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Björn Sandstede

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Weierstraß–Institut
für Angewandte Analysis
und Stochastik
Mohrenstraße 39
D – 10117 Berlin
Germany

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Weierstraß–Institut für Angewandte Analysis und Stochastik (WIAS)
Mohrenstraße 39
D — 10117 Berlin
Germany

Fax: + 49 30 2044975
e–mail (X.400): c=de;a=d400-gw;p=WIAS-BERLIN;s=preprint
e–mail (Internet): preprint@wias-berlin.de

Center Manifolds for Homoclinic Solutions

Björn Sandstede[†]
Weierstraß-Institut für Angewandte Analysis und Stochastik
Mohrenstraße 39
10117 Berlin, Germany

Abstract

In this article, center-manifold theory for homoclinic solutions of ordinary differential equations or semilinear parabolic equations is developed. Here, a center manifold along a homoclinic orbit $q(t)$ is a locally invariant manifold containing all solutions which stay close to $q(t)$ in phase space for all times. Therefore, as usual, the low-dimensional center manifold contains the interesting recurrent dynamics nearby the homoclinic orbit and a considerable reduction of dimension is achieved. The manifold is of class $C^{1,\beta}$ for some $\beta > 0$.
As one application, results of Shilnikov about the occurrence of complicated dynamics nearby homoclinic solutions approaching saddle-foci equilibria are generalized to semilinear parabolic equations.

[†]Current address: Division of Applied Mathematics, Brown University, 182 George Street, Providence, RI 02912, USA

1 Introduction

In order to investigate bifurcations near equilibria or periodic solutions, center-manifold theory provides a powerful tool. The main property of a center manifold can be stated as follows. It is a locally invariant manifold which contains all solutions staying near the equilibrium or the periodic orbit for all times. Hence, the relevant dynamics is preserved if one restricts the semiflow to this manifold, and an enormous reduction of the dimension of the problem is achieved.

The purpose of the present paper is to develop an analogous geometric approach to the bifurcation theory of homoclinic orbits as suggested in [CDF90]. Here, by definition, a homoclinic orbit is a solution converging to the same equilibrium for time tending to $\pm\infty$. Again, a center manifold for a homoclinic solution is a locally invariant manifold containing all solutions which stay nearby the homoclinic orbit for all times. In general, this center manifold will only be of class $C^{1,\rho}$. But, as it turns out, this smoothness is sufficient to prove the manifold to be useful. In fact, we will show a generalization of the so-called Shilnikov chaos to semilinear parabolic equations as a simple application.

Finally, let us point out that the results presented here are new even for finite dimensional systems. Indeed, there are two related but independent results obtained by Brunovsky [Bru91] and Homburg [Hom93]. Both of them are restricted to the special case of a two-dimensional center manifold for homoclinic orbits converging to a hyperbolic equilibrium. Moreover, they are only valid in finite dimensions. Homburg [Hom93] requires the additional hypothesis that the leading eigenvalues have to be of different modulus.

The article is organized as follows. In section 2, the assumptions and the main result are stated, the proof of which is contained in section 3. Some applications of the main results are collected in section 4.

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2 Main results

Consider the semilinear parabolic equations

$$\dot{x} + Ax = f(x) + \mu\, g(x,\mu) \qquad (x,\mu) \in X^\alpha \times \mathbb{R}^p \tag{2.1}$$

and

$$\dot{x} + Ax = f(x) + \mu\, g(t,x,\mu) \qquad (x,\mu) \in X^\alpha \times \mathbb{R}^p. \tag{2.2}$$

Here, A denotes a sectorial operator with associated fractional power spaces X^α, see [Hen81]. The nonlinearities satisfy $f : X^\alpha \to X$ and $g : X^\alpha \times \mathbb{R}^p \to X$ or $g : S^1 \times X^\alpha \times \mathbb{R}^p \to X$, respectively, for some $\alpha \in [0,1)$. We assume that $f,\ g \in C^{r,\rho}$ with $r, \rho > 0$ such that $f(0) = Df(0) = 0$. Let $q(t)$ denote a homoclinic solution of (2.1) (or (2.2)) for $\mu = 0$ satisfying

$$\lim_{t\to\pm\infty} q(t) = 0.$$

We denote the semigroup of the variational equation along $q(t)$

$$\dot{v} + Av = D_x f(q(t))\, v \tag{2.3}$$

by $\mathcal{T}(t,s)$. Next we state the hypotheses needed.

(H1) The spectrum of $-A$ decomposes into three pairwise disjoint spectral sets $\sigma(-A) = \sigma^s \cup \sigma^c \cup \sigma^u$ such that

$$\operatorname{Re}\sigma^s < -\alpha^{ss} < -\alpha^s < \operatorname{Re}\sigma^c < \alpha^u < \alpha^{uu} < \operatorname{Re}\sigma^u < \alpha^r$$

holds for suitable positive constants α^i and $i = ss, s, u, uu$. Now write P_0^s, P_0^c and P_0^u, respectively, for the corresponding spectral projections and $E_0^i := \mathrm{R}P_0^i$ for $i = s, c, u$ for their ranges. We assume that $\dim E_0^c + \dim E_0^u < \infty$.

The next hypothesis is related to the existence of linear invariant foliations of the variational equation along $q(t)$, that is the existence of exponential dichotomies.

(H2) Assume that there exist complementary projections $P^s(t)$, $P^c(t)$ and $P^u(t)$ depending continuously on $t \in \mathbb{R}$ and commuting with the semiflow $\mathcal{T}(t,s)$, that is $\mathcal{T}(t,s)\, P^i(s) = P^i(t)\, \mathcal{T}(t,s)$ for $i = s, c, u$ and $t \geq s$. Moreover, the projections satisfy

$$\begin{aligned} |\mathcal{T}(t,s)P^s(s)|_\alpha &\leq K\, e^{-\alpha^{ss}(t-s)} \\ |\mathcal{T}(t,s)P^c(s)|_\alpha &\leq K\, e^{-\alpha^{u}(t-s)} \\ |\mathcal{T}(s,t)P^c(t)|_\alpha &\leq K\, e^{-\alpha^{s}(s-t)} \\ |\mathcal{T}(s,t)P^u(t)|_\alpha &\leq K\, e^{\alpha^{uu}(t-s)} \end{aligned} \tag{2.4}$$

for all $t \geq s$. In particular, the inverses of the semigroups $\mathcal{T}(t,s)|_{\mathrm{R}P^c(s)}$ and $\mathcal{T}(t,s)|_{\mathrm{R}P^u(s)}$ exist and are denoted by $\mathcal{T}(s,t)$.

Assumption (H2) guarantees the existence of three continuous vector bundles given by $\mathrm{R}P^s(t)$, $\mathrm{R}P^c(t)$ and $\mathrm{R}P^u(t)$ along the orbit of $q(t)$, which extend continuously to the closure of the orbit by the spectral projections P_0^i for $i = s, c, u$. In the appendix, an hypothesis in terms on dichotomies on $\mathbb{R}^+$ and $\mathbb{R}^-$ is given, which is equivalent to (H2). The next assumption is concerned with the exponential behavior of $q(t)$ at infinity. The range of the center projection $P^c(t)$ will be the tangent space of the center manifold at $q(t)$. Thus, $\dot{q}(t)$ should be contained in that space.

(H3) The homoclinic solution $q(t)$ satisfies $\dot{q}(t) \in \mathrm{R}P^c(t)$ for all $t \in \mathbb{R}$.

The last assumption is of technical nature.

(H4) If $q(t) \notin W^s_{loc}(0,0)$ for $t \to \infty$ or $q(t) \notin W^u_{loc}(0,0)$ for $t \to -\infty$, then 0 is a simple eigenvalue of $-A$ and $\sigma(-A) \cap i\mathbb{R} = \{0\}$.

We remark here, that we do not assume that the homoclinic orbit converges along principal eigendirections. Zero is not assumed to be a hyperbolic equilibrium. Furthermore, $q(t)$ is allowed to oscillate at $t = \pm\infty$, i.e., it may converge along eigendirections corresponding to complex eigenvalues. Moreover, these eigenvalues are not assumed to be simple. Now we have the following theorems.

Theorem 1 *Consider equation (2.1) and assume that (H1)-(H4) are satisfied. Choose $\beta \in (0,1]$ and $k \in \mathbb{N}$ such that $\min(\alpha^{ss}/\alpha^s, \alpha^{uu}/\alpha^u, r+\rho) > k+\beta$. Then there exists a manifold W^c_{hom} for all μ with $|\mu| < \mu_0$ with the following properties:*

(i) $W^c_{hom} \in C^{k,\beta}$ *and it depends in a $C^{k,\beta}$-way on μ*

(ii) $\dim W^c_{hom} = \dim E^c_0$

(iii) W^c_{hom} *is normally hyperbolic*

(iv) W^c_{hom} *contains all solutions which stay inside a small tubular neighborhood of $q(t)$ for all times*

(v) W^c_{hom} *is locally invariant under the semiflow of (2.1).*

Theorem 2 *Consider equation (2.2) and assume that (H1)-(H4) are satisfied. Choose $\beta \in (0,1]$ and $k \in \mathbb{N}$ such that $\min(\alpha^{ss}/\alpha^s, \alpha^{uu}/\alpha^u, r+\rho) > k+\beta$. Then there exists a manifold W^c_{hom} for any $|\mu| < \mu_0$ with the following properties:*

(i) $W^c_{hom} \in C^{k,\beta}$ *and it depends in a $C^{k,\beta}$-way on μ*

(ii) $\dim W^c_{hom} = \dim E^c_0$

(iii) W^c_{hom} *is normally hyperbolic*

(iv) W^c_{hom} *contains all orbits under the period map of (2.2) which stay inside a small tubular neighborhood of* $q(t)$ *for all forward and backward iterates.*

(v) W^c_{hom} *is locally invariant under the period map of (2.2).*

Both theorems can be extended to center manifolds for heteroclinic cyles in an obvious way.

3 The proof

Before constructing the center manifold W^c_{hom}, we prove the existence of a center-unstable manifold W^{cu}_{hom}. Then we will restrict the semiflow to this finite-dimensional manifold and reverse time inside W^{cu}_{hom}. We are able to apply the same-procedure as before for the flow on W^{cu}_{hom}. This enables us to prove the existence of a center manifold W^c_{hom}. Before already starting with the proof, let us briefly sketch the main ideas in the following subsection.

3.1 Sketch of the proof

We shall use the graph transform for proving the existence of the invariant manifold. Let M be an approximation of the desired surface W^{cu}_{hom} containing the homoclinic solution $q(t)$ and possessing tangent spaces $T_{q(t)}M$ close to the subspaces $\mathrm{R}P^c(t) \oplus \mathrm{R}P^u(t) = \mathrm{R}P^{cu}(t)$ for all t. Then we can define the stable bundle E over M consisting of the union of the subspaces $\mathrm{R}P^s(t)$ attached to the base space M. We shall find the manifold W^{cu}_{hom} as a Lipschitz continuous section of the bundle E. In other words, W^{cu}_{hom} should be the graph of a Lipschitz continuous function σ defined for $x \in M$ with values in the stable fiber E_x attached to x. The graph transform maps such a section σ to a new one called $\Phi^\#_T(\sigma)$ which is obtained in the following way. First, apply the time T-map Φ_T of the semiflow to the surface graph(σ). The surface obtained will again be represented as a section $\Phi^\#_T(\sigma)$ of E such that graph$(\Phi^\#_T(\sigma)) = \Phi_T(\text{graph}(\sigma))$. Of course, at the present stage, it is not obvious at all whether this is always possible. Actually, the hardest part of the proof is concerned with the question whether $\Phi^\#_T$ is well defined. As it turns out the map $\Phi^\#_T$ is a contraction on the space of Lipschitz continuous sections with Lipschitz constant bounded by one due to the strong contraction of the linearized semiflow along the stable fibers.

In order to show that $\Phi^\#_T$ is well defined, one has to prove that each point in the domain M has a preimage and, moreover, that this preimage is unique. We say that Φ_T is overflowing if for any graph the first property is satisfied. Unfortunately, in our situation, Φ_T is

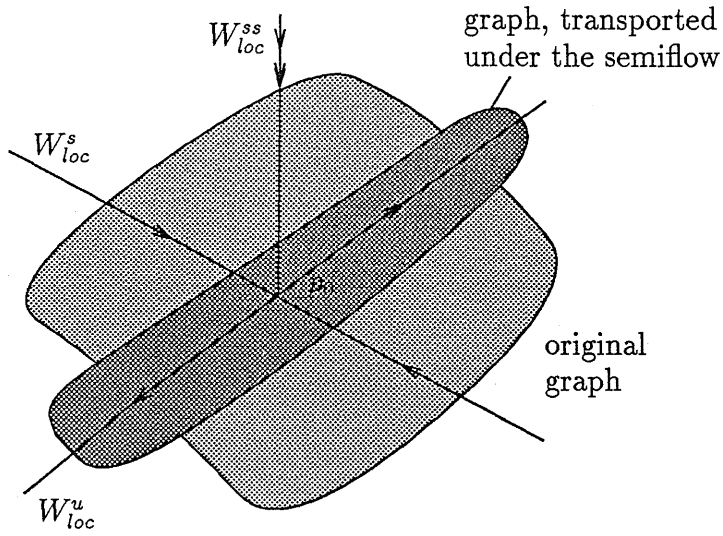

Figure 1: Local invariant manifolds near an equilibrium.

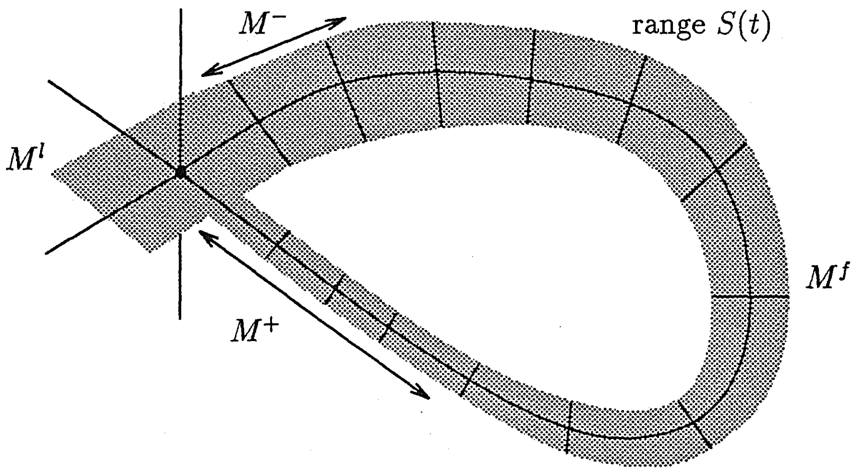

Figure 2: Domain of definition of the graphs.

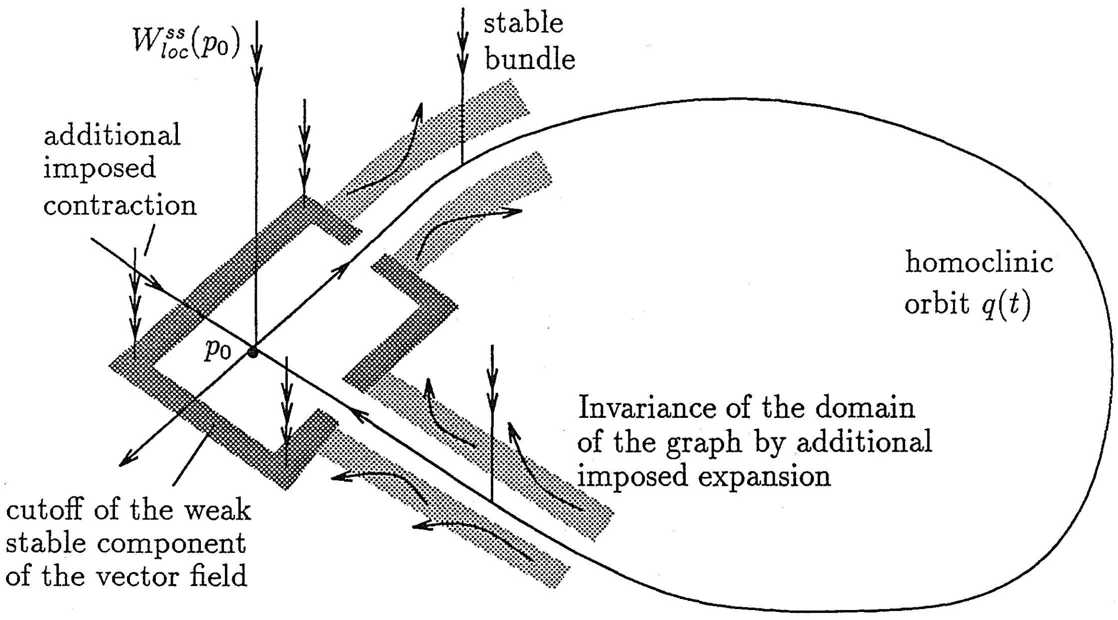

Figure 3: The modified equation.

not overflowing. Indeed, nearby the equilibrium, there are contracting directions present which prevent the overflowing property to hold. In fact, in this region the domain of a transported graph will shrink, see Figure 1. Usually, the use of smooth cutoff functions for the nonlinearity allows to work in the whole space X and thus avoids the difficulties mentioned above. However, here we are interested in the global system possessing a large homoclinic orbit.

Instead, we use the following device. First, we divide the approximation M into four submanifolds, which may overlap each other. We will denote these submanifolds by M^l, $M^\pm$ and M^f, i.e. $M = M^l \cup M^+ \cup M^- \cup M^f$, chosen roughly as in Figure 2. M^l is a neighborhood of the equilibrium of size ϵ, where ϵ is a small scaling parameter. The sets M^+ and M^- are the tails of the homoclinic orbits of fixed size together with a neighborhood in the transverse center-unstable direction of size ϵ. M^f consists of the remaining part of M. The overflowing property on M^f is obtained by choosing this submanifold carefully. The diameter of M^f in the directions transverse to the homoclinic solution has to shrink faster then the semiflow contracts these directions. This guarantees the overflowing. However, this procedure is only possible for bounded time intervals, whence we have to do something differently near the equilibrium on the manifolds $M^\pm$. There, we impose an additional expansion which is effective only in the transverse direction in $M^\pm$, see Figure 3. It turns out that this modification preserves the contraction in the stable fibers and hence the normal hyperbolicity of M, but guarantees the overflowing in this regime. Hence we are left with the submanifold M^l. Here we will smoothly cutoff the whole center-unstable component of the original equation in a self-similar manner, which will not affect a neighborhood of the homoclinic orbit of size ϵ.

From this point on, the graph transform runs as a machinery. Due to the use of the cutoff functions and the quite complicated definition of the submanifolds, the proof will still be quite technical. In order to prove the regularity of the manifolds, we use the following approach. We observe that the ball B_R (in the C^0-norm) of the space of C^1-functions possessing Hölder-continuous derivatives is closed in the set of Lipschitz continuous functions endowed with the C^0-norm, see [Hen81]. Then we prove that this ball will be mapped into itself by the graph transform for a suitable chosen R. Thus, by closeness, the unique fixed point is contained in B_R and therefore is at least $C^{1,\beta}$ for some $\beta > 0$.

3.2 The trial manifold and the stable bundle

First, choose two cutoff functions $\chi(\tau)$ and $\tilde{\chi}(\tau)$ for $\tau \in \mathbb{R}$ such that

$$\chi(\tau) \begin{cases} = 0 & \text{for } \tau \le \frac{1}{3} \\ \in (0,1) & \text{for } \tau \in (\frac{1}{3}, \frac{2}{3}) \\ = 1 & \text{for } \tau \ge \frac{2}{3} \end{cases} \qquad \tilde{\chi}(\tau) \begin{cases} = 0 & \text{for } \tau \le 2 \\ \in (0,1) & \text{for } \tau \in (2,3) \\ = 1 & \text{for } \tau \ge 3. \end{cases} \tag{3.1}$$

Let $\sup(|D\chi| + |D\tilde{\chi}|) =: K_\chi$ and $\chi_\epsilon(\tau) := \chi(\tau/\epsilon)$ as well as $\tilde{\chi}_\epsilon(\tau) := \tilde{\chi}(\tau/\epsilon)$.

We write $-A|_{E_0^{cu}}$ on the finite-dimensional space E_0^{cu} as a matrix $-A|_{E_0^{cu}} = \Delta + J$. Here, Δ denotes the diagonal part and J the Jordan block part of $-A|_{E_0^{cu}}$. Choosing suitable coordinates on E_0^{cu}, we may assume that the inequalities

$$\mathrm{Re}\,\sigma^s + |J| < -\alpha^{ss} < -\alpha^s < \mathrm{Re}\,\sigma^{cu} \pm |J| < \alpha^r$$

hold. We choose a scalar product $\langle\cdot,\cdot\rangle$ for the space E_0^{cu} such that the coordinates chosen above are orthogonal with respect to $\langle\cdot,\cdot\rangle$. In particular, the generalized eigenspaces are perpendicular to each other. By performing this transformation, we may change the constant of the exponential dichotomies, but do not change the assumptions (H1) up to (H4). Moreover, there exist constants $a_1, a_2 > 0$ satisfying $a_1 |x|_\alpha \le |x| \le a_2 |x|_\alpha$ for $x \in E_0^{cu}$. Here we denote the norm induced by the scalar product on E_0^{cu} by $|\cdot|$. Thus we can replace the original norm $|\cdot|_\alpha$ by the product norm $|\cdot|$ on E_0^{cu} and $|\cdot|_\alpha$ on E_0^s.

Definition Suppose that $L : X^\alpha \to X^\alpha$ is a linear, bounded operator. The minimum norm of L is defined by

$$m(L) := \inf_{x \neq 0} \frac{|Lx|_\alpha}{|x|_\alpha}.$$

Lemma 3.1 *There exists a constant K such that the inequalities*

$$\begin{aligned}
|\mathcal{T}(t,s)|_\alpha &\le Ke^{\alpha^r(t-s)} \\
|\mathcal{T}(t,s)P^s(s)|_\alpha &\le Ke^{-\alpha^{ss}(t-s)} \\
|\mathcal{T}(s,t)P^{cu}(t)|_\alpha &\le Ke^{\alpha^s(t-s)} \\
m(\mathcal{T}(t,s)|_{\mathrm{R}P^{cu}(s)}) &\ge K^{-1}e^{-\alpha^s(t-s)}
\end{aligned}$$

hold for $t \ge s$.

Proof. The first inequality follows from [Hen81, sect. 7.1], see also [San93, Lemma 1.1]. The next two inequalities hold by assumption, while the last estimate follows immediately from the third one. Indeed, $\mathcal{T}(t,s)|_{\mathrm{R}P^{cu}(s)}$ is invertible with inverse given by $\mathcal{T}(s,t)|_{\mathrm{R}P^{cu}(t)}$. For invertible operators $m(L)^{-1} = \|L^{-1}\|$ holds. □

Now, choose $\nu > 0$ small and $T > 0$ large enough such that $\alpha^s + \nu < \alpha^{ss}$ and

$$K^2\, e^{-(\alpha^{ss}-\alpha^s-\nu)T} < 1. \tag{3.2}$$

Here, K denotes the same constant as in Lemma 3.1. By hypothesis (H1), such a choice is possible. Throughout, we will use the following notation.

