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Regularity of second derivatives in elliptic transmission problems near an interior regular multiple line of contact

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Abstract

We investigate the regularity of the weak solution to elliptic transmission problems that involve several materials intersecting at a closed interior line of contact. We prove that local weak solutions possess second order generalized derivatives up to the contact line, mainly exploiting their higher regularity in the direction tangential to the line. Moreover we are thus able to characterize the higher regularity of the gradient and the Hölder exponent by means of explicit estimates known in the literature for two dimensional problems. They show that strong regularity properties, for instance the integrability of the gradient to a power larger than the space dimension $d = 3$, are to expect if the oscillations of the diffusion coefficient are moderate (that is for far larger a range than what a theory of small perturbations would allow), or if the number of involved materials does not exceed three.

1 Introduction

The paper is concerned with the regularity of weak solutions to

$$-\operatorname{div}(\kappa\,\nabla u) = f \text{ in } \Omega, \tag{1}$$

$$[u]_S = 0\,, \quad [-\kappa\,\nabla u\cdot\nu]_S = q \text{ on } S = \bigcup_{i=1}^{m} S_i\,, \tag{2}$$

where $\Omega \subset \mathbb{R}^3$ is a bounded domain and $S \subset \Omega$ is a two-dimensional *closed* hypersurface consisting of $m \in \mathbb{N}$ finitely many pieces $S_1, \ldots, S_m$ of class $\mathcal{C}^2$ intersecting at a closed contact line K contained in the interior of the domain Ω. The surface S is assumed inducing a partition of the domain Ω into m open subsets $\Omega_1, \ldots, \Omega_m$.

In the equation (1), the function f is the given right-hand, and the matrix-valued coefficient function κ is assumed to be material-dependent, that means

$$\kappa(x) = \kappa^i \quad \text{if } x \in \Omega_i\,, \tag{3}$$

with elliptic matrices $\kappa^i \in \mathbb{R}^{3\times3}_{\text{sym}}$ for $i = 1, \ldots, m$. The conditions (2) are the transmission conditions: The symbol $[\cdot]_S$ denotes the difference between the values of the enclosed quantity from both sides of S, ν is a unit normal to the surface S, and q is a given function.

We prove the existence up to K of second weak derivatives for weak solutions to (1), (2). The idea is to exploit that the problem is differentiable in the direction tangent to the line K. For appropriate data, this property yields the higher regularity of the solution in this one direction, a fact which allows to locally project the problem (1), (2) on a plane. This complexity reduction yields estimates for u and ∇u in L^2-spaces weighted by some power of the distance to the line K, which afterwards can be converted into regularity results for standard data norms.

This argumentation turns out not only able to deal with very general nonvanishing transmission conditions, but it also leads to a characterization of the integrability of $D^2(u)$ and $D(u)$ as functions of a number $0 < \alpha_{\text{opt}} < 1$ which we could call the best possible interior Hölder exponent for local weak solutions to transmission problems in *two-space dimensions*, for which explicit estimates are known. In the general case, we can choose

$$\alpha_{\text{opt}} = \alpha_{\text{opt}}(\kappa) \geq \sqrt{\inf_{\Omega} \frac{\lambda_{\min}(\kappa)}{\lambda_{\max}(\kappa)}} \, \frac{4}{\pi} \, \arctan \sqrt{\frac{\inf_{\Omega} \lambda_{\min}(\kappa)}{\sup_{\Omega} \lambda_{\max}(\kappa)}} \, . \tag{4}$$

In the case of isotropic diffusion, we even can choose

$$\alpha_{\text{opt}}(\kappa) \geq \max\left\{\frac{4}{\pi} \, \arctan \sqrt{\frac{\inf_{\Omega} \kappa}{\sup_{\Omega} \kappa}}, \, \alpha_{\text{iso}}\right\} \tag{5}$$

where α_{iso} is a constant depending on the number m of materials involved and of the angles of contact of the surfaces $S_1, \ldots, S_m$ at K. In order to estimate α_{iso}, we shall rely on the results attained in the paper [Mer03] for the cases $m = 2, 3$. For $m = 3$, we need to further introduce for $x \in K$ the number $\gamma(x)$ defined as the opening of the region Ω_{i_0} where the coefficient κ takes its intermediate value, that is $i_0 \in \{1, 2, 3\}$ is such that $\max \kappa > \kappa_{i_0} > \min \kappa$. Then

$$\alpha_{\text{iso}} > \frac{1}{2} \text{ if } m = 2 \qquad \alpha_{\text{iso}} > \min\left\{\frac{1}{2}, \, \frac{\pi}{2 \max_{x \in K} \gamma(x)}\right\} \text{ if } m = 3 \, . \tag{6}$$

For further inequalities concerning the number α_{iso} in the case $m \geq 4$, we refer to the Theorem 16 of [Mer03].

In this way, we obtain explicit formula showing in particular that the higher integrability of $\nabla u \in L^s$ can degenerate only for one of the following two reasons:

(1) The ratio of the eigenvalues $\lambda_{\min}(\kappa^i)/\lambda_{\max}(\kappa^i)$ is small for one of the materials $i = 1, \ldots, m$ (strong anisotropy: Example in [ERS07]);

(2) The overall ratio $\lambda_{\min}(\kappa)/\lambda_{\max}(\kappa)$ is small (strong discontinuity: Examples in [Mey63], [Mer03]);

Moreover, we obtain the desirable integrability of $|\nabla u|$ to a power larger than the space dimension $d = 3$ for many a situation where the mentioned ratios are moderate or $m = 2, 3$. The remainder of the Introduction is devoted to the precise formulation of these statements.

The main result Throughout the paper, Ω denote a bounded domain. There are disjoint subdomains $\Omega_i \subset \Omega$, $i = 1, \ldots, m$ such that $\overline{\Omega} = \bigcup_{i=1}^m \overline{\Omega_i}$. The hypersurfaces $\partial\Omega_i \cap \partial\Omega_j$, $i, j = 1, \ldots, m$, $i \neq j$ are assumed to be of class $\mathcal{C}^2$ and intersecting only at a curve K, the triple contact line contained in the interior of the domain.

The surface $S = \bigcup_{i=1}^m S_i \cup K$ with unit normal ν is the interface for the transmission problem. With the superscripts ν^i, we denote the restriction of ν to S_i. We choose the orientation of ν^i according to a counter clockwise circulation around the curve K. A unit (co-)normal to the surface S_i at the curve K is given by

$$T^i := \tau \times \nu^i \,, \tag{7}$$

where τ is the unit directional vector of the curve K. The angle of contact $\gamma \in]0,\, 2\pi[$ between the surfaces S_i and S_{i+1} at the curve K is fixed via the relations

$$\begin{cases} \cos\gamma_i = \nu^i \cdot \nu^{i+1} \\ \sin\gamma_i = \nu^i \cdot T^{i+1} \end{cases} \text{for } i < m, \quad \begin{cases} \cos\gamma_m = \nu^m \cdot \nu^1 \\ \sin\gamma_m = \nu^m \cdot T^1 \end{cases} \text{on } K\,. \tag{8}$$

Due to the fact that there is a common contact line K, the angles must satisfy

$$\sum_{i=1}^m \gamma_i = 2\,\pi\,. \tag{9}$$

We require from the geometrical setting that

$$\inf_{i=1,\ldots,m,\, y\in K} |\sin\alpha_i(y)| > 0\,. \tag{10}$$

Otherwise, the surfaces $S_1, \ldots, S_m$ would be pairwise tangent at some point of K, thus merging to a smooth surface. Beside the geometry, the diffusion coefficient κ is the essential parameter in the regularity discussion. We require for $i = 1, \ldots, m$ that $\kappa^i \in \mathbb{R}^{3\times3}_{\text{sym}}$ and that there are constants $0 < \lambda_{\min}(\kappa^i) \le \lambda_{\max}(\kappa^i) < \infty$ such that

$$\lambda^i_{\min}\,\xi^2 \le \kappa^i\xi\cdot\xi \le \lambda^i_{\max}\,\xi^2 \text{ for all } \xi \in \mathbb{R}^3\,. \tag{11}$$

For the given flux q in the condition (2), we will need well-known trace spaces. We denote R_{S_i} the restriction operator from S onto S_i for $i = 1, \ldots, m$, and we define

$$W^{1/2,2}(S\setminus K) := \{q \in L^2(S) \;:\; q_i := R_{S_i}(q) \in W^{1/2,2}(S_i)\} \tag{12}$$

$$W^{1/2,2}_{00}(S\setminus K) := \{q \in L^2(S) \;:\; q_i := R_{S_i}(q) \in W^{1/2,2}_{00}(S_i)\}\,. \tag{13}$$

Theorem 1.1. *For $i = 1, \ldots, m$, let $\kappa^i \in \mathbb{R}^{3\times3}_{sym}$ satisfy* (11)*. Assume that $f \in L^2(\Omega)$, and $q \in W^{1/2,2}(S\setminus K)$. For $0 \le \alpha \le 1$, define*

$$t(\alpha) := \frac{2}{2-\alpha}, \quad s(\alpha) := 2\,(1+\alpha)\,. \tag{14}$$

Then, every local weak solution $u \in W^{1,2}_{loc}(\Omega)$ for the problem (1), (2) *belongs to the space $W^{2,t(\alpha)}_{loc}(\Omega\setminus S) \cap W^{1,s(\alpha)}_{loc}(\Omega)$ for all $0 < \alpha < \alpha_{opt}$ (cf.* (4)*). The continuity inequality*

$$\|u\|_{W^{2,t}(\Omega\setminus S)} + \|u\|_{W^{1,s}(\Omega)} \le c\,(\|f\|_{L^2(\Omega)} + \|q\|_{W^{1/2,2}(S\setminus K)})\,.$$

holds with a constant c depending only on the maximal main curvatures of the surfaces S_i.

The property $\nabla u \in L^s(\Omega)$ with a $s > 3$ is often desirable from the point of view of applications.

