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Existence of weak solutions for a hyperbolic-parabolic phase field system with mixed boundary conditions on non-smooth domains

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Abstract

The aim of this paper is to prove existence of weak solutions of hyperbolic-parabolic evolution inclusions defined on Lipschitz domains with mixed boundary conditions describing, for instance, damage processes and elasticity with inertial effects. To this end, we first present a suitable weak formulation in order to deal with such evolution inclusions. Then, existence of weak solutions is proven by utilizing time-discretization, H^2–regularization and variational techniques.

1 Introduction

The gradient-of-damage model motivated by Frémond and Nedjar in [FN96] describes the damage progression by microscopic motions in solid structures resulting from the growth of microcracks and microvoids. In this approach, an order parameter z models the degree of damage in every material point. It has the range $[0,1]$ with the following interpretation: the value 1 stands for no damage, a value between 0 and 1 qualifies partial damage and the value 0 indicates maximal damage. Beyond that, elastic deformations are described by a vector function u which specifies the displacement from a given reference configuration Ω. The evolution law for u and z consists of two equations: a hyperbolic equation for the mechanical forces and a parabolic equation for the damage process involving two subgradients. The considered evolution can be summarized in the following PDE system with the unknowns (u, z, ξ, φ):

$$u_{tt} - \operatorname{div}\left(W_{,e}(\epsilon(u), z)\right) = l, \tag{1a}$$
$$z_t - \Delta_p z + W_{,z}(\epsilon(u), z) + f'(z) + \xi + \varphi = 0, \tag{1b}$$
$$\xi \in \partial I_{[0,\infty)}(z), \tag{1c}$$
$$\varphi \in \partial I_{(-\infty,0]}(z_t), \tag{1d}$$

supplemented with the following initial-boundary conditions

$$u = b \qquad \text{on } \Gamma_{\mathrm{D}} \times (0,T), \tag{2a}$$
$$W_{,e}(\epsilon(u), z) \cdot \nu = 0 \qquad \text{on } \Gamma_{\mathrm{N}} \times (0,T), \tag{2b}$$
$$\nabla z \cdot \nu = 0 \qquad \text{on } \partial\Omega, \tag{2c}$$
$$u(0) = u^0 \qquad \text{in } \Omega, \tag{2d}$$
$$u_t(0) = v^0 \qquad \text{in } \Omega, \tag{2e}$$
$$z(0) = z^0 \qquad \text{in } \Omega. \tag{2f}$$

The hyperbolic equation (1a) is the balance equation of forces containing inertial effects modeled by u_{tt}, the parabolic equation (1b) describes the evolution law for the damage processes and (1c) as well as (1d) are sub-gradients corresponding to the constraints that the damage is non-negative ($z \geq 0$) and irreversible ($z_t \leq 0$).

Moreover, l denotes the exterior volume forces, f is a given damage-dependent potential, $\epsilon(u)$ describes the linearized strain tensor, i.e. $\epsilon(u) = \frac{1}{2}(\nabla u + (\nabla u)^{\mathrm{T}})$, and Γ_{D} and Γ_{N} are the Dirichlet part and the Neumann part of the boundary $\partial\Omega$. The elastic energy density W is assumed to be of the form

$$W(e, z) = \frac{1}{2}h(z)\mathbf{C}e : e, \tag{3}$$

where h models the influence of the damage on the stiffness tensor $\mathbf{C}$. We assume that $h' \geq 0$ and that complete damage does not occur, i.e., h is bounded from below by a positive constant. Let us note that an activation threshold for the damage process can be modeled by linear terms in the potential f.

Our goal of this paper is to prove existence of weak solutions for the system (1) on Lipschitz domains Ω. First of all, we would like to mention that because of the Lipschitz regularity of $\partial\Omega$ and the mixed boundary conditions for u, we cannot apply H^2–regularity theory. Furthermore, we do not have viscous terms such as $\epsilon(u_t)$ in the force balance equation (1a) which gives better space-time regularity for u. In the literature (see for instance [Seg04, BSS05, RR12]), H^2–regularity for u as well as viscous regularizations are used to handle the differential inclusion (1b)-(1d) with Yosida regularization techniques. We present a different approach which allows to prove existence in a weak notion. More precisely, we show that the inclusion (1b)-(1d) can be rewritten as a variation inequality and a total energy inequality. For both properties, we need less regularity for the damage variable z. By using variational techniques introduced in [HK13] and H^2–regularization techniques, we are eventually able to show existence of weak solutions.

The paper is structured as follows. In Section 2, we introduce some notation and preliminary mathematical results from [HK13]. The main part is Section 3. We state and justify a notion of weak solutions in Subsection 3.1. The proof of the existence theorem ranges from Subsection 3.2 to Subsection 3.3. In the first instance, we prove existence of weak solution for an H^2–regularized problem by using a time-discretization scheme and by applying variational techniques from Section 2 to pass to the time-continuous system. Finally, we get rid of the regularization by a further limit passage which is performed in Subsection 3.3.

We like to conclude this introduction with a list in Table 1 showing some selected mathematical works of related damage models and their results (the list is, of course, only an excerpt and by far not complete).

Model type	*Boundary conditions*	*Results*	*References*
quasi-static force balance + regularized rate-depend. damage	"smooth" boundary, 0–boundary cond.	local existence +uniqueness (special cases)	[BS04]
viscoelasticity + rate-dependent damage	"smooth" boundary, 0–boundary cond.	local existence +uniqueness (special cases)	[BSS05]
quasi-static force balance + rate-dependent damage + Cahn-Hilliard equation	Lipschitz boundary, mixed boundary cond.	existence +higher integr.	[HK11] [HK13]

quasi-static force balance + rate-dependent damage / rate-indep. damage	Lipschitz boundary, mixed boundary cond.	existence +regularity +uniqueness (special cases)	[KRZ11]
quasi-static force balance + rate-independent damage	Lipschitz boundary, mixed boundary cond.	existence +regularity +uniqueness	[MR06] [BMR09] [MT10]
viscoelasticity + rate-dependent damage + heat equation	$\mathcal{C}^2$–boundary, 0–boundary cond.	existence +regularity +uniqueness (special cases)	[RR12]

Table 1: Selected mathematical results for gradient-of-damage models.

2 Notation and preliminary results

Throughout this work, let $p > n$ be a constant and $p' = p/(p-1)$ its dual and let $\Omega \subseteq \mathbb{R}^n$ ($n = 1, 2, 3$) be a bounded Lipschitz domain. For the Dirichlet boundary Γ_{D} and the Neumann boundary Γ_{N} of $\partial\Omega$, we adopt the assumptions from [Ber11], i.e., Γ_{D} and Γ_{N} are non-empty and relatively open sets in $\partial\Omega$ with finitely many path-connected components such that $\Gamma_{\mathrm{D}} \cap \Gamma_{\mathrm{N}} = \emptyset$ and $\overline{\Gamma_{\mathrm{D}}} \cup \overline{\Gamma_{\mathrm{N}}} = \partial\Omega$.

The considered time interval is denoted by $[0, T]$ and $\Omega_t := \Omega \times [0, t]$ for $t \in [0, T]$. The partial derivative of a function h with respect to a variable s is abbreviated by $h_{,s}$. Furthermore, we define for $k \geq 1$ the spaces

$$
\begin{aligned}
W_+^{k,p}(\Omega) &:= \big\{u \in W^{k,p}(\Omega) \,|\, u \geq 0 \text{ a.e. in } \Omega\big\},\\
W_-^{k,p}(\Omega) &:= \big\{u \in W^{k,p}(\Omega) \,|\, u \leq 0 \text{ a.e. in } \Omega\big\},\\
H_{\Gamma_{\mathrm{D}}}^{k}(\Omega) &:= \big\{u \in H^{k}(\Omega) \,|\, u = 0 \text{ on } \Gamma_{\mathrm{D}} \text{ in the sense of traces}\big\}.
\end{aligned}
$$

In our considerations, we will frequently make use of the compact embedding

$$W^{1,p}(\Omega) \hookrightarrow \mathcal{C}(\overline{\Omega}).$$

The following variational and approximation results are crucial for the proof of the existence theorem in the next chapter.

Lemma 2.1 (See [HK13]) *Let $f \in L^{p'}(\Omega; \mathbb{R}^n)$, $g \in L^1(\Omega)$ and $z \in W_+^{1,p}(\Omega)$ with $f \cdot \nabla z \geq 0$ a.e. in Ω and $\{f = 0\} \supseteq \{z = 0\}$ in an a.e. sense. Furthermore, we assume that*

$$\int_\Omega \big(f \cdot \nabla\zeta + g\zeta\big)\,\mathrm{d}x \geq 0 \quad \textit{for all } \zeta \in W_-^{1,p}(\Omega) \textit{ with } \{\zeta = 0\} \supseteq \{z = 0\}.$$

Then

$$\int_\Omega \big(f \cdot \nabla\zeta + g\zeta\big)\,\mathrm{d}x \geq \int_{\{z=0\}} \max\{0, g\}\zeta\,\mathrm{d}x \quad \textit{for all } \zeta \in W_-^{1,p}(\Omega).$$

Remark 2.2 *In [HK13], g is assumed to be in $L^p(\Omega)$. But the proof extends to $g \in L^1(\Omega)$ without any modifications.*

In the next lemma, the notation $\{\zeta = 0\} \supseteq \{f = 0\}$ for functions in $L^\infty(0,T;W^{1,p}(\Omega))$ should be read as

$$\big\{x \in \overline{\Omega} \,|\, \zeta(x,t) = 0\big\} \supseteq \big\{x \in \overline{\Omega} \,|\, f(x,t) = 0\big\} \qquad \textit{for a.e. } t \in (0,T).$$

In the following, the subscript τ always refers to a sequence τ_k, $k \in \mathbb{N}$, with $\tau_k \searrow 0$ as $k \to \infty$.

Lemma 2.3 (See [HK13]) *Let*

- $f_\tau, f \in L^\infty(0,T;W^{1,p}_+(\Omega))$, $\tau > 0$ *with* $f_\tau(t) \to f(t)$ *weakly in* $W^{1,p}(\Omega)$ *as* $\tau \searrow 0$ *for a.e.* $t \in (0,T)$,
- $\zeta \in L^\infty(0,T;W^{1,p}_+(\Omega))$ *with* $\{\zeta = 0\} \supseteq \{f = 0\}$.

Then, there exist a sequence $\zeta_\tau \in L^\infty(0,T;W^{1,p}_+(\Omega))$ *and constants* $\nu_{\tau,t} > 0$ *such that*

- $\zeta_\tau \to \zeta$ *strongly in* $L^q(0,T;W^{1,p}(\Omega))$ *as* $\tau \to 0^+$ *for all* $q \geq 1$,
- $\zeta_\tau \to \zeta$ *weakly-star in* $L^\infty(0,T;W^{1,p}(\Omega))$ *as* $\tau \to 0^+$,
- $\zeta_\tau \leq \zeta$ *a.e. in* Ω_T *for all* $\tau > 0$ *(in particular* $\{\zeta_\tau = 0\} \supseteq \{\zeta = 0\}$*)*,
- $\nu_{\tau,t}\zeta_\tau(t) \leq f_\tau(t)$ *in* Ω *for a.e.* $t \in (0,T)$ *and for all* $\tau > 0$.

If, in addition, $\zeta \leq f$ *a.e. in* Ω_T *then the last condition can be refined to*

$$\zeta_\tau \leq f_\tau \textit{ a.e. in } \Omega_T \textit{ for all } \tau > 0.$$

3 Analysis of the hyperbolic-parabolic system

3.1 Notion of weak solutions and existence theorem

In this work, we assume $f \in \mathcal{C}^1([0,1],\mathbb{R}_+)$ for the damage potential (see (1b)) and W to be given by (3) with $h \in \mathcal{C}^1([0,1];\mathbb{R})$ and $h \geq c$ on $[0,1]$ for a constant $c > 0$. Furthermore, we will use the assumption $h' \geq 0$ on $[0,1]$.