Definition 3.1

(i) C_0 will denote a constant which depends only on the choices made above but is independent of T. In contrast, $C(T)$ may depend in addition on the choice of T.

(ii) δ denotes a small positive constant to be chosen later on. δ will only depend on the size of a constant $C(T)$.

(iii) Furthermore, $\epsilon > 0$ is a small parameter which plays the role of a scaling factor in the cutoff procedure.

(iv) We denote functions converging to zero for $y \to 0$ by the Landau symbol o_y.

Next, we have to investigate the homoclinic orbit near the equilibrium in order to get an asymptotic expression for it. Assume that the following hypothesis is fulfilled.

(A) Let B be a sectorial operator and $g \in C^{1,\beta}(X^\alpha, X)$ for some $\beta > 0$ such that $g(0) = Dg(0) = 0$. Moreover, the spectrum of $-B$ is given by $\sigma(-B) = \sigma^s \cup \sigma^{ss}$ with $\mathrm{Re}\,\sigma^{ss} < -\lambda^{ss} < \mathrm{Re}\,\sigma^s$ and $\mathrm{Re}\,\sigma^s = -\lambda^s < 0$. We denote the corresponding spectral projections by Q_0^s and Q_0^{ss}, compare [Hen81, ch. 1.5]. Moreover, assume $\dim \mathrm{R}Q_0^s < \infty$.

We define $B^s := BQ_0^s$ and $B^{ss} := BQ_0^{ss}$ and consider the equation

$$\dot{x} + Bx = g(x), \quad x \in X^\alpha. \tag{3.3}$$

Lemma 3.2 *Assume that (A) is satisfied and choose γ such that $0 < \gamma < \lambda^s$. Then there exists an $\eta > 0$ such that the following holds. Take any $x_0 \in U_\eta(0) \subset X^\alpha$ and denote the solution of (3.3) satisfying $x(0) = x_0$ by $x(t)$. Then the limit $v := \lim_{t\to\infty} e^{B^s t} x(t) \in \mathrm{R}Q_0^s$ exists and we have the estimate*

$$|x(t) - e^{-B^s t} v|_\alpha \le C e^{-\min(\lambda^{ss},(1+\beta)(\lambda^s-\gamma))t}.$$

The last inequality is fulfilled for the time derivatives, too.

Proof. Choose $\eta > 0$ sufficiently small such that any solution $x(t)$ with $|x(0)|_\alpha < \eta$ exists for all $t > 0$ and converges to zero like $e^{-(\lambda^s-\gamma)t}$ for $t \to \infty$. We have $|g(x)| \le K\,|x|_\alpha^{1+\beta}$ for any $x \in U_\eta(0)$. Rewriting (3.3) as an integral equation and projecting it in the weak and strong stable subspaces, we get

$$\begin{aligned} Q_0^s\, x(t) &= e^{-B^s(t-s)} Q_0^s\, x(s) + \textstyle\int_s^t e^{-B^s(t-\tau)} Q_0^s\, g(x(\tau))\, d\tau \\ Q_0^{ss}\, x(t) &= e^{-B^{ss}(t-s)} Q_0^{ss}\, x(s) + \textstyle\int_s^t e^{-B^{ss}(t-\tau)} Q_0^{ss}\, g(x(\tau))\, d\tau. \end{aligned} \tag{3.4}$$

Using the inequality for g we obtain for $t \ge s$

$$\begin{aligned} |\textstyle\int_s^t e^{B^s\tau} Q_0^s\, g(x(\tau))\, d\tau|_\alpha &\le |\textstyle\int_s^t C e^{(\lambda^s-\gamma)\tau} K e^{-(1+\beta)(\lambda^s-\gamma)\tau}\, d\tau| \\ &\le K C e^{-\beta(\lambda^s-\gamma)s} (1 - e^{-\beta(\lambda^s-\gamma)(t-s)}) \\ |\textstyle\int_s^t e^{-B^{ss}(t-\tau)} Q_0^{ss}\, g(x(\tau))\, d\tau|_\alpha &\le \textstyle\int_s^t C(t-\tau)^{-\alpha} e^{-\lambda^{ss}(t-\tau)} K e^{-(1+\beta)(\lambda^s-\gamma)\tau}\, d\tau \\ &\le C e^{-\min(\lambda^{ss},(1+\beta)(\lambda^s-\gamma))t} \end{aligned} \tag{3.5}$$

uniformly in s. Here, we may have to make λ^{ss} a little bit smaller. The first inequality in (3.5) implies that $e^{B^s t} x(t)$ is a Cauchy sequence in X^α for $t \to \infty$. We denote the limit by

$$v := \lim_{t\to\infty} e^{B^s t} x(t) = \lim_{t\to\infty} e^{B^s t} Q_0^s\, x(t).$$

Then, $v \in \mathrm{R}Q_0^s$ and

$$Q_0^s\, x(t) = e^{-B^s t} v + \int_\infty^t e^{-B^s(t-\tau)} Q_0^s\, g(x(\tau))\, d\tau$$

by taking the limit $s \to \infty$ in (3.4). Now, we will compare the limit solution $e^{-B^s t} v$ and $x(t)$. We obtain

$$\begin{aligned} &|x(t) - e^{-B^s t} v|_\alpha \\ &\le \int_\infty^t C e^{-(\lambda^s-\gamma)(t-\tau)}\, |x(\tau)|_\alpha^{1+\beta}\, d\tau + \int_0^t C(t-\tau)^{-\alpha}\, e^{-\lambda^{ss}(t-\tau)}\, |x(\tau)|_\alpha^{1+\beta}\, d\tau \\ &\le C e^{-\min(\lambda^{ss},(1+\beta)(\lambda^s-\gamma))t}. \end{aligned} \tag{3.6}$$

The same procedure works for the time derivatives of $x(t)$ using [Hen81, Lemma 3.5.1] and the estimates following (3.4). We will not work out this in detail. □

We will use the above lemma in order to parametrize a neighborhood of the homoclinic solution near the equilibrium. The next lemma as well as the definitions following it will be stated for positive t only in order to avoid unnecessary extended notation. However, we will use them for negative t, too.

Lemma 3.3 *Define*

$$\begin{aligned} V_\epsilon &= U_\epsilon(0) \subset \mathbb{R}^{\dim E_0^{cu}-1} \\ W_\epsilon &= U_\epsilon(0) \subset E_0^s \end{aligned}$$

to be neighborhoods of zero of size ϵ. Then there exist numbers $\epsilon_0, b_0 > 0$ and a map

$$\begin{aligned} G(r,v,w) \ : \ (\epsilon, \epsilon_0) \times V_{b_0\epsilon} \times W_{b_0\epsilon} &\to X^\alpha \\ (r,v,w) &\mapsto r\,G_1(r) + G_2(r)\,v + w + r\,G_3(r) \end{aligned}$$

for $0 < \epsilon < \epsilon_0$ with the following properties.

(i) *The function*

$$G(r,0,0) = r\,(G_1(r) + G_3(r))$$

parametrizes the homoclinic orbit $q(t)$ on a time interval $(t(\epsilon_0), t(\epsilon))$ and $t(\epsilon) \to \infty$ as $\epsilon \to 0$. Moreover, G is a diffeomorphism onto a neighborhood in X^α of the homoclinic orbit restricted to this time interval.

(ii) *We have $G_1(r) \in E_0^{cu}$ and $G_2(r) : V_{b_0\epsilon} \to E_0^{cu}$. Both functions satisfy $D(r\,G_i(r))$, $D(r\,G_i(r))^{-1} \le C$ for $i = 1, 2$ as well as $|G_1(r)|, |G_2(r)| \ge C > 0$. The image $G_2(r)\,v$ is always perpendicular to $G_1(r)$.*

(iii) *The image $G_3(r) \in E_0^s$ is contained in the stable eigenspace E_0^s. Furthermore, $|G_3(r)|_1 \to 0$ tends to zero as $r \to 0$ as a function into X^α.*

The constant C is independent of ϵ.

Let us explain the statement of this lemma. We can parametrize the homoclinic orbit essentially as a graph over the eigenspace E_0^{cu}. Moreover, we can parametrize a neighborhood of the homoclinic orbit by taking the homoclinic solution $q(t)$ and adding vectors in the directions transverse to the time derivative $\dot{q}(t)$ as well as adding vectors which are contained in E_0^s.

Proof. We have to distinguish two cases. First suppose that the homoclinic orbit is contained in the local center manifold $W_{loc}^c(0)$ of the equilibrium. Then, by assumption (H4), we have $\dim W_{loc}^c(0) = 1$ and the statement of the lemma follows from center-manifold theory. Hence, we assume from now on that the homoclinic solution is contained in the local stable manifold. In particular, there exists $\lambda^+ \neq 0$ such that

$$0 < \lim_{t\to\infty} e^{\lambda^+ |t|} |q(t)|_\alpha < \infty$$

and we denote the generalized eigenspace corresponding to all eigenvalues with real part equal to $-\lambda^+$ by E^+. (Similarly, λ^- and E^- are defined for $t \to -\infty$). By assumption, both subspaces are finite-dimensional due to $E^\pm \subset E_0^{cu}$. Restrict the semiflow to the strong stable manifold tangent to the generalized eigenspace associated with eigenvalues whose real part is less or equal to $-\lambda^+$. Applying Lemma 3.2 to the semiflow on that manifold, we see that

$$q(t) = e^{-A^s t} v_0 + \mathcal{O}(e^{-(\lambda^+ + \gamma)t}) \tag{3.7}$$

holds for $t > 0$ and some $\gamma > 0$ with A^s equal to $A|_{E^+}$. The estimate holds for the derivatives, too. We decompose the solution $e^{-A^s t} v_0 = v_1(t) + v_2(t)$ corresponding to $|v_1(t)| \simeq t^k e^{-\lambda^+ t}$ and $|v_2(t)| = o(t^k e^{-\lambda^+ t})$ for some k depending on the size of the Jordan block of A^s. Next subsume $v_2(t)$ into the remainder term $G_3(t)$ which together with its time derivative still satisfies $G_3(t) = o(v_1(t))$. Then define $r(t) = t^k e^{-\lambda^+ t}$ and parametrize $v_1(t)$ as a function of r, whence

$$v_1(t(r)) = r e^{i\vec{\omega} t(r)},$$

where $\vec{\omega}$ denotes the vector of imaginary parts of the eigenvalues in the Jordan block. This proves the claim concerning G_1 and G_3. In order to conclude the statements about G_2 choose suitable vectors which are perpendicular to $v_1(r)$. The claims concerning the existence of the inverse as well as the boundedness of the norms follows easily by using the bounds for v and w, see [San93] for the details. □

Let

$$\tilde{G}(r, v, w) := r\, G_1(r) + G_2(r)\, v + w + \tilde{\chi}_\epsilon(r)\, r\, G_3(r),$$

where $\tilde{\chi}$ has been defined in (3.1). The map $\tilde{G}$ will again be a diffeomorphism, but $\tilde{G}(r, 0, 0)$ will parametrize the homoclinic orbit only for $r \geq 3\epsilon$. Moreover, $\tilde{G}(r, v, 0) \in E_0^{cu}$ for all $\epsilon < r \leq 2\epsilon$. Observe that the norms of $D\tilde{G}(r, v, w)$ and $D\tilde{G}^{-1}(r, v, w)$ are still bounded uniformly in ϵ due to $G_3(r) = o_\epsilon$ for $r \in (\epsilon, 3\epsilon)$. Now we define the following nonlinear mappings which will act afterwards as bundle projections onto the trial manifold and the stable bundle for points near the equilibrium.

Definition Let $\tilde{G}(r, v, w) = x$. Then we define

$$\begin{aligned} \pi_\gamma(x) &:= \tilde{G}(r, 0, 0) \\ \pi_E(x) &:= w \\ \pi_{\gamma^\perp}(x) &:= G_2(r)\, v, \end{aligned} \tag{3.8}$$

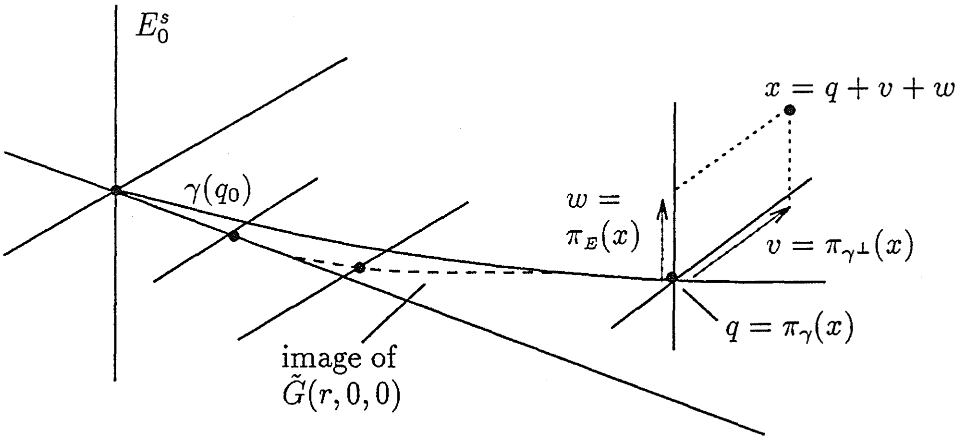

Figure 4: The projections π_γ, $\pi_{\gamma^\perp}$ and π_E.

or, in other words, we have $\pi_\gamma(x) = \tilde{G}(P_1\, G^{-1}(x))$, $\pi_E(x) = \tilde{G}(P_3\, G^{-1}(x))$ and $\pi_{\gamma^\perp}(x) = \tilde{G}(P_2\, G^{-1}(x))$. Here, P_i projects on the ith component of (r, v, w) for $i = 1, 2, 3$. Let $\pi_M(x) = \pi_\gamma(x) + \pi_{\gamma^\perp}(x)$ for x such that $\tilde{G}(r, v, w) = x$ is defined. Then we can extend $\pi_E(x)$ and $\pi_M(x)$ by $P_0^{cu}\, x$ and $\pi_E(x) = P_0^s\, x$, respectively, for all x with $|x|_\alpha \leq 3\epsilon$.

In order to define the graph transform, we need to define global stable and unstable bundles in a continuous way. Thus we have to extend the projections defined above in a global way along the homoclinic solution. First, note that the projections $P^i(t)$ defined in hypothesis (H2) converge to the spectral projections P_0^i for $t \to \pm\infty$ and $i = s, c, u$ by [San93, Lemma 1.1] and the appendix. We choose a continuous scalar product $\langle \cdot, \cdot \rangle_t$ on the bundle given by $\mathrm{R}P^{cu}(t)$ over $q(t)$ which coincides with the scalar product defined on E_0^{cu} for all large $|t|$, that is

$$\langle \cdot, \cdot \rangle_t = \langle P_0^{cu} \cdot, P_0^{cu} \cdot \rangle$$

for $|t|$ sufficiently large. This is always possible, because the bundle $\mathrm{R}P^{cu}(t)$ is continuous and possesses finite-dimensional fibers, see [BJ73, ch. 4.11].

Definition We define

$$\tilde{S}(t)\, x := P^{cu}(t)\, x - \left\langle \frac{\dot{q}(t)}{|\dot{q}(t)|_\alpha}, P^{cu}(t)\, x \right\rangle_t \frac{\dot{q}(t)}{|\dot{q}(t)|_\alpha}$$

to be the continuous projection onto the orthogonal complement - with respect to the scalar product chosen above - of $\operatorname{span} \dot{q}(t)$ in $\mathrm{R}P^{cu}(t)$.

The projections $P^s(t)$ and $\tilde{S}(t)$ are in general only continuous in t in the operator norm, see [San93, Lemma 1.2(v)]. However, we can approximate them arbitrarily close by C^∞-projections. For any given δ, there exist r_0 and T_0 such that

$$\begin{aligned}
&\sup_{|x|_\alpha<r_0} |Df(x)| < \delta && \sup_{r\in(0,r_0]} |G_3(r)|_1 < \delta \\
&\sup_{|t|\geq T_0} |P^s(t)-P_0^s|_\alpha < \tfrac{1}{2}\delta && \sup_{|t|\geq T_0} |P^{cu}(t)-P_0^{cu}|_\alpha < \tfrac{1}{2}\delta.
\end{aligned} \tag{3.9}$$

Hence, δ controls the convergence of the dichotomies towards the spectral projections as well as the bound of the nonlinearity f as mentioned in definition 3.1 above. Then we have the following lemma.

Lemma 3.4 *For any given $\delta > 0$ there exist functions $Q^s(t)$ and $S(t)$ in $C^\infty(\mathbb{R}, L(X^\alpha))$ such that $Q^s(t)$, $S(t)$ are projections and $|Q^s(t)-P^s(t)|_\alpha$, $|S(t)-\tilde{S}(t)|_\alpha < \delta$ for all $t \in \mathbb{R}$. Furthermore, $Q^s(t) = P_0^s$ and*

$$\begin{aligned}
\operatorname{Range} S(t) &= \operatorname{Image} G_2(r) \\
\operatorname{Kernel} S(t) &= \operatorname{Image} D_r\Big(r\, G_1(r)\Big) \oplus E_0^s
\end{aligned}$$

for $t \geq T_0$ and $t \leq -T_0$, respectively. Here $G(r,0,0) = q(t)$.

Proof. Owing to (3.9), we have

$$\frac{\delta}{2} \geq \begin{cases} |P^s(t)-P_0^s|_\alpha & \qquad |t| \geq T_0 \\ |\tilde{S}(t) - P_0^+ P_0^{cu}|_\alpha \qquad \text{for} & \qquad t \geq T_0 \\ |\tilde{S}(t) - P_0^- P_0^{cu}|_\alpha & \qquad t \leq -T_0. \end{cases}$$

As a matter of fact, it is possible to connect any two projections Q_0 and Q_1 satisfying $|Q_0 - Q_1|_\alpha < 1$ by a C^∞-path $Q(\tau)$ of projections for $\tau \in [0,1]$. Indeed, define $Q(\tau) = Q_0$ and $Q(\tau) = Q_1$ for $\tau \leq \frac{1}{4}$ and $\tau \geq \frac{3}{4}$, respectively. Then we interpolate these projections as in [Kat66, exc. I.4.6 (4.13)] by using a cutoff function. Afterwards we divide the interval $[-T_0, T_0]$ in a suitable way. □

Next we parametrize the homoclinic orbit $q(t)$ using the arclength. Define $q(t(\tau)) =: \tilde{q}(\tau)$ for $\tau \in \mathbb{R}$ and $\tilde{q}(\cdot) \in C^{r,\rho}$ such that $|D_\tau \tilde{q}(\tau)|_\alpha \leq 2$. This is possible by approximating the arclength by a C^∞-function $\tilde{t}(\tau)$ satisfying $|\tilde{t}(\tau) - t(\tau)| \leq |\dot{q}(\tilde{t}(\tau))|_\alpha$. Therefore, we can locally parametrize a tubular neighborhood of the homoclinic solution at $q = q(t(\tau_0))$ by the map

$$\begin{aligned}
h_q :\ (\tau_0 - \tau_1, \tau_0 + \tau_1) \times \mathrm{R}S(t(\tau_0)) \times \mathbb{R}Q^s(t(\tau_0)) &\longrightarrow X^\alpha \\
(\tau, v, w) &\longmapsto q(t(\tau)) + S(t(\tau))\, v + Q^s(t(\tau))\, w.
\end{aligned}$$

In particular, $h_q \in C^{r,\rho}$ and

$$Dh_q(\tau_0, 0, 0)(\tilde{\tau}, \tilde{v}, \tilde{w}) = \frac{d}{d\tau}\tilde{q}(\tau_0)\, \tilde{\tau} + S(t(\tau_0))\, \tilde{v} + Q^s(t(\tau_0))\, \tilde{w}.$$

Thus we obtain $|Dh_q(\tau_0,0,0)|_\alpha, |Dh_q(\tau_0,0,0)^{-1}|_\alpha \leq C$ and C depends only on the norms of the projections S and Q^s. These, however, can be estimated by the norms of $\tilde{S}$ and P^s for δ small. Therefore, we conclude that h_q is invertible for all $\tau \in [t^{-1}(-T_1), t^{-1}(T_1)]$ and $|v|_\alpha, |w|_\alpha \leq \eta_0$. Here η_0 depends on the choice of T_1. We fix T_1 by choosing $T_1 := T_0 + 5T$.

Now we are in a position to extend the nonlinear projections π_γ, $\pi_{\gamma^\perp}$, π_E and π_M to a full neighborhood of the homoclinic orbit. We define locally

$$
\begin{aligned}
\pi_\gamma(x) &:= h_q \circ \Big((\tau, v, w) \mapsto (\tau, 0, 0) \Big) \circ h_q^{-1} \\
\pi_{\gamma^\perp}(x) &:= h_q \circ \Big((\tau, v, w) \mapsto (0, v, 0) \Big) \circ h_q^{-1} \\
\pi_E(x) &:= h_q \circ \Big((\tau, v, w) \mapsto (0, 0, w) \Big) \circ h_q^{-1} \\
\pi_M(x) &:= \pi_\gamma(x) + \pi_{\gamma^\perp}(x).
\end{aligned}
\tag{3.10}
$$

It is easy to see, that these mappings are well-defined using the following facts: $\pi_\gamma(x) = q(t) \in \gamma(q(0))$, $\pi_{\gamma^\perp}(x) \in \mathrm{R}S(t)$ and $\pi_E(x) \in \mathrm{R}Q^s(t)$.

In words, $\pi_\gamma(x)$ projects x to the nearest point $q(t)$ on the homoclinic orbit. $\pi_{\gamma^\perp}(x)$ denotes the center-unstable part of x relative to the point $\pi_\gamma(x)$ on the homoclinic orbit and $\pi_E(x)$ equals the stable component again relative to $\gamma(q(0))$. Here, the center-unstable component is contained in $\mathrm{R}P^{cu}(t)$, the stable component in $\mathrm{R}Q^s(t)$.