Corollary 1.2. *Assumptions of Theorem 1.1. Assume that one of the following conditions is valid:*

(1) $r := \sup_{i=1,\ldots,m} \frac{\lambda_{\max}(\kappa^i)}{\lambda_{\min}(\kappa^i)}$ *satisfies* $r < 4$, *and* $\frac{\inf_{i=1,\ldots,m} \lambda_{\min}(\kappa^i)}{\sup_{i=1,\ldots,m} \lambda_{\max}(\kappa^i)} > \tan^2(r^{1/2}\,\frac{\pi}{8})$*;*

(2) The coefficient κ *is a scalar function, and* $m = 2$ *or* $m = 3$ *and the opening of the domain in which* κ *takes its intermediate value is everywhere strictly less than* π *at* K*;*

Then, $\nabla u \in L^s(\Omega)$ *for a* $s > 3$.

Our second statement puts the best Hölder exponent in three dimensions in relationship to the two-dimensional Hölder exponent α_{opt} of the formula (4).

Theorem 1.3. *Assumptions of Theorem 1.1. Assume that* $u \in W^{1,2}_{loc}(\Omega)$ *is a local weak solution to* (1), (2) *with* $f = 0 = q$. *Then,* u *belongs to* $C^{\beta}_{loc}(\Omega)$ *for* $\beta := \frac{\alpha_{opt}}{1+\alpha_{opt}}$. *If in addition at least to of the surfaces* $S_1, \ldots, S_m$ *are of class* $\mathcal{C}^3$, *then* u *belongs to* $C^{\beta}_{loc}(\Omega)$ *for all* $\beta < \alpha_{opt}$.

2 A starting inequality

2.1 An *a priori* estimate in two dimensions

The main idea of the paper is to use a dimension reduction method. In this preliminary section, we consider a bounded domain $G \subset \mathbb{R}^2$ and a one dimensional submanifold $\Gamma \subset G$ consisting of $m \in \mathbb{N}$ curves Γ_i, of class $\mathcal{C}^2$ that meet at an interior point $x_K \in G$. For $i = 1, \ldots, m$ a uniformly elliptic matrix $A^i \in \mathbb{R}^{3\times 3}_{\text{sym}}$ is given. We denote $A := A^i$ in G_i. For $x \in G$, we denote $d_K(x) := |x - x_K|$.

Theorem 2.1. *For* $u \in W^{2,2}(G \setminus \Gamma) \cap W^{1,2}(G)$ *such that* $u\, d_K^{-1} \in W^{1,2}(G)$ *the inequality*

$$\|u\|_{W^{2,2}_{loc}(G\setminus\Gamma)} \leq c\,(\|\operatorname{div}(A\,\nabla u)\|_{L^2(G)} + \|[A\nabla u\cdot\nu]_\Gamma\|_{W^{1/2,2}_{00}(\Gamma\setminus\{x_K\})} + \|u\, d_K^{-1}\|_{W^{1,2}(G)})$$

The constant c *depends only on the curvature of the curves* Γ_i.

Proof. Denote $f(u) := -\operatorname{div}(A\nabla u)$ and $q(u) := [-A\,\nabla u\cdot\nu]_S$. Then, for all $\phi \in C^1_c(G)$

$$\int_G A\nabla u\cdot\nabla\phi + \int_\Gamma q\,\phi = \int_G f\,\phi\,. \tag{15}$$

For $x \in \Gamma_i$, we call $T^i(x)$ the tangential unit vector pointing in the outward direction at x_K. We claim that there is a vector field $T \in L^\infty(\Omega \setminus \mathbb{R}^3)$ such that $T \in W^{1,\infty}_{\text{loc}}(\Omega \setminus \mathbb{R}^3)$ and

$$T = T^i \text{ on } S^i, \quad |\nabla T| \leq c\, d_K^{-1} \text{ in } G \setminus \{x_K\}\,.$$

This claim is indeed readily verified, since the vector field $\bar{T} = d_K\, T$ is Lipschitz continuous on Γ, and thus possess an extension $\bar{T} \in W^{1,\infty}(G;\, \mathbb{R}^3)$. We can find a neighbourhood $B_\rho(x_K)$ such that $|\bar{T}(x)| > 0$ in $B_\rho(x_K) \setminus \{x_K\}$. For $x \in B_\rho(x_K) \setminus \{x_k\}$, we define $T(x) := \bar{T}(x)/|\bar{T}(x)|$, which is a unit vector with the required property.

Our assumptions and the properties of T now clearly imply that the function $u_T := T \cdot \nabla u$ belongs to $W^{1,2}(G)$, and that

$$\|u_T\|_{W^{1,2}(G)} \leq c\,(\|u\|_{W^{2,2}(G\setminus\Gamma)} + \|u\, d_K^{-1}\|_{W^{1,2}(G)})\,.$$

In the relation (15), we now insert a test function of the form $\phi = T \cdot \nabla \eta$ with $\eta \in C^2_c(G)$. A few integration by parts yield for $u_T = T \cdot \nabla u$ the relation

$$\int_G A\nabla u_T \cdot \nabla \eta = \int_G D(T)\nabla u \cdot \nabla \eta + \int_\Gamma q\, T \cdot \nabla \eta - \int_G f\, T \cdot \nabla \eta\,, \tag{16}$$
$$D_{i,j}(T) := A_i \cdot T_{j,x} + A_j \cdot T_{i,x} - \kappa_{i,j}\, \operatorname{div} T\,.$$

Observe that $\|D(T)\nabla u\|_{L^2(G)} \leq c\, \|\nabla u\, d_K^{-1}\|_{L^2(G)}$. Moreover, we will next show in the Lemma 2.2 that

$$\left|\int_\Gamma q\, T \cdot \nabla \eta\right| \leq c\, \|q\|_{W^{1/2,2}_{00}(\Gamma\setminus\{x_K\})}\, \|\eta\|_{W^{1/2,2}(\Gamma\setminus\{x_K\})}\,.$$

The relation (16) now turns out valid for all $\eta \in W^{1,2}_0(G)$. Thus, we can insert a function of the form $\eta = \zeta^2\, u_T$, $\zeta \in C^\infty_c(G)$, $\zeta \equiv 1$ on $B_\rho(x_K)$. After a few calculations, we obtain the estimate

$$\int_G \zeta^2\, |\nabla u_T|^2 \leq c\,(\|f\|_{L^2(G)} + \|q\|_{W^{1/2,2}_{00}(\Gamma\setminus\{x_K\})} + \|u\, d_K^{-1}\|_{W^{1,2}(G)})^2\,.$$

The regularity theory near the interior of each curve Γ_i is well known. We thus obtain that

$$\|u_T\|_{W^{1,2}_{\text{loc}}(G)} \leq c\,(\|f\|_{L^2(G)} + \|q\|_{W^{1/2,2}_{00}(\Gamma\setminus\{x_K\})} + \|u\, d_K^{-1}\|_{W^{1,2}(G)})\,.$$

We next use the validity of the equation $-\operatorname{div}(A\nabla u) = f$ almost everywhere in $G\setminus\Gamma$, and call ν the vector $(-T_2,\, T_1)$ and $u_\nu := \nu\cdot\nabla u$ the part of the gradient orthogonal to u_T. Orthonormal decomposition yields in $B_\rho(x_K) \setminus \Gamma$ the identity

$$\begin{aligned}
&-(A\nu \cdot \nu)\, \nu \cdot \nabla u_\nu = f \\
&+ \nabla u \cdot \{T\,[(T\cdot\nabla)AT \cdot T + (\nu\cdot\nabla)AT\cdot\nu] + \nu\,[(T\cdot\nabla)A\nu\cdot T + (\nu\cdot\nabla)A\nu\cdot\nu]\} \\
&\quad + (T\cdot\nabla u_T)\, AT\cdot T + (T\cdot\nabla u_\nu)\, AT\cdot\nu + (\nu\cdot\nabla u_T)\, AT\cdot T\ .
\end{aligned}$$

We also note the formula $T\cdot\nabla u_\nu = \nu\cdot\nabla u_T + [(T\cdot\nabla)\nu - (\nu\cdot\nabla)T]\cdot\nabla u$. It thus follows that

$$\|\nu\cdot\nabla u_\nu\|_{L^2(B_\rho(x_K))} \leq c\,(\|f\|_{L^2(G)} + \|u_T\|_{W^{1,2}(G)} + \|u\, d_K^{-1}\|_{W^{1,2}(G)})\,.$$

The claim follows. □

Lemma 2.2. *Assume that $q \in W^{1/2,2}_{00}(\Gamma \setminus \{x_K\})$. For $v \in W^{1,2}(\Gamma \setminus \{x_K\})$, define a linear functional $F(v) := \int_S q\, T \cdot \nabla v$. Then, F extends to an element of $[W^{1/2,2}(\Gamma \setminus \{x_K\})]^*$ and*

$$\|F(v)\|_{[W^{1/2,2}(\Gamma\setminus\{x_K\})]^*} \leq c\,\|q\|_{W^{1/2,2}_{00}(\Gamma\setminus\{x_K\})} \,.$$

Proof. For $i := 1, 2, 3$, and for $q \in [C^1_c(\overline{\Gamma \setminus \{x_K\}})]^3$, define $(a^i(q))(v) := \int_{\Gamma_i} q\, T^i \cdot \nabla v$. Then $|(a^i(q))(v)| \leq \|q\|_{L^2(\Gamma_i)}\, \|v\|_{W^{1,2}(\Gamma_i)}$. Since q has a compact support in $\Gamma \setminus \{x_K\}$, integration by parts yields

$$a^i(q)(v) = -\int_{\Gamma_i} \operatorname{div}_\Gamma(q\, T)\, v$$

. Since $|\operatorname{div}_\Gamma T| \leq c\, d_K^{-1}$, the Hardy inequality implies that

$$|a^i(q)(v)| \leq c\,\|q\|_{W^{1,2}_0(\Gamma_i)}\,\|v\|_{L^2(\Gamma_i)} \,.$$

It follows that

$$a^i \in \mathscr{L}(L^2(\Gamma_i),\, [W^{1,2}(\Gamma_i)]^*),\;\; a^i \in \mathscr{L}(W^{1,2}_0(\Gamma_i),\, [L^2(\Gamma_i)]^*) \,.$$

We note the interpolation identities ([LM68], Ch. 1, Section 6.2)

$$[L^2(\Gamma_i),\, W^{1,2}_0(\Gamma_i)]_{1/2} = W^{1/2,2}_{00}(\Gamma_i),\;\; [(W^{1,2}(\Gamma_i))^*,\, (L^2(\Gamma_i))^*]_{1/2} = (W^{1/2,2}(\Gamma_i))^* \,.$$

Thus $\|a^i\|_{\mathscr{L}(W^{1/2,2}_{00}(\Gamma_i),\,(W^{1/2,2}(\Gamma_i))^*)} \leq c$, proving the claim. □

2.2 Directional regularity

We now turn to the three-dimensional problem (1), (2). Observe at first that the vector fields τ, ν^i and T^i possess natural extensions.