The main idea for a weak formulation is to rewrite the doubly nonlinear differential inclusion (1b)-(1d) into a variational inequality and a total energy inequality. This kind of notion was introduced in [HK11] and is adapted to the present situation in the following (see Proposition 3.2 for a justification).

Definition 3.1 (Weak solution) *We consider the following given data:*

external volume forces:	$l \in L^2(0,T;L^2(\Omega;\mathbb{R}^n))$,
Dirichlet boundary data:	$b \in H^1(0,T;H^2(\Omega;\mathbb{R}^n) \cap W^{2,1}(0,T;L^2(\Omega;\mathbb{R}^n))$,
initial values:	$u^0 \in H^1(\Omega;\mathbb{R}^n)$, $v^0 \in L^2(\Omega;\mathbb{R}^n)$, $z^0 \in W^{1,p}(\Omega)$
	with $0 \leq z^0 \leq 1$ *a.e. in* Ω.

A weak solution of the PDE system (1)-(2) *for the data* (f,b,u^0,v^0,z^0) *is a triple* (u,z,ξ) *satisfying the following properties:*

(i) spaces:

$$
\begin{aligned}
&u \in L^\infty(0,T;H^1(\Omega;\mathbb{R}^n)) \cap W^{1,\infty}(0,T;L^2(\Omega;\mathbb{R}^n)) \cap H^2(0,T;(H^1_{\Gamma_\mathrm{D}}(\Omega;\mathbb{R}^n))^*)\\
&\qquad \text{with } u = b \text{ on } \Gamma_\mathrm{D} \times (0,T),\ u(0) = u^0 \text{ a.e. in } \Omega,\ \partial_t u(0) = v^0 \text{ a.e. in } \Omega,\\
&z \in L^\infty(0,T;W^{1,p}(\Omega)) \cap H^1(0,T;L^2(\Omega))\\
&\qquad \text{with } z(0) = z^0 \text{ in } \Omega,\ z \geq 0 \text{ a.e. in } \Omega_T,\ \partial_t z \leq 0 \text{ a.e. in } \Omega_T,\\
&\xi \in L^\infty(0,T;L^1(\Omega)),
\end{aligned}
$$

(ii) for all $\zeta \in H^1_{\Gamma_\mathrm{D}}(\Omega;\mathbb{R}^n)$ and for a.e. $t \in (0,T)$:

$$
\langle \partial_{tt} u, \zeta \rangle_{H^1} + \int_\Omega W_{,e}(\epsilon(u), z) : \epsilon(\zeta)\,\mathrm{d}x = \int_\Omega l \cdot \zeta\,\mathrm{d}x, \tag{4}
$$

(iii) for all $\zeta \in W^{1,p}_-(\Omega)$ and for a.e. $t \in (0,T)$:

$$
0 \leq \int_\Omega \big(|\nabla z|^{p-2} \nabla z \cdot \nabla \zeta + (W_{,z}(\epsilon(u), z) + f'(z) + \xi + \partial_t z)\zeta \big)\,\mathrm{d}x, \tag{5}
$$

(iv) for all $\zeta \in L^\infty_+(\Omega)$ and for a.e. $t \in (0,T)$:

$$
0 \geq \int_\Omega \xi(\zeta - z)\,\mathrm{d}x, \tag{6}
$$

(v) total energy inequality for a.e. $t \in (0,T)$:

$$
\begin{aligned}
&\int_\Omega \left(\frac{1}{p}|\nabla z(t)|^p + W(\epsilon(u(t)), z(t)) + f(z(t)) + \frac{1}{2}|\partial_t u(t)|^2 \right) \mathrm{d}x\\
&\quad + \int_{\Omega_t} |\partial_t z|^2 \,\mathrm{d}x\,\mathrm{d}s - \int_\Omega \partial_t u(t) \cdot \partial_t b(t)\,\mathrm{d}x\\
&\quad \leq \int_\Omega \left(\frac{1}{p}|\nabla z^0|^p + W(\epsilon(u^0), z^0) + f(z^0) + \frac{1}{2}|v^0|^2 \right) \mathrm{d}x\\
&\qquad - \int_\Omega v^0 \cdot \partial_t b^0\,\mathrm{d}x + \int_{\Omega_t} W_{,e}(\epsilon(u), z) : \epsilon(\partial_t b)\,\mathrm{d}x\,\mathrm{d}s - \int_{\Omega_t} \partial_t u \cdot \partial_{tt} b\,\mathrm{d}x\,\mathrm{d}s\\
&\qquad + \int_{\Omega_t} l \cdot (\partial_t u - \partial_t b)\,\mathrm{d}x\,\mathrm{d}s.
\end{aligned} \tag{7}
$$

Proposition 3.2 *Let (u, z, ξ) be a weak solution. Furthermore, if additionally*

$$
u \in H^1(0,T;H^1(\Omega;\mathbb{R}^n)), \quad z \in H^1(0,T;W^{1,p}(\Omega)),
$$

then for a.e. $t \in (0,T)$

$$
\begin{aligned}
&z_t - \Delta_p z + W_{,z}(\epsilon(u), z) + f'(z) + \xi + \varphi = 0 \text{ in } \left(W^{1,p}(\Omega)\right)^*,\\
&\xi \in \partial I_{W^{1,p}_+(\Omega)}(z),\\
&\varphi \in \partial I_{W^{1,p}_-(\Omega)}(z_t).
\end{aligned}
$$

Moreover, the total energy inequality (7) becomes an energy balance.

Proof. Define the free energy functional $\mathcal{E}$ as

$$\mathcal{E}(u,z) := \int_\Omega \left(\frac{1}{p}|\nabla z|^p + W(\epsilon(u),z) + g(z)\right) \mathrm{d}x.$$

The Gâteaux derivatives $\mathrm{d}_u\mathcal{E}$ and $\mathrm{d}_z\mathcal{E}$ are given as follows:

$$\begin{aligned}
\langle \mathrm{d}_u\mathcal{E}(u,z),\zeta\rangle_{H^1} &= \int_\Omega W_{,e}(\epsilon(u),z) : \epsilon(\zeta)\,\mathrm{d}x,\\
\langle \mathrm{d}_z\mathcal{E}(u,z),\zeta\rangle_{W^{1,p}} &= \int_\Omega \left(|\nabla z|^{p-2}\nabla z\cdot\nabla\zeta + W_{,z}(\epsilon(u),z) : \epsilon(\zeta) + f'(z)\right) \mathrm{d}x.
\end{aligned}$$

Testing (4) with $\partial_t u(t) - \partial_t b(t)$ yields

$$\begin{aligned}
\langle \mathrm{d}_u\mathcal{E}(u,z),\partial_t u\rangle_{H^1} = \langle \mathrm{d}_u\mathcal{E}(u,z),\partial_t b\rangle_{H^1} + \int_\Omega l\cdot(\partial_t u - \partial_t b)\,\mathrm{d}x - \frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega \frac{1}{2}|\partial_t u|^2\,\mathrm{d}x&\\
+ \langle \partial_{tt}u, \partial_t b\rangle_{H^1}.& \qquad (8)
\end{aligned}$$

Testing (5) with $\partial_t z(t)$ yields

$$-\int_\Omega |\partial_t z|^2\,\mathrm{d}x \le \langle \mathrm{d}_z\mathcal{E}(u,z),\partial_t z\rangle_{W^{1,p}} + \int_\Omega \xi\,\partial_t z\,\mathrm{d}x. \qquad (9)$$

By using (8), the total energy inequality (7) can be rewritten as

$$\mathcal{E}(u(t),z(t)) - \mathcal{E}(u^0,z^0) \le \int_0^t \langle \mathrm{d}_u\mathcal{E}(u(s),z(s)),\partial_t u(s)\rangle_{H^1}\,\mathrm{d}s - \int_{\Omega_t}|\partial_t z(s)|^2\,\mathrm{d}x\,\mathrm{d}s. \qquad (10)$$

The right hand side of (10) can be estimated by (9) as follows:

$$\begin{aligned}
&\int_0^t \langle \mathrm{d}_u\mathcal{E}(u(s),z(s)),\partial_t u(s)\rangle_{H^1}\,\mathrm{d}s - \int_{\Omega_t}|\partial_t z(s)|^2\,\mathrm{d}x\,\mathrm{d}s\\
&\quad\le \int_0^t \Big(\langle \mathrm{d}_u\mathcal{E}(u(s),z(s)),\partial_t u(s)\rangle_{H^1} + \langle \mathrm{d}_z\mathcal{E}(u(s),z(s)),\partial_t z(s)\rangle_{W^{1,p}}\Big)\,\mathrm{d}s + \int_{\Omega_t}\xi\,\partial_t z\,\mathrm{d}x\,\mathrm{d}s\\
&\quad= \mathcal{E}(u(t),z(t)) - \mathcal{E}(u^0,z^0) + \int_{\Omega_t}\xi\,\partial_t z\,\mathrm{d}x\,\mathrm{d}s. \qquad (11)
\end{aligned}$$

To obtain an energy balance from (10) and (11), we have to show $\int_{\Omega_t}\xi\,\partial_t z\,\mathrm{d}x\,\mathrm{d}s = 0$. Indeed, from (6) we infer $\xi = 0$ a.e. in $\{z>0\}$ and $\xi \le 0$ a.e. in $\{z=0\}$. Therefore, it suffices to prove the following:

$$\textit{for a.e. } (x,s)\in\Omega_t: \quad z(x,s) = 0 \implies \partial_t z(x,s) = 0.$$

This is true because of $\partial_t z \le 0$ a.e. in Ω_T and by Fubini's theorem (we also use $z \in \mathcal{C}(\overline{\Omega_T})$ which follows from an Aubin-Lions type embedding $L^\infty(0,T;W^{1,p}(\Omega)) \cap H^1(0,T;L^2(\Omega)) \hookrightarrow \mathcal{C}(\overline{\Omega_T})$, see [Sim86])

$$\begin{aligned}
\int_{\{z=0\}} \partial_t z(x,s)\,\mathrm{d}(x,s) &= \int_\Omega \int_{\{z(x)=0\}} \partial_t z(x,s)\,\mathrm{d}s\,\mathrm{d}x\\
&= \int_\Omega \Big(z\big(x,T\big) - z\big(x,T_x^{\inf}\big)\Big)\,\mathrm{d}x = 0,
\end{aligned}$$

where $\{z(x)=0\}$ denotes the x-cut of $\{z=0\}$, i.e.

$$\{z(x)=0\}:=\{s\in[0,T]\,|\,\{z(x,s)=0\}\}$$

and

$$T_x^{\inf}:=\begin{cases}\inf\{s\in[0,T]\,|\,z(x,s)=0\} & \text{if } \{z(x,\cdot)=0\}\neq\emptyset,\\ T & \text{else.}\end{cases}$$

Now, we have two ways of expressing the energy differences:

$$\mathcal{E}(u(t),z(t))-\mathcal{E}(u^0,z^0)=\int_0^t\langle \mathrm{d}_u\mathcal{E}(u(s),z(s)),\partial_t u(s)\rangle_{H^1}\,\mathrm{d}s-\int_{\Omega_t}|\partial_t z(s)|^2\,\mathrm{d}x\,\mathrm{d}s$$

and

$$\begin{aligned}&\mathcal{E}(u(t),z(t))-\mathcal{E}(u^0,z^0)\\&\quad=\int_0^t\Big(\langle \mathrm{d}_u\mathcal{E}(u(s),z(s)),\partial_t u(s)\rangle_{H^1}+\langle \mathrm{d}_z\mathcal{E}(u(s),z(s)),\partial_t z(s)\rangle_{W^{1,p}}\Big)\,\mathrm{d}s.\end{aligned}$$

Comparison and adding $\int_{\Omega_t}\xi\,\partial_t z\,\mathrm{d}x\,\mathrm{d}s$ yield for a.e. $t\in(0,T)$:

$$\langle \mathrm{d}_z\mathcal{E}(u(t),z(t))+\partial_t z(t)+\xi(t),\partial_t z(t)\rangle_{W^{1,p}}=0.$$

The variational property (5) can be rewritten for all $\zeta\in W_-^{1,p}(\Omega)$ as

$$\langle -\mathrm{d}_z\mathcal{E}(u(t),z(t))-\partial_t z(t)-\xi(t),\zeta\rangle_{W^{1,p}}\leq 0.$$

Adding both, we get for all $\zeta\in W_-^{1,p}(\Omega)$

$$-\langle \mathrm{d}_z\mathcal{E}(u(t),z(t))+\partial_t z(t)+\xi(t),\zeta-\partial_t z(t)\rangle_{W^{1,p}}\leq 0,$$

which proves the claim. $\square$

The main aim of this work is to prove existence of weak solutions in the sense above.

