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Controlling unstable chaos: Stabilizing chimera states by feedback

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Abstract

We present a control scheme that is able to find and stabilize a chaotic saddle in a system with a large number of interacting particles. This allows us to track a high dimensional chaotic attractor through a bifurcation where it loses its attractivity. Similar to classical delayed feedback control, the scheme is non-invasive, however, only in an appropriately relaxed sense considering the chaotic regime as a statistical equilibrium displaying random fluctuations as a finite size effect. We demonstrate the control scheme for so called chimera states, which are coherence-incoherence patterns in coupled oscillator systems. The control makes chimera states observable close to coherence, for small numbers of oscillators, and for random initial conditions.

Introduction. The classical goal of control is to force a given system to show robustly a behavior a-priori chosen by the engineer (say, track a desired trajectory). However, feedback control can also be an analysis tool in nonlinear dynamics: whenever the feedback input $u(t)$ is zero, i.e the control is *non-invasive*, one can observe natural but dynamically unstable regimes of the uncontrolled nonlinear system such as equilibria or periodic orbits [1]. A famous example is the method of time-delayed feedback control [2], which provides a non-invasive stabilization of unstable periodic orbits and equilibria [3]. In general, a control scheme can be useful for nonlinear analysis if the controlled system converges to an invariant set of the uncontrolled system without requiring particular a-priori knowledge about the location of the invariant set. In this context the term "chaos control" is used to describe the stabilization of an unstable periodic orbit that is embedded into a chaotic attractor. Thus, classical chaos control refers to suppressing chaos [4, 1].

In this paper, we present a control scheme that is able to stabilize a high-dimensional chaotic regime in a system with a large number of interacting particles. Our example is a so called *chimera state*, which is a coherence-incoherence pattern in a system of coupled oscillators. We demonstrate that at its point of disappearance this chaotic attractor turns into a chaotic saddle, which in our numerical simulation we are able to track as a stable object by applying the control scheme. The control scheme is a classical proportional control that acts globally on a spatially extended system, and has been widely used e.g. for the control of reaction-diffusion patterns [5]. For a chaotic regime, it is non-invasive on average in the following sense: (i) $\langle u \rangle \to 0$ for $t \to \infty$: the time average of the control input tends to zero over time intervals of increasing length. (ii) $u \to 0$ for $N \to \infty$: the control becomes small for an increasing number of particles. The limit $N \to \infty$ has been studied in detail for chimera states. Chimera states are stationary solutions of a well-understood continuum limit system [6, 7, 8]. This enables us to compare the chaotic saddle in the finite oscillator system with the corresponding saddle equilibrium in the continuum limit system. However, our control method does not depend on the knowledge of such a limit and it may be useful in general to numerically detect a tipping point of a macroscopic

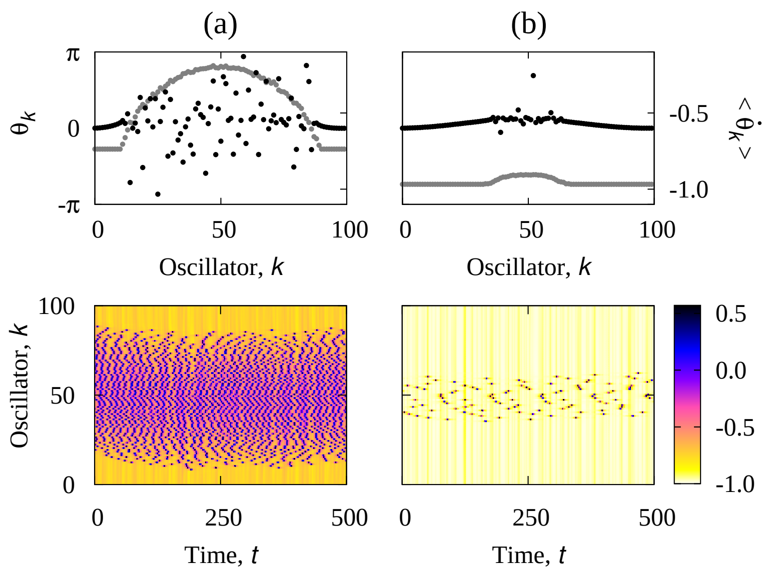

Figure 1: Chimera states far away from complete coherence (a) and close to coherence (b), obtained by numerical simulation of (1), (2) with $A = 0.9$. Upper panels: Snapshot of phases (black) and time-averaged phase velocities (gray). Lower panels: Space-time plots of angular velocities. We require feedback control (6) to observe pattern (b).

state with an irregular motion on a microscopic level. On the other hand, we will show that the proposed control scheme also works for small system size, where the continuum limit provides only a rough qualitative description.

