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Direct and inverse interaction problems with bi-periodic interfaces between acoustic and elastic waves

Guanghui Hu[1], Andreas Kirsch[2]

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[1] Weierstrass Institute
Mohrenstr. 39
10117 Berlin
Germany
E-Mail: guanghui.hu@wias-berlin.de

[2] Department of Mathematics
KIT
76131 Karlsruhe
Germany
E-Mail: andreas.kirsch@kit.edu

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Weierstraß-Institut für Angewandte Analysis und Stochastik (WIAS)
Leibniz-Institut im Forschungsverbund Berlin e. V.
Mohrenstraße 39
10117 Berlin
Germany

Fax: +49 30 20372-303
E-Mail: preprint@wias-berlin.de
World Wide Web: http://www.wias-berlin.de/

Abstract

Consider a time-harmonic acoustic plane wave incident onto a doubly periodic (biperiodic) surface from above. The medium above the surface is supposed to be filled with homogeneous compressible inviscid fluid with a constant mass density, whereas the region below is occupied by an isotropic and linearly elastic solid body characterized by the Lamé constants. This paper is concerned with direct (or forward) and inverse fluid-solid interaction (FSI) problems with unbounded bi-periodic interfaces between acoustic and elastic waves. We present a variational approach to the forward interaction problem with Lipschitz interfaces. Existence of quasi-periodic solutions in Sobolev spaces is established at arbitrary frequency of incidence, while uniqueness is proved only for small frequencies or for all frequencies excluding a discrete set. Concerning the inverse problem, we show that the factorization method by Kirsch (1998) is applicable to the FSI problem in periodic structures. A computational criterion and a uniqueness result are justified for precisely characterizing the elastic body by utilizing the scattered acoustic near field measured in the fluid.

1 Introduction

Consider a time-harmonic acoustic plane wave incident onto an unbounded doubly periodic (or bi-periodic) surface from above; see Figure 1. The medium above the surface is supposed to be filled with homogeneous compressible inviscid fluid with a constant mass density, whereas the region below is occupied by an isotropic and linearly elastic solid body characterized by the Lamé constants. Due to the external incident acoustic field, an elastic wave propagating downward is incited inside the solid, while the incident acoustic wave is scattered back into the fluid. This leads to the fluid-structure interaction (FSI) problem with unbounded bi-periodic interfaces between acoustic and elastic waves, which has many applications in underwater acoustics, sonic and photonic crystals as well as in the field of ultrasonic non-destructive evaluation (NDE); see [10, 7, 22, 35] and references therein. In particular, the investigation of surface (or Rayleigh) waves can be important in developing new surface acoustic wave devices and planar actuators ([10]). Note that periodic interfaces are widely used in the real world, e.g., a material's surface preparation, grain structure, lamimation and fiber reinforecement. These applications motivate us to rigorously investigate FSI problems in periodic structures and the associated inverse problems, for which a vast literature by far has only come from engineering community.

Since Lord Rayleigh's original work [29], grating diffraction problems have received much attention in both the physical and mathematical communities. Consequently, the scattering of pure acoustic, elastic or electromagnetic waves has been studied extensively concerning theoretical analysis and numerical approximation, using integral equation methods (e.g., [34], [31], [1]) or variational methods (e.g., [24], [11], [5], [12]). In particular, the variational approach appears to be well adapted to the analytical and numerical treatment of rather general two-dimensional and three-dimensional periodic diffractive structures involving complex materials and non-smooth interfaces. To investigate the forward FSI problem, we establish an equivalent variational formulation in a bounded periodic cell involving two nonlocal boundary operators. Relying on properties of the Dirichlet-to-Neumann maps for the Helmholtz and Navier equations, we show existence of solutions in quasi-periodic Sobolev spaces by establishing the Fredholmness of the operator generated by the corresponding sesquilinear form. Moreover, uniqueness is proved only for small frequencies or for all frequencies excluding a discrete set. A non-uniqueness example in Lemma

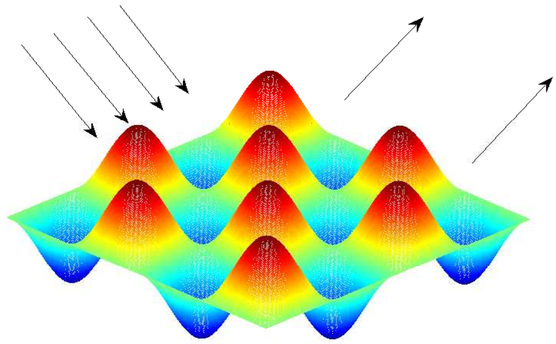

Figure 1: Scattering of plane waves from an egg-crate shaped bi-periodic surface in $\mathbb{R}^3$.