Remark 2.3. *Since for $i = 1, \ldots, m$ the surface S_i is of class $\mathcal{C}^2$, there is an extension surface $\widetilde{S}_i$ of class $\mathcal{C}^2$ as well that contains the curve K in its interior. The unit normal to $\widetilde{S}_i$ can be extended into Ω using the ansatz $\nu^i = \nabla d_i$, where d_i is the signed distance function to $\widetilde{S}_i$. The extension ν^i is a continuously differentiable unit vector in a neighbourhood $B_\rho(\widetilde{S}_i) := \{x \in \Omega \,:\, |d_i(x)| < \rho\}$ of $\widetilde{S}_i$. The size of $\rho > 0$ is determined by the maximal main curvatures of the surface, that is the number $\|\,\delta\,\nu^i\|_{L^\infty(\widetilde{S}_i)}$. From the neighbourhood B_ρ, an arbitrary extension to the entire domain can be constructed so that $\nu^i \in [C^1(\overline{\Omega})]^3$.*

We can construct an extension of the vector τ into Ω by setting

$$\tau(x) := \frac{\nabla d_i(x) \times \nabla d_j(x)}{|\nabla d_i(x) \times \nabla d_j(x)|} \text{ for } x \in B_\rho(K) \cap \Omega_k \,, \tag{17}$$

where Ω_k denotes the part of the domain bounded by the surfaces $S_i,\ S_j$. Observe that

$$|\nabla d_i(x) \times \nabla d_j(x)| = |\nu^i(x) \times \nu^j(x)| = |\sin \alpha^k(x)|$$

on K. Thus, in view of the assumption (10), $|\nabla d_i(x) \times \nabla d_j(x)| > 0$ on K, and owing to Remark 2.3, also $|\nabla d_i(x) \times \nabla d_j(x)| > 0$ in some neighbourhood $B_\rho(K)$ determined by the curvatures of the surfaces. We commence with an auxiliary statement.

Lemma 2.4. *Assume that* $\zeta \in W^{1/2,2}(S\setminus K)$*. For* $v \in W^{1,2}(S\setminus K)$*, define a linear functional* $F(v) := \int_S \zeta\,\tau\cdot\nabla v$*. Then,* F *extends to an element of* $[W^{1/2,2}(S\setminus K)]^*$ *and*

$$\|F(v)\|_{[W^{1,2}(\Omega)]^*} \le c\,\|\zeta\|_{W^{1/2,2}(S\setminus K)}\,.$$

Proof. For $i := 1, 2, 3$, and for $\zeta \in C^1(\overline{S_i})$, define a linear functional $a^i(\zeta)$ via

$$(a^i(\zeta))(v) := \int_{S_i} \zeta\,\tau\cdot\nabla v, \quad v \in W^{1,2}(S\setminus K)\,.$$

Clearly, $|(a^i(\zeta))(v)| \le \|\zeta\|_{L^2(S_i)}\,\|v\|_{W^{1,2}(S_i)}$. Using integration by parts, and the fact that τ is tangent across K, observe that $a^i(\zeta)(v) = -\int_{S_i}\{\mathrm{div}_S\,\tau\,\zeta\,v + \tau\cdot\nabla\zeta\,v\}$, entailing that

$$|a^i(\zeta)(v)| \le c(\tau)\,\|\zeta\|_{W^{1,2}(S_i)}\,\|v\|_{L^2(S_i)}\,,$$

where c depends only on τ. Since $C^1(\overline{S_i})$ is dense in $L^2(S_i)$ as well as in $W^{1,2}(S_i)$, the operator a^i extends to an element of

$$\mathscr{L}(L^2(S_i),\,[W^{1,2}(S_i)]^*) \cap \mathscr{L}(W^{1,2}(S_i),\,[L^2(S_i)]^*)$$

with the inequalities

$$\|a^i\|_{\mathscr{L}(L^2(S_i),\,[W^{1,2}(S_i)]^*)} \le 1, \quad \|a^i\|_{\mathscr{L}(W^{1,2}(S_i),\,[L^2(S_i)]^*)} \le c(\tau)\,.$$

Note that ([LM68], Ch. 1, Section 6.2)

$$[L^2(S_i),\,W^{1,2}(S_i)]_{1/2} = W^{1/2,2}(S_i), \quad [(W^{1,2}(S_i))^*,\,(L^2(S_i))^*]_{1/2} = (W^{1/2,2}(S_i))^*\,.$$

Thus $\|a^i\|_{\mathscr{L}(W^{1/2,2}(S_i),\,(W^{1/2,2}(S_i))^*)} \le c\,\sqrt{c(\tau)}$ proving that

$$|F(v)| = |\sum_{i=1}^{3}(a^i(\zeta))(v)| \le c\,\|\zeta\|_{W^{1/2,2}(S\setminus K)}\,\|v\|_{W^{1/2,2}(S\setminus K)}\,.$$

The claim follows. □

In order to prove a basic result on directional regularity for the problem (1), (2), we next need a technical description of how to locally flatten the curve K in order to reduce the problem to a reference configuration where it is easier to separate variables. For $x_0 \in K$ and $t, \rho > 0$, we define a curvilinear cylinder $Z_{t,\rho}(x_0) \subset B_\rho(K)$, and for $j = 1, 2$ also surfaces $\Gamma^j_{t,\rho}(x_0)$ via

$$\begin{aligned}
Z_{t,\rho}(x_0) &:= \{x \in \Omega \,:\, |(x - x_0)\cdot\tau(x_0)| < t,\; d_K(x) < \rho\}\,,\\
\Gamma^1_{t,\rho}(x_0) &:= \{x \in \Omega \,:\, |(x - x_0)\cdot\tau(x_0)| < t,\; d_K(x) = \rho\}\,,\\
\Gamma^2_{t,\rho}(x_0) &:= \{x \in \Omega \,:\, |(x - x_0)\cdot\tau(x_0)| = t,\; d_K(x) < \rho\}\,.
\end{aligned}$$

where d_K is the distance to the curve K. We define $K_{t,\rho}(x_0) := Z_{t,\rho}(x_0) \cap K$, the piece of the curve K contained in this cylinder. We introduce a reference domain, surfaces and curve via

$$\begin{aligned}
&Z :=]-1,1[\times B_1,\\
&\Gamma^1_0 :=]-1,1[\times\partial B_1, \quad \Gamma^2_0 := \{-1,1\}\times\partial B_1\\
&K_0 := \{z \in Z \,:\, z_2 = 0 = z_3\}\,.
\end{aligned}$$

Here and in the remainder of the section, we use the notation B_r only in connection with the two dimensional ball of radius r centred at zero for the reference coordinates. Since K is a curve of class $\mathcal{C}^2$, we claim that there is a diffeomorphism $\Phi = \Phi^{t,\rho,x_0}$ of class $C^2(\overline{Z})$ mapping Z onto $Z_{t,\rho}(x_0)$, and moreover such that

$$\Phi(\Gamma_0^j) = \Gamma_{t,\rho}^j(x_0) \text{ for } j=1,2, \quad \Phi(K_0) = K_{t,\rho}(x_0), \quad \Phi(0) = x_0\,.$$

Moreover, for $t+\rho$ sufficiently small, the piece of curve $K_{t,\rho}(x_0)$ is almost flat. Thus, there is an orthogonal matrix O mapping the standard unit vector $\mathbf{e}^1$ onto $\tau(x_0)$, and for $z \in Z$ a transformation $\Phi^0(z) := Oz + x_0$, such that

$$\|\Phi^{t,\rho} - \Phi^0\|_{C^1(\overline{Z})} \to 0 \text{ for } t+\rho \to 0\,. \tag{18}$$

In order to save space, we do not attempt to describe the detailed construction of the mapping Φ here. We then define a surface $S_0 \subset Z$ via

$$S_0 = \Phi^{-1}(S), \quad \nu^0(z) = \frac{(d\Phi(z))^T \nu(\Phi(z))}{|(d\Phi(z))^T \nu(\Phi(z))|} \text{ unit normal to } S_0\,. \tag{19}$$

Observe that the unit tangent vector of the line K_0 is the standard unit vector $\mathbf{e}^1$, and that the ansatz

$$\tau(\Phi(z)) = \frac{d\Phi(z)\mathbf{e}^1}{|d\Phi(z)\mathbf{e}^1|}\,,$$

provides an extension of class C^1 for the vector τ into Ω

Lemma 2.5. *Let $u \in W^{1,2}_{loc}(\Omega)$ be a local weak solution to* (1), (2). *Assume that $f \in L^2(\Omega)$ and $q \in W^{1/2,2}(S \setminus K)$. Then, the function $u_\tau = \tau \cdot \nabla u$ belongs to $W^{1,2}_{loc}(\Omega)$, and it satisfies for all $v \in C^1_c(\Omega)$ the relation*

$$\int_\Omega \kappa \nabla u_\tau \cdot \nabla v = \int_\Omega \{\kappa\, D(\tau)\nabla u \cdot \nabla v - f\,\tau\} \cdot \nabla v + \int_S q\,\tau \cdot \nabla v \tag{20}$$
$$D_{i,j}(\tau) := A_i \cdot \tau_{j,x} + A_j \cdot \tau_{i,x} - \kappa_{i,j}\,\operatorname{div}\tau\,.$$

Moreover, a continuity estimate is valid:

$$\|u_\tau\|_{W^{1,2}_{loc}(\Omega)} \leq c\,(\|f\|_{L^2(\Omega)} + \|q\|_{W^{1/2,2}(S\setminus K)})\,.$$

