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**A new model for quantum dot light emitting-absorbing
devices**

Dedicated to the memory of Pierre Duclos

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Abstract

Motivated by the Jaynes-Cummings (JC) model, we consider here a quantum dot coupled simultaneously to a reservoir of photons and to two electric leads (free-fermion reservoirs). This Jaynes-Cummings-Leads (JCL) model makes possible that the fermion current through the dot creates a photon flux, which describes a light-emitting device. The same model also describes a transformation of the photon flux into a fermion current, i.e. a quantum dot light-absorbing device. The key tool to obtain these results is an abstract Landauer-Büttiker formula.

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1 Introduction

The Landauer-Büttiker formula is widely used for the analysis of the steady state current flowing through a quantum device. It goes back to [18] and [7] and was initially developed based on phenomenological arguments for non-interacting electrons (free-fermions). The essential idea was to describe a quantum system as an inner or sample system (dot) with left and right leads attached to it, i.e. free-fermion reservoirs with two different electro-chemical potentials. The goal was to calculate the steady electron current going from one lead through the dot to another one.

It was Landauer and Büttiker who found that this current is directly related to the *transmission coefficients* of some natural scattering system related to this particle transport problem. The phenomenological approach of Landauer and Büttiker later has been justified in several papers by deriving the formula from fundamental concepts of the Quantum Mechanics, see the series of papers [1, 5, 8, 9, 10, 11, 12, 13, 14] and [19].

Note that this quantum mechanical approach is possible since for the case of free-fermion reservoirs the corresponding transport problem reduces to study the Hamiltonian dynamics of extended “one-particle” system. During last decade there has been an important progress in rigorous development of the Quantum Statistical Mechanics of Open Systems [2, 3, 4]. This is a many-body approach adapted for interacting systems. It also allows, besides the Hamiltonian [2], to develop a Markovian description of effective microscopic dynamics of the sample system (dot) connected to environment of external reservoirs [3]. Then evolution the sample system is governed by a quantum Master Equation. Although powerful and useful the Markovian approach needs a microscopic Hamiltonian justification, which is a nontrivial problem [3].

In the present paper we follow the one-particle quantum mechanical Hamiltonian approach. Motivated by the quantum optics Jaynes-Cummings (JC) model, we consider here a *two-level* quantum dot coupled *simultaneously* to environment of *three* external reservoirs. The first is the standard JC one-mode photon resonator, which makes the JC quantum dot an *open system* [16]. Two others are *free-fermion* reservoirs coupled to the quantum dot. They mimic two electric leads. This new Jaynes-Cummings-Leads (*JCL*-) model makes possible that the fermion current through the dot creates a photon flux into the resonator, i.e. it describes a *light-emitting* device. The same model is also able to describe a transformation of the external photon flux into a current of fermions, which corresponds to a quantum dot *light-absorbing* device.

The aim of the paper is to analyze the fermion current going through the dot as a function of electro-chemical potentials on leads and the contact with the photon reservoir. Although the latter is the canonical *JC*-interaction, the coupling of the JC model with leads needs certain precautions, if we like to stay in the framework of one-particle quantum mechanical Hamiltonian approach and the scattering theory.

We discuss the construction of our *JCL*-model in Sections 2.2-2.7. For simplicity, we choose for the leads Hamiltonians the one-particle discrete Schrödinger operators with constant one-site (electric) potentials on each of leads. Notice that these Hamiltonians are one band bounded self-adjoint operators. The advantage is that one can easily adjust the leads band spectra positions (and consequently the dot-leads transmission coefficients) shifting them with respect to the two-point quantum dot spectrum by varying the one-site electric potentials (voltage). In Section 2.5 we show that the our model fits into framework of *trace-class scattering* and in Section 2.7 we verify the important property that the coupled Hamiltonian has no singular continuous spectrum.

Our main tool is an abstract Landauer-Büttiker-type formula applied in Sections 3.1 and 3.2 to the case of the *JCL*-model. Note that this abstract formula allows to calculate not only the electron current but also fluxes for other quantities, such as photon or energy/entropy currents. In particular, we calculate the outgoing flux of photons induced by electric current via leads. This corresponds to a *light-emitting* device. We also found that pumping the JCL quantum dot by photon flux from resonator may induce current of fermions into leads. This reversing imitates a quantum *light-absorbing* cell device. These are the main properties of our model and the main application of the Landauer-Büttiker-type formula of

Sections 3.1 and 3.2. They are presented in Sections 4 and 5, where we distinguish *contact-induced* and *photon-induced* fermion currents.

To describe the results of Sections 4 and 5 note that in our setup the sample Hamiltonian is a *two-level* quantum dot *decoupled* from the one-mode resonator. Then the unperturbed Hamiltonian H_0 describes is a collection of four totally decoupled sub-systems: the sample, the resonator and the two leads. The perturbed Hamiltonian H is a fully coupled system and the feature of our model is that it is *totally* (i.e. including the leads) *embedded* into the external electromagnetic field of resonator. This allows a systematic application of the abstract Landauer-Büttiker-type formula, c.f. Sections 3.1 and 3.2.

As we see there is a variety of possibilities to switch on interactions between sub-systems, i.e. to produce intermediate Hamiltonians. We distinguish the following two of them:

- (a) First to switch on the coupling between sample and resonator: the standard JC model H_{JC} , see e.g. [16]. Then to connect it to leads, which gives the Hamiltonian $H_{JCL} := H$ of the fully coupled system.
- (b) First to couple the sample to leads: the corresponding Hamiltonian H_{SL} is a standard “Black Box” SL -model for free-fermion current, see [1], [4]. Then to embed it into resonator and to couple the sample with electromagnetic field by the JC -interaction. This again produces our JCL -model with $H_{JCL} = H$.

Similar to the SL -model $\{H_{SL}, H_0\}$, it turns out that the JCL -model also fits into the framework of the abstract Landauer-Büttiker formula, and in particular, is a trace-class scattering system $\{H_{JCL} = H, H_{SL}\}$. The current in the SL -model is called the *contact-induced* current J_{el}^c . It was a subject of numerous papers, see e.g. [1, 5], or [4] and references quoted there. Note that the current J_{el} is due to the difference of electro-chemical potentials between two leads, but it may be *zero* even if this difference is not null [12, 13].

The fermion current in the JCL -model, takes into account the effect of the electron-photon interaction under the assumption that the leads are already coupled. It is called the *photon-induced* component J_{el}^{ph} of the total current. Up to our knowledge the present paper is the first, where it is studied rigorously. We show that the total free-fermion current J in the JCL -model decomposes into a sum of the contact- and the photon-induced currents: $J_{el} := J_{el}^c + J_{el}^{ph}$. An extremal case is, when the contact-induced current is zero, but the photon-induced component is not, c.f. Section 5.1. In this case the flux of photons J_{ph} out of the quantum dot (sample) is also non-zero, i.e. the dot serves as the light emitting device, c.f. Section 5.2. In general the $J_{ph} \neq 0$ only when the photon-induced component $J_{el}^{ph} \neq 0$.

In this paper we derive explicit formulas for these currents in the following three cases which are important for the understanding of the JCL -model:

- (i) The electro-chemical potentials of fermions in the left and right leads are equal. Note that in this case the (contact-induced) current in the JCL -model is zero.
- (ii) The spectrum of the left and right lead Hamiltonians do not overlap. Again, in this case the contact-induced electron current J_{el}^c of the current in the JCL -model is zero, and only the *photon-induced electron current* J_{el}^{ph} of the total current is possible.
- (iii) The leads are coupled to the Jaynes-Cummings model such that left and right leads interact only by virtue of the photon interaction in the Jaynes-Cummings model. Then the contact-induced electron current J_{el}^c is also zero.

For these cases we find that the photon induced electron current $J_{\alpha,el}^{ph}$ entering the left ($\alpha = l$) or right

($\alpha = r$) lead is given by

$$J_{\alpha,el}^{ph} = - \sum_{\substack{m,n \in \mathbb{N}_0 \\ \varkappa \in \{l,r\}}} \frac{e}{2\pi} \int_{\mathbb{R}} d\lambda \widehat{\sigma}_{n\varkappa m\alpha}^{ph}(\lambda) \times \\ (\rho^{ph}(n) f_{FD}(\lambda - \mu_\alpha - n\omega) - \rho^{ph}(m) f_{FD}(\lambda - \mu_\varkappa - m\omega)).$$

where $\widehat{\sigma}_{n\varkappa m\alpha}^{ph}(\lambda) \geq 0$ is a partial scattering cross-section between the left channel with m -photons and the \varkappa -channel with n -photons at energy $\lambda \in \mathbb{R}$. By $e > 0$ the magnitude of the electron charge is denoted. The photon current is given by

$$J_{ph} = \sum_{\substack{m,n \in \mathbb{N}_0 \\ \alpha, \varkappa \in \{l,r\}}} (n - m) \rho^{ph}(m) \frac{1}{2\pi} \int_{\mathbb{R}} d\lambda f_{FD}(\lambda - \mu_\alpha - m\omega) \widehat{\sigma}_{n\varkappa m\alpha}^{ph}(\lambda).$$

Both formulas become simpler if it is assumed that the *JCL*-model is time reversible symmetric. In this case we get

$$J_{l,el}^{ph} = - \sum_{m,n \in \mathbb{N}_0} \frac{e}{2\pi} \int_{\mathbb{R}} d\lambda \widehat{\sigma}_{nr,ml}^{ph}(\lambda) \times \\ (\rho^{ph}(n) f_{FD}(\lambda - \mu_l - n\omega) - \rho^{ph}(m) f_{FD}(\lambda - \mu_r - m\omega)),$$

and

$$J_{ph} = \sum_{\substack{m,n \in \mathbb{N}_0, n > m \\ \varkappa, \alpha \in \{l,r\}}} \frac{1}{2\pi} \int_{\mathbb{R}} d\lambda \widehat{\sigma}_{n\varkappa m\alpha}^{ph}(\lambda) \times \\ (n - m) (\rho^{ph}(m) f_{FD}(\lambda - \mu_\alpha - m\omega) - \rho^{ph}(n) f_{FD}(\lambda - \mu_\varkappa - n\omega)).$$

It turns out that choosing the parameters of the model in a suitable manner one gets either a photon emitting or a photon absorbing system. Hence *JCL*-model can be used either as a light emission device or as a light-cell. Proofs of explicit formulas for fermion and photon currents $J_{l,el}^{ph}$, J_{ph} is the contents of Sections 4 and 5.

Note that the *JCL*-model is called *mirror symmetric* if (roughly speaking) one can interchange left and right leads and the *JCL*-model remains unchanged. In Section 5 we discuss a surprising example of a mirror symmetric *JCL*-model such that the free-fermion current is *zero* but the model is photon emitting. This peculiarity is due to a specific choice of the *photon-fermion* interaction in our model.

2 Jaynes-Cummings quantum dot coupled to leads

2.1 Jaynes-Cummings model

The starting point for construction of our *JCL*-model is the quantum optics Jaynes-Cummings Hamiltonian H^{JC} . Its simplest version is a *two-level* system (quantum dot) with the energy spacing ε , defined by Hamiltonian h_S on the Hilbert space $\mathfrak{h}_S = \mathbb{C}^2$, see e.g. [16]. It is assumed that this system is "open" and interacts with the one-mode ω photon resonator with Hamiltonian h^{ph} .

Since mathematically h^{ph} coincides with quantum harmonic oscillator, the Hilbert space of the resonator is the boson Fock space $\mathfrak{h}^{ph} = \mathfrak{F}_+(\mathbb{C})$ over \mathbb{C} and

$$h^{ph} = \omega b^* b. \quad (2.1)$$

Here b^* and b are verifying the Canonical Commutation Relations (*CCR*) creation and annihilation operators with domains in $\mathfrak{F}_+(\mathbb{C}) \simeq \ell^2(\mathbb{N}_0)$. Operator (2.1) is self-adjoint on its domain

$$\text{dom}(h^{ph}) = \left\{ (k_0, k_1, k_2, \dots) \in \ell^2(\mathbb{N}_0) : \sum_{n \in \mathbb{N}_0} n^2 |k_n|^2 < \infty \right\}.$$

Note that canonical basis $\{\phi_n := (0, 0, \dots, k_n = 1, 0, \dots)\}_{n \in \mathbb{N}_0}$ in $\ell^2(\mathbb{N}_0)$ consists of eigenvectors of operator (2.1): $h^{ph} \phi_n = n\omega \phi_n$.

To model the *two-level* system with the energy spacing ε , one fixes in \mathbb{C}^2 two ortho-normal vectors $\{e_0^S, e_1^S\}$, for example

$$e_0^S := \begin{pmatrix} 0 \\ 1 \end{pmatrix} \quad \text{and} \quad e_1^S := \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \quad (2.2)$$

which are eigenvectors of Hamiltonian h_S with eigenvalues $\{\lambda_0^S = 0, \lambda_1^S = \varepsilon\}$. To this end we put

$$h_S := \varepsilon \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \quad (2.3)$$

and we introduce two *ladder* operators:

$$\sigma^+ := \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \quad \sigma^- := \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}. \quad (2.4)$$

Then one gets $h_S = \varepsilon \sigma^+ \sigma^-$ as well as

$$e_1^S = \sigma^+ e_0^S, \quad e_0^S = \sigma^- e_1^S \quad \text{and} \quad \sigma^- e_0^S = \begin{pmatrix} 0 \\ 0 \end{pmatrix}. \quad (2.5)$$

So, e_0^S is the ground state of Hamiltonian h_S . Note that *non-interacting* Jaynes-Cummings Hamiltonian H_0^{JC} lives in the space $\mathfrak{H}^{JC} = \mathfrak{h}_S \otimes \mathfrak{h}^{ph} = \mathbb{C}^2 \otimes \mathfrak{F}_+(\mathbb{C})$ and it is defined as the *matrix* operator

$$H_0^{JC} := h_S \otimes I_{\mathfrak{h}^{ph}} + I_{\mathfrak{h}_S} \otimes h^{ph}. \quad (2.6)$$

Here $I_{\mathfrak{h}^{ph}}$ denotes the unit operator in the Fock space \mathfrak{h}^{ph} , whereas $I_{\mathfrak{h}_S}$ stays for the unit matrix in the space \mathfrak{h}_S .

With operators (2.4) the interaction V_{Sb} between quantum dot and photons (bosons) in the resonator is defined (in the rotating-wave approximation [16]) by the operator

$$V_{Sb} := g_{Sb} (\sigma^+ \otimes b + \sigma^- \otimes b^*). \quad (2.7)$$

Operators (2.6) and (2.7) define the Jaynes-Cummings model Hamiltonian

$$H_{JC} := H_0^{JC} + V_{Sb}, \quad (2.8)$$

which is self-adjoint operator on the common domain $\text{dom}(H_0^{JC}) \cap \text{dom}(V_{Sb})$. The standard interpretation of H_{JC} is that (2.8) describes an “open” two-level system interacting with external one-mode electromagnetic field [16].

Since the one-mode resonator is able to absorb *infinitely* many bosons this interpretation sounds reasonable, but one can see that the spectrum $\sigma(H_{JC})$ of the Jaynes-Cummings model is *discrete*. To this end note that the so-called number operator

$$\mathfrak{N}_{JC} := \sigma^+ \sigma^- \otimes I_{\mathfrak{h}^{ph}} + I_{\mathfrak{h}_S} \otimes b^* b$$

commutes with H_{JC} . Then, since for any $n \geq 0$

$$\mathfrak{H}_{n>0}^{JC} := \{\zeta_0 e_0^S \otimes \phi_n + \zeta_1 e_1^S \otimes \phi_{n-1}\}_{\zeta_{0,1} \in \mathbb{C}}, \quad \mathfrak{H}_{n=0}^{JC} := \{\zeta_0 e_0^S \otimes \phi_0\}_{\zeta_0 \in \mathbb{C}}, \quad (2.9)$$

are eigenspaces of operator \mathfrak{N}_{JC} , they reduce H_{JC} , i.e. $H_{JC} : \mathfrak{H}_n^{JC} \rightarrow \mathfrak{H}_n^{JC}$. Note that $\mathfrak{H}^{JC} = \bigoplus_{n \geq 0} \mathfrak{H}_n^{JC}$, where each \mathfrak{H}_n^{JC} is invariant subspace of operator (2.8). Therefore, it has the representation

$$H_{JC} = \bigoplus_{n \in \mathbb{N}_0} H_{JC}^{(n)}, \quad n > 1, \quad H_{JC}^{(0)} = 0. \quad (2.10)$$

Here operators $H_{JC}^{(n)}$ are the restrictions of H_{JC} , which act in each \mathfrak{H}_n^{JC} as

$$\begin{aligned} H_{JC}^{(n)}(\zeta_0 e_0^S \otimes \phi_n + \zeta_1 e_1^S \otimes \phi_{n-1}) = \\ [\zeta_0 n\omega + \zeta_1 g_{Sb}\sqrt{n}] e_0^S \otimes \phi_n + [\zeta_1(\varepsilon + (n-1)\omega) + \zeta_0 g_{Sb}\sqrt{n}] e_1^S \otimes \phi_{n-1} . \end{aligned} \quad (2.11)$$

Hence, the spectrum $\sigma(H_{JC}) = \bigcup_{n \geq 0} \sigma(H_{JC}^{(n)})$. By virtue of (2.11) the spectrum $\sigma(H_{JC}^{(n)})$ is defined for $n \geq 1$ by eigenvalues $E(n)$ of two-by-two matrix $\widehat{H}_{JC}^{(n)}$ acting on the coefficient space $\{\zeta_0, \zeta_1\}$:

$$\widehat{H}_{JC}^{(n)} \begin{pmatrix} \zeta_1 \\ \zeta_0 \end{pmatrix} = \begin{pmatrix} \varepsilon + (n-1)\omega & g_{Sb}\sqrt{n} \\ g_{Sb}\sqrt{n} & n\omega \end{pmatrix} \begin{pmatrix} \zeta_1 \\ \zeta_0 \end{pmatrix} = E(n) \begin{pmatrix} \zeta_1 \\ \zeta_0 \end{pmatrix} . \quad (2.12)$$

Then (2.10) and (2.12) imply that the spectrum of the Jaynes-Cummings model Hamiltonian H_{JC} is *pure point*:

$$\begin{aligned} \sigma(H_{JC}) = \sigma_{p.p.}(H_{JC}) = \\ \{0\} \cup \bigcup_{n \in \mathbb{N}} \left\{ n\omega + \frac{1}{2}(\varepsilon - \omega) \pm \sqrt{(\varepsilon - \omega)^2/4 + g_{Sb}^2 n} \right\} . \end{aligned} \quad (2.13)$$

This property is evidently persists for any system Hamiltonian h_S with discrete spectrum and linear interaction (2.7) with a finite mode photon resonator [16].

We resume the above observations concerning the Jaynes-Cummings model, which is our starting point, by following remarks:

- (a) The standard Hamiltonian (2.8) describes instead of *flux* only oscillations of photons between resonator and quantum dot, i.e. the system h_S is not “open” enough.
- (b) Since one our aim is to model a *light-emitting* device, the system h_S needs an *external* source of energy to pump it into dot, which then be transformed by interaction (2.7) into the outgoing *photon current* pumping the resonator.
- (c) To reach this aim we extend the standard Jaynes-Cummings model to our *JCL*-model by attaching to the quantum dot h_S (2.3) two *leads*, which are (infinite) reservoirs of *free* fermions. Manipulating with *electro-chemical* potentials of fermions in these reservoirs we can force one of them to inject fermions in the quantum dot, whereas another one to absorb the fermions out the quantum dot with the same rate. This current of fermions throughout the dot would pump it and produce the photon current according scenario (b).
- (d) The most subtle point is to invent a *leads-dot* interaction V_{lS} , which ensures the above mechanism and which is simple enough that one still be able to treat this *JCL*-model using our extension of the Landauer-Büttiker formalism.

2.2 The JCL-model

First let us make some general remarks and formulate certain conditions indispensable when one follows the modeling (d).

- (1) Note that since the Landauer-Büttiker formalism [13] is essentially a scattering theory on a contact between two subsystems, it is developed only on a “one-particle” level. This allows to study with this formalism only ideal (*non-interacting*) many-body systems. This condition we impose on many-body fermion systems (electrons) in two leads. Thus, only direct interaction between different components of the system: dot-photons V_{Sb} and electron-dot V_{lS} are allowed.

- (2) It is well-known that fermion reservoirs are technically simpler to treat than boson one [13]. Moreover, in the framework of our model it is also very natural since we study electric current although produced by “non-interacting electrons”. So, below we use fermions/electrons as synonymous.
- (3) In spite of precautions formulated above, the first difficulty to consider an ideal many-body system interacting with quantized electromagnetic field (photons) is induced *indirect* interaction. If electrons can emit and absorb photons, it is possible for one electron to emit a photon that another electron absorbs, thus creating the indirect photon-mediated electron-electron interaction. This interaction makes impossible to develop the Landauer-Büttiker formula, which requires non-interacting framework.

Assumption 2.1 To solve this difficulty we forbid in our model the photon-mediated interaction. To this end we suppose that every electron (in leads and in dot) interacts with its *own* distinct copy of the electromagnetic field. So, to consider electrons together with its photon fields as non-interacting “composed particles”, which allows to apply the Landauer-Büttiker approach. Formally it corresponds to the “one-electron” Hilbert space $\mathfrak{h}^{el} \otimes \mathfrak{h}^{ph}$, where \mathfrak{h}^{ph} is the Hilbert space of the individual photon field. The fermion description of composed-particles $\mathfrak{h}^{el} \otimes \mathfrak{h}^{ph}$ corresponds to the antisymmetric Fock space $\mathfrak{F}_-(\mathfrak{h}^{el} \otimes \mathfrak{h}^{ph})$.

The composed-particle assumption 2.1 allows us to use the Landauer-Büttiker formalism developed for ideal many-body fermion systems. Now we come closer to the formal description of our JCL-model with two (infinite) leads and a one-mode quantum resonator.

Recall that the Hilbert space of the Jaynes-Cummings Hamiltonian with two energy levels is $\mathfrak{H}^{JC} = \mathbb{C}^2 \otimes \mathfrak{F}_+(\mathbb{C})$. The boson Fock space is constructed from a one-dimensional Hilbert space since we consider only photons of a single fixed frequency. We model the electrons in the leads as free fermions living on a discrete semi-infinite lattices. Thus

$$\mathfrak{h}^{el} = \ell^2(\mathbb{N}) \oplus \mathbb{C}^2 \oplus \ell^2(\mathbb{N}) = \mathfrak{h}_l^{el} \oplus \mathfrak{h}_S \oplus \mathfrak{h}_r^{el} \quad (2.14)$$

is the one-particle Hilbert space for electrons and for the dot. Here, \mathfrak{h}_α^{el} , $\alpha \in \{l, r\}$, are the Hilbert spaces of the *left* respectively *right* lead and $\mathfrak{h}_S = \mathbb{C}^2$ is the Hilbert space of the quantum dot. We denote by

$$\{\delta_n^\alpha\}_{n \in \mathbb{N}}, \quad \{\delta_n^S\}_{j=0}^1$$

the canonical basis consisting of individual lattice sites of \mathfrak{h}_α^{el} , $\alpha \in \{l, r\}$, and of \mathfrak{h}_S , respectively. With the Hilbert space for photons, $\mathfrak{h}^{ph} = \mathfrak{F}_+(\mathbb{C}) \simeq \ell^2(\mathbb{N}_0)$, we define the Hilbert space of the *full* system, i.e. quantum dot with leads and with the photon field, as

$$\mathfrak{H} = \mathfrak{h}^{el} \otimes \mathfrak{h}^{ph} = (\ell^2(\mathbb{N}) \oplus \mathbb{C}^2 \oplus \ell^2(\mathbb{N})) \otimes \ell^2(\mathbb{N}_0). \quad (2.15)$$

Remark 2.2 Note that the structure of full space (2.15) takes into account the condition 2.1 and produces composed fermions via the last tensor product. It also manifests that electrons *in the dot* as well as those *in the leads* are composed with photons. This makes difference with the picture imposed by the the Jaynes-Cummings model, when *only dot* is composed with photons:

$$\mathfrak{H} = \ell^2(\mathbb{N}) \oplus \mathbb{C}^2 \otimes \ell^2(\mathbb{N}_0) \oplus \ell^2(\mathbb{N}), \quad \mathfrak{H}^{JC} = \mathbb{C}^2 \otimes \ell^2(\mathbb{N}_0), \quad (2.16)$$

see (2.6), (2.7) and (2.8), where $\mathfrak{H}^{JC} = \mathfrak{h}_S \otimes \mathfrak{h}^{ph}$. The next step is a choice of interactions between subsystems: dot-resonator-leads.

According to (2.14) the decoupled leads-dot Hamiltonian is the matrix operator

$$h_0^{el} = \begin{pmatrix} h_l^{el} & 0 & 0 \\ 0 & h_S & 0 \\ 0 & 0 & h_r^{el} \end{pmatrix} \quad \text{on} \quad u = \begin{pmatrix} u_l \\ u_S \\ u_r \end{pmatrix}, \quad \{u_\alpha \in \ell^2(\mathbb{N})\}_{\alpha \in \{l, r\}}, \quad u_S \in \mathbb{C}^2,$$

where $h_\alpha^{el} = -\Delta^D + v_\alpha$ with a constant potential bias $v_\alpha \in \mathbb{R}$, $\alpha \in \{l, r\}$, and h_S can be any self-adjoint two-by-two matrix with eigenvalues $\{\lambda_0^S, \lambda_1^S := \lambda_0^S + \varepsilon\}$, $\varepsilon > 0$, and eigenvectors $\{e_0^S, e_1^S\}$, cf (2.3). Here, Δ^D denotes the discrete Laplacian on $\ell^2(\mathbb{N})$ with homogeneous Dirichlet boundary conditions given by

$$\begin{aligned} (\Delta^D f)(x) &:= f(x+1) - 2f(x) + f(x-1), \quad x \in \mathbb{N}, \\ \text{dom}(\Delta^D) &:= \{f \in \ell^2(\mathbb{N}_0) : f(0) := 0\}, \end{aligned}$$

which is obviously a bounded self-adjoint operator. Notice that $\sigma(\Delta^D) = [0, 4]$.

