Weierstraß-Institut
für Angewandte Analysis und Stochastik
Leibniz-Institut im Forschungsverbund Berlin e. V.

Preprint ISSN 2198-5855

Existence and asymptotic stability of a periodic solution with boundary layers of reaction-diffusion equations with singularly perturbed Neumann boundary conditions

Valentin F. Butuzov[1], Nikolai N. Nefedov[1], Lutz Recke[2], Klaus R. Schneider[3]

submitted: December 10, 2013

[1] Moscow State University
Faculty of Physics
Department of Mathematics
Vorob'jovy Gory
19899 Moscow, Russia
E-Mail: butuzov@phys.msu.ru
nefedovov@phys.msu.ru

[2] Institut für Mathematik
Humboldt Universität zu Berlin
Rudower Chausse 25
12489 Berlin, Germany
E-Mail: recke@mathemtik.hu-berlin.de

[3] Weierstraß-Institut
Mohrenstraße 39
10117 Berlin, Germany
E-Mail: Klaus.Schneider@wias-berlin.de

No. 1893

Berlin 2014

2010 *Mathematics Subject Classification.* 35B25 35B35 35B12 35K57.

Key words and phrases. singularly perturbed parabolic periodic problems, exponential asymptotic stability, Krein-Rutman theorem, lower and upper solutions.

Acknowledgements. This work was partially supported by RFBR grants N 12-01-00387, N 13-01-00200 and the program of cooperation of the Moscow State University and the Humboldt University of Berlin.

Edited by
Weierstraß-Institut für Angewandte Analysis und Stochastik (WIAS)
Leibniz-Institut im Forschungsverbund Berlin e. V.
Mohrenstraße 39
10117 Berlin
Germany

Fax: +49 30 20372-303
E-Mail: preprint@wias-berlin.de
World Wide Web: http://www.wias-berlin.de/

Abstract

We consider singularly perturbed reaction-diffusion equations with singularly perturbed Neumann boundary conditions. We establish the existence of a time-periodic solution $u(x,t,\varepsilon)$ with boundary layers and derive conditions for their asymptotic stability The boundary layer part of $u(x,t,\varepsilon)$ is of order one, which distinguishes our case from the case of regularly perturbed Neumann boundary conditions, where the boundary layer is of order ε. Another peculiarity of our problem is that - in contrast to the case of Dirichlet boundary conditions - it may have several asymptotically stable time-periodic solutions, where these solutions differ only in the desribtion of the boundary layers. Our approach is based on the construction of sufficiently precise lower and upper solutions.

1 Statement of the problem.

We consider the following singularly perturbed parabolic periodic boundary value problem with singularly perturbed Neumann conditions

$$
\begin{aligned}
&N_\varepsilon(u) := \varepsilon^2\left(\frac{\partial^2 u}{\partial x^2} - \frac{\partial u}{\partial t}\right) - f(u,x,t,\varepsilon) = 0 \\
&\quad \text{for} \quad (x,t) \in \mathcal{D} := \{(x,t) \in \mathbb{R}^2 : -1 < x < 1, t \in \mathbb{R}\}, \\
&\varepsilon\,\frac{\partial u}{\partial x}(-1,t,\varepsilon) = u^{(-)}(t), \quad \varepsilon\,\frac{\partial u}{\partial x}(1,t,\varepsilon) = u^{(+)}(t) \quad \text{for} \quad t \in \mathbb{R}, \\
&u(x,t,\varepsilon) = u(x,t+T,\varepsilon) \quad \text{for} \quad t \in \mathbb{R}, \quad -1 \le x \le 1
\end{aligned}
\tag{1.1}
$$

for $\varepsilon \in I_{\varepsilon_0} := \{0 < \varepsilon \le \varepsilon_0\},\ 0 < \varepsilon_0 \ll 1,\ f,\ u^{(-)}$ and $u^{(+)}$ are sufficiently smooth and T-periodic in t.

Our interest in such problems is motivated by reaction-diffusion problems with a strong flow on the boundary. This fact is described in the paper by Nesterov [1], where a linear reaction-diffusion equation was considered.
Another motivation comes from the study of singularly perturbed problems with multiple roots of the degenerate equation. We illustrate this fact by the following problem with a double root of

the degenerate equation (see a related problem in [2]).

$$\begin{aligned}
&\varepsilon^3\left(\frac{\partial^2 v}{\partial x^2}-\frac{\partial v}{\partial t}\right)- \\
&\Big[v-(\varphi(x,t)+\varepsilon\varphi_1(x,t))\Big]\Big[v-(\varphi(x,t)+\varepsilon\varphi_2(x,t))\Big]=0 \\
&\quad \text{for} \quad (x,t)\in\mathcal{D}, \\
&\frac{\partial v}{\partial x}(-1,t,\varepsilon)=v^{(-)}(t), \quad \frac{\partial v}{\partial x}(1,t,\varepsilon)=v^{(+)}(t) \quad \text{for} \quad t\in\mathbb{R}, \\
&v(x,t,\varepsilon)=v(x,t+T,) \quad \text{for} \quad t\in\mathbb{R}, \quad -1\le x\le 1.
\end{aligned} \tag{1.2}$$

It is obvious that the substitution $u=(v-\varphi(x,t))/\varepsilon$ transforms problem (1.2) into a problem of the type (1.1).