Proof. There are several ways to prove the claim. A possibility is to use the coordinate transformation $\Phi \in C^2(Z)$ that maps a neighbourhood of the line K onto a reference configuration, where the image of K is a line. In the reference coordinates, we obtain that

$$\int_Z A\nabla\hat{u} \cdot \nabla\phi + \int_{S_0} \hat{q}\,\phi = \int_Z \hat{f}\,\phi, \quad \phi \in W^{1,2}_0(Z)\,, \tag{21}$$

where

$$A(z) := |\det d\Phi(z)|\,(d\Phi(z))^{-1} \circ \kappa \circ (d\Phi(z))^{-T} \in C^1(\overline{Z \setminus S_0}; \mathbb{R}^{3\times 3}_{\text{sym}})\,,$$
$$\hat{q} := |\det d\Phi|\, q \circ \Phi \in W^{1/2,2}(S_0 \setminus K_0), \quad \hat{f} = |\det d\Phi|\, f \circ \Phi \in L^2(Z)\,. \tag{22}$$

Here we can apply the method of finite differences. For $h > 0$ and $\phi \in W^{1,2}_0(Z)$ such that $\text{supp}(\phi) \subset Z_h := \{z \in \overline{Z} \,:\, |z_1| \leq 1-h$, we can define the averaging

$$\phi_h(z) := h^{-1} \int_{z_1}^{z_1+h} \phi(t,\, \bar{z})\, dt\,,$$

and we choose in (21) the test function $\phi = \partial_{z_1}\phi_h = h^{-1}\,(\phi(z_1+h,\, \bar{z}) - \phi(z))$. Owing to the assumptions, we obtain from Lemma 2.4 the estimate

$$\left|\int_{S_0} \hat{q}\, \partial_{z_1}\phi_h\right| \leq c\, \|\hat{q}\|_{W^{1/2,2}(S_0\setminus K_0)}\, \|\phi_h\|_{W^{1/2,2}(S_0\setminus K_0)}\,.$$

Thus, the function $\hat{u}_h \in W^{1,2}_{\text{loc}}(Z)$ satisfies for all $\phi \in W^{1,2}_0(Z)$ such that $\text{supp}(\phi) \subset Z_h$

$$\left|\int_Z A\nabla\hat{u}_h \cdot \nabla\phi\right| \leq c\,(\|\hat{q}\|_{W^{1/2,2}(S_0\setminus K_0)} + \|\hat{f}\|_{L^2(Z)})\, \|\phi_h\|_{W^{1,2}(Z)}\,.$$

We choose ϕ of the form $\eta^2\, \hat{u}_h$, $\eta \in C^\infty_c(Z_h)$ arbitrary, and after a few standard estimates, we obtain that

$$\int_Z |\nabla\hat{u}_h|^2\, \eta^2 \leq c_\eta\, (\|\nabla\hat{u}\|_{L^2(Z)} + \|\hat{q}\|_{W^{1/2,2}(S_0\setminus K_0)} + \|\hat{f}\|_{L^2(Z)})^2\,.$$

It follows that $\hat{u}_{z_1} \in W^{1,2}_{\text{loc}}(Z)$. Translated in the original coordinates, $u_\tau \in W^{1,2}_{\text{loc}}(\Omega)$. Thus, we can use in (15) a test function of the form $\phi = \tau \cdot \nabla\eta$ with $\eta \in C^2_c(\Omega)$. The claim easily follows (cp. (16)). ☐

3 Dimension reduction

Consider the local coordinate transformation Φ in the previous section. Then, the transformed $\hat{u} := u \circ \Phi$ for a local weak solution u to (1), (2) satisfies

$$\int_Z A\nabla\hat{u} \cdot \nabla\phi + \int_{S_0} \hat{q}\, \phi = \int_Z \hat{f}\, \phi \quad \forall\, \phi \in W^{1,2}_0(Z)\,, \tag{23}$$

with A, $\hat{q}$ and $\hat{f}$ according to (22). Observe that the unit tangent vector of the line K_0 is the standard unit vector $\mathbf{e}^1$, and that $\tau(\Phi(z)) = d\Phi(z)\mathbf{e}^1/|d\Phi(z)\mathbf{e}^1|$.

Thus, we easily verify that $\nu^0_1(z) = 0$ for all $z \in S_0$. For $t \in]-1,\, 1[$, the intersection of the surface S_0 with the plane $\{z_1 = t\}$ is a one-dimensional submanifold of class $\mathcal{C}^2$. We set

$$S_0(t) := \{\bar{z} \in B_1 \,:\, (t,\, \bar{z}) \in S_0\}\,.$$

The curve $S_0(t)$ consists of the three smooth parts $S_{0,i}(t)$ analogously defined and intersecting at the point $\{t, 0, 0\}$. Owing to $\nu^0_1 = 0$, the separation of variables formula

$$\int_S \phi\, dS = \int_{-1}^{1} \int_{S_0(t)} \phi(t)\, ds\, dt$$

is valid. We further observe that

$$\partial_{z_1}\hat{u}(z) = (d\Phi(z))^T\nabla u(\Phi(z))\cdot\frac{(d\Phi(z))^{-1}\,\tau(\Phi(z))}{|(d\Phi(z))^{-1}\,\tau(\Phi(z))|} = \frac{u_\tau(\Phi(z))}{|(d\Phi(z))^{-1}\,\tau(\Phi(z))|}\,.$$

Therefore, the Lemma 2.5 yields $\hat{u}_{z_1}\in W^{1,2}(Z)$ together with the inequality

$$\|\hat{u}_{z_1}\|_{W^{1,2}(Z)} \leq c_\Phi\,\|u_\tau\|_{W^{1,2}(Z_{t,\rho}(x_0))} \leq c_\Phi\,(\|f\|_{L^2(\Omega)} + \|q\|_{W^{1/2,2}(S\setminus K)})\,. \tag{24}$$

The theory of Bochner-measurable vector-valued functions implies the existence of linear isometries

$$\begin{aligned} L^2(Z) &\cong L^2(-1,1;\;L^2(B_1))\\ W^{1,2}(Z) &\cong L^2(-1,1;\;W^{1,2}(B_1))\cap W^1_2(-1,1;\;L^2(B_1))\,. \end{aligned} \tag{25}$$

We identify $\nabla\hat{u}_{z_1}$ with a function of the class $L^2(-1,1;\;L^2(B_1))$, and

$$\|\nabla\hat{u}_{z_1}\|_{L^2(-1,1;\,L^2(B_1))} \leq c\,(\|f\|_{L^2(\Omega)} + \|q\|_{W^{1/2,2}(S\setminus K)})\,.$$

This regularity allows to further modify the integral relation (21). We denote

$$\begin{aligned} \overline{\nabla} := (0,\,\partial_{z_2},\,\partial_{z_3}),\quad g &:= \hat{f} + (A\nabla\hat{u}\cdot\mathbf{e}^1)_{z_1} + \overline{\operatorname{div}}(A\mathbf{e}^1\,\hat{u}_{z_1})\\ h &:= \hat{q} + [A]_S\mathbf{e}^1\cdot\nu\,u_{z_1}\,. \end{aligned} \tag{26}$$

and a few integration by parts in (23) show that

$$\int_Z A\overline{\nabla}\hat{u}\cdot\overline{\nabla}\phi + \int_{S_0} h\,\phi = \int_Z g\,\phi \quad \forall\;\phi\in W^{1,2}_0(Z)\,.$$

The function g belongs to $L^2(-1,1;\;L^2(B_1))$ and it satisfies the inequality

$$\begin{aligned} \|g\|_{L^2(-1,1;\,L^2(B_1))} &\leq c\,(\|\hat{f}\|_{L^2(-1,1;\,L^2(B_1))} + \|\hat{u}_{z_1}\|_{W^{1,2}(Z)})\\ &\leq c\,(\|f\|_{L^2(\Omega)} + \|q\|_{W^{1/2,2}(S\setminus K)})\,. \end{aligned} \tag{27}$$

The function h satisfies for almost all $z_1\in(-1,\;1)$ the inequality

$$\begin{aligned} &\|h(z_1)\|_{W^{1/2,2}(S_0(z_1)\setminus\{z_1,0,0\})}\\ &\qquad\leq c\,(\|\hat{q}(z_1)\|_{W^{1/2,2}(S_0(z_1)\setminus\{z_1,0,0\})} + \|\hat{u}_{z_1}(z_1)\|_{W^{1,2}(B_1)})\,. \end{aligned}$$

Thus, due also to (24)

$$\int_{-1}^1 \|h(z_1)\|^2_{W^{1/2,2}(S_0(z_1)\setminus\{z_1,0,0\})}\,dz_1 \leq c\,(\|f\|_{L^2(\Omega)} + \|q\|_{W^{1/2,2}(S\setminus K)})^2\,. \tag{28}$$

Now we choose ϕ of the form $\phi(z) = \psi(z_1)\,\eta(\bar{z})$, $\bar{z} := (z_2,\;z_3)$, where $\eta(\bar{z}) = 0$ for $|\bar{z}| = 1$, and $\psi(z_1) = 0$ for $|z_1| = 1$. For almost all $z_1\in]-1,\;1[$ we see that the function $\hat{u}(z_1)\in W^{1,2}(B_1)$ satisfies

$$\int_{B_1} A(z_1)\overline{\nabla}\hat{u}(z_1)\cdot\overline{\nabla}\eta + \int_{S_0(z_1)} h(z_1)\,\eta = \int_{B_1} g(z_1)\,\eta\,,\quad \eta\in W^{1,2}_0(B_1)\,. \tag{29}$$

