going to show that there is a subsequence such that $u_n \wedge \mathbb{1} \to u \wedge \mathbb{1}$ strongly in $\mathrm{Dom}(\mathfrak{t})$. Here and below, subsequences of u_n will not be relabeled.

Step 2. We make some general observations for $\mathrm{Dom}(\mathfrak{t})$. Let $w \in \mathrm{Dom}(\mathfrak{t})$ and $w_n \in C_D^\infty(\Omega)$ such that $w_n \to w$ in $\mathrm{Dom}(\mathfrak{t})$. Then $w_n \to w$ in $W^{1,2}(\Omega)$, and by the continuity of the traces from (3.1), we also have $w_n \to w$ in $L^2(\Gamma_\mathrm{d})$ and $w_n \to w$ in $L^2(\Sigma)$. Moreover, $\nabla_{\Gamma_\mathrm{d}} w_n$ is a Cauchy sequence in $L^2(\Gamma_\mathrm{d}, \mu^*_{\Gamma_\mathrm{d}})$ and $\nabla_\Sigma w_n$ is a Cauchy sequence in $L^2(\Sigma, \mu^*_\Sigma)$. With abuse of notation we denote the limits by $\nabla_{\Gamma_\mathrm{d}} w \in L^2(\Gamma_\mathrm{d}, \mu^*_{\Gamma_\mathrm{d}})$ and $\nabla_\Sigma w \in L^2(\Sigma, \mu^*_\Sigma)$ (depending on the support of $\mu^*_{\Gamma_\mathrm{d}}$ and μ^*_Σ, the traces do not have to be weakly differentiable on the surfaces). Note that the maps $w \mapsto \nabla_{\Gamma_\mathrm{d}} w$ and $w \mapsto \nabla_\Sigma w$ are linear. For the norm of w we have

$$\|w\|^2_{\mathrm{Dom}(\mathfrak{t})} = \|w\|^2_{W^{1,2}(\Omega)} + \|\nabla_{\Gamma_{\mathrm{d}}} w\|^2_{L^2(\Gamma_{\mathrm{d}},\mu^*_{\Gamma_{\mathrm{d}}})} + \|\nabla_\Sigma w\|^2_{L^2(\Sigma,\mu^*_\Sigma)}.$$

The assertion $u_n \wedge \mathbb{1} \to u \wedge \mathbb{1}$ strongly in $\mathrm{Dom}(\mathfrak{t})$ is thus equivalent to $u_n \wedge \mathbb{1} \to u \wedge \mathbb{1}$ in $W^{1,2}(\Omega)$, $\nabla_{\Gamma_{\mathrm{d}}}(u_n \wedge \mathbb{1}) \to \nabla_{\Gamma_{\mathrm{d}}}(u \wedge \mathbb{1})$ in $L^2(\Gamma_{\mathrm{d}}, \mu^*_{\Gamma_{\mathrm{d}}})$, and $\nabla_\Sigma(u_n \wedge \mathbb{1}) \to \nabla_\Sigma(u \wedge \mathbb{1})$ in $L^2(\Sigma, \mu^*_\Sigma)$. In the sequel we show that $\nabla_\Sigma(u_n \wedge \mathbb{1}) \to \nabla_\Sigma(u \wedge \mathbb{1})$ in $L^2(\Sigma, \mu^*_\Sigma)$, the arguments for the other assertions are similar. We shall follow the proof of [33, Theorem 1].

Step 3. Let $h(\tau) = \tau \wedge \mathbb{1} - \frac{\tau}{2}$, such that $h \circ u = u \wedge \mathbb{1} - \frac{u}{2}$. We show that $\mu^*_\Sigma d\mathcal{H}_{d-1}$-almost everywhere one has

$$|\nabla_\Sigma(h \circ u)| = \frac{1}{2}|\nabla_\Sigma u|. \tag{3.3}$$

Indeed, $\nabla_\Sigma(u \wedge \mathbb{1})(x)$ only takes the values $\nabla_\Sigma u(x)$ or zero. To see this, as in Step 1 we note that

$$\nabla_\Sigma(u_n \wedge \mathbb{1})(x) = \begin{cases} \nabla_\Sigma u_n(x), & u_n(x) < 1, \\ 0, & u_n(x) \geq 1. \end{cases}$$

Since $u_n \to u$ in $L^2(\Sigma)$, for a subsequence we have $u_n(x) \to u(x)$ for $\mathcal{H}_{d-1}$-almost every x. Similarly, $\nabla_\Sigma u_n \to \nabla_\Sigma u$ in $L^2(\Sigma, \mu^*_\Sigma)$ implies that $\nabla_\Sigma u_n(x) \to \nabla_\Sigma u(x)$ for $\mu^*_\Sigma d\mathcal{H}_{d-1}$-almost every x. Thus $\nabla_\Sigma(u_n \wedge \mathbb{1})(x)$ converges to $\nabla_\Sigma u(x)$ or zero. On the other hand, since each functional on $L^2(\Sigma, \mu^*_\Sigma)$ induces a functional on $\mathrm{Dom}(\mathfrak{t})$, we have $\nabla_\Sigma(u_n \wedge \mathbb{1}) \to \nabla_\Sigma(u \wedge \mathbb{1})$ weakly in $L^2(\Sigma, \mu^*_\Sigma)$. Now it is a consequence of the Banach-Saks theorem that the weak and the pointwise limit must coincide. Hence $\nabla_\Sigma(u \wedge \mathbb{1})(x)$ only takes the values $\nabla_\Sigma u(x)$ or zero, and (3.3) follows.

Step 4. Since $\nabla_\Sigma u_n \to \nabla_\Sigma u$, the assertion $\nabla_\Sigma(u_n \wedge \mathbb{1}) \to \nabla_\Sigma(u \wedge \mathbb{1})$ is equivalent to $\nabla_\Sigma(h \circ u_n) \to \nabla_\Sigma(h \circ u)$ in $L^2(\Sigma, \mu^*_\Sigma)$. We have $\nabla_\Sigma(h \circ u_n) \to \nabla_\Sigma(h \circ u)$ weakly by Step 1. By (3.3) we also have convergence of the norms, since

$$\|\nabla_\Sigma(h \circ u_n)\|_{L^2(\Sigma,\mu^*_\Sigma)} = \frac{1}{2}\|\nabla_\Sigma u_n\|_{L^2(\Sigma,\mu^*_\Sigma)} \to \frac{1}{2}\|\nabla_\Sigma u\|_{L^2(\Sigma,\mu^*_\Sigma)} = \|\nabla_\Sigma(h \circ u)\|_{L^2(\Sigma,\mu^*_\Sigma)}.$$

We therefore conclude that $\nabla_\Sigma(u_n \wedge \mathbb{1}) \to \nabla_\Sigma(u \wedge \mathbb{1})$ strongly in $L^2(\Sigma, \mu^*_\Sigma)$. $\square$

Now we may argue as in [9, Proposition 2.16] to obtain the following properties of the semigroup $T_2(\cdot)$. By $\mathbb{L}^2_\mathbb{R}$ we denote the subspace of real-valued elements of $\mathbb{L}^2$.