Our goal is to establish the existence of a T-periodic solution of problem (1.1) with a boundary layer, and to investigate the stability of this solution in the Lyapunov sense. Our approach is based on the asymptotic method of differential inequalities which has been applied successfully to different classes of singularly perturbed problems (see the survey paper [3]). The main idea of this method is to construct sufficiently precise lower and upper solutions of the problem by means of formal asymptotic expansions and to apply the results from [4], where we have developed an approach to investigate the asymptotic stability of periodic solutions to singularly perturbed reaction-advection-diffusion equations by using the theorem of Krein-Rutman. The construction of lower and upper solutions to problem (1.1) extends the approaches in the papers [5, 6, 7, 3].

2 Assumptions. Main result

We consider problem (1.1) under the following assumptions:

(A_0). *f, $u^{(-)}$ and $u^{(+)}$ are sufficiently smooth and T-periodic in t.*

If we put $\varepsilon=0$ in (1.1) we get the so-called degenerate equation

$$f(u,x,t,0)=0.$$

Concerning this equation we assume

(A_1). *The degenerate equation has a solution $u=\varphi(x,t)$ such that*

$$f_u(\varphi(x,t),x,t,0)>0 \qquad \textit{for} \quad (x,t)\in\overline{\mathcal{D}}.$$

For a formal asymptotic solution $U(x,t,\varepsilon)$ of (1.1) we use the ansatz

$$\begin{aligned}
U(x,t,\varepsilon)&=\overline{U}(x,t,\varepsilon)+Q(\xi,t,\varepsilon)+R(\eta,t,\varepsilon) \\
&=\sum_{i=0}^{\infty}\varepsilon^i\left(\overline{U}_i(x,t)+Q_i(\xi,t)+R_i(\eta,t)\right),
\end{aligned} \tag{2.1}$$

where $\overline{U}$ denotes the regular part and Q and R denote the boundary layer parts of the formal asymptotic solution U, the stretched variables ξ and η are defined as

$$\xi = \frac{1+x}{\varepsilon}, \quad \eta = \frac{1-x}{\varepsilon}.$$

To formulate the next assumptions we consider the boundary value problems

$$\begin{aligned} &\frac{\partial^2 Q_0}{\partial \xi^2} = f\Big(\varphi(-1,t) + Q_0, -1, t, 0\Big) \quad \text{for} \quad \xi > 0, \quad t \in \mathbb{R}, \\ &\frac{\partial Q_0}{\partial \xi}(0,t) = u^{(-)}(t), \quad Q_0(\infty, t) = 0, \end{aligned} \tag{2.2}$$

$$\begin{aligned} &\frac{\partial^2 R_0}{\partial \eta^2} = f\Big(\varphi(1,t) + R_0, 1, t, 0\Big) \quad \text{for} \quad \eta > 0, \quad t \in \mathbb{R}, \\ &\frac{\partial R_0}{\partial \eta}(0,t) = u^{(+)}(t), \quad R_0(\infty, t) = 0, \end{aligned} \tag{2.3}$$

which determine the zeroth order terms of the boundary layer functions near the left and the right boundary, respectively.

(A_2). *For any fixed t, the boundary value problem (2.2) has a solution monotone in ξ satisfying*

$$\frac{\partial^2 Q_0}{\partial \xi^2}(0,t) = f\Big(\varphi(-1,t,0) + Q_0(0,t), -1, t, 0\Big) > 0 (< 0)$$
$$\text{for} \quad Q_0(0,t) > 0 \ (Q_0(0,t) < 0).$$

(A_3). *For any fixed t, the boundary value problem (2.3) has a solution monotone in η satisfying*

$$\frac{\partial^2 R_0}{\partial \eta^2}(0,t) = f\Big(\varphi(1,t,0) + R_0(0,t), 1, t, 0\Big) > 0 \ (< 0)$$
$$\text{for} \quad R_0(0,t) > 0 \ (R_0(0,t) < 0).$$

Our main result on the existence and stability of a boundary layer solution to (1.1) reads as follows.