In order to define a modified vector field later on, we need some further definitions. First of all, we can estimate the contraction rate of $\mathcal{T}(-T_1, T_1)$ restricted to the range $\mathrm{R}P^{cu}(-T_1)$ of the center-unstable projection from below by

$$
\kappa_0 := (Ke^{(\alpha^s+\nu)T})^{-\frac{2T_1}{T}}
\tag{3.11}
$$

using Lemma 3.1. Note that κ_0 depends only on the choices of T and T_1, that is on δ. Next, consider Figure 5. Take the ball $U^{cu}_{2\epsilon}(0)$ with radius 2ϵ at zero in E^{cu}_0. Then choose two subsets $\mathcal{A}$ and $\mathcal{B}$ which are contained in the spherical annulus $U^{cu}_{2\epsilon, 2(\epsilon-\kappa_4)}(0)$, that is the set of $x \in U^{cu}_{2\epsilon}(0)$ satisfying $2(\epsilon - \kappa_4) \leq |x|_\alpha \leq 2\epsilon$, and fulfill $\mathcal{A} \subset \mathcal{B}$. In addition, $\mathcal{B}$ coincides with the spherical annulus except for a κ_3-neighborhood of the images of the maps $r\,G_1^\pm(r)$ parametrizing the homoclinic orbit up to the error term $r\,G_3(r)$. We choose $\kappa_2 < \frac{1}{3}\kappa_0$ as the minimal distance of $\mathcal{A}$ to the image of $r\,G_1^\pm(r)$, see Figure 5. Then we define cutoff functions $\vartheta^{cu}(x)$ and $\vartheta^s(x)$ by

$$
\left\{
\begin{array}{rclrcl}
 & & \multicolumn{4}{l}{\vartheta^{cu}(x),\, \vartheta^s(x) \in [0,1]} \\
\vartheta^{cu}|_{\mathcal{C}U_\eta(\mathcal{A})} &=& 1 & \vartheta^{cu}|_{\mathcal{A}} &=& 0 \\
\vartheta^s|_{\mathcal{B}} &=& 1 & \vartheta^s|_{\mathcal{C}U_\eta(\mathcal{B})} &=& 0
\end{array}
\right.
\tag{3.12}
$$

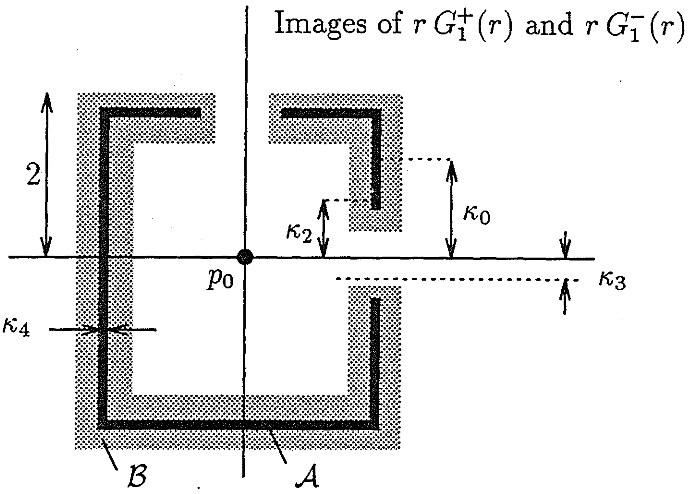

Figure 5: Cutoff regions of ϑ^{cu} and ϑ^s.

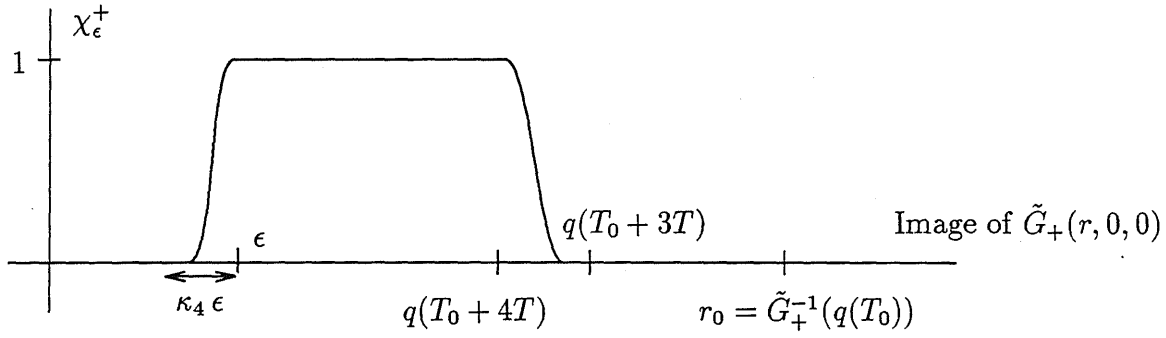

Figure 6: The cutoff function χ_ϵ^+. Similarly, χ_ϵ^- is defined for $t \to -\infty$.

for $x \in E_0^{cu}$. Here, we choose $\eta > 0$ sufficiently small such that $\kappa_3 \geq \eta$ and therefore $U_\eta(\mathcal{A}) \subset \mathcal{B}$ holds. Then, $U_\eta(\mathcal{B})$ does not intersect the image of $r\, G_1^{\pm}(r)$. In particular, we obtain $\{x \,|\, \vartheta^{cu}(x) < 1\} \subset \{x \,|\, \vartheta^s(x) = 1\}$. Define $\max(|D\vartheta^{cu}|, |D\vartheta^s|) =: K_\vartheta$. Observe that these norms depend only on the choice of T and r_0 via κ_0 defined in (3.11). The cutoff functions can be defined in such a way that they are invariant under rotations of each coordinate separately outside neighborhoods of the incoming and outcoming eigenspaces $E^{\pm}$ defined in the proof of Lemma 3.3. Then we can define $\vartheta_\epsilon^s(x)$ and $\vartheta_\epsilon^s(x)$ by replacing x by x/ϵ and furthermore rotating each coordinate of x/ϵ individually by $e^{i\tilde{\omega}t(\epsilon)}$, see the proof of Lemma 3.3. Here, the angle depends on ϵ only. This guarantees that the orbit $\gamma(v_1)$ of the linear part is mapped onto itself by the scaling. Thus, the scaled cutoff does not affect the homoclinic orbit $\gamma(q(0))$.

Furthermore, choose two cutoff functions $\chi_\epsilon^{\pm}$ defined on the image of $\tilde{G}_{\pm}(r,0,0)$ by using the fixed function χ defined in (3.1) and the diffeomorphism $\tilde{G}_{\pm}(r,0,0)$. See Figure 6 for the cutoff properties of $\chi_\epsilon^{\pm}$. Note that $\tilde{G}_{\pm}(r,0,0)$ has bounds on the derivatives and their inverses independent on ϵ by Lemma 3.3. Hence, the cutoff functions satisfy $|D\chi_\epsilon^{\pm}(G_{\pm}(r,0,0))| \leq K_\chi$ for $r \geq \epsilon$.

Now, we shall define our trial manifold M_ϵ:

$$
\begin{aligned}
&M_\epsilon := \\
&\Big\{x \in E_0^{cu} \,\big|\, |x|_\alpha < 2\epsilon\Big\} \cup \\
&\Big\{x = \tilde{G}_+(r,v,0) \,\big|\, r \in (\epsilon, r^+),\ |v|_\alpha < \kappa_0\epsilon\Big\} \cup \\
&\Big\{x = \tilde{G}_-(r,v,0) \,\big|\, r \in (\epsilon, r^-),\ |v|_\alpha < \epsilon\Big\} \cup \\
&\Big\{x = \pi(x) + \pi_{\gamma^\perp}(x) \,\big|\, q(t) = \pi(x) \text{ for } t \in (-T_1, T_1),\ |\pi_{\gamma^\perp}(x)|_\alpha < (Ke^{(\alpha^s+\nu)T})^{-\frac{t+T_1}{T}}\, \epsilon\Big\}.
\end{aligned}
$$

Here $(r^{\pm},0,0) = G_{\pm}^{-1}(q(\pm T_1))$. Moreover, we define a decomposition of M_ϵ into not necessarily disjoint sets M^l, $M^{\pm}$ and M^f, see Figure 7:

$$
\begin{aligned}
M_\epsilon^l &:= \Big\{x \in E_0^{cu} \,\big|\, |x|_\alpha < 2\epsilon\Big\} \cup \Big\{x = \tilde{G}_+(r,v,0) \,\big|\, r \in (\epsilon, 3\epsilon e^{\alpha^s T}),\ |v|_\alpha < \kappa_0\epsilon\Big\} \cup \\
&\quad\ \Big\{x = \tilde{G}_-(r,v,0) \,\big|\, r \in (\epsilon, 3\epsilon e^{\alpha^r T}),\ |v|_\alpha < \epsilon\Big\} \\
M_\epsilon^+ &:= \Big\{x = \tilde{G}_+(r,v,0) \,\big|\, r \in (3\epsilon, r_l^+),\ |v|_\alpha < \kappa_0\epsilon\Big\} \\
M_\epsilon^- &:= \Big\{x = \tilde{G}_-(r,v,0) \,\big|\, r \in (3\epsilon, r_l^-),\ |v|_\alpha < \epsilon\Big\} \\
M_\epsilon^f &:= \Big\{x \in M_\epsilon \,\big|\, q(t) = \pi(x) \text{ for } t \in (-T_0 - 2T, T_0 + 2T)\Big\}
\end{aligned}
$$

with $(r_l^{\pm},0,0) = G^{-1}(q(\pm T_0))$. Furthermore, we define the stable bundle with base space M_ϵ in the following way. Choose T_2 such that $|DG_3|_\alpha, |D_x f| \leq \kappa_1\,\delta$ holds for $|x|_\alpha \leq |q(\pm T_2)|_\alpha$ and let T_3 be a number satisfying $T_2 < T_3$ and $(Ke^{(\alpha^s+\nu)T})^{-\frac{T_3-T_2}{T}}\,\kappa_0 = \kappa_1$. The

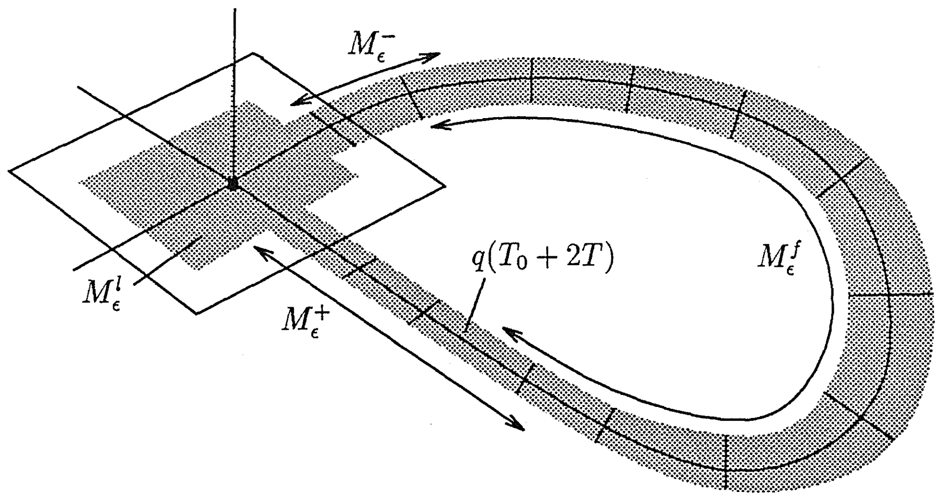

Figure 7: The submanifolds M_ϵ^l, $M_\epsilon^\pm$ and M_ϵ^f.

number κ_1 is a constant independent of ϵ which will be fixed later on. The let

$$R(q(t)) := \begin{cases} \epsilon & \text{for } t \in [-T_1 - T, T_1] \\ \epsilon\,(Ke^{(\alpha^s+\nu)T})^{-\frac{t+T_1}{T}} & \text{for } t \in [-T_1, T_1] \\ \kappa_0\,\epsilon & \text{for } t \in [T_1, T_2] \\ \kappa_0\,\epsilon\,(Ke^{(\alpha^s+\nu)T})^{-\frac{t-T_2}{T}} & \text{for } t \in [T_2, T_3] \\ \kappa_1\,\epsilon & \text{otherwise} \end{cases} \tag{3.13}$$

and define the fiber $E_\epsilon^s(x)$ for $x \in M_\epsilon$ by

$$E_\epsilon^s(x) := \begin{cases} \left\{w \in \mathrm{R}Q^s(\pi_\gamma(x)) \,|\, |w|_\alpha < R(\pi_\gamma(x))\right\} & \text{for } x \in M_\epsilon\ |x|_\alpha > 2\epsilon \\ \left\{w \in E_0^s \,|\, |w|_\alpha < \kappa_1\,\epsilon\right\} & \text{otherwise.} \end{cases}$$

This very technical definition is forced by the requirement that M_ϵ must be overflowing. Hence, the norm of vectors in the bundle has to be restricted.

Lemma 3.5 *The set*

$$E_\epsilon := \left\{x \,|\, \pi_M(x) \in M_\epsilon, \pi_E(x) \in E_\epsilon(x)\right\}$$

is a $C^{r,\rho}$ bundle in X^α with base space M_ϵ. Furthermore, the homoclinic orbit $\gamma(q_0) \subset E_\epsilon$ is contained in the bundle for all sufficiently small ϵ.

Proof. This follows from the definition of the mappings π_M, π_E and the construction of the bundle. The homoclinic orbit is surely contained in the bundle except possibly in the domain with $\tilde{\chi}_\epsilon < 1$. But in this region we have $|G_3(r)|_\alpha = o_\epsilon$. Thus the error in the norm of $\pi_E(x)$ caused by the cutoff of $r\,G_3(r)$ is small compared with $\kappa_1\epsilon$. Indeed, κ_1 is independent of ϵ. $\square$

Lemma 3.6 *(i) Let $\pi_\gamma(x) = q(t)$. Then we have the following estimates for $D\pi_M(x)$ and $D\pi_E(x)$ uniformly in $x \in M_\epsilon$*

$$\left.\begin{array}{l} |D\pi_M(q(t)) - P^{cu}(t)|_\alpha \\ |D\pi_E(q(t)) - P^s(t)|_\alpha \end{array}\right\} < C(T)\,\delta + o_\epsilon,$$

and the o-term depends only on the norms of P^s and P^{cu}. We define $D\pi_M(q(t)) =: Q^{cu}(t)$

(ii) It is possible to parametrize M_ϵ^f locally near each $q = q(t(\tau_0))$ by

$$\begin{aligned} (id + h_q^M):\ & (\tau_0 - \tau_1, \tau_0 + \tau_1) \times \mathbb{R}S(t(\tau_0)) \longrightarrow X^\alpha \\ & (\tau, v) \longmapsto q(t(\tau_0)) + D_\tau\tilde{q}(\tau_0)\,(\tau - \tau_0) + v + h_q^M(\tau, v), \end{aligned}$$

where h_q^M is given by

$$h_q^M(\tau, v) = q(t(\tau)) - q(t(\tau_0)) - D_\tau\tilde{q}(\tau_0)\,(\tau - \tau_0) + \Big(S(t(\tau)) - S(t(\tau_0))\Big)\,v.$$

Moreover, $|Dh_q^M|_\alpha < C_0\,\epsilon$ in M_ϵ^f.

Proof. The first claim (i) follows from the definitions, Lemma 3.4 and $|x - \pi_\gamma(x)|_\alpha < \epsilon$. For the second claim we use the definition of h_q^M and compactness of the time interval $[-T_1, T_1]$ under consideration. $\square$

3.3 The graph transform

We will first set $\mu = 0$, whence (2.1) and (2.2) coincide. The case $\mu \neq 0$ is investigated in section 3.6. Define a new system of equations for $x \in E_\epsilon$ in the following way:

$$\begin{aligned} \dot{x} &= F^s(x) + F^{cu}(x) \\ &= -A\,P_0^s\,x + P_0^s\,f(x) - \beta_1\,\vartheta_\epsilon^s(P_0^{cu}\,x)\,P_0^s\,x + \\ &\quad \Big(-A\,P_0^{cu}\,x + P_0^{cu}\,f(x) + B_\epsilon^+(x) + B_\epsilon^-(x)\Big)\,\vartheta_\epsilon^{cu}(P_0^{cu}\,x) \\ &:= -A\,P_0^s\,x + P_0^s\,f(x) - \beta_1\,\vartheta_\epsilon^s(P_0^{cu}\,x)\,P_0^s\,x + \\ &\quad \bigg(-A\,P_0^{cu}\,x + P_0^{cu}\,f(x) + \beta_0\Big(\chi_{\kappa_0\epsilon}(|\pi_{\gamma^\perp}(x)|)\,\chi_\epsilon^+(\pi_\gamma(x)) + \\ &\quad \chi_\epsilon(|\pi_{\gamma^\perp}(x)|)\,\chi_\epsilon^-(\pi_\gamma(x))\Big)\pi_{\gamma^\perp}(x)\bigg)\,\vartheta_\epsilon^{cu}(P_0^{cu}\,x). \end{aligned} \tag{3.14}$$

The maps B^+ and B^- are defined by

$$\begin{aligned} B^+_\epsilon(x) &:= \beta_0\, \chi_{\kappa_0\epsilon}(|\pi_{\gamma^\perp}(x)|)\, \chi^+_\epsilon(\pi_\gamma(x))\, \pi_{\gamma^\perp}(x) \\ B^-_\epsilon(x) &:= \beta_0\, \chi_\epsilon(|\pi_{\gamma^\perp}(x)|)\, \chi^-_\epsilon(\pi_\gamma(x))\, \pi_{\gamma^\perp}(x). \end{aligned}$$

Here, β_1 is chosen such that

$$\beta_1 > \gamma_1 = C(T, \kappa_0)\, K_\vartheta, \tag{3.15}$$

where γ_1 is constant depending on $C(T, \kappa_0)\, K_\vartheta$, but is independent on ϵ. We will specify γ_1 in equation (3.34). Moreover, we fix β_0 such that $\beta_0 > \alpha^s$.

We should explain the meaning of the terms in (3.14). The first part

$$F^s(x) = -A\, P^s_0\, x + P^s_0\, f(x) - \beta_1\, \vartheta^s_\epsilon(P^{cu}_0\, x)\, P^s_0\, x$$

consists of the stable part of the differential equation (2.1) and the additional contraction $-\beta_1\, \vartheta^s_\epsilon(P^{cu}_0\, x)\, P^s_0\, x$, which is effective only near the equilibrium. The second term $F^{cu}(x)$ is the unstable component of the vector field. We explain the different terms in this expressions separately. First of all, we cutoff the whole vector field (and not just the nonlinearity) in the unstable direction for $x \in \mathcal{A}$. This is realized by the expression $(\dots)\, \vartheta^{cu}_\epsilon(P^{cu}_0\, x)$. The term in parentheses is the original unstable component of (2.1) together with

$$\beta_0\, \chi_{\kappa_0\epsilon}(|\pi_{\gamma^\perp}(x)|)\, \chi^\pm_\epsilon(\pi_\gamma(x))\, \pi_{\gamma^\perp}(x). \tag{3.16}$$

The norm used in the argument for $\chi_{\kappa_0\epsilon}$ is induced by the scalar product on the finite-dimensional space E^{cu}_0. Hence, it is differentiable. Observe, that the terms $B^\pm$ are zero for $|x|_\alpha < \epsilon$. Indeed, for $|x|_\alpha \leq \epsilon$, we have

$$\left.\begin{array}{l} \chi_{\kappa_0\epsilon} \cdot \chi^+_\epsilon \cdot \vartheta^{cu}_\epsilon \\ \chi_\epsilon \cdot \chi^-_\epsilon \cdot \vartheta^{cu}_\epsilon \end{array}\right\} \equiv 0$$

by definition. The new nonlinearities $B^\pm_\epsilon(x)$ introduce an additional expansion for $x \in M^+$, which make the manifold M_ϵ overflowing due to $\beta_0 > \alpha^s$, see Figure 3. Equation (3.14) is well defined and we denote the corresponding nonlinear semiflow by $\Phi(t)$. Next we estimate the norm of the linearization of (3.14).

Lemma 3.7 *The linearization of (3.14) along a solution $x(t)$ staying in E_ϵ for $t \in [0, T]$ is given by*

$$\begin{aligned} DF(x)\, y &= \Big(-A\, P^s_0 + P^s_0\, Df(x)\Big)y - \beta_1\, \tfrac{1}{\epsilon}\Big(D\vartheta^s(P^{cu}_0\, x)\, P^{cu}_0\, y\Big)\, P^s_0\, x + \\ &\quad -\beta_1\, \vartheta^s_\epsilon(P^{cu}_0\, x)\, P^s_0\, y + \\ &\quad \Big(-A\, P^{cu}_0 + P^{cu}_0\, Df(x) + DB^+_\epsilon(x) + DB^-_\epsilon(x)\Big)\, \vartheta^{cu}_\epsilon(P^{cu}_0\, x)\, y + \\ &\quad \Big(-A\, P^{cu}_0\, x + P^{cu}_0\, f(x) + B^+_\epsilon(x) + B^-_\epsilon(x)\Big)\, \tfrac{1}{\epsilon} D\vartheta^{cu}(P^{cu}_0\, x)\, P^{cu}_0\, y \\ &=: -A\, P^s_0\, y + V(t)\, y. \end{aligned} \tag{3.17}$$

The linear operator $V(t)$ is bounded in $L(X^\alpha, X)$ with norm $|V(t)| \le C(\delta, \kappa_0, \beta_1)$ independent of ϵ.

Proof. First, the operator AP_0^{cu} is defined on the finite-dimensional eigenspace E_0^{cu}. The derivatives of the cutoff functions are bounded, because we have already extracted the factor $\frac{1}{\epsilon}$ in the equation above. If $D\vartheta^s(x)$ and $D\vartheta^{cu}(x)$ are nonzero, we have $|x|_\alpha \le \epsilon$. Thus the terms involving these derivatives are bounded uniformly in ϵ. Furthermore, we have

$$
\begin{aligned}
&DB^\pm(x)\, y = \\
&\quad D\chi_\epsilon^\pm(\pi_\gamma(x))\, D\pi_\gamma(x)\, y\, \chi_{\kappa_0\epsilon}\, \beta_0\, \pi_{\gamma^\perp}(x) + \chi_\epsilon^\pm\, \chi_{\kappa_0\epsilon}\, \beta_0\, D\pi_{\gamma^\perp}(x)\, y + \\
&\quad \chi_\epsilon^\pm\, D\chi\Big(\frac{1}{\kappa_0\epsilon}|\pi_{\gamma^\perp}(x)|\Big)\, \frac{1}{\kappa_0\, \epsilon}\, \beta_0 \cdot \Big\langle |\pi_{\gamma^\perp}(x)|^{-1} \cdot \pi_{\gamma^\perp}(x), D\pi_{\gamma^\perp}(x)\, y \Big\rangle \cdot \pi_{\gamma^\perp}(x).
\end{aligned}
$$

By using $|\pi_{\gamma^\perp}(x)|_\alpha \le \epsilon$ this derivative is easily seen to be bounded uniformly in ϵ. □

In order to define the graph transform, we introduce the set of Lipschitz continuous sections of the bundle E_ϵ

$$\Sigma_\epsilon := \Big\{\sigma : M_\epsilon \to X^\alpha \,|\, L(\sigma) \le 1,\, \sigma(x) \in E_\epsilon(x)\; \forall x \in M_\epsilon \Big\},$$

where we have used the local and global Lipschitz constants

$$
\begin{aligned}
L_x(\sigma) &:= \overline{\lim_{y\to x,\, y\in M_\epsilon}} \frac{|\sigma(y) - \sigma(x)|_\alpha}{|y - x|_\alpha} \\
L(\sigma) &:= \sup_{x\in M_\epsilon} L_x(\sigma).
\end{aligned}
$$

Σ_ϵ is a complete metric space if endowed with the C^0-norm $\|\sigma\| = \sup_{x\in M_\epsilon} |\sigma(x)|_\alpha$. The graph transform is defined for graphs σ contained in Σ_ϵ. Denote by g_σ a right inverse of $\pi_M \circ \Phi_T \circ (id + \sigma)$ defined on M_ϵ, that is

$$\pi_M\Big(\Phi_T(y + \sigma(y))\Big)\Big|_{y=g_\sigma(x)} = x \qquad \text{for } x \in M_\epsilon.$$

Here, Φ_T denotes the time T-map of the semiflow of (3.14). The graph transform is the map

$$\Phi_T^\#(\sigma) := \pi_E \circ \Phi_T \circ (id + \sigma) \circ g_\sigma.$$

In words, the graph of the image $\Phi_T^\#(\sigma)$ describes the surface $\Phi_T\,(id + \sigma)(M_\epsilon)$. At this point it is not clear whether the map $\Phi_T^\#$ is defined at all. We have to verify the following points:

- $\Phi_T^\#$ is well defined, i.e. the right inverse of g_σ is defined for each $\sigma \in \Sigma_\epsilon$,

- $\Phi_T^\# : \Sigma_\epsilon \to \Sigma_\epsilon$,
- $\Phi_T^\#$ is a contraction.