Proposition 3.6. *The semigroup $T_2(\cdot)$ generated by $-A_2$ leaves $\mathbb{L}^2_\mathbb{R}$ invariant, it is $\mathbb{L}^\infty$-contractive and positive.*

Proof. The set $\mathbb{L}^2_\mathbb{R}$ is closed an convex, and $\varphi \mapsto \mathrm{Re}\,\varphi$ is the orthogonal projection onto $\mathbb{L}^2_\mathbb{R}$. For $u \in C^\infty_D(\Omega)$ we have $\mathrm{Re}\,\mathfrak{t}(u, u - \mathrm{Re}\,u) \geq 0$, and this inequality carries over to all $u \in \mathrm{Dom}(\mathfrak{t})$ by density. Hence each $T_2(t)$ leaves $\mathbb{L}^2_\mathbb{R}$ invariant by [3, Proposition 2.9(iii)].

For the $\mathbb{L}^\infty$-contractivity and the positivity, as in [9] it suffices to show that $T_2(\cdot)$ leaves the closed and convex set $\mathcal{C} = \{\varphi \in \mathbb{L}^2_{\mathbb{R}} : \varphi \le \mathbb{1}\}$ invariant. The aim is to apply the invariance criterion of [3, Proposition 2.9(ii)].

The orthogonal projection P onto $\mathcal{C}$ is given by $P\varphi = (\operatorname{Re}\varphi)\wedge\mathbb{1}$. For $u \in \operatorname{Dom}(\mathfrak{t})$, Lemma 3.5 shows that $(\operatorname{Re} u)\wedge\mathbb{1} \in \operatorname{Dom}(\mathfrak{t})$. Moreover, for $u \in C_D^\infty(\Omega)$ one has $P\mathfrak{J}u = \mathfrak{J}((\operatorname{Re} u)\wedge\mathbb{1})$ and

$$\operatorname{Re}\mathfrak{t}((\operatorname{Re} u)\wedge\mathbb{1}, u - (\operatorname{Re} u)\wedge\mathbb{1}) = 0.$$

Given $u \in \operatorname{Dom}(\mathfrak{t})$, this identity remains valid since we can find a sequence of $C_D^\infty(\Omega)$-functions as in Lemma 3.5. Hence the criterion of [3] applies to $\mathcal{C}$ as desired. $\square$

Now standard interpolation and duality arguments together with [35, Proposition 3.12] allow to extend $T_2(\cdot)$ to the entire $\mathbb{L}^p$-scale as follows.

Proposition 3.7. *For all $p \in [1,\infty]$ the semigroup $T_2(\cdot)$ generated by $-A_2$ extends consistently to a contraction semigroup $T_p(\cdot)$ on $\mathbb{L}^p$, which is strongly continuous for $p \in [1,\infty)$ and analytic for $p \in (1,\infty)$.*

We define

$$A_p \text{ is the negative generator of } T_p(\cdot).$$

Then A_p coincides with A_2 on $\operatorname{Dom}(A_p)\cap\operatorname{Dom}(A_2)$. Let the relaxation coefficient $\zeta \in \mathbb{L}^\infty$ be as in Assumption 3.1. Rescaling in measure as in the proof of [9, Theorem 2.21] and using [9, Proposition 2.20], we obtain that the operators $-\zeta^{-1}A_p$ generate consistent contractive C_0-semigroups on $\mathbb{L}^p$ for $p \in [1,\infty]$, which are analytic for $p \in (1,\infty)$.

We can thus apply [31, Proposition 2.2] to obtain the main result of this section.

Theorem 3.8. *For each $p \in (1,\infty)$ the operator $\zeta^{-1}A_p$ with domain $\operatorname{Dom}(A_p)$ admits a bounded holomorphic functional calculus on $\mathbb{L}^p$, with angle strictly smaller than $\frac{\pi}{2}$. As a consequence, $\zeta^{-1}A_p$ enjoys maximal parabolic L^s-regularity for all $s \in (1,\infty)$ and $-\zeta^{-1}A_p$ generates an analytic C_0-semigroup on $\mathbb{L}^p$. Furthermore, the fractional power domains are given by complex interpolation, i.e.,*

$$\operatorname{Dom}(A_p^\theta) = [\mathbb{L}^p, \operatorname{Dom}(A_p)]_\theta, \qquad \theta \in [0,1],$$

and the resolvent of $\zeta^{-1}A_p$ is compact.

4. Degenerate bulk diffusion

In this section we generalize the above setting and allow for degeneracies in the bulk diffusion coefficient μ_Ω. Of special interest is the case when the degeneracy takes place at the dynamic boundary part Γ_{d} or the dynamic interface Σ. In this case the continuity of the map $\mathfrak{J} : \mathrm{Dom}(\mathfrak{t}) \to \mathbb{L}^2$, which is crucial for our approach, depends on the degeneracy of the bulk diffusion.

Throughout we keep Assumption 3.1, but we replace the uniform ellipticity of μ_Ω by the assumption that there are constants $c_1, c_2 > 0$ such that

$$\big(\mu_\Omega(x)v, v\big) \geq c_1 \mu_\Omega^*(x)|v|^2, \qquad \|\mu_\Omega(x)\|_{\mathcal{L}(\mathbb{R}^d)} \leq c_2 \mu_\Omega^*(x), \qquad x \in \Omega, \quad v \in \mathbb{R}^d, \tag{4.1}$$

where $\mu_\Omega^*(x) = \mathrm{dist}(x, S)^\gamma$ for a compact $(d-k)$-dimensional Lipschitz submanifold $S \subset \overline{\Omega}$, $1 \leq k \leq d$, and $0 < \gamma < k$ for the distance exponent.