For $1 < r < 2$ arbitrary, the Sobolev theorem for two dimensional domains implies the following inequalities:

$$\begin{aligned} \left| \int_{S_0(z_1)} h(z_1)\, \eta \right| &\leq \|h(z_1)\|_{L^{r/2(r-1)}(S_0(z_1))}\, \|\eta\|_{L^{r/(2-r)}(S_0(z_1))} \\ &\leq c_r\, \|h(z_1)\|_{L^{r/2(r-1)}(S_0(z_1))}\, \|\eta\|_{W^{1,r}(B_1)} \\ \left| \int_{B_1} g(z_1)\, \eta \right| &\leq c\, \|g(z_1)\|_{L^{2r/(3r-2)}(B_1)}\, \|\eta\|_{L^{2r/(2-r)}(B_1)} \\ &\leq c_r\, \|g(z_1)\|_{L^{2r/(3r-2)}(B_1)}\, \|\eta\|_{W^{1,r}(B_1)}\, . \end{aligned}$$

As $g(z_1) \in L^2(B_1)$ and $h(z_1) \in L^t(S)$, $1 \leq t < +\infty$ arbitrary, we thus obtain that the right-hand of (29) generates a linear continuous functional on $W^{1,r}(B_1)$ for all $r > 1$. Owing to an inequality due originally to deGiorgi and Nash (see [Tro87], Section 2.4 or [GT01], Section 8.9 for more recent detailed proofs), we obtain first for a certain $0 < \alpha(A) < 1$ the interior Hölder regularity

$$[\hat{u}(z_1)]_{C^\alpha_{\mathsf{loc}}(B_1)} \leq c\, \big(\|g(z_1)\|_{L^{\frac{2}{2-\alpha}}(B_1)} + \|h(z_1)\|_{L^{\frac{1}{1-\alpha}}(S_0(z_1))}\big)\, . \tag{30}$$

Using (27) and (28), it follows that the function $\hat{u}$ satisfies the inequality

$$\|\hat{u}\|_{L^2(-1,1;\, C^\alpha_{\mathsf{loc}}(B_1))} \leq c\, \big(\|f\|_{L^2(\Omega)} + \|q\|_{W^{1/2,2}(S\setminus K)}\big)\, . \tag{31}$$

These preliminary considerations allow to state the following result.

Lemma 3.1. *For all* $0 < \alpha < \alpha_{\mathit{opt}}$ *(cf.* (4)*) the function* $\hat{u}$ *satisfying* (29) *belongs to the space* $L^2(-1,1;\, C^\alpha_{\mathit{loc}}(B_1))$ *and it satisfies the continuity estimate*

$$\|\hat{u}\|_{L^2(-1,1;\, C^\alpha_{\mathit{loc}}(B_1))} \leq c_\alpha\, \big(\|f\|_{L^2(\Omega)} + \|q\|_{W^{1/2,2}(S\setminus K)}\big)\, .$$

Proof. We first consider the general case of a matrix valued diffusion coefficient. The best (actually the worst possible) Hölder exponent for the problem (29) with $\hat{f} = 0 = \hat{q}$ is estimated by the formula (cf. [Ric08], Th. 1 and [PS72])

$$\alpha_{\mathsf{opt}} \geq \left(\sup_{B_\rho(\bar{z}) \subset B_1} \frac{\frac{1}{2\pi\rho} \int_{B_\rho(\bar{z})} \frac{A\nu\cdot\nu}{\sqrt{\det(A)}}}{\frac{4}{\pi}\, \arctan \left(\frac{\inf_{B_\rho(\bar{z})} \sqrt{\det(A)}}{\sup_{B_\rho(\bar{z})} \sqrt{\det(A)}} \right)^{1/2}} \right)^{-1} . \tag{32}$$

Here $B_\rho(\bar{z})$ denotes arbitrary a disk contained in the interior of the domain B_1. For $\varepsilon > 0$ arbitrary, the property (18), implies that there are $t,\, \rho > 0$ such that $\|A - \kappa\|_{C(\overline{Z})} < \varepsilon$. Thus, we can obtain the estimate (31) with $\alpha = \alpha_{\mathsf{opt}} - o(\varepsilon)$ (cf. (4)).

As second, we consider the case where the matrix κ is isotropic. In this case, the property (18) implies that $\|A - \kappa\, \mathsf{Id}\|_{C(\overline{Z})} < \varepsilon$, and therefore, the formula (32) directly implies that (31) is valid for all $\alpha < 4/\pi\, \arctan \sqrt{\inf \kappa / \sup \kappa}$.

To proceed, we consider the method of proof for the Hölder continuity of $\hat{u}$ in the paper [PS72]. It consists in showing the existence of constants $c > 0$ (the Hölder constant) and $0 < \alpha < 1$ (the Hölder exponent) such that

$$\int_{B_\rho(\bar{z})} |\overline{\nabla}\hat{u}|^2 \leq c\,\rho^{2\alpha}, \quad \text{for all } B_\rho(\bar{z}) \subset B_1 \text{ arbitrary}\,. \tag{33}$$

In fact, in the precise geometrical situation here under investigation, it is not necessary to consider every $B_\rho(\bar{z}) \subset B_1$ in (33) for the following reason: If a disk $B_\rho(\bar{z})$ intersects several different pieces among the curves $S_{0,1}(z_1), \ldots, S_{0,m}(z_1)$, then the distance $|\bar{z} - z_K|$ is proportional to ρ by a fixed factor γ depending only on the openings of these curves at z_K and on their curvature. Thus, if (33) is valid for balls centred at z_K, then also

$$\int_{B_\rho(\bar{z})} |\overline{\nabla}\hat{u}|^2 \leq \int_{B_{2\gamma\rho}(z_K)} |\overline{\nabla}\hat{u}|^2 \leq c\,(2\gamma\rho)^{1+2\alpha} \leq \tilde{c}\,\rho^{1+2\alpha}\,.$$

It is therefore possible to prove (33) considering only discs $B_\rho(\bar{z})$ that intersect at most one of the curves $S_{0,i}(z_1)$ for $i = 1, \ldots, m$, or discs centred at the point z_K that intersect every curve among $S_{0,1}(z_1), \ldots, S_{0,m}(z_1)$.

We now prove (33) for such balls with $\alpha = \alpha_{\text{opt}} - o(\varepsilon)$ where ε is arbitrarily small. Since $\hat{u}$ satisfies (29) with $\hat{f} = 0 = \hat{q}$, the Gauss integral theorem yields

$$\int_{S_\rho} A\overline{\nabla}\hat{u} \cdot \overline{\nabla}\hat{u} = \int_{\partial S_\rho} A\overline{\nabla}\hat{u} \cdot \nu\,(\hat{u} - c), \quad c \in \mathbb{R} \text{ arbitrary}\,.$$

We choose $c := (\int_{\partial S_\rho} \kappa)^{-1} \int_{\partial S_\rho} \kappa\,\hat{u}$, which ensures that $\int_{\partial S_\rho} \kappa\,(\hat{u}-c) = 0$ and that $\hat{u}(y) = c$ at some point $y \in \partial S_\rho$. Thus, denoting $B := (\kappa^{-1} A - \text{Id})$, it follows that

$$\int_{S_\rho} \kappa\,|\overline{\nabla}\hat{u}|^2 + \int_{S_\rho} \kappa\,B\overline{\nabla}\hat{u} \cdot \overline{\nabla}\hat{u} = \int_{\partial S_\rho} \kappa\,\nabla\hat{u} \cdot \nu\,(\hat{u} - c) + \int_{\partial S_\rho} \kappa\,B\overline{\nabla}\hat{u} \cdot \nu\,(\hat{u} - c)\,.$$

Since (18) ensures that $\|B\|_\infty < \varepsilon$

$$\begin{aligned}\left|\int_{\partial S_\rho} B\overline{\nabla}\hat{u} \cdot \nu\,(\hat{u} - c)\right| &\leq \varepsilon\,\sup\kappa\,\int_{\partial S_\rho} |\overline{\nabla}\hat{u} \cdot \nu|\,\max_{y\in\partial S_\rho} |\hat{u}(y) - c| \\ &\leq \varepsilon\,\frac{\sup\kappa}{\inf\kappa}\,\int_{\partial S_\rho} |\overline{\nabla}\hat{u} \cdot \nu|\,\int_{\partial S_\rho} |\,\delta_s\,\hat{u}| \\ &\leq c\,\varepsilon\,\rho\,\int_{\partial S_\rho} \kappa\,|\overline{\nabla}\hat{u}|^2\,.\end{aligned}$$

Therefore, we easily obtain that

$$(1 - c\,\varepsilon)\,\int_{S_\rho} \kappa\,|\overline{\nabla}\hat{u}|^2 \leq \left|\int_{\partial S_\rho} \kappa\,\nabla\hat{u} \cdot \nu\,(\hat{u} - c)\right| + c\,\varepsilon\,\rho\,\int_{\partial S_\rho} \kappa\,|\overline{\nabla}\hat{u}|^2\,. \tag{34}$$

We now use the inequality (10) in Lemma 1 of [PS72], and obtain that

$$\left|\int_{\partial S_\rho(\bar z)} \kappa\, \overline{\nabla}\hat u\cdot\nu\,(\hat u - c)\right| \le \left(\int_{\partial S_\rho(\bar z)} \kappa\,|\overline{\nabla}\hat u|^2\right)^{1/2} \left(\int_{\partial S_\rho(\bar z)} \kappa\,|\hat u - c|^2\right)^{1/2}$$
$$\le \frac{\rho}{\sqrt{\lambda}} \int_{\partial S_\rho(\bar z)} \kappa\,|\overline{\nabla}\hat u|^2\,,$$

where $\lambda > 0$ is the smallest eigenvalue for the $1-d$ transmission problem

$$\begin{cases} (a(t)\,w')' + \lambda\, a(t)\, w = 0, \quad t \in]0,\, 2\pi[\,, \\ w \text{ periodic of period } 2\pi\,. \end{cases} \tag{35}$$

where the function $a(t)$ is defined as $\kappa(\bar z + \rho\,(\cos t,\, \sin t))$. In the case that the the disk S_ρ is centred at z_K, the function a takes $m \in \mathbb{N}$ values, where m is the number of smooth pieces $S_{0,1}, \ldots, S_{0,m}$. Instead of using the estimate of the paper [PS72] for the number λ, we use the estimates obtained for the same problem in [Mer03], Th. 16. For the cases $m = 2$ and $m = 3$ mentionned in the condition (6), it follows that $\lambda^{1/2} \ge \alpha_{\text{opt}}$. If instead the disk S_ρ crosses at most one of the $S_{0,1}(z_1), \ldots, S_{0,m}(z_1)$, then, again according to the Theorem 16 of [Mer03], we obtain that $\lambda^{1/2} > 1/2$. From (34), we can conclude as in the proof of [PS72], Th. 1 that

$$(1 - c\,\varepsilon) \int_{S_\rho(\bar z)} \kappa\,|\overline{\nabla}\hat u|^2 \le \frac{\rho}{2}\left(\frac{1}{\sqrt{\lambda}} + c\,\varepsilon\right) \int_{\partial S_\rho(\bar z)} \kappa\,|\overline{\nabla}\hat u|^2\,.$$