3.4 shows that uniqueness does not hold in general, even if the interface is given by the graph of some smooth bi-periodic function. This is in sharp difference from the result in [24] for the pure Helmholtz equation and that in [13] for the pure Lamé system, where the uniqueness is proved via periodic Rellich's identities under the assumption that the underlying scattering interface is given by a graph. The variational argument developed for the forward scattering problem extends to the quasi-periodic boundary value problems introduced in Section 4.2 for all but possibly a discrete set of frequencies. These boundary value problems will play important roles in justifying the factorization method for the inverse FSI problem.

Our inverse FSI problem consists of recovering the bi-periodic interface from the scattered acoustic near field measured in the fluid. We show that the factorization method, which was first put forward by Kirsch [23] for identifying bounded obstacles, is applicable to the inverse FSI problem under consideration. Such a method requires neither computation of direct solutions nor initial guesses, and it provides a sufficient and necessary condition for precisely characterizing the unknown interface; see Section 4.5. Our theoretical justification of the factorization method combines the original ideas in [3, 4] for inverse grating diffraction problems modeled by the Helmholtz equation and the novel idea used in [26] for inverse FSI problems with an bounded elastic body. In the present study, the auxiliary boundary value problem (III) (see Section 4.2) will play the same role as the interior transmission eigenvalue problem occurring in [26]. In contrast to the normality of the far-field operator in [26], the near-field operator arising from grating diffraction problems fails to be normal. Thanks to properties of the middle operator shown in Lemma 4.4, we can still apply the widely used range identity [25, Theorem 2.5.1]. A novelty of our analysis is that the denseness of the data-to-pattern operator G is proved in a non-trivial way; see Lemma 4.3. In addition, a small gap in the proof of [26, Lemma 2.5] is filled in Lemma 4.2 of the present paper.

Inversion schemes for locating bounded elastic bodies immersed in fluid can be also found in [14, 15] where an optimization technique was applied and in [32, 33] using the Reciprocity Gap (RG) method and the Linear Sampling Method (LSM). We also refer to [28, 21] for the factorization method in inverse electromagnetic and elastic scattering from diffraction gratings.

The paper is organized as follows. In Section 2 we rigorously formulate the direct and inverse interaction problems with bi-periodic Lipschitz surfaces between acoustic and elastic waves. Section 3 is devoted to the solvability of the forward FSI problem through the variational approach. In Section 4, we present a theoretical justification of the factorization method for precisely characterizing the elastic body in terms of the spectrum of the near-field operator.

We end up this section by introducing some notation that will be used throughout the paper. Denote by $(\cdot)^\top$ the transpose of a vector or a matrix, and by $(\cdot)^*$ the adjoint of an operator. The symbols $e_j, j = 1,2,3$, denote the Cartesian unit vectors in $\mathbb{R}^3$. For $a \in \mathbb{C}$, let $|a|$ denote its modulus, and for **a** $\in \mathbb{R}^3$, let $|$**a**$|$ denote its Euclidean norm. The notation **a** $\cdot$ **b** stands for the inner product $\sum_{j=1}^3 a_j b_j$ of **a** $= (a_1, a_2, a_3)$, **b** $= (b_1, b_2, b_3) \in \mathbb{C}^3$. For $x = (x_1, x_2, x_3) \in \mathbb{R}^3$, we write $\tilde{x} = (x_1, x_2)$ so that $x = (\tilde{x}, x_3)$.

2 Formulations of direct and inverse scattering problems

We assume an incident acoustic wave v^{in} is incident onto a bi-periodic Lipschitz surface $\Gamma \subset \mathbb{R}^3$ from above. Without loss of generality we suppose that Γ is 2π-periodic in x_1 and x_2, i.e.,

$$x = (\tilde{x}, x_3) \in \Gamma \quad \Rightarrow \quad (\tilde{x} + 2n\pi, x_3) \in \Gamma, \quad \text{for all} \quad n = (n_1, n_2) \in \mathbb{Z}^2.$$