By a compact 0-dimensional Lipschitz submanifold S we mean a finite union of points. In case $1 \leq k \leq d-1$, we say that S is a $(d-k)$-dimensional *Lipschitz submanifold* if for all $x \in S$ there is an open neighbourhood V of x in $\mathbb{R}^d$ and a bi-Lipschitz mapping φ from V to $\mathbb{R}^d$ such that $\varphi(S \cap V) \subset \mathbb{R}^{d-k} \times \{0_{\mathbb{R}^k}\}$.

Observe that for $\gamma = 0$ we are in the nondegenerate situation of the previous section. We are particularly interested in the case when $S \cap \overline{\Gamma_{\mathrm{d}} \cup \Sigma} \neq \emptyset$, i.e., bulk diffusion is impossible on the dynamic surfaces.

4.1. **Weighted function spaces.** In order to incorporate the degeneracy of μ_Ω into the domain of the sesquilinear form $\mathfrak{t}$ we have to deal with weighted function spaces.

We define $W_D^{1,2}(\Omega, \mu_\Omega^*)$ as the closure of $C_D^\infty(\Omega)$ with respect to the norm $\|\cdot\|_{W_D^{1,2}(\Omega,\mu_\Omega^*)}$, which is given by

$$\|u\|^2_{W_D^{1,2}(\Omega,\mu_\Omega^*)} = \|u\|^2_{L^2(\Omega)} + \|\nabla u\|^2_{L^2(\Omega,\mu_\Omega^*)}.$$

As before, here we write

$$\|f\|^2_{L^2(\Omega,\mu_\Omega^*)} = \int_\Omega |f|^2 \mu_\Omega^* \, dx.$$

Note that μ_Ω^* appears as a weight only in the gradient, the $L^2(\Omega)$-norm remains unweighted.

We record the following properties. For the general theory of *Muckenhoupt weights* we refer to [17, Chapter 9].

Lemma 4.1. (a) *The weight μ_Ω^* belongs to the Muckenhoupt class $\mathcal{A}_2$.*

(b) *One has the continuous embedding $W_D^{1,2}(\Omega, \mu_\Omega^*) \subset W^{1,1}(\Omega)$.*

(c) *$W_D^{1,2}(\Omega, \mu_\Omega^*)$ is a Hilbert space with scalar product*

$$(u, v)_{W_D^{1,2}(\Omega,\mu_\Omega^*)} = (u, v)_{L^2(\Omega)} + \int_\Omega (\nabla u, \nabla v)\, \mu_\Omega^* \, dx.$$

Proof. Assertion (a) follows from our assumption $0 < \gamma < k$, see [13, Lemma 2.2]. Using Hölder's inequality, it is straightforward to check that $L^2(\Omega, \mu_\Omega^*) \subset L^1(\Omega)$ (see also [17, Exercise 9.3.6]), which yields (b). Then (c) follows from (b). $\square$

To prove the continuity of $\mathrm{tr}_{\Gamma_\mathrm{d}}$ and tr_Σ on $W^{1,2}_D(\Omega,\mu^*_\Omega)$ we start with an extension operator of this space to $W^{1,2}(\mathbb{R}^d,\mu^*_\Omega)$. Here the norm is given by

$$\|u\|^2_{W^{1,2}(\mathbb{R}^d,\mu^*_\Omega)} = \|u\|^2_{L^2(\mathbb{R}^d)} + \|\nabla u\|^2_{L^2(\mathbb{R}^d,\mu^*_\Omega)}.$$

Lemma 4.2. *There is a continuous extension operator* $\mathcal{E} : W^{1,2}_D(\Omega,\mu^*_\Omega) \to W^{1,2}(\mathbb{R}^d,\mu^*_\Omega)$. *For any* $u \in W^{1,2}_D(\Omega,\mu^*_\Omega)$ *we have that* $\operatorname{supp}\mathcal{E}u \subset B(0,2R)$, *where* $R = \sup\{|x| : x \in \Omega\}$.

Proof. Step 1. From Assumption 3.1 we find a finite open cover $\bigcup_{\alpha=1}^N V_\alpha \subset B(0,2R)$ of $\overline{\Omega}$ with the following properties. For $\alpha = 1,\dots,N_\Omega$ the sets V_α are strictly contained in Ω; for $\alpha = N_\Omega+1,\dots,N_D$ we have $V_\alpha \cap \Gamma_D \neq \emptyset$ and $V_\alpha \cap (\overline{\Gamma_N \cup \Gamma_\mathrm{d}}) = \emptyset$; for $\alpha = N_D+1,...,N$ there is a bi-Lipschitz map φ_α from V_α to the open unit cube Q in $\mathbb{R}^d$ such that

$$\varphi_\alpha(\Omega \cap V_\alpha) = Q_-, \qquad \varphi_\alpha(\partial\Omega \cap V_\alpha) = Q_0,$$

where $Q_- \subset Q$ is the open lower half-cube in $\mathbb{R}^d$ and $Q_0 = \{x \in Q : x_d = 0\}$. We further take a smooth partition of unity $(\psi_\alpha)_\alpha$ for $\overline{\Omega}$ subordinate to the cover $\bigcup_\alpha V_\alpha$, i.e., such that $\operatorname{supp}\psi_\alpha$ is contained in V_α.

Step 2. For any $u \in C^\infty_D(\Omega)$ and $\alpha = N_D+1,\dots,N$ we have that $\psi_\alpha u$ is compactly supported in $\overline{\Omega} \cap V_\alpha$, where the support only depends on ψ_α. Choose an open subcube $\widetilde{Q} \subset Q$ such that $\varphi_\alpha(\operatorname{supp}\psi_\alpha) \subset \widetilde{Q}$. Then $W_\alpha = \varphi_\alpha^{-1}(\widetilde{Q}_-)$ is a domain with Lipschitz boundary which contains $\operatorname{supp}\psi_\alpha$. Finally, take smooth cut-off functions ϕ_α such that $\phi_\alpha \equiv 1$ on $\operatorname{supp}\psi_\alpha$ and $\operatorname{supp}\phi_\alpha \subset V_\alpha$.