We define the *lead-dot interaction* for coupling $g_{el} \in \mathbb{R}$ by the matrix operator acting in (2.14) as

$$v_{el} = g_{el} \begin{pmatrix} 0 & \langle \cdot, \delta_0^S \rangle \delta_1^l & 0 \\ \langle \cdot, \delta_1^l \rangle \delta_0^S & 0 & \langle \cdot, \delta_1^r \rangle \delta_1^S \\ 0 & \langle \cdot, \delta_1^S \rangle \delta_1^r & 0 \end{pmatrix}, \quad (2.17)$$

where non-trivial off-diagonal entries are *projection* operators in the Hilbert space (2.14) with the scalar product $u, v \mapsto \langle u, v \rangle$ for $u, v \in \mathfrak{h}^{el}$. Here $\{\delta_0^S, \delta_1^S\}$ is ortho-normal basis in \mathfrak{h}_S^{el} , which in general may be different from $\{e_0^S, e_1^S\}$. Hence, interaction (2.17) describes quantum *tunneling* between leads and the dot via contact sites of the leads, which are supports of δ_1^l and δ_1^r .

Then Hamiltonian for the system of interacting leads and dot we define as $h^{el} := h_0^{el} + v_{el}$. Here both h_0^{el} and h^{el} are bounded self-adjoint operators on \mathfrak{h}^{el} .

Recall that photon Hamiltonian in the one-mode resonator is defined by operator $h^{ph} = \omega b^* b$ with domain in the Fock space $\mathfrak{F}_+(\mathbb{C}) \simeq \ell^2(\mathbb{N}_0)$, (2.1). We denote the canonical basis in $\ell^2(\mathbb{N}_0)$ by $\{\Upsilon_n\}_{n \in \mathbb{N}_0}$. Then for the spectrum of h^{ph} one obviously gets

$$\sigma(h^{ph}) = \sigma_{pp}(h^{ph}) = \bigcup_{n \in \mathbb{N}_0} \{n\omega\}. \quad (2.18)$$

We introduce the following decoupled Hamiltonian H_0 , which describes the system when the leads are decoupled from the quantum dot and the electron does not interact with the photon field.

$$H_0 := H_0^{el} + H^{ph}, \quad (2.19)$$

where

$$H_0^{el} := h_0^{el} \otimes I_{\mathfrak{h}^{ph}} \quad \text{and} \quad H^{ph} := I_{\mathfrak{h}^{el}} \otimes h^{ph}.$$

The operator H_0 is self-adjoint on $\text{dom}(H_0) = \text{dom}(I_{\mathfrak{h}^{el}} \otimes h^{ph})$. Recall that h_0^{el} and h^{ph} are bounded self-adjoint operators. Hence H_0^{el} and H^{ph} are semi-bounded from below which yields that H_0 is semi-bounded from below.

The interaction of the photons and the electrons in the quantum dot is given by the coupling of the dipole moment of the electrons to the electromagnetic field in the rotating wave approximation. Namely,

$$V_{ph} = g_{ph} \left((\cdot, e_0^S) e_1^S \otimes b + (\cdot, e_1^S) e_0^S \otimes b^* \right) \quad (2.20)$$

for some coupling constant $g_{ph} \in \mathbb{R}$. The total Hamiltonian is given by

$$H := H^{el} + H^{ph} + V_{ph} = H_0 + V_{el} + V_{ph}, \quad (2.21)$$

where $H^{el} := h^{el} \otimes I_{\mathfrak{h}^{ph}}$ and $V_{el} := v_{el} \otimes I_{\mathfrak{h}^{ph}}$.

In the following we call $\mathcal{S} = \{H, H_0\}$ the Jaynes-Cummings-leads system, in short *JCL-model*, which we are going to analyze. In particular, we are interested in the electron and photon currents for that system. The analysis will be based on the abstract Landauer-Büttiker formula, cf. [1, 13].

Lemma 2.3 *H is bounded from below self-adjoint such that $\text{dom}(H) = \text{dom}(H_0)$.*

Proof. Let $c \geq 2$. Then

$$\|b\Upsilon_n\|^2 \leq \|b^*\Upsilon_n\|^2 = n + 1 \leq c^{-1}n^2 + c, \quad n \in \mathbb{N}_0.$$

Consider elements $f \in \mathfrak{h}_S \otimes \mathfrak{h}^{ph} \cap \text{dom}(I_{\mathfrak{h}^{el}} \otimes h^{ph})$ with

$$f = \sum_{j,l} \beta_{jl} e_j \otimes \Upsilon_l, \quad j \in \{0, 1\}, \quad l \in \mathbb{N}_0,$$

which are dense in $\mathfrak{H}^{JC} := \mathfrak{h}_S^{el} \otimes h^{ph}$. Then $\|f\|^2 = \sum_{j,l} |\beta_{jl}|^2$ and $\|(I_{\mathfrak{h}^{el}} \otimes b^*b)f\|^2 = \sum_{j,l=1} |\beta_{jl}|^2 l^2$. We obtain

$$\begin{aligned} \|((\cdot, e_1^S)e_0^S \otimes b)f\|^2 &\leq \sum_{j,l} |\beta_{jl}|^2 \|b\Upsilon_l\|^2 \leq \\ &\sum_{j,l} |\beta_{jl}|^2 (c^{-1}l^2 + c) = c^{-1} \|(I_{\mathfrak{h}^{el}} \otimes b^*b)f\|^2 + c\|f\|^2 \end{aligned}$$

Similarly,

$$\|((\cdot, e_1^S)e_0^S \otimes b^*)f\|^2 \leq c^{-1} \|(I_{\mathfrak{h}^{el}} \otimes b^*b)f\|^2 + c\|f\|^2.$$

If $c \geq 2$ is large enough, then we obtain that V_{ph} is dominated by H^{ph} with relative bound less than one. Hence H is self-adjoint and $\text{dom}(H_0) = \text{dom}(H)$. Since H_0^{el} and V_{el} are bounded and H^{ph} is self-adjoint and bounded from below, it follows that $H = H_0^{el} + H^{ph} + V_{el} + V_{ph}$ is bounded from below [17, Thm. V.4.1]. \square

2.3 Time reversible symmetric systems

A system described by the Hamiltonian H is called time reversible symmetric if there is a conjugation Γ defined on \mathfrak{H} such that $\Gamma H = H\Gamma$. Recall that Γ is a conjugation if the conditions $\Gamma^2 = I$ and $(\Gamma f, \Gamma g) = \overline{(f, g)}$, $f, g \in \mathfrak{H}$.

Let \mathfrak{h}_n^{ph} , $n \in \mathbb{N}_0$, the subspace spanned by the eigenvector Υ_n in \mathfrak{h}^{ph} . We set

$$\mathfrak{H}_{n_\alpha} := \mathfrak{h}_\alpha^{el} \otimes \mathfrak{h}_n^{ph}, \quad n \in \mathbb{N}_0, \quad \alpha \in \{l, r\}. \quad (2.22)$$

Notice that

$$\mathfrak{H} = \bigoplus_{n \in \mathbb{N}_0, \alpha \in \{l, r\}} \mathfrak{H}_{n_\alpha}$$

Definition 2.4 The *JCL*-model is called time reversible symmetric if there is a conjugation Γ acting on \mathfrak{H} such that H and H_0 are time reversible symmetric and the subspaces \mathfrak{H}_{n_α} , $n \in \mathbb{N}_0$, $\alpha \in \{l, r\}$, reduces Γ .

Example 2.5 Let γ_α^{el} and γ_S^{el} be conjugations defined by

$$\gamma_\alpha^{el} f_\alpha := \overline{f_\alpha} := \{\overline{f_\alpha(k)}\}_{k \in \mathbb{N}}, \quad f_\alpha \in \mathfrak{h}_\alpha^{el}, \quad \alpha \in \{l, r\},$$

and

$$\gamma_S^{el} f_S = \gamma_S^{el} \begin{pmatrix} f_S(0) \\ f_S(1) \end{pmatrix} := \begin{pmatrix} \overline{f_S(0)} \\ \overline{f_S(1)} \end{pmatrix}$$

We set $\gamma^{el} := \gamma_l^{el} \oplus \gamma_S^{el} \oplus \gamma_r^{el}$. Further, we set

$$\gamma^{ph} \psi := \overline{\psi} = \{\overline{\psi(n)}\}_{n \in \mathbb{N}_0}, \quad \psi \in \mathfrak{h}^{ph}.$$

We set $\Gamma := \gamma^{el} \otimes \gamma^{ph}$. One easily checks that Γ is a conjugation on $\mathfrak{H} = \mathfrak{h}^{el} \otimes \mathfrak{h}^{ph}$.

Lemma 2.6 Let γ_α^{el} , $\alpha \in \{S, l, r\}$, and γ^{ph} be given by Example 2.5.

- (i) If the conditions $\gamma_S^{el} e_0^S = e_0^S$ and $\gamma_S^{el} e_1^S = e_1^S$ are satisfied, then H_0 is time reversible symmetric with respect to Γ and, moreover, the subspaces \mathfrak{H}_{n_α} , $n \in \mathbb{N}_0$, $\alpha \in \{l, r\}$, reduces Γ .
- (ii) If in addition the conditions $\gamma_S^{el} \delta_0^S = \delta_0^S$ and $\gamma_S^{el} \delta_1^S = \delta_1^S$ are satisfied, then JCL -model is time reversible symmetric.

Proof. (i) Obviously we have

$$\gamma_\alpha^{el} h_\alpha^{el} = h_\alpha^{el} \gamma_\alpha^{el}, \quad \alpha \in \{l, r\}, \quad \text{and} \quad \gamma^{ph} h^{ph} = h^{ph} \gamma^{ph}.$$

If $\gamma_S^{el} e_0^S = e_0^S$ and $\gamma_S^{el} e_1^S = e_1^S$ is satisfied, then $\gamma_S^{el} h_S^{el} = h_S^{el} \gamma_S^{el}$ which yields $\gamma^{el} h_0^{el} = h_0^{el} \gamma^{el}$ and, hence, $\Gamma H_0 = \Gamma H_0$. Since $\gamma^{el} \mathfrak{h}_\alpha^{el} = \mathfrak{h}_\alpha^{el}$ and $\gamma^{ph} \mathfrak{h}^{ph} = \mathfrak{h}^{ph}$ one gets $\Gamma \mathfrak{H}_{n_\alpha} = \mathfrak{H}_{n_\alpha}$ which shows that \mathfrak{H}_{n_α} reduces Γ .

(ii) Notice that $\gamma_\alpha^{el} \delta_1^\alpha = \delta_1^\alpha$, $\alpha \in \{l, r\}$. If in addition the conditions $\gamma_S^{el} \delta_0^S = \delta_0^S$ and $\gamma_S^{el} \delta_1^S = \delta_1^S$ are satisfied, then $\gamma^{el} v_{el} = v_{el} \gamma^{el}$ is valid which yields $\gamma^{el} h^{el} = h^{el} \gamma^{el}$. Hence $\Gamma H = H \Gamma$. Together with (i) this proves that the JCL -model is time reversible symmetric. \square

Choosing

$$e_0^S := \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \quad e_1^S := \begin{pmatrix} 0 \\ 1 \end{pmatrix}, \quad \delta_0^S := \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ 1 \end{pmatrix}, \quad \delta_1^S := \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ -1 \end{pmatrix} \quad (2.23)$$

one satisfies the condition $\gamma_S^{el} e_0^S = e_0^S$ and $\gamma_S^{el} e_1^S = e_1^S$ as well as $\gamma_S^{el} \delta_0^S = \delta_0^S$ and $\gamma_S^{el} \delta_1^S = \delta_1^S$.

2.4 Mirror symmetric systems

A unitary operator U acting on \mathfrak{H} is called a mirror symmetry if the conditions

$$U \mathfrak{H}_{n_\alpha} = \mathfrak{H}_{n_{\alpha'}}, \quad \alpha, \alpha' \in \{l, r\}, \quad \alpha \neq \alpha'$$

are satisfied. In particular, this yields $U \mathfrak{H}^{JC} = \mathfrak{H}^{JC}$, $\mathfrak{H}^{JC} := \mathfrak{h}_S^{el} \otimes \mathfrak{h}^{ph}$.

Definition 2.7 The JCL -model is called mirror symmetric if there is a mirror symmetry commuting with H_0 and H .

One easily verifies that if H_0 is mirror symmetric, then

$$H_{n_{\alpha'}} U = U H_{n_\alpha}, \quad n \in \mathbb{N}_0, \quad \alpha, \alpha' \in \{l, r\}, \quad \alpha \neq \alpha',$$

where

$$H_{n_\alpha} := h_\alpha^{el} \otimes I_{\mathfrak{h}^{ph}} + I_{\mathfrak{h}_\alpha^{el}} \otimes h_n^{ph} = h_\alpha^{el} + n\omega, \quad n \in \mathbb{N}_0, \quad \alpha, \alpha' \in \{l, r\}, \quad \alpha \neq \alpha'.$$

In particular, this yields that $v_\alpha = v_{\alpha'}$. Moreover, one gets $U H_S = H_S U$ where $H_S := h_S^{el} \otimes I_{\mathfrak{h}^{ph}} + I_{\mathfrak{h}_S^{el}} \otimes h^{ph}$.

Notice that if H and H_0 commute with the same mirror symmetry U , then also the operator $H_c := h^{el} \otimes I_{\mathfrak{h}^{ph}} + I_{\mathfrak{h}^{el}} \otimes h^{ph}$ commutes with U , i.e. is mirror symmetric.

Example 2.8 Let $\mathcal{S} = \{H, H_0\}$ be the JCL -model. Let $v_l = v_r$ and let e_0^S and e_1^S as well as δ_0^S and δ_1^S be given by (2.23). We set

$$u_S^{el} e_0^S := e_0^S \quad \text{and} \quad u_S^{el} e_1^S = -e_1^S \quad (2.24)$$

as well as

$$u^{ph}\Upsilon_n = e^{-in\pi}\Upsilon_n, \quad n \in \mathbb{N}_0. \quad (2.25)$$

Obviously, $U_S := u_S^{el} \otimes u^{ph}$ defines a unitary operator on \mathfrak{H}^{JC} . A straightforward computation shows that

$$U_S H_S = H_S U_S \quad \text{and} \quad U_S V_{ph} = V_{ph} U_S. \quad (2.26)$$

Furthermore, we set

$$u_{rl}^{el} \delta_n^l := \delta_n^r, \quad \text{and} \quad u_{lr}^{el} \delta_n^r = \delta_n^l, \quad n \in \mathbb{N}, \quad (2.27)$$

and

$$u^{el} := \begin{pmatrix} 0 & 0 & u_{lr}^{el} \\ 0 & u_S^{el} & 0 \\ u_{lr}^{el} & 0 & 0 \end{pmatrix}.$$

We have

$$v_{el} u^{el} \begin{pmatrix} f_l \\ f_S \\ f_r \end{pmatrix} = \begin{pmatrix} \langle f_S, (u_S^{el})^* \delta_0^S \rangle \delta_1^l \\ \langle f_r, (u_{lr}^{el})^* \delta_1^l \rangle \delta_0^S + \langle f_l, (u_{rl}^{el})^* \delta_1^r \rangle \delta_1^S \\ \langle f_S, (u_S^{el})^* \delta_1^S \rangle \delta_1^r \end{pmatrix} \quad (2.28)$$

Since $\delta_0^S := \frac{1}{\sqrt{2}}(e_0^S + e_1^S)$ and $\delta_1^S := \frac{1}{\sqrt{2}}(e_0^S - e_1^S)$ we get from (2.24)

$$(u_S^{el})^* \delta_0^S = \delta_1^S \quad \text{and} \quad (u_S^{el})^* \delta_1^S = \delta_0^S. \quad (2.29)$$

Obviously we have

$$(u_{lr}^{el})^* \delta_1^l = \delta_1^r \quad (u_{rl}^{el})^* \delta_1^r = \delta_1^l. \quad (2.30)$$

Inserting (2.29) and (2.30) into (2.28) we find

$$v_{el} u^{el} \begin{pmatrix} f_l \\ f_S \\ f_r \end{pmatrix} = \begin{pmatrix} \langle f_S, \delta_1^S \rangle \delta_1^l \\ \langle f_r, \delta_1^r \rangle \delta_0^S + \langle f_l, \delta_1^l \rangle \delta_1^S \\ \langle f_S, \delta_0^S \rangle \delta_1^r \end{pmatrix} \quad (2.31)$$

us Further we have

$$u^{el} v_{el} \begin{pmatrix} f_l \\ f_S \\ f_r \end{pmatrix} = \begin{pmatrix} \langle f_S, \delta_1^S \rangle \delta_1^l \\ \langle f_l, \delta_1^l \rangle \delta_1^S + \langle f_r, \delta_1^r \rangle \delta_0^S \\ \langle f_S, \delta_0^S \rangle \delta_1^r \end{pmatrix}. \quad (2.32)$$

Comparing (2.31) and (2.32) we get $u^{el} v_{el} = v_{el} u^{el}$. Setting $U := u^{el} \otimes u^{ph}$ one immediately proves that $UH_0 = H_0U$ and $UH = HU$. Since $U\mathfrak{H}_{n\alpha} = \mathfrak{H}_{n_{ga}}$, it is satisfied \mathcal{S} is mirror symmetric.

Notice that in addition the Example 2.8 \mathcal{S} is time reversible symmetric.

2.5 Spectral properties of H : first part

In the following our goal is to apply the Landauer-Büttiker formula to the JCL -model. By $\mathfrak{L}_p(\mathfrak{H})$, $1 \leq p \leq \infty$, we denote in the following the Schatten-v. Neumann ideals.

Proposition 2.9 *If $\mathcal{S} = \{H, H_0\}$ is the JCL -model, then $(H+i)^{-1} - (H_0+i)^{-1} \in \mathfrak{L}_1(\mathfrak{H})$. In particular, the absolutely continuous parts H^{ac} and H_0^{ac} are unitarily equivalent.*

Proof. We have

$$\begin{aligned} (H+i)^{-1} - (H_0+i)^{-1} &= (H_0+i)^{-1}V(H+i)^{-1} = \\ &= (H_0+i)^{-1}V(H_0+i)^{-1} - (H_0+i)^{-1}V(H_0+i)^{-1}V(H+i)^{-1} \end{aligned}$$

where $V = H - H_0 = V_{el} + V_{ph}$. Taking into account Lemma 2.3 it suffices to prove that $(H_0 + i)^{-1}V(H_0 + i)^{-1} \in \mathfrak{L}_1(\mathfrak{H})$. Using the spectral decomposition of h^{ph} with respect to $\mathfrak{h}^{ph} = \bigoplus_{n \in \mathbb{N}_0} \mathfrak{h}_n^{ph}$, where \mathfrak{h}_n^{ph} are the subspaces spanned by Υ_n , we obtain

$$(H_0 + i)^{-1} = \bigoplus_{n \in \mathbb{N}_0} (h_0^{el} + n\omega + i)^{-1} \otimes I_{\mathfrak{h}_n^{ph}}. \quad (2.33)$$

We have $(H_0 + i)^{-1}V(H_0 + i)^{-1} = (H_0 + i)^{-1}(V_{el} + V_{ph})(H_0 + i)^{-1}$. Since v_{el} is a finite rank operator we have $\|v_{el}\|_{\mathfrak{L}_1} < \infty$. Furthermore, \mathfrak{h}_n^{ph} is obviously one-dimensional for any $n \in \mathbb{N}_0$. Hence $\|I_{\mathfrak{h}_n^{ph}}\|_{\mathfrak{L}_1} = 1$. From (2.33) and $V_{el} = v_{el} \otimes I_{\mathfrak{h}^{ph}}$ we obtain

$$\begin{aligned} \|(H_0 + i)^{-1}V_{el}(H_0 + i)^{-1}\|_{\mathfrak{L}_1} &= \sum_{n \in \mathbb{N}_0} \|(h_0^{el} + n\omega + i)^{-1}v_{el}(h_0^{el} + n\omega + i)^{-1}\|_{\mathfrak{L}_1} \\ &\leq \sum_{n \in \mathbb{N}_0} \|(h_0^{el} + n\omega + i)^{-2}\| \|v_{el}\|_{\mathfrak{L}_1} \end{aligned}$$

Since h_0^{el} is bounded we get

$$\|(h_0^{el} + n\omega + i)^{-1}\| = \sup_{\lambda \in \sigma(h_0^{el})} (\sqrt{(\lambda + n\omega)^2 + 1})^{-1} \leq c(n+1)^{-1} \quad (2.34)$$

for some $c > 0$. This immediately implies $\|(H_0 + i)^{-1}V_{el}(H_0 + i)^{-1}\|_{\mathfrak{L}_1} < \infty$.

We are going to handle $(H_0 + i)^{-1}V_{ph}(H_0 + i)^{-1}$. Let p_n^{ph} be the projection from \mathfrak{h}^{ph} onto \mathfrak{h}_n^{ph} . We have

$$\begin{aligned} &(H_0 + i)^{-1}(\cdot, e_0^S)e_1^S \otimes b(H_0 + i)^{-1} \\ &= \sum_{m, n \in \mathbb{N}_0} (h_0^{el} + m\omega + i)^{-1}(\cdot, e_0^S)e_1^S (h_0^{el} + n\omega + i)^{-1} \otimes p_m^{ph} b p_n^{ph} \\ &= \sum_{n \in \mathbb{N}} (h_0^{el} + (n-1)\omega + i)^{-1}(\cdot, e_0^S)e_1^S (h_0^{el} + n\omega + i)^{-1} \otimes \sqrt{n}\Upsilon_{n-1}(\cdot, \Upsilon_n) \end{aligned}$$

From (2.34) we get

$$\begin{aligned} &\|(h_0^{el} + (n-1)\omega + i)^{-1}(\cdot, e_0^S)e_1^S (h_0^{el} + n\omega + i)^{-1} \otimes \sqrt{n}\Upsilon_n(\cdot, \Upsilon_n)\|_{\mathfrak{L}_1} \\ &\leq c^2 \frac{\sqrt{n}}{n(n+1)}, \end{aligned}$$

$n \in \mathbb{N}$, which yields

$$\|(H_0 + i)^{-1}(\cdot, e_0^S)e_1^S \otimes b(H_0 + i)^{-1}\|_{\mathfrak{L}_1} \leq c^2 \sum_{n \in \mathbb{N}} \frac{\sqrt{n}}{n(n+1)} < \infty.$$

Since

$$\|(H_0 + i)^{-1}(\cdot, e_1^S)e_0^S \otimes b^*(H_0 + i)^{-1}\|_{\mathfrak{L}_1} = \|(H_0 + i)^{-1}(\cdot, e_0^S)e_1^S \otimes b(H_0 + i)^{-1}\|_{\mathfrak{L}_1}$$

one gets $(H_0 + i)^{-1}V_{ph}(H_0 + i)^{-1} \in \mathfrak{L}_1(\mathfrak{H})$ which completes the proof. \square

Thus, the *JCL*-model $\mathcal{S} = \{H, H_0\}$ is a \mathfrak{L}_1 -scattering system. Let us recall that $h_\alpha^{el} = -\Delta^D + v_\alpha$, $\alpha \in \{l, r\}$, on $\mathfrak{h}_l^{el} = \mathfrak{h}_r^{el} = \ell^2(\mathbb{N})$.

Lemma 2.10 *Let $\alpha \in \{l, r\}$. We have*

$$\sigma(h_\alpha^{el}) = \sigma_{ac}(h_\alpha^{el}) = [v_\alpha, 4 + v_\alpha].$$

The normalized generalized eigenfunctions of h_α^{el} are given by

$$g_\alpha(x, \lambda) = \pi^{-\frac{1}{2}} (1 - (-\lambda + 2 + v_\alpha)^2/4)^{-\frac{1}{4}} \sin(\arccos((-\lambda + 2 + v_\alpha)/2)x)$$

for $x \in \mathbb{N}$, $\lambda \in (v_\alpha, 4 + v_\alpha)$.