Applying the control scheme permits us to study the macroscopic state in regions of the phase and parameter space that are inaccessible in conventional simulations or experiments. In the coupled oscillator system this reveals several interesting properties of the stabilized chimera states. In the controlled system, we observe a stable branch of chimera states bifurcating from the completely coherent (synchronized) solution. This represents a new mechanism for the emergence of a self-organized pattern from a spatially homogeneous state. We will show that the dynamical regime of a chimera state close to complete coherence can be described as a state of *self-modulated excitability*. Moreover, it turns out that also the chimera states on the primarily stable branch change their stability properties under the influence of the control. It is known that in the uncontrolled system the chimera states have a dormant instability that will lead eventually to a sudden collapse of the pattern [9]. We will show that this collapse can be successfully suppressed by the control. Since the chimera's life-span as a chaotic super-transient [10] increases exponentially with the system size, this collapse suppression provides stable chimera states also for very small system size. In addition to the collapse suppression, the control enlarges the basin of attraction such that random initial conditions converge almost surely to the chimera state, which is both of particular importance for experimental realizations [11, 12, 13, 14, 15].

Chimera states in coupled oscillator systems. A chimera state is a regime of spatially extended chaos [16] that can be observed in large systems of oscillators [17, 18] with non-local coupling. It has the peculiarity that the chaotic motion of incoherently rotating oscillators is confined to a certain region by a self-organized process of pattern formation whereas other oscillators oscillate in a phase-locked coherent manner (see Fig. 1(a)). The prototypical model of coupled phase oscillators has the form

$$\frac{d\theta_k}{dt} = \omega - \frac{2\pi}{N}\sum_{j=1}^{N} G_{kj}\sin(\theta_k - \theta_j + \alpha), \quad k = 1\dots N \tag{1}$$

where the coupling matrix G determines the spatial arrangement of the oscillators. Well-studied cases are rings [17, 18, 19, 7, 16, 9, 20], two-tori [21, 22] and the plane [23, 24]. We choose here a ring of oscillators and

$$G_{kj} = G(x_k - x_j) = \frac{1}{2\pi}[1 + A\cos(x_k - x_j)], \tag{2}$$

where $x_k = 2k\pi/N - \pi$ is the location of oscillator k on the ring and $\theta_k \in [0, 2\pi)$ is its phase. Considering x as a continuous spatial variable, one can derive the continuum limit equation

$$\frac{dz}{dt} = \mathrm{i}\omega z + \frac{1}{2}e^{-\mathrm{i}\alpha}\mathcal{G}z - \frac{1}{2}e^{\mathrm{i}\alpha}z^2\mathcal{G}\overline{z} \tag{3}$$

for the complex local order parameter $z(x,t)$, see [6, 7, 8] for details. The non-local coupling is here given by the integral convolution

$$(\mathcal{G}\varphi)(x) := \int_{-\pi}^{\pi} G(x-y)\varphi(y)\mathrm{d}y.$$

In this limit a chimera state is represented by a uniformly rotating solution of the form

$$z(x,t) = a(x)e^{\mathrm{i}\Omega t}, \tag{4}$$

where Ω is a constant frequency and $a(x)$ is a constant non-uniform spatial profile including coherent regions characterized by $|a(x)| = 1$ and incoherent regions where $|a(x)| < 1$, cf. Fig. 1.