Step 3. Now for $u \in C^\infty_D(\Omega)$ we define $\mathcal{E}u$ by

$$\mathcal{E}u = \sum_{\alpha=1}^{N_\Omega} \psi_\alpha u + \sum_{\alpha=N_\Omega+1}^{N_D} \mathcal{E}_\alpha(\psi_\alpha u) + \sum_{\alpha=N_D+1}^{N} \phi_\alpha \mathcal{E}_\alpha(\psi_\alpha u|_{W_\alpha}),$$

where the extensions $\mathcal{E}_\alpha$ are given as follows. For $\alpha = N_\Omega+1,\dots,N_D$ we define $\mathcal{E}_\alpha(\psi_\alpha u)$ as the trivial extension of $\psi_\alpha u$ from $V_\alpha \cap \Omega$ to $\mathbb{R}^d$. Since $V_\alpha \cap (\overline{\Gamma_N \cup \Gamma_\mathrm{d}}) = \emptyset$ and u is supported away from Γ_D, for those α we have

$$\|\mathcal{E}_\alpha(\psi_\alpha u)\|_{W^{1,2}(\mathbb{R}^d,\mu^*_\Omega)} = \|\psi_\alpha u\|_{W^{1,2}(\Omega,\mu^*_\Omega)} \le C\|u\|_{W^{1,2}(\Omega,\mu^*_\Omega)}.$$

For $\alpha = N_D+1,\dots N$ we let $\mathcal{E}_\alpha$ be the extension operator from [6] for the Lipschitz domain W_α. Then

$$\begin{aligned}\|\phi_\alpha\mathcal{E}_\alpha(\psi_\alpha u|_{W_\alpha})\|_{W^{1,2}(\mathbb{R}^d,\mu^*_\Omega)} &\le C\|\mathcal{E}_\alpha(\psi_\alpha u|_{W_\alpha})\|_{W^{1,2}(\mathbb{R}^d,\mu^*_\Omega)} \le C\|\psi_\alpha u|_{W_\alpha}\|_{W^{1,2}(W_\alpha,\mu^*_\Omega)} \\ &= C\|\psi_\alpha u\|_{W^{1,2}(\Omega,\mu^*_\Omega)} \le C\|u\|_{W^{1,2}(\Omega,\mu^*_\Omega)}.\end{aligned}$$

Therefore $\mathcal{E}$ extends continuously from $C^\infty_D(\Omega)$ to $\mathcal{E} : W^{1,2}_D(\Omega,\mu^*_\Omega) \to W^{1,2}(\mathbb{R}^d,\mu^*_\Omega)$, which gives the desired extension operator. $\square$

In a next step we prove Sobolev embeddings of $W^{1,2}(\mathbb{R}^d,\mu_\Omega^*)$ into unweighted Slobodetskii spaces $W^{\theta,q}(\mathbb{R}^d)$.

Proposition 4.3. *Assume $q\in[2,\infty)$ and $\theta\in(0,1)$ are such that $1-\frac{d+\gamma}{2}\geq\theta-\frac{d}{q}$. Then*

$$W^{1,2}(\mathbb{R}^d,\mu_\Omega^*)\subset W^{\theta,q}(\mathbb{R}^d).$$

Proof. Step 1. Let $B^1_{2,2}(\mathbb{R}^d,\mu_\Omega^*)$ be the Besov space with respect to the weight μ_Ω^*. Since μ_Ω^* belongs to the Muckenhoupt class $\mathcal{A}_2$ by Lemma 4.1, it follows from Remark 1.7 and Proposition 1.8 of [19] that

$$W^{1,2}(\mathbb{R}^d,\mu_\Omega^*)\subset B^1_{2,2}(\mathbb{R}^d,\mu_\Omega^*).$$

Moreover, $W^{\theta,q}(\mathbb{R}^d)=B^\theta_{q,q}(\mathbb{R}^d)$ for $\theta\in(0,1)$ by [39, Section 2.3.1]. The asserted embedding is thus equivalent to

$$B^1_{2,2}(\mathbb{R}^d,\mu_\Omega^*)\subset B^\theta_{q,q}(\mathbb{R}^d).$$

Step 2. We derive this embedding from the sufficient condition given in [19, Proposition 2.1(i)]. Let $Q(x,r)$ be the cube in $\mathbb{R}^d$ with edges parallel to the coordinate axes, centered at $x\in\mathbb{R}^d$ with edge length $r>0$. According to [19], for all $l\in\mathbb{N}_0$ and $m\in\mathbb{Z}^d$ we have to show that

$$\sup_{l\in\mathbb{N}_0,m\in\mathbb{Z}^d}2^{-l(1-\theta+\frac{d}{q})}\Big(\int_{Q(2^{-l}m,2^{-l})}\operatorname{dist}(x,S)^\gamma\,dx\Big)^{-1/2}<\infty.$$

By the assumption $1-\frac{d+\gamma}{2}\geq\theta-\frac{d}{q}$, this will be a consequence of the estimate

$$\int_{Q(2^{-l}m,2^{-l})}\operatorname{dist}(x,S)^\gamma\,dx\geq c2^{-l(d+\gamma)},\qquad l\in\mathbb{N},\qquad m\in\mathbb{Z}^d,\tag{4.2}$$

where $c>0$ is independent of l and m. In the sequel we prove (4.2).

Step 3. Since S is Lipschitzian, there is a tube S_κ of width $\kappa>0$ around S such that every $Q(2^{-l}m,2^{-l})\subset S_\kappa$ lies in a neighbourhood V of S which is mapped to the unit cube in $\mathbb{R}^d$ by a bi-Lipschitz map ψ such that $\psi(S\cap V)=(-1,1)^{d-k}\times\{0_{\mathbb{R}^k}\}$.

Choose $l_0\in\mathbb{N}$ such that $2^{-l_0\gamma}+2^{-l_0}\leq\kappa$. We claim that it suffices to prove (4.2) for $l\geq l_0$ and m such that $Q(2^{-l}m,2^{-l})\subset S_\kappa$, where c is independent of those l and m.

Assume this assertion is proved. Let $l\geq l_0$ and m be such that $Q(2^{-l}m,2^{-l})$ is not contained in S_κ. Then we trivially have

$$\int_{Q(2^{-l}m,2^{-l})}\operatorname{dist}(x,S)^\gamma\,dx\geq c2^{-ld}2^{-l_0\gamma}\geq c2^{-l(d+\gamma)}.$$

This yields (4.2) for $l\geq l_0$ and arbitrary m. Let $l<l_0$. Then

$$\int_{Q(2^{-l}m,2^{-l})}\operatorname{dist}(x,S)^\gamma\,dx\geq\int_{Q(2^{-l_0}(2^{l_0-l}m),2^{-l_0})}\operatorname{dist}(x,S)^\gamma\,dx\geq c2^{-l_0(d+\gamma)}\geq\tilde{c}2^{-l(d+\gamma)},$$

where $\tilde{c} = 2^{-l_0(d+\gamma)}$ is independent of l and m.