Denote by Ω^+ the region above Γ, which is filled with a homogeneous compressible inviscid fluid with the constant mass density $\rho_f > 0$. Let v^{in} be a time-harmonic plane wave with frequency $\omega > 0$ and speed of sound $c_0 > 0$, taking the form

$$v^{in} = \exp(ik\hat{\theta} \cdot x), \quad \hat{\theta} = (\sin\theta_1 \cos\theta_2, \sin\theta_1 \sin\theta_2, -\cos\theta_1) \in \mathbb{S}^2 := \{x \in \mathbb{R}^3 : |x| = 1\}, \tag{1}$$

where, $\hat{\theta}$ denotes the incident direction with the incident angles $\theta_1 \in [0, \pi/2), \theta_2 \in [0, 2\pi)$, and $k = \omega/c_0$ is the wave number in the fluid. We assume the region below Γ, denoted by Ω^-, is occupied by an isotropic and linearly elastic solid body characterized by the real valued constant mass density $\rho > 0$ and the Lamé constants $\lambda, \mu \in \mathbb{R}$ satisfying $\mu > 0, 3\lambda + 2\mu > 0$. The domain Ω^- is supposed to satisfy a cone condition. Note that this assumption will only be used in the proof of Lemma 4.2 for the inverse problem.

Under the hypothesis of small amplitude oscillations both in the solid and the fluid, the direct or forward scattering problem can be formulated as the following boundary value problem: Find the total acoustic field $v = v^{in} + v^{sc}$ and the transmitted elastic field u generated from a known (prescribed) incident wave v^{in} such that (see e.g. [30, 20, 35])

$$\left\{ \begin{array}{lll} (\Delta + k^2)\, v = 0 & \text{in} \;\; \Omega^+, & \\ (\Delta^* + \omega^2 \rho)\, u = 0 & \text{in} \;\; \Omega^-, & \Delta^* := \mu\Delta + (\lambda + \mu)\, \mathrm{grad}\, \mathrm{div}\,, \\ \eta\, u \cdot \nu = \partial_\nu v, & \text{on} \;\; \Gamma, & \eta := \rho_f \omega^2 > 0, \\ Tu = -v\, \nu, & \text{on} \;\; \Gamma. & \end{array} \right. \tag{2}$$

Here, the notation $\nu = (\nu_1, \nu_2, \nu_3) \in \mathbb{S}^2$ denotes the unit normal vector on Γ pointing into Ω^- and $\partial_\nu u = \nu \cdot \nabla u$. Similarly, we shall use the symbol $\partial_j u$ to denote $\partial u / \partial x_j$. In (2), Tu stands for the *stress vector* or *traction* having the form:

$$Tu = T(\lambda, \mu) u := 2\mu\, \partial_\nu u + \lambda (\mathrm{div}\, u)\, \nu + \mu\, \nu \times \mathrm{curl}\; u, \quad \text{on} \quad \Gamma. \tag{3}$$

By Betti's formula (see e.g. [27]), the above stress operator plays the role of the normal derivative in the scalar Helmholtz equation.

Throughout the paper, we write $\alpha = (\alpha_1, \alpha_2) := k(\sin\theta_1\cos\theta_2, \sin\theta_1\sin\theta_2) \in \mathbb{R}^2$. Obviously, the incident field v^{in} is α-quasiperiodic in the sense that $v^{in}(x)\exp(-i\alpha\cdot\tilde{x})$ is 2π-periodic with respect to x_1 and x_2. The periodicity of the structure together with the form of the incident wave implies that the solution (v, u) must also be α-quasiperiodic, i.e., for $w = v$ in Ω^+ and $w = u$ in Ω^- it holds that

$$w(\tilde{x} + 2n\pi, x_3) = \exp(2i\alpha\cdot n\pi)\, w(x_1, x_2, x_3), \qquad \forall\, n = (n_1, n_2) \in \mathbb{Z}^2. \tag{4}$$

Since the domain $\Omega^{\pm}$ is unbounded in the $\pm x_3$-direction, a radiation condition must be imposed at infinity to ensure well-posedness of the boundary value problem (2). Let

$$\Gamma^+ := \max_{x\in\Gamma}\{x_3\}, \quad \Gamma^- := \min_{x\in\Gamma}\{x_3\}.$$

Following [24], we require the scattered acoustic field v^{sc} to satisfy the upward Rayleigh expansion (see also [16, 2, 5])

$$v^{sc} \;=\; \sum_{n\in\mathbb{Z}^2} v_n \exp(i\alpha_n\cdot\tilde{x} + i\eta_n x_3), \quad x_3 > \Gamma^+, \tag{5}$$

with the Rayleigh coefficients $v_n \in \mathbb{C}$. The parameters $\alpha_n = (\alpha_n^{(1)}, \alpha_n^{(2)}) \in \mathbb{R}^2$ and $\eta_n \in \mathbb{C}$ given respectively by

$$\alpha_n = \alpha + n \in \mathbb{R}^2, \quad \eta_n = \begin{cases} (k^2 - |\alpha_n|^2)^{\frac{1}{2}} & \text{if } |\alpha_n| \le k, \\ i(|\alpha_n|^2 - k^2)^{\frac{1}{2}} & \text{if } |\alpha_n| > k, \end{cases} \qquad \text{for } n \in \mathbb{Z}^2. \tag{6}$$