A chimera state with finite N shows temporal and spatial fluctuations around the corresponding stationary limiting profile. The color/shade patterns in Fig. 2(a) show the stationary densities of the global order parameter

$$r(t) = \frac{1}{N}\left|\sum_{k=1}^{N} e^{\mathrm{i}\theta_k(t)}\right|$$

fluctuating around its mean value for a series of chimera trajectories with varying parameter $\beta = \pi/2 - \alpha$. For the continuum limit (3) we obtain a continuous branch of chimera solutions (4) shown as a blue curve in Fig. 2, using the continuum version

$$r(t) = \frac{1}{2\pi}\left|\int_{-\pi}^{\pi} z(x,t)\mathrm{d}x\right| \tag{5}$$

for the global order parameter. Fig. 2(a) shows, that increasing the parameter β beyond 0.22, the chimera state disappears. In the context of the continuum limit $N \to \infty$ this corresponds to a classical fold of the solution branch, which continues as an unstable solution up to the completely coherent state at $(\alpha = \pi/2, r = 1)$.

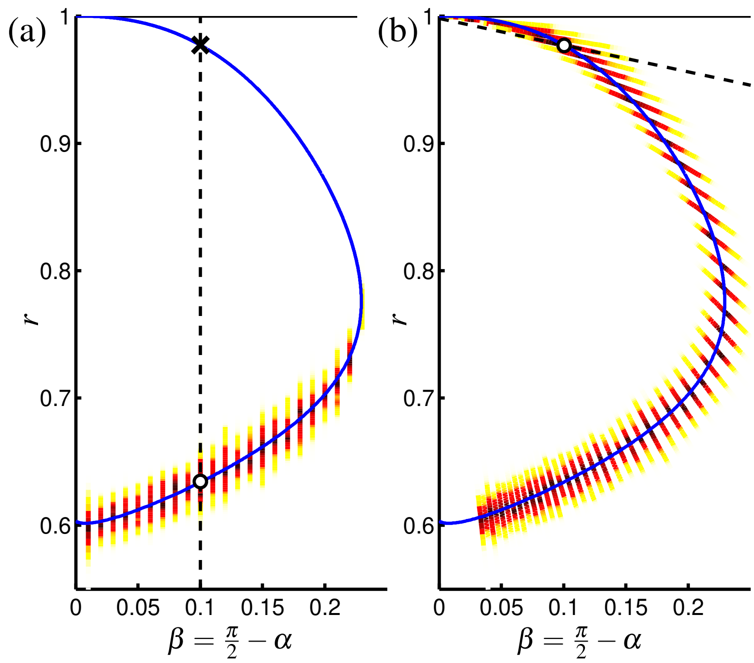

Figure 2: Chimera states projected to the (β, r) plane ($N = 400$, $A = 0.9$). Panel (a): Uncontrolled chimeras; sequence of simulation runs with stepwise increasing parameter β. Panel (b): Controlled chimeras; sequence of simulation runs with stepwise increasing control gain K. Blue curve: numerically computed chimera solution of the continuum limit equation (3). Color/shade patterns: observed density in each run (darker=higher density, see also histograms in Fig. 3). Highlighted runs along dashed lines correspond to the parameter values used in Fig. 1 and Fig. 3.

Control scheme. In order to study this unstable branch in more detail for moderately sized N without relying on the continuum limit, we employ the proportional control scheme

$$\alpha(t) = \alpha_0 + K(r(t) - r_0), \tag{6}$$

where (α_0, r_0) is a reference point in the (α, r)-plane shown in Fig. 2, and the control gain K determines the slope of a straight line along which the controlled system evolves in time (see dashed lines in Fig. 2); $K = 0$ corresponds to a vertical line, $K \to \infty$ to a horizontal line. Note that already in [25] chimera states in a system with a nonlinear state-dependent phase-lag parameter have been investigated. However, depending on the local order parameter this feedback cannot be interpreted as a global non-invasive control of the original system. As the input enters the system parameter α, control (6) also respects all symmetries of the original system. The controlled system (1), (6) has the same symmetries as the uncontrolled system (1): rotational in phase $\theta_k \mapsto \theta_k + \phi$, rotational in space $k \mapsto k + \ell$, reflection in space $k \mapsto N - k$.