Step 4. It remains to prove (4.2) for $l \geq l_0$ and m such that $Q(2^{-l}m, 2^{-l}) \subset S_\kappa$. The integral in (4.2) transforms as

$$\int_{Q(2^{-l}m,2^{-l})} \operatorname{dist}(x,S)^\gamma \, dx = \int_{\psi(Q(2^{-l}m,2^{-l}))} \operatorname{dist}(\psi^{-1}(y),S)^\gamma |\det \psi'(y)|^{-1} \, dy,$$

where $|\det \psi'|^{-1} \geq c$ can be uniformly chosen by compactness of S. From the bi-Lipschitz property of ψ it follows that $\operatorname{dist}(\psi^{-1}(y),S) \eqsim \operatorname{dist}(y,\psi(S))$. Since $\psi(S) \subset \mathbb{R}^{d-k} \times \{0_{\mathbb{R}^k}\}$, we thus get

$$\int_{Q(2^{-l}m,2^{-l})} \operatorname{dist}(x,S)^\gamma \, dx \geq c \int_{\psi(Q(2^{-l}m,2^{-l}))} \big(|y_{d-k+1}|^\gamma + \ldots + |y_d|^\gamma\big) \, dy.$$

Again the bi-Lipschitz property of ψ yields $\delta > 0$, independent of l and m, such that $Q(\psi(2^{-l}m), \delta 2^{-l})$ is contained in $\psi(Q(2^{-l}m, 2^{-l}))$. It therefore remains to estimate

$$\int_{Q(\psi(2^{-l}m),\delta 2^{-l})} \big(|y_{d-k+1}|^\gamma + \ldots + |y_d|^\gamma\big) \, dy = 2^{-l(d-1)} \sum_{j=0}^{k-1} \int_{\psi_j(2^{-l}m)-\delta 2^{-l}}^{\psi_j(2^{-l}m)+\delta 2^{-l}} |y_{d-j}|^\gamma \, dy_{d-j}.$$

For each j, the here the integral is given by

$$\eta(s,t) := \frac{1}{\gamma+1}(\operatorname{sign}(s+t)|s+t|^{\gamma+1} - \operatorname{sign}(s-t)|s-t|^{\gamma+1}),$$

where $s = \psi_j(2^{-l}m) \in \mathbb{R}$ and $t = \delta 2^{-l} > 0$. By distinguishing the three cases $s \geq t$, $s \in (-t,t)$ and $s \leq -t$ and using the triangle inequality for the γ-norm in $\mathbb{R}^2$, we see that $\eta(s,t) \geq ct^{\gamma+1}$, where c is independent of s. We thus obtain the estimate

$$\int_{Q(\psi(2^{-l}m),\delta 2^{-l})} \big(|y_{d-k+1}|^\gamma + \ldots + |y_d|^\gamma\big) \, dy \geq c 2^{-l(d+\gamma)},$$

independently of m, and this gives (4.2). $\square$

We combine the above results to obtain the following properties of the traces.

Proposition 4.4. *For $1 < r < \frac{2(d-1)}{d+\gamma-2}$ the trace operators $\operatorname{tr}_{\Gamma_{\mathrm{d}}}$ and tr_Σ are continuous and compact*

$$\operatorname{tr}_{\Gamma_{\mathrm{d}}} : W^{1,2}(\Omega, \mu_\Omega^*) \to L^r(\Gamma_{\mathrm{d}}, d\mathcal{H}_{d-1}), \qquad \operatorname{tr}_\Sigma : W^{1,2}(\Omega, \mu_\Omega^*) \to L^r(\Sigma, d\mathcal{H}_{d-1}).$$

Proof. We consider Σ, the arguments for Γ_{d} are the same. Let $\mathcal{E}$ be the extension operator for $W^{1,2}(\Omega, \mu_\Omega^*)$ from Lemma 4.2. As in the proof of [9, Proposition 2.8] one can show that $\operatorname{tr}_\Sigma = \operatorname{tr}_\Sigma \mathcal{E}$. Proposition 4.3 together with the support property of $\mathcal{E}$ implies that there is $\varepsilon > 0$ such that $\mathcal{E}$ maps $W^{1,2}(\Omega, \mu_\Omega^*)$ compactly into $W^{1/r+\varepsilon,r}(\mathbb{R}^d)$ for $r > 1$, provided $1 - \frac{d+\gamma}{2} > \frac{1-d}{r}$. Since $d \geq 2$ and $\gamma > 0$ we have $1 - \frac{d+\gamma}{2} < 0$, such that this inequality is equivalent to $r < \frac{2(d-1)}{d+\gamma-2}$. Now [9, Lemma 2.7] implies that tr_Σ maps $W^{1/r+\varepsilon,r}(\mathbb{R}^d)$ continuously into $L^r(\Sigma, d\mathcal{H}_{d-1})$ for those r. Altogether, tr_Σ is continuous and compact. □

4.2. **The operators A_p on $\mathbb{L}^p$.** We modify $\mathrm{Dom}(\mathfrak{t})$ from Definition 3.2 to take into account the degeneracy of the diffusion coefficient μ_Ω. We set

$$(u,v)_{\mathrm{Dom}(\mathfrak{t})} = (u,v)_{W^{1,2}(\Omega,\mu_\Omega^*)} + \int_{\Gamma_\mathrm{d}} \big(\nabla_{\Gamma_\mathrm{d}} u, \overline{\nabla_{\Gamma_\mathrm{d}} v}\big)\, \mu_{\Gamma_\mathrm{d}}^*\, d\mathcal{H}_{d-1} + \int_\Sigma \big(\nabla_\Sigma u, \overline{\nabla_\Sigma v}\big)\, \mu_\Sigma^*\, d\mathcal{H}_{d-1},$$

and define as before $\mathrm{Dom}(\mathfrak{t})$ as the closure of $C_D^\infty(\Omega)$ with respect to the corresponding Hilbert norm $\|\cdot\|_{\mathrm{Dom}(\mathfrak{t})}$. It is now given by

$$\|u\|^2_{\mathrm{Dom}(\mathfrak{t})} = \|u\|^2_{W^{1,2}(\Omega,\mu_\Omega^*)} + \|\nabla_{\Gamma_\mathrm{d}} u\|^2_{L^2(\Gamma_\mathrm{d},\mu^*_{\Gamma_\mathrm{d}})} + \|\nabla_\Sigma u\|^2_{L^2(\Sigma,\mu_\Sigma^*)}.$$