To see the corresponding expansion of the elastic field, we decompose it into the compressional and shear parts,

$$u = \frac{1}{i}\left(\operatorname{grad}\varphi + \operatorname{curl}\psi\right) \quad \text{with} \quad \varphi := -\frac{i}{k_p^2}\operatorname{div} u\,, \quad \psi := \frac{i}{k_s^2}\operatorname{curl} u\,, \tag{7}$$

where the scalar functions φ, ψ satisfy the homogeneous Helmholtz equations

$$(\Delta + k_p^2)\,\varphi = 0 \quad \text{and} \quad (\Delta + k_s^2)\,\psi = 0 \quad \text{in} \quad \Omega^-\,. \tag{8}$$

In (8), k_p and k_s denote respectively the compressional and shear wave numbers defined as

$$k_p := \omega\rho/\sqrt{2\mu + \lambda}\,, \quad k_s := \omega\rho/\sqrt{\mu}.$$

Applying the Rayleigh expansion for the scalar Helmholtz equation to φ and ψ, respectively, we finally obtain a corresponding expansion of u into downward propagating plane elastic waves

$$u = \sum_{n\in\mathbb{Z}^2}\left\{A_{p,n}\begin{pmatrix}\alpha_n^\top \\ -\beta_n\end{pmatrix}\exp(i\alpha_n\cdot\tilde{x} - i\beta_n x_3) + \mathbf{A}_{s,n}\exp(i\alpha_n\cdot\tilde{x} - i\gamma_n x_3)\right\},\ x_3 < \Gamma^-, \tag{9}$$

with the Rayleigh coefficients $A_{p,n} \in \mathbb{C}$ and $\mathbf{A}_{s,n} = (A_{s,n}^{(1)}, A_{s,n}^{(2)}, A_{s,n}^{(3)}) \in \mathbb{C}^3$ satisfying the orthogonality

$$\mathbf{A}_{s,n}\cdot(\alpha_n, -\gamma_n)^\top = 0, \qquad \text{for all } n \in \mathbb{Z}^2. \tag{10}$$

The parameters β_n resp. γ_n occurring (9) are defined analogously to η_n with k replaced by k_p resp. k_s. Denote by u_p and u_s the compressional and shear parts of u, respectively, i.e., for $x_3 < \Gamma^-$,

$$u_p = \sum_{n\in\mathbb{Z}^2} \left\{ A_{p,n} \left(\begin{array}{c} \alpha_n^\top \\ -\beta_n \end{array} \right) \exp(i\alpha_n \cdot \tilde{x} - i\beta_n x_3) \right\}, \quad u_s = \sum_{n\in\mathbb{Z}^2} \left\{ \mathbf{A}_{s,n} \exp(i\alpha_n \cdot \tilde{x} - i\gamma_n x_3) \right\}.$$

Then, it is obvious that $u = u_p + u_s$ and

$$(\Delta + k_p^2)\, u_p = 0, \quad \operatorname{curl} u_p = 0, \quad (\Delta + k_s^2)\, u_s = 0, \quad \operatorname{div} u_s = 0 \quad \text{in} \quad \Omega^-.$$

Since η_n, β_n and γ_n are real for at most finitely many indices $n \in \mathbb{Z}^2$, we observe that only the finite number of plane waves in (5) corresponding to $|\eta_n| \le k$ and those in (9) corresponding to $|\beta_n| \le k_p$ or $|\gamma_n| \le k_s$ propagate into the far field, while the remaining part consists of evanescent (or surface) waves decaying exponentially as $|x_3| \to +\infty$. Thus, the above Rayleigh expansion (5) resp. (9) converges uniformly with all derivatives in the half-space $\{x_3 > b\}$ for any $b > \Gamma^+$ resp. $\{x_3 < a\}$ for any $a < \Gamma^-$.

Now, we can formulate our direct scattering problem as the following boundary value problem, in which the interface Γ is not necessarily the graph of a bi-periodic function.

(DP): Given a bi-periodic Lipschitz surface $\Gamma \subset \mathbb{R}^3$ (which is 2π-periodic in x_1 and x_2) and an incident field v^{in} of the form (1), find a scalar function $v = v^{in} + v^{sc} \in H^1_{loc}(\Omega^+)$ and a vector function $u \in H^1_{loc}(\Omega^-)^3$ that satisfy the equations and transmission conditions in (2), the quasi-periodic condition (4) and the radiation conditions (5) and (9).