Theorem 2.1 *Let the assumptions (A_0) – (A_3) be satisfied. Then, for sufficiently small ε, there exists a solution $u(x,t,\varepsilon)$ of problem (1.1) satisfying*

$$\Big| u(x,t,\varepsilon) - \Big[\varphi(x,t) + Q_0\Big(\frac{1+x}{\varepsilon}, t\Big) + R_0\Big(\frac{1-x}{\varepsilon}, t\Big)\Big]\Big|$$
$$\leq c\varepsilon \quad \textit{for} \quad x \in [-1,1], \ t \in \mathbb{R},$$

where c is a constant independent of ε, which is asymptotically stable.

As mentioned above, our approach to construct lower and upper solutions consists of two steps:
1. Construction of a formal asymptotic expansion for a boundary layer solution.
2. Modification of this formal solution to get lower and upper solutions of the problem. These steps will be described in what follows.

3 Construction of a formal asymptotic solution

In order to construct a formal solution $U(x,t,\varepsilon)$ of problem (1.1) we use the standard procedure proposed by Vasil'eva (see e.g. [9]):

To determine the terms in the expansions (2.1) we represent the nonlinear function $f(u,x,t,\varepsilon)$ in a form which is similar to (2.1). According to [9], $f(u,x,t,\varepsilon)$ will be represented in $\overline{\mathcal{D}}$ in the form

$$\begin{aligned} & f(u,x,t,\varepsilon) = f(\overline{U}(x,t,\varepsilon),x,t,\varepsilon) + \\ & \Big[f\Big(\overline{U}(x,t,\varepsilon) + Q(\xi,t,\varepsilon),x,t,\varepsilon\Big) - f\Big(\overline{U}(x,t,\varepsilon),x,t,\varepsilon\Big)\Big]_{|x=-1+\varepsilon\xi} + \\ & \Big[f\Big(\overline{U}(x,t,\varepsilon) + Q(\xi,t,\varepsilon) + R(\eta,t,\varepsilon),x,t,\varepsilon\Big) \\ & - f\Big(\overline{U}(x,t,\varepsilon) + Q(\xi,t,\varepsilon),x,t,\varepsilon\Big)\Big]_{|x=1-\varepsilon\eta}. \end{aligned} \tag{3.1}$$

We also represent the differential operator

$$L_\varepsilon = \varepsilon^2 \frac{\partial^2}{\partial x^2} - \varepsilon^2 \frac{\partial}{\partial t}$$

when it acts on the boundary layer functions Q and R by using the stretched variables ξ and η respectively. Thus, for the Q-function we have

$$L_\varepsilon = \frac{\partial^2}{\partial \xi^2} - \varepsilon^2 \frac{\partial}{\partial t}.$$

For the R-function the differential operator L_ε is transformed similarly.
Substituting the representations for U, f, and for L_ε into equation (1.1) and equating separately the parts depending on x, ξ and η, we obtain equations determining the terms of the asymptotic expansion (2.1).

For the regular part we have

$$\varepsilon^2 \left(\frac{\partial^2 \overline{U}(x,t,\varepsilon)}{\partial x^2} - \frac{\partial \overline{U}(x,t,\varepsilon)}{\partial t} \right) - f(\overline{U}(x,t,\varepsilon),x,t,\varepsilon) = 0 \quad \text{for} \quad (x,t) \in \mathcal{D}. \tag{3.2}$$

For $\varepsilon = 0$ we get from (3.2) the degenerate equation and therefore by hypothesis (A_1) we have

$$\overline{U}_0(x,t) = \varphi(x,t) \quad \text{for} \quad (x,t) \in \mathcal{D}.$$

We can use (3.2) to derive linear algebraic equations to determine $\overline{U}_k(x,t)$ for $k = 1, 2, \ldots$

$$f_u(\varphi(x,t),x,t,0))\, \overline{U}_k = f_k(x,t) \quad \text{for} \quad (x,t) \in \mathcal{D}. \tag{3.3}$$

The functions $f_k(x,t)$ are determined by the functions $\overline{U}_j(x,t)$ with $j < k$, in particular we have

$$f_1(x,t) = -f_\varepsilon(\varphi(x,t), x, t, 0).$$

According to assumption (A_1) the problems (3.3) can be solved uniquely.