The unique fixed point σ_* of $\Phi_T^\#(\sigma)$ guaranteed by Banach's fixed point theorem will then be identified with the invariant manifold W_{hom}^{cu}. Before we start proving these claims we state some basic well-known lemmata needed in the following.

Lemma 3.8 *Consider the equations*

$$\dot{x} + Ax = f(x) \tag{3.18}$$

$$\dot{x} + Ax = f(x) + g(x,t) \tag{3.19}$$

with $f, g \in C^{1,\rho}(X^\alpha, X)$ *and* $|g|_1 < \eta$. *Then the estimates*

$$\left.\begin{array}{l} |x_1(T,\cdot) - x_2(T,\cdot)|_\alpha \\ |D_x(x_1(T,\cdot) - x_2(T,\cdot))|_\alpha \end{array}\right\} \le C(T)\, C(A,f)\, \mathrm{o}_\eta.$$

hold for the differences $x_1(T,x_0) - x_2(T,x_0)$ *of two solutions* x_1 *and* x_2 *for (3.18) and (3.19) to the same initial point* x_0, *respectively.*

Proof. See [Hen81, Thm. 3.4.1, 3.4.4 and Lem. 7.1.1]. □

Lemma 3.9 *For a differentiable function* g *we have*

$$g(x+y) - g(x+\tilde{y}) = Dg(x)\,(y-\tilde{y}) + |y-\tilde{y}|\, \mathrm{o}_{|y|+|\tilde{y}|}.$$

Proof. Indeed

$$\begin{aligned} g(x+y) - g(x+\tilde{y}) &= \textstyle\int_0^1 Dg(x+\tilde{y}+s(y-\tilde{y}))\,ds\,(y-\tilde{y}) \\ &= Dg(x)\,(y-\tilde{y}) + \textstyle\int_0^1 \left[Dg(x+\tilde{y}+s(y-\tilde{y})) - Dg(x)\right] ds\,(y-\tilde{y}) \end{aligned}$$

and $Dg(x+\tilde{y}+s(y-\tilde{y})) - Dg(x)$ is of the order $\mathrm{o}_{|y|+|\tilde{y}|}$. □

The function $\pi_M \circ \Phi_T \circ (id+\sigma)$ will now be investigated separately on each M_ϵ^i for $i = f, \pm, l$. The strategy is to divide the map Φ_T into a "nice" and a "small" part.

3.3.1 The global part of M: $\pi_M(x) \in M_\epsilon^f$

In this region, the original and the modified equations (2.1) and (3.14) coincide. Thus $D\Phi_T(q(t)) = \mathcal{T}(t+T,t)$ and we can apply Lemma 3.1 in the following. The first lemma will guarantee that the manifold M_ϵ^f overflows. Remember the definition (3.13)

$$R(t) = \epsilon(Ke^{(\alpha^s+\nu)T})^{-\frac{t+T_1}{T}}$$

for $t \in [-T_1, T_1]$.

Lemma 3.10 *There exists numbers $\delta_0 > 0$ and $\epsilon_0 > 0$, such that for all $\delta < \delta_0$, $\epsilon < \epsilon_0$ and $t \in [-T_1, T_1]$*

$$U_{R(t+T)}(q(t+T)) \cap M_\epsilon^f \;\subset\; \pi_M \circ \Phi_T \circ (id + \sigma)\, \Big(U_{R(t)}(q(t)) \cap M_\epsilon^f\Big)$$
$$\pi_E \circ \Phi_T \circ (id + \sigma)\, \Big(U_{R(t)}(q(t)) \cap M_\epsilon^f\Big) \;\subset\; U_{R(t+T)}(q(t+T)) \cap E_\epsilon.$$

Moreover, $\delta_0 > 0$ depends only on $C(T)$.

Proof. Throughout, the index i equals $i = cu, s$. We decompose

$$\begin{aligned}
\pi_M \circ \Phi_T(q(t)+v) &= \pi_M(q(t+T)) + D\pi_M(q(t+T))\, D\Phi_T(q(t))\, v + N(t,v) \\
&= \pi_M(q(t+T)) + Q^{cu}(t+T)\, A(t)\, v + N(t,v), \\
\pi_E \circ \Phi_T(q(t)+v) &= \pi_E(q(t+T)) + D\pi_E(q(t+T))\, D\Phi_T(q(t))\, v + N(t,v) \\
&= \pi_E(q(t+T)) + Q^{s}(t+T)\, A(t)\, v + N(t,v),
\end{aligned}$$

see Lemma 3.6 for the definition of Q^{cu}. Here, the Lipschitz constant $L(N) \le C_0\, o_\epsilon$ of the nonlinearity N is small by Lemma 3.9. Moreover, by definition $|Q^i(t) - P^i(t)|_\alpha \le \delta$. We will prove the lemma using the mappings $Q^{cu}(t+T)\, A(t) + N(t,x)$ and $Q^s(t+T)\, A(t) + N(t,v)$, respectively, with

$$\{v \mid |Q^{cu}(t)\, v|_\alpha \le R(t)\}$$
$$\{v \mid |Q^{s}(t)\, v|_\alpha \le R(t)\}.$$

This is sufficient, because $M_\epsilon^f \cap U_\epsilon(q)$ and the tangent space $T_q M_\epsilon^f$ of M_ϵ^f at q are ϵ-close in the C^1-norm and the estimates proved below are independent of ϵ.
We have $|P^i(t)\, v|_\alpha \ge (1-\delta)\, |v|_\alpha$ for any v, such that $Q^i(t)\, v = v$. Indeed,

$$\delta\, |v|_\alpha \ge |(P^i(t) - Q^i(t))\, v|_\alpha \ge |v|_\alpha - |P^i(t)\, v|_\alpha.$$

We observe that $R(t+T) = R(t)\, K^{-1}\, e^{-(\alpha^s+\nu)T}$. Furthermore

$$\begin{aligned}
Q^i(t+T)\, (A(t) + N(t,\cdot)) \;=\;& A(t)\, P^i(t)\, Q^i(t) + \Big(Q^i(t+T) - P^i(t+T)\Big)\, A(t) + \\
& Q^i(t+T)\, N(t,\cdot) + A(t)\, \Big(Q^i(t) - P^i(t)\Big).
\end{aligned}$$

Thus we obtain

$$\begin{aligned}
|Q^{cu}(t+T)\, (A(t)\, v + N(t,v))|_\alpha \;\ge\;& K^{-1} e^{-\alpha^s T}\, |Q^{cu}(t)\, v|_\alpha \\
& -2\, R(t)\, \delta\, \Big(2\, |A(t)|_\alpha + K^{-1} e^{-\alpha^s T} + |Q^{cu}(t)|_\alpha\Big) \\
|Q^{s}(t+T)\, (A(t)\, v + N(t,v))|_\alpha \;\le\;& K e^{-\alpha^{ss} T}\, |Q^{s}(t)\, v|_\alpha \\
& +2\, R(t)\, \delta\, \Big(2\, |A(t)|_\alpha + K e^{-\alpha^{ss} T} + |Q^{s}(t)|_\alpha\Big).
\end{aligned}$$

We define

$$C = 2 \sup_{t \in [-T_1, T_1]} \Big(\max(K^{-1} e^{-\alpha^s T} + |Q^{cu}(t)|_\alpha,\, K e^{-\alpha^{ss} T} + |Q^{s}(t)|_\alpha) + 2\, |A(t)|_\alpha\Big)$$

and observe that C depends only on the constants mentioned in the statement of the lemma. Therefore

$$\begin{aligned}
|Q^{cu}(t+T)\,(A(t)\,v+N(t,v))|_\alpha &\geq K^{-1}e^{-\alpha^s T}\,|Q^{cu}(t)\,v|_\alpha - C\,R(t)\,\delta \\
|Q^{s}(t+T)\,(A(t)\,v+N(t,v))|_\alpha &\leq K e^{-\alpha^{ss} T}\,|Q^{s}(t)\,v|_\alpha + C\,R(t)\,\delta.
\end{aligned}$$

Now the first inclusion of the claim is equivalent to

$$|Q^{cu}(t)\,v|_\alpha = R(t) \implies |Q^{cu}(t+T)\,(A(t)\,v+N(t,v))|_\alpha > R(t+T).$$

Substituting the above proved estimate and using the definition of $R(t)$ we obtain

$$\begin{aligned}
K^{-1}e^{-\alpha^s T}\,R(t)\,(1-C\,\delta) &> R(t+T) && \iff \\
K^{-1}e^{-\alpha^s T}\,(1-C\,\delta) &> K^{-1}e^{-(\alpha^s+\nu)T} && \iff \\
e^{\nu T}\,(1-C\,\delta) &> 1
\end{aligned}$$

and the last inequality is satisfied for $\delta < \delta_0 := C^{-1}(1-e^{-\nu T})$. Here $\delta_0 > 0$ is positive due to the assumptions on ν, α^s and T. Likewise the second inequality is equivalent to

$$|Q^{s}(t)\,v|_\alpha < R(t) \implies |Q^{s}(t+T)\,(A(t)\,v+N(t,v))|_\alpha < R(t+T)$$

and thus to

$$\begin{aligned}
K e^{-\alpha^{ss}T}\,R(t)\,(1+C\,\delta) &< R(t+T) && \iff \\
K e^{-\alpha^{ss}T}\,(1+C\,\delta) &< K^{-1}e^{-(\alpha^s+\nu)T} && \iff \\
K^2 e^{(\alpha^s+\nu-\alpha^{ss})T}\,(1+C\,\delta) &< 1.
\end{aligned}$$

Again the last inequality is fulfilled for all $\delta < \delta_0$, where $\delta_0 = C^{-1}(K^{-2}e^{(\alpha^{ss}-\alpha^s-\nu)T}-1) > 0$. Positiveness of δ_0 follows as above. □

Lemma 3.11 *The mapping $\pi_M \circ \Phi_T \circ (id+\sigma)$ possesses a Lipschitz continuous right inverse g_σ defined on $U_{2\epsilon}(q(t)) \cap M^f_\epsilon$ for each $q(t) \in M^f_\epsilon$ uniformly in $\sigma \in \Sigma_\epsilon$. Moreover,*

$$L_y(g_\sigma) \leq K e^{\alpha^s T} + \mathrm{o}_\epsilon + C(T)\,\delta,$$

for each $y = \pi_M \circ \Phi_T \circ (id+\sigma)(x)$ such that $x, y \in M^f_\epsilon$.

Proof. By using Lemma 3.9 we obtain

$$\begin{aligned}
&\Big|\pi_M\Big(\Phi_T(q+u+\sigma(q+u))\Big) - \pi_M\Big(\Phi_T(q+\tilde u+\sigma(q+\tilde u))\Big)\Big|_\alpha \\
&\geq \Big|D\pi_M(\Phi_T(q))\,D\Phi_T(q)\Big(u+\sigma(q+u)-\tilde u-\sigma(q+\tilde u)\Big)\Big|_\alpha - \mathrm{o}_\epsilon\,|u-\tilde u|_\alpha \\
&\geq \Big|P^{cu}(t+T)\,D\Phi_T(q(t))\,\Big(u+\sigma(q+u)-\tilde u-\sigma(q+\tilde u)\Big)\Big|_\alpha - (\mathrm{o}_\epsilon + C(T)\,\delta)\,|u-\tilde u|_\alpha \\
&\geq \Big|D\Phi_T(q(t))\,P^{cu}(t)\,\Big(u+\sigma(q+u)-\tilde u-\sigma(q+\tilde u)\Big)\Big|_\alpha - (\mathrm{o}_\epsilon + C(T)\,\delta)\,|u-\tilde u|_\alpha \\
&\geq \Big|D\Phi_T(q(t))\,P^{cu}(t)\,(u-\tilde u)\Big|_\alpha - \Big|D\Phi_T(q(t))\,P^{cu}(t)\,(\sigma(q+u)-\sigma(q+\tilde u))\Big|_\alpha - \\
&\quad (\mathrm{o}_\epsilon + C(T)\,\delta)\,|u-\tilde u|_\alpha.
\end{aligned}$$

Now we estimate the term involving the graph σ using the facts $Q^{cu}(\pi_\gamma(q+u))\,\sigma(q+u) = 0$ and $|\sigma(q+u)|_\alpha < \epsilon$ together with Lemma 3.6(i).

$$\begin{aligned}
&\left|D\Phi_T(q(t))\,P^{cu}(t)\,(\sigma(q+u)-\sigma(q+\tilde{u}))\right|_\alpha \\
&\quad\leq\; C(T)\left(\left|\left(P^{cu}(q)-Q^{cu}(\pi_\gamma(q+\tilde{u}))\right)(\sigma(q+u)-\sigma(q+\tilde{u}))\right|_\alpha + \right.\\
&\qquad\left.\left|\left(Q^{cu}(\pi_\gamma(q+u))-Q^{cu}(\pi_\gamma(q+\tilde{u}))\right)\sigma(q+u)\right|_\alpha\right)\\
&\quad\leq\; (\mathrm{o}_\epsilon + C(T)\,\delta)\,|u-\tilde{u}|_\alpha.
\end{aligned}$$

Moreover, we have

$$\left|D\Phi_T(q(t))\,P^{cu}(t)\,(u-\tilde{u})\right|_\alpha \geq K^{-1}e^{-\alpha^s T}\,(1-C(T)\,\delta)\,|u-\tilde{u}|_\alpha$$

using Lemma 3.1 and 3.6(ii). Therefore we conclude

$$\begin{aligned}
&\left|\pi_M\left(\Phi_T(q+u+\sigma(q+u))\right)-\pi_M\left(\Phi_T(q+\tilde{u}+\sigma(q+\tilde{u}))\right)\right|_\alpha\\
&\quad\geq\; K^{-1}e^{-\alpha^s T}\,|u-\tilde{u}|_\alpha + (\mathrm{o}_\epsilon + C(T)\,\delta)\,|u-\tilde{u}|_\alpha.
\end{aligned}$$

The use of the constant $C(T)$ is justified. Indeed, the estimates for these terms depend only on the norms of $D\Phi_T$ and the projections. The lemma follows now from the Lipschitz inverse function theorem [Shu80, ch.5 Thm. I.1]. □

Next we compute the Lipschitz constant of $\pi_E \circ \Phi_T \circ (id+\sigma)$ on M^f_ϵ.

Lemma 3.12 *For any $x \in M^f_\epsilon$ and $\epsilon \leq \epsilon_0$ such that $\epsilon_0 > 0$ is small the following estimate hold*

$$L_x(\pi_E \circ \Phi_T \circ (id+\sigma)) \leq Ke^{-\alpha^{ss}T} + \mathrm{o}_\epsilon + C(T)\,\delta.$$

Proof. Using Lemma 3.6 we obtain near $q = q(t(\tau_0)) \in M^f_\epsilon$

$$\begin{aligned}
&\left|\pi_E\left(\Phi_T(q+u+\sigma(q+u))\right)-\pi_E\left(\Phi_T(q+\tilde{u}+\sigma(q+\tilde{u}))\right)\right|_\alpha\\
&\quad\leq\; \left|D\pi_E(\Phi_T(q))\,D\Phi_T(q)\,(u+\sigma(q+u)-\tilde{u}-\sigma(q+\tilde{u}))\right|_\alpha + \mathrm{o}_\epsilon\,|u-\tilde{u}|_\alpha\\
&\quad\leq\; \left|P^s(t+T)\,D\Phi_T(q)\,(u+\sigma(q+u)-\tilde{u}-\sigma(q+\tilde{u}))\right|_\alpha + (\mathrm{o}_\epsilon + C(T)\,\delta)\,|u-\tilde{u}|_\alpha\\
&\quad\leq\; \left|D\Phi_T(q)\,P^s(t)\,(u+\sigma(q+u)-\tilde{u}-\sigma(q+\tilde{u}))\right|_\alpha + (\mathrm{o}_\epsilon + C(T)\,\delta)\,|u-\tilde{u}|_\alpha\\
&\quad\leq\; \left|D\Phi_T(q)\,P^s(t)\,(\sigma(q+u)-\sigma(q+\tilde{u}))\right|_\alpha + (\mathrm{o}_\epsilon + C(T)\,\delta)\,|u-\tilde{u}|_\alpha\\
&\quad\leq\; Ke^{-\alpha^{ss}T}\,|u-\tilde{u}|_\alpha + (\mathrm{o}_\epsilon + C(T)\,\delta)\,|u-\tilde{u}|_\alpha.
\end{aligned}$$

The estimate for $D\Phi_T(q(t))\,P^s(t)$ follows from Lemma 3.1. □

Now, we consider the composition $\pi_E \circ \Phi_T \circ (id+\sigma)\circ g_\sigma$ locally near each $q(t) \in M^f_\epsilon$. Here, we choose any right inverse g_σ guaranteed by Lemma 3.11, which might be not unique at this moment.

Lemma 3.13 *Choose $\delta < C(T)^{-1}$ and $\epsilon \leq \epsilon_0$. Then, we have for each $y \in M_\epsilon^f$*

$$L_y(\pi_E \circ \Phi_T \circ (id + \sigma) \circ g_\sigma) \leq 1.$$

Moreover, the mapping $\pi_E \circ \Phi_T \circ (id + \sigma) \circ g_\sigma$ is a local section of $E_\epsilon^s|_{M_\epsilon^f}$.

Proof. We obtain the estimate for the Lipschitz constant by using (3.2) and the Lemmata 3.11 and 3.12. Indeed,

$$\begin{aligned} L_y(\Phi_T^\#(\sigma)) &\leq L_{g_\sigma(y)}(\pi_E \circ \Phi_T \circ (id + \sigma))\, L_y(g_\sigma) \\ &\leq (Ke^{-\alpha^{ss}T} + o_\epsilon + C(T)\,\delta)\,(Ke^{\alpha^s T} + o_\epsilon + C(T)\,\delta) \\ &\leq K^2\, e^{-(\alpha^{ss}-\alpha^s)T} + o_\epsilon + C(T)\,\delta \\ &< 1. \end{aligned}$$

The mapping $\pi_E \circ \Phi_T \circ (id + \sigma) \circ g_\sigma$ is a section, because g_σ is a right inverse of $\pi_M \circ \Phi_T \circ (id + \sigma)$. By Lemma 3.10 we know that the image of $\pi_E \circ \Phi_T \circ (id + \sigma) \circ g_\sigma$ is contained in $E_\epsilon^s|_{M_\epsilon^f}$. This proves the lemma. □

3.3.2 The tail parts of M: $\pi_M(x) \in M_\epsilon^\pm$

We will restrict ourselves on the case $\pi_M(x) \in M_\epsilon^+$. The same result with analogous proofs is valid for $\pi_M(x) \in M_\epsilon^-$. The differential equation (3.14) on M_ϵ^+ is given by

$$\begin{aligned} \dot{x} &= -Ax + f(x) + B_\epsilon^+(x) \\ &= -Ax + f(x) + \chi_\epsilon^+(\pi_\gamma(x))\, \chi_{\kappa_0 \epsilon}(|\pi_{\gamma^\perp}(x)|)\, \beta_0\, \pi_{\gamma^\perp}(x), \end{aligned} \tag{3.20}$$

for $\pi_M(x) \in M_\epsilon^+$. In the subset M_ϵ^+, the mapping $\pi_M(x)$ satisfies $|D\pi_M(x) - P_0^{cu}|_\alpha \leq \delta$ by definition. Thus, the tangent spaces of M_ϵ^+ and the eigenspace E_0^{cu} are δ-close to each other. The linearization along a solution $x(t) \in E_\epsilon$ of (3.20) for $t \in [0, T]$ is given by

$$\dot{y} = \Big(-A + DB_\epsilon^+(x(t)) + Df(x(t)) \Big) y. \tag{3.21}$$

We will first consider the equation

$$\begin{aligned} (3.22)\ \dot{y} &= -Ay + DB_\epsilon^+(x)\, y \\ &= -Ay + D\chi_\epsilon^+(\pi_\gamma(x))\, D\pi_\gamma(x)\, y\, \chi_{\kappa_0 \epsilon}\, \beta_0\, \pi_{\gamma^\perp}(x) + \chi_\epsilon^+\, \chi_{\kappa_0\epsilon}\, \beta_0\, D\pi_{\gamma^\perp}(x)\, y \,+ \\ &\qquad \chi_\epsilon^+\, D\chi\Big(\frac{1}{\kappa_0 \epsilon} |\pi_{\gamma^\perp}(x)|\Big)\, \frac{1}{\kappa_0\, \epsilon}\, \beta_0 \cdot \Big\langle |\pi_{\gamma^\perp}(x)|^{-1} \cdot \pi_{\gamma^\perp}(x), D\pi_{\gamma^\perp}(x)\, y \Big\rangle \cdot \pi_{\gamma^\perp}(x), \end{aligned}$$

where the term coming from the nonlinearity is removed. We denote the linear semiflow of (3.22) by $\tilde{T}(t, x_0)$.