This implies for the function $g(\rho) := \int_{S_\rho} \kappa\,|\overline{\nabla}\hat u|^2$ a differential inequality

$$g(\rho) \le \frac{1}{2(1 - c\,\varepsilon)}\left(\frac{1}{\sqrt{\lambda}} + c\,\varepsilon\right)\rho\, g'(\rho)\,,$$

from which we can conclude in well-known manner to (33) with

$$\alpha := \frac{1}{2(1 - c\,\varepsilon)}\left(\frac{1}{\sqrt{\lambda}} + c\,\varepsilon\right), \quad \varepsilon > 0 \text{ arbitrary.}$$

The claim follows. □

On the footing of Lemma 3.1, we can introduce for $z_1 \in]-1, 1[$ the auxiliary function $w := u(z_1) - u(z_1, 0, 0) \in W^{1,2}(B_1) \cap C^{\alpha}(B_1)$. Note that $w = 0$ at $z_K = \{z_1, 0, 0\}$. The following statement on weighted regularity is now easily established.

Proposition 3.2. *For all $0 \le \alpha < \alpha_{opt}$, the functions $w\, d_K^{-1-\alpha}$ and $|\nabla w|\, d_K^{-\alpha}$ belong to $L^2(B_1)$, $d_K(z) := |z - z_K|$), with the inequality*

$$\int_{B_1} \{\frac{w^2}{d_K^{2+2\alpha}} + \frac{|\nabla w|^2}{d_K^{2\alpha}}\} \le c\, \left(\|g(z_1)\|_{L^2(B_1)} + \|h(z_1)\|_{W^{1/2,2}(S_0(z_1)\setminus\{z_K\})}\right)^2 .$$

Proof. We choose $\alpha < \alpha_0 < \alpha_{\text{opt}}$. Note that

$$\int_{B_1} \frac{w^2}{d_K^{2+2\alpha}} \le c\,[w(z_1)]^2_{C^{\alpha_0}(\overline{B_{1/2}})} \int_{B_{1/2}} \frac{1}{|\bar z|^{2+2(\alpha-\alpha_0)}} d\bar z + c\,\|w(z_1)\|^2_{L^2(B_{1/2,2})}$$
$$\le c\,([w(z_1)]^2_{C^{\alpha_0}(\overline{B_{1/2}})} + \|w(z_1)\|_{L^2(B_1)})^2\,.$$

At second, the function w satisfies for all $\phi \in W^{1,2}_0(\Omega)$ (cf. (29))

$$\int_{B_1} A\nabla w\cdot\nabla\phi = -\int_{S_0(z_1)} h(z_1)\,\phi + \int_{B_1} g(z_1)\,\phi\,.$$

We fix a $\zeta \in C^\infty_c(B_1)$ such that $\zeta \equiv 1$ on $B_{1/2}$, and $\zeta = 0$ outside of $B_{2/3}$. Using elementary truncature techniques arguments, we can insert $\phi = w\,d_K^{-2\alpha}\,\zeta^2$ in the latter relation. We obtain that

$$\int_{B_1} \zeta^2\,d_K^{-2\alpha}\,A\nabla w\cdot\nabla w - 2\,\alpha \int_{B_1} A\nabla w\cdot\nabla d_K\,\zeta^2\,\frac{w}{d_K^{2\alpha+1}}$$
$$+\,2\int_{B_1} \zeta\,d_K^{-2\alpha}\,w\,A\nabla w\cdot\nabla\zeta = -\int_{S_0(z_1)} h(z_1)\,w\,d_K^{-2\alpha}\,\zeta^2 + \int_{B_1} g(z_1)\,w\,d_K^{-2\alpha}\,\zeta^2\,.$$

We note the inequality

$$\left|2\,\alpha \int_{B_1} A\nabla w\cdot\nabla d_K\,\zeta^2\,\frac{w}{d_K^{2\alpha+1}}\right| \le \frac{1}{2}\int_{B_1} \zeta^2\,d_K^{-2\alpha}\,A\nabla w\cdot\nabla w$$
$$+\,2\,\alpha^2 \int_{B_1} A\nabla d_K\cdot\nabla d_K\,\frac{w^2}{d_K^{2+2\alpha}}\,\zeta^2$$
$$\le \frac{1}{2}\int_{B_1} \zeta^2\,d_K^{-2\alpha}\,A\nabla w\cdot\nabla w + 2\,\alpha^2 \int_{B_1} w^2\,d_K^{-[2+2\alpha]}\,. \tag{36}$$

Moreover

$$2\left|\int_{B_1} \zeta\,d_K^{-2\alpha}\,w\,A\nabla w\cdot\nabla\zeta\right| \le \frac{1}{4}\int_{B_1} \zeta^2\,d_K^{-2\alpha}\,A\nabla w\cdot\nabla w + c_\zeta \int_{B_1} w^2\,d_K^{-2\alpha}\,.$$

It follows that

$$\frac{1}{4}\int_{B_1} \zeta^2\,d_K^{-2\alpha}\,A\nabla w\cdot\nabla w \le \int_{S_0(z_1)} |h(z_1)|\,|w|\,d_K^{-2\alpha}$$
$$+\int_{B_1} |g(z_1)|\,|w|\,d_K^{-2\alpha} + c\,\|w\,d_K^{-1-\alpha}\|^2_{L^2(B_1)}\,.$$

To further estimate the right-hand side, we note that

$$\left|\int_{S_0(z_1)} |h(z_1)|\,|w|\,d_K^{-2\alpha}\,\zeta^2\right| \le c\,[w]_{C^{\alpha_0}(B_{2/3})}\left|\int_{S_0(z_1)} |h(z_1)|\,d_K^{-\alpha}\right|$$
$$\le c\;[w]_{C^{\alpha_0}(B_{1-\delta/2})}\,\|h(z_1)\|_{W^{1/2,2}(S_0(z_1)\setminus\{z_K\})}$$
$$\left|\int_{B_1} |g(z_1)|\,|w|\,d_K^{-2\alpha}\,\zeta^2\right| \le c\;[w]_{C^{\alpha_0}(B_{2/3})}\,\|g(z_1)\|_{L^2(B_1)}\,.$$

The claim follows. □

We now turn our attention to the proof of the main Theorem 1.1. For $n \in \mathbb{N}$ we can choose a function $\psi_n \in C^2_c([0,\, \infty])$ satisfying

$$\psi_n(t) \begin{cases} = 0 & \text{for } 0 \le t \le 1/n \\ \in [0,\, 1] & \text{for } 1/n \le t \le 2/n\,, \\ = 1 & \text{for } 2/n \le t \end{cases} \quad \begin{aligned} |\psi_n'(t)| &\le c\, n\, \chi_{[1/n,\, 2/n]}(t), \\ |\psi_n''(t)| &\le c\, n^2\, \chi_{[1/n,\, 2/n]}(t)\,. \end{aligned} \tag{37}$$

For $z \in B_1$, we denote $d_K(z) := \mathrm{dist}(z,\, z_K)$.

Lemma 3.3. *Let $0 < \alpha < \alpha_{opt}$. Then, $w\, d_K^{1-\alpha} \in W^{2,2}(B_1 \setminus S_0(z_1))$, and*

$$\|d_K^{1-\alpha}\, w\|_{W^{2,2}(B_1\setminus S_0)} \le c\,(\|g(z_1)\|_{L^2(B_1)} + \|h(z_1)\|_{W^{1/2,2}(S_0(z_1)\setminus\{z_K\})})\,.$$

Proof. Since the function w is a modification of $\hat{u}(z_1)$ by a constant, it satisfies (29) as well. Classical results for transmission problems near smooth surfaces yield $w \in W^{2,2}_{\text{loc}}(B_{1,i} \setminus \{z_K\})$ for $i = 1, \ldots, m$. Here, the domains $B_{1,i}$ are generated by a partition of B_1 due to $S_0(z_1)$.

We choose functions ψ_n, $n \in \mathbb{N}$ according to (37), and implicitely assume for ease of writing throughout the proof that these functions and their derivatives are evaluated at the point $d_K(x)$. Define $u_n := d_K^{1-\alpha}\, \psi_n\, w$. Clearly, $u_n \in W^{2,2}(B_1 \setminus S_0)$. Moreover, we can easily compute that

$$\|u_n\, d_K^{-1}\|_{W^{1,2}(B_1)} \le c\,(\|w\, d_K^{-1-\alpha}\|_{L^2(B_1)} + \|\nabla w\, d_K^{-\alpha}\|_{L^2(B_1)})\,.$$

For simplicity, we denote throughout the proof div and ∇ instead of $\overline{\mathrm{div}}$ and $\overline{\nabla}$ the differential operators with respect to the $z-$ variable in the *two-dimensional* domain B_1. We compute the operator

$$\begin{aligned} -\,\mathrm{div}(\kappa\, \nabla u_n) = f_n + \tilde{f}_n, \quad f_n &= -\,\mathrm{div}(\kappa\, \nabla(w\, d_K^{1-\alpha}))\, \psi_n \\ \tilde{f}_n &= -2\, \nabla(w\, d_K^{1-\alpha}) \cdot \nabla\psi_n - \mathrm{div}(\kappa \nabla\psi_n)\, w\, d_K^{1-\alpha}\,. \end{aligned}$$

The pointwise majoration $\tilde{f}_n \le c\,(|w|\, d_K^{-1-\alpha} + |\nabla w|\, d_K^{-\alpha})$ is easily shown to be valid. Thus, $\tilde{f}_n \to 0$ in $L^2(B_1)$. On the other hand

$$-\,\mathrm{div}(\kappa\, \nabla(w\, d_K^{1-\alpha})) = \hat{f}(z_1)\, d_K^{1-\alpha} - 2\,(1-\alpha)\, \nabla w \cdot \nabla d_K\, d_K^{-\alpha} - \mathrm{div}(\kappa \nabla d_K^{1-\alpha})\, w\,,$$

and therefore $|f_n| \le c\,(|g(z_1)| + |w|\, d_K^{-1-\alpha} + |\nabla w|\, d_K^{-\alpha})$.