Proof. We prove the absolute continuity of the spectrum by showing that

$$\{g_\alpha(x, \lambda) \mid \lambda \in (-2, 2)\}$$

is a complete set of generalized eigenfunctions. Note that it suffices to prove the lemma for

$$((\Delta^D + 2)f)(x) = f(x+1) + f(x-1), \quad f(0) = 0.$$

The lemma then follows by replacing λ with $-\lambda + 2 + v_\alpha$. Let $\lambda \in (-2, 2)$ and

$$g_{\Delta^D}(x, \lambda) = \pi^{-\frac{1}{2}} (1 - \lambda^2/4)^{-\frac{1}{4}} \sin(\arccos(\lambda/2)x)$$

Note that $g_{\Delta^D}(0, \lambda) = 0$, whence the boundary condition is satisfied. We substitute $\mu = \arccos(\lambda/2) \in (0, \pi)$, i.e. $\lambda = 2 \cos(\mu)$ and obtain

$$\sin(\mu(x+1)) + \sin(\mu(x-1)) = 2 \sin(\mu x) \cos(\mu),$$

whence $g_{\Delta^D}(x, \lambda)$ satisfies the eigenvalue equation. It is obvious that $g_{\Delta^D}(\cdot, \lambda) \notin \ell^2(\mathbb{N}_0)$ for $\lambda \in (-2, 2)$. To complete the proof of the lemma, it remains to show the ortho-normality and the completeness. For the ortho-normality, we have to show that

$$\sum_{x \in \mathbb{N}} g_{\Delta^D}(x, \lambda) g_{\Delta^D}(x, \nu) = \delta(\lambda - \nu).$$

Let $\psi \in C_0^\infty((-2, 2))$. We use the substitution $\mu = \arccos(\nu/2)$ and the relation

$$\sin(\arccos(y)) = (1 - y^2)^{-\frac{1}{2}}$$

to obtain

$$\begin{aligned} & \int_{-2}^2 d\nu \sum_{x \in \mathbb{N}} g_{\Delta^D}(x, \lambda) g_{\Delta^D}(x, \nu) \psi(\nu) \\ &= 2\pi^{-1} \int_0^\pi d\mu \sum_{x \in \mathbb{N}} \frac{\sin(\mu) \sin(\arccos(\lambda/2)x) \sin(\mu x)}{(\sin(\mu))^{\frac{1}{2}} (\sin(\arccos(\lambda/2)))^{\frac{1}{2}}} \psi(2 \cos(\mu)) \\ &= (2\pi)^{-1} \int_0^\pi d\mu \sum_{x \in \mathbb{N}} \frac{(\sin(\mu))^{\frac{1}{2}}}{(\sin(\arccos(\lambda/2)))^{\frac{1}{2}}} \left(e^{i(\arccos(\lambda/2) - \mu)x} + \right. \\ & \quad \left. e^{-i(\arccos(\lambda/2) - \mu)x} - e^{i(\arccos(\lambda/2) + \mu)x} - e^{-i(\arccos(\lambda/2) + \mu)x} \right) \psi(2 \cos(\mu)) \end{aligned}$$

Observe that for the Dirichlet kernel

$$\sum_{x \in \mathbb{N}_0} (e^{ixy} + e^{-ixy}) - 1 = 2\pi \delta(y),$$

whence

$$\begin{aligned} & \int_{-2}^2 d\nu \sum_{x \in \mathbb{N}} g_{\Delta^D}(x, \lambda) g_{\Delta^D}(x, \nu) \psi(\nu) \\ &= \int_0^\pi d\mu \frac{(\sin(\mu))^{\frac{1}{2}}}{(\sin(\arccos(\lambda/2)))^{\frac{1}{2}}} \left(\delta(\arccos(\lambda/2) - \mu) + \right. \\ & \quad \left. \delta(\arccos(\lambda/2) + \mu) \right) \psi(2 \cos(\mu)) = \psi(\lambda). \end{aligned}$$

In the second equality we use that the summand containing $\delta(\arccos(\lambda/2) + \mu)$ is zero since both $\arccos(\lambda/2) > 0$ and $\mu > 0$. Thus, the generalized eigenfunctions are orthonormal. Finally, using once more the substitution $\mu = \arccos(\nu/2)$, we get

$$\begin{aligned} & \int_{-2}^2 d\nu g_{\Delta^D}(x, \nu) g_{\Delta^D}(y, \nu) \\ &= \int_{-2}^2 d\nu (1 - (\nu/2)^2)^{-\frac{1}{2}} \sin(\arccos(\nu/2)x) \sin(\arccos(\nu/2)y) \\ &= 2\pi^{-1} \int_0^\pi d\mu (\sin(\mu))^{-1} \sin(\mu) \sin(\mu x) \sin(\mu y) \\ &= \delta_{xy} \end{aligned}$$

for $x, y \in \mathbb{N}$, whence the family of generalized eigenfunctions is also complete. \square

From these two lemmas we obtain the following corollary that gives us the spectral properties of H_0 .

Proposition 2.11 *Let $S = \{H, H_0\}$ be the JCL-model. Then $\sigma(H_0) = \sigma_{ac}(H_0) \cup \sigma_{pp}(H_0)$, where*

$$\sigma_{ac}(H_0) = \bigcup_{n \in \mathbb{N}_0} [v_l + n\omega, v_l + 4 + n\omega] \cup [v_r + n\omega, v_r + 4 + n\omega]$$

and

$$\sigma_{pp}(H_0) = \bigcup_{n \in \mathbb{N}_0} \{\lambda_j^S + n\omega : j = 0, 1\}.$$

The eigenvectors are given by $\tilde{g}(m, n) = e_m^S \otimes \Upsilon_n$, $m = 0, 1$, $n \in \mathbb{N}_0$. The generalized eigenfunctions are given by $\tilde{g}_\alpha(\cdot, \lambda, n) = g_\alpha(\cdot, \lambda - n\omega) \otimes \Upsilon_n$ for $\lambda \in \sigma_{ac}(H_0)$, $n \in \mathbb{N}_0$, $\alpha \in \{l, r\}$.

Proof. It is well known (see e.g. [15]) that for two self-adjoint operators A and B with $\sigma_{sc}(A) = \sigma_{sc}(B) = \emptyset$, we have $\sigma_{sc}(A \otimes 1 + 1 \otimes B) = \emptyset$,

$$\sigma_{ac}(A \otimes 1 + 1 \otimes B) = (\sigma_{ac}(A) + \sigma(B)) \cup (\sigma(A) + \sigma_{ac}(B))$$

and

$$\sigma_{pp}(A \otimes 1 + 1 \otimes B) = \sigma_{pp}(A) + \sigma_{pp}(B).$$

Furthermore, if $\psi_A(\lambda_A)$ and $\psi_B(\lambda_B)$ are (generalized) eigenfunctions of A and B , respectively, then $\psi_A(\lambda_A) \otimes \psi_B(\lambda_B)$ is a (generalized) eigenfunction of $A \otimes I + I \otimes B$ for the (generalized) eigenvalue $\lambda_A + \lambda_B$.

The lemma follows now with $A = h_0^{el}$ and $B = h^{ph}$ using Lemmata 2.10 and (2.18) and the fact that h_S has eigenvectors $\{e_0^S, e_1^S\}$ with eigenvalues $\{\lambda_0^S, \lambda_1^S = \lambda_0^S + \varepsilon\}$. \square

2.6 Spectral representation

For the convenience of the reader we define here what we mean under a spectral representation of the absolutely continuous part K_0^{ac} of a self-adjoint operator K_0 on a separable Hilbert space \mathfrak{K} . Let \mathfrak{k} be an auxiliary separable Hilbert space. We consider the Hilbert space $L^2(\mathbb{R}, d\lambda, \mathfrak{k})$. By \mathcal{M} we define the multiplication operator induced by the independent variable λ in $L^2(\mathbb{R}, d\lambda, \mathfrak{k})$. Let $\Phi : \mathfrak{K}^{ac}(K_0) \rightarrow L^2(\mathbb{R}, d\lambda, \mathfrak{k})$ be an isometry acting from $\mathfrak{K}^{ac}(K_0)$ into $L^2(\mathbb{R}, d\lambda, \mathfrak{k})$ such that $\Phi \text{dom}(K_0^{ac}) \subseteq \text{dom}(\mathcal{M})$ and

$$\mathcal{M}\Phi f = \Phi K_0^{ac} f, \quad f \in \text{dom}(K_0^{ac}).$$

Obviously, the orthogonal projection $P := \Phi\Phi^*$ commutes with \mathcal{M} which yields the existence of a measurable family $\{P(\lambda)\}_{\lambda \in \mathbb{R}}$ such that

$$(P\widehat{f})(\lambda) = P(\lambda)\widehat{f}(\lambda), \quad \widehat{f} \in L^2(\mathbb{R}, \mathfrak{k}).$$

We set $L^2(\mathbb{R}, d\lambda, \mathfrak{k}(\lambda)) := PL^2(\mathbb{R}, \lambda, \mathfrak{k})$, $\mathfrak{k}(\lambda) := P(\lambda)\mathfrak{k}$, and call the triplet

$$\Pi(K_0^{ac}) := \{L^2(\mathbb{R}, d\lambda, \mathfrak{k}(\lambda)), \mathcal{M}, \Phi\}$$

a spectral representation of K_0^{ac} . If $\{L^2(\mathbb{R}, d\lambda, \mathfrak{k}(\lambda)), \mathcal{M}, \Phi\}$ is a spectral representation of K^{ac} , then K^{ac} is unitarily equivalent $\mathcal{M}_0 := \mathcal{M} \upharpoonright L^2(\mathbb{R}, d\lambda, \mathfrak{k}(\lambda))$. Indeed, one has $\Phi K_0^{ac} \Phi^* = \mathcal{M}_0$. The function $\xi_{K_0^{ac}}(\lambda) := \dim(\mathfrak{k}(\lambda))$, $\lambda \in \mathbb{R}$, is called the spectral multiplicity function of K_0^{ac} . Notice that $0 \leq \xi_{K_0^{ac}}(\lambda) \leq \infty$ for $\lambda \in \mathbb{R}$.

For $\alpha \in \{l, r\}$ the generalized eigenfunctions of h_α^{el} define generalized Fourier transforms by $\phi_\alpha^{el} : \mathfrak{h}_\alpha^{el} = \mathfrak{h}_\alpha^{el, ac}(h_\alpha^{el}) \rightarrow L^2([v_\alpha, v_\alpha + 4])$ and

$$(\phi_\alpha^{el} f_\alpha)(\lambda) = \sum_{x \in \mathbb{N}_0} g_\alpha(x, \lambda) f_\alpha(x), \quad f_\alpha \in \mathfrak{h}_\alpha^{el}. \quad (2.35)$$

Setting

$$\mathfrak{h}_\alpha^{el}(\lambda) := \begin{cases} \mathbb{C} & \lambda \in [v_\alpha, v_\alpha + 4] \\ 0 & \lambda \in \mathbb{R} \setminus [v_\alpha, v_\alpha + 4]. \end{cases} \quad (2.36)$$

one easily verifies that $\Pi(h_\alpha^{el}) = \{L^2(\mathbb{R}, d\lambda, \mathfrak{h}_\alpha^{el}(\lambda)), \mathcal{M}, \phi_\alpha^{el}\}$ is a spectral representation of $h_\alpha^{el} = h_\alpha^{el, ac}$, $\alpha = l, r$, where we always assumed implicitly that $(\phi_\alpha^{el} f_\alpha)(\lambda) = 0$ for $\lambda \in \mathbb{R} \setminus [v_\alpha, v_\alpha + 4]$. Setting

$$\mathfrak{h}^{el}(\lambda) := \begin{matrix} \mathfrak{h}_l^{el}(\lambda) \\ \oplus \\ \mathfrak{h}_r^{el}(\lambda) \end{matrix} \subseteq \mathbb{C}^2, \quad \lambda \in \mathbb{R}, \quad (2.37)$$

and introducing the map

$$\phi^{el} : \mathfrak{h}^{el, ac}(h_0^{el}) = \begin{matrix} \mathfrak{h}_l^{el} \\ \oplus \\ \mathfrak{h}_r^{el} \end{matrix} \longrightarrow L^2(\mathbb{R}, d\lambda, \mathfrak{h}^{el}(\lambda)) \quad (2.38)$$

defined by

$$\phi^{el} f := \begin{pmatrix} \phi_l^{el} f_l \\ \phi_r^{el} f_r \end{pmatrix}, \quad \text{where } f := \begin{pmatrix} f_l \\ f_r \end{pmatrix} \quad (2.39)$$

we obtain a spectral representation $\Pi(h_0^{el, ac}) = \{L^2(\mathbb{R}, d\lambda, \mathfrak{h}^{el}(\lambda)), \mathcal{M}, \phi^{el}\}$ of the absolutely continuous part $h_0^{el, ac} = h_l^{el} \oplus h_r^{el}$ of h_0^{el} . One easily verifies that $0 \leq \xi_{h_0^{el, ac}}(\lambda) \leq 2$ for $\lambda \in \mathbb{R}$. Introducing

$$\lambda_{\min}^{el} := \min\{v_l, v_r\} \quad \text{and} \quad \lambda_{\max}^{el} := \max\{v_l + 4, v_r + 4\} \quad (2.40)$$

one easily verifies that $\xi_{h_0^{el, ac}}(\lambda) = 0$ for $\lambda \in \mathbb{R} \setminus [\lambda_{\min}^{el}, \lambda_{\max}^{el}]$.

Notice, if $v_r + 4 \leq v_l$, then

$$\mathfrak{h}^{el}(\lambda) = \begin{cases} \mathbb{C}, & \lambda \in [v_r, v_r + 4] \cup [v_l, v_l + 4], \\ \{0\}, & \text{otherwise} \end{cases}$$

which shows that h_0^{el} has simple spectrum. In particular, it holds $\xi_{h_0^{el, ac}}(\lambda) = 1$ for $\lambda \in [v_r, v_r + 4] \cup [v_l, v_l + 4]$ and otherwise $\xi_{h_0^{el, ac}}(\lambda) = 0$.

Let us introduce the Hilbert space $\mathfrak{h} := l^2(\mathbb{N}_0, \mathbb{C}^2) = \bigoplus_{n \in \mathbb{N}_0} \mathfrak{h}_n$, $\mathfrak{h}_n := \mathbb{C}^2$, $n \in \mathbb{N}_0$. Regarding $\mathfrak{h}^{el}(\lambda - n\omega)$ as a subspace of \mathfrak{h}_n one regards

$$\mathfrak{h}(\lambda) := \bigoplus_{n \in \mathbb{N}_0} \mathfrak{h}_n(\lambda), \quad \mathfrak{h}_n(\lambda) := \mathfrak{h}^{el}(\lambda - n\omega), \quad \lambda \in \mathbb{R}, \quad (2.41)$$

as a measurable family of subspaces in \mathfrak{h} . Notice that $0 \leq \dim(\mathfrak{h}(\lambda)) < \infty$, $\lambda \in \mathbb{R}$. We consider the Hilbert space $L^2(\mathbb{R}, d\lambda, \mathfrak{h}(\lambda))$.

Furthermore, we introduce the isometric map $\Phi : \mathfrak{H}(H_0^{ac}) \longrightarrow L^2(\mathbb{R}, d\lambda, \mathfrak{h}(\lambda))$ defined by

$$(\Phi f)(\lambda) = \bigoplus_{n \in \mathbb{N}_0} \begin{pmatrix} (\phi_l^{el} f_l(n))(\lambda - n\omega) \\ (\phi_r^{el} f_r(n))(\lambda - n\omega) \end{pmatrix}, \quad \lambda \in \mathbb{R} \quad (2.42)$$

where

$$\bigoplus_{n \in \mathbb{N}_0} \begin{pmatrix} f_l(n) \\ f_r(n) \end{pmatrix} \in \bigoplus_{n \in \mathbb{N}_0} \mathfrak{h}^{el,ac}(h_0^{el}) \otimes \mathfrak{h}_n^{ph} = \bigoplus_{n \in \mathbb{N}} \begin{pmatrix} \mathfrak{h}_l^{el} \otimes h_n^{ph} \\ \mathfrak{h}_r^{el} \otimes h_n^{ph} \end{pmatrix}$$

where $\mathfrak{h}_{ph} = \bigoplus_{n \in \mathbb{N}_0} \mathfrak{h}_n^{ph}$ and \mathfrak{h}_n^{ph} is the subspace spanned by the eigenvectors Υ_n of h^{ph} . One easily verifies that Φ is an isometry acting from $\mathfrak{H}^{ac}(H_0^{ac})$ onto $L^2(\mathbb{R}, d\lambda, \mathfrak{h}(\lambda))$.

Lemma 2.12 *The triplet $\{L^2(\mathbb{R}, d\lambda, \mathfrak{h}(\lambda)), \mathcal{M}, \Phi\}$ forms a spectral representation of H_0^{ac} , that is, $\Pi(H_0^{ac}) = \{L^2(\mathbb{R}, d\lambda, \mathfrak{h}(\lambda)), \mathcal{M}, \Phi\}$ where there is a constant $d \in \mathbb{N}_0$ such that $0 \leq \xi_{H_0^{ac}}^{ac}(\lambda) \leq 2d_{\max}$ for $\lambda \in \mathbb{R}$ where $d_{\max} := \frac{\lambda_{\max}^{el} - \lambda_{\min}^{el}}{\omega}$ and λ_{\max}^{el} and λ_{\min}^{el} are given by (2.40).*

Proof. It remains to show that Φ transform H_0^{ac} into the multiplication operator \mathcal{M} . We have

$$H_0^{ac} f = \bigoplus_{n \in \mathbb{N}_0} \begin{pmatrix} (h_l^{el} f_l)(n) + n\omega f_l(n) \\ (h_r^{el} f_r)(n) + n\omega f_r(n) \end{pmatrix}$$

which yields

$$\begin{aligned} & (\Phi H_0^{ac} f)(\lambda) \\ &= \bigoplus_{n \in \mathbb{N}_0} \begin{pmatrix} (\phi_l^{el} (h_l^{el} f_l)(n))(\lambda - n\omega) + n\omega (\phi_l^{el} f_l(n))(\lambda - n\omega) \\ (\phi_r^{el} (h_r^{el} f_r)(n))(\lambda - n\omega) + n\omega (\phi_r^{el} f_r(n))(\lambda - n\omega) \end{pmatrix} \\ &= \bigoplus_{n \in \mathbb{N}_0} \begin{pmatrix} \lambda (\phi_l^{el} f_l(n))(\lambda - n\omega) \\ \lambda (\phi_r^{el} f_r(n))(\lambda - n\omega) \end{pmatrix} = (\mathcal{M}\Phi f)(\lambda), \quad \lambda \in \mathbb{R}. \end{aligned}$$

which proves the desired property.

One easily checks that $\mathfrak{h}(\lambda)$ might be only non-trivial if $\lambda - n\omega \in [\lambda_{\min}^{el}, \lambda_{\max}^{el}]$. Hence we get that $\mathfrak{h}(\lambda)$ is non-trivial if the condition

$$\frac{\lambda - \lambda_{\max}^{el}}{\omega} \leq n \leq \frac{\lambda - \lambda_{\min}^{el}}{\omega}$$

is satisfied. Hence

$$0 \leq \xi_{H_0^{ac}}^{ac}(\lambda) \leq 2 \text{card} \left\{ n \in \mathbb{N}_0 : \frac{\lambda - \lambda_{\max}^{el}}{\omega} \leq n \leq \frac{\lambda - \lambda_{\min}^{el}}{\omega} \right\}, \quad \lambda \in \mathbb{R}.$$

or

$$0 \leq \xi_{H_0^{ac}}^{ac}(\lambda) \leq 2 \text{card} \left\{ n \in \mathbb{N}_0 : 0 \leq n \leq \frac{\lambda_{\max}^{el} - \lambda_{\min}^{el}}{\omega} \right\}, \quad \lambda \in \mathbb{R}.$$

Hence $0 \leq \xi_{H_0^{ac}}^{ac}(\lambda) \leq d_{\max}$ for $\lambda \in \mathbb{R}$. □

In the following we denote the orthogonal projection from $\mathfrak{h}(\lambda)$ onto $\mathfrak{h}_n(\lambda)$ by $P_n(\lambda)$, $\lambda \in \mathbb{R}$, cf (2.41). Since $\mathfrak{h}(\lambda) = \bigoplus_{n \in \mathbb{N}_0} \mathfrak{h}_n(\lambda)$ we have $I_{\mathfrak{h}(\lambda)} = \sum_{n \in \mathbb{N}_0} P_n(\lambda)$, $\lambda \in \mathbb{R}$. Further, we introduce the subspaces

$$\mathfrak{h}_{n_\alpha}(\lambda) := \mathfrak{h}_\alpha^{el}(\lambda - n\omega), \quad \lambda \in \mathbb{R}, \quad n \in \mathbb{N}_0.$$

Notice that

$$\mathfrak{h}_n(\lambda) = \bigoplus_{\alpha \in \{l, r\}} \mathfrak{h}_{n_\alpha}(\lambda), \quad \lambda \in \mathbb{R}, \quad n \in \mathbb{N}_0.$$

By $P_{n_\alpha}(\lambda)$ we denote the orthogonal projection from $\mathfrak{h}(\lambda)$ onto $\mathfrak{h}_{n_\alpha}(\lambda)$, $\lambda \in \mathbb{R}$. Obviously, we have $P_n(\lambda) = \sum_{\alpha \in \{l, r\}} P_{n_\alpha}(\lambda)$, $\lambda \in \mathbb{R}$.

Example 2.13 In general the direct integral $\Pi(H_0^{ac})$ can be very complicated, in particular, the structure of $\mathfrak{h}(\lambda)$ given by (2.41) is difficult to analyze. However, there are interesting simple cases:

(i) Let $v = v_l = v_r$ and $4 \leq \omega$. In this case we have $\mathfrak{h}^{el}(\lambda) = \mathbb{C}^2$ for $[v, v + 4]$ and

$$\mathfrak{h}(\lambda) = \begin{cases} \mathbb{C}^2, & \lambda \in [v + n\omega, v + n\omega + 4], \quad n \in \mathbb{N}_0, \\ \{0\}, & \text{otherwise.} \end{cases}$$

(ii) Let $v_r = 0$, $v_l = 4$, $\omega_0 = 4$. Then

$$\mathfrak{h}(\lambda) = \begin{cases} \mathfrak{h}_r^{el}(\lambda) = \mathbb{C}, & \lambda \in [0, 4), \\ \mathfrak{h}_{lr}^{el}(\lambda) = \mathbb{C}^2, & \lambda \in [4, 8), \\ \mathfrak{h}_{rl}^{el}(\lambda) = \mathbb{C}^2, & \lambda \in [8, 12), \\ \dots & \end{cases}$$

where

$$\mathfrak{h}_{\alpha\alpha'}^{el}(\lambda) = \begin{matrix} \mathfrak{h}_\alpha^{el}(\lambda) \\ \oplus \\ \mathfrak{h}_{\alpha'}^{el}(\lambda) \end{matrix}, \quad \alpha, \alpha' \in \{l, r\}, \quad \alpha \neq \alpha'.$$

Hence $\dim(\mathfrak{h}(\lambda)) = 2$ for $\lambda \geq 4$. □

Let Z be a bounded operator acting on $\mathfrak{H}^{ac}(H_0)$ and commuting with H_0^{ac} . Since Z commutes with H_0^{ac} there is a measurable family $\{Z(\lambda)\}_{\lambda \in \mathbb{R}}$ of bounded operators acting on $\mathfrak{h}(\lambda)$ such that Z is unitarily equivalent to the multiplication operator induced by $\{Z(\lambda)\}_{\lambda \in \mathbb{R}}$ in $\Pi(H_0^{ac})$. We set

$$Z_{m_\alpha n_\varkappa}(\lambda) := P_{m_\alpha}(\lambda)Z(\lambda) \upharpoonright \mathfrak{h}_{n_\varkappa}(\lambda), \quad \lambda \in \mathbb{R}, \quad m, n \in \mathbb{N}_0, \quad \alpha, \varkappa \in \{l, r\}.$$

Let $Z_{m_\alpha n_\varkappa} := P_{m_\alpha} Z P_{n_\varkappa}$ where P_{m_α} is the orthogonal projection from \mathfrak{H} onto $\mathfrak{H}_{m_\alpha} \subseteq \mathfrak{H}^{ac}(H_0)$, cf. (2.22). Obviously, the multiplication operator induced $\{Z_{m_\alpha n_\varkappa}(\lambda)\}_{\lambda \in \mathbb{R}}$ in $\Pi(H_0^{ac})$ is unitarily equivalent to $Z_{m_\alpha n_\varkappa}$.

Since by Lemma 2.12 $\mathfrak{h}(\lambda)$ is a finite dimensional space, the operators $Z(\lambda)$ are finite dimensional ones and we can introduce the quantity

$$\sigma_{m_\alpha n_\varkappa}(\lambda) = \text{tr}(Z_{m_\alpha n_\varkappa}(\lambda)^* Z_{m_\alpha n_\varkappa}(\lambda)), \quad \lambda \in \mathbb{R}, \quad m, n \in \mathbb{N}_0, \quad \alpha, \varkappa \in \{l, r\}.$$

Lemma 2.14 Let H_0 be the self-adjoint operator defined by (2.19) on \mathfrak{H} . Further let Z be a bounded operator on $\mathfrak{H}^{ac}(H_0)$ commuting with H_0^{ac}

(i) Let Γ be a conjugation on \mathfrak{H} , cf. Section 2.3. If Γ commutes with H_0 and P_{n_α} , $n \in \mathbb{N}_0$, $\alpha \in \{l, r\}$ and $\Gamma Z \Gamma = Z^*$ holds, then $\sigma_{m_\alpha n_\varkappa}(\lambda) = \sigma_{n_\varkappa m_\alpha}(\lambda)$, $\lambda \in \mathbb{R}$.

(ii) Let U be a mirror symmetry on \mathfrak{H} . If U commutes with H_0 and Z , then $\sigma_{m_\alpha n_\varkappa}(\lambda) = \sigma_{m_{\alpha'} n_{\varkappa'}}(\lambda)$, $\lambda \in \mathbb{R}$, $m, n \in \mathbb{N}_0$, $\alpha, \alpha', \varkappa, \varkappa' \in \{l, r\}$, $\alpha \neq \alpha'$, $\varkappa \neq \varkappa'$.