Lemma 3.14 *The derivative $DB^{\pm}(x)y = DB^{\pm}(x)\, P_0^{cu}\, y + o_\delta y$ depends only on $P_0^{cu}y$ up to an error of the size δ. Moreover, $|Df(x)| < \delta$ as an element of $L(X^\alpha, X)$. Thus, we subsume the term $o_\delta y$ into the nonlinearity $Df(x)$ and continue by considering $DB^{\pm}(x)\, P_0^{cu}\, y$. Remember, that $|D\chi_\epsilon^{\pm}(x)|$ does not depend on ϵ for $x \in M_\epsilon^{\pm}$. Moreover, $DB^{\pm}(x)$ is bounded uniformly in ϵ, i.e. $|DB_\epsilon^{+}(x)|_\alpha \le C(\kappa_0)$. Indeed, $|\pi_{\gamma^\perp}(x)| \le \kappa_0\, \epsilon$ for $x \in U_{2\epsilon}(0)$. Therefore the semiflow $\tilde{T}(t, x_0)$ is bounded by $|\tilde{T}(t, x_0)|_\alpha \le C(T, \kappa_0)$ for $t \in [0, T]$.*

Now we have to estimate equation (3.22). Observe, that the stable and unstable parts of (3.22) decouple due to Lemma 3.14. The stable part is given by

$$\frac{d}{dt} P_0^s\, y + A\, P_0^s\, y = 0. \tag{3.23}$$

The unstable part is given by the expression

$$\begin{aligned}
(3.24)\ \dot{u} &= -Au + DB_\epsilon^{+}(x)\, u \\
&= -Au + D\chi_\epsilon^{+}(\pi_\gamma(x))\, D\pi_\gamma(x)\, u\, \chi_{\kappa_0\epsilon}\, \beta_0\, \pi_{\gamma^\perp}(x) + \chi_\epsilon^{+}\, \chi_{\kappa_0\epsilon}\, \beta_0\, D\pi_{\gamma^\perp}(x)\, u\, + \\
&\quad \chi_\epsilon^{+}\, D\chi\Big(\frac{1}{\kappa_0 \epsilon} |\pi_{\gamma^\perp}(x)|\Big)\, \frac{1}{\kappa_0\, \epsilon}\, \beta_0 \cdot \Big\langle |\pi_{\gamma^\perp}(x)|^{-1} \cdot \pi_{\gamma^\perp}(x), D\pi_{\gamma^\perp}(x)\, u \Big\rangle \cdot \pi_{\gamma^\perp}(x) \\
&=: A(t)u,
\end{aligned}$$

where we have defined $P_0^{cu}\, y =: u$.

Lemma 3.15 *The minimum norm of the fundamental matrix $U(T)$ of $\dot{u} = A(t)\, u$ defined on E_0^{cu} can be estimated by*

$$m(U(T)) \ge e^{-\alpha^s T}.$$

Proof. Observe that the equation (3.24) is defined on the finite-dimensional space E_0^{cu}. Thus, we can take the scalar product $\langle A(t)\, u, u\rangle$ and obtain using the decomposition of $-A\, P_0^{cu}$ in the Jordan block and the diagonal part

$$\begin{aligned}
\langle A(t)\, u, u\rangle &= \langle \Delta u, u\rangle + \langle J\, u, u\rangle + D\chi_\epsilon^{+}(\pi_\gamma(x))\, D\pi_\gamma(x)\, u\, \chi_{\kappa_0\epsilon}\, \beta_0 \Big\langle \pi_{\gamma^\perp}(x), u\Big\rangle + \\
&\quad \chi_\epsilon^{+}\, D\chi\left(\frac{|\pi_{\gamma^\perp}(x)|}{\kappa_0\, \epsilon}\right) \frac{1}{\kappa_0\, \epsilon}\, \beta_0 \Big\langle \pi_{\gamma^\perp}(x), u\Big\rangle \cdot \Big\langle |\pi_{\gamma^\perp}(x)|^{-1} \cdot \pi_{\gamma^\perp}(x), D\pi_{\gamma^\perp}(x)\, u\Big\rangle + \\
&\quad \chi_\epsilon^{+}\, \chi_{\kappa_0\epsilon}\, \beta_0 \Big\langle D\pi_{\gamma^\perp}(x)\, u, u\Big\rangle.
\end{aligned}$$

Now, $\langle \pi_{\gamma^\perp}(x), D\pi_{\gamma^\perp}(x)\, u\rangle = \langle \pi_{\gamma^\perp}(x), u\rangle$ due to the definition of $\pi_{\gamma^\perp}$. In fact, for $x \in E_\epsilon^l$, the projection $D\pi_{\gamma^\perp}(x)$ is given by P_0^{cu} followed by a further projection in E_0^{cu} plus a small map in E_0^s. Moreover, $\pi_{\gamma^\perp}(x) \in E^{cu}$. This proves the claim about the scalar product.

Therefore, using $\mathrm{Re}\,\langle \Delta\, u, u\rangle - \|J\| \geq -\tilde{\alpha}^s > -\alpha^s$, we obtain

$$
\begin{aligned}
\frac{1}{|u|^2}\Big\langle A(t)\,u, u\Big\rangle &\geq -\tilde{\alpha}^s - C_0\,\epsilon\,(1 + C(T)\,K_\chi\,\beta_0) + \\
&\quad \chi_\epsilon^+\,D\chi\,\beta_0\,(\kappa_0\,\epsilon)^{-1}\,|\pi_{\gamma^\perp}(x)|^{-1}\Big\langle \pi_{\gamma^\perp}(x), u\Big\rangle^2 \frac{1}{|u|^2} - C(T)\,K_\chi\,\beta_0\,\delta + \beta_0\,(1 - C_0\,\delta) \\
&\geq\ -\tilde{\alpha}^s + C(T)\,\delta + o_\epsilon \ \geq\ -\alpha^s
\end{aligned}
$$

if only δ and ϵ are sufficiently small, the former only compared with the constant $C(T)$. For the last estimate we have used the orthogonality of the spectral projections. By [Har82, Lemma IV.4.2] the estimate $\frac{1}{|u|^2}\big\langle A(t)\,u, u\big\rangle \geq -\alpha^s$ implies the claimed estimate $m(U(t)) \geq e^{-\alpha^s t}$ for the minimum norm of the fundamental matrix $U(t)$ uniformly in $t \in [0, T]$. □

Now we have all the necessary informations to prove the existence of a local right inverse near $x \in E_\epsilon^+$.

Lemma 3.16 *For all $\delta < C(T)^{-1}$ and $\epsilon \leq \epsilon_0$ sufficiently small there exists a local right inverse g_σ of $\pi_M \circ \Phi_T \circ (id + \sigma)$ uniformly in $\sigma \in \Sigma_\epsilon$. Moreover,*

$$
\begin{aligned}
L_y(g_\sigma) &\leq e^{\alpha^s T} + C(T)\,\delta + \kappa_0\,\epsilon \\
L_x\Big(\pi_E \circ \Phi_T \circ (id + \sigma)\Big) &\leq K e^{-\alpha^{ss} T} + C(T)\,\delta
\end{aligned}
$$

holds for each $y = \pi_M \circ \Phi_T \circ (id + \sigma)(x)$ such that $x, y \in M_\epsilon^+$. Furthermore, the composition is Lipschitz continuous, satisfies

$$
L_x\Big(\pi_E \circ \Phi_T \circ (id + \sigma) \circ g_\sigma\Big) \leq 1
$$

and $\pi_E \circ \Phi_T \circ (id + \sigma) \circ g_\sigma$ is a local section of the bundle E_ϵ.

Proof. First, we show that the manifold M_ϵ^+ is overflowing. Choose $x \in M_\epsilon^+$ such that $|\pi_{\gamma^\perp}(x)|_\alpha = \kappa_0\,\epsilon$. Then, we obtain

$$
\begin{aligned}
\Big\langle \frac{d}{dt}\pi_{\gamma^\perp}(x), \pi_{\gamma^\perp}(x)\Big\rangle &= \Big\langle D\pi_{\gamma^\perp}(x)\Big(-Ax + f(x) + B_\epsilon^+(x)\Big), \pi_{\gamma^\perp}(x)\Big\rangle \\
&= \Big\langle D\pi_{\gamma^\perp}(x)\Big(-A(\pi_\gamma(x) + \pi_{\gamma^\perp}(x) + \pi_E(x)) + f(x) + B_\epsilon^+(x)\Big), \pi_{\gamma^\perp}(x)\Big\rangle \\
&= \Big\langle D\pi_{\gamma^\perp}(x)\Big(-A\pi_{\gamma^\perp}(x) + f(x) - f(\pi_\gamma(x)) + B_\epsilon^+(x)\Big) + \\
&\qquad D\pi_{\gamma^\perp}(x)\Big(-A\pi_\gamma(x) + f(\pi_\gamma(x))\Big), \pi_{\gamma^\perp}(x)\Big\rangle \\
&= \Big\langle D\pi_{\gamma^\perp}(x)\Big(-A\pi_{\gamma^\perp}(x) + f(x) - f(\pi_\gamma(x)) + B_\epsilon^+(x)\Big), \pi_{\gamma^\perp}(x)\Big\rangle \\
&= \Big\langle P_0^{cu}\,P_0^+\Big(id + Dh(Q_0^+\,x)\,Q_0^+\Big)\Big(-A\,P_0^{cu}\,P_0^+\,(x + h(Q_0^+\,x)) + \\
&\qquad f(x) - f(\pi_\gamma(x)) + \beta_0\,P_0^{cu}\,P_0^+\,(x + h(Q_0^+\,x))\Big), \pi_{\gamma^\perp}(x)\Big\rangle \\
&= \Big\langle -A\,\pi_{\gamma^\perp}(x) + \beta_0\,\pi_{\gamma^\perp}(x) + D\pi_{\gamma^\perp}(x)\,\Big(f(x) - f(\pi_\gamma(x))\Big), \pi_{\gamma^\perp}(x)\Big\rangle \\
&\geq (\beta_0 - \alpha^s - C(T)\,\delta)\,(\kappa_0\,\epsilon)^2.
\end{aligned}
$$

Indeed, the term $D\pi_{\gamma^\perp}(x)\big(-A\pi_\gamma(x)+f(\pi_\gamma(x))\big)$ vanishes, because $-A\pi_\gamma(x)+f(\pi_\gamma(x))$ is contained in the image of $D\pi_\gamma(x)$ and $D\pi_{\gamma^\perp}(x)\,D\pi_\gamma(x)=0$ is zero by definition. Furthermore, $|Df|\le\delta$ and thus $|f(x)-f(\pi_\gamma(x))|\le\delta\,|x-\pi_\gamma(x)|\le\delta\,\kappa_0\,\epsilon$. Therefore, the norm of $\pi_{\gamma^\perp}(x)$ grows and thus $x(t)$ have to leave the bundle E_ϵ. This proves, that M_ϵ^+ is overflowing.

Next, we consider the linearization along a solution $x(t)$ of (3.14), which stays in E^+ for all $t\in[0,T]$. Due to the fact $|Df|\le\delta$ and Lemma 3.14, the difference of the semiflows of the full linearized equation (3.21) and the pseudo equation (3.22) can be estimated by

$$|\mathcal{T}(T,x_0)-\tilde{\mathcal{T}}(T,x_0)|_\alpha\le C(T)\,\delta. \tag{3.25}$$

Furthermore, the tangent space of M_ϵ^+ and the space E_0^{cu} are δ-close. This yields for $u_1,u_2\in M_\epsilon^+$ close to x_0

$$\begin{aligned}(3.26)\qquad &|\pi_M\circ\Phi_T\circ(u_1+\sigma(u_1))-\pi_M\circ\Phi_T\circ(u_2+\sigma(u_2))|_\alpha\\ \ge\ &|D\pi_M(x_0)\,\mathcal{T}(T,x_0)\,(u_1+\sigma(u_1)-u_2-\sigma(u_2))|_\alpha-\delta\,C(T)\,|u_1-u_2|_\alpha\\ \ge\ &|P_0^{cu}\,\mathcal{T}(T,x_0)\,(u_1+\sigma(u_1)-u_2-\sigma(u_2))|_\alpha-\delta\,C(T)\,|u_1-u_2|_\alpha\\ \ge\ &|\tilde{\mathcal{T}}(T,x_0)\,P_0^{cu}\,(u_1+\sigma(u_1)-u_2-\sigma(u_2))|_\alpha-\delta\,C(T)\,|u_1-u_2|_\alpha\\ \ge\ &e^{-\alpha^sT}\,|u_1-u_2|_\alpha-\delta\,C(T)\,|u_1-u_2|_\alpha,\end{aligned}$$

Therefore, a local right inverse exists by [Shu80, ch.5 Thm. I.1]. Moreover, we have

$$\begin{aligned}(3.27)\qquad &|\pi_E\circ\Phi_T\circ(u_1+\sigma(u_1))-\pi_E\circ\Phi_T\circ(u_2+\sigma(u_2))|_\alpha\\ \le\ &|D\pi_E(x_0)\,\mathcal{T}(T,x_0)\,(u_1+\sigma(u_1)-u_2-\sigma(u_2))|_\alpha+\delta\,C(T)\,|u_1-u_2|_\alpha\\ \le\ &|P_0^s\,\mathcal{T}(T,x_0)\,(u_1+\sigma(u_1)-u_2-\sigma(u_2))|_\alpha+\delta\,C(T)\,|u_1-u_2|_\alpha\\ \le\ &|\tilde{\mathcal{T}}(T,x_0)\,P_0^s\,(u_1+\sigma(u_1)-u_2-\sigma(u_2))|_\alpha+\delta\,C(T)\,|u_1-u_2|_\alpha\\ \le\ &e^{-\alpha^{ss}T}\,|u_1-u_2|_\alpha+\delta\,C(T)\,|u_1-u_2|_\alpha.\end{aligned}$$

Indeed, $|P_0^s\,(u_1-u_2)|_\alpha\le\delta\,|u_1-u_2|_\alpha$ and the estimate for $\tilde{\mathcal{T}}(T,x_0)\,P_0^s$ follows from the decoupled equation (3.23). Finally, we have to estimate the Lipschitz constant of $\Phi_T^\#(\sigma)$. This follows now easily using (3.26) and (3.27). By (3.23), the image of $\Phi_T^\#(\sigma)$ is again contained in E_ϵ. □

We have not proved that $\Phi_T^\#(\sigma)$ is well defined, because uniqueness of the right inverse is still not clear.

3.3.3 The local part of M: $\pi_M(x) \in M^l_\epsilon$

We repeat equation (3.14) on M^l_ϵ:

$$
(3.28) \qquad \begin{aligned}
\dot{x} &= -A\,P_0^s\,x + P_0^s\,f(x) - \beta_1\,\vartheta^s_\epsilon(P_0^{cu}\,x)\,P_0^s\,x + \\
&\quad \Big(-A\,P_0^{cu}\,x + P_0^{cu}\,f(x) + B^+_\epsilon(x) + B^-_\epsilon(x)\Big)\,\vartheta^{cu}_\epsilon(P_0^{cu}\,x) \\
&:= -A\,P_0^s\,x + P_0^s\,f(x) - \beta_1\,\vartheta^s_\epsilon(P_0^{cu}\,x)\,P_0^s\,x + \\
&\quad \bigg(-A\,P_0^{cu}\,x + P_0^{cu}\,f(x) + \beta_0\Big(\chi_{\kappa_0\epsilon}(|\pi_{\gamma^\perp}(x)|)\,\chi^+_\epsilon(\pi_\gamma(x)) + \\
&\quad \chi_\epsilon(|\pi_{\gamma^\perp}(x)|)\,\chi^-_\epsilon(\pi_\gamma(x))\Big)\,\pi_{\gamma^\perp}(x)\bigg)\,\vartheta^{cu}_\epsilon(P_0^{cu}\,x).
\end{aligned}
$$

Assume that $x(t) \in M^l_\epsilon$ is a solution of (3.28). Then we consider the variational equation (3.17) along $x(t)$. First, we will again neglect the influence of the nonlinearity

$$
(3.29) \qquad \begin{aligned}
&\Big(P_0^s\,Df(x) + P_0^{cu}\,Df(x)\,\vartheta^{cu}_\epsilon(P_0^{cu}\,x) + P_0^{cu}\,f(x)\,\epsilon^{-1}\,D\vartheta^{cu}(\epsilon^{-1}P_0^{cu}\,x)\,P_0^{cu}\Big)\,y - \\
&\beta_1\,\epsilon^{-1}\,\Big(D\vartheta^s(\epsilon^{-1}\,P_0^{cu}\,x)\,P_0^{cu}\,y\Big)\,P_0^s\,x
\end{aligned}
$$

in the variational equation along $x(t)$. For this term, we have the following estimate.

Lemma 3.17 *The operators in (3.29) are bounded by*

$$
\Big|D\Big(P_0^s\,f(x) + P_0^{cu}\,f(x)\,\vartheta^{cu}_\epsilon(P_0^{cu}\,x)\Big)\Big|_\alpha \le o_\epsilon.
$$

and

$$
\Big|\beta_1\,\epsilon^{-1}\,\Big(D\vartheta^s(\epsilon^{-1}\,P_0^{cu}\,x)\,y\Big)\,P_0^s\,x\Big|_\alpha \le \beta_1\,\kappa_1\,K_\vartheta|y|_\alpha
$$

uniformly in ϵ.

Proof. First of all, we conclude

$$
\Big|P_0^s\,Df(x) + P_0^{cu}\,Df(x)\,\vartheta^{cu}_\epsilon(P_0^{cu}\,x) + P_0^{cu}\,f(x)\,\epsilon^{-1}\,D\vartheta^{cu}(\epsilon^{-1}\,P_0^{cu}\,x)\,P_0^{cu}\Big|_\alpha \le C_0\,K_\vartheta\,\epsilon^\rho = o_\epsilon.
$$

Indeed, for $x \in M^l_\epsilon$ we have $|f(x)| \le C_0\,|x|_\alpha^{1+\rho}$, $|Df(x)| \le C_0\,|x|^\rho_\alpha$ and $|x|_\alpha \le 3\epsilon$. This proves the first claim. The second claim follows from

$$
\Big|\beta_1\,\epsilon^{-1}\,\Big(D\vartheta^s(\epsilon^{-1}\,P_0^{cu}\,x)\,y\Big)\,P_0^s\,x\Big|_\alpha \le \beta_1\,\epsilon^{-1}\,K_\vartheta\,|P_0^s\,x|_\alpha|P_0^s\,y|_\alpha \le \beta_1\,\kappa_1\,K_\vartheta\,|P_0^s\,y|_\alpha.
$$

□

The modified linearized equation without the operator in (3.29) is then given by

$$
(3.30) \qquad \begin{aligned}
\dot{y} &= -A\,P_0^s\,y - \beta_1\,\vartheta^s_\epsilon(P_0^{cu}\,x)\,P_0^s\,y + \\
&\quad \Big(-A\,P_0^{cu} + DB^+_\epsilon(x) + DB^-_\epsilon(x)\Big)\,\vartheta^{cu}_\epsilon(P_0^{cu}\,x)\,y + \\
&\quad \Big(-A\,P_0^{cu}\,x + B^+_\epsilon(x) + B^-_\epsilon(x)\Big)\,\epsilon^{-1}\,D\vartheta^{cu}(\epsilon^{-1}\,P_0^{cu}\,x)\,P_0^{cu}\,y.
\end{aligned}
$$

For this equation, the minimum norm will depend on the initial point $x(0)$, i.e. we have a relative and not an absolute normal hyperbolic equation in the terminology of [HPS77]. We define

$$a(x_0) := \int_{[0,T]} \varphi_{I(x_0)}(t)\, dt.$$

Here, $\varphi_{I(x_0)}$ denotes the characteristic function of the set $I(x_0)$ given by

$$I(x_0) := \Big\{ t \in [0,T] \,|\, P_0^{cu}\, x(t) \in \mathcal{C}\, U_\eta(\mathcal{A}) \Big\}.$$

Hence, $I(x_0)$ is the set of those time points, for which the projection of the solution $x(t)$ onto E_0^{cu} is contained in the set $\mathcal{C}\, U_\eta(\mathcal{A})$, see (3.12). In other words, for these points $\vartheta_\epsilon^{cu}(P_0^{cu}\, x(t)) \equiv 1$.

Lemma 3.18 *For the linear semiflow $\tilde{\mathcal{T}}(T, x_0)$ of the modified equation (3.30), the following estimates hold*

$$\begin{aligned} m(\tilde{\mathcal{T}}(T, x_0)\, P_0^{cu}) &\geq e^{-\alpha^s(T - a(x_0))}\, e^{-\gamma_1\, a(x_0)} \\ |\tilde{\mathcal{T}}(T, x_0)\, P_0^s|_\alpha &\leq K e^{-\alpha^{ss} T}\, e^{-\beta_1\, a(x_0)}. \end{aligned}$$

Here, $\gamma_1 = C_0\, K_\vartheta\, C(\kappa_0)$ (see (3.34)) and β_1 has already been defined in (3.15).

Proof. We rewrite equation (3.30) using the coordinates $v = P_0^{cu}\, y$, $w = P_0^s\, y$

$$\begin{aligned} \dot{v} &= \vartheta_\epsilon^{cu}(P_0^{cu}\, x)\Big(-A + DB_\epsilon^+(x) + DB_\epsilon^-(x)\Big)\, v\, + \\ &\quad \Big(-A\, P_0^{cu}\, x + B_\epsilon^+(x) + B_\epsilon^-(x)\Big)\, \epsilon^{-1}\, D\vartheta^{cu}(\epsilon^{-1}\, P_0^{cu}\, x)\, v \end{aligned} \tag{3.31}$$

and

$$\dot{w} = -Aw - \beta_1\, \vartheta_\epsilon^s(P_0^{cu}\, x)\, w. \tag{3.32}$$

Here, we have used Lemma 3.14. Hence, (3.30) decouples into (3.31) and (3.32). We will first compute the minimum norm of $\tilde{\mathcal{T}}(T, x_0)$ on E_0^{cu}. To that end, we estimate (3.31) separately for $t \notin I(x_0)$ and $t \in I(x_0)$.

As long as $t \notin I(x_0)$, the equalities $\vartheta_\epsilon^{cu} \equiv 1$ and $D\vartheta_\epsilon^{cu} \equiv 0$ hold. Thus, (3.31) transforms into

$$\dot{v} = \Big(-A + DB_\epsilon^+(x) + DB_\epsilon^-\Big)\, v.$$

For $0 \leq t \leq t + \tau \leq T$ and $[t, t+\tau] \cap I(x_0) = \emptyset$, this yields the estimate

$$m(\tilde{\mathcal{T}}(t+\tau, t, x_0)) \geq e^{-\alpha^s \tau} \tag{3.33}$$

for the minimum norm as in the paragraph 3.3.2. Now suppose $t \in I(x_0)$. Then

$$\begin{aligned}(3.34)\qquad & \Big|\vartheta_\epsilon^{cu}(P_0^{cu}\,x)\left(-A+DB_\epsilon^+(x)+DB_\epsilon^-\right)v+ \\ & \qquad \left(-A\,P_0^{cu}\,x+B_\epsilon^+(x)+B_\epsilon^-(x)\right)\epsilon^{-1}\,D\vartheta^{cu}(\epsilon^{-1}\,P_0^{cu}\,x)\,v\Big| \\ & \le\; C_0\,K_\chi\,K_\vartheta\,|v| \;=:\; \gamma_1\,|v|\end{aligned}$$

due to $|P_0^{cu}\,x| \le 3\epsilon$. Observe that the norm of $DB_\epsilon^\pm$ is bounded by a constant $C(T)$ independently of ϵ. Indeed, the support of the derivative of $\chi_\epsilon^\pm$, which depends on ϵ, is contained in $\mathcal{A}$, see Figure 6. But inside $\mathcal{A}$, we have $\vartheta_\epsilon^{cu} \equiv 0$ by definition. Thus, the term $D\chi_\epsilon^\pm$ vanishes completely and the norm is indeed independent of ϵ. Therefore, we obtain the crude estimate

$$m(\tilde{T}(t+\tau,t,x_0)\,P_0^{cu}) \ge e^{-\gamma_1\,\tau} \tag{3.35}$$

for all values of t and τ satisfying $0 \le t \le t+\tau \le T$ and $[t,t+\tau] \subset I(x_0)$.