We also compute the operator

$$\begin{aligned} [-\kappa \nabla u_n \cdot \nu]_{S_0(z_1)} = q_n + \tilde{q}_n, \quad q_n &:= [-\kappa \nabla(w\, d_K^{1-\alpha}) \cdot \nu]_{S_0(z_1)}\, \psi_n \\ \tilde{q}_n &= [-\kappa \nabla\psi_n \cdot \nu]_{S_0(z_1)}\, w\, d_K^{1-\alpha}\,. \end{aligned}$$

The functions $\tilde{q}^i_n := R_{S_{0,i}(z_1)} \tilde{q}_n$ possess a natural extension into B_1 (cf. the Remark 2.3). We note that

$$\partial_{z_j} \tilde{q}^i_n = [-\kappa \nu^i] \nabla\psi_{n,z_j} w\, d_K^{1-\alpha} + [-\kappa \nu^i_{z_j}] \nabla\psi_n w\, d_K^{1-\alpha} + [-\kappa \nu^i] \nabla\psi_n (w\, d_K^{1-\alpha})_{z_j}\,.$$

Thus, owing to the properties (37), the majoration

$$|\nabla \tilde{q}_n^i| \leq c\,(|w|\,d_K^{-1-\alpha} + |\nabla w|\,d_K^{-\alpha}) \in L^2(B_1)\,,$$

is valid, and we easily show that $\tilde{q}_n \to 0$ in $W^{1/2,2}(S_0(z_1) \setminus \{z_K\})$. Moreover, choosing a $\alpha < \alpha_0 < \alpha_{\text{opt}}$, we can show that there is a sequence $c_n \to 0$ such that

$$\begin{aligned}\int_{S_0(z_1)} \frac{|\tilde{q}_n|^2}{d_K} &\leq c \int_{S_0(z_1)} |\psi_n'|^2\,|w|^2\,d_K^{1-2\alpha}\\ &\leq c\,[w]_{C^{\alpha_0}_{\text{loc}}(B_1)} \int_{S_0(z_1)\cap B_{2/n}(z_K)} \frac{1}{d_K^{1+2(\alpha-\alpha_0)}} \leq c_n\,[w]_{C^{\alpha_0}_{\text{loc}}(B_1)}\,.\end{aligned}$$

Thus, owing to a well-known characterisation of the $W^{1/2,2}_{00}-$norm ([LM68], Th. 11.7), we see that $\tilde{q}_n \to 0$ even in $W^{1/2,2}_{00}(S_0(z_1) \setminus \{z_K\})$. On the other hand

$$[-\kappa\nabla(w\,d_K^{1-\alpha})\cdot\nu]_{S_0(z_1)}\,\psi_n = h(z_1)\,d_K^{1-\alpha}\,\psi_n + (1-\alpha)\,w\,d_K^{-\alpha}\,[-\kappa\nabla d_K\cdot\nu]_{S_0(z_1)}\,\psi_n\,.$$

We observe that $h(z_1) \in W^{1/2,2}(S_0(z_1) \setminus \{z_K\})$. Thus, we can regard without loss of generality the funcitions $h_i := R_{S_{0,i}(z_1)} h(z_1)$ as elements of $W^{1,2}(B_1)$. Note that $h_i\,d_K^{-\alpha}$ belongs to $L^2(B_1)$ for all $\alpha < 1$. Therefore, it is readily shown that

$$\|q_n\|_{W^{1/2,2}(S_0(z_1)\setminus\{z_K\})} \leq c\,(\|h(z_1)\|_{W^{1/2,2}(S_0(z_1)\setminus\{z_K\})} + \|w\,d_K^{-\alpha}\|_{W^{1,2}(B_1)})\,.$$

On the other hand

$$\begin{aligned}\int_{S_0(z_1)} \frac{|q_n|^2}{d_K} &\leq c \int_{S_0(z_1)} \{h^2(z_1)\,d_K^{1-2\alpha} + |w|^2\,d_K^{-1-2\alpha}\}\\ &\leq c\,(\|h(z_1)\|_{W^{1/2,2}(S_0(z_1)\setminus\{z_K\})} + [w]_{C^{\alpha_0}_{\text{loc}}(B_1)})\,,\end{aligned}$$

showing even that q_n is uniformly bounded in $W^{1/2,2}_{00}(S_0(z_1) \setminus \{z_K\})$. We now apply the Theorem 2.1 with $G = B_1$, $\Gamma := S_0(z_1)$ and $x_K := \{z_1, 0, 0\}$. We obtain that According to the Theorem 2.1, we obtain that

$$\begin{aligned}\|u_n\|_{W^{2,2}(B_1)} &\leq c\,(\|g(z_1)\|_{L^2(B_1)} + \|h(z_1)\|_{W^{1/2,2}(S_0(z_1)\setminus\{z_K\})} + [w]_{C^{\alpha_0}_{\text{loc}}(B_1)}) + c_n\\ &\leq c\,(\|g(z_1)\|_{L^2(B_1)} + \|h(z_1)\|_{W^{1/2,2}(S_0(z_1)\setminus\{z_K\})})\,,\end{aligned}$$

with $c_n \to 0$. The claim follows. □

Corollary 3.4. *Assumptions of Lemma 3.3. The second derivatives of u belong to $L^{t(\alpha)}(\Omega)$, the first derivatives belong to $L^{s(\alpha)}(\Omega)$ for all $0 < \alpha < \alpha_{\text{opt}}$, where the functions $t,\ s$ are given by Theorem 1.1. Moreover,*

$$\|u\|_{W^{2,t}(\Omega\setminus S)} + \|u\|_{W^{1,s}(\Omega)} \leq c\,(\|f\|_{L^2(\Omega)} + \|q\|_{W^{1/2,2}(S\setminus K)})\,.$$

Proof. We choose $\alpha < \alpha_0 < \alpha_{\text{opt}}$. Then, in the reference coordinates

$$\int_{B_1} |D^2_{\bar z}(\hat u(z_1))|^{\frac{2}{2-\alpha}} \le \int_{B_1} (|D^2_{\bar z}(\hat u(z_1))|\, d_K^{1-\alpha_0})^{\frac{2}{2-\alpha}}\, d_K^{-\frac{2-2\alpha_0}{2-\alpha}}$$
$$\le \|d_K^{1-\alpha}\, D^2_{\bar z}(\hat u(z_1))\|^{\frac{2}{2-\alpha}}_{L^2(B_1)} \left(\int_\Omega d_K^{-\frac{2-2\alpha_0}{1-\alpha}}\right)^{\frac{1-\alpha}{2-\alpha}} \le c\,\|d_K^{1-\alpha}\, D^2_{\bar z}(\hat u(z_1))\|^{\frac{2}{2-\alpha}}_{L^2(B_1)}\,.$$

Thus, $D^2_{\bar z}(\hat u) \in L^{2/(2-\alpha)}(Z \setminus S_0)$, and

$$\|D^2_{\bar z}(\hat u)\|^2_{L^2(-1,1;\, L^{\frac{2}{2-\alpha}}(B_1))}$$
$$\le c\left(\|g\|^2_{L^2(-1,1;\, L^2(B_1))} + \int_{-1}^1 \|h(z_1)\|^2_{W^{1/2,2}(S_0(z_1)\setminus\{z_1,0,0\})}\, dz_1\right)$$
$$\le c\,(\|f\|_{L^2(\Omega)} + \|q\|_{W^{1/2,2}(S\setminus K)})^2\,.$$

Since $\nabla \hat u_{z_1} \in L^2(Z)$ (Proposition 2.5), the claim follows for the second derivatives.

In order to prove the higher-integrability of ∇u, we again use to the reference coorindates in the cylinder Z. Owing to Sobolev embedding theorem in two-dimesions

$$\|\nabla_{\bar z}\hat u(z_1)\|_{L^{\frac{2}{1-\alpha}}(B_1)} \le c\,\|D^2_{\bar z}(\hat u(z_1))\|_{L^{\frac{2}{2-\alpha}}(B_1\setminus S_0(z_1))}$$
$$\le c\,(\|g(z_1)\|_{L^2(B_1)} + \|h(z_1)\|_{W^{1/2,2}(S_0(z_1)\setminus\{z_K\})})\,.$$

Thus, we obtain that

$$\|\nabla_{\bar z}\hat u\|_{L^2(-1,1;\, L^{\frac{2}{1-\alpha}}(B_1))} \le c\,(\|f\|_{L^2(\Omega)} + \|q\|_{W^{1/2,2}(S\setminus K)})\,.$$

On the other hand, $D^2_{z_1,\bar z}\hat u \in L^2(Z)$, and therefore $\nabla_{\bar z}\hat u$ belongs to $W^1_2(-1,1;\, L^2(B_1))$. Since $\nabla \hat u_{z_1} \in L^2(-1,\,1;\, L^2(B_1))$, we obtain that $\nabla \hat u$ belongs to $W^1_2(-1,1;\, L^2(B_1))$, which altogether implies that

$$\nabla \hat u \in L^2(-1,\,1;\, L^{\frac{2}{1-\alpha}}(B_1)) \cap L^\infty(-1,1;\, L^2(B_1))\,. \tag{38}$$

Using well-known interpolation formulas, we obtain that

$$\nabla \hat u \in L^s(Z), \quad s = 2\,(1+\alpha)\,.$$

□

4 The Hoelder exponent in three dimensions

In this last section, we want to determine the relationship between the Hölder exponent for local weak solutions to

$$-\operatorname{div}(\kappa\,\nabla u) = 0 \text{ in } \Omega, \tag{39}$$
$$[u]_S = 0\,, \quad [-\kappa\,\nabla u\cdot \nu]_S = 0 \text{ on } S\,, \tag{40}$$

and the number α_{opt}. We shall here distinguish between the cases $K \in \mathcal{C}^2$ and $K \in \mathcal{C}^k$, $k \geq 3$. We define a number $p > 2$ as the largest $2 < p = p(\kappa, \Omega) \leq 3$ such that:

$$-\operatorname{div}(\kappa \nabla \cdot) : W_0^{1,p}(\Omega) \to [W_0^{1,p'}(\Omega)]^* \text{ is continuously invertible}. \tag{41}$$

The existence of $p > 2$ was proved in [Mey63].