Since the evanescent (or surface) waves in (5) can be hardly measured in the fluid far away from the interface, we shall use near-field rather than far-field data to recover the interface. Our concern on the inverse problem is to detect Γ from knowledge of the scattered acoustic near field $v^{sc}(\tilde{x}, b)$ ($b > \Gamma^+$) measured above the interface. In the inversion algorithms, we shall send several incident waves from the admissible set $\mathcal{I} = \{v_j^{in} : -M \le j \le M\}$ for some $M \in \mathbb{N}^+$, and then record the corresponding near-field data for each incident wave. More precisely, the inverse problem under consideration can be formulated as follows:

(IP): Recover the scattering interface Γ from the scattered near-field data $\{v_j^{sc}(\tilde{x}, b)\colon |j| \le M\}$ for some $b > \Gamma^+$ and $M \in \mathbb{N}^+$, where $v_j^{sc}(\tilde{x}, x_3)$ denotes the scattered acoustic field in the fluid generated by the incident wave $v_j^{in} \in \mathcal{I}$.

3 Solvability of direct problem

In this section we propose an equivalent variational formulation of (DP), based on the approach of [24, 16] and [12, 13] for the scattering of acoustic and elastic waves by diffraction gratings. Thanks to the periodicity of the unbounded domains $\Omega^\pm$, we will restrict ourselves to one single periodic cell $(0, 2\pi) \times (0, 2\pi)$ where the compact imbedding of Sobolev spaces can be applied. This, together with Friedrich's inequality for the Helmholtz equation and Korn's inequality for the Navier equation, enables us to justify strong ellipticity of the sesquilinear form generated by the variational formulation.

We begin with introducing artificial boundaries

$$\Gamma_b^\pm := \{(x_1, x_2, \pm b) :\ 0 \le x_1, x_2 \le 2\pi\}, \quad \pm b \gtrless \Gamma^\pm,$$

and the bounded domains

$$\Omega_b^\pm := \{x \in \Omega^\pm :\ 0 < x_1, x_2 < 2\pi,\ x_3 \lessgtr \pm b\}\,.$$

For simplicity we still use Γ to denote one period of the grating surface; see Figure 2. Since Γ is a Lipschitz surface, $\Omega_b^\pm$ are bounded Lipschitz domains in $\mathbb{R}^3$. Let $H_\alpha^1(\Omega_b^\pm)$ denote the Sobolev space of

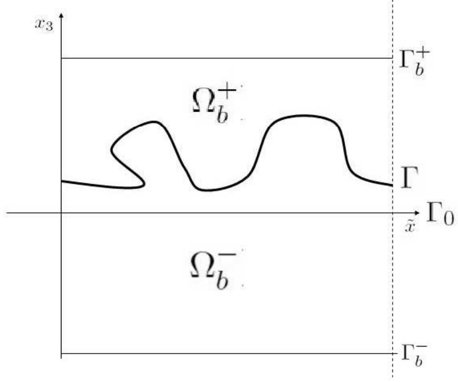

Figure 2: The geometry settings in one periodic cell. For $\tilde{x} = (x_1, x_2) \in (0, 2\pi) \times (0, 2\pi)$, $\Gamma_b^\pm := \{(\tilde{x}, \pm b)\}$. $\Omega_b^\pm$ denotes the domain between $\Gamma_b^\pm$ and Γ.

scalar functions on $\Omega_b^\pm$ which are α-quasiperiodic with respect to x_1 and x_2. Introduce the energy space

$$V = V(\alpha) := V^+ \times V^-, \quad V^+ := H_\alpha^1(\Omega_b^+), \quad V^- := H_\alpha^1(\Omega_b^-)^3.$$

equipped with the norm in the usual product space of $H^1(\Omega_b^+) \times H^1(\Omega_b^-)^3$. Using the transmission conditions in (2), it follows from Green's and Betti's formulas that for $(\varphi, \psi) \in V$,

$$\begin{aligned} -\int_{\Omega_b^+} (\Delta + k^2) v\,\overline{\varphi}\,dx &= \int_{\Omega_b^+} \left[\nabla v \cdot \nabla\overline{\varphi} - k^2 v\overline{\varphi}\right] dx - \eta \int_\Gamma u \cdot \nu\overline{\varphi}\,ds - \int_{\Gamma_b^+} \partial_\nu v\overline{\varphi}\,ds\,, \\ -\int_{\Omega_b^-} (\Delta^* + \omega^2\rho) u \cdot \overline{\psi}\,dx &= \int_{\Omega_b^-} \left[\mathcal{E}(u, \overline{\psi}) - \omega^2\rho u \cdot \overline{\psi}\right] dx - \int_\Gamma v\,\nu \cdot \overline{\psi} ds - \int_{\Gamma_b^-} Tu \cdot \overline{\psi}\,ds\,, \end{aligned} \tag{11}$$

where the bar indicates the complex conjugate, T is the stress vector defined by (3) and

$$\mathcal{E}(u, \psi) = 2\mu \left(\sum_{i,j=1}^{3} \partial_i u_j\ \partial_i \psi_j\right) + \lambda\,(\operatorname{div} u)(\operatorname{div} \psi) - \mu \operatorname{curl} u \cdot \operatorname{curl} \psi. \tag{12}$$