In Fig. 2(b) we show another sequence of stationary densities for chimera states in the plane $\beta = \pi/2 - \alpha$ vs. global order parameter r, obtained from numerical simulations of (1), now with control (6) and stepwise increasing control gain K. The reference point has been fixed to $(\alpha_0, r_0) = (\pi/2 + 0.01, 1)$. In this way, we find stabilized chimera states along the whole branch of equilibria from the continuum limit. Fig. 3 shows in more detail the invasiveness of the control for the runs highlighted in Figs. 2(a) and 2(b) by the dashed lines. Whereas for the uncontrolled run the global order parameter r fluctuates around the corresponding equilibrium value from the continuum limit (Fig. 3(a)), in the controlled run both r and β fluctuate around their mean values (Figs. 3(c) and (d)). These fluctuations decrease for an increasing number of oscillators (compare histograms for $N = 100$ and $N = 400$ in Fig. 3). Since for a finite N system the invasiveness of the control is given by the fluctuations of these global quantities, it is non-invasive on average satisfying conditions (i)–(ii) stated above. In the continuum limit (3), the control (6), (5) acts on the solutions (4) in an exactly non-invasive manner and the stabilization can be shown by a classical stability analysis.

Proportional control (6) is only one option to achieve non-invasive control for a chaotic saddle in the relaxed sense of conditions (i)–(ii). Alternatives are any non-invasive methods for stabilization of unknown equilibria. For example, a PI (proportional-integral) control was used in [26] to explore the saddle-type branch of a partially synchronized regime in a small-world network in the continuum limit. PI control adjusts the system parameter to achieve a prescribed output (requiring to find two control gains). Thus, it is also non-invasive in a branch tracking context in the relaxed sense. Time-delayed feedback or wash-out filters [27] are suitable near instabilities other than folds of the continuum-limit equilibrium; for instance in [28], time-delayed feedback has been used to suppress or enhance synchronization in a system of globally coupled oscillators.

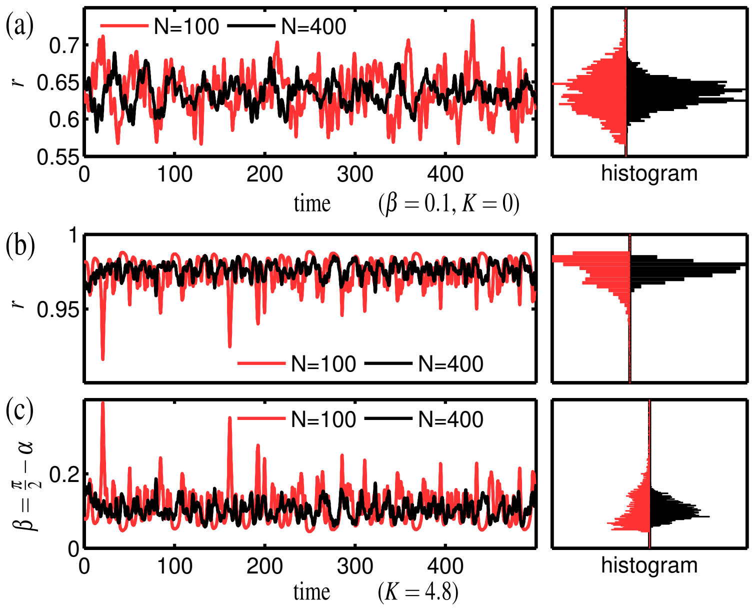

Figure 3: Time profiles and histograms of global order parameter r for chimera without control (a), and r and β for chimera with feedback control (b) and (c), for $N = 100$ and $N = 400$ oscillators ($K = 4.8$ for (b,c), $A = 0.9$).