Recall that the map $\mathfrak{J}$ is for $u \in C_D^\infty(\Omega)$ given by $\mathfrak{J}u = (u, u_{\Gamma_\mathrm{d}}, u_\Sigma)$. In the following we distinguish between the cases when the surface S, where the bulk diffusion degenerates, is away from Γ_d and Σ, and where the relation between these sets is arbitrary. In the first case we have to restrict to $\gamma < 1$ for the distance exponent to obtain the continuity of $\mathfrak{J}$ into $\mathbb{L}^2$.

Lemma 4.5. *Assume either* $0 < \gamma < d-k$ *and* $S \cap (\overline{\Gamma_\mathrm{d} \cup \Sigma}) = \emptyset$, *or assume* $0 < \gamma < 1$. *Then* $\mathfrak{J} : \mathrm{Dom}(\mathfrak{t}) \to \mathbb{L}^2$ *is continuous and has dense range. If (additionally)* $0 < \gamma < 2$, *then* $\mathfrak{J}$ *is compact.*

Proof. Step 1. Since $\mathrm{Dom}(\mathfrak{t}) \subset W_D^{1,2}(\Omega, \mu_\Omega^*)$, for continuity and compactness it suffices to consider $\mathfrak{J}$ on $W_D^{1,2}(\Omega,\mu_\Omega^*)$ instead of $\mathrm{Dom}(\mathfrak{t})$.

By definition we have $W_D^{1,2}(\Omega,\mu_\Omega^*) \subset L^2(\Omega)$. We claim that the latter embedding is also compact if $\gamma < 2$. Decompose the embedding into the extension $\mathcal{E}$ to $W^{1,2}(\mathbb{R}^d,\mu_\Omega^*)$ from Lemma 4.2 and the restriction to Ω. By Proposition 4.3 we have $W^{1,2}(\mathbb{R}^d,\mu_\Omega^*) \subset W^{\theta,2}(\mathbb{R}^d)$ for some $\theta > 0$, provided $\gamma < 2$. Its support property yields that $\mathcal{E}$ is compact if θ is chosen slightly smaller. Hence $W_D^{1,2}(\Omega,\mu_\Omega^*)$ embeds compactly into $L^2(\Omega)$ for $\gamma < 2$.

Step 2. We show that the traces at Γ_d and Σ are continuous and compact from $W_D^{1,2}(\Omega,\mu_\Omega^*)$ into $L^2(\Gamma_\mathrm{d})$ and $L^2(\Sigma)$, respectively. Assume $\gamma < 1$. Then $\frac{2(d-1)}{d+\gamma-2} > 2$, and the assertion follows from Proposition 4.4. Next assume $S \cap (\overline{\Gamma_\mathrm{d} \cup \Sigma}) = \emptyset$. Choose a smooth cut-off ψ such that $\psi \equiv 0$ on S and $\psi \equiv 1$ in a neighbourhood of $\overline{\Gamma_\mathrm{d} \cup \Sigma}$. Then $\mathrm{tr}_\Sigma u = \mathrm{tr}_\Sigma(\psi u)$ for all $u \in W_D^{1,2}(\Omega,\mu_\Omega^*)$. The multiplication with ψ is continuous from $W_D^{1,2}(\Omega,\mu_\Omega^*)$ into the unweighted space $W_D^{1,2}(\Omega)$, and tr_Σ is continuous and compact from $W_D^{1,2}(\Omega)$ to $L^2(\Sigma)$ by [9, Lemma 2.10].

Step 3. By the proof of [9, Lemma 2.10] we have that $\mathfrak{J}C_D^\infty(\Omega)$ is dense in $\mathbb{L}^2$. Hence $\mathfrak{J}W_D^{1,2}(\Omega,\mu_\Omega^*)$ is dense since $C_D^\infty(\Omega)\subset W_D^{1,2}(\Omega,\mu_\Omega^*)$. $\square$

Now one can argue in the same way as in Lemma 3.3 to show that the sesquilinear form

$$\mathfrak{t}(u,v)=\int_\Omega(\mu_\Omega\nabla u,\overline{\nabla v})\,dx+\int_{\Gamma_\mathrm{d}}\big(\mu_{\Gamma_\mathrm{d}}\nabla_{\Gamma_\mathrm{d}}u,\overline{\nabla_{\Gamma_\mathrm{d}}v}\big)\,d\mathcal{H}_{d-1}+\int_\Sigma\big(\mu_\Sigma\nabla_\Sigma u,\overline{\nabla_\Sigma v}\big)\,d\mathcal{H}_{d-1}$$

extends continuously from $C_D^\infty(\Omega)$ to $\mathrm{Dom}(\mathfrak{t})$, and that it is $\mathfrak{J}$-elliptic. Therefore, as in Proposition 3.4 we obtain a closed and densely defined operator A_2 associated to $\mathfrak{t}$, which is the negative generator of an analytic C_0-semigroup $T_2(\cdot)$ on $\mathbb{L}^2$. To show that $T_2(\cdot)$ is $\mathbb{L}^\infty$-contractive, we need the following, which is analogous to Lemma 3.5.