Concerning the boundary layer function Q we get

$$\begin{aligned}
&\frac{\partial^2 Q}{\partial \xi^2} - \varepsilon^2 \frac{\partial Q}{\partial t} = f\Big(\overline{U}(-1+\varepsilon\xi, t, \varepsilon) + Q, -1+\varepsilon\xi, t, \varepsilon\Big) \\
&-f\Big(\overline{U}^{(\pm)}(-1+\varepsilon\xi, t, \varepsilon), -1+\varepsilon\xi, t, \varepsilon\Big), \\
&\frac{\partial Q}{\partial \xi}(0,t,\varepsilon) + \varepsilon\frac{\partial \overline{U}}{\partial x}(-1,t,\varepsilon) = u^{(-)}(t).
\end{aligned} \tag{3.4}$$

For $Q_k(\xi,t)$ we also use the additional condition

$$Q_k(\infty, t) = 0. \tag{3.5}$$

For the zeroth order boundary layer function Q_0 we obtain from (3.4) the boundary value problem (see (2.2))

$$\begin{aligned}
&\frac{\partial^2 Q_0}{\partial \xi^2} = f\Big(\varphi(-1,t) + Q_0, -1, t, 0\Big) \quad \text{for} \quad \xi > 0, \quad t \in \mathbb{R}, \\
&\frac{\partial Q_0}{\partial \xi}(0,t) = u^{(-)}(t), \quad Q_0(\infty, t) = 0.
\end{aligned} \tag{3.6}$$

The differential equation in (3.6) is a second order autonomous ordinary differential equations (t is a parameter), which can be analyzed in the phase plane (Q_0, Q_0'), where the origin $(0,0)$ is a saddle point of equation (3.6). From the phase plane analysis it follows that problem (3.6) has a solution in the case when the straight line (for a fixed t) $Q_0' = u^{(-)}(t)$ intersects the separatrix tending to the rest point (0,0). It is known that the boundary layer function Q_0 satisfies the estimate

$$|Q_0(\xi,t)| \le c\,exp(-\kappa\xi) \quad \text{for} \quad \xi > 0,\ t \in \mathbb{R}, \tag{3.7}$$

where κ and c are some positive numbers. Different from the case of the Dirichlet problem, (3.6) can have several solutions. Our choice of a solution is determined by the assumption (A_2).

Using (3.4) we get that the boundary layer functions $Q_1(\xi,t)$ can be determined from the problem

$$\begin{aligned}
&\frac{\partial^2 Q_1}{\partial \xi^2} - f_u\Big(\overline{U_0}(-1,t) + Q_0, -1, t, 0\Big) Q_1 = q_1(\xi,t) \quad \text{for} \quad \xi > 0, \quad t \in \mathbb{R}, \\
&\frac{\partial Q_1}{\partial \xi}(0,t) = -\frac{\partial \overline{U_0}}{\partial x}(-1,t), \quad Q_1(\infty,t) = 0,
\end{aligned} \tag{3.8}$$

where

$$
\begin{aligned}
q_1(\xi,t) = & \Big[f_u\Big(\overline{U_0}(-1,t)+Q_0,-1,t,0\Big) - f_u\Big(\overline{U_0}(-1,t),-1,t,0\Big)\Big] \\
& \times\Big(\frac{\partial\overline{U_0}}{\partial x}(-1,t)\xi+\overline{U_1}(-1,t,0)\Big) \\
& +\Big[f_x\Big(\overline{U_0}(-1,t)+Q_0,-1,t,0\Big) - f_x\Big(\overline{U_0}(-1,t),-1,t,0\Big)\Big]\xi \\
& +\Big[f_\varepsilon\Big(\overline{U_0}(-1,t)+Q_0,-1,t,0\Big) - f_\varepsilon\Big(\overline{U_0}(-1,t),-1,t,0\Big)\Big].
\end{aligned}
\tag{3.9}
$$

The solution of this problem can be given explicitly

$$
\begin{aligned}
Q_1(\xi,t) = & \frac{z(\xi,t)}{\frac{\partial z}{\partial\xi}(0,t)}\left\{-\frac{\partial\overline{U_0}}{\partial x}(-1,t)+\frac{1}{z(0,t)}\int\limits_0^\infty z(\chi,t)q_1(\chi,t)d\chi\right\} \\
& - z(\xi,t)\int\limits_0^\xi \frac{1}{z^2(\eta,t)}\left[\int\limits_\eta^\infty z(\chi,t)q_1(\chi,t)d\chi\right]d\eta.
\end{aligned}
\tag{3.10}
$$

where

$$
z(\xi,t)\equiv\frac{\partial Q_0}{\partial\xi}(\xi,t).
$$

From (3.9) and (3.7) it follows

$$
|q_1(\xi,t)|\le c_1\,exp(-\kappa_1\xi).
$$

Therefore, we get from (3.10) that the function Q_1 satisfies the estimate

$$
|Q_1(\xi,t)|\le c\,exp(-\kappa\xi) \quad \text{for} \quad \xi\ge 0\,t\in\mathbb{R},
$$

where c and κ are some positive numbers.