Theorem 3.3 *To the given data (l,b,u^0,v^0,z^0), there exists a weak solution of system* (1)-(2) *in the sense of Definition 3.1.*

3.2 Existence of weak solutions for an H^2–regularized system

Here, we first solve a regularized version of our introduced damage model. The passage to the limit system is performed in the next subsection.

Regularization

The regularized PDE system is given in a classical notion by a quadruple (u,z,ξ,φ) satisfying

$$\begin{aligned}&u_{tt}-\mathrm{div}W_{,e}(\epsilon(u),z)+\delta Au=l,\\&z_t-\Delta_p z+W_{,z}(\epsilon(u),z)+f'(z)+\xi+\varphi=0,\end{aligned}$$

$\xi \in \partial I_{[0,\infty)}(z),$
$\varphi \in \partial I_{(-\infty,0]}(z_t),$

where the linear operator $A : H^2(\Omega;\mathbb{R}^n) \to (H^2(\Omega;\mathbb{R}^n))^*$ is defined as

$$\langle Au, v\rangle_{H^2} := \int_\Omega \langle \nabla(\nabla u), \nabla(\nabla v)\rangle_{\mathbb{R}^{n\times n\times n}} \,\mathrm{d}x := \sum_{1\le i,j,k\le n} \int_\Omega \frac{\mathrm{d}^2 u_k}{\mathrm{d}x_i \mathrm{d}x_j} \frac{\mathrm{d}^2 v_k}{\mathrm{d}x_i \mathrm{d}x_j} \,\mathrm{d}x.$$

For the PDE system, we modify Definition 3.1 as follows.

Definition 3.4 (Weak solution for the regularized system) *We consider the given data as in Definition 3.1. A weak solution of the regularized PDE system for the data (l, b, u^0, v^0, z^0) is a triple (u, z, ξ) satisfying the following properties:*

(i) spaces:

$$\begin{aligned} &u \in L^\infty(0,T;H^2(\Omega;\mathbb{R}^n)) \cap W^{1,\infty}(0,T;L^2(\Omega;\mathbb{R}^n)) \cap H^2(0,T;(H^2_{\Gamma_\mathrm{D}}(\Omega;\mathbb{R}^n))^*),\\ &\qquad \text{with } u = b \text{ on } \Gamma_\mathrm{D}\times(0,T),\ u(0) = u^0 \text{ a.e. in } \Omega,\ \partial_t u(0) = v^0 \text{ a.e. in } \Omega,\\ &z \in L^\infty(0,T;W^{1,p}(\Omega)) \cap H^1(0,T;L^2(\Omega)),\\ &\qquad \text{with } z(0) = z^0 \text{ in } \Omega,\ z \ge 0 \text{ a.e. in } \Omega_T,\ \partial_t z \le 0 \text{ a.e. in } \Omega_T,\\ &\xi \in L^\infty(0,T;L^1(\Omega)),\end{aligned}$$

(ii) for all $\zeta \in H^2_{\Gamma_\mathrm{D}}(\Omega;\mathbb{R}^n)$ and for a.e. $t \in (0,T)$:

$$\langle \partial_{tt} u, \zeta\rangle_{H^2} + \int_\Omega W_{,e}(\epsilon(u), z) : \epsilon(\zeta)\,\mathrm{d}x + \delta\langle Au, \zeta\rangle_{H^2} = \int_\Omega l\cdot\zeta\,\mathrm{d}x, \tag{12}$$

(iii) for all $\zeta \in W^{1,p}_-(\Omega)$ and for a.e. $t \in (0,T)$:

$$0 \le \int_\Omega \big(|\nabla z|^{p-2}\nabla z\cdot\nabla\zeta + (W_{,z}(\epsilon(u), z) + f'(z) + \xi + \partial_t z)\zeta\big)\,\mathrm{d}x, \tag{13}$$

(iv) for all $\zeta \in L^\infty_+(\Omega)$ and for a.e. $t \in (0,T)$:

$$0 \ge \int_\Omega \xi(\zeta - z)\,\mathrm{d}x, \tag{14}$$

(v) full energy inequality for a.e. $t \in (0,T)$:

$$\begin{aligned} &\int_\Omega \left(\frac{1}{p}|\nabla z(t)|^p + W(\epsilon(u(t)), z(t)) + f(z(t)) + \frac12|\partial_t u(t)|^2\right)\mathrm{d}x + \int_{\Omega_t} |\partial_t z|^2\,\mathrm{d}x\,\mathrm{d}s\\ &+ \frac{\delta}{2}\langle Au(t), u(t)\rangle_{H^2} - \int_\Omega \partial_t u(t)\cdot\partial_t b(t)\,\mathrm{d}x\\ &\qquad \le \int_\Omega \left(\frac1p|\nabla z^0|^p + W(\epsilon(u^0), z^0) + f(z^0) + \frac12|v^0|^2\right)\mathrm{d}x + \frac{\delta}{2}\langle Au^0, u^0\rangle_{H^2}\\ &\qquad - \int_\Omega v^0\cdot\partial_t b^0\,\mathrm{d}x + \int_{\Omega_t} W_{,e}(\epsilon(u), z) : \epsilon(\partial_t b)\,\mathrm{d}x\,\mathrm{d}s - \int_{\Omega_t} \partial_t u\cdot\partial_{tt} b\,\mathrm{d}x\,\mathrm{d}s\\ &\qquad + \int_{\Omega_t} l\cdot(\partial_t u - \partial_t b)\,\mathrm{d}x\,\mathrm{d}s + \delta\int_0^t \langle Au(t), \partial_t b(t)\rangle_{H^2}\,\mathrm{d}t. \end{aligned} \tag{15}$$

The existence proof for the regularized system is carried out in five steps in the following and is based on a time-discretization scheme.

Step 1: Time-discretization

Let $\tau > 0$ denote the discretization fineness and let $M_\tau := \lfloor T/\tau \rfloor$ be the number of discrete time points.

We fix an $m \in 1, \ldots, M_\tau$ and define the functional $\mathcal{F}_\tau^m : H^2(\Omega;\mathbb{R}^n) \times W^{1,p}(\Omega) \to \mathbb{R}$ by

$$\mathcal{F}_\tau^m(u,z) := \int_\Omega \left(\frac{1}{p}|\nabla z|^p + W(\epsilon(u), z) + f(z) - l(m\tau)\cdot u \right) \mathrm{d}x + \frac{\delta}{2}\langle Au, u\rangle_{H^2} \\ + \frac{\tau}{2}\left\| \frac{z - z^{m-1}}{\tau} \right\|_{L^2}^2 + \frac{\tau^2}{2}\left\| \frac{u - 2u^{m-1} + u^{m-2}}{\tau^2} \right\|_{L^2}^2 .$$

A minimizer of $\mathcal{F}_\tau^m$ in the subspace

$$\left\{ u \in H^2(\Omega;\mathbb{R}^n) \mid u|_{\Gamma_\mathrm{D}} = b(\tau m)|_{\Gamma_\mathrm{D}} \right\} \times \left\{ z \in W^{1,p}(\Omega) \mid 0 \le z \le z^{m-1} \right\}$$

obtained by the direct method is denoted by (u_τ^m, z_τ^m). By a recursive minimization procedure starting from the initial values (u^0, z^0) and $u^{-1} := u^0 - \tau v^0$, we obtain functions (u_τ^m, z_τ^m) for $m = 1, \ldots, M_\tau$. The velocity field v_τ^m is set to $(u_\tau^m - u_\tau^{m-1})/\tau$ and b_τ^m and l_τ^m are given by $b(\tau m)$ and $l(\tau m)$.

Let $w_\tau^m \in \{l_\tau^m, b_\tau^m, u_\tau^m, v_\tau^m, z_\tau^m\}$, we introduce the piecewise constant interpolations w_τ, w_τ^- and the linear interpolation $\widehat{w}_\tau$ w.r.t. time as

$$\begin{aligned} w_\tau(t) &:= w_\tau^m && \text{with } m = \lceil t/\tau \rceil, \\ w_\tau^-(t) &:= w_\tau^{\max\{0,m-1\}} && \text{with } m = \lceil t/\tau \rceil, \\ \widehat{w}_\tau(t) &:= \beta w_\tau^m + (1-\beta) w_\tau^{\max\{0,m-1\}} && \text{with } m = \lceil t/\tau \rceil,\ \beta = \frac{t-(m-1)\tau}{\tau} \end{aligned}$$

and the piecewise constant functions t_τ and t_τ^- as

$$\begin{aligned} t_\tau &:= \lceil t/\tau \rceil \tau = \min\{m\tau \mid m \in \mathbb{N}_0 \text{ and } m\tau \ge t\}, \\ t_\tau^- &:= \max\{0, t_\tau - \tau\}. \end{aligned}$$

We would like to remark that, by definition, $w_\tau(t) = w_\tau(t_\tau)$ for all $t \in [0,T]$ and

$$\partial_t \widehat{v}_\tau(t) = \frac{u_\tau^m - 2u_\tau^{m-1} + u_\tau^{m-2}}{\tau^2}$$

for $t \in \lceil t/\tau \rceil$.