Proof. (i) Since Γ commutes with H_0 the conjugation Γ is reduce by $\mathfrak{H}^{ac}(H_0)$. So without loss of generality we assume that Γ acts on $\mathfrak{H}^{ac}(H_0)$. We set $\Gamma_{n_\alpha} := \Gamma \upharpoonright \mathfrak{H}_{n_\alpha}$. Notice that

$$\Gamma = \bigoplus_{n \in \mathbb{N}_0, \alpha \in \{l, r\}} \Gamma_{n_\alpha}.$$

There is a measurable family $\{\Gamma(\lambda)\}_{\lambda \in \mathbb{R}}$ of conjugations such that the multiplication operator induced by $\{\Gamma(\lambda)\}_{\lambda \in \mathbb{R}}$ in $\Pi(H_0^{ac})$ is unitarily equivalent to Γ . Moreover, since Γ commutes with P_{n_α} we get that the multiplication operator induced by the measurable family

$$\Gamma_{n_\alpha}(\lambda) := \Gamma(\lambda) \upharpoonright \mathfrak{h}_{n_\alpha}(\lambda), \quad \lambda \in \mathbb{R}, \quad m \in \mathbb{N}_0, \quad \alpha \in \{l, r\},$$

is unitarily equivalent to Γ_{n_α} . Using $\Gamma Z \Gamma = Z^*$ we get $\Gamma_{m_\alpha} Z_{m_\alpha n_\alpha} \Gamma_{n_\alpha} = Z_{n_\alpha m_\alpha}^*$. Hence

$$\Gamma_{m_\alpha}(\lambda) Z_{m_\alpha n_\alpha}(\lambda) \Gamma_{n_\alpha}(\lambda) = Z_{n_\alpha m_\alpha}(\lambda)^*, \quad \lambda \in \mathbb{R}. \quad (2.43)$$

If X is trace class operator, then $\text{tr}(\Gamma X \Gamma) = \overline{\text{tr}(X)}$. Using that we find

$$\sigma_{m_\alpha n_\alpha}(\lambda) = \frac{\text{tr}(\Gamma_{n_\alpha}(\lambda) Z_{m_\alpha n_\alpha}(\lambda)^* Z_{m_\alpha n_\alpha}(\lambda) \Gamma_{n_\alpha}(\lambda))}{\text{tr}(\Gamma_{n_\alpha}(\lambda) Z_{m_\alpha n_\alpha}(\lambda)^* \Gamma_{m_\alpha} \Gamma_{m_\alpha} Z_{m_\alpha n_\alpha}(\lambda) \Gamma_{n_\alpha}(\lambda))}$$

From (2.43) we obtain

$$\sigma_{m_\alpha n_\alpha}(\lambda) = \overline{\text{tr}(Z_{n_\alpha m_\alpha}(\lambda) Z_{n_\alpha m_\alpha}(\lambda)^*)} = \sigma_{n_\alpha m_\alpha}(\lambda), \quad \lambda \in \mathbb{R},$$

which proves (i).

(ii) Again without loss of generality we can assume that U acts only $\mathfrak{H}^{ac}(H_0)$. Since U commutes with H_0 there is a measurable family $\{U(\lambda)\}_{\lambda \in \mathbb{R}}$ of unitary operators acting on $\mathfrak{h}(\lambda)$ such that the multiplication operator induced by $\{U(\lambda)\}_{\lambda \in \mathbb{R}}$ is unitarily equivalent to U . Since $U \mathfrak{H}_{n_\alpha} = \mathfrak{H}_{n_{\alpha'}}$ we have $U(\lambda) \mathfrak{h}_{n_\alpha}(\lambda) = \mathfrak{h}_{n_{\alpha'}}(\lambda)$, $\lambda \in \mathbb{R}$. Hence

$$\begin{aligned} \sigma_{m_\alpha n_\alpha}(\lambda) &= \text{tr}(U(\lambda) Z_{m_\alpha n_\alpha}(\lambda)^* Z_{m_\alpha n_\alpha}(\lambda) U(\lambda)^*) = \\ &= \text{tr}(U(\lambda) Z_{m_\alpha, n_\alpha}(\lambda)^* U(\lambda)^* U(\lambda) Z_{m_\alpha, n_\alpha}(\lambda) U(\lambda)^*). \end{aligned}$$

Hence

$$\sigma_{m_\alpha n_\alpha}(\lambda) = \text{tr}(P_{n_{\alpha'}} U(\lambda) Z(\lambda)^* U(\lambda)^* P_{m_{\alpha'}}(\lambda) U(\lambda) Z(\lambda) U(\lambda)^* P_{n_{\alpha'}}(\lambda)).$$

Since U commutes with Z we find

$$\sigma_{m_\alpha n_\alpha}(\lambda) = \text{tr}(P_{n_{\alpha'}} Z(\lambda)^* P_{m_{\alpha'}}(\lambda) Z(\lambda) P_{n_{\alpha'}}(\lambda)) = \sigma_{m_{\alpha'} n_{\alpha'}}(\lambda), \quad \lambda \in \mathbb{R}.$$

which proves (ii). □

2.7 Spectral properties of H : second part

Since we have full information on the spectral properties of H_0 we can use this to show that H has no singular continuous spectrum. Crucial for that is the following lemma: with the help of [6, Cor. IV.15.19], which establishes existence and completeness of wave operators and absence of singular continuous spectrum through a time-falloff method. We cite it as a Lemma for convenience, with slight simplifications that suffice for our purpose.

Lemma 2.15 ([6, Corollary IV.15.19]) *Let $\{H_0, H\}$ be a scattering system and let Λ be a closed countable set. Let F_+ and F_- be two self-adjoint operators such that $F_+ + F_- = P_{H_0}^{ac}$ and*

$$s - \lim_{t \rightarrow \infty} e^{\mp it H_0} F_{\pm} e^{\pm it H_0} = 0.$$

If $(H - i)^{-1} - (H_0 - i)^{-1} \in \mathfrak{L}_\infty(\mathfrak{H})$, $(1 - P_{H_0}^{ac})\gamma(H_0) \in \mathfrak{L}_\infty(\mathfrak{H})$, and

$$\left| \int_0^{\pm\infty} dt \left\| \left((H_0 - i)^{-1} - (H - i)^{-1} \right) e^{-itH_0} \gamma(H_0) F_\pm \right\| \right| < \infty$$

for all $\gamma \in C_0^\infty(\mathbb{R} \setminus \Lambda)$, then $W_\pm(H, H_0)$ exist and are complete and $\sigma_{sc}(H) = \sigma_{sc}(H_0) = \emptyset$. Furthermore, each eigenvalue of H and H_0 in $\mathbb{R} \setminus \Lambda$ is of finite multiplicity and these eigenvalues accumulate at most at points of Λ or at $\pm\infty$.

We already know that the wave operators exist and are complete since the resolvent difference is trace class. Hence, we need Lemma 2.15 only to prove the following proposition.

Proposition 2.16 *The Hamiltonian H defined by (2.21) has no singular continuous spectrum, that is, $\sigma_{sc}(H) = \emptyset$.*

Proof. At first we have to construct the operators F_\pm . To this end, let $\mathcal{F} : L^2(\mathbb{R}) \rightarrow L^2(\mathbb{R})$ be the usual Fourier transform, i.e

$$(\mathcal{F}f)(\mu) := \widehat{f}(\mu) := \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} e^{-i\mu x} f(x) dx, \quad f \in L^2(\mathbb{R}, dx), \quad \mu \in \mathbb{R}.$$

Further, let Π_\pm be the orthogonal projection onto $L^2(\mathbb{R}_\pm)$ in $L^2(\mathbb{R})$. We set

$$F_\pm = \Phi^* \mathcal{F} \Pi_\pm \mathcal{F}^* \Phi$$

where Φ is given by (2.42). We immediately obtain $F_- + F_+ = P_{ac}(H_0)$. We still have to show that

$$s - \lim_{t \rightarrow \infty} \| e^{\mp itH_0} \Phi^* \mathcal{F} \Pi_\pm \mathcal{F}^* \Phi e^{\pm itH_0} f \| = 0$$

for $f \in \mathfrak{H}^{ac}(H_0)$. We prove the relation only for F_+ since the proof for F_- is essentially identical. We have

$$(\Pi_+ \mathcal{F}^* \Phi e^{itH_0} f)(x) = (2\pi)^{-\frac{1}{2}} \chi_{\mathbb{R}_+}(x) \int_{\mathbb{R}} d\mu e^{i(x+t)\mu} \widehat{f}(\mu) = \chi_{\mathbb{R}_+}(x) \psi(x+t)$$

with $\psi = \mathcal{F}\widehat{f}$. Now

$$\begin{aligned} \| e^{-itH_0} \Phi^* \mathcal{F} \Pi_+ \mathcal{F}^* \Phi e^{itH_0} f \|^2 &= \\ \| \Pi_+ \mathcal{F}^* \Phi e^{itH_0} f \|^2 &= \int_{\mathbb{R}_+} dx |\psi(x+t)|^2 = \int_t^\infty dx |\psi(x)|^2 \xrightarrow{t \rightarrow \infty} 0. \end{aligned}$$

Concerning the compactness condition, we already know that $(H - i)^{-1} - (H_0 - i)^{-1} \in \mathfrak{L}_1(\mathfrak{H}) \subset \mathfrak{L}_\infty(\mathfrak{H})$ from Proposition 2.9. Let

$$\Lambda = \bigcup_{n \in \mathbb{N}_0} \{v_l + n\omega, v_r + n\omega, v_l + 4 + n\omega, v_r + 4 + n\omega\},$$

which is closed and countable. We know from Corollary 2.11 that H_0 has no singular continuous spectrum and the eigenvalues are of finite multiplicity. It follows that $(1 - P_{ac}(H_0))\gamma(H_0)$ is compact for every $\gamma \in C_0^\infty(\mathbb{R} \setminus \Lambda)$. The remaining assumption of Lemma 2.15 is

$$\left| \int_0^{\pm\infty} dt \left\| \left((H - i)^{-1} - (H_0 - i)^{-1} \right) \gamma(H_0) e^{-itH_0} F_\pm \right\| \right| < \infty.$$

If we can prove this, then we immediately obtain that H has no singular continuous spectrum. Now $(H - i)^{-1} - (H_0 - i)^{-1} = (H - i)^{-1} (V_{el} + V_{ph})(H_0 - i)^{-1}$. But $(H - i)^{-1}$ is bounded,

$$\text{ran}(F_\pm) \subset \mathfrak{H}^{ac}(H_0) = (\mathfrak{h}_l^{el} \oplus \mathfrak{h}_r^{el}) \otimes \mathfrak{h}^{ph},$$

and $V_{ph}P^{ac}(H_0) = 0$. Also, $V_{el} = v_{el} \otimes I_{\mathfrak{h}^{ph}}$ and

$$\ker(v_{el})^\perp \subset \mathbb{C}\delta_1^l \oplus \mathfrak{h}_S \oplus \mathbb{C}\delta_1^r.$$

Hence, it suffices to prove

$$\left| \int_0^{\pm\infty} dt \left\| P_1^\alpha (H_0 - i)^{-1} \gamma(H_0) e^{-itH_0} F_\pm \right\| \right| < \infty,$$

$\alpha \in \{l, r\}$, where $P_1^\alpha = p_1^\alpha \otimes I_{\mathfrak{h}^{ph}}$ and p_1^α is the orthogonal projection onto \mathfrak{h}_α^{el} . In the following we treat only the case F_+ . The calculations for F_- are completely analogous. We use that Φ maps H_0^{ac} into the multiplication operator \mathcal{M} induced by λ . Hence we get

$$\begin{aligned} & \left\| P_1^\alpha \tilde{\gamma}(H_0) e^{-itH_0} \Phi^* \mathcal{F} f \right\| = \left\| P_1^\alpha \Phi^* \Phi \tilde{\gamma}(H_0) e^{-itH_0} \Phi^* \mathcal{F} f \right\| = \\ & = (2\pi)^{-\frac{1}{2}} \left(\sum_{n \in \mathbb{N}_0} \left| \int_{\delta_{\alpha,n}} d\lambda g_\alpha(1, \lambda - n\omega) \tilde{\gamma}(\lambda) \int_{\mathbb{R}_+} dx e^{-i\lambda(x+t)} f(x) \right|^2 \right)^{\frac{1}{2}} \end{aligned}$$

where $\text{supp}(f) \subseteq \mathbb{R}_+$, $\tilde{\gamma}(\lambda) := (\lambda - i)^{-1} \gamma(\lambda)$, $\lambda \in \mathbb{R}$, and $\delta_{\alpha,n} := [v_\alpha + n\omega_0, v_\alpha + n\omega + 4]$. Notice that $\tilde{\gamma}(\lambda) \in C_0^\infty(\mathbb{R} \setminus \Lambda)$. We find

$$\begin{aligned} & \int_{\delta_{j,n}} d\lambda g_\alpha(1, \lambda - n\omega) \tilde{\gamma}(\lambda) \int_{\mathbb{R}_+} dx e^{-i\lambda(x+t)} f(x) = \\ & \int_{v_\alpha}^{v_\alpha+4} d\lambda g_\alpha(1, \lambda) \tilde{\gamma}(\lambda + n\omega) \int_{\mathbb{R}_+} dx e^{-i(\lambda+n\omega)(x+t)} f(x) \end{aligned}$$

which yields

$$\begin{aligned} & \left\| P_1^\alpha \Phi^* \Phi \tilde{\gamma}(H_0) e^{-itH_0} \Phi^* \mathcal{F} f \right\| = \\ & = (2\pi)^{-\frac{1}{2}} \left(\sum_{n \in \mathbb{N}_0} \left| \int_{v_\alpha}^{v_\alpha+4} d\lambda g_\alpha(1, \lambda) \tilde{\gamma}(\lambda + n\omega) \times \right. \right. \\ & \left. \left. \int_{\mathbb{R}_+} dx e^{-i(\lambda+n\omega)(x+t)} f(x) \right|^2 \right)^{\frac{1}{2}}. \end{aligned}$$

Since the support of $\gamma(\lambda)$ is compact we get that the sum $\sum_{n \in \mathbb{N}_0}$ is finite. Changing the integrals we get

$$\begin{aligned} & \int_{\delta_{\alpha,n}} d\lambda g_\alpha(1, \lambda - n\omega) \tilde{\gamma}(\lambda) \int_{\mathbb{R}_+} dx e^{-i\lambda(x+t)} f(x) = \\ & \int_{\mathbb{R}_+} dx f(x) e^{-in\omega_0(x+t)} \int_{v_\alpha}^{v_\alpha+4} d\lambda g_\alpha(1, \lambda) \tilde{\gamma}(\lambda + n\omega) e^{-i\lambda(x+t)} \end{aligned}$$

Integrating by parts m -times we obtain

$$\begin{aligned} & \int_{\delta_{\alpha,n}} d\lambda g_\alpha(1, \lambda - n\omega) \tilde{\gamma}(\lambda) \int_{\mathbb{R}_+} dx e^{-i\lambda(x+t)} f(x) = \\ & (-i)^m \int_{\mathbb{R}_+} dx f(x) \frac{e^{-in\omega_0(x+t)}}{(x+t)^m} \int_{v_\alpha}^{v_\alpha+4} d\lambda e^{-i\lambda(x+t)} \frac{d^m}{d\lambda^m} (g_\alpha(1, \lambda) \tilde{\gamma}(\lambda + n\omega)) \end{aligned}$$

Hence

$$\begin{aligned} & \left| \int_{\delta_{\alpha,n}} d\lambda g_\alpha(1, \lambda - n\omega) \tilde{\gamma}(\lambda) \int_{\mathbb{R}_+} dx e^{-i\lambda(x+t)} f(x) \right|^2 \\ & \leq C_n^2 \left(\int_{\mathbb{R}_+} dx |f(x)| \frac{1}{(x+t)^m} \right)^2 \end{aligned}$$

which yields

$$\begin{aligned} & \left| \int_{\delta_{\alpha,n}} d\lambda g_{\alpha}(1, \lambda - n\omega) \tilde{\gamma}(\lambda) \int_{\mathbb{R}_+} dx e^{-i\lambda(x+t)} f(x) \right|^2 \\ & \leq C_n^2 \frac{1}{t^{(2m-1)}} \|f\|^2 \end{aligned}$$

for $m \in \mathbb{N}$ where

$$C_n := \int_{v_{\alpha}}^{v_{\alpha}+4} d\lambda \left| \frac{d^m}{d\lambda^m} \left(g_{\alpha}(1, \lambda) \tilde{\gamma}(\lambda + n\omega) \right) \right|.$$

Notice that $C_n = 0$ for sufficiently large $n \in \mathbb{N}$. Therefore

$$\|P_1^{\alpha} \tilde{\gamma}(H_0) e^{-itH_0} \Phi^* \mathcal{F}f\| \leq \left(\sum_{n \in \mathbb{N}_0} C_n^2 \right)^{1/2} \frac{1}{t^{m-1/2}} \|f\|, \quad f \in L^2(\mathbb{R}_+, dx),$$

which shows that $\|P_1^{\alpha} \tilde{\gamma}(H_0) e^{-itH_0} F_+\| \in L^1(\mathbb{R}_+, dt)$ for $m \geq 2$. \square

3 Landauer-Büttiker formula and applications

3.1 Landauer-Büttiker formula

The abstract Landauer-Büttiker formula can be used to calculate flows through devices. Usually one considers a pair $\mathcal{S} = \{K, K_0\}$ be of self-adjoint operators where the unperturbed Hamiltonian K_0 describes a totally decoupled system, that means, the inner system is closed and the leads are decoupled from it, while the perturbed Hamiltonian K describes the system where the leads are coupled to the inner system. An important ingredient is system $\mathcal{S} = \{K, K_0\}$ is represents a complete scattering or even a trace class scattering system.

In [1] an abstract Landauer-Büttiker formula was derived in the framework of a trace class scattering theory for semi-bounded self-adjoint operators which allows to reproduce the results of [18] and [7] rigorously. In [13] the results of [1] were generalized to non-semi-bounded operators. Following [1] we consider a trace class scattering system $\mathcal{S} = \{K, K_0\}$. We recall that $\mathcal{S} = \{K, K_0\}$ is called a trace class scattering system if the resolvent difference of K and K_0 belongs to the trace class. If $\mathcal{S} = \{K, K_0\}$ is a trace class scattering system, then the wave operators $W_{\pm}(K, K_0)$ exists and are complete. The scattering operator is defined by $S(K, K_0) := W_+(K, K_0)^* W_-(K, K_0)$. The main ingredients besides the trace class scattering system $\mathcal{S} = \{K, K_0\}$ are the density and the charge operators ρ and Q , respectively.

The density operator ρ is a non-negative bounded self-adjoint operator commuting with K_0 . The charge Q is a bounded self-adjoint operator commuting also with K_0 . If K has no singular continuous spectrum, then the current related to the density operator ρ and the charge Q is defined by

$$J_{\rho, Q}^{\mathcal{S}} = -i \operatorname{tr} (W_-(K, K_0) \rho W_-(K, K_0)^* [K, Q]) \quad (3.1)$$

where $[K, Q]$ is the commutator of K and Q . In fact, the commutator $[K, Q]$ might be not defined. In this case the regularized definition

$$J_{\rho, Q}^{\mathcal{S}} = -i \operatorname{tr} \left(W_-(K, K_0) (I + K_0^2) \rho W_-(K, K_0)^* \frac{1}{K - i} [K, Q] \frac{1}{K + i} \right) \quad (3.2)$$

is used where it is assumed that $(I + K_0^2) \rho$ is a bounded operator. Since the condition $(H - i)^{-1} [H, Q] (H + i)^{-1} \in \mathfrak{L}_1(\mathfrak{H})$ is satisfied the definition (3.2) makes sense. By $\mathfrak{L}_1(\mathfrak{H})$ is the ideal of trace class operators is denoted.

Let K_0 be self-adjoint operator on the separable Hilbert space \mathfrak{K} . We call ρ be a density operator for K_0 if ρ is a bounded non-negative self-adjoint operator commuting with K_0 . Since ρ commutes with K_0 one gets that ρ leave invariant the subspace $\mathfrak{K}^{ac}(K_0)$. We set

$$\rho_{ac} := \rho \upharpoonright \mathfrak{K}^{ac}(K_0).$$

call ρ_{ac} the ac -density part of ρ .

A bounded self-adjoint operator Q commuting with K_0 is called a charge. If Q is a charge, then

$$Q_{ac} := Q \upharpoonright \mathfrak{K}^{ac}(K_0).$$

is called its ac -charge part.

Let $\Pi(K_0^{ac}) = \{L^2(\mathbb{R}, d\lambda, \mathfrak{k}(\lambda)), \mathcal{M}, \Phi\}$ be a spectral representation of K_0^{ac} . If ρ is a density operator, then there is a measurable family $\{\rho_{ac}(\lambda)\}_{\lambda \in \mathbb{R}}$ of bounded self-adjoint operators such that the multiplication operator

$$(\mathcal{M}_{\rho_{ac}} \widehat{f})(\lambda) := \rho_{ac}(\lambda) \widehat{f}(\lambda), \quad \widehat{f} \in \text{dom}(\mathcal{M}_{\rho_{ac}}) := L^2(\mathbb{R}, d\lambda, \mathfrak{k}(\lambda)),$$

is unitarily equivalent to ac -part ρ_{ac} , that is, $\mathcal{M}_{\rho_{ac}} = \Phi \rho_{ac} \Phi^*$. In particular this yields that $\text{ess-sup}_{\lambda \in \mathbb{R}} \|\rho_{ac}(\lambda)\|_{\mathcal{B}(\mathfrak{k}(\lambda))} = \|\rho_{ac}\|_{\mathcal{B}(\mathfrak{K}^{ac}(K_0))}$. In the following we call $\{\rho_{ac}(\lambda)\}_{\lambda \in \mathbb{R}}$ the density matrix of ρ_{ac} .

Similarly, one gets that if Q , then there is a measurable family $\{Q_{ac}(\lambda)\}_{\lambda \in \mathbb{R}}$ of bounded self-adjoint operators such that the multiplication operator

$$\begin{aligned} (\mathcal{M}_{Q_{ac}} \widehat{f})(\lambda) &:= Q_{ac}(\lambda) \widehat{f}(\lambda), \\ \widehat{f} \in \text{dom}(Q_{ac}) &:= \{f \in L^2(\mathbb{R}, d\lambda, \mathfrak{k}(\lambda)) : Q_{ac}(\lambda) \widehat{f}(\lambda) \in L^2(\mathbb{R}, d\lambda, \mathfrak{k}(\lambda))\}, \end{aligned}$$

is unitarily equivalent to Q_{ac} , i.e. $\mathcal{M}_{Q_{ac}} = \Phi Q_{ac} \Phi^*$. In particular, one has

$$\text{ess-sup}_{\lambda \in \mathbb{R}} \|Q_{ac}(\lambda)\|_{\mathcal{B}(\mathfrak{k}(\lambda))} = \|Q_{ac}\|_{\mathcal{B}(\mathfrak{K}^{ac}(K_0))}. \quad (3.3)$$

If Q is a charge, then the family $\{Q_{ac}(\lambda)\}_{\lambda \in \mathbb{R}}$ is called the charge matrix of the ac -part of Q .

Let $S = \{K, K_0\}$ be a trace scattering system. By $\{S(\lambda)\}_{\lambda \in \mathbb{R}}$ we denote the scattering matrix which corresponds to the scattering operator $S(K, K_0)$ with respect to the spectral representation $\Pi(K_0^{ac})$. The operator $T := S(K, K_0) - P^{ac}(K_0)$ is called the transmission operator. By $\{T(\lambda)\}_{\lambda \in \mathbb{R}}$ we denote the transmission which is related to the transmission operator. Scattering and transmission matrix are related by $S(\lambda) = T_{\mathfrak{k}(\lambda)} + T(\lambda)$ for a.e. $\lambda \in \mathbb{R}$. Notice that $T(\lambda)$ belongs for to the trace class a.e. $\lambda \in \mathbb{R}$.

Theorem 3.1 ([13, Corollary 2.14]) *Let $S := \{K, K_0\}$ be a trace class scattering system and let $\{S(\lambda)\}_{\lambda \in \mathbb{R}}$ be the scattering matrix of S with respect to the spectral representation $\Pi(K_0^{ac})$. Further let ρ and Q be density and charge operators and let $\{\rho_{ac}(\lambda)\}_{\lambda \in \mathbb{R}}$ and $\{Q_{ac}(\lambda)\}_{\lambda \in \mathbb{R}}$ be the density and charge matrices of the ac -parts ρ_{ac} and charge Q_{ac} with respect to $\Pi(K_0^{ac})$, respectively. If $(I + K_0^2)\rho$ is bounded, then the current $J_{\rho, Q}^S$ defined by (3.2) admits the representation*

$$J_{\rho, Q}^S = \frac{1}{2\pi} \int_{\mathbb{R}} \text{tr}(\rho_{ac}(\lambda)(Q_{ac}(\lambda) - S^*(\lambda)Q_{ac}(\lambda)S(\lambda))) d\lambda \quad (3.4)$$

where the integrand on the right hand side and the current $J_{\rho, Q}^S$ satisfy the estimate

$$|\text{tr}(\rho_{ac}(\lambda)(Q_{ac}(\lambda) - S^*(\lambda)Q_{ac}(\lambda)S(\lambda)))| \leq 4\|\rho(\lambda)\|_{\mathfrak{L}(\mathfrak{k}(\lambda))} \|T(\lambda)\|_{\mathfrak{L}_1(\mathfrak{k}(\lambda))} \|Q(\lambda)\|_{\mathfrak{L}(\mathfrak{k}(\lambda))} \quad (3.5)$$

for a.e. $\lambda \in \mathbb{R}$ and

$$|J_{\rho, Q}^S| \leq C_0 \|(H + i)^{-1} - (H_0 + i)^{-1}\|_{\mathfrak{L}_1(\mathfrak{K})} \quad (3.6)$$

where $C_0 := \frac{2}{\pi} \|(1 + H_0^2)\rho\|_{\mathfrak{L}(\mathfrak{K})}$.

In applications not every charge Q is a bounded operator. We say the self-adjoint operator Q commuting with K_0 is a p -tempered charge if $Q(H_0 - i)^{-p}$ is a bounded operator for $p \in \mathbb{N}_0$. As above we can introduce $Q_{ac} := Q \upharpoonright \text{dom}(Q) \cap \mathfrak{K}^{ac}(K_0)$. It turns out that $QE_{K_0}(\Delta)$ is a bounded operator for any bounded Borel set Δ . This yields that the corresponding charge matrix $\{Q_{ac}(\lambda)\}_{\lambda \in \mathbb{R}}$ is a measurable family of bounded self-adjoint operators such that

$$\text{ess-sup}_{\lambda \in \mathbb{R}} (1 + \lambda^2)^{p/2} \|Q_{ac}(\lambda)\|_{\mathfrak{L}(\mathfrak{k}(\lambda))} < \infty.$$

To generalize the current $J_{\rho, Q}^S$ to tempered charges Q one uses the fact that $Q(\Delta) := QE_{K_0}(\Delta)$ is a charge for any bounded Borel set Δ . Hence the current $J_{\rho, Q(\Delta)}^S$ is well-defined by (3.2) for any bounded Borel set Δ . Using Theorem 3.1 one gets that for p -tempered charges the limit

$$J_{\rho, Q}^S := \lim_{\Delta \rightarrow \mathbb{R}} J_{\rho, Q(\Delta)}^S \quad (3.7)$$

exists provided $(H_0 - i)^{p+2}\rho$ is a bounded operator. This gives rise for the following corollary.