Hence, we conclude

$$m(\tilde{T}(T,x_0)\,P_0^{cu}) \ge e^{-\alpha^s(T-a(x_0))}\,e^{-\gamma_1\,a(x_0)}.$$

The crucial point in the above computations is that the constant K appearing in the exponential trichotomies is absent here due to the decomposition $-A\,P_0^{cu} = \Delta + J$ and the choice of coordinates which guarantees that $\|J\|$ is small.

Next, we consider (3.32) on the stable subspace E_0^s

$$\dot{w} = -Aw - \beta_1\,\vartheta_\epsilon^s(P_0^{cu}\,x)\,w,$$

which possesses the explicit solution

$$w(t) = e^{-\beta_1\int_0^t \vartheta_\epsilon^s(P_0^{cu}\,x(\tau))\,d\tau}\,e^{-At}\,w_0.$$

Thus we obtain the estimate

$$|\tilde{T}(T,x_0)\,P_0^s)|_\alpha \le K e^{-\alpha^{ss}T}\,e^{-\beta_1\,a(x_0)}.$$

Indeed, $\int_0^T \vartheta_\epsilon^s(P_0^{cu}\,x(\tau))\,d\tau \ge a(x_0)$ holds due to $\{\vartheta^{cu} > 0\} \subset \{\vartheta^s = 1\}$ and $\vartheta^s \ge 0$. $\square$

We will now investigate the semiflow of the full variational equation (3.17). To achieve this, we have to incorporate the influence of the operators coming from the nonlinearities handled in Lemma 3.17.

Lemma 3.19

$$
\begin{aligned}
m(P_0^{cu}\,\mathcal{T}(T,x_0)|_{E_0^{cu}}) &\geq e^{-\alpha^s(T-a(x_0))}\,e^{-\gamma_1\,a(x_0)} + o_\epsilon \\
|P_0^s\,\mathcal{T}(T,x_0)|_{E_0^s}|_\alpha &\leq K e^{-\alpha^{ss}T}\,e^{-\beta_1\,a(x_0)} + \kappa_1\,C(\kappa_0)\,C(T)\,K_\vartheta + o_\epsilon \\
|P_0^s\,\mathcal{T}(T,x_0)|_{E_0^{cu}}|_\alpha &\leq o_\epsilon \\
|P_0^{cu}\,\mathcal{T}(T,x_0)|_{E_0^s}|_\alpha &\leq o_\epsilon.
\end{aligned}
$$

Proof. This follows immediately from Lemma 3.18 and Lemma 3.17. Indeed, we have $\Big|\mathcal{T}(t,x_0)\Big|_\alpha \leq e^{\gamma_1 t}$ by Lemma 3.7 and the definition of γ_1. Then the claim follows from the variation of constant formula. □

Now, we can prove the existence of local right inverses on M_ϵ^l.

Lemma 3.20 *For $x \in E_\epsilon^l$, there exists a local right inverse g_σ of $\pi_M \circ \Phi_T \circ (id+\sigma)$ uniformly in $\sigma \in \Sigma_\epsilon$ if $\epsilon < \epsilon_0$ is sufficiently small. Moreover,*

$$
\begin{aligned}
L_y(g_\sigma) &\leq e^{\alpha^s(T-a(x))}\,e^{\gamma_1\,a(x)} + o_\epsilon \\
L_x(\pi_E \circ \Phi_T \circ (id+\sigma)) &\leq K e^{-\alpha^{ss}T}\,e^{-\beta_1\,a(x)} + \kappa_1\,C(\kappa_0)\,C(T)\,K_\vartheta + o_\epsilon.
\end{aligned}
$$

for each $y = \pi_M \circ \Phi_T \circ (id+\sigma)(x)$ such that $x, y \in M_\epsilon^l$. Furthermore, we have

$$
L_{x_0}(\pi_E \circ \Phi_T \circ (id+\sigma) \circ g_\sigma) \leq 1
$$

for $\kappa_1 < \Big(C(\kappa_0)\,C(T)\,K_\vartheta\Big)^{-1}$ and $\pi_E \circ \Phi_T \circ (id+\sigma) \circ g_\sigma$ is a local section of the bundle E_ϵ.

Proof. Choose any $\sigma \in \Sigma_\epsilon$. We will first show that M_ϵ^l is overflowing. The component of the right hand side of (3.14) in the unstable space E_0^{cu} vanishes identically if $P_0^{cu}\,x \in \mathcal{A}$. On the other hand, for $x \in M_\epsilon^+$ and $|\pi_{\gamma^\perp}(x)| = \kappa_0\,\epsilon$, the manifold is overflowing due to

$$
\begin{aligned}
&\Big\langle \vartheta_\epsilon^{cu}(P_0^{cu}\,x)\Big((\Delta + J + \beta_0\,\pi_{\gamma^\perp}(x) + f(\pi_{\gamma^\perp}(x))\Big), \pi_{\gamma^\perp}(x)\Big\rangle \\
&\qquad \geq (\beta_0 - \alpha^s - O(\epsilon))\,|\pi_{\gamma^\perp}(x)|^2.
\end{aligned}
$$

Next, we have to invert the mapping $\pi_M \circ \Phi_T \circ (id+\sigma)$ locally. This, as well as the remainder part of the claim, follows easily as in the previous lemmata using Lemma 3.19. □

3.3.4 The global injectivity of the right inverse g_σ

We will show here that the mapping $\pi_M \circ \Phi_T \circ (id+\sigma)$ is globally invertible on M_ϵ, which is in fact an easy consequence of the following lemma.

Lemma 3.21 ([For77, Satz 4.22]) *Let X and Y two locally compact spaces and $p : Y \to X$ a proper map with the following property: each point $y \in Y$ possesses a neighborhood $V \subset Y$ such that $p|_V$ is injective. Then p is a covering map.*

Lemma 3.22 $\pi_M \circ \Phi_T \circ (id + \sigma)$ *possesses a right inverse defined on M_ϵ.*

Proof. We apply Lemma 3.21 to $p = \pi_M \circ \Phi_T \circ (id + \sigma)$, $X = M_\epsilon$ and $Y = p^{-1}(X)$. We can assume that M_ϵ is compact. Now, $Y \subset X$ whence Y is compact, too, which implies that p is indeed proper. Therefore, the number of preimages of $\pi_M \circ \Phi_T \circ (id + \sigma)$ is constant, because the map is a covering map of a connected space. But the points $q(t) \in M_\epsilon^f$ admit precisely one preimage for sufficiently small ϵ, namely some point near $q(t - T)$ due to compactness of $\gamma(q(0))$. $\square$

By the series of Lemmata 3.13, 3.16, 3.20 and 3.22, we conclude that the map $\Phi_T^\# : \Sigma_\epsilon \to \Sigma_\epsilon$ is well defined.

3.3.5 The existence of a fixed point of $\Phi_T^\#$

In this paragraph, we will show that the mapping $\Phi_T^\# : \Sigma_\epsilon \to \Sigma_\epsilon$ is a contraction. From that we conclude the existence of a unique fixed point.

Definition We define

$$m_x(g) = \varliminf_{y \to x,\, y \in D_\epsilon} \frac{|g(x) - g(y)|_\alpha}{|x - y|_\alpha}.$$

to be the minimum norm $m(g)$ of a map $g : M_\epsilon \to X^\alpha$.

Lemma 3.23 *For each $\eta_0 > 0$ and L_0, $m_0 > 0$, there exists an $\epsilon_0 > 0$, such that for any $\epsilon \le \epsilon_0$ and $x, y \in M_\epsilon$ satisfying $|x - y|_\alpha \le \epsilon_0$ the following holds:*

(i) *For each $g : M_\epsilon \to X^\alpha$ with $L(g) \le L_0$ we have*

$$|g(x) - g(y)|_\alpha \le L_0 (1 + \eta_0) |x - y|_\alpha.$$

(ii) *Each $g : M_\epsilon \to X^\alpha$ with $m(g) \ge m_0$ fulfills*

$$|g(x) - g(y)|_\alpha \ge m_0 (1 - \eta_0) |x - y|_\alpha.$$

The proof is straightforward, see [San93].

Lemma 3.24 *The mapping $\Phi_T^\# : \Sigma_\epsilon \to \Sigma_\epsilon$ is a contraction with Lipschitz constant*

$$K e^{-\alpha^{ss} T} + C(T)\,\delta + o_\epsilon + \kappa_1\, C(\kappa_0)\, C(T)\, K_\vartheta.$$

Thus, it possesses a unique fixed point $\sigma_* \in \Sigma_\epsilon$.

Proof. We have to estimate the difference $\|\Phi_T^\#(\sigma) - \Phi_T^\#(\tilde\sigma)\|$ for $\sigma, \tilde\sigma \in \Sigma_\epsilon$. Observe that

$$
\begin{aligned}
(3.36)\qquad & |\Phi_T^\#(\sigma)(x) - \Phi_T^\#(\tilde\sigma)(x)|_\alpha \\
&= |\pi_E \circ \Phi_T \circ (id+\sigma) \circ g_\sigma(x) - \pi_E \circ \Phi_T \circ (id+\tilde\sigma) \circ g_{\tilde\sigma}(x)|_\alpha \\
&\leq |\pi_E \, \Phi_T \, (id+\sigma) \, g_\sigma(x) - \pi_E \, \Phi_T \, (id+\tilde\sigma) \, g_\sigma(x)|_\alpha + \\
&\quad\ |\pi_E \, \Phi_T \, (id+\tilde\sigma) \, g_\sigma(x) - \pi_E \, \Phi_T \, (id+\tilde\sigma) \, g_{\tilde\sigma}(x)|_\alpha \\
(3.37)\qquad &\leq |D(\pi_E \, \Phi_T)\,(g_\sigma x)\,(\sigma(g_\sigma x) - \tilde\sigma(g_\sigma x))|_\alpha + o_\epsilon\, |\sigma(g_\sigma x) - \tilde\sigma(g_\sigma x)|_\alpha + \\
&\quad\ |\pi_E \, \Phi_T \, (id+\tilde\sigma) \, g_\sigma(x) - \pi_E \, \Phi_T \, (id+\tilde\sigma) \, g_{\tilde\sigma}(x)|_\alpha .
\end{aligned}
$$

We consider the first term in (3.37). By the proofs of the Lemmata 3.12, 3.16 and 3.20 applied to $x \in M_\epsilon^f$, $M_\epsilon^\pm$ and M_ϵ^l, respectively, the following estimate holds

$$
(3.38)\qquad |D(\pi_E \, \Phi_T)\,(g_\sigma x)|_\alpha \leq \Big(K e^{-\alpha^{ss} T} + o_\epsilon + C(T)\,\delta + \kappa_1\, C(T, \kappa_0)\, K_\vartheta\Big).
$$

Thus, we obtain

$$
\begin{aligned}
(3.39)\qquad & |D(\pi_E \, \Phi_T)\,(g_\sigma x)\,(\sigma(g_\sigma x) - \tilde\sigma(g_\sigma x))|_\alpha \\
&\leq \Big(K e^{-\alpha^{ss} T} + o_\epsilon + C(T)\,\delta + \kappa_1\, C(T, \kappa_0)\, K_\vartheta\Big)\, |\sigma(g_\sigma x) - \tilde\sigma(g_\sigma x)|.
\end{aligned}
$$

It remains to estimate the second term in (3.37). Here, the difference $|g_\sigma(x) - g_{\tilde\sigma}(x)|_\alpha$ is the term we have to deal with. By definition

$$
\pi_M \circ \Phi_T \circ (id+\sigma) \circ g_\sigma = \pi_M \circ \Phi_T \circ (id+\tilde\sigma) \circ g_{\tilde\sigma} = id_{M_\epsilon}
$$

and thus

$$
\begin{aligned}
& \pi_M \, \Phi_T \, (id+\sigma) \, g_\sigma(x) - \pi_M \, \Phi_T \, (id+\tilde\sigma) \, g_{\tilde\sigma}(x) \\
&= \pi_M \, \Phi_T \, (id+\sigma) \, g_\sigma(x) - \pi_M \, \Phi_T \, (id+\tilde\sigma) \, g_\sigma(x) + \\
&\quad\ \pi_M \, \Phi_T \, (id+\tilde\sigma) \, g_\sigma(x) - \pi_M \, \Phi_T \, (id+\tilde\sigma) \, g_{\tilde\sigma}(x).
\end{aligned}
$$

From this identities, we conclude

$$
\begin{aligned}
& |\pi_M \, \Phi_T \, (id+\tilde\sigma) \, g_\sigma(x) - \pi_M \, \Phi_T \, (id+\tilde\sigma) \, g_{\tilde\sigma}(x)|_\alpha \\
&= |\pi_M \, \Phi_T \, (id+\sigma) \, g_\sigma(x) - \pi_M \, \Phi_T \, (id+\tilde\sigma) \, g_\sigma(x)|_\alpha .
\end{aligned}
$$

Now $|g_\sigma(x) - g_{\tilde\sigma}(x)|_\alpha \leq \epsilon$, because $|\pi_M \, \Phi_T \, (id+\sigma)\,(x) - \pi_M \, \Phi_T \, (id+\tilde\sigma)\,(x)|_\alpha = o_\epsilon$ due to Lemma 3.9. The claim follows now from Lipschitz continuity of the considered mappings. Therefore, again by the Lemmata 3.12, 3.16 and 3.20 and Lemma 3.23,

$$
\begin{aligned}
& |\pi_M \, \Phi_T \, (id+\tilde\sigma) \, g_\sigma(x) - \pi_M \, \Phi_T \, (id+\tilde\sigma) \, g_{\tilde\sigma}(x)|_\alpha \\
&\geq \Big(L_{g_\sigma x}(\pi_M \, \Phi_T) + o_\epsilon\Big)\, |g_\sigma(x) - g_{\tilde\sigma}(x)|_\alpha .
\end{aligned}
$$

Thus, we obtain

$$
\begin{aligned}
|g_\sigma(x) - g_{\tilde\sigma}(x)|_\alpha \leq \Big(L_{g_\sigma x}(\pi_M \Phi_T) + o_\epsilon\Big)^{-1} \cdot \\
\Big|\pi_M \Phi_T (id+\sigma)\, g_\sigma(x) - \pi_M \Phi_T (id+\tilde\sigma)\, g_\sigma(x)\Big|_\alpha .
\end{aligned}
$$

It remains to consider the term

$$|\pi_M \Phi_T (id+\sigma)(x_0) - \pi_M \Phi_T (id+\tilde\sigma)(x_0)|_\alpha$$

with $y := g_\sigma(x)$. Now

$$
\begin{aligned}
& |\pi_M \Phi_T (id+\sigma)(y) - \pi_M \Phi_T (id+\tilde\sigma)(y)|_\alpha \\
= \; & \Big| \int_0^1 D(\pi_M \Phi_T)\Big(y + \tilde\sigma(y) + \tau\,(\sigma(y) - \tilde\sigma(y))\Big)\,(\sigma(y) - \tilde\sigma(y))\, d\tau \Big|_\alpha \\
\leq \; & \Big(|D(\pi_M \Phi_T)(y)\, D\pi_E(y)|_\alpha + o_\epsilon\Big)\, |\sigma(y) - \tilde\sigma(y)|_\alpha
\end{aligned}
$$

and an application of the Lemmata 3.12, 3.16 and 3.20 yields

$$
\begin{aligned}
& \Big(L_{g_\sigma x}(\pi_M \Phi_T) + o_\epsilon\Big)^{-1} \Big|D(\pi_M \Phi_T)(g_\sigma x)\, D\pi_E(g_\sigma x)\Big|_\alpha \\
& \qquad \leq o_\epsilon + C(T)\,\delta + \kappa_1\, C(T,\kappa_0)\, K_\vartheta .
\end{aligned} \tag{3.40}
$$

Thus we obtain finally

$$
\begin{aligned}
& |g_\sigma(x) - g_{\tilde\sigma}(x)|_\alpha \\
\leq \; & \Big(o_\epsilon + C(T)\,\delta + \kappa_1\, C(T,\kappa_0)\, K_\vartheta\Big)\, |\sigma(g_\sigma x) - \tilde\sigma(g_\sigma x)|_\alpha .
\end{aligned} \tag{3.41}
$$

We substitute the resulting inequalities (3.39) and (3.41) into (3.36) and conclude

$$
\begin{aligned}
& |\Phi_T^\#(\sigma)(x) - \Phi_T^\#(\tilde\sigma)(x)|_\alpha \\
\leq \; & |D(\pi_E \Phi_T)\,(g_\sigma x)\,(\sigma(g_\sigma x) - \tilde\sigma(g_\sigma x))|_\alpha + o_\epsilon\, |\sigma(g_\sigma x) - \tilde\sigma(g_\sigma x)|_\alpha + \\
& |\pi_E \Phi_T (id+\tilde\sigma)\, g_\sigma(x) - \pi_E \Phi_T (id+\tilde\sigma)\, g_{\tilde\sigma}(x)|_\alpha \\
\leq \; & |D(\pi_E \Phi_T)\,(g_\sigma x)\,(\sigma(g_\sigma x) - \tilde\sigma(g_\sigma x))|_\alpha + o_\epsilon\, |\sigma(g_\sigma x) - \tilde\sigma(g_\sigma x)|_\alpha + \\
& \Big(|D(\pi_E \Phi_T)(g_\sigma x)|_\alpha + o_\epsilon\Big)\, |g_\sigma(x) - g_{\tilde\sigma}(x)|_\alpha \\
\leq \; & \Big(\Big(K e^{-\alpha^{ss} T} + o_\epsilon + C(T)\,\delta + \kappa_1\, C(T,\kappa_0)\, K_\vartheta\Big) + \\
& \Big(|D(\pi_E \Phi_T)(g_\sigma x)|_\alpha + o_\epsilon\Big)\,\Big(o_\epsilon + C(T)\,\delta + \kappa_1\, C(T,\kappa_0)\, K_\vartheta\Big)\Big)\, |\sigma(g_\sigma x) - \tilde\sigma(g_\sigma x)|_\alpha \\
\leq \; & \Big(K e^{-\alpha^{ss} T} + o_\epsilon + C(T)\,\delta + \kappa_1\, C(T,\kappa_0)\, K_\vartheta\Big)\, |\sigma(g_\sigma x) - \tilde\sigma(g_\sigma x)|_\alpha .
\end{aligned}
$$

Here, we have estimated the expression

$$|\pi_E \Phi_T (id+\tilde\sigma)\, g_\sigma(x) - \pi_E \Phi_T (id+\tilde\sigma)\, g_{\tilde\sigma}(x)|_\alpha$$

in the same manner as before. This finishes the proof of the lemma. □

Thus, essentially the contraction of Φ_T in the fibers gives the contraction property of the mapping $\Phi_T^{\#}$. We denote the graph of the fixed point σ_* by W^{cu}_{hom}. Therefore, $W^{cu}_{hom} \subset \Phi_T W^{cu}_{hom}$ holds. We will now characterize this surface by another property.

Lemma 3.25 *Take a sequence of points $x_{-n} \in E_\epsilon$ for $n \in \mathbb{N}_0$ such that $\Phi_T(x_{-n}) = x_{-(n-1)}$. Then $x_0 \in W^{cu}_{hom}$.*

Proof. The sequence x_{-n} fulfills in particular $\operatorname{dist}(x_{-n}, W^{cu}_{hom}) \leq 2\epsilon$. Indeed, $|\pi_E(x)|_\alpha \leq \epsilon$ for all $x \in E_\epsilon$. The contraction property of $\Phi_T^{\#}$ yields

$$\operatorname{dist}(\Phi_T^n(x), W^{cu}_{hom}) \leq \kappa^n \operatorname{dist}(x, W^{cu}_{hom})$$

for some $\kappa < 1$ and all $x \in E_\epsilon$ such that $\Phi_T^j(x) \in E_\epsilon$ for $1 \leq j \leq n$. Using $\Phi_T^n(x_{-n}) = x_0$ we obtain for the sequence x_{-n}

$$\operatorname{dist}(x_0, W^{cu}_{hom}) = \operatorname{dist}(\Phi_T^n(x_{-n}), W^{cu}_{hom}) \leq \kappa^n \operatorname{dist}(x_{-n}, W^{cu}_{hom}) \leq 2\,\kappa^n\,\epsilon.$$

Thus by taking the limit $n \to \infty$ we conclude $\operatorname{dist}(x_0, W^{cu}_{hom}) = 0$. □

Therefore, we have constructed a manifold W^{cu}_{hom} which contains all points staying in a neighborhood of the homoclinic orbit for all backward iterates of the time T-map Φ_T. Moreover, by construction, this manifold is locally invariant under Φ_T. The next lemma shows that the manifold W^{cu}_{hom} is actually locally invariant under the semiflow Φ_t for $t \geq 0$.

Lemma 3.26 *W^{cu}_{hom} is locally invariant under Φ_t for $t \geq 0$.*

Proof. Instead of carrying out the graph transform using Φ_T, it is possible to use the map $\Phi_{\tilde{T}}$ provided $|\tilde{T} - T| \leq \eta$ for a sufficiently small $\eta > 0$. Then $\Phi_{\tilde{T}}^{\#}(\sigma_*)$ is contained in Σ_ϵ, too. Moreover, the semiflow property implies

$$\Phi_T \Phi_{\tilde{T}}\, W^{cu}_{hom} = \Phi_{\tilde{T}}\, \Phi_T\, W^{cu}_{hom} = \Phi_{\tilde{T}}\, W^{cu}_{hom}.$$

Thus, $\Phi_{\tilde{T}}^{\#}(\sigma_*) = \Phi_{\tilde{T}}\, W^{cu}_{hom}$ is another fixed point of $\Phi_T^{\#}$ and by uniqueness we conclude $\Phi_{\tilde{T}}^{\#}(\sigma_*) = \sigma_*$. Substituting $\tilde{T} = T + t$ for $0 \leq t \leq \eta$ yields

$$W^{cu}_{hom} = \Phi_{\tilde{T}}\, W^{cu}_{hom} = \Phi_{T+t}\, W^{cu}_{hom} = \Phi_t\, W^{cu}_{hom}.$$

Iterating this argument proves the lemma. □

3.4 Regularity

Up to now, we have constructed a Lipschitz continuous manifold W^{cu}_{hom}. In this section, we will show that W^{cu}_{hom} actually possesses more regularity. In fact, W^{cu}_{hom} will be at least $C^{1,\rho}$. For simplicity, we will only prove this regularity. The case $C^{k,\beta}$ for $\beta > 0$ is proved in a similar way. We remark that the proof of C^k-regularity instead of an additional Hölder continuity of the kth-derivative is much more complicated, see e.g. [HPS77, Thm. 4.1]. Hence, let us prove that W^{cu}_{hom} is contained in $C^{1,\rho}$.

First, we show that we can assume that the manifold M_ϵ as well as the bundle E_ϵ are C^∞ smooth. By construction, they are only in $C^{1,\rho}$ a priori.

Lemma 3.27 *There exists a C^∞-bundle $\hat{E}_\epsilon$ possessing a C^∞-manifold $\hat{M}_\epsilon$ as base space, such that the following holds*

(i) $\hat{M}_\epsilon$ and $\hat{E}_\epsilon$ are C^1-close to M_ϵ and E_ϵ.