Now we introduce the Dirichlet-to-Neumann (DtN) maps $\mathcal{T}^\pm$ on the artificial boundaries $\Gamma_b^\pm$.

Definition 3.1. *For any $w \in H_\alpha^{1/2}(\Gamma_b^+)$, the DtN operator $\mathcal{T}^+ w$ is defined as $\partial_\nu v^{sc}|_{\Gamma_b^+}$, where v^{sc} is the unique α-quasiperiodic solution of the homogeneous Helmholtz equation in $x_3 > b$ which satisfies the upward radiation condition (5) and the Dirichlet boundary value $v^{sc} = w$ on Γ_b^+.*

Analogously, for any $w \in H_\alpha^{1/2}(\Gamma_b^-)^3$ the DtN operator $\mathcal{T}^- w$ is defined as $Tu|_{\Gamma_b^-}$, where u is the unique α-quasiperiodic solution of the homogeneous Navier equation in $x_3 < -b$ which satisfies the downward radiation condition (9) and the Dirichlet boundary value $u = w$ on Γ_b^-.

In this paper we employ the following equivalent norm on $H^s_\alpha(\mathbb{R}^2)$:

$$\|w\|_{H^s_\alpha(\mathbb{R}^2)} = \Big(\sum_{n\in\mathbb{Z}^2} (1+|n|)^{2s}\,|\hat{w}_n|^2 \Big)^{1/2}, \quad s\in\mathbb{R}, \tag{13}$$

where $\hat{w}_n \in \mathbb{C}$ are the Fourier coefficients of $\exp(-i\alpha\cdot\tilde{x})\,w(\tilde{x})$, that is,

$$w = \sum_{n\in\mathbb{Z}^2} \hat{w}_n \exp(i\alpha_n\cdot\tilde{x}).$$

Letting $w \in H^{1/2}_\alpha(\Gamma^\pm_b)$ be given as above, we can derive explicit expressions of the DtN maps from the definitions of $\mathcal{T}^\pm$. Actually, we have

$$\mathcal{T}^+ w = \sum_{n\in\mathbb{Z}^2} i\eta_n\,\hat{w}_n \exp(i\alpha_n\cdot\tilde{x}), \quad \mathcal{T}^- w = \sum_{n\in\mathbb{Z}^2} iW_n\,\hat{w}_n \exp(i\alpha_n\cdot\tilde{x}), \tag{14}$$

where, η_n is defined as in (6) and W_n is the 3×3 matrix taking the form

$$W_n = W_n(\omega,\rho,\alpha) := \frac{1}{|\alpha_n|^2+\beta_n\gamma_n} \begin{pmatrix} a_n & b_n & c_n \\ b_n & d_n & e_n \\ -c_n & -e_n & f_n \end{pmatrix}, \tag{15}$$

with

$$\begin{aligned} a_n &:= \mu[(\gamma_n-\beta_n)(\alpha_n^{(2)})^2 + k_s^2\beta_n], & b_n &:= -\mu\alpha_n^{(1)}\alpha_n^{(2)}(\gamma_n-\beta_n), \\ c_n &:= (2\mu\alpha_n^2 - \omega^2\rho + 2\mu\gamma_n\beta_n)\alpha_n^{(1)}, & e_n &:= (2\mu\alpha_n^2 - \omega^2\rho + 2\mu\gamma_n\beta_n)\alpha_n^{(2)}, \\ d_n &:= \mu[(\gamma_n-\beta_n)(\alpha_n^{(1)})^2 + k_s^2\beta_n], & f_n &:= \gamma_n\omega^2\rho. \end{aligned}$$