Corollary 3.2 *Let the assumptions of the Theorem 3.1 be satisfied. If for some $p \in \mathbb{N}_0$ the operator $(H_0 - i)^{p+2}\rho$ is bounded and Q is a p -tempered charge for K_0 , then the current defined by (3.7) admits the representation (3.4) where the right hand side of (3.4) satisfies the estimate (3.5). Moreover, the current $J_{\rho, Q}^S$ can be estimated by*

$$|J_{\rho, Q}^S| \leq C_p \|(H + i)^{-1} - (H_0 + i)^{-1}\|_{\mathfrak{L}_1(\mathfrak{K})} \quad (3.8)$$

where $C_p := \frac{2}{\pi} \|(1 + H_0^2)^{p+2/2}\rho\|_{\mathfrak{L}(\mathfrak{K})} \|Q(I + H_0^2)^{-p/2}\|_{\mathfrak{L}(\mathfrak{K})}$.

At first glance the formula (3.4) is not very similar to the original Landauer-Büttiker formula of [7, 18]. To make the formula more convenient we recall that a standard application example for the Landauer-Büttiker formula is the so-called black-box model, cf. [1]. In this case the Hilbert space \mathfrak{K} is given by

$$\mathfrak{K} = \mathfrak{K}_S \oplus \bigoplus_{j=1}^N \mathfrak{K}_j, \quad 2 \leq N < \infty. \quad (3.9)$$

and K_0 by

$$K_0 = K_S \oplus \bigoplus_{j=1}^N K_j, \quad 2 \leq N < \infty. \quad (3.10)$$

The Hilbert space \mathfrak{K}_S is called the sample or dot and K_S is the sample or dot Hamiltonian. The Hilbert spaces \mathfrak{K}_j are called reservoirs or leads and K_j are the reservoir or lead Hamiltonians. For simplicity we assume that the reservoir Hamiltonians K_j are absolutely continuous and the sample Hamiltonian K_S has point spectrum. A typical choice for the density operator is

$$\rho = f_S(K_S) \oplus \bigoplus_{j=1}^N f_j(K_j), \quad (3.11)$$

where $f_S(\cdot)$ and $f_j(\cdot)$ are non-negative bounded Borel functions, and for the charge

$$Q = g_S(H_S) \oplus \bigoplus_{j=1}^N g_j(H_j), \quad (3.12)$$

where $g_S(\cdot)$ and $g_j(\cdot)$ a bounded Borel functions. Making this choice the Landauer-Büttiker formula (3.4) takes the form

$$J_{\rho, Q}^S = \frac{1}{2\pi} \sum_{j, k=1}^N \int_{\mathbb{R}} (f_j(\lambda) - f_k(\lambda)) g_j(\lambda) \sigma_{jk}(\lambda) d\lambda \quad (3.13)$$

where

$$\sigma_{jk}(\lambda) := \text{tr}(T_{jk}(\lambda)^* T_{jk}(\lambda)), \quad j, k = 1, \dots, N, \quad \lambda \in \mathbb{R}, \quad (3.14)$$

are called the *total transmission probability* from reservoir k to reservoir j , cf. [1]. We call it the *cross-section* of the scattering process going from channel k to channel j at energy $\lambda \in \mathbb{R}$. $\{T_{jk}(\lambda)\}_{\lambda \in \mathbb{R}}$ is called the transmission matrix from channel k to channel j at energy $\lambda \in \mathbb{R}$ with respect to the spectral representation $\Pi(K_0^{ac})$. We note that $\{T_{jk}(\lambda)\}_{\lambda \in \mathbb{R}}$ corresponds to the transmission operator

$$T_{jk} := P_j T(K, K_0) P_k, \quad T(K, K_0) := S(K, K_0) - P^{ac}(K_0), \quad (3.15)$$

acting from the reservoir k to reservoir j where $T(K, K_0)$ is called the transmission operator. Let $\{T(\lambda)\}_{\lambda \in \mathbb{R}}$ be the transmission matrix. Following [1] the current $J_{\rho, Q}^S$ given by (3.13) is directed from the reservoirs into the sample.

The quantity $\|T(\lambda)\|_{\mathfrak{L}_2} = \text{tr}(T(\lambda)^* T(\lambda))$ is well-defined and is called the cross-section of the scattering system \mathcal{S} at energy $\lambda \in \mathbb{R}$. Notice that

$$\sigma(\lambda) = \|T(\lambda)\|_{\mathfrak{L}_2} = \text{tr}(T(\lambda)^* T(\lambda)) = \sum_{j,k=1}^N \sigma_{jk}(\lambda). \quad \lambda \in \mathbb{R},$$

We point out that the channel cross-sections $\sigma_{jk}(\lambda)$ admit the property

$$\sum_{j=1}^N \sigma_{jk}(\lambda) = \sum_{j=1}^N \sigma_{kj}(\lambda), \quad \lambda \in \mathbb{R}, \quad (3.16)$$

which is a consequence of the unitarity of the scattering matrix. Moreover, if there is a conjugation J such that $KJ = JK$ and $K_0J = JK_0$ holds, that is, if the scattering system \mathcal{S} is time reversible symmetric, then we have even more, namely, it holds

$$\sigma_{jk}(\lambda) = \sigma_{kj}(\lambda), \quad \lambda \in \mathbb{R}. \quad (3.17)$$

Usually the Landauer-Büttiker formula (3.13) is used to calculate the electron current entering the reservoir j from the sample. In this case one has to choose $Q := Q_j^{\epsilon l} := -\epsilon P_j$ where P_j is the orthogonal projection from \mathfrak{K} onto \mathfrak{K}_j and $\epsilon > 0$ is the magnitude of the elementary charge. This is equivalent to choose $g_j(\lambda) = -\epsilon$ and $g_k(\lambda) = 0$ for $k \neq j$, $\lambda \in \mathbb{R}$. Doing so we get the Landauer-Büttiker formula simplifies to

$$J_{\rho, Q_j^{\epsilon l}}^S = -\frac{\epsilon}{2\pi} \sum_{k=1}^N \int_{\mathbb{R}} (f_j(\lambda) - f_k(\lambda)) \sigma_{jk}(\lambda) d\lambda. \quad (3.18)$$

To restore the original Landauer-Büttiker formula one sets

$$f_j(\lambda) = f(\lambda - \mu_j), \quad \lambda \in \mathbb{R}, \quad (3.19)$$

where μ_j is the chemical potential of the reservoir \mathfrak{K}_j and $f(\cdot)$ is a bounded non-negative Borel function called the distribution function. This gives to the formula

$$J_{\rho, Q_j^{\epsilon l}}^S = -\frac{\epsilon}{2\pi} \sum_{k=1}^N \int_{\mathbb{R}} (f(\lambda - \mu_j) - f(\lambda - \mu_k)) \sigma_{jk}(\lambda) d\lambda. \quad (3.20)$$

In particular, if we choose one

$$f(\lambda) := f_{FD}(\lambda) := \frac{1}{1 + e^{\beta\lambda}}, \quad \beta > 0, \quad \lambda \in \mathbb{R}, \quad (3.21)$$

where $f_{FD}(\cdot)$ is the Fermi-Dirac distribution function, and inserting (3.21) into (3.20) we obtain

$$J_{\rho, Q_j^{\epsilon l}}^S = -\frac{\epsilon}{2\pi} \sum_{k=1}^N \int_{\mathbb{R}} (f_{FD}(\lambda - \mu_j) - f_{FD}(\lambda - \mu_k)) \sigma_{jk}(\lambda) d\lambda. \quad (3.22)$$

If we have only two reservoirs, then they are usually denoted by l (left) and r (right). Let $j = l$ and $k = r$. Then

$$J_{\rho, Q_l}^S = -\frac{e}{2\pi} \int_{\mathbb{R}} (f_{FD}(\lambda - \mu_l) - f_{FD}(\lambda - \mu_r)) \sigma_{lr}(\lambda) d\lambda. \quad (3.23)$$

One easily checks that $J_{\rho, Q_l}^S \leq 0$ if $\mu_l \geq \mu_r$. That means, the current is leaving the left reservoir and is entering the right one which is accordance with physical intuition.

Example 3.3 Notice that $s_c := \{h^{el}, h_0^{el}\}$ is a \mathfrak{L}_1 scattering system. The Hamiltonian h^{el} takes into account the effect of coupling of reservoirs or leads $\mathfrak{h}_l := l^2(\mathbb{N})$ and $\mathfrak{h}_r := l^2(\mathbb{N})$ to the sample $\mathfrak{h}_S = \mathbb{C}^2$ which is also called the quantum dot. The leads Hamiltonian are given by $h_\alpha^{el} = -\Delta^D + v_\alpha$, $\alpha = l, r$. The sample or quantum dot Hamiltonian is given by h_S^{el} . The wave operators are given by

$$w_\pm(h^{el}, h_0^{el}) := s\text{-}\lim_{t \rightarrow \infty} e^{ith^{el}} e^{-ith_0^{el}} P^{ac}(h_0^{el}) \quad (3.24)$$

The scattering operator is given by $s_c := w_+(h^{el}, h_0^{el})^* w_-(h^{el}, h_0^{el})$. Let $\Pi(h_0^{el, ac})$ the spectral representation of $h_0^{el, ac}$ introduced in Section 2.6. If ρ^{el} and q^{el} are density and charge operators for h_0^{el} , then the Landauer-Büttiker formula takes the form

$$J_{\rho^{el}, q^{el}}^{s_c} = \frac{1}{2\pi} \int_{\mathbb{R}} \text{tr}(\rho_{ac}^{el}(\lambda) (q_{ac}^{el} - s_c(\lambda)^* q_{ac}^{el}(\lambda) s_c(\lambda))) \quad (3.25)$$

where $\{s_c(\lambda)\}_{\lambda \in \mathbb{R}}$, $\{q^{el}(\lambda)\}_{\lambda \in \mathbb{R}}$ and $\{\rho^{el}(\lambda)\}_{\lambda \in \mathbb{R}}$ are the scattering, charge and density matrices with respect to $\Pi(h_0^{el, ac})$, respectively. The condition that $((h_0^{el})^2 + I_{\mathfrak{h}^{el}})\rho^{el}$ is a bounded operator is superfluous because h_0^{el} is a bounded operator. For the same reason we have that every p -tempered charge q^{el} is in fact a charge, that means, q^{el} is a bounded self-adjoint operator.

The scattering system s_c is a black-box model with reservoirs \mathfrak{h}_l^{el} and \mathfrak{h}_r^{el} . Choosing

$$\rho^{el} = f_l(h^{el}) \oplus f_S(h_S^{el}) \oplus f_r(h_r^{el})$$

where $f_\alpha(\cdot)$, $\alpha = l, r$, are bounded Borel functions, and

$$q^{el} = g_l(h_l^{el}) \oplus g_S(h_S^{el}) \oplus g_r(h_r^{el}),$$

where $g_\alpha(\cdot)$, $\alpha \in \{l, r\}$, are locally bounded Borel functions, then from (3.13) it follows that

$$J_{\rho^{el}, q^{el}}^{s_c} = \frac{1}{2\pi} \sum_{\substack{\alpha, \varkappa \in \{l, r\} \\ \alpha \neq \varkappa}} \int_{\mathbb{R}} (f_\alpha(\lambda) - f_\varkappa(\lambda)) g_\alpha(\lambda) \sigma_c(\lambda) d\lambda$$

where $\{\sigma_c(\lambda)\}_{\lambda \in \mathbb{R}}$ is the channel cross-section from left to right and vice versa. Indeed, let $\{t_c(\lambda)\}_{\lambda \in \mathbb{R}}$ the transition matrix which corresponds to the transition operator $t_c := s_c - I_{\mathfrak{h}^{el}}$. Obviously, one has $t_c(\lambda) = I_{\mathfrak{h}(\lambda)} - s_c(\lambda)$, $\lambda \in \mathbb{R}$. Let $\{p_\alpha^{el}(\lambda)\}_{\lambda \in \mathbb{R}}$ be the matrix which corresponds to the orthogonal projection p_α^{el} from \mathfrak{h}^{el} onto \mathfrak{h}_α^{el} . Further, let $t_{rl}^c(\lambda) := p_r^{el}(\lambda) t_c(\lambda) p_l^{el}$ and $t_{lr}^c := p_l^{el}(\lambda) t_c(\lambda) p_r^{el}$. Notice that both quantities are in fact scalar functions. Obviously, the channel cross-sections $\sigma_{lr}^c(\lambda)$ and $\sigma_{rl}^c(\lambda)$ at energy $\lambda \in \mathbb{R}$ are given by $\sigma_c(\lambda) := \sigma_{lr}^c(\lambda) = |t_{lr}^c(\lambda)|^2 = |t_{rl}^c(\lambda)|^2 = \sigma_{rl}^c(\lambda)$, $\lambda \in \mathbb{R}$.

In particular, if $g_l(\lambda) = 1$ and $g_r = 0$, then

$$J_{\rho^{el}, q^{el}}^{s_c} = \frac{1}{2\pi} \int_{\mathbb{R}} (f_l(\lambda) - f_r(\lambda)) \sigma_c(\lambda) d\lambda, \quad (3.26)$$

and $q_l^{el} := p_l^{el}$. Following [1] $J_{\rho^{el}, q^{el}}^{s_c}$ denotes the current entering the quantum dot from the left lead.

3.2 Application to the JCL-model

Let $S = \{H, H_0\}$ be now the JCL-model. Further, let ρ and Q be a density operator and a charge for H_0 , respectively. Under these assumptions the current $J_{\rho, Q}^S$ is defined by

$$J_{\rho, Q}^S := -i \operatorname{tr} \left(W_-(H, H_0)(I + H_0^2)\rho W_-(H, H_0)^* \frac{1}{H - i} [H, Q] \frac{1}{H + i} \right), \quad (3.27)$$

and admits representation (3.4). If Q is a p -tempered charge and $(H_0 - i)^{p+2}\rho$ is a bounded operator, then the current $J_{\rho, Q}^S$ is defined in accordance with (3.7) and the Landauer-Büttiker formula (3.4) is valid, too.

We introduce the intermediate scattering system $S_c := \{H, H_c\}$ where

$$H_c := h^{el} \otimes I_{\mathfrak{h}^{ph}} + I_{\mathfrak{h}^{el}} \otimes h^{ph} = H_0 + V_{el}.$$

The Hamiltonian H_c describes the coupling of the leads to quantum dot but under the assumption that the photon interaction is not switched on.

Obviously, $S_{ph} := \{H, H_c\}$ and $S_c := \{H_c, H_0\}$ are \mathfrak{L}_1 -scattering systems. The corresponding scattering operators are denote by S_{ph} and S_c , respectively. Let $\Pi(H_c^{ac}) = \{L^2(\mathbb{R}, d\lambda, \mathfrak{h}_c(\lambda)), \mathcal{M}, \Phi_c\}$ of H_c^{ac} be a spectral representation of H_c . The scattering matrix of the scattering system $\{H, H_c\}$ with respect to $\Pi(H_c^{ac})$ is denoted by $\{S_{ph}(\lambda)\}_{\lambda \in \mathbb{R}}$. The scattering matrix of the scattering system $\{H_c, H_0\}$ with respect to $\Pi(H_0^{ac}) = \{L^2(\mathbb{R}, d\lambda, \mathfrak{h}_0(\lambda)), \mathcal{M}, \Phi_0\}$ is denoted by $\{S_c(\lambda)\}_{\lambda \in \mathbb{R}}$.

Since S_c is a \mathfrak{L}_1 -scattering system the wave operators $W_{\pm}(H_c, H_0)$ exists and are complete and since $\Phi_c W_{\pm}(H_c, H_0) \Phi_0^*$ commute with \mathcal{M} , there is a measurable families $\{W_{\pm}(\lambda)\}_{\lambda \in \mathbb{R}}$ of isometries acting from $\mathfrak{h}_0(\lambda)$ onto $\mathfrak{h}_c(\lambda)$ for a.e. $\lambda \in \mathbb{R}$ such that

$$(\Phi_c W_{\pm}(H_c, H_0) \Phi_0^* \widehat{f})(\lambda) = W_{\pm}(\lambda) \widehat{f}(\lambda), \quad \lambda \in \mathbb{R}, \quad \widehat{f} \in L^2(\mathbb{R}, d\lambda, \mathfrak{h}_0(\lambda)).$$

The families $\{W_{\pm}(\lambda)\}_{\lambda \in \mathbb{R}}$ are called wave matrices.

A straightforward computation shows that $\widehat{S}_{ph} := W_+(H_c, H_0)^* S_{ph} W_+(H_c, H_0)$ commutes with H_0 . Hence, with respect to the spectral representation $\Pi(H_0^{ac})$ the operator \widehat{S}_{ph} is unitarily equivalent to a multiplication induced by a measurable family $\{\widehat{S}_{ph}(\lambda)\}_{\lambda \in \mathbb{R}}$ of unitary operators in $\mathfrak{h}_0(\lambda)$. A straightforward computation shows that

$$\widehat{S}_{ph}(\lambda) = W_+(\lambda)^* S_{ph}(\lambda) W_+(\lambda) \quad (3.28)$$

for a.e. $\lambda \in \mathbb{R}$. Roughly speaking, $\{\widehat{S}_{ph}(\lambda)\}_{\lambda \in \mathbb{R}}$ is the scattering matrix of S_{ph} with respect to the spectral representation $\Pi(H_0^{ac})$.

Furthermore, let

$$\rho^c := W_-(H_c, H_0)\rho W_-(H_c, H_0)^* \quad (3.29)$$

and

$$Q^c := W_+(H_c, H_0)QW_+(H_c, H_0)^*. \quad (3.30)$$

The operators ρ^c and Q^c are density and tempered charge operators for the scattering system S_{ph} . Indeed, one easily verifies that ρ^c and Q^c are commute with H_c . Moreover, ρ^c is non-negative. Furthermore, if Q is a charge, then Q^c is a charge, too. This gives rise to introduce the currents $J_{\rho^c, Q^c}^c := J_{\rho^c, Q^c}^{S_c}$,

$$J_{\rho^c, Q^c}^c := -i \operatorname{tr} \left(W_-(H_c, H_0)\rho W_-(H_c, H_0)^* \frac{1}{H_c - i} [H_c, Q] \frac{1}{H_c + i} \right), \quad (3.31)$$

and $J_{\rho, Q}^{ph} := J_{\rho^c, Q^c}^{S_{ph}}$

$$J_{\rho, Q}^{ph} := -i \operatorname{tr} \left(W_-(H, H_c)\rho^c W_-(H, H_c)^* \frac{1}{H - i} [H, Q^c] \frac{1}{H + i} \right) \quad (3.32)$$

which are well defined. If Q is p -tempered charge and $(H_0 - i)^{p+2}\rho$ is a bounded operator, then one easily checks that Q^c is a p -tempered charge and $(H_c - i)^{p+2}\rho^c$ is a bounded operator. Hence the definition of the currents $J_{\rho^c, Q^c}^{S_c}$ can be extended to this case and the Landauer-Büttiker formula (3.4) holds.

Finally we note that the corresponding matrices $\{\rho_{ac}^c(\lambda)\}_{\lambda \in \mathbb{R}}$ and $\{Q_{ac}^c(\lambda)\}_{\lambda \in \mathbb{R}}$ are related to the matrices $\{\rho_{ac}(\lambda)\}_{\lambda \in \mathbb{R}}$ and $\{Q_{ac}(\lambda)\}_{\lambda \in \mathbb{R}}$ by

$$\rho_{ac}^c(\lambda) = W_-(\lambda)\rho_{ac}(\lambda)W_-(\lambda)^* \quad \text{and} \quad Q_{ac}^c(\lambda) = W_+(\lambda)Q_{ac}(\lambda)W_+(\lambda)^* \quad (3.33)$$

for a.e. $\lambda \in \mathbb{R}$.

Proposition 3.4 (Current decomposition) *Let $S = \{H, H_0\}$ be the JCL-model. Further, let ρ and Q be a density operator and a p -tempered charge, $p \in \mathbb{N}_0$, for H_0 , respectively. If $(H_0 - i)^{p+2}\rho$ is a bounded operator, then the decomposition*

$$J_{\rho, Q}^S = J_{\rho, Q}^c + J_{\rho, Q}^{ph} \quad (3.34)$$

holds where $J_{\rho, Q}^c$ and $J_{\rho, Q}^{ph}$ are given by (3.31) and (3.32).

In particular, let $\{S_c(\lambda)\}_{\lambda \in \mathbb{R}}$, $\{\rho_{ac}(\lambda)\}_{\lambda \in \mathbb{R}}$ and $\{Q_{ac}(\lambda)\}_{\lambda \in \mathbb{R}}$ be scattering, density and charge matrices of S_c , ρ and Q with respect to $\Pi(H_0^{ac})$ and let $\{S_{ph}(\lambda)\}_{\lambda \in \mathbb{R}}$, $\{\rho_{ac}^c(\lambda)\}_{\lambda \in \mathbb{R}}$ and $\{Q_{ac}^c(\lambda)\}_{\lambda \in \mathbb{R}}$ be the scattering, density and charge matrices of the scattering operator S_{ph} , density operator ρ^c , cf. (3.29), and charge operator Q^c , cf. (3.30), with respect to the spectral representation $\Pi(H_c^{ac})$. Then the representations

$$J_{\rho, Q}^c := \frac{1}{2\pi} \int_{\mathbb{R}} \text{tr}(\rho_{ac}(\lambda)(Q_{ac}(\lambda) - S_c(\lambda)^*Q_{ac}(\lambda)S_c(\lambda)))d\lambda, \quad (3.35)$$

$$J_{\rho, Q}^{ph} := \frac{1}{2\pi} \int_{\mathbb{R}} \text{tr}(\rho_{ac}^c(\lambda)(Q_{ac}^c(\lambda) - S_{ph}(\lambda)^*Q_{ac}^c(\lambda)S_{ph}(\lambda)))d\lambda, \quad (3.36)$$

take place.

Proof. Since S_c and S_{ph} are \mathfrak{L}_1 -scattering systems from Theorem 3.1 the representations (3.35) and (3.36) are easily follow. Taking into account (3.33) we get

$$\begin{aligned} \text{tr}(\rho_{ac}^c(\lambda)(Q_{ac}^c(\lambda) - S_{ph}(\lambda)^*Q_{ac}^c(\lambda)S_{ph}(\lambda))) &= \\ \text{tr}(W_-(\lambda)\rho_{ac}W_-(\lambda)^*(W_+(\lambda)Q_{ac}(\lambda)W_+(\lambda) - S_{ph}(\lambda)^*Q_{ac}^c(\lambda)S_{ph}(\lambda))). \end{aligned}$$

Using $S_c(\lambda) = W_+(\lambda)^*W_-(\lambda)$ we find

$$\begin{aligned} \text{tr}(\rho_{ac}^c(\lambda)(Q_{ac}^c(\lambda) - S_{ph}(\lambda)^*Q_{ac}^c(\lambda)S_{ph}(\lambda))) &= \text{tr}(\rho_{ac}(\lambda) \times \\ (S_c(\lambda)^*Q_{ac}(\lambda)S_c(\lambda) - W_-(\lambda)^*S_{ph}(\lambda)^*W_+(\lambda)Q_{ac}(\lambda)W_+(\lambda)^*S_{ph}(\lambda)W_-(\lambda))). \end{aligned} \quad (3.37)$$

Since $\{H_c, H_0\}$ and $\{H, H_c\}$ are \mathfrak{L}_1 -scattering systems the existence of the wave operators $W_{\pm}(H, H_c)$ and $W_{\pm}(H_c, H_0)$ follows. Using the chain rule we find $W_{\pm}(H, H_0) = W_{\pm}(H, H_c)W_{\pm}(H_c, H_0)$ which yields

$$\begin{aligned} S &= W_+(H, H_0)^*W_+(H, H_0) \\ &= W_+(H_c, H_0)^*W_+(H, H_c)W_-(H, H_c)W_-(H_c, H_0) \\ &= W_+(H_c, H_0)^*S_{ph}W_-(H_c, H_0). \end{aligned}$$

Hence the scattering matrix $\{S(\lambda)\}_{\lambda \in \mathbb{R}}$ of $\{H, H_0\}$ admits the representation

$$S(\lambda) = W_+(\lambda)^*S_{ph}(\lambda)W_-(\lambda), \quad \lambda \in \mathbb{R}. \quad (3.38)$$

Inserting (3.38) into (3.37) we get

$$J_{\rho,Q}^{ph} = \frac{1}{2\pi} \int_{\mathbb{R}} \text{tr}(\rho_{ac}(\lambda)(S_c(\lambda)^* Q_{ac}(\lambda) S_c(\lambda) - S(\lambda)^* Q_{ac}(\lambda) S(\lambda))) d\lambda \quad (3.39)$$

Using (3.39) we obtain

$$J_{\rho,Q}^c + J_{\rho,Q}^{ph} = \frac{1}{2\pi} \int_{\mathbb{R}} \text{tr}(\rho_{ac}(\lambda)(Q_{ac}(\lambda) - S(\lambda)^* Q_{ac}(\lambda) S(\lambda))) d\lambda.$$

Finally, taking into account (3.4) we obtain (3.34). \square

Remark 3.5

(i) The current $J_{\rho,Q}^c$ is due to the coupling of the leads to the quantum dot and is therefore called the *contact induced current*.

(ii) The current $J_{\rho,Q}^{ph}$ is due to the interaction of photons with electrons and is therefore called the *photon induced current*. Notice that this current is calculated under the assumption that the leads already contacted to the dot.