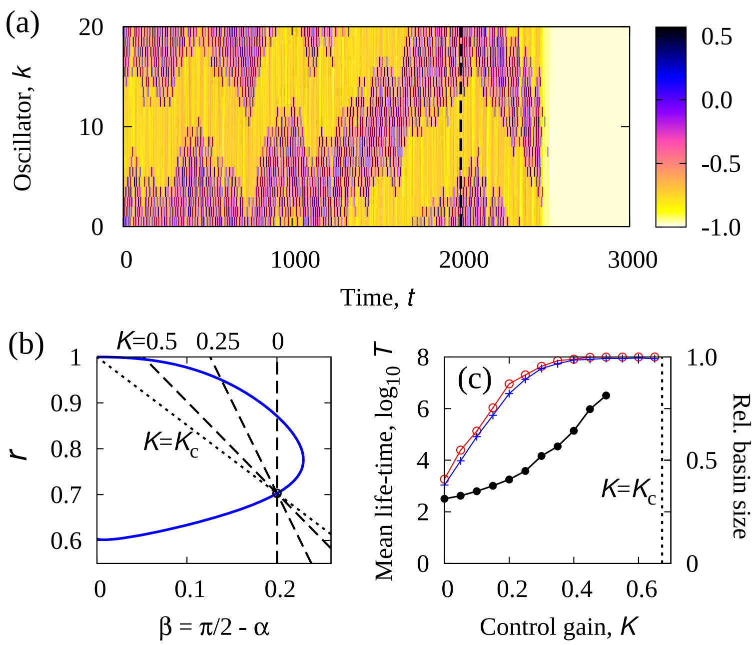

Figure 4: Influence of the control on a stable chimera state. Panel (a): Switching of the control with $K = 1$ at $t = 2000$ permits the subsequently observed collapse for $N = 20$. Panel (b): Controlling the same chimera state with increasing values of the control gain K. Panel (c): mean life-time before collapse for $N = 20$ (dots); fraction of random initial conditions attracted by the chimera state for $N = 20$ (circles) and $N = 100$ (crosses)

Suppression of collapse and enlarged basins. We study now the influence of the control scheme on the lower branch of classical chimera states far from complete coherence, which are already stable in the continuum limit without the control. As described in [9], the classical chimera states from time to time show a sudden transition to the completely coherent state and have to be considered as weakly chaotic type-II supertransients [10]. The life-time before collapse increases exponentially with the system size which implies that chimera states disappear quickly for $N \approx 20$ (cf. Fig. 4(a)), whereas they typically appear as stable objects for any observable time-span if $N > 100$. The collapse process can be understood as follows. Driven by finite size fluctuations, the trajectory can tunnel the barrier represented by the chimera on the unstable branch and eventually reach the stable coherent state. Applying the control, this scenario changes drastically: Increasing the control gain K, the mean life-time before collapse increases by several orders of magnitude and, at the same time, the basin of attraction of the chimera state grows correspondingly. Fig. 4(c) shows the average observed life-times for increasing values of K. In our simulations over 10^7 time units, which we performed for each K, the number of observed collapses decreased successively until for $K > 0.5$, we did not observe a single collapse event during this time span. Finally, for $K \geq K_c \approx 0.67$ the chaotic saddle acting as a barrier disappears and the completely coherent state becomes unstable, which ultimately prevents a collapse to this state. Accordingly, all random initial data converged to the chimera state. Note that we have chosen the reference point on the chimera branch, see Fig. 4(b), such that the given chimera state exists for all values of the control gain K. Hence, with feedback control stable chimera states can be observed for considerably smaller values of N, and arbitrary initial conditions, which is of particular importance for experimental realizations.

Self-modulated excitability close to coherence. Up to now, stable chimera states have been observed only far from the completely coherent solution, except for the results in [29] where the onset of incoherence has been triggered by an inhomogeneous stimulation profile. In the controlled system (1), (6) there is a stable branch of chimera states bifurcating from complete coherence in a homogeneous setting. This is another example of a pattern forming bifurcation mechanism in a system with diffusive coupling. The chimera states close to complete coherence display particular properties distinguishing them from classical chimera states. Fig. 1(b) shows that the onset of incoherence manifests itself as the emergence of isolated excitation bursts caused by phase slips of single or few oscillators, which appear irregular in space and time but are confined by a process of self-localization to a certain region. Indeed, close to the bifurcation point the dynamics of each single oscillator is close to a saddle-node-on-limit-cycle bifurcation. Hence, the emergence of a chimera state can be understood as a transition from quiescent to oscillatory behavior, which happens in a self-localized excitation region within a discrete excitable medium. At the same time, the isolated phase slipping events are no more well described by the average quantities from the continuum limit, which are continuous in space and constant in time.

Conclusion. We demonstrate, that a feedback control that is non-invasive in our relaxed sense is useful to explore complex dynamical regimes in large coupled systems. In particular, it can be used to classify the disappearance of a chaotic attractor as a transition to a chaotic saddle, which is the classical scenario for so-called *tipping*, without relying on a closed-form continuum limit. Specific to partial coherence, feedback control is applicable to existing experimental setups of coupled oscillators [12, 13, 15] as the coupling is often computer controlled. Feedback control makes it possible to study the phenomenon of partial coherence for much smaller N, close to complete coherence, and without specially prepared initial conditions.

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