Since the functions (u_τ^m, z_τ^m) are minimizers, we obtain the following necessary conditions (Euler-Lagrange equations):

- $u_\tau, v_\tau \in L^\infty(0,T;H^2(\Omega;\mathbb{R}^n))$, $\widehat{u}_\tau, \widehat{v}_\tau \in W^{1,\infty}(0,T;H^2(\Omega;\mathbb{R}^n))$, $z_\tau \in L^\infty(0,T;W^{1,p}(\Omega))$, $\widehat{z}_\tau \in W^{1,\infty}(0,T;W^{1,p}(\Omega))$,
- for all $\zeta \in H^2_{\Gamma_\mathrm{D}}(\Omega;\mathbb{R}^n)$ and for a.e. $t \in (0,T)$:

$$\int_\Omega \partial_t \widehat{v}_\tau \cdot \zeta \,\mathrm{d}x + \int_\Omega W_{,e}(\epsilon(u_\tau), z_\tau) : \epsilon(\zeta)\,\mathrm{d}x + \delta\langle Au_\tau, \zeta\rangle_{H^2} = \int_\Omega l_\tau \cdot \zeta \,\mathrm{d}x, \tag{16}$$

- for a.e. $t \in (0,T)$ and for all $\zeta \in W^{1,p}(\Omega)$ with $0 \le \zeta + z_\tau(t) \le z_\tau^-(t)$:

$$0 \le \int_\Omega \big(|\nabla z_\tau|^{p-2}\nabla z_\tau \cdot \nabla\zeta + (W_{,z}(\epsilon(u_\tau), z_\tau) + f'(z_\tau) + \partial_t\widehat{z}_\tau)\zeta\big)\,\mathrm{d}x. \tag{17}$$

Step 2: A priori estimates

- Testing (16) with $u_\tau - u_\tau^- - (b_\tau - b_\tau^-)$, and using the estimate

$$\int_\Omega \partial_t\widehat{v}_\tau \cdot (u_\tau - u_\tau^-)\,\mathrm{d}x \ge \frac{1}{2}\,\|v_\tau\|_{L^2}^2 - \frac{1}{2}\,\|v_\tau^-\|_{L^2}^2$$

as well as the convexity estimates (note that $z_\tau^- \ge z_\tau$)

$$\int_\Omega W_{,e}(\epsilon(u_\tau), z_\tau) : \epsilon(u_\tau - u_\tau^-)\,\mathrm{d}x \ge \int_\Omega \big(W(\epsilon(u_\tau), z_\tau) - W(\epsilon(u_\tau^-), z_\tau^-)\big)\,\mathrm{d}x \tag{18}$$

$$\delta\langle Au_\tau, u_\tau - u_\tau^-\rangle_{H^2} \ge \frac{\delta}{2}\langle Au_\tau, u_\tau\rangle_{H^2} - \frac{\delta}{2}\langle Au_\tau^-, u_\tau^-\rangle_{H^2}, \tag{19}$$

yield

$$\begin{aligned}
&\frac{1}{2}\,\|v_\tau(t)\|_{L^2}^2 - \frac{1}{2}\,\|v_\tau^-(t)\|_{L^2}^2 + \frac{\delta}{2}\langle Au_\tau(t), u_\tau(t)\rangle_{H^2} - \frac{\delta}{2}\langle Au_\tau^-(t), u_\tau^-(t)\rangle_{H^2}\\
&+ \int_\Omega \big(W(\epsilon(u_\tau(t)), z_\tau(t)) - W(\epsilon(u_\tau^-(t)), z_\tau^-(t))\big)\,\mathrm{d}x - \int_\Omega \partial_t\widehat{v}_\tau(t)\cdot\big(b_\tau(t) - b_\tau^-(t)\big)\,\mathrm{d}x\\
&\quad \le \int_\Omega l_\tau(t)\cdot\big(u_\tau(t) - u_\tau^-(t) - (b_\tau(t) - b_\tau^-(t))\big)\,\mathrm{d}x\\
&\qquad + \int_\Omega W_{,e}(\epsilon(u_\tau(t)), z_\tau(t)) : \epsilon(b_\tau(t) - b_\tau^-(t))\,\mathrm{d}x + \delta\langle Au_\tau(t), b_\tau(t) - b_\tau^-(t)\rangle_{H^2}.
\end{aligned} \tag{20}$$

The right hand side can be estimated as follows ($\eta > 0$ has to be chosen suitably small)

$$\begin{aligned}
\text{r.h.s.} &\le \tau\|l_\tau(t)\|_{L^2}\Big(\|v_\tau(t)\|_{L^2} + \|\partial_t\widehat{b}_\tau(t)\|_{L^2}\Big) + C\tau\|\epsilon(u_\tau(t))\|_{L^2}\|\epsilon(\partial_t\widehat{b}_\tau(t))\|_{L^2}\\
&\quad + \delta\tau\langle Au_\tau(t), u_\tau(t)\rangle_{H^2}^{1/2}\langle A\partial_t\widehat{b}_\tau(t), \partial_t\widehat{b}_\tau(t)\rangle_{H^2}^{1/2}\\
&\le C\tau\Big(\|l_\tau(t)\|_{L^2}^2 + \|v_\tau(t)\|_{L^2}^2 + \|\partial_t\widehat{b}_\tau(t)\|_{L^2}^2 + \eta\|\epsilon(u_\tau(t))\|_{L^2}^2 + C_\eta\|\epsilon(\partial_t\widehat{b}_\tau(t))\|_{L^2}^2\\
&\quad + \delta\langle Au_\tau(t), u_\tau(t)\rangle_{H^2} + \delta\langle A\partial_t\widehat{b}_\tau(t), \partial_t\widehat{b}_\tau(t)\rangle_{H^2}\Big).
\end{aligned} \tag{21}$$

Summing (20) over the discrete time points $t = \tau, 2\tau, \ldots, m\tau$ with $m \in \{1, \ldots, M_\tau\}$, taking (21) and the regularity assumptions for l and b into account, we obtain

$$\begin{aligned}
&\frac{1}{2}\,\|v_\tau(t)\|_{L^2}^2 + \frac{\delta}{2}\langle Au_\tau(t), u_\tau(t)\rangle_{H^2} + c\|\epsilon(u_\tau(t))\|_{L^2}^2 - \int_0^{t_\tau}\int_\Omega \partial_t\widehat{v}_\tau\cdot\partial_t\widehat{b}_\tau\,\mathrm{d}x\,\mathrm{d}s\\
&\quad \le \frac{1}{2}\,\|v^0\|_{L^2}^2 + \frac{\delta}{2}\langle Au^0, u^0\rangle_{H^2} + \int_\Omega W(\epsilon(u^0), z^0)\,\mathrm{d}x\\
&\qquad + C\int_0^{t_\tau}\big(\|v_\tau(s)\|_{L^2}^2 + \eta\|\epsilon(u_\tau(t))\|_{L^2}^2 + \delta\langle Au_\tau(s), u_\tau(s)\rangle_{H^2} + 1 + C_\eta\big)\,\mathrm{d}s
\end{aligned}$$

for all $t \in [0,T]$. The discrete integration by parts formula yields for all $t \in [0,T]$

$$\int_0^{t_\tau}\int_\Omega \partial_t\widehat{v}_\tau \cdot \partial_t\widehat{b}_\tau \,\mathrm{d}x\,\mathrm{d}s = \int_\Omega v_\tau(t)\cdot\partial_t\widehat{b}_\tau(t)\,\mathrm{d}x - \int_\Omega v^0\cdot\partial_t\widehat{b}_\tau(0)\,\mathrm{d}x \\ - \int_0^{t_\tau}\int_\Omega v_\tau^-(s)\cdot\frac{\partial_t\widehat{b}_\tau(s)-\partial_t\widehat{b}_\tau(s-\tau)}{\tau}\,\mathrm{d}x\,\mathrm{d}s. \tag{22}$$

We eventually obtain for all $t \in [0,T]$

$$\begin{aligned}
&\frac{1}{2}\|v_\tau(t)\|_{L^2}^2 + \frac{\delta}{2}\langle Au_\tau(t),u_\tau(t)\rangle_{H^2} + c\|\epsilon(u_\tau(t))\|_{L^2}^2\\
&\quad\le \widehat{C}\Big(1+\int_0^{t_\tau}\big(\|v_\tau(s)\|_{L^2}^2 + \eta\|\epsilon(u_\tau(s))\|_{L^2}^2 + \delta\langle Au_\tau(s),u_\tau(s)\rangle_{H^2}\big)\,\mathrm{d}s\\
&\qquad - \int_0^{t_\tau}\int_\Omega v_\tau^-(s)\cdot\frac{\partial_t\widehat{b}_\tau(s)-\partial_t\widehat{b}_\tau(s-\tau)}{\tau}\,\mathrm{d}x\,\mathrm{d}s\Big)\\
&\quad\le \widetilde{C}\Big(1+\int_0^{t_\tau}\big(\|v_\tau(s)\|_{L^2}^2 + \|v_\tau^-(s)\|_{L^2}^2 + \eta\|\epsilon(u_\tau(s))\|_{L^2}^2 + \delta\langle Au_\tau(s),u_\tau(s)\rangle_{H^2}\big)\,\mathrm{d}s\Big).
\end{aligned}$$

Applying Gronwall's inequality for piecewise constant functions shows

$$\begin{aligned}
\|\nabla(\nabla u_\tau)\|_{L^\infty(0,T;L^2(\Omega;\mathbb{R}^{n\times n\times n}))} &< C,\\
\|\epsilon(u_\tau)\|_{L^\infty(0,T;L^2(\Omega;\mathbb{R}^{n\times n}))} &< C,
\end{aligned}$$

and

$$\|v_\tau\|_{L^\infty(0,T;L^2(\Omega;\mathbb{R}^n))} < C,$$

where $C>0$ is independent of τ. Combining these estimates with Korn's inequality, we obtain

$$\|u_\tau\|_{L^\infty(0,T;H^2(\Omega;\mathbb{R}^n))} < C.$$

Consequently, by noticing $v_\tau = \partial_t\widehat{u}_\tau$,

$$\|\widehat{u}_\tau\|_{L^\infty(0,T;H^2(\Omega;\mathbb{R}^n))\cap W^{1,\infty}(0,T;L^2(\Omega;\mathbb{R}^n))} < C.$$

A comparison argument in (16) also gives

$$\|\widehat{v}_\tau\|_{L^\infty(0,T;L^2(\Omega;\mathbb{R}^n))\cap H^1(0,T;(H^2_{\Gamma_{\mathrm{D}}}(\Omega;\mathbb{R}^n))^*)} < C.$$

- Testing (17) with $z_\tau^- - z_\tau$ and using the convexity estimate

$$\int_\Omega |\nabla z_\tau|^{p-2}\nabla z_\tau\cdot\nabla(z_\tau - z_\tau^-)\,\mathrm{d}x \ge \frac{1}{p}\|\nabla z_\tau\|_{L^p}^p - \frac{1}{p}\|\nabla z_\tau^-\|_{L^p}^p$$

yield

$$\begin{aligned}
&\frac{1}{p}\|\nabla z_\tau(t)\|_{L^p}^p - \frac{1}{p}\|\nabla z_\tau^-(t)\|_{L^p}^p + \tau\,\|\partial_t\widehat{z}_\tau(t)\|_{L^2}^2\\
&\qquad\le \int_\Omega \big(W_{,z}(\epsilon(u_\tau(t)),z_\tau(t)) + f'(z_\tau(t))\big)\,(z_\tau^-(t) - z_\tau(t))\,\mathrm{d}x.
\end{aligned} \tag{23}$$

Thus ($\eta > 0$ has to be chosen suitably small)

$$
\begin{aligned}
&\frac{1}{p}\|\nabla z_\tau(t)\|_{L^p}^p - \frac{1}{p}\|\nabla z_\tau^-(t)\|_{L^p}^p + \tau\,\|\partial_t \widehat{z}_\tau(t)\|_{L^2}^2 \\
&\quad \le \tau\eta\,\|\partial_t \widehat{z}_\tau(t)\|_{L^2}^2 + C_\eta\|W_{,z}(\epsilon(u_\tau(t)), z_\tau(t)) + f'(z_\tau(t))\|_{L^2}^2.
\end{aligned}
$$

Summing over the discrete time points $t = 1, \ldots, m\tau$, $m \in \{0, \ldots, M_\tau\}$, and using $0 \le z_\tau \le 1$ as well as $\|u_\tau\|_{L^\infty(0,T;H^2(\Omega;\mathbb{R}^n))} < C$, we end up with

$$
\begin{aligned}
\|z_\tau\|_{L^\infty(0,T;W^{1,p}(\Omega))} &< C, \\
\|\widehat{z}_\tau\|_{L^\infty(0,T;W^{1,p}(\Omega)) \cap H^1(0,T;L^2(\Omega))} &< C.
\end{aligned}
$$

In conclusion, we obtain the following a priori estimates.