Lemma 4.1. *Let $u \in W^{1,2}_{loc}(\Omega)$ satisfy* (39), (40). *Then, $u,\ u_\tau \in W^{1,p}_{loc}(\Omega)$ with $p > 2$ according to* (41). *Assume in addition that the curve K is of class $\mathcal{C}^3$. Then, the function $u_{\tau,\tau} = \tau \cdot \nabla u_\tau$ also belong to $W^{1,p}_{loc}(\Omega)$.*

Proof. By the definition of p, we directly obtain that $u \in W^{1,p}_{\text{loc}}(\Omega)$ and that $\|u\|_{W^{1,p}_{\text{loc}}(\Omega)} \leq c\,\|u\|_{W^{1,2}_{\text{loc}}(\Omega)}$. Now observe that u_τ satisfies (20) with $f = 0 = q$. Since $\nabla u \in L^p(\Omega)$, we obtain in the same way that $u_\tau \in W^{1,p}_{\text{loc}}(\Omega)$ and that $\|u_\tau\|_{W^{1,p}_{\text{loc}}(\Omega)} \leq c\,\|u\|_{W^{1,p}_{\text{loc}}(\Omega)}$.

If $K \in \mathcal{C}^3$, then $\tau \in C^2(\Omega)$. We are allowed to differentiate in (20) once more in the direction of τ, and we obtain that

$$\int_\Omega \kappa \nabla u_{\tau,\tau} \cdot \nabla\phi = \int_\Omega \{\kappa\, D(\tau) \nabla u_\tau + M(\tau)\nabla u\} \cdot \nabla\phi\,, \tag{42}$$

where the matrix $M(\tau)$ satisfies $\|M\|_{L^\infty(\Omega;\,\mathbb{R}^{3\times3})} \leq c\,\|\tau\|_{W^{2,\infty}(\Omega;\mathbb{R}^3)}$. Thus, $u_{\tau,\tau} \in W^{1,p}_{\text{loc}}(\Omega)$. □

Using the same localisation and flattening technique as above, we denote (cp. (26))

$$\overline{\nabla} := (0,\, \partial_{z_2},\, \partial_{z_3}), \quad g := \kappa\,(A\nabla\hat{u}\cdot\mathbf{e}^1)_{z_1}\,, \quad G := -\kappa\,\hat{u}_{z_1}\,A\mathbf{e}^1\,, \tag{43}$$

The function g belongs to $L^p(-1,1;\, L^p(B_1))$. The vector field G belongs to the space $L^p(-1,1;\, W^{1,p}(B_1))$. Owing to the embedding $W^{1,p}(B_1) \hookrightarrow L^\infty(B_1)$ continuously for $p > 2$, the field G therefore also belongs to $L^p(-1,1\ L^\infty(B_1))$. Moreover, owing to (24)

$$\|g\|_{L^p(-1,1;\,L^p(B_1))} + \|G\|_{L^p(-1,1\ L^\infty(B_1))} \leq c\,\|\hat{u}_{z_1}\|_{W^{1,p}(Z)}\,. \tag{44}$$

If $K \in \mathcal{C}^3$, we see in addition that g belongs to $W^1_p(-1,1;\, L^p(B_1))$. The vector field G belongs to $W^1_p(-1,1;\, W^{1,p}(B_1))$, with the estimates

$$\|g\|_{W^1_p(-1,1;\,L^p(B_1))} + \|G\|_{W^1_p(-1,1\ W^{1,p}(B_1))} \leq c\,(\|\hat{u}_{z_1}\|_{W^{1,p}(Z)} + \|\hat{u}_{z_1,z_1}\|_{W^{1,p}(Z)})\,. \tag{45}$$

For almost all $z_1 \in]-1,\,1[$ we see that the function $\hat{u}(z_1) \in W^{1,2}(B_1)$ satisfies

$$\int_{B_1} A(z_1)\overline{\nabla}\hat{u}(z_1)\cdot\overline{\nabla}\eta = \int_{B_1}\{g(z_1)\,\eta + G(z_1)\cdot\overline{\nabla}\eta\}\,, \quad \eta \in W^{1,2}_0(B_1)\,. \tag{46}$$

The Sobolev theorem for two dimensional domains implies the following inequalities:

$$\left|\int_{B_1} g(z_1)\,\eta\right| \leq c\,\|g(z_1)\|_{L^2(B_1)}\,\|\eta\|_{W^{1,1}(B_1)}$$
$$\left|\int_{B_1} G(z_1)\cdot\overline{\nabla}\eta\right| \leq \|G(z_1)\|_{L^\infty(B_1)}\,\|\eta\|_{W^{1,1}(B_1)}\,.$$

Thus, the right-hand of (29) generates a linear continuous functional on $W^{1,r}(B_1)$ for all $r \geq 1$. Thus, we obtain an estimate

$$\|\hat{u}\|_{L^s(-1,1;\,C^\alpha(B_1))} \leq c\,\|u\|_{W^{1,2}_{\text{loc}}(\Omega)}\,,$$

with $s = p$ for $K \in \mathcal{C}^2$, and even $s = +\infty$ if $K \in \mathcal{C}^3$.

Corollary 4.2. *Assume that $K \in \mathcal{C}^3$. Let $u \in W^{1,2}_{loc}(\Omega)$ satisfy* (39), (40). *Then, $u \in C^\alpha_{loc}(\Omega)$ for all $\alpha < \alpha_{opt}$.*

Proof. In the reference coordinates, the function $\hat{u}_{z_1}$ belongs to $W^1_p(-1,1;\,W^{1,p}(B_1))$. Since $p > 2$, this is embedded in the space $C^{1-1/p}(-1,1;\,C^{1-2/p}(B_1))$, and therefore $\hat{u}_{z_1}$ is bounded. Thus, for $y, z \in Z$

$$\begin{aligned}\frac{|\hat{u}(y) - \hat{u}(z)|}{|y-z|^\alpha} &\leq \frac{\|u_{z_1}\|_{L^\infty(Z)}\,|z_1 - y_1| + [\hat{u}(z_1)]_{C^\alpha(B_1)}\,|\bar{y} - \bar{z}|^\alpha}{|y-z|^\alpha}\\ &\leq c\left(\|u_{z_1}\|_{L^\infty(Z)} + \|\hat{u}\|_{L^\infty(-1,1;\,C^\alpha(B_1))}\right).\end{aligned}$$

The claim follows. □

If K is only of class $\mathcal{C}^2$, we have to work a little more. First, it is possible to obtain additional regularity for the function u_τ.

Lemma 4.3. *The function u_τ belongs to $C^{1-2/p}_{loc}(\Omega)$.*

Proof. We recall that u satisfies the relation

$$\int_\Omega \kappa \nabla u_\tau \cdot \nabla v = \int_\Omega \kappa\, D(\tau) \nabla u \cdot \nabla v, \quad v \in C^1_c(\Omega)\,. \tag{47}$$

In the reference coordinates, $\nabla_{\bar{z}} u$ belongs to $W^1_p(-1,1;\,L^p(B_1))$. Choosing $B_\rho(\bar{z}) \subset B_1$ arbitrary, Hölder's inequality yields

$$\int_{B_\rho(\bar{z})} |\nabla_{\bar{z}} u(z_1)|^2 \leq \|\nabla_{\bar{z}} u(z_1)\|^2_{L^p(B_1)}\ \text{meas}(B_\rho)^{1-2/p}\,.$$

For $t \in]-1,1[$ arbitrary, it follows that

$$\int_{t-\rho}^{t+\rho} \int_{B_\rho(\bar{z})} |\nabla_{\bar{z}} u(z_1)|^2 \leq c\,\|\nabla_{\bar{z}} u(z_1)\|^2_{L^\infty(-1,1;\,L^p(B_1))}\,\rho^{3-2/p}\,.$$

We can show that the $\nabla_{\bar{z}}(u)$ belongs to the Campanato space $L^{2,\mu}(Z)$, $\mu = 3-2/p$. In curved coordinates, observe that $u_\tau \in L^{3p/(3-p)}_{\text{loc}}(\Omega)$, which embeds into $L^{2,\mu}_{\text{loc}}(\Omega)$ for $\mu = 5 - 6/p$ $\nabla u \in L^{2,\mu}_{\text{loc}}(\Omega)$, $\mu 3 - 2/p$. Applying standard results ([Tro87], Lemma 3.3), the function u_τ satisfying (47) is in $C^\delta_{\text{loc}}(\Omega)$ with $\delta = 1 - 2/p$. □

We argue like in the preceding section. Without entering every detail, we obtain for the homogeneous problem $q = 0 = f$ a slightly improved version of the relation (38) in the form of

$$\nabla\hat{u} \in L^p(-1,\, 1;\, L^{\frac{2}{1-\alpha}}(B_1)) \cap L^\infty(-1, 1;\, L^p(B_1))\,, \tag{48}$$

for all $\alpha < \alpha_{\mathrm{opt}}$. Applying interpolation, this yields

$$\nabla\hat{u} \in L^s(Z),\; s = p\,(2 - (1-\alpha)\,\frac{p}{2})\,.$$

We now apply the embedding result of [KP11], Th. A.1 for the anisotropic Sobolev spaces. Choosing $\alpha < \alpha_{\mathrm{opt}}$ near enough, we obtain that

$$u \in C^{\beta}_{\mathrm{loc}},\; \beta = 1 - \frac{2}{s} \geq \frac{\alpha_{\mathrm{opt}}}{1 + \alpha_{\mathrm{opt}}}\,.$$

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