The higher order terms Q_k can be determined by problems, which have the same structure as (3.8) (the index 1 has to be replaced by the index k and q_k is a known function) and can be represented explicitly analogously to Q_1 satisfying

$$
|Q_k(\xi,t)|\le c\,exp(-\kappa\xi) \quad \text{for} \quad \xi\ge 0,\ t\in\mathbb{R}.
$$

If we take into account this estimate we have (see (3.1))

$$
\begin{aligned}
& \Big[f\Big(\overline{U}(x,t,\varepsilon)+Q(\xi,t,\varepsilon)+R(\eta,t,\varepsilon),x,t,\varepsilon\Big) \\
& -f\big(\overline{U}(x,t,\varepsilon)+Q(\xi,t,\varepsilon),x,t,\varepsilon\big)\Big]_{|x=1-\varepsilon\eta} = \\
& \Big[f\Big(\overline{U}(x,t,\varepsilon)+R(\eta,t,\varepsilon),x,t,\varepsilon\Big)-f\big(\overline{U}(x,t,\varepsilon),x,t,\varepsilon\big)\Big]_{|x=1-\varepsilon\eta}+\Pi(\eta,t,\varepsilon),
\end{aligned}
$$

where Π is exponentially small in η. Thus the boundary layer functions R_k can be determined analogously to the functions Q_k . In particular, R_0 is determined by the problem (2.3) under assumption (A_3).

Since $f, u^{(\pm)}$ are sufficiently smooth, the formal asymptotics can be constructed to any order n. From these constructions it follows that the corresponding approximations satisfy (1.1) up to order ε^{n+1}.

4 Construction of upper and lower solutions, existence result

For the convenience of the reader we recall the definition of lower and upper solutions to the boundary value problem (1.1) and the corresponding basic result.

Definition 4.1 *Let α and β be functions mapping $\overline{\mathcal{D}} \times \overline{I}_{\varepsilon_0}$ continuously into R, which are twice continuously differentiable in x, continuously differentiable in t and T-periodic in t. Then α and β are called ordered lower and upper solutions of* (1.1) *for $\varepsilon \in I_{\varepsilon_0}$, if they satisfy for $\varepsilon \in I_{\varepsilon_0}$ the following conditions:*

$$
\begin{aligned}
&1^\circ. \quad \alpha(x,t,\varepsilon) \le \beta(x,t,\varepsilon) \quad \text{for} \quad (x,t) \in \overline{\mathcal{D}},\\
&2^\circ. \quad N_\varepsilon(\alpha) \ge 0 \ge N_\varepsilon(\beta) \ \text{ for } (x,t) \in \mathcal{D},\\
&3^\circ. \quad \varepsilon \frac{\partial \alpha}{\partial x}(-1,t,\varepsilon) \ge u^{(-)}(t) \ge \varepsilon \frac{\partial \beta}{\partial x}(-1,t,\varepsilon) \ \text{ for } t \in \mathbb{R},\\
&\qquad\ \varepsilon \frac{\partial \alpha}{\partial x}(1,t,\varepsilon) \le u^{(+)}(t) \le \varepsilon \frac{\partial \beta}{\partial x}(1,t,\varepsilon) \ \text{ for } t \in \mathbb{R}.
\end{aligned}
$$

We also call the functions α and β as ordered lower and upper solutions asymptotic order of q if they additionally satisfy the inequalities

$$
N_\varepsilon(\alpha) \ge c\varepsilon^q, \quad N_\varepsilon(\beta) \le -c\varepsilon^q,
$$

where q and c are some positive constants independent of ε.

Remark 4.1 *It is known (see, e.g., [10]) that the existence of ordered lower and upper solutions implies the existence of a solution $u(x,t,\varepsilon)$ of (1.1) satisfying*

$$
\alpha(x,t,\varepsilon) \le u(x,t,\varepsilon) \le \beta(x,t,\varepsilon) \quad \text{for} \quad (x,t) \in \mathcal{D} \quad \text{and} \quad \varepsilon \in I_{\varepsilon_0}.
$$

In what follows we describe a method to construct upper and lower solutions to (1.1) by some modification of the formal asymptotic solution to (1.1).