(ii) The fixed point σ_* is a section $\hat{\sigma}$ of the bundle $\hat{E}_\epsilon$ with domain $\hat{M}_\epsilon$.

Proof. Fix $\eta > 0$. Then we can approximate the homoclinic orbit $q(\cdot) : \mathbb{R} \to X^\alpha$ with $q(\cdot) \in C^1$ by a C^∞-function $\hat{q}(\cdot)$ such that

$$\begin{aligned} |q(t) - \hat{q}(t)|_\alpha &\leq \eta && \text{for } t \in \mathbb{R} \\ |\tfrac{d}{dt}(q(t) - \hat{q}(t))|_\alpha &\leq \eta && \text{for } t \in \mathbb{R} \\ \hat{q}(t) &\in E_0^{cu} && \text{for } |t| \geq \eta^{-1} \end{aligned}$$

holds. This can be achieved by a convolution with a smooth mollifier as in the finite-dimensional case. Moreover, we can define smooth projections $\hat{\pi}$, $\hat{\pi}_M$ and $\hat{\pi}_E$ which are η-close to the corresponding original projections in the C^1-norm. Then the graph transform is well defined for the new bundle provided we choose η sufficiently small. Indeed, the terms occuring in addition are of order η in C^1. The overflowing property is again fulfilled for small η by continuity. This proves the lemma. □

In the following, we will denote the new C^∞-approximations again by M_ϵ and E_ϵ. Next, we define Hölder regularity for maps on submanifolds of Banach spaces. To this end, we parametrize the tangent spaces $T_z M_\epsilon$ for all $z \in M_\epsilon$ close to $x \in M_\epsilon$ over the tangent space $T_x M_\epsilon$ in x by the C^∞-map

$$\tilde{H}_x : U_{\epsilon_0} \times T_x M_\epsilon \to E_\epsilon(x)$$

which is linear in the second variable such that

$$H_x(z)\, u := u + \tilde{H}_x(z, u) = u + \tilde{H}_x(z)\, u \in T_z M_\epsilon$$

defined for any $z \in U_{\epsilon_0}(x) \cap M_\epsilon$. The image of this map is $\mathrm{R}(id + \tilde{H}_x(z)) = \mathrm{R}H_x(z) = T_z M_\epsilon$. Thus $|\tilde{H}_x(z)|_\alpha \to 0$ for $z \to x$.

Definition The derivative of a map $g : M_\epsilon \to X^\alpha$ is called Hölder continuous with Hölderexponent β at $x \in M_\epsilon$ iff the map $Dg(z) \circ H_x(z)$ is Hölder continuous in the usual sense as a map $T_x M_\epsilon \to X^\alpha$, i.e.

$$\overline{\lim_{z,\tilde{z} \to x,\, z,\tilde{z} \in M_\epsilon}} \frac{|Dg(z) \circ H_x(z) - Dg(\tilde{z}) \circ H_x(\tilde{z})|_\alpha}{|z - \tilde{z}|_\alpha^\beta} =: R_x(g) \leq \sup_{x \in M} R(g) < \infty.$$

Here, the norm in the numerator is given by $|\cdot|_{L(T_x M_\epsilon, X^\alpha)}$ with the norm induced on $T_x M_\epsilon$ by $T_x M_\epsilon \hookrightarrow X^\alpha$.

In the present situation, the manifold M_ϵ is flat near the origin. Therefore, the definitions given above coincide with the usual definition of Hölder continuity in $\mathbb{R}^n$. Indeed, $H_x(\cdot) \equiv id$ there.

We denote by $B_R := B_R(C^{1,\beta}(M_\epsilon))$ the subset $\Sigma_\epsilon \cap \left\{\sigma \in C^{1,\beta}(M_\epsilon) \big/ R(\sigma) \leq R\right\}$. Then, we have the following result.

Lemma 3.28 *$B_R(C^{1,\beta}(M_\epsilon))$ is closed in Σ_ϵ for $\beta > 0$.*

Proof. This follows by applying the result [Hen81, Lemma 6.1.6] in our specific context. □

Of course, the lemma is wrong for $\beta = 0$. The reason for the importance of the closeness of $B_R = B_R(C^{1,\beta}(M_\epsilon))$ is stated below.

Lemma 3.29 *In order to prove $\sigma_* \in C^{1,\beta}$ it is enough to show that*

$$\Phi_T^\#(B_R) \subset B_R \subset \Sigma_\epsilon$$

for a suitable chosen $R > 0$. Indeed, this implies that the fixed point σ_* has to be contained in $B_R(C^{1,\beta}(M_\epsilon))$ due to the closeness of this set.

Lemma 3.30 *Consider mappings $f \in C^{1,\beta}(M_\epsilon, X^\alpha)$ and $g \in C^{1,\beta}(M_\epsilon)$ such that $g : M_\epsilon \to M_\epsilon$. Then we have*

$$R_x(D(f \circ g)) \leq R_{g(x)}(Df)\, L_x(g)^{1+\beta} + L_{g(x)}(f)\, R_x(Dg) + L_{g(x)}(f)\, L_x(g)^{1+\beta}\, R_{gx}(H).$$

Proof. For arbitrary function $\tilde{f} \in C^\beta(M_\epsilon, X^\alpha)$ and $\tilde{g} \in C^{0,1}(M_\epsilon, X^\alpha)$, we have

$$\frac{|(\tilde{f} \circ \tilde{g})(x) - (\tilde{f} \circ \tilde{g})(z)|_\alpha}{|x - z|_\alpha^\beta} \leq \frac{|\tilde{f}(\tilde{g}(x)) - \tilde{f}(\tilde{g}(z))|_\alpha}{|\tilde{g}(x) - \tilde{g}(z)|_\alpha^\beta} \left(\frac{|\tilde{g}(x) - \tilde{g}(z)|_\alpha}{|x - z|_\alpha}\right)^\beta$$

and therefore,

$$R(\tilde{f} \circ \tilde{g}) \leq R(\tilde{f})\, L^\beta(\tilde{g}). \tag{3.42}$$

Using this inequality, we obtain

$$\begin{aligned} |D(f \circ g)(x) - D(f \circ g)(z)\, H_x(z)|_\alpha &\leq \\ \leq\ & \left|\left(Df(g(x)) - Df(g(z))\, H_{g(x)}(g(z))\right) Dg(x)\right|_\alpha + \\ & \left|Df(g(z))\left(H_{g(x)}(g(z))\, Dg(x) - Dg(z)\, H_x(z)\right)\right|_\alpha \\ \leq\ & |Df(g(x)) - Df(g(z))\, H_{g(x)}(g(z))|_\alpha\, |Dg(x)|_\alpha + \\ & |Df(g(z))|_\alpha \cdot \left(|H_{g(x)}(g(z)) - id_{T_{g(x)}M_\epsilon}|_\alpha\, |Dg(x)|_\alpha + \right. \\ & \left. |Dg(x) - Dg(z)\, H_x(z)|_\alpha\right). \end{aligned} \tag{3.43}$$

Multiplying with $|x-z|_\alpha^{-\beta}$ and taking the limit $z \to x$ yields

$$R_x(D(f \circ g)) \leq R_{g(x)}(Df)\, L_x(g)^{1+\beta} + L_{g(x)}(f)\left(R_{g(x)}(H)\, L_x(g)^{1+\beta} + R_x(g)\right),$$

which proves the lemma. □

Lemma 3.31 *There exists an $R_0 > 0$ such that $\Phi_T^\#(B_{R_0}) \subset B_{R_0}$. In general, R_0 will depend on ϵ.*

Proof. Take any $\sigma \in B_R$. Then $\Phi_T^\#(\sigma) \in C^1(M_\epsilon)$ and the derivative fulfills $|D(\Phi_T^\#(\sigma))|_\alpha \leq 1$. Thus it remains to show that $D(\Phi_T^\#(\sigma)) \in B_R$. We have

$$D(\Phi_T^\#(\sigma)) = D\pi_E \circ D\Phi_T \circ (id + D\sigma) \circ Dg_\sigma$$

and the time T-map Φ_T is contained in $C^{1,\beta}(U, X^\alpha)$ for some neighborhood U of M_ϵ. But in general the Hölder constant $R(D\Phi_T) \to \infty$ will tend to infinity as $\epsilon \to 0$ due to the use of cutoff functions.

We will first show that the derivatives of the right inverses g_σ are Hölder continuous and will compute their Hölder constant. We remark that we can estimate the minimum norm of the derivative $D(\pi_M \circ \Phi_T \circ (id + \sigma))$ due to the Lemmata 3.11, 3.16 and 3.19. Indeed, the following holds

$$\begin{aligned} m_0(x) &:= m\Big(D(\pi_M \circ \Phi_T \circ (id+\sigma))(x)\Big) \\ &\geq \begin{cases} K e^{-\alpha^s T} + o_\epsilon + C(T)\,\delta & \text{for } x \in M_\epsilon^f \\ e^{-\alpha^s T} + C(T, \kappa_0)\,\epsilon + C(T)\,\delta & \text{for } x \in M_\epsilon^\pm \\ e^{-\alpha^s (T - a(x))}\, e^{-\gamma_1 a(x)} + o_\epsilon & \text{for } x \in M_\epsilon^l. \end{cases} \end{aligned} \tag{3.44}$$

Next, we will use an implicit characterization of the right inverses. By definition, we have

$$\pi_M \circ \Phi_T \circ (id + \sigma) \circ g_\sigma = id_{M_\epsilon}$$

and thus

$$D\pi_M \circ D\Phi_T \circ (id + D\sigma) \circ Dg_\sigma = id_{T_x M_\epsilon}.$$

As in equation (3.43), we conclude

$$\begin{aligned}
&\Big|id_{T_x M_\epsilon} - id_{T_z M_\epsilon} H_x(z)\Big|_\alpha = \\
&= \Big|D(\pi_M \circ \Phi_T \circ (id+\sigma))(g_\sigma x)\, Dg_\sigma(x) - \\
&\quad D(\pi_M \circ \Phi_T \circ (id+\sigma))(g_\sigma z)\, Dg_\sigma(z)\, H_x(z)\Big|_\alpha \\
&\geq -\Big|\Big(D(\pi_M \circ \Phi_T \circ (id+\sigma))(g_\sigma x) - D(\pi_M \circ \Phi_T \circ (id+\sigma))(g_\sigma z)\, H_{g_\sigma x}(g_\sigma z)\Big) \cdot \\
&\quad Dg_\sigma(x)\Big|_\alpha + \Big|D(\pi_M \circ \Phi_T \circ (id+\sigma))(g_\sigma z)\, \Big(H_{g_\sigma x}(g_\sigma z)\, Dg_\sigma(x) - Dg_\sigma(z)\, H_x(z)\Big)\Big|_\alpha \\
&\geq m_0(g_\sigma z)\, \Big|H_{g_\sigma x}(g_\sigma z)\, Dg_\sigma(x) - Dg_\sigma(z)\, H_x(z))\Big|_\alpha - \\
&\quad \Big|D(\pi_M \circ \Phi_T \circ (id+\sigma))(g_\sigma x) - D(\pi_M \circ \Phi_T \circ (id+\sigma))(g_\sigma z)\, H_{g_\sigma x}(g_\sigma z)\Big|_\alpha \Big|Dg_\sigma(x)\Big|_\alpha \\
&\geq m_0(g_\sigma z)\, \Big|Dg_\sigma(x) - Dg_\sigma(z)\, H_x(z))\Big|_\alpha - m_0(g_\sigma z)\, \Big|H_{gx}(gz) - id_{T_{gx} M_\epsilon}\Big|_\alpha \Big|Dg_\sigma(x)\Big|_\alpha - \\
&\quad \Big|D(\pi_M \circ \Phi_T \circ (id+\sigma))(g_\sigma x) - D(\pi_M \circ \Phi_T \circ (id+\sigma))(g_\sigma z)\, H_{g_\sigma x}(g_\sigma z)\Big|_\alpha \Big|Dg_\sigma(x)\Big|_\alpha.
\end{aligned}$$

Therefore, we obtain

$$\begin{aligned}
&\Big|Dg_\sigma(x) - Dg_\sigma(z)\, H_x(z)\Big|_\alpha \\
&\leq m_0(g_\sigma z)^{-1} \Big(\Big|id_{T_x M_\epsilon} - H_x(z)\Big|_\alpha + \Big|D(\pi_M \circ \Phi_T \circ (id+\sigma))(g_\sigma x) - \\
&\quad D(\pi_M \circ \Phi_T \circ (id+\sigma))(g_\sigma z)\, H_{g_\sigma x}(g_\sigma z)\Big|_\alpha \Big|Dg_\sigma(x)\Big|_\alpha\Big) + \\
&\quad \Big|H_{gx}(gz) - id_{T_{gx} M_\epsilon}\Big|_\alpha \Big|Dg_\sigma(x)\Big|_\alpha.
\end{aligned}$$

The identity $\pi_E(x + \sigma(x)) = \sigma(x)$ yields

$$\begin{aligned}
&\Big|D\sigma(x) - D\sigma(z)\, H_x(z)\Big|_\alpha \\
&\leq \Big|D\pi_E(x + \sigma(x))\, \Big(D\sigma(x) - D\sigma(z)\, H_x(z)\Big)\Big|_\alpha + \\
&\quad \Big|\Big(D\pi_E(x + \sigma(x)) - D\pi_E(z + \sigma(z))\Big)\Big|_\alpha \Big|D\sigma(z)\, H_x(z))\Big|_\alpha \\
&\leq \Big|D\pi_E(x + \sigma(x))\Big|_\alpha \Big|D\sigma(x + \sigma(x)) - D\sigma(z + \sigma(z))\, H_x(z)\Big|_\alpha + \\
&\quad \Big|D\pi_E(x + \sigma(x)) - D\pi_E(z + \sigma(z))\Big|_\alpha \Big|D\sigma(z)\, H_x(z))\Big|_\alpha.
\end{aligned}$$

Thus, Dg_σ is Hölder continuous and we obtain the estimate

$$\begin{aligned}
&R_x(Dg_\sigma) \leq L_x(g_\sigma)^{1+\beta}\, R(H) + \\
&\quad m_0(g_\sigma x)^{-1} \Big(R(H) + L_x(g_\sigma)^\beta \Big(R(D(\pi_M\, \Phi_T))\, L(id + \sigma)^{1+\beta} + \\
&\quad L(D(\pi_M\, \Phi_T))\, (R(H) + R(D\pi_E)) + |D(\pi_M\, \Phi_T)(y)\, D\pi_E(y)|_\alpha\, R_{g_\sigma x}(D\sigma)\Big)\Big)
\end{aligned}$$

for the Hölder constant of Dg_σ using the definition $y := g_\sigma(x) + \sigma(g_\sigma(x))$. As for equation (3.40), we obtain

$$|D(\pi_M \Phi_T)(y)\, D\pi_E(y)|_\alpha \le o_\epsilon + C(T)\,\delta + \kappa_1\, C(T,\kappa_0)\, K_\vartheta.$$

Thus, we end up with the inequality

$$\begin{aligned}(3.45)\qquad & R_x(Dg_\sigma) \le L_x(g_\sigma)^{1+\beta}\, R(H) + \\ & m_0(g_\sigma x)^{-1} \Big(R(H) + L_x(g_\sigma)^\beta \Big(R(D(\pi_M\, \Phi_T))\, L(id+\sigma)^{1+\beta} + \\ & L(D(\pi_M\, \Phi_T))\, (R(H) + R(D\pi_E)) + \\ & (o_\epsilon + C(T)\,\delta + \kappa_1\, C(T,\kappa_0)\, K_\vartheta)\, R_{g_\sigma x}(D\sigma)\Big)\Big).\end{aligned}$$

Finally, we can estimate the Hölder constant of $\Phi_T^\#(\sigma)$. To that end, we have to consider the difference

$$|D(\pi_E \circ \Phi_T \circ (id+\sigma) \circ g_\sigma)(x) - D(\pi_E \circ \Phi_T \circ (id+\sigma) \circ g_\sigma)(z)|_\alpha.$$

The corresponding estimates follow the same lines as those for $D(\pi_M \circ \Phi_T \circ (id+\sigma) \circ g_\sigma)$, whence we will only give the result

$$\begin{aligned} & R_x(D(\pi_E \circ \Phi_T \circ (id+\sigma) \circ g_\sigma)) \\ \le\ & R(D(\pi_E\, \Phi_T)\, D\pi_E)\, L(id+\sigma)^{1+\beta}\, L(g_\sigma) + \\ & |D(\pi_E\, \Phi_T)(y)\, D\pi_E(y)|_\alpha\, R_{g_\sigma x}(D\sigma)\, L_x(g_\sigma)^{1+\beta} + \\ & R_x(Dg_\sigma)\, |D(\pi_E\, \Phi_T)|_\alpha\, (C_0\, L(\sigma)+1) + |D(\pi_E\, \Phi_T)|_\alpha\, L(\sigma)\, L_x(g_\sigma)^{1+\beta}\, R(H).\end{aligned}$$

Here, $y = g_\sigma(x) + \sigma(g_\sigma(x))$ as above. We substitute equation (3.45) for $R(g_\sigma)$ into this inequality. Because we are only interested in a bound for $R(D\sigma)$, we will denote the bounds for the Hölder constants of $D(\pi_M\, \Phi_T)$, $D(\pi_E\, \Phi_T)$ and H as well as for all Lipschitz constants by C_ϵ. However, C_ϵ will not depend on $R(D\sigma)$. Note that C_ϵ might tend to infinity as ϵ tends to zero. Then, we obtain

$$\begin{aligned} & R_x(D\Phi_T^\#(\sigma)) \\ \le\ & C_\epsilon + |D(\pi_E\, \Phi_T)(y)\, D\pi_E(y)|_\alpha\, R_{g_\sigma x}(D\sigma)\, L_x(g_\sigma)^{1+\beta} + \\ & R_x(Dg_\sigma)\, |D(\pi_E\, \Phi_T)|_\alpha\, (C_0\, L(\sigma)+1) \\ \le\ & C_\epsilon + R_{g_\sigma x}(D\sigma)\Big(|D(\pi_E\, \Phi_T)(y)\, D\pi_E(y)|_\alpha\, L_x(g_\sigma)^{1+\beta} + \\ & \Big(o_\epsilon + C(T)\,\delta + \kappa_1\, C(T,\kappa_0)\, K_\vartheta\Big)\, m_0(g_\sigma x)^{-1}\, L_x(g_\sigma)^\beta\Big).\end{aligned}$$

Observe that the estimates for $L(g_\sigma)$ and $m_0(x)$ are uniform in ϵ. This yields

$$(3.46)\qquad R_x(D\Phi_T^\#(\sigma))$$

$$\leq \quad C_\epsilon + R_{g_\sigma x}(D\sigma)\Big(o_\epsilon + C(T)\,\delta + \kappa_1\, C(T,\kappa_0)\, K_\vartheta + \Big|D(\pi_E\Phi_T)\big(g_\sigma x + \sigma(g_\sigma x)\big)\; D\pi_E\big(g_\sigma x + \sigma(g_\sigma x)\big)\Big|_\alpha \, L_x(g_\sigma)^{1+\beta}\Big).$$

By definition, we have

$$L_x(g_\sigma) \leq m_0(g_\sigma x).$$

Summarizing the Lemmata 3.12, 3.16 and 3.19 together with (3.44) we obtain

$$|D(\pi_E\,\Phi_T)(g_\sigma x + \sigma(g_\sigma x))\, D\pi_E(g_\sigma x + \sigma(g_\sigma x))|_\alpha\, L_x(g_\sigma)^{1+\beta} \leq$$

$$\leq \begin{cases} \Big(K e^{-\alpha^{ss}T} + o_\epsilon + C(T)\,\delta\Big)\cdot\Big(K e^{\alpha^s T} + o_\epsilon + C(T)\,\delta\Big)^{1+\beta} & \text{for } \; x \in M_\epsilon^f \\ \Big(K e^{-\alpha^{ss}T} + o_\epsilon + C(T)\,\delta\Big)\cdot\Big(e^{\alpha^s T} + C(T,\kappa_0)\,\epsilon + C(T)\,\delta\Big)^{1+\beta} & \text{for } \; x \in M_\epsilon^\pm \\ \Big(K e^{-\alpha^{ss}T}\, e^{-\gamma_1\, a(g_\sigma x)} + \kappa_1\, C(\kappa_0)\, C(T)\, K_\vartheta + o_\epsilon\Big)\cdot & \\ \quad \Big(e^{\alpha^s(T-a(g_\sigma x))}\, e^{\gamma_1\, a(g_\sigma x)} + o_\epsilon\Big)^{1+\beta} & \text{for } \; x \in M_\epsilon^l \end{cases}$$

$$\leq \begin{cases} K e^{-(\alpha^{ss}-(1+\beta)\,\alpha^s)T} + o_\epsilon + C(T)\,\delta & \text{for } \; x \in M_\epsilon^f \\ K e^{-(\alpha^{ss}-(1+\beta)\,\alpha^s)T} + o_\epsilon + C(T)\,\delta & \text{for } \; x \in M_\epsilon^\pm \\ K e^{-(\alpha^{ss}-(1+\beta)\,\alpha^s)T}\, e^{-(\gamma_1+(1+\beta)\,\alpha^s-(1+\beta)\,\gamma_1)\,a(g_\sigma x)} + & \\ \quad \kappa_1\, C(\kappa_0)\, C(T)\, K_\vartheta\, e^{\gamma_1\,(1+\beta)\,T} + o_\epsilon. & \text{for } \; x \in M_\epsilon^l \end{cases}$$

By assumption, we have $\alpha^{ss} > (1+\beta)\alpha^s$. Therefore, there exists $\eta < 1$ satisfying

$$\Big|D(\pi_E\,\Phi_T)\big(g_\sigma x + \sigma(g_\sigma x)\big)\; D\pi_E\big(g_\sigma x + \sigma(g_\sigma x)\big)\Big|_\alpha\, L_x(g_\sigma)^{1+\beta} < \eta.$$

Finally we conclude by using (3.46)

$$R_x(D\Phi_T^\#(\sigma)) \leq C_\epsilon + \eta\, R_{g_\sigma x}(D\sigma).$$

Now suppose $R(D\sigma) \leq R_0$. Then, we have to show that $R(D\Phi_T^\#(\sigma)) \leq R_0$ is satisfied either. Hence, it is sufficient to choose $R_0 \geq (1-\eta)^{-1}C_\epsilon > 0$. This proves the lemma. □

3.5 The existence of W^c_{hom}

The series of lemmata above proves the existence of a locally invariant manifold W^{cu}_{hom} possessing all the properties stated in Theorem 1. There are mainly two different strategies in order to show the existence of the proper center manifold W^c_{hom}.