Lemma 4.6. *Let $u\in\mathrm{Dom}(\mathfrak{t})$ be real-valued. Then $u\wedge\mathbb{1}\in\mathrm{Dom}(\mathfrak{t})$, and there is a sequence $(u_n)_n\subset C_D^\infty(\Omega)$ such that $u_n\to u$ and $u_n\wedge\mathbb{1}\to u\wedge\mathbb{1}$ in $\mathrm{Dom}(\mathfrak{t})$.*

Proof. By Lemma 4.1(b) we have $\mathrm{Dom}(\mathfrak{t})\subset W_D^{1,2}(\Omega,\mu_\Omega^*)\subset W^{1,1}(\Omega)$, which implies that

$$\nabla(u\wedge\mathbb{1})(x)=\begin{cases}\nabla u(x), & u(x)<1,\\ 0, & u(x)\ge 1,\end{cases}$$

for all $u\in\mathrm{Dom}(\mathfrak{t})$ and $x\in\Omega$. For $u_n\in C_D^\infty(\Omega)$ such that $u_n\to u$ in $\mathrm{Dom}(\mathfrak{t})$ as $n\to\infty$, as in Step 1 of the proof of Lemma 3.5 this gives $u_n\wedge\mathbb{1}\to u\wedge\mathbb{1}$ weakly in $\mathrm{Dom}(\mathfrak{t})$. Strong convergence of a subsequence can then be shown as in the Steps 2-4. $\square$

As in Section 3.3, the semigroup $T_2(\cdot)$ on $\mathbb{L}^2$ extends consistently to $T_p(\cdot)$ on $\mathbb{L}^p$ for $p\in[1,\infty]$, and for the generators A_p and the relaxation coefficient ζ we obtain our main result.

Theorem 4.7. *Assume either $0<\gamma<d-k$ and $S\cap(\overline{\Gamma_\mathrm{d}\cup\Sigma})=\emptyset$, or assume $0<\gamma<1$. Then for each $p\in(1,\infty)$ the operator $\zeta^{-1}A_p$ with domain $\mathrm{Dom}(A_p)$ admits a bounded holomorphic functional calculus on $\mathbb{L}^p$, with angle strictly smaller than $\frac{\pi}{2}$. As a consequence, $\zeta^{-1}A_p$ enjoys maximal parabolic L^s-regularity for all $s\in(1,\infty)$ and $-\zeta^{-1}A_p$ generates an analytic C_0-semigroup on $\mathbb{L}^p$. Furthermore, the fractional power domains are given by complex interpolation, i.e.,*

$$\mathrm{Dom}(A_p^\theta)=[\mathbb{L}^p,\mathrm{Dom}(A_p)]_\theta,\qquad \theta\in[0,1].$$

The resolvent of $\zeta^{-1}A_p$ is compact if $\gamma<2$.

5. Embeddings for fractional power domains

Let A_p be the operator from Theorem 3.8 or 4.7. In this section we investigate conditions on $p \in (2,\infty)$ and $\theta \in (0,1)$ such that for the domain of the fractional power A_p^θ we have

$$\operatorname{Dom}(A_p^\theta) \hookrightarrow \mathbb{L}^\infty. \tag{5.1}$$

We in particular aim to quantify the conditions in dependence on whether diffusion is degenerate or not, and where it degenerates.

Here our motivation are semilinear versions of (1.1)–(1.6), i.e., where the right-hand side $(f_{\Omega\setminus\Sigma}, f_{\Gamma_\mathrm{d}}, f_\Sigma)$ depends nonlinearly on the solution itself. If (5.1) holds true, then the Nemytzkii operator induced by a nonlinearity is well-defined on $\operatorname{Dom}(A_p^\theta)$ with values in $\mathbb{L}^p$, which in principle allows to apply the standard theory for semilinear parabolic equations to obtain local-in-time well-posedness (see the discussion in the introduction).

The key to the embedding (5.1) is the regularity of the image of $\mathfrak{J}$.

Lemma 5.1. *Let $p, r \in (2,\infty)$ and $\theta \in (0,1)$ such that $\theta > \frac{r}{(r-2)p}$. Assume*

$$\mathfrak{J}\operatorname{Dom}(\mathfrak{t}) \subset \mathbb{L}^r. \tag{5.2}$$

Then $\operatorname{Dom}(A_p^\theta) \subset \mathbb{L}^\infty$.

Proof. Let $T_p(\cdot)$ be the semigroup on $\mathbb{L}^p$ generated by $-A_p$. The arguments given in the proof of [9, Lemma 2.19] show that there is $C > 0$ such that

$$\|e^{-t}T_2(t)\varphi\|_{\mathbb{L}^\infty} \le Ct^{-\frac{r}{(r-2)2}}\|\varphi\|_{\mathbb{L}^2}, \qquad t>0, \qquad \varphi \in \mathbb{L}^2.$$

Interpolating this inequality with the $\mathbb{L}^\infty$-contractivity of $T_2(\cdot)$, we obtain that

$$\|e^{-t}T_p(t)\varphi\|_{\mathbb{L}^\infty} \le Ct^{-\frac{r}{(r-2)p}}\|\varphi\|_{\mathbb{L}^p}, \qquad t>0, \qquad \varphi \in \mathbb{L}^p. \tag{5.3}$$

Since $1 + A_p$ is invertible, we have that

$$u \mapsto \|(A_p+1)^\theta u\|_{\mathbb{L}^p}$$

defines an equivalent norm on $\operatorname{Dom}(A_p^\theta)$. For $\theta \in (0,1)$ it is further well-known that

$$(A_p+1)^{-\theta} = \frac{1}{\Gamma(\theta)}\int_0^\infty t^{\theta-1}e^{-t}T_p(t)\,dt.$$

Using (5.3) for $t \in (0,1)$ and the contractivity of $T_p(\cdot)$ for $t > 1$, for $u \in \operatorname{Dom}(A_p^\theta)$ we obtain

$$\|u\|_{\mathbb{L}^\infty} \le C\|u\|_{\operatorname{Dom}(A_p^\theta)}\int_0^1 t^{\theta-1-\frac{r}{(r-2)p}}\,dt + C\|u\|_{\operatorname{Dom}(A_p^\theta)}\int_1^\infty e^{-t}\,dt.$$

Here the first integral is finite if $\theta > \frac{r}{(r-2)p}$. In this case the embedding $\operatorname{Dom}(A_p^\theta) \subset \mathbb{L}^\infty$ follows. □

In the sequel we determine $r_0 > 2$ as large as possible such that (5.2) holds for all $2 < r < r_0$. Since

$$\mathbb{L}^r = L^r(\Omega) \oplus L^r(\Gamma_\mathrm{d}) \oplus L^r(\Sigma),$$

the number r_0 depends on how large r can be such that

$$\mathrm{Dom}(\mathfrak{t}) \subset L^r(\Omega), \qquad \mathrm{tr}_{\Gamma_\mathrm{d}} : \mathrm{Dom}(\mathfrak{t}) \to L^r(\Gamma_\mathrm{d}), \qquad \mathrm{tr}_\Sigma : \mathrm{Dom}(\mathfrak{t}) \to L^r(\Sigma),$$

are simultaneously continuous. In turn, this depends on whether the bulk diffusion degenerates or not, if it degenerates at $\Gamma_\mathrm{d} \cup \Sigma$ where traces are taken, and where the diffusion on Γ_d and Σ degenerates.