Corollary 3.5 *There exists a $C > 0$ such that for all $\tau > 0$*

$$
\begin{aligned}
\|v_\tau\|_{L^\infty(0,T;L^2(\Omega;\mathbb{R}^n))} &< C, && (24)\\
\|u_\tau\|_{L^\infty(0,T;H^2(\Omega;\mathbb{R}^n))} &< C, && (25)\\
\|\widehat{u}_\tau\|_{L^\infty(0,T;H^2(\Omega;\mathbb{R}^n)) \cap W^{1,\infty}(0,T;L^2(\Omega;\mathbb{R}^n))} &< C, && (26)\\
\|\widehat{v}_\tau\|_{L^\infty(0,T;L^2(\Omega;\mathbb{R}^n)) \cap H^1(0,T;(H^2_{\Gamma_{\mathrm{D}}}(\Omega;\mathbb{R}^n))^*)} &< C, && (27)\\
\|z_\tau\|_{L^\infty(0,T;W^{1,p}(\Omega))} &< C, && (28)\\
\|\widehat{z}_\tau\|_{L^\infty(0,T;W^{1,p}(\Omega)) \cap H^1(0,T;L^2(\Omega))} &< C. && (29)
\end{aligned}
$$

Step 3: Compactness

Standard weakly and weakly-star compactness results applied to the a priori estimates (24)-(29) reveal existence of functions

$$
\begin{aligned}
&u, u^- \in L^\infty(0,T;H^2(\Omega;\mathbb{R}^n)),\\
&\widehat{u} \in L^\infty(0,T;H^2(\Omega;\mathbb{R}^n)) \cap W^{1,\infty}(0,T;L^2(\Omega;\mathbb{R}^n)),\\
&v, v^- \in L^\infty(0,T;L^2(\Omega;\mathbb{R}^n)),\\
&\widehat{v} \in L^\infty(0,T;L^2(\Omega;\mathbb{R}^n)) \cap W^{1,\infty}(0,T;(H^2_{\Gamma_{\mathrm{D}}}(\Omega;\mathbb{R}^n))^*),\\
&z, z^- \in L^\infty(0,T;W^{1,p}(\Omega)),\\
&\widehat{z} \in L^\infty(0,T;W^{1,p}(\Omega)) \cap H^1(0,T;L^2(\Omega))
\end{aligned}
$$

and subsequences indexed by τ_k such that

$$
\begin{aligned}
&u_{\tau_k} \to u && \text{weakly-star in } L^\infty(0,T;H^2(\Omega;\mathbb{R}^n)), && (30)\\
&u_{\tau_k}^- \to u^- && \text{weakly-star in } L^\infty(0,T;H^2(\Omega;\mathbb{R}^n)), && (31)\\
&\widehat{u}_{\tau_k} \to \widehat{u} && \text{weakly-star in } L^\infty(0,T;H^2(\Omega;\mathbb{R}^n)) \cap W^{1,\infty}(0,T;L^2(\Omega;\mathbb{R}^n)), && (32)\\
&v_{\tau_k} \to v && \text{weakly-star in } L^\infty(0,T;L^2(\Omega;\mathbb{R}^n)), && (33)\\
&v_{\tau_k}^- \to v^- && \text{weakly-star in } L^\infty(0,T;L^2(\Omega;\mathbb{R}^n)), && (34)\\
&\widehat{v}_{\tau_k} \to \widehat{v} && \text{weakly-star in } L^\infty(0,T;L^2(\Omega;\mathbb{R}^n)) \cap H^1(0,T;(H^2_{\Gamma_{\mathrm{D}}}(\Omega;\mathbb{R}^n))^*), && (35)\\
&z_{\tau_k} \to z && \text{weakly-star in } L^\infty(0,T;W^{1,p}(\Omega)), && (36)
\end{aligned}
$$

$$z^-_{\tau_k} \to z^- \qquad \text{weakly-star in } L^\infty(0,T;W^{1,p}(\Omega)), \tag{37}$$

$$\widehat{z}_{\tau_k} \to \widehat{z} \qquad \text{weakly-star in } L^\infty(0,T;W^{1,p}(\Omega)) \text{ and weakly in } H^1(0,T;L^2(\Omega)). \tag{38}$$

Taking into account $u_{\tau_k} - u^-_{\tau_k} = \tau_k \partial_t \widehat{u}_{\tau_k}$ and $\tau_k \partial_t \widehat{u}_{\tau_k} \to 0$ strongly in $L^\infty(0,T;L^2(\Omega;\mathbb{R}^n))$, we obtain $u = u^- = \widehat{u}$. Analogously, we get $v = v^- = \widehat{v}$ and $z = z^- = \widehat{z}$. The identity $\partial_t \widehat{u}_{\tau_k} = v_{\tau_k}$ implies $\partial_t u = v$. Thus

$$\begin{aligned}
&u \in L^\infty(0,T;H^2(\Omega;\mathbb{R}^n)) \cap W^{1,\infty}(0,T;L^2(\Omega;\mathbb{R}^n)) \cap H^2(0,T;(H^2_{\Gamma_{\mathrm{D}}}(\Omega;\mathbb{R}^n))^*),\\
&z \in L^\infty(0,T;W^{1,p}(\Omega)) \cap H^1(0,T;L^2(\Omega)).
\end{aligned}$$

Applying standard compactness arguments (Aubin-Lion theorem, see [Sim86], and the compact embedding $W^{1,p}(\Omega) \hookrightarrow L^\infty(\Omega)$) show

$$u_{\tau_k}, u^-_{\tau_k}, \widehat{u}_{\tau_k} \to u \qquad \text{strongly in } L^q(0,T;H^1(\Omega;\mathbb{R}^n)) \text{ for every } q \geq 1, \tag{39}$$

$$z_{\tau_k}, z^-_{\tau_k}, \widehat{z}_{\tau_k} \to z \qquad \text{strongly in } \mathcal{C}(\overline{\Omega_T}). \tag{40}$$

By choosing further subsequences (we omit the additional subscript), we also obtain pointwise a.e. in Ω_T convergence of u_{τ_k}.

Strong convergence of of $\{\nabla z_{\tau_k}\}$ in $L^p(\Omega_T;\mathbb{R}^n)$ can be shown by a tricky approximation argument.

Lemma 3.6 *There exists a subsequence of τ_k (omitting the additional subscript) such that $z_{\tau_k} \to z$ in $L^p(0,T;W^{1,p}(\Omega))$.*

Proof. According to Lemma 2.3, there exists an approximation sequence $\{\zeta_{\tau_k}\} \subseteq L^\infty(0,T;W^{1,p}_+(\Omega))$ with the properties:

$$\zeta_{\tau_k} \to z \qquad \text{strongly in } L^p(0,T;W^{1,p}(\Omega)), \tag{41}$$

$$0 \leq \zeta_{\tau_k} \leq z^-_{\tau_k} \qquad \text{a.e. in } \Omega_T \text{ for all } k \in \mathbb{N}. \tag{42}$$

We omit the subscript k for notational convenience. Due to (42), we obtain

$$0 \leq (\zeta_\tau - z_\tau) + z_\tau \leq z^-_\tau$$

and, therefore, we can test (17) with $\zeta_\tau - z_\tau$ and integrate in time:

$$\int_{\Omega_T} |\nabla z_\tau|^{p-2} \nabla z_\tau \cdot \nabla (z_\tau - \zeta_\tau)\,\mathrm{d}x\,\mathrm{d}t \leq \int_{\Omega_T} \big(W_{,z}(\epsilon(u_\tau), z_\tau) + f'(z_\tau) + \partial_t \widehat{z}_\tau\big)(\zeta_\tau - z_\tau)\,\mathrm{d}x\,\mathrm{d}t.$$

A uniform p-convexity argument and the above estimate show

$$\begin{aligned}
c\|\nabla z - \nabla z_\tau\|^p_{L^p} \leq{}& \int_{\Omega_T} \big(|\nabla z|^{p-2}\nabla z - |\nabla z_\tau|^{p-2}\nabla z_\tau\big) \cdot \nabla(z - z_\tau)\,\mathrm{d}x\,\mathrm{d}t\\
={}& \int_{\Omega_T} |\nabla z|^{p-2}\nabla z \cdot \nabla(z - z_\tau)\,\mathrm{d}x\,\mathrm{d}t + \int_{\Omega_T} |\nabla z_\tau|^{p-2}\nabla z_\tau \cdot \nabla(z_\tau - \zeta_\tau)\,\mathrm{d}x\,\mathrm{d}t\\
&+ \int_{\Omega_T} |\nabla z_\tau|^{p-2}\nabla z_\tau \cdot \nabla(\zeta_\tau - z)\,\mathrm{d}x\,\mathrm{d}t\\
\leq{}& \int_{\Omega_T} \big(W_{,z}(\epsilon(u_\tau), z_\tau) + f'(z_\tau) + \partial_t \widehat{z}_\tau\big)(\zeta_\tau - z_\tau)\,\mathrm{d}x\,\mathrm{d}t\\
&+ \int_{\Omega_T} |\nabla z|^{p-2}\nabla z \cdot \nabla(z - z_\tau)\,\mathrm{d}x\,\mathrm{d}t + \int_{\Omega_T} |\nabla z_\tau|^{p-2}\nabla z_\tau \cdot \nabla(\zeta_\tau - z)\,\mathrm{d}x\,\mathrm{d}t.
\end{aligned} \tag{43}$$

In the following, we prove that every term on the right hand side converges to 0 as $\tau \searrow 0$.

- The first integral on the r.h.s of (43) can be estimated as follows

$$\begin{aligned}\int_{\Omega_T} &\big(W_{,z}(\epsilon(u_\tau), z_\tau) + f'(z_\tau) + \partial_t \widehat{z}_\tau\big)\,(\zeta_\tau - z_\tau)\,\mathrm{d}x\,\mathrm{d}t \\ &\le \big\|W_{,z}(\epsilon(u_\tau), z_\tau) + f'(z_\tau)\big\|_{L^\infty(0,T;L^1(\Omega))}\|\zeta_\tau - z_\tau\|_{L^1(0,T;L^\infty(\Omega))} \\ &\quad + \big\|\partial_t \widehat{z}_\tau\big\|_{L^2(\Omega_T)}\big\|\zeta_\tau - z_\tau\big\|_{L^2(\Omega_T)}. \end{aligned} \tag{44}$$

  Note that for this estimate it suffices to have boundedness of u_τ in $L^\infty(0,T;H^1(\Omega;\mathbb{R}^n))$ instead of the much stronger result (25) (the situation will change for the passage $\delta \searrow 0$ in Subsection 3.3). By using the boundedness of u_τ in $L^\infty(0,T;H^1(\Omega;\mathbb{R}^n))$, boundedness properties (28)-(29) and convergence properties (40)-(41), we get convergence to 0^+ of the two summands on the right hand side of (44).

- Due to (36), the second integral on the r.h.s. of (43) converges to 0+.

- We estimate the third integral on the r.h.s. of (43):

$$\int_\Omega |\nabla z_\tau|^{p-2}\nabla z_\tau \cdot \nabla(\zeta_\tau - z)\,\mathrm{d}x\,\mathrm{d}t \le \|\nabla z_\tau\|_{L^p}^{p-1}\|\nabla(\zeta_\tau - z)\|_{L^p}.$$

  Because of the boundedness property (28) and the convergence property (41), we obtain convergence to 0^+ of the integral term above.