The expression of $\mathcal{T}^+$ is well-known (see [24, 16]), whereas that of $\mathcal{T}^-$ is derived in [13]. Making use of the norm (13) and the asymptotic behavior $\eta_n, \beta_n, \gamma_n \sim i|n|$ as $|n|\to\infty$, one can straightforwardly verify that

$$\mathcal{T}^+ : H^{1/2}_\alpha(\mathbb{R}^2) \to H^{-1/2}_\alpha(\mathbb{R}^2), \quad \mathcal{T}^- : H^{-1/2}_\alpha(\mathbb{R}^2)^3 \to H^{1/2}_\alpha(\mathbb{R}^2)^3$$

are both bounded operators. It is worth pointing that the positivity of $-\mathrm{Re}\,\mathcal{T}^+$, i.e., the inequality

$$-\mathrm{Re}\int_{\Gamma_b^+} \mathcal{T}^+ w\,\overline{w}\,ds = (4\pi)^2 \sum_{|\alpha_n|\ge k} |\eta_n|^2\,|\hat{w}_n|^2 \ge 0 \quad \text{for all} \quad w\in H^{1/2}_\alpha(\Gamma_b^+), \tag{16}$$

does not apply to the operator $\mathcal{T}^-$ on $H^{1/2}_\alpha(\Gamma_b^-)^3$ (see [12, 13]). With the definitions of $\mathcal{T}^\pm$, the terms $\partial_\nu v$ and Tu occurring on the right hand sides of (11) can be reformulated as

$$(\partial_\nu v)|_{\Gamma_b^+} = f_0 + \mathcal{T}^+(v|_{\Gamma_b^+}), \quad (Tu)|_{\Gamma_b^-} = \mathcal{T}^-(u|_{\Gamma_b^-}), \tag{17}$$

with

$$f_0 := (\partial_\nu v^{in})|_{\Gamma_b^+} - \mathcal{T}^+(v^{in}|_{\Gamma_b^+}) = -2i\eta_0 \exp(i\alpha\cdot\tilde{x} - i\eta_0 b) \in H^{-1/2}_\alpha(\Gamma_b^-), \tag{18}$$

which follows from the expression of v^{in} in (1). Combining (1) and (11), we obtain the following variational formulation of (DP): Find $(v,u)\in V$ such that

$$A((v,u),(\varphi,\psi)) = \int_{\Gamma_b^+} f_0\,\overline{\varphi}\,ds \quad \text{for all } (\varphi,\psi)\in V, \tag{19}$$

where the sesquilinear form $A: V\times V\to\mathbb{C}$ is defined as

$$\begin{aligned} A((v,u),(\varphi,\psi)) \;&:=\; \int_{\Omega_b^+}\left[\nabla v\cdot\nabla\overline{\varphi} - k^2 v\overline{\varphi}\right]dx - \eta\int_\Gamma u\cdot\nu\overline{\varphi}\,ds - \int_{\Gamma_b^+}\mathcal{T}^+ v\overline{\varphi}\,ds \\ &+\; \eta\left[\int_{\Omega_b^-}\left[\mathcal{E}(u,\overline{\psi}) - \omega^2\rho u\cdot\overline{\psi}\right]dx - \int_\Gamma v\,\nu\cdot\overline{\psi}ds - \int_{\Gamma_b^-}\mathcal{T}^- u\cdot\overline{\psi}ds\right] \end{aligned} \tag{20}$$

for all $(\varphi,\psi)\in V$. The above sesquilinear form obviously generates a continuous linear operator $\mathcal{A}: V\to V'$ such that

$$A((v,u),(\varphi,\psi)) = \langle \mathcal{A}(v,u),(\varphi,\psi)\rangle \quad \text{for all } (\varphi,\psi)\in V. \tag{21}$$

Here V' denotes the dual space of V with respect to the duality $\langle\cdot,\cdot\rangle$ extending the product in $L^2(\Omega_b^+)\times L^2(\Omega_b^-)^3$. Next, we show the strong ellipticity of A in the following lemma.

Lemma 3.2. *The sesquilinear form A defined in (20) is strongly elliptic over V, and the operator $\mathcal{A}$ defined by (21) is always a Fredholm operator with index zero.*

Proof. By [13], the operator $-\mathrm{Re}\,(\mathcal{T}^-)$ can be decomposed into the sum of a positive definite operator $\mathcal{T}_1$ and a finite rank operator $\mathcal{T}_2$ over $H_\alpha^{-1/2}(\Gamma_b^-)^3$. Introduce the sesquilinear forms

$$\begin{aligned} A_1((v,u),(\varphi,\psi)) \;&:=\; \int_{\Omega_b^+}\left[\nabla v\cdot\nabla\overline{\varphi} + v\overline{\varphi}\right]dx - \int_{\Gamma_b^+}\mathcal{T}^+ v\overline{\varphi}\,ds \\ &\quad +\eta\left[\int_{\Omega_b^-}\left[\mathcal{E}(u,\overline{\psi}) + u\cdot\overline{\psi}\right]dx + \int_{\Gamma_b^-}\mathcal{T}_1\,u\cdot\overline{\psi}ds\right], \\ A_2((v,u),(\varphi,\psi)) \;&:=\; -\int_{\Omega_b^+}\left[(1+k^2)v\overline{\varphi}\right]dx - \eta\int_\Gamma u\cdot\nu\overline{\varphi}\,ds \\ &\quad +\eta\left[\int_{\Omega_b^-}\left[-(1+\omega^2\rho)u\cdot\overline{\psi}\right]dx - \int_\Gamma v\,\nu\cdot\overline{\psi}ds + \int_{\Gamma_b^-}\mathcal{T}_2 u\cdot\overline{\psi}ds\right]. \end{aligned}$$