In finite-dimensional spaces, we can reverse time and repeat the procedure described above. Then, we obtain a center-stable manifold W^{cs}_{hom} containing all those solutions which stay near the homoclinic orbit for all positive times. The intersection of W^{cu}_{hom} and W^{cs}_{hom} yields the desired center manifold W^{c}_{hom}. However, this strategy fails in infinite-dimensional spaces for the obvious reason that the semiflow cannot be extended to backward time. Instead we restrict the semiflow to the finite-dimensional manifold W^{cu}_{hom} the existence of which was obtained previously. Now we can try to reverse time on this invariant manifold and repeat the graph transform by replacing the space X^α by W^{cu}_{hom}.

We will realize this second strategy as follows. First, parametrize the manifold W^{cu}_{hom} as a graph $\hat{\sigma}_*$ over the C^∞-manifold $\hat{M}_\epsilon$ in the bundle $\hat{E}_\epsilon$ using Lemma 3.27. Then, we take the pullback of the semiflow restricted to W^{cu}_{hom} onto $\hat{M}_\epsilon$. To that end, we decompose any solution $x(t) \in W^{cu}_{hom}$ into the two components

$$\begin{aligned} u(t) := \pi_M(x(t)) &\in \hat{M}_\epsilon \\ \hat{\sigma}_*(u(t)) &\in \hat{E}(u(t)). \end{aligned}$$

By the invariance of W^{cu}_{hom}, we conclude $u(t) + \hat{\sigma}_*(u(t)) = x(t)$. Below we will prove that the vector field

$$\dot{u} = D\pi_M(u + \hat{\sigma}(u))\Big(- A(u + \hat{\sigma}(u)) + f(u + \hat{\sigma}(u))\Big). \tag{3.47}$$

is well defined and Lipschitz continuous on $\hat{M}_\epsilon$, see Lemma 3.32. Assume for a moment that this claim is true. Then $u(t)$ is a solution of the differential equation (3.47). At the converse assume that $\hat{u}(t)$ solves (3.47) on $\hat{M}_\epsilon$. Then $\hat{x}(t) = \hat{u}(t) + \hat{\sigma}(\hat{u}(t))$ is a solution of the original equation satisfying $\hat{x}(0) = \hat{u}(0) + \hat{\sigma}(\hat{u}(0)) \in W^{cu}_{hom}$. Indeed, denote the solution with initial point $x(0) = \hat{x}(0)$ contained in W^{cu}_{hom} by $x(t)$. Then $\pi_M(x(t))$ is a solution of (3.47) with the same initial point as $\hat{x}(t)$. Due to the uniqueness of solutions of (3.47) proved in Lemma 3.32 below, these solution curves have to coincide. Thus, the differential equation (3.47) is equivalent to the flow on W^{cu}_{hom}. The regularity of the vector field is identical to the regularity of σ_*.
The point in the above argument is that we will not lose any regularity when restricting the vector field to a less smooth manifold. Originally, W^{cu}_{hom} is in $C^{1,\beta}$. Thus, the tangent bundle and hence the vector field restricted to W^{cu}_{hom} are a priori only C^β. The procedure mentioned above will provide a vector field which is as smooth as the manifold, namely $C^{1,\beta}$.

Now, we can reverse time in the manifold $\hat{M}_\epsilon$ and consider (3.47) in negative time direction. Next, we apply Theorem 1 to W^{cu}_{hom}. The assumptions (H1) up to (H4) are easily seen to be satisfied. Of course, Theorem 1 is valid on finite dimensional manifolds as well. Thus, we obtain a locally invariant graph σ^c_* of equation (3.47). Taking the composition with σ^{cu}_*

yields a graph $(id_{cu} + \sigma_*^{cu}) \circ (id_c + \sigma_*^c)$, which represents the locally invariant surface W^c_{hom} in X^α.

It remains to prove the claim about the Lipschitz continuity of the vector field (3.47).

Lemma 3.32 *Assume that $f \in C^{k,\beta}(X, X^\alpha)$ with $k, \beta > 0$. Then the vector field induced on $\hat{M}_\epsilon$ is of class $C^{k,\tilde{\beta}}$ for any $\tilde{\beta} < \beta$.*

Proof. We will restrict ourselves to the case $k = 1$. We have to show that the derivative of

$$D\pi_M(u + \hat{\sigma}(u))\Big(- A(u + \hat{\sigma}_*(u)) + f(u + \hat{\sigma}_*(u))\Big)$$

is Hölder continuous. In fact, it is even not clear that this expression is well defined. However, note that the semiflow restricted to graph $\hat{\sigma}_*$ is invertible owing to the existence of the right inverse $g_{\hat{\sigma}_*}$. Thus, it is sufficient to show Hölder continuity of the derivative of

$$D_x\Big(D\pi_M(\Phi(T,x))\,\frac{d}{dt}\Phi(t,x)\Big|_{t=T}\Big)$$

at $x = u + \hat{\sigma}_*(u)$, because composition with $g_{\hat{\sigma}_*}$ yields the vector field at x. First, we prove that

$$\dot{D}_x\Phi(t,x)\Big|_{t=T} \in C^{\tilde{\beta}}(X^\alpha, X) \tag{3.48}$$

holds for any $\tilde{\beta} < \beta$. By [Hen81, Lemma 3.5.1], we obtain

$$\frac{d}{dt}\Phi_t(x) = -A\,e^{-At} + e^{-At} f(\Phi_t\,x) + \int_0^t A\,e^{-A(t-s)}\Big(f(\Phi_t\,x) - f(\Phi_s\,x)\Big)\,ds \tag{3.49}$$

for any $t > 0$. We claim that

$$\begin{aligned} D\dot{\Phi}_t(x) \;&=\; -A\,e^{-At} + e^{-At}\,Df(\Phi_t\,x)\,D\Phi_t(x) + \\ &\quad \textstyle\int_0^t A\,e^{-A(t-s)}\Big(Df(\Phi_t\,x)\,D\Phi_t(x) - Df(\Phi_s\,x)\,D\Phi_s(x)\Big)\,ds \end{aligned}$$

and $D\dot{\Phi}_T(\cdot) : X^\alpha \to L(X^\alpha, X) \in C^{\tilde{\beta}}$ holds. To prove the claim, consider the estimate

$$\begin{aligned} &\Big|f(\Phi_t(x+h)) - f(\Phi_t\,x) - Df(\Phi_t\,x)\,D\Phi_t(x)\,h + \\ &\qquad Df(\Phi_s\,x)\,D\Phi_s(x)\,h + f(\Phi_s\,x) - f(\Phi_s(x+h))\Big| \\ &\le \int_0^1 \Big|Df(\Phi_t(x+\tau h))\,D\Phi_t(x+\tau h) - Df(\Phi_s(x+\tau h))\,D\Phi_s(x+\tau h) + \\ &\qquad Df(\Phi_s\,x)\,D\Phi_s(x) - Df(\Phi_t\,x)\,D\Phi_t(x)\Big|\,d\tau\,|h|_\alpha \\ &\le |t-s|^{\eta\frac{1}{n}}\,|h|_\alpha^{1+\beta\frac{n-1}{n}}. \end{aligned}$$

Then, we obtain by denoting $F(t,h) := Df(\Phi_t(x+\tau h))\,D\Phi_t(x+\tau h)$

$$\begin{aligned} |F(t,h) - F(s,h) + F(s,0) - F(t,0)| &\le C\,|t-s|^\eta \\ |F(t,h) - F(t,0) + F(s,0) - F(s,h)| &\le C\,|h|_\alpha^\beta, \end{aligned}$$

because $f \in C^{1,\beta}$, $\Phi_t \in C^{1,\beta}$ and the solutions are of class C^η for some $\eta > 0$, see [Hen81, Lemma 3.4.4]. Therefore, we can estimate the difference in the integral in (3.49) yielding

$$\begin{aligned}
&\int_0^t \left|A\, e^{-A(t-s)}\right| \Big| f(\Phi_t(x+h)) - f(\Phi_t x) - Df(\Phi_t\, x)\, D\Phi_t(x) h + \\
&\qquad Df(\Phi_s\, x)\, D\Phi_s(x)\, h + f(\Phi_s\, x) - f(\Phi_s(x+h)) \Big|\, ds \\
\leq\;& \int_0^t (t-s)^{-1}\, e^{\alpha(t-s)} \int_0^1 \Big| Df(\Phi_t(x+\tau h))\, D\Phi_t(x+\tau h) - \\
&\qquad Df(\Phi_s(x+\tau h))\, D\Phi_s(x+\tau h) + Df(\Phi_s\, x)\, D\Phi_s(x) - Df(\Phi_t\, x)\, D\Phi_t(x) \Big|\, d\tau\, |h|_\alpha\, ds \\
\leq\;& \int_0^t C\,(t-s)^{\eta\frac{1}{n}-1}\, e^{\alpha(t-s)}\, ds\, |h|_\alpha^{1+\beta\frac{n-1}{n}} \\
\leq\;& C\, |h|_\alpha^{1+\beta\frac{n-1}{n}}.
\end{aligned}$$

This proves our claim (3.48).

Finally, we extend the mappings $D\pi_M(x) : X^\alpha \to X^\alpha$ for fixed x to mappings from X to X. Indeed, we can extend the projections of the exponential dichotomies to these spaces, see [Hen81] or [San93, Lemma 1.2(v)]. The same holds for the approximations of those mappings defined in Lemma 3.4 and hence in turn for $\pi_M(\cdot)$ by definition. Thus, the map $D\pi_M(\Phi_T(x))\, D\dot{\Phi}_T(x) : X^\alpha \to X$ is well defined and Hölder continuous with exponent $\tilde{\beta} < \beta$. The image of this operator is contained in the tangent space $T_{\Phi_T(x)}\hat{M}_\epsilon$ for any $x \in M_\epsilon$. Moreover, $\hat{M}_\epsilon \subset X$ endowed with the induced topology is diffeomorphic to $\hat{M}_\epsilon \subset X^\alpha$ as the identity is an injective immersion and M_ϵ is compact. This implies that $D\pi_M(\Phi_T(x))\, D\dot{\Phi}_T(x)$ composed with the inverse of the identity is Hölder continuous on $\hat{M}_\epsilon$ as a subset of X^α. The lemma is proved. □

The lemma can be proved much easier if one requires more regularity for the nonlinearity $f : X^\alpha \to X$. Indeed, it is easy to prove using [Hen81, Lemma 3.5.1] that $D\dot{\Phi}_T(\cdot) : X^\alpha \to X^\alpha$ is contained in $C^{1,\beta}$ provided $f \in C^{2,\beta}$. Then, of course, the restriction to $\hat{M}_\epsilon$ admits the same regularity.

3.6 The parameter dependent version

Now in fact we desire a center manifold for the equation

$$\dot{x} + Ax = f(x) + \mu\, g(x,\mu) \tag{3.50}$$

or

$$\dot{x} + Ax = f(x) + \mu\, g(t,x,\mu) \tag{3.51}$$

such that g is either independent of time or else periodic in t with period P. So far, we have considered (3.50) or (3.51) with $\mu = 0$. But actually all the results obtained up to

now are stable with respect to C^1 perturbations. Indeed, if we replace f by $f + \mu g$, we just have to add terms which are bounded by μ uniformly in ϵ. Thus, we end up with the same result as before if only μ is sufficiently small. By the uniform contraction mapping theorem, see [CH82, Thm. 2.2.2], the fixed point W^c_{hom} depends as smoothly on μ as the mapping $\Phi^{\#}_T$ does, that is $C^{k,\beta}$.
In case g is periodic in time, there are some other modifications in order. First of all, we have to choose T as a multiple nP of the period with $n \in \mathbb{N}$. Then, we need to prove Lemma 3.26 in this context. But we can carry out the graph transform in such a way that the whole analysis is valid for time steps T and $T+P$ without any further modifications. Indeed, P is a constant and the interval $[T, T+P]$ is compact. Then we can proceed as in Lemma 3.26 in order to prove that the center manifold is invariant under the Poincaré map Φ_P.

This finishes the proof of the Theorems 1 and 2.

4 Applications

In this section, we will give one application of our main result. In fact, we will generalize the well-known Shilnikov chaos induced by a homoclinic orbit to a saddle-focus to semilinear parabolic equations. Usually, one needs a C^1-linearization in order to prove this result. An application of a result of Belitskii provides us with the necessary lemma which can be used to linearize the vector field restricted to the center manifold. Consider

$$\dot{x} + Ax = f(x), \qquad x \in X^\alpha \tag{4.1}$$

with A sectorial and $f : X^\alpha \to X$ in $C^{1,\beta}$ such that $f(0) = Df(0) = 0$ and $\alpha \in [0,1)$. Assume that $q(t)$ is a homoclinic solution of (4.1) converging to zero.

(H) Suppose that the spectrum of $-A$ decomposes into $\sigma(-A) = \sigma^s \cup \sigma^c \cup \sigma^u$ such that $\operatorname{Re}\sigma^s < -\lambda^s < 0 < \lambda^u < \operatorname{Re}\sigma^u$. Moreover, σ^c is given by one of the following sets

(i) $\sigma^c = \{-\lambda^s \pm i\beta^s, \lambda^u \pm i\beta^u\}$ with $\beta^s, \beta^u \neq 0$ and $\lambda^s \neq \lambda^u$,
(ii) $\sigma^c = \{-\lambda^s, \lambda^u \pm i\beta^u\}$ with $\beta^u \neq 0$ and $\lambda^s > \lambda^u$,
(iii) $\sigma^c = \{-\lambda^s \pm i\beta^s, \lambda^u\}$ with $\beta^s \neq 0$ and $\lambda^u > \lambda^s$.

Here, all eigenvalues are counted with multiplicity, that is they are all simple.

We denote the corresponding spectral projections by P^s_0, P^c_0 and P^u_0. The first lemma is concerned with linearizing the vector field on the center manifold.

Lemma 4.1 *Consider*

$$\dot{x} = Bx + f(x), \qquad x \in \mathbb{R}^n \tag{4.2}$$

with $f \in C^{1,\beta}$ for $\beta > 0$ such that $f(0) = Df(0) = 0$. We denote the flow by $\Phi(t,x)$. Assume that the spectrum of B is given by σ^c counted with multiplicity, see hypothesis (H). In particular, $2 \leq n \leq 4$. Then, there exists a map $h : U_\delta(0) \to \mathbb{R}^n \in C^1$ with $h(0) = Dh(0) = 0$ for $\delta > 0$ sufficiently small such that $id + h$ conjugates the nonlinear flow $\Phi(t,\cdot)$ and the linearized flow e^{-Bt} in $U_\delta(0)$.

Proof. Consider the local stable and unstable manifolds $W^s_{loc}(0)$ and $W^u_{loc}(0)$, which are of class $C^{1,\beta}$. Thus, there exists a $C^{1,\beta}$-map $g : U_\delta \to \mathbb{R}^n$ satisfying $g(0) = Dg(0) = 0$, such that the stable and unstable eigenspaces are invariant under the transformed flow $\hat{\Phi}_t := (id+g)^{-1} \circ \Phi_t \circ (id+g)$. In general, the transformed vector field would be of class C^β only. However, $\hat{\Phi}_t$ is still of class $C^{1,\beta}$. Therefore, by the results of Belitskii, we obtain the existence of a conjugacy $id+\hat{h}$ conjugating e^{-B} and $\hat{\Phi}(1,\cdot)$, which fulfills $\hat{h}(0) = D\hat{h}(0) = 0$. Indeed, using the notation in [Bel73] we can apply [Bel73, Thm. A, p. 276] by defining $G = \Lambda = F'(0) := e^{-B}$ and $F := \hat{\Phi}(1,\cdot)$. Moreover, observe that (still in the notation in [Bel73])

$$I(\Gamma, \Lambda; 1, \beta) = [e^{(-\lambda^s + \beta\lambda^u)}, e^{(\lambda^u - \beta\lambda^s)}]$$

holds using the definition $\Gamma := \{\{1\},\{2\},\{1,2\}\}$. Thus, $-\lambda^s, \lambda^u \notin I(\Gamma,\Lambda;1,\beta)$ and the assumptions of [Bel73, Thm. A, p. 276] are fulfilled. The result for flows follows now as in [Har82, IX.9]. Define

$$id + \tilde{h} := \int_0^1 e^{Bs} \circ (id + \hat{h}) \circ \hat{\Phi}(s,\cdot)\, ds.$$

Then, we have $\tilde{h}(0) = 0$ and $id + D\tilde{h}(0) = \int_0^1 e^{Bs}\,(id + D\hat{h}(0))\, D\hat{\Phi}(s,0)\, ds = id$. Therefore, $id+\tilde{h}$ is invertible on $U_\delta(0)$ for all sufficiently small $\delta > 0$. Moreover, $(id+\tilde{h})$ is a conjugacy of e^{-Bt} and $\hat{\Phi}(t,\cdot)$, see [Har82, IX.9]. Hence, the desired conjugacy of Φ_t and e^{-Bt} is given by $(id+g) \circ \tilde{h} \in C^1$. □

Using this result, we obtain the generalization of the Shilnikov-chaos to semilinear parabolic equation. Indeed, by the results of Deng [Den93, Rem.(c) following Thm. 2.1] or Tresser [Tre84] it is sufficient to have a C^1-linearization result available in order to prove the existence of shift-dynamics near a homoclinic orbit to a saddle-focus. In fact, we have proved the following result.

Theorem 3 *Consider equation (4.1) and assume (H), (H2) and (H3) with $\alpha^s = \lambda^s + \gamma$, $\alpha^{ss} = \lambda^s + 2\gamma$, $\alpha^u = \lambda^u + \gamma$ and $\alpha^{uu} = \lambda^u + 2\gamma$ for $\gamma > 0$ small. Then there exists a* *center manifold W^c_{hom}. Moreover, the results of Tresser [Tre84] and Deng [Den93] apply to the flow on the center manifold W^c_{hom}. In particular, there exists horseshoes nearby the homoclinic orbit.*

It is easy to see that hypothesis (H2) is equivalent to the strong inclination property needed in [Den93] in the finite-dimensional setting. Hence, the assumptions needed here are not stronger than the ones used in [Den93]. Note that even in the finite-dimensional case, the center manifold provides some more insight in the dynamics as it shows that the horseshoe dynamics is confined to an invariant three- or four-dimensional manifold.

We end by stating that there exists a three-dimensional center manifold in the inclination-flip and orbit-flip bifurcations investigated by [Yan87], [HKK94] and [San93]. The details will appear elsewhere.

A Exponential dichotomies

The variational equation admits exponential trichotomies for $t \leq -t_0$ and $t \geq t_0$ for some large t_0, that is there exists complementary projections $P^i(t)$ and $\hat{P}^i(t)$ for $t \geq t_0$ and $t \leq -t_0$, respectively, and $i = s, c, u$ with the properties

$$\text{(A.1)} \qquad \begin{array}{rclr} |\mathcal{T}(t,s)P^s(s)|_\alpha & \leq & K\, e^{-\alpha^{ss}(t-s)} & t \geq s \geq t_0 \\ |\mathcal{T}(t,s)P^c(s)|_\alpha & \leq & K\, e^{-\alpha^{u}(t-s)} & t \geq s \geq t_0 \\ |\mathcal{T}(t,s)P^c(s)|_\alpha & \leq & K\, e^{-\alpha^{s}(s-t)} & s \geq t \geq t_0 \\ |\mathcal{T}(t,s)P^u(s)|_\alpha & \leq & K\, e^{\alpha^{uu}(t-s)} & s \geq t \geq t_0 \end{array}$$

and similar for negative times

$$\text{(A.2)} \qquad \begin{array}{rclr} |\mathcal{T}(t,s)\hat{P}^s(s)|_\alpha & \leq & K\, e^{-\alpha^{ss}(t-s)} & s \leq t \leq -t_0 \\ |\mathcal{T}(t,s)\hat{P}^c(s)|_\alpha & \leq & K\, e^{-\alpha^{u}(t-s)} & s \leq t \leq -t_0 \\ |\mathcal{T}(t,s)\hat{P}^c(s)|_\alpha & \leq & K\, e^{-\alpha^{s}(s-t)} & t \leq s \leq -t_0 \\ |\mathcal{T}(t,s)\hat{P}^u(s)|_\alpha & \leq & K\, e^{\alpha^{uu}(t-s)} & t \leq s \leq -t_0. \end{array}$$

Moreover, the projections commute with the semiflow $\mathcal{T}(t,s)$ in the usual way. The next hypothesis is concerned with the matching of these projections at $t = 0$.

($\widetilde{\text{H2}}$) The projections $P^i(t)$ and $\hat{P}^i(t)$ can be continued as exponential trichotomies up to $t = 0$ for $i = s, c, u$ and they satisfy

$$\begin{array}{rcl} \mathrm{R}\hat{P}^u(0) \oplus \big(\mathrm{R}P^s(0) \oplus \mathrm{R}P^c(0)\big) & = & X^\alpha \\ \mathrm{R}P^s(0) \oplus \big(\mathrm{R}\hat{P}^u(0) \oplus \mathrm{R}\hat{P}^c(0)\big) & = & X^\alpha. \end{array}$$

The extension up to $t = 0$ is always possible for ordinary differential equations. For parabolic equations a sufficient condition is backward uniqueness, see [Lin86] or [San93]. Next, we modify the projections of the exponential dichotomies in order to obtain projections defined on the whole real line $\mathbb{R}$. Note that these new projections will not give rise to dichotomies in the sense of [Pal84]. Indeed, we only have a pseudo-hyperbolic structure given by $-\alpha^{ss} < -\alpha^{s} < 0$, while in [Pal84] a separation in stable and unstable exponential rates is required.

Lemma A.1 *There exist projections $\tilde{P}^s(t)$, $\tilde{P}^u(t)$ and $\tilde{P}^c(t)$ for $t \in \mathbb{R}$ possessing the same properties as the original projections $P^i(t)$ and $\hat{P}^i(t)$ for $t \geq 0$ and $t \leq 0$, respectively. Moreover, the inequalities (A.1) and (A.2) still hold and the projections $\tilde{P}^i(t)$ converge to P_0^i for $i = s, c, u$ and $t \to \pm\infty$.*

Proof. The subspaces $\mathrm{R}P^s(0)$, $\mathrm{R}\hat{P}^u(0)$ and $\mathrm{R}(\hat{P}^u(0) + \hat{P}^c(0)) \cap \mathrm{R}(P^s(0) + P^c(0))$ define a decomposition of X^α in complementary and closed subspaces. Using this decomposition we can define complementary projections $\tilde{P}^s(0)$, $\tilde{P}^u(0)$ and $\tilde{P}^c(0)$. By [San93, Lemma 1.2] the claim follows. □

Note that the subspaces $\mathrm{R}P^s(0)$, $\mathrm{R}\hat{P}^u(0)$ and $\mathrm{R}(\hat{P}^u(0) + \hat{P}^c(0)) \cap \mathrm{R}(P^s(0) + P^c(0))$ are unique under the exponential growth conditions stated in (A.1) and (A.2). Therefore, hypothesis (H3) can be replaced by

($\widetilde{\mathrm{H3}}$) There exists a constant $C > 0$ such that $|q(t)|_\alpha \geq Ce^{-\alpha^s t}$ and $|q(-t)|_\alpha \geq Ce^{-\alpha^u t}$ for $t \to \infty$, that is $q(t)$ converges with an exponential rate less than α^s or α^u to zero.

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