In conclusion, we obtain $z_{\tau_k} \to z$ strongly in $L^p(0,T;W^{1,p}(\Omega))$. □

Step 4: Energy inequality

By using the sharper estimate

$$\begin{aligned}\int_\Omega &W_{,e}(\epsilon(u_\tau), z_\tau) : \epsilon(u_\tau - u_\tau^-)\,\mathrm{d}x \\ &\ge \int_\Omega \big(W(\epsilon(u_\tau), z_\tau) - W(\epsilon(u_\tau^-), z_\tau^-)\big)\,\mathrm{d}x \\ &\quad + \int_\Omega \frac{1}{2}\big(h(z_\tau^-) - h(z_\tau)\big)\,\mathbf{C}\epsilon(u_\tau^-) : \epsilon(u_\tau^-)\,\mathrm{d}x \end{aligned}$$

than the convexity estimate (18), we obtain by testing (16) with $u_\tau - u_\tau^- - (b_\tau - b_\tau^-)$ (cf. (20)):

$$\begin{aligned}&\frac{1}{2}\|v_\tau(t)\|_{L^2}^2 - \frac{1}{2}\big\|v_\tau^-(t)\big\|_{L^2}^2 + \frac{\delta}{2}\langle Au_\tau(t), u_\tau(t)\rangle_{H^2} - \frac{\delta}{2}\langle Au_\tau^-(t), u_\tau^-(t)\rangle_{H^2} \\ &+ \int_\Omega \big(W(\epsilon(u_\tau(t)), z_\tau(t)) - W(\epsilon(u_\tau^-(t)), z_\tau^-(t))\big)\,\mathrm{d}x - \int_\Omega \partial_t \widehat{v}_\tau(t)\cdot\big(b_\tau(t) - b_\tau^-(t)\big)\,\mathrm{d}x \\ &+ \int_\Omega \frac{1}{2}\big(h(z_\tau^-(t)) - h(z_\tau(t))\big)\,\mathbf{C}\epsilon(u_\tau^-(t)) : \epsilon(u_\tau^-(t))\,\mathrm{d}x \\ &\quad \le \int_\Omega l_\tau(t)\cdot\big(u_\tau(t) - u_\tau^-(t) - (b_\tau(t) - b_\tau^-(t))\big)\,\mathrm{d}x \\ &\qquad + \int_\Omega W_{,e}(\epsilon(u_\tau(t)), z_\tau(t)) : \epsilon(b_\tau(t) - b_\tau^-(t))\,\mathrm{d}x + \delta\langle Au_\tau(t), b_\tau(t) - b_\tau^-(t)\rangle_{H^2}. \end{aligned} \tag{45}$$

Adding the estimates (45) and (23), we end up with

$$
\begin{aligned}
&\frac{1}{2}\left\|v_\tau(t)\right\|_{L^2}^2-\frac{1}{2}\left\|v_\tau^-(t)\right\|_{L^2}^2+\frac{\delta}{2}\langle Au_\tau(t),u_\tau(t)\rangle_{H^2}-\frac{\delta}{2}\langle Au_\tau^-(t),u_\tau^-(t)\rangle_{H^2}\\
&+\frac{1}{p}\|\nabla z_\tau(t)\|_{L^p}^p-\frac{1}{p}\|\nabla z_\tau^-(t)\|_{L^p}^p+\tau\left\|\partial_t\widehat{z}_\tau(t)\right\|_{L^2}^2-\int_\Omega\partial_t\widehat{v}_\tau(t)\cdot\big(b_\tau(t)-b_\tau^-(t)\big)\,\mathrm{d}x\\
&+\int_\Omega\big(W(\epsilon(u_\tau(t)),z_\tau(t))-W(\epsilon(u_\tau^-(t)),z_\tau^-(t))\big)\,\mathrm{d}x\\
&+\int_\Omega f(z_\tau(t))\,\mathrm{d}x-\int_\Omega f(z_\tau^-(t))\,\mathrm{d}x+\tau e_\tau^1(t)+\tau e_\tau^2(t)\\
&\qquad\le\int_\Omega l_\tau(t)\cdot\big(u_\tau(t)-u_\tau^-(t)-(b_\tau(t)-b_\tau^-(t))\big)\,\mathrm{d}x\\
&\qquad\quad+\int_\Omega W_{,e}(\epsilon(u_\tau(t)),z_\tau(t)):\epsilon(b_\tau(t)-b_\tau^-(t))\,\mathrm{d}x+\delta\langle Au_\tau(t),b_\tau(t)-b_\tau^-(t)\rangle_{H^2}
\end{aligned}
$$

with the error terms

$$
\begin{aligned}
e_\tau^1(t)&:=\int_\Omega\frac{1}{2}\frac{h(z_\tau^-(t))-h(z_\tau(t))}{\tau}\mathbf{C}\epsilon(u_\tau^-(t)):\epsilon(u_\tau^-(t))\,\mathrm{d}x\\
&\quad+\int_\Omega W_{,z}(\epsilon(u_\tau(t)),z_\tau(t))\,\partial_t\widehat{z}_\tau(t)\,\mathrm{d}x\\
e_\tau^2(t)&:=-\int_\Omega\frac{f(z_\tau(t))-f(z_\tau^-(t))}{\tau}\,\mathrm{d}x+\int_\Omega f'(z_\tau(t))\,\partial_t\widehat{z}_\tau(t)\,\mathrm{d}x.
\end{aligned}
$$

Summing over the discrete time points and taking into account formula (22), we finally obtain

$$
\begin{aligned}
&\int_\Omega\left(\frac{1}{p}|\nabla z_\tau(t)|^p+W(\epsilon(u_\tau(t)),z_\tau(t))+f(z_\tau(t))+\frac{1}{2}|v_\tau(t)|^2\right)\mathrm{d}x+\frac{\delta}{2}\langle Au_\tau(t),u_\tau(t)\rangle_{H^2}\\
&+\int_0^{t_\tau}\int_\Omega|\partial_t\widehat{z}_\tau|^2\,\mathrm{d}x\,\mathrm{d}s+\int_0^{t_\tau}\big(e_\tau^1(s)+e_\tau^2(s)\big)\,\mathrm{d}s-\int_\Omega v_\tau(t)\cdot\partial_t\widehat{b}_\tau(t)\,\mathrm{d}x\\
&\quad\le\int_\Omega\left(\frac{1}{p}|\nabla z^0|^p+W(\epsilon(u^0),z^0)+f(z^0)+\frac{1}{2}|v^0|^2\right)\mathrm{d}x+\frac{\delta}{2}\langle Au^0,u^0\rangle_{H^2}\\
&\quad+\int_0^{t_\tau}\int_\Omega l_\tau\cdot\Big(\partial_t\widehat{u}_\tau-\partial_t\widehat{b}_\tau\Big)\,\mathrm{d}x\,\mathrm{d}s+\int_0^{t_\tau}\int_\Omega W_{,e}(\epsilon(u_\tau),z_\tau):\epsilon(\partial_t\widehat{b}_\tau)\,\mathrm{d}x\,\mathrm{d}s\\
&\quad+\delta\int_0^{t_\tau}\langle Au_\tau(s),\partial_t\widehat{b}_\tau(s)\rangle_{H^2}\,\mathrm{d}s-\int_\Omega v^0\cdot\partial_t\widehat{b}_\tau(0)\,\mathrm{d}x\\
&\quad-\int_0^{t_\tau}\int_\Omega v_\tau^-(s)\cdot\frac{\partial_t\widehat{b}_\tau(s)-\partial_t\widehat{b}_\tau(s-\tau)}{\tau}\,\mathrm{d}x\,\mathrm{d}s.
\end{aligned}\tag{46}
$$

Step 5: Continuous limit

We are going to establish the equalities and inequalities of the weak formulation (12)-(15).

- By using the canonical embedding $L^2(\Omega;\mathbb{R}^n)\hookrightarrow(H^2_{\Gamma_\mathrm{D}}(\Omega;\mathbb{R}^n))^*$, it follows for all $\zeta\in H^2_{\Gamma_\mathrm{D}}(\Omega;\mathbb{R}^n)$

$$\int_\Omega\partial_t\widehat{v}_\tau(t)\cdot\zeta\,\mathrm{d}x=\langle\partial_t\widehat{v}_\tau(t),\zeta\rangle_{H^2}.$$

  Keeping this identity in mind, integrating (16) from $t=0$ to $t=T$ and passing to the limit $\tau\searrow 0$ by using (30), (35), (39) and (40), we obtain (12).

- The main difficulty is to obtain the variational inequalities (13) and (14). The proof is performed in two steps.
  - Let $\zeta \in L^\infty(0,T;W^{1,p}_-(\Omega))$ with $\{\zeta = 0\} \supseteq \{z = 0\}$. By Lemma 2.3, we obtain a sequence $\{\zeta_{\tau_k}\} \subseteq L^\infty(0,T;W^{1,p}_-(\Omega))$ (we omit k) and constants $\nu_{\tau,t} > 0$ with the properties:

$$\zeta_\tau \to \zeta \qquad \text{strongly in } L^p(0,T;W^{1,p}(\Omega)), \tag{47}$$

$$0 \geq \nu_{\tau,t}\zeta_\tau(t) \geq -z_\tau(t) \text{ in } \Omega \qquad \text{for a.e. } t \text{ and all } \tau. \tag{48}$$

    Due to (48) and $z_\tau \leq z_\tau^-$, we also have for a.e. t:

$$0 \leq \nu_{\tau,t}\zeta_\tau(t) + z_\tau(t) \leq z_\tau^-(t) \text{ in } \Omega.$$

    In consequence, for a.e. t, we can test (17) with $\nu_{\tau,t}\zeta_\tau(t)$. Dividing the resulting inequality by $\nu_{\tau,t}$, integrating in time and passing to the limit $\tau \searrow 0$ by using Lemma 3.6 and the convergence properties (38), (39) and (47) yield

$$0 \leq \int_{\Omega_T} \big(|\nabla z|^{p-2}\nabla z \cdot \nabla\zeta + (W_{,z}(\epsilon(u),z) + f'(z) + \partial_t z)\zeta\big)\,\mathrm{d}x.$$

    In particular, we get an a.e. in time t formulation.
  - We may apply Lemma 2.1 to the above variational inequality. Then, we obtain for all $\zeta \in W^{1,p}_-(\Omega)$ the inequality

$$0 \leq \int_{\Omega_T} \Big(|\nabla z|^{p-2}\nabla z \cdot \nabla\zeta + (W_{,z}(\epsilon(u),z) + f'(z) + \widehat{\xi} + \partial_t z)\zeta\Big)\,\mathrm{d}x \tag{49}$$

    with $\widehat{\xi} \in L^1(\Omega_T)$ given by

$$\widehat{\xi} = -\chi_{\{z=0\}}\max\Big\{0, W_{,z}(e(u),z) + f'(z) + \partial_t z\Big\}.$$

    Due to $\partial_t z \leq 0$ a.e. in Ω_T, we may replace $\widehat{\xi}$ by $\xi \in L^\infty(0,T;L^1(\Omega))$ in (49), where ξ is given by

$$\xi = -\chi_{\{z=0\}}\max\Big\{0, W_{,z}(e(u),z) + f'(z)\Big\}. \tag{50}$$

    In particular, (13) is valid for a subgradient ξ satisfying (14).