Corollary 3.6 *Let the assumptions of Proposition 3.4 be satisfied. With respect to the spectral representation $\Pi(H_0^{ac})$ of H_0^{ac} the photon induced current $J_{\rho,Q}^{ph}$ can be represented by*

$$J_{\rho,Q}^{ph} := \frac{1}{2\pi} \int_{\mathbb{R}} \text{tr}(\widehat{\rho}_{ac}(\lambda)(Q_{ac}(\lambda) - \widehat{S}_{ph}(\lambda)^* Q_{ac}(\lambda) \widehat{S}_{ph}(\lambda))) d\lambda \quad (3.40)$$

where the measurable families $\{\widehat{S}_{ph}(\lambda)\}_{\lambda \in \mathbb{R}}$ and $\{\widehat{\rho}_{ac}(\lambda)\}_{\lambda \in \mathbb{R}}$ are given by (3.28) and

$$\widehat{\rho}_{ac}(\lambda) := S_c(\lambda) \rho_{ac}(\lambda) S_c(\lambda)^* \quad \lambda \in \mathbb{R}, \quad (3.41)$$

respectively.

Proof. Using (3.33) and $S_c(\lambda) = W_+(\lambda)^* W_-(\lambda)$ we find

$$\begin{aligned} \text{tr}(\rho_{ac}^c(\lambda)(Q_{ac}^c(\lambda) - S_{ph}(\lambda)^* Q_{ac}^c(\lambda) S_{ph}(\lambda))) &= \text{tr}(S_c(\lambda) \rho_{ac}(\lambda) S_c(\lambda)^* \times \\ &\quad (Q_{ac}(\lambda) - W_+(\lambda)^* S_{ph}(\lambda)^* W_+(\lambda) Q_{ac}(\lambda) W_+(\lambda)^* S_{ph}(\lambda) W_+(\lambda))). \end{aligned}$$

Taking into account the representations (3.28) and (3.41) we get

$$\begin{aligned} \text{tr}(\rho_{ac}^c(\lambda)(Q_{ac}^c(\lambda) - S_{ph}(\lambda)^* Q_{ac}^c(\lambda) S_{ph}(\lambda))) &= \\ \text{tr}(S_c(\lambda) \rho_{ac}(\lambda) S_c(\lambda)^* (Q_{ac}(\lambda) - \widehat{S}_{ph}(\lambda)^* Q_{ac}(\lambda) \widehat{S}_{ph}(\lambda))) & \end{aligned}$$

which immediately yields (3.40). \square

Remark 3.7 In the following we call $\{\widehat{\rho}_{ac}(\lambda)\}_{\lambda \in \mathbb{R}}$, cf. (3.41), the photon modified electron density matrix. Notice that $\{\widehat{\rho}_{ac}(\lambda)\}_{\lambda \in \mathbb{R}}$ might be non-diagonal even if the electron density matrix $\{\rho_{ac}(\lambda)\}_{\lambda \in \mathbb{R}}$ is diagonal.

4 Analysis of currents

In the following we analyze currents $J_{\rho, Q}^c$ and $J_{\rho, Q}^{ph}$ under the assumption that ρ and Q have the tensor product structure

$$\rho = \rho^{el} \otimes \rho^{ph} \quad \text{and} \quad Q = q^{el} \otimes q^{ph} \quad (4.1)$$

where ρ^{el} and ρ^{ph} as well as q^{el} and q^{ph} are density operators and (tempered) charges for h_0^{el} and h^{ph} , respectively. Since ρ^{ph} commutes with h^{ph} , which is discrete, the operator ρ^{ph} has the form

$$\rho^{ph} = \rho^{ph}(n)(\cdot, \Upsilon_n)\Upsilon_n, \quad n \in \mathbb{N}_0, \quad (4.2)$$

where $\rho^{ph}(n)$ are non-negative numbers. Similarly, q^{ph} can be represented by

$$q^{ph} = q^{ph}(n)(\cdot, \Upsilon_n)\Upsilon_n, \quad n \in \mathbb{N}_0, \quad (4.3)$$

where $q^{ph}(n)$ are real numbers.

Lemma 4.1 *Let $S = \{H, H_0\}$ be the JCL-model. Assume that $\rho \neq 0$ and Q have the structure (4.1) where ρ^{el} is a density operator and q^{el} is a charge for h_0^{el} .*

(i) *The operator $(H_0 - i)^{p+2}\rho$, $p \in \mathbb{N}_0$, is bounded if and only if the condition*

$$\sup_{n \in \mathbb{N}_0} \rho^{ph}(n)n^{p+2} < \infty \quad (4.4)$$

is satisfied.

(ii) *The charge Q is p -tempered if and only if*

$$\sup_{n \in \mathbb{N}} |q^{ph}(n)|n^{-p} < \infty. \quad (4.5)$$

is valid

Proof. (i) The operator $(H_0 - i)^{p+2}\rho$ admits the representation

$$(H_0 - i)^{p+2}\rho = \bigoplus_{p \in \mathbb{N}_0} \rho^{ph}(n)(h_0^{el} + n\omega - i)^{p+2}\rho^{el}.$$

We have

$$\begin{aligned} \|(H_0 - i)^{p+2}\rho\|_{\mathfrak{L}(S)} &= \sup_{p \in \mathbb{N}_0} \rho^{ph}(n)\|(h_0^{el} + n\omega - i)^{p+2}\rho^{el}\|_{\mathfrak{L}(h^{el})} \\ &= \sup_{p \in \mathbb{N}_0} \rho^{ph}(n)n^{p+2}n^{-(p+2)}\|(h_0^{el} + n\omega - i)^{p+2}\rho^{el}\|_{\mathfrak{L}(h^{el})}. \end{aligned} \quad (4.6)$$

Since $\lim_{n \rightarrow \infty} n^{-(p+2)}\|(h_0^{el} + n\omega - i)^{p+2}\rho^{el}\|_{\mathfrak{L}(h^{el})} = \omega^{p+2}\|\rho^{el}\|_{\mathfrak{L}(h^{el})}$ we get for sufficiently large $n \in \mathbb{N}_0$ that

$$\frac{\omega^{p+2}}{2}\|\rho^{el}\|_{\mathfrak{L}(h^{el})} \leq n^{-(p+2)}\|(h_0^{el} + n\omega - i)^{p+2}\rho^{el}\|_{\mathfrak{L}(h^{el})}.$$

Using that and (4.6) we immediately obtain (4.4). Conversely, from (4.6) and (4.4) we obtain that $(H_0 - i)^{p+2}\rho$ is a bounded operator.

(ii) As above we have

$$Q(H_0 - i)^{-p} = \bigoplus_{n \in \mathbb{N}_0} q^{ph}(n)q^{el}$$

Hence

$$\|Q(H_0 - i)^{-p}\|_{\mathfrak{L}(S)} = \sup_{n \in \mathbb{N}_0} |q^{ph}(n)|\|q^{el}(h_0^{el} + n\omega - i)^{-p}\|_{\mathfrak{L}(h^{el})}.$$

Since $\lim_{n \rightarrow \infty} n^p\|(h_0^{el} + n\omega - i)^{-p}\|_{\mathfrak{L}(h^{el})} = \omega^{-p}\|q^{el}\|_{\mathfrak{L}(h^{el})}$ we get similarly as above that (4.5) holds. The converse is obvious. \square

4.1 Contact induced current

Let us recall that $S_c = \{H_c, H_0\}$ is a \mathfrak{L}_1 -scattering system. An obvious computations shows that

$$W_{\pm}(H_c, H_0) = w_{\pm}(h^{el}, h_0^{el}) \otimes I_{\mathfrak{h}^{ph}}$$

where $w_{\pm}(h^{el}, h_0^{el})$ is given by (3.24). Hence

$$S_c = s_c \otimes I_{\mathfrak{h}^{ph}}, \quad \text{where } s_c := w_{+}(h_c^{el}, h_0^{el})^* w_{-}(h_c^{el}, h_0^{el}).$$

Proposition 4.2 *Let $S = \{H, H_0\}$ be the JCL-model. Assume that ρ and Q are given by (4.1) where ρ^{el} and q^{el} are density and charge operators for h_0^{el} and ρ^{ph} and q^{ph} for h^{ph} , respectively. If for some $p \in \mathbb{N}_0$ the conditions (4.4) and (4.5) are satisfied, then the current $J_{\rho, Q}^c$ is well defined and admits the representation*

$$J_{\rho, Q}^c = \gamma J_{\rho^{el}, q^{el}}^{s_c}, \quad \gamma := \sum_{n \in \mathbb{N}_0} q^{ph}(n) \rho^{ph}(n) \quad (4.7)$$

where $J_{\rho^{el}, q^{el}}^{s_c}$ is defined by (3.2). In particular, if $\text{tr}(\rho^{ph}) = 1$ and $q^{ph} = I_{\mathfrak{h}^{ph}}$, then $J_{\rho, Q}^c = J_{\rho^{el}, q^{el}}^{s_c}$.

Proof. First of all we note that by lemma 4.1 the operator $(H_0 - i)^{p+2} \rho$ is bounded and Q is p -tempered. Hence the current $J_{\rho, Q}^{s_c}$ is correctly defined and the Landauer-Büttiker formula (3.4) is valid.

With respect to the spectral representation $\Pi(H_0^{ac})$ of Lemma 2.12 the charge matrix $\{Q_{ac}(\lambda)\}_{\lambda \in \mathbb{R}}$ of $Q_{ac} = q_{ac}^{el} \otimes q^{ph}$ admits the representation

$$Q_{ac}(\lambda) = \bigoplus_{n \in \mathbb{N}_0} q_{ac}^{el}(\lambda - n\omega) q^{ph}(n), \quad \lambda \in \mathbb{R}. \quad (4.8)$$

Since $S_c = s_c \otimes I_{\mathfrak{h}^{ph}}$ the scattering matrix $\{S_c(\lambda)\}_{\lambda \in \mathbb{R}}$ admits the representation

$$S_c(\lambda) = \bigoplus_{n \in \mathbb{N}_0} s_c(\lambda - n\omega), \quad \lambda \in \mathbb{R}.$$

Hence

$$\begin{aligned} Q_{ac}(\lambda) - S_c(\lambda)^* Q_{ac}(\lambda) S_c(\lambda) = \\ \bigoplus_{n \in \mathbb{N}_0} q^{ph}(n) (q_{ac}^{el}(\lambda - n\omega) - s_c(\lambda - n\omega)^* q_{ac}^{el}(\lambda - \omega n) s_c(\lambda - n\omega)). \end{aligned} \quad (4.9)$$

Moreover, the density matrix $\{\rho_{ac}(\lambda)\}_{\lambda \in \mathbb{R}}$ admits the representation

$$\rho_{ac}(\lambda) = \bigoplus_{n \in \mathbb{N}_0} \rho^{ph}(n) \rho_{ac}^{el}(\lambda - n\omega) \quad (4.10)$$

Inserting (4.10) into (4.9) we find

$$\begin{aligned} \rho^{ac}(\lambda) (Q_{ac}(\lambda) - S_c(\lambda)^* Q_{ac}(\lambda) S_c(\lambda)) = \bigoplus_{n \in \mathbb{N}_0} q^{ph}(n) \rho^{ph}(n) \times \\ \rho_{ac}^{el}(\lambda - n\omega) (q_{ac}^{el}(\lambda - \omega n) - s_c(\lambda - n\omega)^* q_{ac}^{el}(\lambda - \omega n) s_c(\lambda - n\omega)) \end{aligned}$$

Since $\gamma = \sum_{n \in \mathbb{N}_0} q^{ph}(n) \rho^{ph}(n)$ is absolutely convergent by (4.4) and (4.5) we obtain that

$$\begin{aligned} \text{tr}(\rho^{ac}(\lambda) (Q_{ac}(\lambda) - S_c(\lambda)^* Q_{ac}(\lambda) S_c(\lambda))) = \sum_{n \in \mathbb{N}_0} q^{ph}(n) \rho^{ph}(n) \times \\ \text{tr}(\rho_{ac}^{el}(\lambda - n\omega) (q_{ac}^{el}(\lambda - \omega n) - s_c(\lambda - n\omega)^* q_{ac}^{el}(\lambda - \omega n) s_c(\lambda - n\omega))) \end{aligned} \quad (4.11)$$

Obviously, we have

$$\begin{aligned} & \left| \operatorname{tr} \left(\rho_{ac}^{el}(\lambda - n\omega) \left(q_{ac}^{el}(\lambda - \omega n) - s_c(\lambda - n\omega)^* q_{ac}^{el}(\lambda - \omega n) s_c(\lambda - n\omega) \right) \right) \right| \leq \\ & 4 \left\| \rho_{ac}^{el}(\lambda - n\omega) \right\|_{\mathfrak{L}(\mathfrak{h}_n(\lambda))} \left\| q_{ac}^{el}(\lambda - n\omega) \right\|_{\mathfrak{L}(\mathfrak{h}_n(\lambda))}, \quad \lambda \in \mathbb{R}. \end{aligned}$$

We insert (4.11) into the Landauer-Büttiker formula (3.35). Using (4.4) and (4.5) as well as

$$\int_{\mathbb{R}} \left\| \rho_{ac}^{el}(\lambda) \right\|_{\mathfrak{L}(\mathfrak{h}_n(\lambda))} \left\| q_{ac}^{el}(\lambda) \right\|_{\mathfrak{L}(\mathfrak{h}_n(\lambda))} d\lambda < \infty$$

we see that we can interchange the integral and the sum. Doing so we get

$$\begin{aligned} J_{\rho, Q}^c &= \sum_{n \in \mathbb{N}_0} q^{ph}(n) \rho^{ph}(n) \frac{1}{2\pi} \int_{\mathbb{R}} \operatorname{tr} \left(\rho_{ac}^{el}(\lambda - n\omega) \times \right. \\ & \left. \left(q_{ac}^{el}(\lambda - \omega n) - s_c(\lambda - n\omega)^* q_{ac}^{el}(\lambda - \omega n) s_c(\lambda - n\omega) \right) \right) d\lambda. \end{aligned}$$

Using (3.25) we prove (4.7).

If $\operatorname{tr}(\rho^{ph}) = 1$, then $\sum_{\mathbb{N}_0} \rho^{ph}(n) = 1$. Further, if $\rho^{ph} = I_{\mathfrak{h}^{ph}}$, then $q^{ph}(n) = 1$. Hence $\gamma = 1$. \square

4.2 Photon induced current

To calculate the current $J_{\rho, Q}^{ph}$ we used the representation (3.40). We set

$$\widehat{S}_{mn}^{ph}(\lambda) := P_m(\lambda) \widehat{S}_{ph}(\lambda) \upharpoonright \mathfrak{h}_n(\lambda), \quad \lambda \in \mathbb{R}.$$

where $\{\widehat{S}_{ph}(\lambda)\}_{\lambda \in \mathbb{R}}$ is defined by (3.28) and $P_m(\lambda)$ is the orthogonal projection from $\mathfrak{h}(\lambda)$, cf. (2.41), onto $\mathfrak{h}_m(\lambda) := \mathfrak{h}^{el}(\lambda - m\omega)$, $\lambda \in \mathbb{R}$.

Proposition 4.3 *Let $S = \{H, H_0\}$ be the JCL-model. Assume that ρ and Q are given by (4.1) where ρ^{el} and q^{el} are density and charge operators for \mathfrak{h}_0^{el} and ρ^{ph} and q^{ph} for \mathfrak{h}^{ph} , respectively. If for some $p \in \mathbb{N}_0$ the conditions (4.4) and (4.5) are satisfied, then the current $J_{\rho, Q}^{ph}$ is well-defined and admits the representation*

$$\begin{aligned} J_{\rho, Q}^{ph} &= \sum_{m \in \mathbb{N}_0} \rho^{ph}(m) \sum_{n \in \mathbb{N}_0} q^{ph}(n) \frac{1}{2\pi} \int_{\mathbb{R}} d\lambda \operatorname{tr} \left(\widehat{\rho}_{ac}^{el}(\lambda - m\omega) \times \right. \\ & \left. \left(q_{ac}^{el}(\lambda - n\omega) \delta_{mn} - \widehat{S}_{nm}^{ph}(\lambda)^* q_{ac}^{el}(\lambda - n\omega) \widehat{S}_{nm}^{ph}(\lambda) \right) \right). \end{aligned} \quad (4.12)$$

where $\{\widehat{\rho}_{ac}^{el}(\lambda)\}_{\lambda \in \mathbb{R}}$ is the photon modified electron density defined, cf. (3.41), which takes the form

$$\widehat{\rho}_{ac}^{el}(\lambda) = s_c(\lambda) \rho^{el}(\lambda) s_c(\lambda)^*, \quad \lambda \in \mathbb{R}. \quad (4.13)$$

Proof. By Lemma 4.1 we get that that the charge Q is p -tempered and $(H_0 - i)^p \rho$ is a bounded operator. By Corollary 3.2 the current $J_{\rho, Q}^{ph} := J_{\rho^e, Q^e}^{S_{ph}}$ is well-defined.

Since $\left(Q_{ac}(\lambda) - \widehat{S}_{ph}(\lambda)^* Q_{ac}(\lambda) \widehat{S}_{ph}(\lambda) \right)$ is a trace class operator for $\lambda \in \mathbb{R}$ we get from (3.40) and (4.10) that

$$\begin{aligned} & \operatorname{tr} \left(\widehat{\rho}_{ac}^{el}(\lambda) \left(Q_{ac}(\lambda) - \widehat{S}_{ph}(\lambda)^* Q_{ac}(\lambda) \widehat{S}_{ph}(\lambda) \right) \right) = \sum_{m \in \mathbb{N}_0} \rho^{ph}(m) \times \\ & \operatorname{tr} \left(\widehat{\rho}^{el}(\lambda - m\omega) P_m(\lambda) \left(Q_{ac}(\lambda) - \widehat{S}_{ph}(\lambda)^* Q_{ac}(\lambda) \widehat{S}_{ph}(\lambda) \right) P_m(\lambda) \right) \end{aligned}$$

Further we have

$$\begin{aligned}
& P_m(\lambda) \left(Q_{ac}(\lambda) - \widehat{S}_{ph}(\lambda)^* Q_{ac}(\lambda) \widehat{S}_{ph}(\lambda) \right) P_m(\lambda) \\
&= q^{ph}(m) \left(q^{el}(\lambda - m\omega) - P_m(\lambda) \widehat{S}_{ph}(\lambda)^* Q_{ac}(\lambda) \widehat{S}_{ph}(\lambda) \right) P_m(\lambda) \\
&= q^{ph}(m) q^{el}(\lambda - m\omega) - \sum_{n \in \mathbb{N}_0} q^{ph}(n) \widehat{S}_{nm}^{ph}(\lambda)^* q^{el}(\lambda - n\omega) \widehat{S}_{nm}^{ph}(\lambda)
\end{aligned}$$

for $\lambda \in \mathbb{R}$ where $\widehat{S}_{nm}^{ph}(\lambda)^* := P_n(\lambda) \widehat{S}_{ph}(\lambda) P_m(\lambda)$, $\lambda \in \mathbb{R}$. Notice that $\sum_{n \in \mathbb{N}_0}$ is a sum with a finite number of summands. Hence

$$\begin{aligned}
& \text{tr} \left(\widehat{\rho}_{ac}(\lambda) \left(Q_{ac}(\lambda) - \widehat{S}_{ph}(\lambda)^* Q_{ac}(\lambda) \widehat{S}_{ph}(\lambda) \right) \right) = \sum_{m \in \mathbb{N}_0} \rho^{ph}(m) \sum_{n \in \mathbb{N}_0} q^{ph}(n) \times \\
& \text{tr} \left(\widehat{\rho}^{el}(\lambda - m\omega) \left(q^{el}(\lambda - m\omega) \delta_{mn} - \widehat{S}_{nm}^{ph}(\lambda)^* q^{el}(\lambda - n\omega) \widehat{S}_{nm}^{ph}(\lambda) \right) \right)
\end{aligned}$$

We are going to show that

$$\begin{aligned}
& \sum_{m \in \mathbb{N}_0} \rho^{ph}(m) \sum_{n \in \mathbb{N}_0} |q^{ph}(n)| \int_{\mathbb{R}} \left| \text{tr} \left(\widehat{\rho}^{el}(\lambda - m\omega) \times \right. \right. \\
& \left. \left. \left(q^{el}(\lambda - m\omega) \delta_{mn} - \widehat{S}_{nm}^{ph}(\lambda)^* q^{el}(\lambda - n\omega) \widehat{S}_{nm}^{ph}(\lambda) \right) \right) \right| d\lambda < \infty.
\end{aligned}$$

Obviously one has the estimate

$$\begin{aligned}
& \left| \text{tr} \left(\widehat{\rho}^{el}(\lambda - m\omega) \left(q^{el}(\lambda - m\omega) \delta_{mn} - \widehat{S}_{nm}^{ph}(\lambda)^* q^{el}(\lambda - n\omega) \widehat{S}_{nm}^{ph}(\lambda) \right) \right) \right| \leq \\
& 2 \|\widehat{\rho}^{el}(\lambda - m\omega)\|_{\mathfrak{L}(\mathfrak{h}_m(\lambda))} \left(\|q^{el}(\lambda - m\omega)\|_{\mathfrak{L}(\mathfrak{h}_m(\lambda))} \delta_{nm} + \|q^{el}(\lambda - n\omega)\|_{\mathfrak{L}(\mathfrak{h}_n(\lambda))} \right).
\end{aligned}$$

Further, we get

$$\begin{aligned}
& \int_{\lambda \in \mathbb{R}} \|\widehat{\rho}^{el}(\lambda - m\omega)\|_{\mathfrak{L}(\mathfrak{h}_m(\lambda))} \|q^{el}(\lambda - m\omega)\|_{\mathfrak{L}(\mathfrak{h}_m(\lambda))} \delta_{nm} \leq \\
& \int_{\lambda \in \mathbb{R}} \|\widehat{\rho}^{el}(\lambda)\|_{\mathfrak{L}(\mathfrak{h}_m(\lambda))} \|q^{el}(\lambda)\|_{\mathfrak{L}(\mathfrak{h}_m(\lambda))} d\lambda
\end{aligned}$$

and

$$\begin{aligned}
& \int_{\mathbb{R}} \|\widehat{\rho}^{el}(\lambda - m\omega)\|_{\mathfrak{L}(\mathfrak{h}_m(\lambda))} \|q^{el}(\lambda - n\omega)\|_{\mathfrak{L}(\mathfrak{h}_n(\lambda))} d\lambda \leq \\
& \|q_{ac}^{el}\|_{\mathfrak{L}(\mathfrak{h}^{el})} \int_{\lambda \in \mathbb{R}} \|\widehat{\rho}^{el}(\lambda - (m-n)\omega)\|_{\mathfrak{L}(\mathfrak{h}_{m-n}(\lambda))} d\lambda
\end{aligned}$$

If the conditions (4.4) and (4.5) are satisfied, then

$$\sum_{m \in \mathbb{N}_0} \rho^{ph}(m) |q^{ph}(m)| \int_{\mathbb{R}} \|\widehat{\rho}^{el}(\lambda)\|_{\mathfrak{L}(\mathfrak{h}_m(\lambda))} \|q^{el}(\lambda)\|_{\mathfrak{L}(\mathfrak{h}_m(\lambda))} d\lambda < \infty$$

Further, we have

$$\begin{aligned}
& \sum_{m \in \mathbb{N}_0} \rho^{ph}(m) \sum_{n \in \mathbb{N}_0} |q^{ph}(n)| \int_{\lambda \in \mathbb{R}} \|\widehat{\rho}^{el}(\lambda - (m-n)\omega)\|_{\mathfrak{L}(\mathfrak{h}_{m-n}(\lambda))} d\lambda \leq \\
& (v_{\max} - v_{\min} + 4) \|\rho_{ac}^{el}\|_{\mathfrak{L}(\mathfrak{h}^{el})} \sum_{m \in \mathbb{N}_0} \rho^{ph}(m) \sum_{|m-n| \leq d_{\max}} |q^{ph}(n)| < \infty
\end{aligned}$$

where d_{\max} is introduced by Lemma 2.12. To prove

$$\sum_{m \in \mathbb{N}_0} \rho^{ph}(m) \sum_{|m-n| \leq d_{\max}} |q^{ph}(n)| < \infty$$

we use again (4.4) and (4.5). The last step admits to interchange the integral and the sums which immediately proves (4.12) \square

Corollary 4.4 *Let $S = \{H, H_0\}$ be the JCL-model. Assume that ρ and Q are given by (4.1) where ρ^{el} and q^{el} are density and charge operators for h_0^{el} and ρ^{ph} and q^{ph} for h^{ph} , respectively. If ρ^{el} is an equilibrium state, i.e. $\rho^{el} = f^{el}(h_0^{el})$, then*

$$J_{\rho, Q}^{ph} = \sum_{m, n \in \mathbb{N}_0} q^{ph}(n) \frac{1}{2\pi} \int_{\mathbb{R}} (\rho^{ph}(n) f^{el}(\lambda - n\omega) - \rho^{ph}(m) f^{el}(\lambda - m\omega)) \times \\ \text{tr} \left(\widehat{S}_{nm}^{ph}(\lambda) * q_{ac}^{el}(\lambda - n\omega) \widehat{S}_{nm}^{ph}(\lambda) \right) d\lambda. \quad (4.14)$$

Proof. From (4.12) we get

$$J_{\rho, Q}^{ph} = \sum_{n \in \mathbb{N}_0} q^{ph}(n) \sum_{m \in \mathbb{N}_0} \rho^{ph}(m) \frac{1}{2\pi} \int_{\mathbb{R}} d\lambda f^{el}(\lambda - m\omega) \times \\ \text{tr} \left(q_{ac}^{el}(\lambda - n\omega) \delta_{mn} - \widehat{S}_{nm}^{ph}(\lambda) * q_{ac}^{el}(\lambda - n\omega) \widehat{S}_{nm}^{ph}(\lambda) \right).$$

Hence

$$J_{\rho, Q}^{ph} = \sum_{n \in \mathbb{N}_0} q^{ph}(n) \frac{1}{2\pi} \int_{\mathbb{R}} d\lambda \sum_{m \in \mathbb{N}_0} \rho^{ph}(m) f^{el}(\lambda - m\omega) \times \\ \text{tr} \left(q_{ac}^{el}(\lambda - n\omega) \delta_{mn} - \widehat{S}_{nm}^{ph}(\lambda) * q_{ac}^{el}(\lambda - n\omega) \widehat{S}_{nm}^{ph}(\lambda) \right).$$

This gives

$$J_{\rho, Q}^{ph} = \sum_{n \in \mathbb{N}_0} q^{ph}(n) \frac{1}{2\pi} \int_{\mathbb{R}} d\lambda (\rho^{ph}(n) f^{el}(\lambda - n\omega) \text{tr} (q_{ac}^{el}(\lambda - n\omega)) - \\ \sum_{m \in \mathbb{N}_0} \rho^{ph}(m) f^{el}(\lambda - m\omega) \text{tr} \left(\widehat{S}_{nm}^{ph}(\lambda) * q_{ac}^{el}(\lambda - n\omega) \widehat{S}_{nm}^{ph}(\lambda) \right)). \quad (4.15)$$

Since

$$\sum_{m \in \mathbb{N}_0} \rho^{ph}(m) f^{el}(\lambda - m\omega) \text{tr} \left(\widehat{S}_{nm}^{ph}(\lambda) * q_{ac}^{el}(\lambda - n\omega) \widehat{S}_{nm}^{ph}(\lambda) \right) = \\ \sum_{m \in \mathbb{N}_0} (\rho^{ph}(m) f^{el}(\lambda - m\omega) - \rho^{ph}(n) f^{el}(\lambda - n\omega)) \times \\ \text{tr} \left(\widehat{S}_{nm}^{ph}(\lambda) * q_{ac}^{el}(\lambda - n\omega) \widehat{S}_{nm}^{ph}(\lambda) \right) + \\ \rho^{ph}(n) f^{el}(\lambda - n\omega) \sum_{m \in \mathbb{N}_0} \text{tr} \left(\widehat{S}_{nm}^{ph}(\lambda) * q_{ac}^{el}(\lambda - n\omega) \widehat{S}_{nm}^{ph}(\lambda) \right)$$

Inserting this into (4.15) we obtain (4.14). \square

5 Electron and photon currents

5.1 Electron current

To calculate the electron current induced by contacts and photons contact we make the following choice throughout this section. We set

$$Q_{\alpha}^{el} := q_{\alpha}^{el} \otimes q^{ph}, \quad q_{\alpha}^{el} := -\epsilon p_{\alpha}^{el} \quad \text{and} \quad q^{ph} := I_{h^{ph}}, \quad \alpha \in \{l, r\}, \quad (5.1)$$

where p_α^{el} denotes the orthogonal projection from \mathfrak{h}_α^{el} onto \mathfrak{h}_α^{el} . By $\epsilon > 0$ we denote the magnitude of the elementary charge. Since p_α^{el} commutes with h_α^{el} one easily verifies that Q_α^{el} commutes with H_0 which shows that Q_α^{el} is a charge. Following [1] the flux related to Q_α^{el} gives us the electron current $J_{\rho, Q_\alpha^{el}}^S$ entering the lead α from the sample. Notice $Q_\alpha^{el} = -\epsilon P_\alpha$ where P_α is the orthogonal projection from \mathfrak{H} onto $\mathfrak{H}_\alpha := \mathfrak{h}_\alpha^{el} \otimes \mathfrak{h}^{ph}$. Since $q^{ph} = I_{\mathfrak{h}^{ph}}$ the condition (4.5) is immediately satisfied for any $p \geq 0$.