Let U_n be the partial sums of order n of the expansions (2.1), that is

$$
U_n(x,t,\varepsilon) = \sum_{i=0}^{n} \varepsilon^i \left(\overline{U}_i(x,t) + Q_i(\xi,t) + R_i(\eta,t) \right).
$$

We define an upper solution $\beta(x,t,\varepsilon) = \beta_n(x,t,\varepsilon)$ and a lower solution $\alpha(x,t,\varepsilon) = \alpha_n(x,t,\varepsilon)$ to (1.1) for $\varepsilon \in I_{\varepsilon_0}$ in $\mathcal{D}$ by

$$
\begin{aligned}
\beta_n(x,t,\varepsilon) = U_n(x,t,\varepsilon) + \varepsilon^{n+1}\Big(&\bar{U}_{n+1}(x,t) + \gamma \\
&+ Q_{n+1}(\xi,t) + Q_\beta(\xi,t) + R_{n+1}(\eta,t) + R_\beta(\eta,t)\Big)
\end{aligned}
\tag{4.1}
$$

and

$$\begin{aligned}\alpha_n(x,t,\varepsilon) &= U_n(x,t,\varepsilon) + \varepsilon^{n+1}\Big(\bar{U}_{n+1}(x,t) - \gamma \\ &+ Q_{n+1}(\xi,t) + Q_\alpha(\xi,t) + R_{n+1}(\eta,t) + R_\alpha(\eta,t)\Big)\end{aligned} \tag{4.2}$$

Here, $\gamma > 0$ is a number independent of ε . The functions Q_α and Q_β, $R_\alpha(\eta,t)$ and $R_\beta(\eta,t)$ are chosen in such a way that α_n and β_n fulfill the conditions in Definition 4.1.

We define the function $Q_\beta(\xi,t)$ by the boundary value problems

$$\begin{aligned}&\frac{\partial^2 Q_\beta}{\partial \xi^2} - f_u\Big(\overline{U_0}(-1,t) + Q_0, -1, t, 0\Big)Q_\beta = q_\beta(\xi,t) \quad \text{for} \quad \xi > 0, \quad t \in \mathbb{R}, \\ &\frac{\partial Q_\beta}{\partial \xi}(0,t) = -\delta, \quad Q_\beta(\infty,t) = 0,\end{aligned} \tag{4.3}$$

where δ is some positive constant and

$$q_\beta(\xi,t) = \Big[f_u\Big(\overline{U_0}(-1,t) + Q_0, -1, t, 0\Big) - f_u\Big(\overline{U_0}(-1,t), -1, t, 0\Big)\Big]\gamma - M\exp(-\kappa\xi).$$

From the exponential estimate of Q_0 in (3.7) it follows that the first term in the representation for q_β satisfies an exponential estimate. Therefore, we can choose M sufficiently large and κ sufficiently small such that q_β is an exponentially decaying negative function. The function $Q_\alpha(\xi,t)$ is defined by the problem

$$\begin{aligned}&\frac{\partial^2 Q_\alpha}{\partial \xi^2} - f_u\Big(\overline{U_0}(-1,t) + Q_0, -1, t, 0\Big)Q_\alpha = q_\alpha(\xi,t) \quad \text{for} \quad \xi > 0, \quad t \in \mathbb{R}, \\ &\frac{\partial Q_\alpha}{\partial \xi}(0,t) = \delta, \quad Q_\alpha(\infty,t) = 0,\end{aligned} \tag{4.4}$$

where

$$q_\alpha(\xi,t) = -[f_u\Big(\overline{U_0}(-1,t) + Q_0, -1, t, 0\Big) - f_u\Big(\overline{U_0}(-1,t), -1, t, 0\Big)]\gamma + M\exp(-\kappa\xi).$$

It is clear that q_α is an exponentially decaying positive function.

The functions R_α and R_β are defined analogously.

The solutions of the problems (4.4), and (4.3) can be given explicitly analogously to the problem (3.8). We have

$$\begin{aligned}Q_\alpha(\xi,t) &= \frac{z(\xi,t)}{\frac{\partial z}{\partial \xi}(0,t)}\delta + \frac{z(\xi,t)}{\frac{\partial z}{\partial \xi}(0,t)}\frac{1}{z(0,t)}\int_0^\infty z(\chi,t)q_\alpha(\chi,t)d\chi \\ &- z(\xi,t)\int_0^\xi \frac{1}{z^2(\eta,t)}\left[\int_\eta^\infty z(\chi,t)q_\alpha(\chi,t)d\chi\right]d\eta.\end{aligned} \tag{4.5}$$

and

$$
\begin{aligned}
Q_\beta(\xi,t) = & \frac{z(\xi,t)}{\frac{\partial z}{\partial \xi}(0,t)}(-\delta) + \frac{z(\xi,t)}{\frac{\partial z}{\partial \xi}(0,t)} \frac{1}{z(0,t)} \int\limits_0^\infty z(\chi,t) q_\beta(\chi,t) d\chi \\
& - z(\xi,t) \int\limits_0^\xi \frac{1}{z^2(\eta,t)} \left[\int\limits_\eta^\infty z(\chi,t) q_\beta(\chi,t) d\chi \right] d\eta.
\end{aligned}
\tag{4.6}
$$