- To treat the energy inequality in (46), we set

$$\begin{aligned}
A_\tau(t) := {}& \int_\Omega \left(\frac{1}{p}|\nabla z_\tau(t)|^p + W(\epsilon(u_\tau(t)), z_\tau(t)) + f(z_\tau(t)) + \frac{1}{2}|v_\tau(t)|^2\right)\mathrm{d}x \\
& - \int_\Omega \left(\frac{1}{p}|\nabla z^0|^p + W(\epsilon(u^0), z^0) + f(z^0) + \frac{1}{2}|v^0|^2\right)\mathrm{d}x \\
& + \frac{\delta}{2}\langle Au_\tau(t), u_\tau(t)\rangle_{H^2} - \frac{\delta}{2}\langle Au^0, u^0\rangle_{H^2} - \int_\Omega v_\tau(t)\cdot\partial_t\widehat{b}_\tau(t)\,\mathrm{d}x \\
& + \int_\Omega v^0 \cdot \partial_t\widehat{b}_\tau(0)\,\mathrm{d}x
\end{aligned}$$

$$
\begin{aligned}
B_\tau(t) :=& \int_0^{t_\tau}\int_\Omega |\partial_t \widehat{z}_\tau|^2 \,\mathrm{d}x\,\mathrm{d}s - \int_0^{t_\tau}\int_\Omega l_\tau \cdot \left(\partial_t \widehat{u}_\tau - \partial_t \widehat{b}_\tau\right) \mathrm{d}x\,\mathrm{d}s \\
& - \int_0^{t_\tau}\int_\Omega W_{,e}(\epsilon(u_\tau), z_\tau) : \epsilon(\partial_t \widehat{b}_\tau) \,\mathrm{d}x\,\mathrm{d}s - \delta \int_0^{t_\tau} \langle A u_\tau(s), \partial_t \widehat{b}_\tau(s)\rangle_{H^2} \,\mathrm{d}s \\
& + \int_0^{t_\tau}\int_\Omega v_\tau^-(s) \cdot \frac{\partial_t \widehat{b}_\tau(s) - \partial_t \widehat{b}_\tau(s-\tau)}{\tau} \,\mathrm{d}x\,\mathrm{d}s, \\
E_\tau^1(t) :=& \int_0^{t_\tau} e_\tau^1(s)\,\mathrm{d}s, \\
E_\tau^2(t) :=& \int_0^{t_\tau} e_\tau^2(s)\,\mathrm{d}s.
\end{aligned}
$$

Then, (46) is equivalent to

$$
A_\tau(t) + B_\tau(t) + E_\tau^1(t) + E_\tau^2(t) \le 0. \tag{51}
$$

Furthermore, by the a priori estimates, we observe that

$$
|A_\tau(t)| + |B_\tau(t)| + |E_\tau^1(t)| + |E_\tau^2(t)| < C \tag{52}
$$

for all $t \in [0, T]$ and for all $\tau > 0$ (along a subsequence τ_k). Next, we consider the $liminf_{\tau \searrow 0}$ of each term in (51) separately.

- By the already proven convergence properties and by lower semi-continuity arguments, we obtain

$$
\liminf_{\tau \searrow 0} \int_{t_1}^{t_2} A_\tau(t)\,\mathrm{d}t \ge \int_{t_1}^{t_2} A(t)\,\mathrm{d}t \text{ for all } 0 \le t_1 \le t_2 \le T, \tag{53}
$$

  where A is defined as A_τ but u_τ, z_τ, v_τ and $\widehat{b}_\tau$ are substituted by their continuous limits. Note that this lim inf–estimate does not necessarily hold pointwise a.e. in t because we do not know $v_\tau(t) \to v(t)$ weakly in $L^2(\Omega;\mathbb{R}^n)$ for a.e. t (see (33)).
- Let $0 \le t_1 \le t_2 \le T$ be arbitrary. By Fatou's lemma, by (38) and by a lower semi-continuity argument, we obtain

$$
\begin{aligned}
\liminf_{\tau \searrow 0} \int_{t_1}^{t_2}\int_0^{t_\tau}\int_\Omega |\partial_t \widehat{z}_\tau(s)|^2 \,\mathrm{d}x\,\mathrm{d}s\,\mathrm{d}t &\ge \int_{t_1}^{t_2} \left(\liminf_{\tau \searrow 0} \int_0^t \int_\Omega |\partial_t \widehat{z}_\tau(s)|^2 \,\mathrm{d}x\,\mathrm{d}s\right) \mathrm{d}t \\
&\ge \int_{t_1}^{t_2}\int_0^t\int_\Omega |\partial_t z(s)|^2 \,\mathrm{d}x\,\mathrm{d}s\,\mathrm{d}t.
\end{aligned} \tag{54}
$$

  Taking also (52) and the already known convergence properties into account, we obtain

$$
\liminf_{\tau \searrow 0} \int_{t_1}^{t_2} B_\tau(t)\,\mathrm{d}t \ge \int_{t_1}^{t_2} B(t)\,\mathrm{d}t, \tag{55}
$$

  where B is defined as B_τ but u_τ, $\widehat{u}_\tau$, v_τ^-, z_τ, $\widehat{z}_\tau$ and $\widehat{b}_\tau$ are substituted by their continuous counterparts and $\frac{\partial_t \widehat{b}_\tau(t) - \partial_t \widehat{b}_\tau(t-\tau)}{\tau}$ by $\partial_{tt} b(t)$.

- It holds

$$h(z_\tau^-) = h(z_\tau) + h'(z_\tau)(z_\tau^- - z_\tau) + r(z_\tau^- - z_\tau), \ \frac{r(\eta)}{\eta} \to 0 \text{ as } \eta \to 0$$

by the differentiability of h. We then get

$$\begin{aligned}\int_0^{t_\tau}\int_\Omega \frac{1}{2}\frac{h(z_\tau^-)-h(z_\tau)}{\tau}\mathbf{C}\epsilon(u_\tau^-):\epsilon(u_\tau^-)\,\mathrm{d}x\,\mathrm{d}s \\ = \int_0^{t_\tau}\int_{\{z_\tau^-(s)\neq z_\tau(s)\}}\frac{1}{2}\left(h'(z_\tau)\frac{z_\tau^- - z_\tau}{\tau} + \frac{r(z_\tau^- - z_\tau)}{z_\tau^- - z_\tau}\frac{z_\tau^- - z_\tau}{\tau}\right)\times \\ \times\mathbf{C}\epsilon(u_\tau^-):\epsilon(u_\tau^-)\,\mathrm{d}x\,\mathrm{d}s.\end{aligned} \tag{56}$$

Because of

$$\begin{aligned}&\left\|\frac{r(z_\tau^- - z_\tau)}{z_\tau^- - z_\tau}\right\|_{L^\infty(\{z_\tau^-\neq z_\tau\})} \\ &\quad\leq \left\|\frac{h(z_\tau^-)-h(z_\tau)}{z_\tau^- - z_\tau}\right\|_{L^\infty(\{z_\tau^-\neq z_\tau\})} + \left\|h'(z_\tau)\frac{z_\tau^- - z_\tau}{z_\tau^- - z_\tau}\right\|_{L^\infty(\{z_\tau^-\neq z_\tau\})} < C,\end{aligned}$$

and $\frac{r(z_\tau^- - z_\tau)}{|z_\tau^- - z_\tau|} \to 0$ as $\tau \searrow 0$ a.e. in Ω_T as $\tau \searrow 0$, we conclude by Lebesgue's generalized convergence theorem

$$\left\|\frac{r(z_\tau^- - z_\tau)}{z_\tau^- - z_\tau}\right\|_{L^q(\{z_\tau^-\neq z_\tau\})} \to 0 \text{ for every } q \geq 1.$$

Using this and the already known convergence properties, we end up with

$$\textit{left hand side of } (56) \to \int_{\Omega_t} W_{,z}(\epsilon(u), z)\partial_t z\,\mathrm{d}x\,\mathrm{d}s$$

and, consequently, $E_\tau^1(t) \to 0$ as $\tau \searrow 0$. Together with the uniform boundedness (52), this implies

$$\int_{t_1}^{t_2} E_\tau^1(t)\,\mathrm{d}t \to 0 \text{ as } \tau \searrow 0 \text{ for all } 0 \leq t_1 \leq t_2 \leq T. \tag{57}$$

- The convergence

$$\int_{t_1}^{t_2} E_\tau^2(t)\,\mathrm{d}s \to 0 \text{ as } \tau \searrow 0 \text{ for all } 0 \leq t_1 \leq t_2 \leq T, \tag{58}$$

can also be shown as above.

If we combine (53), (55), (57) and (58) we finally obtain

$$\begin{aligned}0 &\geq \liminf_{\tau\searrow 0}\int_{t_1}^{t_2}\big(A_\tau(t) + B_\tau(t) + E_\tau^1(t) + E_\tau^2(t)\big)\,\mathrm{d}t \\ &\geq \int_{t_1}^{t_2}(A(t) + B(t))\,\mathrm{d}t.\end{aligned}$$

for all $0 \leq t_1 \leq t_2 \leq T$. Thus, $A(t) + B(t) \leq 0$ for a.e. $t \in (0, T)$ which is the desired energy inequality (15).

Hence, we have established existence of weak solutions in the sense of Definition 3.4.

3.3 Passing to the limit system

We now study the limit $\delta \searrow 0$. For each $\delta > 0$, we obtain a solution $(u_\delta, z_\delta, \xi_\delta)$ in the sense of Definition 3.4.

Step 1: A priori estimates

From the energy inequality (15), we infer

$$\|u_\delta\|_{L^\infty(0,T;H^1(\Omega;\mathbb{R}^n))\cap W^{1,\infty}(0,T;L^2(\Omega;\mathbb{R}^n))} < C, \tag{59}$$

$$\|z_\delta\|_{L^\infty(0,T;W^{1,p}(\Omega))\cap H^1(0,T;L^2(\Omega))} < C, \tag{60}$$

$$\sqrt{\delta}\|u_\delta\|_{L^\infty(0,T;H^2(\Omega;\mathbb{R}^n))} < C. \tag{61}$$

By considering (12), we get

$$\begin{aligned}\langle \partial_{tt} u_\delta(t), \zeta\rangle_{H^2} \leq{}& C(\|\epsilon(u_\delta(t))\|_{L^2}+1)\|\epsilon(\zeta)\|_{L^2} + \delta\|\nabla\left(\nabla u_\delta(t)\right)\|_{L^2}\|\nabla\left(\nabla\zeta\right)\|_{L^2}\\ &+\|l\|_{L^2}\|\zeta\|_{L^2}\end{aligned}$$

and, therefore,

$$\|u_\delta\|_{H^2(0,T;(H^2_{\Gamma_\mathrm{D}}(\Omega;\mathbb{R}^n))^*)} < C. \tag{62}$$

Step 2: Compactness

As in the previous section, the a priori estimates (59)-(62) reveal existence of functions

$$\begin{aligned}&u \in L^\infty(0,T;H^1(\Omega;\mathbb{R}^n)) \cap W^{1,\infty}(0,T;L^2(\Omega;\mathbb{R}^n)) \cap H^2(0,T;(H^2_{\Gamma_\mathrm{D}}(\Omega;\mathbb{R}^n))^*),\\ &z \in L^\infty(0,T;W^{1,p}(\Omega)) \cap H^1(0,T;L^2(\Omega))\end{aligned}$$

and subsequences indexed by δ_k such that

$$\begin{aligned}u_{\delta_k} &\to u && \text{weakly-star in } L^\infty(0,T;H^1(\Omega;\mathbb{R}^n)) \cap W^{1,\infty}(0,T;L^2(\Omega;\mathbb{R}^n))\\ &&& \qquad\qquad \cap H^2(0,T;(H^2_{\Gamma_\mathrm{D}}(\Omega;\mathbb{R}^n))^*), && (63)\\ z_{\delta_k} &\to z && \text{weakly-star in } L^\infty(0,T;W^{1,p}(\Omega)) \text{ and weakly in } H^1(0,T;L^2(\Omega)). && (64)\end{aligned}$$

By applying the same technique as in the previous section, strong convergence of ∇z_{δ_k} in $L^p(\Omega_T;\mathbb{R}^n)$ can be obtained. We conclude that

$$z_{\delta_k} \to z \qquad \text{strongly in } L^p(0,T;W^{1,p}(\Omega)). \tag{65}$$

Furthermore, by (60), we find

$$z_{\delta_k} \to z \qquad \text{strongly in } \mathcal{C}(\overline{\Omega_T}) \tag{66}$$

for a subsequence by an Aubin-Lions type compactness result (cf. [Sim86]). As usual, we omit the subscript k.