Let $f(\cdot) : \mathbb{R} \rightarrow \mathbb{R}$ be a non-negative bounded measurable function. We set

$$\rho^{el} = \rho_l^{el} \oplus \rho_S^{el} \oplus \rho_r^{el}, \quad \rho_\alpha^{el} := f(h_\alpha^{el} - \mu_\alpha), \quad \alpha \in \{l, r\}. \quad (5.2)$$

and $\rho = \rho^{el} \otimes \rho^{ph}$. By μ_α the chemical potential of the lead α is denoted. In applications one sets $f(\lambda) := f_{FD}(\lambda)$, $\lambda \in \mathbb{R}$, where $f_{FD}(\lambda)$ is the so-called Fermi-Dirac distribution given by (3.21). If $\beta = \infty$, then $f_{FD}(\lambda) := \chi_{\mathbb{R}_-}(\lambda)$, $\lambda \in \mathbb{R}$. Notice that $[\rho^{el}, p^{el}] = 0$. For ρ^{ph} we choose the Gibbs state

$$\rho^{ph} := \frac{1}{Z} e^{-\beta h^{ph}}, \quad Z = \text{tr}(e^{-\beta h^{ph}}) = \frac{1}{1 - e^{-\beta \omega}}, \quad (5.3)$$

Hence $\rho^{ph} = (1 - e^{-\beta \omega}) e^{-\beta h^{ph}}$. If $\beta = \infty$, then $\rho^{ph} := (\cdot, \Upsilon_0) \Upsilon_0$. Obviously, $\text{tr}(\rho^{ph}) = 1$. We note that $\rho^{ph}(n) = (1 - e^{-\beta \omega}) e^{-n\beta \omega}$, $n \in \mathbb{N}_0$, satisfies the condition (4.4) for any $p \geq 0$. Obviously, $\rho_0 = \rho^{el} \otimes \rho^{ph}$ is a density operator for H_0 .

Definition 5.1 Let $S = \{H, H_0\}$ be the *JCL*-model. If $Q := Q_\alpha^{el}$, where Q_α^{el} is given by (5.1), and $\rho := \rho_0 := \rho^{el} \otimes \rho^{ph}$, where ρ^{el} and ρ^{ph} are given by (5.2) and (5.3), then $J_{\rho_0, Q_\alpha^{el}}^{el} := J_{\rho_0, Q_\alpha^{el}}^S$ is called the electron current entering the lead α . The currents $J_{\rho_0, Q_\alpha^{el}}^c$ and $J_{\rho_0, Q_\alpha^{el}}^{ph}$ are called the contact induced and photon induced electron currents.

5.1.1 Contact induced electron current

The following proposition immediately follows from Proposition 4.2.

Proposition 5.2 Let $S = \{H, H_0\}$ be the *JCL*-model. Then the contact induced electron current $J_{\rho_0, Q_\alpha^{el}}^c$, $\alpha \in \{l, r\}$, is given by $J_{\rho_0, Q_\alpha^{el}}^c = J_{\rho^{el}, q_\alpha^{el}}^{sc}$. In particular, one has

$$J_{\rho_0, Q_\alpha^{el}}^c = -\frac{\epsilon}{2\pi} \int_{\mathbb{R}} (f(\lambda - \mu_\alpha) - f(\lambda - \mu_\varkappa)) \sigma_c(\lambda) d\lambda, \quad \alpha, \varkappa \in \{l, r\}, \quad \alpha \neq \varkappa, \quad (5.4)$$

where $\{\sigma_c(\lambda)\}_{\lambda \in \mathbb{R}}$ is the channel cross-section from left to the right of the scattering system $s_c = \{h^{el}, h_0^{el}\}$, cf. Example 3.3.

Proof. Since $\text{tr}(\rho^{ph}) = 1$ it follows from Proposition 4.2 that $J_{\rho_0, Q_\alpha^{el}}^c = J_{\rho^{el}, q_\alpha^{el}}^{sc}$. From (3.26), cf. Example 3.3, we find (5.4). \square

If $\mu_l > \mu_r$ and $f(\cdot)$ is decreasing, then $J_{\rho_0, Q_\alpha^{el}}^c < 0$. Hence the electron contact current is going from the left lead to the right which is in accordance with the physical intuition. In particular, this is valid for the Fermi-Dirac distribution.

Proposition 5.3 Let $S = \{H, H_0\}$ be the *JCL*-model. Further, let ρ^{el} and ρ^{ph} be given by (5.2) and (5.3), respectively. If the charge Q_α^{el} is given by (5.1), then the following holds:

(E) If $\mu_l = \mu_r$, then $J_{\rho_0, Q_\alpha^{el}}^c = 0$, $\alpha \in \{l, r\}$.

(S) If $v_l \geq v_r + 4$, then $J_{\rho_0, Q_\alpha^{el}}^c = 0$, $\alpha \in \{l, r\}$, even if $\mu_l \neq \mu_r$.

(C) If $e_0^S = \delta_0^S$ and $e_1^S = \delta_1^S$, then $J_{\rho_0, Q_\alpha^{el}}^c = 0$, $\alpha \in \{l, r\}$, even if $\mu_l \neq \mu_r$.

Proof. (E) If $\mu_l = \mu_r$, then $f(\lambda - \mu_l) = f(\lambda - \mu_r)$. Applying formula (5.4) we obtain $J_{\rho_0, Q_\alpha^{el}}^c = 0$.

(S) If $v_l \geq v_r + 4$, then $h_0^{el, ac}$ has simple spectrum. Hence the scattering matrix $\{s_c(\lambda)\}_{\lambda \in \mathbb{R}}$ of the scattering system $s_c = \{h^{el}, h_0^{el}\}$ is a scalar function which immediately yields $\sigma_c(\lambda) = 0$, $\lambda \in \mathbb{R}$, which yields $J_{\rho_0, Q_\alpha^{el}}^c = 0$.

(C) In this case the Hamiltonian h^{el} decomposes into a direct sum of two Hamiltonians which do not interact. Hence the scattering matrix of $\{s_c(\lambda)\}_{\lambda \in \mathbb{R}}$ of the scattering system $s_c = \{h^{el}, h_0^{el}\}$ is diagonal which immediately yields $J_{\rho_0, Q_\alpha^{el}}^c = 0$. \square

5.1.2 Photon induced electron current

To analyze (4.12) is hopeless if we make no assumptions concerning ρ^{el} and the scattering operator s_c . The simplest assumption is that ρ^{el} and s_c commute. In this case we get $\widehat{\rho}^{el}(\lambda) = \rho^{el}(\lambda)$, $\lambda \in \mathbb{R}$.

Lemma 5.4 *Let $S = \{H, H_0\}$ be the JCL-model. Further let ρ^{el} be given by (5.2). If one of the cases (E), (S) or (C) of Proposition 5.3 is realized, then the ρ^{el} and s_c commute.*

Proof. If (E) holds, then $\rho^{el} = f(h_0^{el})$ which yields $[\rho^{el}, s_c] = 0$. If (S) is valid, then the scattering matrix $\{s_c(\lambda)\}_{\lambda \in \mathbb{R}}$ is a scalar function which shows $[\rho^{el}, s_c] = 0$. Finally, if (C) is realized, then the scattering matrix $\{s_c(\lambda)\}_{\lambda \in \mathbb{R}}$ is diagonal. Since the ρ^{el} is given by (5.2) we get $[\rho^{el}, s_c] = 0$. \square

We are going to calculate the current $J_{\rho_0, Q_\alpha^{el}}^{ph}$, see (4.12). Obviously, we have $P_\alpha(\lambda) = \sum_{n \in \mathbb{N}_0} p_\alpha^{el}(\lambda - n\omega)$ and $I_{\mathfrak{h}(\lambda)} = P_l(\lambda) + P_r(\lambda)$, $\lambda \in \mathbb{R}$. We set

$$P_{n_\alpha}(\lambda) := P_\alpha(\lambda)P_n(\lambda) = P_n(\lambda)P_\alpha(\lambda) = p_\alpha^{el}(\lambda - n\omega), \quad \alpha \in \{l, r\},$$

$n \in \mathbb{N}_0$, $\lambda \in \mathbb{R}$. In the following we use the notation $\widehat{T}_{ph}(\lambda) = \widehat{S}_{ph}(\lambda) - I_{\mathfrak{h}(\lambda)}$, $\lambda \in \mathbb{R}$, where $\{\widehat{T}_{ph}(\lambda)\}_{\lambda \in \mathbb{R}}$ is called the transition matrix and $\{\widehat{S}_{ph}(\lambda)\}_{\lambda \in \mathbb{R}}$ is given by (3.28). We set

$$\widehat{T}_{k_\alpha m_\varkappa}^{ph}(\lambda) := P_{k_\alpha}(\lambda)\widehat{T}_{ph}(\lambda)P_{m_\varkappa}(\lambda), \quad \lambda \in \mathbb{R}, \quad \alpha, \varkappa \in \{l, r\}, \quad k, m \in \mathbb{N}_0.$$

and

$$\widehat{\sigma}_{k_\alpha m_\varkappa}^{ph}(\lambda) = \text{tr}(\widehat{T}_{k_\alpha m_\varkappa}^{ph}(\lambda)^* \widehat{T}_{k_\alpha m_\varkappa}^{ph}(\lambda)), \quad \lambda \in \mathbb{R}, \quad (5.5)$$

which is the cross-section between the channels k_α and m_\varkappa .

Proposition 5.5 *Let $S = \{H, H_0\}$ be the JCL-model.*

(i) *If ρ^{el} commutes with the scattering operator s_c and q^{el} , then*

$$J_{\rho_0, Q_\alpha^{el}}^{ph} = - \sum_{\substack{m, n \in \mathbb{N}_0 \\ \varkappa \in \{l, r\}}} \frac{\mathfrak{e}}{2\pi} \int_{\mathbb{R}} \times \quad (5.6)$$

$$(\rho^{ph}(n)f(\lambda - \mu_\alpha - n\omega) - \rho^{ph}(m)f(\lambda - \mu_\varkappa - m\omega)) \widehat{\sigma}_{n_\alpha m_\varkappa}^{ph}(\lambda) d\lambda.$$

(ii) *If in addition $S = \{H, H_0\}$ is time reversible symmetric, then*

$$J_{\rho_0, Q_\alpha^{el}}^{ph} = - \sum_{m, n \in \mathbb{N}_0} \frac{\mathfrak{e}}{2\pi} \int_{\mathbb{R}} \times \quad (5.7)$$

$$(\rho^{ph}(n)f(\lambda - \mu_\alpha - n\omega) - \rho^{ph}(m)f(\lambda - \mu_{\alpha'} - m\omega)) \widehat{\sigma}_{n_\alpha m_{\alpha'}}^{ph}(\lambda) d\lambda,$$

$\alpha, \alpha' \in \{l, r\}$, $\alpha \neq \alpha'$.

Proof. (i) Let us assume that

$$q^{el} = \sum_{\varkappa \in \{l, r\}} g_{\varkappa}(h_{\varkappa}^{el}),$$

Notice that

$$q_{ac}^{el}(\lambda) = \sum_{\varkappa \in \{l, r\}} g_{\varkappa}(\lambda) p_{\varkappa}^{el}(\lambda), \quad \lambda \in \mathbb{R}. \quad (5.8)$$

Inserting (5.8) into (4.12) and using $q^{ph} = I_{\mathfrak{h}^{ph}}$ we get

$$\begin{aligned} J_{\rho_0, Q}^{ph} &= \sum_{\substack{m \in \mathbb{N}_0 \\ \alpha \in \{l, r\}}} \rho^{ph}(m) \sum_{\substack{n \in \mathbb{N}_0 \\ \varkappa \in \{l, r\}}} \frac{1}{2\pi} \int_{\mathbb{R}} d\lambda \phi_{\alpha}(\lambda - m\omega) g_{\varkappa}(\lambda - n\omega) \times \\ &\quad \text{tr} \left(p_{\alpha}^{el}(\lambda - m\omega) \left(p_{\varkappa}^{el}(\lambda - n\omega) \delta_{mn} - \widehat{S}_{nm}^{ph}(\lambda)^* p_{\varkappa}^{el}(\lambda - n\omega) \widehat{S}_{nm}^{ph}(\lambda) \right) \right) \end{aligned}$$

where for simplicity we have set

$$\phi_{\alpha}(\lambda) := f(\lambda - \mu_{\alpha}), \quad \lambda \in \mathbb{R}, \quad n \in \mathbb{N}_0, \quad \alpha \in \{l, r\}. \quad (5.9)$$

Obviously, we have

$$\begin{aligned} J_{\rho_0, Q}^{ph} &= \sum_{\substack{n \in \mathbb{N}_0 \\ \varkappa \in \{l, r\}}} \rho^{ph}(n) \frac{1}{2\pi} \int_{\mathbb{R}} d\lambda \phi_{\varkappa}(\lambda - n\omega) g_{\varkappa}(\lambda - n\omega) \text{tr} \left(p_{\varkappa}^{el}(\lambda - n\omega) \right) - \\ &\quad \sum_{\substack{n \in \mathbb{N}_0 \\ \varkappa \in \{l, r\}}} \sum_{\substack{m \in \mathbb{N}_0 \\ \alpha \in \{l, r\}}} \rho^{ph}(m) \frac{1}{2\pi} \int_{\mathbb{R}} d\lambda \phi_{\alpha}(\lambda - m\omega) g_{\varkappa}(\lambda - n\omega) \times \\ &\quad \text{tr} \left(p_{\alpha}^{el}(\lambda - m\omega) \widehat{S}_{nm}^{ph}(\lambda)^* p_{\varkappa}^{el}(\lambda - n\omega) \widehat{S}_{nm}^{ph}(\lambda) p_{\alpha}^{el}(\lambda - m\omega) \right). \end{aligned} \quad (5.10)$$

Since the scattering matrix $\{\widehat{S}^{ph}(\lambda)\}_{\lambda \in \mathbb{R}}$ is unitary we have

$$p_{\varkappa}^{el}(\lambda - n\omega) = \sum_{\substack{m \in \mathbb{N}_0 \\ \alpha \in \{l, r\}}} p_{\varkappa}^{el}(\lambda - n\omega) \widehat{S}_{mn}^{ph}(\lambda)^* p_{\alpha}^{el}(\lambda - m\omega) \widehat{S}_{mn}^{ph}(\lambda) p_{\varkappa}^{el}(\lambda - n\omega) \quad (5.11)$$

for $n \in \mathbb{N}_0$ and $\varkappa \in \{l, r\}$. Inserting (5.11) into (5.10) we find

$$\begin{aligned} J_{\rho_0, Q}^{ph} &= \sum_{\substack{n \in \mathbb{N}_0 \\ \varkappa \in \{l, r\}}} \sum_{\substack{m \in \mathbb{N}_0 \\ \alpha \in \{l, r\}}} \rho^{ph}(n) \frac{1}{2\pi} \int_{\mathbb{R}} d\lambda \phi_{\varkappa}(\lambda - n\omega) g_{\varkappa}(\lambda - n\omega) \times \\ &\quad \text{tr} \left(p_{\varkappa}^{el}(\lambda - n\omega) \widehat{S}_{nm}^{ph}(\lambda)^* p_{\alpha}^{el}(\lambda - m\omega) \widehat{S}_{mn}^{ph}(\lambda) p_{\varkappa}^{el}(\lambda - n\omega) \right) - \\ &\quad \sum_{\substack{n \in \mathbb{N}_0 \\ \varkappa \in \{l, r\}}} \sum_{\substack{m \in \mathbb{N}_0 \\ \alpha \in \{l, r\}}} \rho^{ph}(m) \frac{1}{2\pi} \int_{\mathbb{R}} d\lambda \phi_{\alpha}(\lambda - m\omega) g_{\varkappa}(\lambda - n\omega) \times \\ &\quad \text{tr} \left(p_{\alpha}^{el}(\lambda - m\omega) \widehat{S}_{nm}^{ph}(\lambda)^* p_{\varkappa}^{el}(\lambda - n\omega) \widehat{S}_{nm}^{ph}(\lambda) p_{\alpha}^{el}(\lambda - m\omega) \right). \end{aligned}$$

Using the notation (5.5) we find

$$\begin{aligned} J_{\rho_0, Q}^{ph} &= \sum_{\substack{n \in \mathbb{N}_0 \\ \varkappa \in \{l, r\}}} \sum_{\substack{m \in \mathbb{N}_0 \\ \alpha \in \{l, r\}}} \rho^{ph}(n) \frac{1}{2\pi} \int_{\mathbb{R}} d\lambda \phi_{\varkappa}(\lambda - n\omega) g_{\varkappa}(\lambda - n\omega) \widehat{\sigma}_{m_{\alpha} n_{\varkappa}}^{ph}(\lambda) - \\ &\quad \sum_{\substack{n \in \mathbb{N}_0 \\ \varkappa \in \{l, r\}}} \sum_{\substack{m \in \mathbb{N}_0 \\ \alpha \in \{l, r\}}} \rho^{ph}(m) \frac{1}{2\pi} \int_{\mathbb{R}} d\lambda \phi_{\alpha}(\lambda - m\omega) g_{\varkappa}(\lambda - n\omega) \widehat{\sigma}_{n_{\varkappa} m_{\alpha}}^{ph}(\lambda) : \end{aligned}$$

By (3.16) we find

$$\sum_{\substack{m \in \mathbb{N}_0 \\ \alpha \in \{l, r\}}} \widehat{\sigma}_{m_\alpha n_\alpha}^{ph}(\lambda) = \sum_{\substack{m \in \mathbb{N}_0 \\ \alpha \in \{l, r\}}} \widehat{\sigma}_{n_\alpha m_\alpha}^{ph}(\lambda) \quad \lambda \in \mathbb{R}.$$

Using that we get

$$J_{\rho_0, Q}^{ph} = \sum_{\substack{m, n \in \mathbb{N}_0 \\ \alpha, \varkappa \in \{l, r\}}} \frac{1}{2\pi} \int_{\mathbb{R}} \times \quad (5.12)$$

$$(\rho^{ph}(n)\phi_\varkappa(\lambda - n\omega) - \rho^{ph}(m)\phi_\alpha(\lambda - m\omega)) g_\varkappa(\lambda - n\omega) \widehat{\sigma}_{n_\alpha m_\alpha}^{ph}(\lambda) d\lambda.$$

Setting $g_\alpha(\lambda) = -\epsilon$ and $g_\varkappa(\lambda) \equiv 0$, $\varkappa \neq \alpha$, we obtain (5.6).

(ii) A straightforward computation shows that

$$\sum_{n, m \in \mathbb{N}_0} \int_{\mathbb{R}} (\rho^{ph}(n)f(\lambda - \mu_\alpha - n\omega) - \rho^{ph}(m)f(\lambda - \mu_\alpha - m\omega)) \widehat{\sigma}_{n_\alpha m_\alpha}^{ph}(\lambda) d\lambda =$$

$$\sum_{n, m \in \mathbb{N}_0} \int_{\mathbb{R}} (\rho^{ph}(m)f(\lambda - \mu_\alpha - m\omega) - \rho^{ph}(n)f(\lambda - \mu_\alpha - n\omega)) \widehat{\sigma}_{m_\alpha n_\alpha}^{ph}(\lambda) d\lambda$$

Since $\widehat{\sigma}_{m_\alpha n_\alpha}^{ph}(\lambda) = \widehat{\sigma}_{n_\alpha m_\alpha}^{ph}(\lambda)$, $\lambda \in \mathbb{R}$, we get

$$\sum_{n, m \in \mathbb{N}_0} \int_{\mathbb{R}} (\rho^{ph}(n)f(\lambda - \mu_\alpha - n\omega) - \rho^{ph}(m)f(\lambda - \mu_\alpha - m\omega)) \widehat{\sigma}_{n_\alpha m_\alpha}^{ph}(\lambda) d\lambda =$$

$$- \sum_{n, m \in \mathbb{N}_0} \int_{\mathbb{R}} (\rho^{ph}(n)f(\lambda - \mu_\alpha - n\omega) - \rho^{ph}(m)f(\lambda - \mu_\alpha - m\omega)) \widehat{\sigma}_{n_\alpha m_\alpha}^{ph}(\lambda) d\lambda$$

which yields

$$\sum_{n, m \in \mathbb{N}_0} \int_{\mathbb{R}} (\rho^{ph}(n)f(\lambda - \mu_\alpha - n\omega) - \rho^{ph}(m)f(\lambda - \mu_\alpha - m\omega)) \widehat{\sigma}_{n_\alpha m_\alpha}^{ph}(\lambda) d\lambda = 0.$$

Using that we get immediately the representation (5.7) from (5.6). \square

Corollary 5.6 *Let $S = \{H, H_0\}$ be the JCL-model.*

(i) *If the cases (E), (S) or (C) of Proposition 5.3 are realized, then the representation (5.6) holds.*

(ii) *If the case (E) of Proposition 5.3 is realized and the system $S = \{H, H_0\}$ is time reversible symmetric, then*

$$J_{\rho_0, Q_\alpha^{e'l}}^{ph} = \quad (5.13)$$

$$- \sum_{m, n \in \mathbb{N}_0} \frac{\epsilon}{2\pi} \int_{\mathbb{R}} (\rho^{ph}(n)f(\lambda - \mu - n\omega) - \rho^{ph}(m)f(\lambda - \mu - m\omega)) \widehat{\sigma}_{n_\alpha m_{\alpha'}}^{ph}(\lambda) d\lambda$$

$n \in \mathbb{N}_0$, $\alpha \in \{l, r\}$ where $\mu := \mu_l = \mu_r$ and $\alpha \neq \alpha'$.

(iii) *If the case (E) of Proposition 5.3 is realized and the system $S = \{H, H_0\}$ is time reversible and mirror symmetric, then $J_{\rho_0, Q_\alpha^{e'l}}^{ph} = 0$.*

Proof. (i) The statement follows from Proposition 5.5(i) and Lemma 5.4.

(ii) Setting $\mu_\alpha = \mu_{\alpha'}$ formula (5.13) follows (5.7).