From (4.5) and (4.6) it follows that functions $Q_\alpha(\xi,t)$ and $Q_\beta(\xi,t)$ are exponentially decaying. Using also that $q_\beta < 0$ and $q_\alpha > 0$ and assumption (A_3) which says that $\frac{\partial^2 Q_0}{\partial \xi^2}(0,t) = \frac{\partial z}{\partial \xi}(0,t) > 0$ and $z(\xi,t) < 0$ for $Q_0(0,t) > 0$ and $\frac{\partial^2 Q_0}{\partial \xi^2}(0,t) = \frac{\partial z}{\partial \xi}(0,t) < 0$ and $z(\xi,t) > 0$ for $Q_0(0,t) < 0$, we have that $Q_\beta > 0$ and $Q_\alpha < 0$. Similarly we have that $R_\beta > 0$ and $R_\alpha < 0$. Therefore the lower solution $\alpha_n(x,t,\varepsilon)$ and the upper solution $\beta_n(x,t,\varepsilon)$ satisfy condition 1^0 in Definition 4.1, that is they are ordered.

Now we have to check that from the representation for $\alpha_n(x,t,\varepsilon)$ and $\beta_n(x,t,\varepsilon)$ it also follows that they satisfy the boundary conditions. We just check it for $\beta_n(x,t,\varepsilon)$ at the left boundary. For sufficiently small ε we have

$$
\begin{aligned}
& \varepsilon \frac{\partial \beta}{\partial x}(-1,t,\varepsilon) = u^{(-)}(t) + \varepsilon^{n+1} \frac{\partial \bar{U}_{n+1}}{\partial x}(x,t) + \varepsilon^n \frac{\partial Q_\beta}{\partial \xi}(0,t) = \\
& u^{(-)}(t) + \varepsilon^{n+1} \frac{\partial \bar{U}_{n+1}}{\partial x}(x,t) + \varepsilon^n(-\delta) < u^{(-)}(t).
\end{aligned}
$$

The other inequalities at the boundaries can be checked similarly. Hence, the conditions in 3^0 of in Definition 4.1 are fulfilled. Finally, we have to check the condition 2^0 in Definition 4.1. For this purpose, we substitute the expressions for α_n and β_n into the operator $N_\varepsilon(u)$ defined in (1.1). We get

$$
\begin{aligned}
& N_\varepsilon(\beta_n(x,t,\varepsilon)) = \varepsilon^{n+1} \Big[- f_u(\varphi(x,t),x,t)\gamma + q_\beta + r_\beta \Big] \\
& + O(\varepsilon^{n+2}) \quad \text{for} \quad (x,t) \in \mathcal{D}.
\end{aligned}
\tag{4.7}
$$

Using that $q_\beta < 0$, $r_\beta < 0$ by the construction and $f_u(\varphi(x,t),x,t) > 0$ by assumption A_1 we get that the coefficient of ε^{n+1} in (4.7) is negative and therefore we have for sufficiently small ε

$$
N_\varepsilon(\beta_n(x,t,\varepsilon)) < -\gamma_1 \varepsilon^{n+1},
$$

where γ_1 is some positive number. Similarly we obtain

$$
N_\varepsilon(\alpha_n(x,t,\varepsilon)) > \gamma_2 \varepsilon^{n+1}.
$$

Thus conditions 2^0 are satisfied. We summarize the results of our construction of upper and lower solutions in the following lemma.

Lemma 4.1 *The functions $\beta_n(x,t,\varepsilon)$ and $\alpha_n(x,t,\varepsilon)$ defined by the expressions (4.1) and (4.2) respectively satisfies the Definition 4.1, and therefore they are ordered upper and lower solutions of problem (1.1). Moreover they obey the following estimate*

$$\beta_n(x,t,\varepsilon) - \alpha_n(x,t,\varepsilon) = O(\varepsilon^{n+1}) \quad \textit{for} \quad x \in [0,1],\ t \in \mathbb{R}. \tag{4.8}$$

Taking into account Remark 4.1 and the estimates (4.8), and the form of α_n and β_n (see (4.2) and (4.1)) we get the following result.