Step 3: Passing to the limit $\delta \searrow 0$

- Integrating (12) from 0 to T and using (63), (66) and $\int_0^T \delta\langle Au_\delta,\zeta\rangle_{H^2}\,\mathrm{d}t \to 0$ due to (61), we conclude

$$\int_0^T \langle\partial_{tt}u,\zeta\rangle_{H^2}\,\mathrm{d}t + \int_{\Omega_T} W_{,e}(\epsilon(u),z):\epsilon(\zeta)\,\mathrm{d}x\,\mathrm{d}t = \int_{\Omega_T} l\cdot\zeta\,\mathrm{d}x\,\mathrm{d}t$$

for all $\zeta \in L^\infty(0,T;H^2_{\Gamma_\mathrm{D}}(\Omega;\mathbb{R}^n))$. Therefore, (4) is true for all $\zeta \in H^2_{\Gamma_\mathrm{D}}(\Omega;\mathbb{R}^n)$ and a.e. $t\in(0,T)$. Using the density of the set $H^2_{\Gamma_\mathrm{D}}(\Omega;\mathbb{R}^n)$ in $H^1_{\Gamma_\mathrm{D}}(\Omega;\mathbb{R}^n)$ (here we need the assumption that the boundary parts Γ_D and Γ_N have finitely many path-connected components, see [Ber11]), we can identify $\partial_{tt}u(t)\in(H^1_{\Gamma_\mathrm{D}}(\Omega;\mathbb{R}^n))^*$ and (4) is true for all $\zeta\in H^1_{\Gamma_\mathrm{D}}(\Omega;\mathbb{R}^n)$ and a.e. $t\in(0,T)$. Furthermore, $\partial_{tt}u\in L^\infty(0,T;(H^1_{\Gamma_\mathrm{D}}(\Omega;\mathbb{R}^n))^*)$.

- We choose the following cluster points w.r.t. a subsequence

$$\chi_\delta := \chi_{\{z_\delta>0\}} \to \chi \qquad \text{weakly-star in } L^\infty(\Omega_T), \tag{67}$$

$$\eta_\delta := \chi_{\{z_\delta=0\}\cap\{W_{,z}(\epsilon(u_\delta),z_\delta)+f'(z_\delta)\le 0\}} \to \eta \qquad \text{weakly-star in } L^\infty(\Omega_T), \tag{68}$$

$$F_\delta := \chi_{\{z_\delta>0\}}\epsilon(u_\delta) \to F \qquad \text{weakly in } L^2(\Omega_T;\mathbb{R}^{n\times n}), \tag{69}$$

$$G_\delta := \chi_{\{z_\delta=0\}\cap\{W_{,z}(\epsilon(u_\delta),z_\delta)+f'(z_\delta)\le 0\}}\epsilon(u_\delta) \to G \qquad \text{weakly in } L^2(\Omega_T;\mathbb{R}^{n\times n}). \tag{70}$$

By (66) and (63), we obtain for a.e. $x\in\{z>0\}$

$$\chi(x)=1,\ \eta(x)=0,\ F(x)=\epsilon(u)(x),\ G(x)=0, \tag{71}$$

because of the following reason:

Let $\zeta\in L^2(\Omega_T;\mathbb{R}^{n\times n})$ with $\mathrm{supp}(\zeta)\subseteq\{z>0\}$. Then, by (66), we obtain $\mathrm{supp}(\zeta)\subseteq\{z_\delta>0\}$ for all sufficiently small $\delta>0$. On the one hand, we find by (69)

$$\int_{\Omega_T} F_\delta:\zeta\,\mathrm{d}x\,\mathrm{d}t \to \int_{\Omega_T} F:\zeta\,\mathrm{d}x\,\mathrm{d}t.$$

On the other hand, by (63), (note that δ can be chosen arbitrarily small)

$$\int_{\Omega_T} F_\delta:\zeta\,\mathrm{d}x\,\mathrm{d}t = \int_{\Omega_T}\epsilon(u_\delta):\zeta\,\mathrm{d}x\,\mathrm{d}t \to \int_{\Omega_T}\epsilon(u):\zeta\,\mathrm{d}x\,\mathrm{d}t.$$

Thus, $\int_{\Omega_T}\epsilon(u):\zeta\,\mathrm{d}x\,\mathrm{d}t = \int_{\Omega_T}F:\zeta\,\mathrm{d}x\,\mathrm{d}t$. The other identities in (71) follow analogously.

Let $\zeta\in L^\infty(0,T;W^{1,p}_-(\Omega))$. Taking (50) into account, inequality (13) becomes by integration over time

$$\begin{aligned}0 \le{}& \int_{\Omega_T}\big(|\nabla z_\delta|^{p-2}\nabla z_\delta\cdot\nabla\zeta + \partial_t z_\delta\zeta\big)\,\mathrm{d}x\,\mathrm{d}t + \int_{\{z_\delta>0\}}\big(W_{,z}(\epsilon(u_\delta),z_\delta)+f'(z_\delta)\big)\zeta\,\mathrm{d}x\,\mathrm{d}t\\ &+\int_{\{z_\delta=0\}\cap\{W_{,z}(\epsilon(u_\delta),z_\delta)+f'(z_\delta)\le 0\}}\big(W_{,z}(\epsilon(u_\delta),z_\delta)+f'(z_\delta)\big)\zeta\,\mathrm{d}x\,\mathrm{d}t.\end{aligned}$$

Applying $limsup_{\delta\searrow 0}$ on both sides and multiplying by -1 yield

$$0 \ge \lim_{\delta\searrow 0}\int_{\Omega_T}\big(|\nabla z_\delta|^{p-2}\nabla z_\delta\cdot\nabla(-\zeta)+\partial_t z_\delta(-\zeta)\big)\,\mathrm{d}x\,\mathrm{d}t$$

$$+\liminf_{\delta\searrow 0}\int_{\Omega_T} h'(z_\delta)(F_\delta)^2(-\zeta)\,\mathrm{d}x\,\mathrm{d}t+\liminf_{\delta\searrow 0}\int_{\Omega_T}\chi_\delta\, f'(z_\delta)(-\zeta)\,\mathrm{d}x\,\mathrm{d}t$$
$$+\liminf_{\delta\searrow 0}\int_{\Omega_T} h'(z_\delta)(G_\delta)^2(-\zeta)\,\mathrm{d}x\,\mathrm{d}t+\liminf_{\delta\searrow 0}\int_{\Omega_T}\eta_\delta\, f'(z_\delta)(-\zeta)\,\mathrm{d}x\,\mathrm{d}t.$$

Weakly l.s.c. arguments, the uniformly convergence property (66) and the properties listed in (71) give

$$\begin{aligned}0\geq&\int_{\Omega_T}\left(|\nabla z|^{p-2}\nabla z\cdot\nabla(-\zeta)+\partial_t z(-\zeta)\right)\,\mathrm{d}x\,\mathrm{d}t\\&+\int_{\{z>0\}}\left(W_{,z}(\epsilon(u),z)+f'(z)\right)(-\zeta)\,\mathrm{d}x\,\mathrm{d}t\\&+\int_{\{z=0\}}\left((F^2+G^2)h'(z)+(\chi+\eta)f'(z)\right)(-\zeta)\,\mathrm{d}x\,\mathrm{d}t.\end{aligned}$$

This inequality may also be rewritten in the following form:

$$\begin{aligned}0\leq&\int_{\Omega_T}\left(|\nabla z|^{p-2}\nabla z\cdot\nabla\zeta+\left(W_{,z}(\epsilon(u),z)+f'(z)+\partial_t z\right)\zeta\right)\,\mathrm{d}x\,\mathrm{d}t\\&+\int_{\{z=0\}}\left((F^2+G^2)h'(z)+(\chi+\eta)f'(z)-W_{,z}(\epsilon(u),z)-f'(z)\right)\zeta\,\mathrm{d}x\,\mathrm{d}t.\end{aligned}$$

Therefore,

$$0\leq\int_{\Omega_T}\left(|\nabla z|^{p-2}\nabla z\cdot\nabla\zeta+\left(W_{,z}(\epsilon(u),z)+f'(z)+\partial_t z+\xi\right)\zeta\right)\,\mathrm{d}x\,\mathrm{d}t$$

with

$$\xi:=\chi_{\{z=0\}}\min\Big\{0,(F^2+G^2)h'(z)+(\chi+\eta-1)f'(z)-W_{,z}(\epsilon(u),z)\Big\}.$$

This proves (5) and (6).

- To prove the energy inequality (7), we can proceed as in Subsection 3.2: Integrating (15) with respect to time on $[t_1,t_2]$ yields ($0\leq t_1\leq t_2\leq T$)

$$\int_{t_1}^{t_2}\left(A_\delta(t)+B_\delta(t)+C_\delta(t)\right)\,\mathrm{d}t\leq 0\tag{72}$$

with

$$\begin{aligned}A_\delta(t):=&\int_\Omega\left(\frac{1}{p}|\nabla z_\delta(t)|^p+W(\epsilon(u_\delta(t)),z_\delta(t))+f(z_\delta(t))+\frac{1}{2}|\partial_t u_\delta(t)|^2\right)\,\mathrm{d}x\\&-\int_\Omega\partial_t u_\delta(t)\cdot\partial_t b(t)\,\mathrm{d}x-\int_\Omega\left(\frac{1}{p}|\nabla z^0|^p+W(\epsilon(u^0),z^0)+f(z^0)+\frac{1}{2}|v^0|^2\right)\,\mathrm{d}x\\&+\int_\Omega v^0\cdot\partial_t b^0\,\mathrm{d}x,\\B_\delta(t):=&\int_{\Omega_t}|\partial_t z_\delta|^2\,\mathrm{d}x\,\mathrm{d}s-\int_{\Omega_t}W_{,e}(\epsilon(u_\delta),z_\delta):\epsilon(\partial_t b)\,\mathrm{d}x\,\mathrm{d}s+\int_{\Omega_t}\partial_t u_\delta\cdot\partial_{tt}b\,\mathrm{d}x\,\mathrm{d}s\\&-\int_{\Omega_t}l\cdot(\partial_t u_\delta-\partial_t b)\,\mathrm{d}x\,\mathrm{d}s,\end{aligned}$$

$$C_\delta(t) := \frac{\delta}{2}\langle Au_\delta(t), u_\delta(t)\rangle_{H^2} - \frac{\delta}{2}\langle Au^0, u^0\rangle_{H^2} - \delta\int_0^t \langle Au_\delta(t), \partial_t b(t)\rangle_{H^2}\,\mathrm{d}t.$$

Let A and B be the corresponding functions to A_δ and B_δ, where u_δ and z_δ are replaced by u and z, respectively. The limit passage in (72) can be performed as follows.

- Weakly lower semi-continuity arguments show

$$\liminf_{\delta\searrow 0}\int_{t_1}^{t_2} A_\delta(t)\,\mathrm{d}t \geq \int_{t_1}^{t_2} A(t)\,\mathrm{d}t.$$

- Fatou's lemma and weakly l.s.c. arguments as well as the convergence property for z_δ (see (65)) show (cf. (54))

$$\liminf_{\delta\searrow 0}\int_{t_1}^{t_2} B_\delta(t)\,\mathrm{d}t \geq \int_{t_1}^{t_2} B(t)\,\mathrm{d}t.$$

- We have

$$C_\delta(t) \geq -\frac{\delta}{2}\langle Au^0, u^0\rangle_{H^2} - \delta\|u_\delta(t)\|_{H^2(\Omega;\mathbb{R}^n)}\|\partial_t b(t)\|_{H^2(\Omega;\mathbb{R}^n)}.$$

  By (61), we obtain

$$\liminf_{\delta\searrow 0}\int_{t_1}^{t_2} C_\delta(t)\,\mathrm{d}t \geq 0.$$

We end up with $\int_{t_1}^{t_2} A(t) + B(t)\,\mathrm{d}t \leq 0$ for all $0 \leq t_1 \leq t_2 \leq T$. This proves (7).

Proof of Theorem 3.3
Putting all steps of Section 3.3 together, Theorem 3.3 is proven. □

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