(iii) If $\mathcal{S} = \{H, H_0\}$ is time reversible and mirror symmetric we get from Lemma 2.14(ii) that $\widehat{\sigma}_{n_\alpha m_{\alpha'}}^{ph}(\lambda) = \widehat{\sigma}_{n_{\alpha'} m_\alpha}^{ph}(\lambda)$, $\lambda \in \mathbb{R}$, $n, m \in \mathbb{N}_0$, $\alpha, \alpha' \in \{l, r\}$, $\alpha \neq \alpha'$. Using that we get from (5.13) that

$$\begin{aligned} J_{\rho_0, Q_\alpha^{el}}^{ph} &= \\ &- \sum_{m, n \in \mathbb{N}_0} \frac{\epsilon}{2\pi} \int_{\mathbb{R}} (\rho^{ph}(n)f(\lambda - \mu - n\omega) - \rho^{ph}(m)f(\lambda - \mu - m\omega)) \widehat{\sigma}_{n_\alpha m_\alpha}^{ph}(\lambda) d\lambda. \end{aligned}$$

Interchanging m and n we get

$$\begin{aligned} J_{\rho_0, Q_\alpha^{el}}^{ph} &= \\ &- \sum_{m, n \in \mathbb{N}_0} \frac{\epsilon}{2\pi} \int_{\mathbb{R}} (\rho^{ph}(m)f(\lambda - \mu - m\omega) - \rho^{ph}(n)f(\lambda - \mu - n\omega)) \widehat{\sigma}_{m_\alpha n_\alpha}^{ph}(\lambda) d\lambda. \end{aligned}$$

Using that \mathcal{S} is time reversible symmetric we get from Lemma 2.14(i) that

$$\begin{aligned} J_{\rho_0, Q_\alpha^{el}}^{ph} &= \\ &- \sum_{m, n \in \mathbb{N}_0} \frac{\epsilon}{2\pi} \int_{\mathbb{R}} (\rho^{ph}(m)f(\lambda - \mu - m\omega) - \rho^{ph}(n)f(\lambda - \mu - n\omega)) \widehat{\sigma}_{n_\alpha m_\alpha}^{ph}(\lambda) d\lambda. \end{aligned}$$

which shows that $J_{\rho_0, Q_\alpha^{el}}^{ph} = -J_{\rho_0, Q_\alpha^{el}}^{ph}$. Hence $J_{\rho_0, Q_\alpha^{el}}^{ph} = 0$. \square

We note that by Proposition 5.3 the contact induced current is zero, i.e. $J_{\rho_0, Q_\alpha^{el}}^c = 0$. Hence, if the \mathcal{S} is time reversible and mirror symmetric, then the total current is zero, i.e. $J_{\rho_0, Q_\alpha^{el}}^S = 0$.

Remark 5.7 Let the case (E) of Proposition 5.3 be realized, that is, $\mu_l = \mu_r$. Moreover, we assume for simplicity that $0 =: v_r \leq v := v_l$.

- (i) If $\beta = \infty$, then $\rho^{ph}(n) = \delta_{0n}$, $n \in \mathbb{N}_0$. From (5.6) we immediately get that $J_{\rho^{el}, Q_\alpha^{el}}^{ph} = 0$. That means, if the temperature is zero, then the photon induced electron current is zero.
- (ii) The photon induced electron current might be zero even if $\beta < \infty$. Indeed, let $\mathcal{S} = \{H, H_0\}$ be time reversible symmetric and let the case (E) be realized. If $\omega \geq v + 4$ and , then $\mathfrak{h}^{el}(\lambda) := \mathfrak{h}_n^{el}(\lambda) = \mathfrak{h}^{el}(\lambda - n\omega)$, $n \in \mathbb{N}_0$. Hence one always has $n = m$ in formula (5.13) which immediately yields $J_{\rho_0, Q_\alpha^{el}}^{ph} = 0$.
- (iii) The photon induced electron current might be different from zero. Indeed, let $\mathcal{S} = \{H, H_0\}$ be time reversible symmetric and let $v = 2$ and $\omega = 4$, then one gets that to calculate the $J_{\rho_0, Q_l^{el}}^{ph}$ one has to take into account $m = n + 1$ in formula (5.13). Therefore we find

$$\begin{aligned} J_{\rho_0, Q_l^{el}}^{ph} &= - \sum_{n \in \mathbb{N}_0} \frac{\epsilon}{2\pi} \int_{\mathbb{R}} d\lambda \times \\ &(\rho^{ph}(n)f(\lambda - \mu - n\omega) - \rho^{ph}(n+1)f(\lambda - \mu - (n+1)\omega)) \widehat{\sigma}_{n_l (n+1)_r}^{ph}(\lambda). \end{aligned}$$

If ρ^{ph} is given by (5.3) and $f(\lambda) = f_{FD}(\lambda)$, cf. (3.21), then one easily verifies that

$$\frac{\partial}{\partial x} \rho^{ph}(x) f_{FD}(\lambda - \mu - x\omega) < 0, \quad x, \mu, \lambda \in \mathbb{R}.$$

Hence $\rho^{ph}(n)f_{FD}(\lambda - \mu - n\omega)$ is decreasing in $n \in \mathbb{N}_0$ for $\lambda, \mu \in \mathbb{R}$ which yields $(\rho^{ph}(n)f(\lambda - \mu - n\omega) - \rho^{ph}(n+1)f(\lambda - \mu - (n+1)\omega)) \geq 0$. Therefore $J_{\rho_0, Q_i^{el}}^{ph} \leq 0$ which means that the photon induced current leaves the left-hand side and enters the right-hand side. In fact $J_{\rho_0, Q_i^{el}}^{ph} = 0$ implies that $\hat{\sigma}_{n_l(n+1)_r}^{ph}(\lambda) = 0$ for $n \in \mathbb{N}_0$ and $\lambda \in \mathbb{R}$ which means that there is no scattering from the left-hand side to the right one and vice versa which can be excluded generically.

5.2 Photon current

The photon current is related to the charge

$$Q := Q^{ph} = -I_{\mathfrak{h}^{el}} \otimes \mathfrak{n},$$

where $\mathfrak{n} = d\Gamma(1) = b^*b$ is the photon number operator on $\mathfrak{h}^{ph} = \mathfrak{F}_+(\mathbb{C})$, which is self-adjoint and commutes with h^{ph} . It follows that Q^{ph} is also self-adjoint and commutes with H_0 . It is not bounded, but since $\text{dom}(\mathfrak{n}) = \text{dom}(h^{ph})$, it is immediately obvious that $Q^{ph}(H_0 + \theta)^{-1}$ is bounded, whence \mathfrak{N} is a tempered charge. Its charge matrix with respect to the spectral representation $\Pi(H_0^{ac})$ of Lemma 2.12 is given by

$$Q_{ac}^{ph}(\lambda) = - \bigoplus_{n \in \mathbb{N}_0} n P_n(\lambda).$$

We recall that $P_n(\lambda)$ is the orthogonal projection from $\mathfrak{h}(\lambda)$ onto $\mathfrak{h}_n(\lambda) = \mathfrak{h}^{el}(\lambda - n\omega)$, $\lambda \in \mathbb{R}$. We are going to calculate the photon current or, how it is also called, the photon production rate.

5.2.1 Contact induced photon current

The following proposition is in accordance with the physical intuition.

Proposition 5.8 *Let $S = \{H, H_0\}$ be the JCL-model. Then $J_{\rho_0, Q^{ph}}^c = 0$.*

Proof. We note that $q_{ac}^{el}(\lambda) = I_{\mathfrak{h}^{el}(\lambda)}$, $\lambda \in \mathbb{R}$. Inserting this into (3.25) we get $J_{\rho_0, q^{el}}^{sc} = 0$. Applying Proposition 4.2 we prove $J_{\rho_0, Q^{ph}}^c = 0$. \square

The result reflects the fact that the lead contact does not contribute to the photon current which is plausible.

5.2.2 Photon current

From the Proposition 5.8 we get that only the photon induced photon current $J_{\rho_0, Q^{ph}}^{ph}$ contributes to the photon current $J_{\rho_0, Q^{ph}}^S$. Since $J_{\rho_0, Q^{ph}}^S = J_{\rho_0, Q^{ph}}^{ph}$ we call $J_{\rho_0, Q^{ph}}^{ph}$ simply the *photon current*.

Using the notation $\hat{T}_{nm}^{ph}(\lambda) := P_n(\lambda) \hat{T}_{ph}(\lambda) \upharpoonright \mathfrak{h}^{el}(\lambda - m\omega)$, $\lambda \in \mathbb{R}$, $m, n \in \mathbb{N}_0$. We set

$$\tilde{T}_{nm}^{ph}(\lambda) = \hat{T}_{nm}^{ph}(\lambda) s_c(\lambda - m\omega), \quad \lambda \in \mathbb{R}, \quad m, n \in \mathbb{N}_0 \quad (5.14)$$

and

$$\tilde{T}_{n\alpha m_\alpha}^{ph}(\lambda) := P_{n_\alpha}(\lambda) \tilde{T}_{nm}^{ph}(\lambda) \upharpoonright \mathfrak{h}_\alpha^{el}(\lambda - m\omega), \quad \lambda \in \mathbb{R}, \quad (5.15)$$

$m, n \in \mathbb{N}_0$, $\alpha, \alpha' \in \{l, r\}$, as well as $\tilde{\sigma}_{n_\alpha m_\alpha}^{ph}(\lambda) := \text{tr}(\tilde{T}_{n_\alpha m_\alpha}^{ph}(\lambda) * \tilde{T}_{n_\alpha m_\alpha}^{ph}(\lambda))$, $\lambda \in \mathbb{R}$.

Proposition 5.9 *Let $S = \{H, H_0\}$ be the JCL-model.*

(i) Then

$$J_{\rho_0, Q^{ph}}^{ph} = \sum_{\substack{m, n \in \mathbb{N}_0 \\ \alpha, \alpha' \in \{l, r\}}} (n - m) \rho^{ph}(m) \frac{1}{2\pi} \int_{\mathbb{R}} f(\lambda - \mu_\alpha - m\omega) \widehat{\sigma}_{n\alpha' m_\alpha}^{ph}(\lambda) d\lambda \quad (5.16)$$

(ii) If ρ^{el} commutes with s_c , then

$$J_{\rho_0, Q^{ph}}^{ph} = \sum_{\substack{m, n \in \mathbb{N}_0 \\ \alpha, \alpha' \in \{l, r\}}} (n - m) \rho^{ph}(m) \frac{1}{2\pi} \int_{\mathbb{R}} f(\lambda - \mu_\alpha - m\omega) \widehat{\sigma}_{n\alpha' m_\alpha}^{ph}(\lambda) d\lambda \quad (5.17)$$

(iii) If ρ^{el} commutes with s_c and $\mathcal{S} = \{H, H_0\}$ is time reversible symmetric, then

$$J_{\rho_0, Q^{ph}}^{ph} = \sum_{\substack{m, n \in \mathbb{N}_0, n > m \\ \alpha, \alpha' \in \{l, r\}}} \frac{1}{2\pi} \int_{\mathbb{R}} d\lambda \times \quad (5.18)$$

$$(n - m) (\rho^{ph}(m) f(\lambda - \mu_\alpha - m\omega) - \rho^{ph}(n) f(\lambda - \mu_{\alpha'} - n\omega)) \widehat{\sigma}_{n\alpha' m_\alpha}^{ph}(\lambda)$$

where $\alpha' \in \{l, r\}$ and $\alpha' \neq \alpha$.

Proof. (i) From (4.12) we get

$$J_{\rho_0, Q^{ph}}^{ph} = - \sum_{m, n \in \mathbb{N}_0} n \rho^{ph}(m) \frac{1}{2\pi} \int_{\mathbb{R}} d\lambda \operatorname{tr} \left(\widehat{\rho}_{ac}^{el}(\lambda - m\omega) \times \right. \\ \left. \left(P_n(\lambda) \delta_{mn} - \widehat{S}_{nm}^{ph}(\lambda)^* q_{ac}^{el}(\lambda - n\omega) \widehat{S}_{nm}^{ph}(\lambda) \right) \right).$$

Hence

$$J_{\rho_0, Q^{ph}}^{ph} = - \sum_{m \in \mathbb{N}_0} m \rho^{ph}(m) \times \\ \frac{1}{2\pi} \int_{\mathbb{R}} \operatorname{tr} \left(\widehat{\rho}_{ac}^{el}(\lambda - m\omega) \left(P_m(\lambda) - \widehat{S}_{mm}^{ph}(\lambda)^* P_m(\lambda) \widehat{S}_{mm}^{ph}(\lambda) \right) \right) d\lambda + \\ \sum_{\substack{m, n \in \mathbb{N}_0 \\ m \neq n}} n \rho^{ph}(m) \frac{1}{2\pi} \int_{\mathbb{R}} \operatorname{tr} \left(\widehat{\rho}_{ac}^{el}(\lambda - m\omega) \widehat{S}_{nm}^{ph}(\lambda)^* P_n(\lambda) \widehat{S}_{nm}^{ph}(\lambda) \right) d\lambda.$$

Using the relation $P_m(\lambda) = I_{\mathfrak{h}(\lambda)} - \sum_{n \in \mathbb{N}_0, m \neq n} P_n(\lambda)$, $\lambda \in \mathbb{R}$, we get

$$J_{\rho_0, Q^{ph}}^{ph} = \\ - \sum_{\substack{m, n \in \mathbb{N}_0 \\ m \neq n}} m \rho^{ph}(m) \frac{1}{2\pi} \int_{\mathbb{R}} \operatorname{tr} \left(\widehat{\rho}_{ac}^{el}(\lambda - m\omega) \left(\widehat{S}_{nm}^{ph}(\lambda)^* P_n(\lambda) \widehat{S}_{nm}^{ph}(\lambda) \right) \right) d\lambda + \\ \sum_{\substack{m, n \in \mathbb{N}_0 \\ m \neq n}} n \rho^{ph}(m) \frac{1}{2\pi} \int_{\mathbb{R}} \operatorname{tr} \left(\widehat{\rho}_{ac}^{el}(\lambda - m\omega) \widehat{S}_{nm}^{ph}(\lambda)^* P_n(\lambda) \widehat{S}_{nm}^{ph}(\lambda) \right) d\lambda.$$

Since $\widehat{T}_{ph}(\lambda) = \widehat{S}_{ph}(\lambda) - I_{\mathfrak{h}(\lambda)}$, $\lambda \in \mathbb{R}$, we find

$$J_{\rho_0, Q^{ph}}^{ph} = \\ - \sum_{m, n \in \mathbb{N}_0} (m - n) \rho^{ph}(m) \frac{1}{2\pi} \int_{\mathbb{R}} \operatorname{tr} \left(\widehat{\rho}_{ac}^{el}(\lambda - m\omega) \widehat{T}_{nm}^{ph}(\lambda)^* \widehat{T}_{nm}^{ph}(\lambda) \right) d\lambda.$$

Using (4.13) and definition (5.14) one gets

$$J_{\rho_0, Q^{ph}}^{ph} = - \sum_{m, n \in \mathbb{N}_0} (m - n) \rho^{ph}(m) \frac{1}{2\pi} \int_{\mathbb{R}} \text{tr} \left(\rho_{ac}^{el}(\lambda - m\omega) \tilde{T}_{nm}^{ph}(\lambda) * \tilde{T}_{nm}^{ph}(\lambda) \right) d\lambda.$$

Since $\rho_{ac}^{el} = \rho_l^{el} \oplus \rho_r^{el}$ where ρ_α^{el} is given by (5.2) we find

$$J_{\rho_0, Q^{ph}}^{ph} = - \sum_{\substack{m, n \in \mathbb{N}_0 \\ \alpha, \varkappa \in \{l, r\}}} (m - n) \rho^{ph}(m) \frac{1}{2\pi} \int_{\mathbb{R}} f(\lambda - \mu_\alpha - m\omega) \text{tr} \left(\tilde{T}_{n\varkappa m_\alpha}^{ph}(\lambda) * \tilde{T}_{n\varkappa m_\alpha}^{ph}(\lambda) \right) d\lambda$$

where we have used (5.15). Using $\tilde{\sigma}_{n\varkappa m_\alpha}^{ph}(\lambda) = \text{tr}(\tilde{T}_{n\varkappa m_\alpha}^{ph}(\lambda) * \tilde{T}_{n\varkappa m_\alpha}^{ph}(\lambda))$ we prove (5.16).

(ii) If ρ_{ac}^{el} commutes with s_c , then $\hat{\rho}_{ac}^{el}(\lambda) = \rho_{ac}^{el}(\lambda)$, $\lambda \in \mathbb{R}$ which yields that one can replace $\tilde{\sigma}_{n\varkappa m_\alpha}^{ph}(\lambda)$ by $\hat{\sigma}_{n\varkappa m_\alpha}^{ph}(\lambda)$, $\lambda \in \mathbb{R}$. Therefore (5.17) holds.

(iii) Obviously we have

$$J_{\rho_0, Q^{ph}}^{ph} = \sum_{\substack{m, n \in \mathbb{N}_0, n > m \\ \alpha, \varkappa \in \{l, r\}}} (n - m) \rho^{ph}(m) \frac{1}{2\pi} \int_{\mathbb{R}} f(\lambda - \mu_\alpha - m\omega) \hat{\sigma}_{n\varkappa m_\alpha}^{ph}(\lambda) d\lambda + \sum_{\substack{m, n \in \mathbb{N}_0, n < m \\ \alpha, \varkappa \in \{l, r\}}} (n - m) \rho^{ph}(m) \frac{1}{2\pi} \int_{\mathbb{R}} f(\lambda - \mu_\alpha - m\omega) \hat{\sigma}_{n\varkappa m_\alpha}^{ph}(\lambda) d\lambda. \quad (5.19)$$

Moreover, a straightforward computation shows that

$$\sum_{\substack{m, n \in \mathbb{N}_0, n < m \\ \alpha, \varkappa \in \{l, r\}}} (n - m) \rho^{ph}(m) \frac{1}{2\pi} \int_{\mathbb{R}} f(\lambda - \mu_\alpha - m\omega) \hat{\sigma}_{n\varkappa m_\alpha}^{ph}(\lambda) d\lambda = \sum_{\substack{m, n \in \mathbb{N}_0, n > m \\ \alpha, \varkappa \in \{l, r\}}} (m - n) \rho^{ph}(n) \frac{1}{2\pi} \int_{\mathbb{R}} f(\lambda - \mu_\varkappa - n\omega) \hat{\sigma}_{m_\alpha n_\varkappa}^{ph}(\lambda) d\lambda.$$

Since $\mathcal{S} = \{H, H_0\}$ is time reversible symmetric we find

$$\sum_{\substack{m, n \in \mathbb{N}_0, n < m \\ \alpha, \varkappa \in \{l, r\}}} (n - m) \rho^{ph}(m) \frac{1}{2\pi} \int_{\mathbb{R}} f(\lambda - \mu_\alpha - m\omega) \hat{\sigma}_{m_\alpha n_\varkappa}^{ph}(\lambda) d\lambda = \sum_{\substack{m, n \in \mathbb{N}_0, n > m \\ \alpha, \varkappa \in \{l, r\}}} (m - n) \rho^{ph}(n) \frac{1}{2\pi} \int_{\mathbb{R}} f(\lambda - \mu_\varkappa - n\omega) \hat{\sigma}_{n_\varkappa m_\alpha}^{ph}(\lambda) d\lambda. \quad (5.20)$$

Inserting (5.20) into (5.19) we obtain (5.18). \square

Corollary 5.10 *Let $\mathcal{S} = \{H, H_0\}$ be the JCL-model and let $f = f_{FD}$. If case (E) of Proposition 5.3 is realized and $\mathcal{S} = \{H, H_0\}$ is time reversible symmetric, then $J_{\rho_0, Q^{ph}}^{ph} \geq 0$.*

Proof. We set $\mu := \mu_l = \mu_r$. One has

$$\rho^{ph}(m) f(\lambda - \mu - m\omega) - \rho^{ph}(n) f(\lambda - \mu - n\omega) = e^{-m\beta\omega} (1 - e^{-(n-m)\beta\omega}) f_{FD}(\lambda - \mu - m\omega) f_{FD}(\lambda - \mu - n\omega) \geq 0$$

for $n > m$. From (5.18) we get $J_{\rho_0, Q^{ph}}^{ph} \geq 0$. □

Remark 5.11 Let us comment the results. If $J_{\rho_0, Q^{ph}}^{ph} \geq 0$, then system \mathcal{S} is called light emitting. Similarly, if $J_{\rho_0, Q^{ph}}^{ph} \leq 0$, then we call it light absorbing. Of course if \mathcal{S} is light emitting and absorbing, then $J_{\rho_0, Q^{ph}}^{ph} = 0$.

(i) If $\beta = \infty$, then $\rho^{ph}(m) = \delta_{0m}$, $m \in \mathbb{N}_0$. Inserting this into (5.16) we get

$$J_{\rho_0, Q^{ph}}^{ph} = \sum_{\substack{n \in \mathbb{N}_0 \\ \alpha, \varkappa \in \{l, r\}}} n \frac{1}{2\pi} \int_{\mathbb{R}} f(\lambda - \mu_\alpha) \tilde{\sigma}_{n, \varkappa 0_\alpha}^{ph}(\lambda) d\lambda \geq 0$$

Hence \mathcal{S} is light emitting.

(ii) Let us show \mathcal{S} might be light emitting even if $\beta < \infty$. We consider the case (E) of Proposition 5.3. If \mathcal{S} is time reversible symmetric, then it follows from Corollary 5.10 that the system is light emitting.

If the system \mathcal{S} is time reversible and mirror symmetric, then $J_{\rho_0, Q_\alpha^{el}}^{ph} = 0$, $\alpha \in \{l, r\}$, by Corollary 5.6(iii). Since $J_{\rho_0, Q^{el}}^c = 0$ by Proposition 5.3 we get that $J_{\rho_0, Q_\alpha^{el}}^S = 0$ but the photon current is larger than zero. So our JCL -model is light emitting by a zero total electron current $J_{\rho_0, Q_\alpha^{el}}^S$.

Let $v_r = 0$, $v_l = 2$ and $\omega = 4$. Hence \mathcal{S} is not mirror symmetric. Then we get from Remark 5.7(iii) that $J_{\rho_0, Q_l^{el}}^{ph} = -J_{\rho_0, Q_r^{el}}^{ph} \leq 0$. Hence there is an electron current from the left to the right lead. Notice that by Proposition 5.3 $J_{\rho_0, Q_l^{el}}^c = 0$. Hence $J_{\rho_0, Q_l^{el}}^S \leq 0$.

(iii) To realize a light absorbing situation we consider the case (S) of Proposition 5.3 and assume that \mathcal{S} is time reversible symmetric. Notice that by Lemma 5.4 s_c commutes with ρ^{el} . We make the choice

$$v_r = 0, \quad v_l \geq 4, \quad \omega = v_l, \quad \mu_l = 0, \quad \mu_r = \omega = v_l.$$

It turns out that with respect to the representation (5.18) one has only to $m = n - 1$, $\varkappa = r$ and $\alpha = l$. Hence

$$J_{\rho_0, Q^{ph}}^{ph} = \sum_{n \in \mathbb{N}} \frac{1}{2\pi} \int_{\mathbb{R}} d\lambda \times \\ (\rho^{ph}(n-1)f(\lambda - (n-1)\omega) - \rho^{ph}(n)f(\lambda - (n+1)\omega)) \hat{\sigma}_{n_l(n-1)_r}^{ph}(\lambda)$$

Since $f(\lambda) = f_{FD}(\lambda)$ we find

$$\rho^{ph}(n-1)f(\lambda - (n-1)\omega) - \rho^{ph}(n)f(\lambda - (n+1)\omega) = \\ \rho^{ph}(n-1)f(\lambda - (n-1)\omega)f(\lambda - (n+1)\omega) \times \\ \left(1 + e^{\beta(\lambda - (n+1)\omega)} - e^{-\beta\omega}(1 + e^{\beta(\lambda - \omega(n-1))})\right)$$

or

$$\rho^{ph}(n-1)f(\lambda - (n-1)\omega) - \rho^{ph}(n)f(\lambda - (n+1)\omega) = \\ \rho^{ph}(n-1)f(\lambda - (n-1)\omega)f(\lambda - (n+1)\omega)(1 - e^{-\beta\omega})(1 - e^{\beta(\lambda - \omega n)}).$$

Since $\lambda - n\omega \geq 0$ we find $\rho^{ph}(n-1)f(\lambda - (n-1)\omega) - \rho^{ph}(n)f(\lambda - (n+1)\omega) \leq 0$ which yields $J_{\rho_0, Q^{ph}}^{ph} \leq 0$.

To calculate $J_{\rho_0, Q_i^{e_l}}^{ph}$ we use formula (5.7). Setting $\alpha = l$ we get $\alpha' = r$ which yields

$$J_{\rho_0, Q_i^{e_l}}^{ph} = - \sum_{m, n \in \mathbb{N}_0} \frac{\epsilon}{2\pi} \int_{\mathbb{R}} d\lambda \times \\ (\rho^{ph}(n)f(\lambda - \mu_r - n\omega) - \rho^{ph}(m)f(\lambda - \mu_l - m\omega)) \widehat{\sigma}_{n_l m_r}^{ph}(\lambda),$$

One checks that $\widehat{\sigma}_{0_l 0_r}^{ph}(\lambda) = 0$ and $\widehat{\sigma}_{n_l m_r}^{ph}(\lambda) = 0$ for $m \neq n + 1$, $n \in \mathbb{N}$. Hence

$$J_{\rho_0, Q_i^{e_l}}^{ph} = - \sum_{n \in \mathbb{N}} \frac{\epsilon}{2\pi} \int_{\mathbb{R}} d\lambda \times \\ (\rho^{ph}(n)f(\lambda - \mu_r - n\omega) - \rho^{ph}(n-1)f(\lambda - \mu_l - (n+1)\omega)) \widehat{\sigma}_{n_l (n+1)_r}^{ph}(\lambda),$$

Since $\mu_r = \omega$ and $\mu_l = 0$ we find

$$J_{\rho_0, Q_i^{e_l}}^{ph} = - \sum_{n \in \mathbb{N}} \frac{\epsilon}{2\pi} \int_{\mathbb{R}} \times \\ f(\lambda - (n+1)\omega) \rho^{ph}(n-1) (1 - e^{-\beta\omega}) \widehat{\sigma}_{n_l (n+1)_r}^{ph}(\lambda) d\lambda \leq 0.$$

Hence there is a current going from the left to right induced by photons. We recall that $J_{\rho_0, Q_i^{e_l}}^c = 0$.

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