Theorem 4.1 *Suppose the assumptions $(A_0) - (A_3)$ to be valid. Then, for sufficiently small ε, there exists a solution $u(x,t,\varepsilon)$ of (1.1) which has a boundary layer near $x = -1$ and $x = 1$ and satisfies for $\varepsilon \in I_{\varepsilon_0}$ the estimates*

$$|u(x,t,\varepsilon) - U_n(x,t,\varepsilon)| \ \leq \ c_n\varepsilon^{n+1} \quad \textit{for} \quad (x,t) \in \overline{\mathcal{D}},$$

where the positive constant c_n does not depend on ε,

$$\alpha_n(x,t,\varepsilon) \leq u(x,t,\varepsilon) \leq \beta_n(x,t,\varepsilon) \quad \textit{for} \quad (x,t) \in \overline{\mathcal{D}}.$$

5 Stability results

In this section we investigate the stability (in the sense of Lyapunov) of the periodic solution $u(x,t,\varepsilon)$ established by Theorem 4.1 in [4]. For convenience we recall the theorem which we will apply to obtain the asymptotic stability of our periodic solution $u(x,t,\varepsilon)$ as the following lemma.

Lemma 5.1 *Let $\alpha(x,t,\varepsilon)$ and $\beta(x,t,\varepsilon)$ be ordered lower and upper solutions of asymptotic order $q > 0$ to (1.1,) and let $u(x,t,\varepsilon)$ be the periodic solution to (1.1) corresponding to them. Suppose that for sufficiently small ε and $(x,t) \in \overline{\mathcal{D}}$ it holds*

$$|u(x,t,\varepsilon)| + |\alpha(x,t,\varepsilon)| + |\beta(x,t,\varepsilon)| \leq \kappa_1$$

and

$$|\beta(x,t,\varepsilon) - u(x,t,\varepsilon)| + |\alpha(x,t,\varepsilon) - u(x,t,\varepsilon)| \leq \kappa_2\varepsilon^{\frac{p}{2}}$$

where κ_1, κ_2 and $p > q$ are constants. Then, for sufficiently small $\varepsilon > 0$, the solution $u(x,t,\varepsilon)$ to (1.1) is asymptotically stable in the sense of Lyapunov.

In our case it follows from Lemma 4.1 that $q = n + 1$ and $p = 2n + 2$. Thefore condition $p > q$ in Lemma 4.1 leads to the condition for the order of our lower and upper solution. We get $n > -1$. Therefore, applying Lemma 4.1 we can state our theorem on the stability of the periodic solution $u(x,t,\varepsilon)$.

Theorem 5.1 *Suppose the assumptions $(A_0) - (A_3)$ to be satisfied. Then, for sufficiently small ε, the periodic solution $u(x,t,\varepsilon)$ of problem (1.1) with boundary layer is asymptotically stable.*

References

[1] A. V. NESTEROV, *The asymptotic behavior of the solution to a parabolic equation with singularly perturbed boundary conditions*, Computational Mathematics and Mathematical Physics, 37(9) (1997), 1051-5057.

[2] V. F. BUTUZOV, N. N. NEFEDOV, L. RECKE, AND K. R. SCHNEIDER, *On a singularly perturbed initial value problem in the case of a double root of the degenerate equation*, Nonlinear Analysis, Theory, Methods and Applications 83 (2013), 1-11.

[3] A. B. VASIL'EVA, V. F. BUTUZOV, N. N. NEFEDOV, *Singularly Perturbed Problems with Boundary and Internal Layers*, Proccedingsof the Steklov Institute of Mathematics, 268 (2010), 258-273.

[4] N. N. NEFEDOV, L. RECKE, K.R. SCHNEIDER , *Asymptotic Stability via the Krein-Rutman Theorem for Singularly Perturbed Parabolic Periodic-Dirichlet Problems*, Regular and chaotic dynamics, 15(2-3) 2010, 382-389.

[5] N. N. NEFEDOV, *The method of differential inequalities for some classes of nonlinear singularly perturbed problems with internal layers*, Differ. Uravn., 31(7) (1995), 1142-1149.

[6] N. N. NEFEDOV, *An Asymptotic method of differential inequalities for the investigation of periodic contrast Structures: Existence,Asymptotics, and Stability*, Differ. Uravn. 36(2) (2000), 262-269.

[7] V. T. VOLKOV, N.N. NEFEDOV, *Development of the asymptotic method of differential inequalities for investigation of periodic contrast Structures in reacton-diffusion-advection equations*. Computational Mathematics and Mathematical Physics. 46(4) (2006), 585-593.

[8] C. V. PAO, *Nonlinear parabolic and elliptic equations*, Plenum Press, New York and London, 1992.

[9] A. B. VASIL'EVA, V. F. BUTUZOV, *Asymptotic Methods in the Theory of Singular Perturbation*, (in Russian), Vysshaya Shkola, Moscow, 1990.

[10] P. Hess, *Periodic-Parabolic Boundary Value Problems and Positivity*, Pitman Resaerch Notes in in Math. Series **247**, Longman Scientific&Technical, 1991.