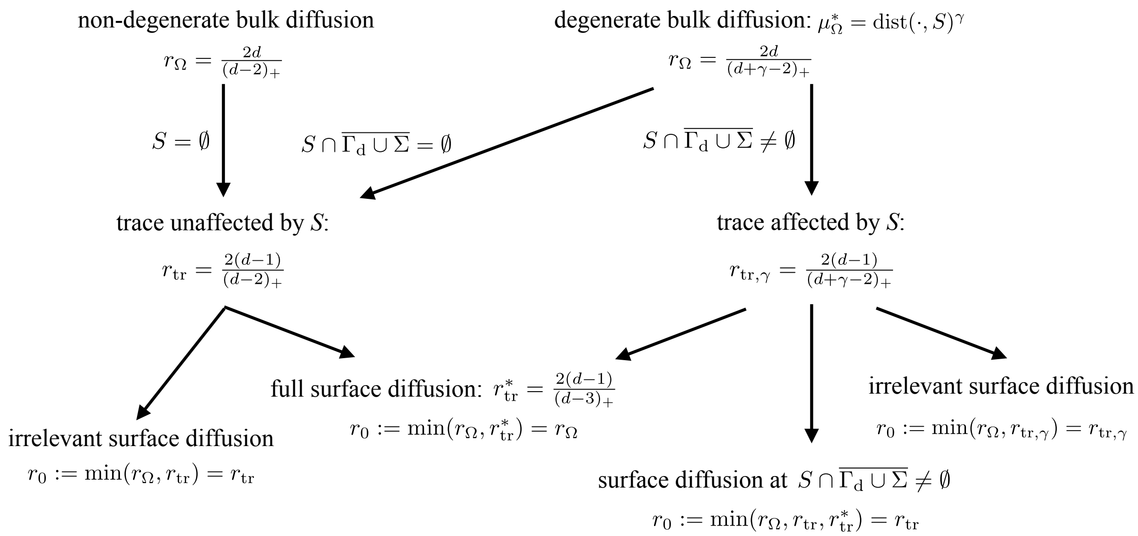
It follows from Lemma 4.2 and Proposition 4.3 that

$$\mathrm{Dom}(\mathfrak{t}) \subset W^{1,2}_D(\Omega) \subset L^r(\Omega)$$

for $r < r_\Omega := \frac{2d}{(d+\gamma-2)_+}$. If $S = \emptyset$ or $S \cap \overline{\Gamma_\mathrm{d} \cup \Sigma} = \emptyset$, then by [9, Proposition 2.8] the traces are continuous from $\mathrm{Dom}(\mathfrak{t}) \subset W^{1,2}_D(\Omega)$ into $L^r(\Gamma_\mathrm{d})$ and $L^r(\Sigma)$ for all $r < r_\mathrm{tr} := \frac{2(d-1)}{(d-2)_+}$. In case $S \cap \overline{\Gamma_\mathrm{d} \cup \Sigma} \neq \emptyset$, where in Theorem 4.7 it is assumed that $\gamma < 1$, Proposition 4.4 shows that the traces are continuous only for $r < r_{\mathrm{tr},\gamma} := \frac{2(d-1)}{(d+\gamma-2)_+}$.

The regularity of the trace improves if surface diffusion is present. Assume that the surface diffusion is uniformly nondegenerate, i.e., $\mu^*_{\Gamma_\mathrm{d}}, \mu^*_\Sigma \geq \eta > 0$. Then the traces belong to $W^{1,2}(\Gamma_\mathrm{d})$ and $W^{1,2}(\Sigma)$. By Sobolev embeddings, the traces are thus continuous into L^r for $r < r^*_\mathrm{tr} := \frac{2(d-1)}{(d-3)_+}$. Observe that $r^*_\mathrm{tr} > r_\mathrm{tr}$, which quantifies the regularity improvement obtained from surface diffusion. Finally, assume that $S \cap \overline{\Gamma_\mathrm{d} \cup \Sigma} \neq \emptyset$ and that $\mu^*_{\Gamma_\mathrm{d}}, \mu^*_\Sigma \geq \eta > 0$ in a neighbourhood of $S \cap \overline{\Gamma_\mathrm{d} \cup \Sigma}$. Then the traces belong to $W^{1,2}$ in this neighbourhood, such that they belong to L^r for $r < \min(r_\mathrm{tr}, r^*_\mathrm{tr}) = r_\mathrm{tr}$. This improves the case without diffusion on the critical set $S \cap \overline{\Gamma_\mathrm{d} \cup \Sigma}$ since $r_\mathrm{tr} > r_{\mathrm{tr},\gamma}$.

Now the number r_0 can be chosen as the minimum of r_Ω and r_tr, $r_{\mathrm{tr},\gamma}$ or r^*_tr according to the cases described above. The following figure gives an overview.



One can check that if $0 \le \gamma < 1$, in any case we have $r_0 > 2$. Together with Lemma 5.1 we thus have the following result, which opens the door to treat semilinear versions of (1.1)–(1.6) as explained above.

Theorem 5.2. *Assume $0 \le \gamma < 1$. Then there are $\theta_0 \in (0,1)$ and $p_0 \in (2,\infty)$ such that $\mathrm{Dom}(A_p^\theta) \hookrightarrow \mathbb{L}^\infty$ for all $\theta \in (\theta_0, 1)$ and $p \in (p_0, \infty)$.*

It is interesting to note that if diffusion is nowhere degenerate, then one can take $r_0 = \frac{2d}{(d-2)_+}$. In this case, by Lemma 5.1 we have $\mathrm{Dom}(A_p^\theta) \hookrightarrow \mathbb{L}^\infty$ provided

$$2\theta > \frac{d}{p}.$$

This is precisely the optimal relation for the embedding of $H^{2\theta,p}(\Omega)$ into $L^\infty(\Omega)$. In a smooth situation one indeed expects that $\mathrm{Dom}(A_p) \subset H^{2,p}(\Omega)$ and thus $\mathrm{Dom}(A_p^\theta) \subset H^{2\theta,p}(\Omega)$, which shows that the above considerations are optimal at least in this case.

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