Then we see $A = A_1 + A_2$, and by (16) and Korn's inequality (see e.g., [20, Chap. 5.4] or [12]),

$$\mathrm{Re}\,A_1((v,u),(v,u)) \ge c_1\left(||v||^2_{V^+} + ||u||^2_{V^-}\right) \quad \text{for all } (v,u)\in V,$$

with some constant $c_1>0$. Moreover, applying Cauchy-Schwarz inequality yields

$$\begin{aligned} \mathrm{Re}\,A_2((v,u),(v,u)) \;&\ge\; -c_2\left(||v||^2_{L^2(\Omega_b^+)} + ||v||^2_{L^2(\Gamma)} + ||u||^2_{L^2(\Omega_b^-)^3} + ||u||^2_{L^2(\Gamma)^3}\right) \\ &\quad +\eta\,\mathrm{Re}\,(\mathcal{T}_2 u,u)_{L^2(\Gamma_b^-)^3}, \end{aligned}$$

for some constant $c_2>0$. From the compact imbeddings $H^1(\Omega_b^\pm)\hookrightarrow L^2(\Omega_b^\pm)$, $H^{1/2}(\Gamma)\hookrightarrow L^2(\Gamma)$ and the compactness of $\mathcal{T}_2$, we conclude that the sesquilinear from A is strongly elliptic over $V\times V$. Consequently, the operator $\mathcal{A}$ defined by (21) is always a Fredholm operator with index zero. ☐

From Lemma 3.2 and the Fredholm alternative, it follows that the variational formulation (19) is uniquely solvable if the homogeneous operator equation $\mathcal{A}(v,u)=0$ has only the trivial solutions $v=0, u=0$. However, it is shown below that only the upward resp. downward propagating modes of v^{sc} resp. u can be uniquely determined other than the evanescent modes.

Lemma 3.3. *Assume $(v^{sc},u)\in V$ is a radiating solution pair to the variational problem (19) with $v^{in}=0$ (or equivalently, $f_0=0$). Then there holds*

$$v_n=0 \quad \textit{for} \quad |\alpha_n|<k, \quad A_{p,n}=0 \quad \textit{for} \quad |\alpha_n|<k_p, \quad |\mathbf{A}_{s,n}|=0 \quad \textit{for} \quad |\alpha_n|<k_s,$$

where v_n, $A_{p,n}$ and $\mathbf{A}_{s,n}$ denote the Rayleigh coefficients of v^{sc} and u.

Proof. Taking the imaginary part of (19) with $\varphi=v^{sc}, \psi=u, v^{in}=0$ and using the fact that $\eta>0$, we get

$$-\mathrm{Im}\left(\mathcal{T}^+ v^{sc}, v^{sc}\right)_{L^2(\Gamma_b^+)} - \eta\,\mathrm{Im}\left(\mathcal{T}^- u, u\right)_{L^2(\Gamma_b^-)^3}=0. \tag{22}$$

From the explicit expressions for $\mathcal{T}^+$ and $\mathcal{T}^-$, one can derive that (see [13] for the second identity)

$$\begin{aligned}
&\mathrm{Im}\left(\mathcal{T}^+ v^{sc}, v^{sc}\right)_{L^2(\Gamma_b^+)} = 4\pi^2 \sum_{n:|\alpha_n|<k} \eta_n\,|v_n|^2,\\
&\mathrm{Im}\left(\mathcal{T}^- u, u\right)_{L^2(\Gamma_b^-)} = 4\pi^2\Big(\sum_{n:|\alpha_n|<k_p} \beta_n\,|A_{p,n}|^2\,\omega^2\rho + \sum_{n:|\alpha_n|<k_s} \gamma_n\,|\mathbf{A}_{s,n}|^2\mu\Big).
\end{aligned} \tag{23}$$

Since $\eta_n>0$ for $|\alpha_n|<k$, $\beta_n>0$ for $|\alpha_n|<k_p$ and $\gamma_n>0$ for $|\alpha_n|<k_s$, we complete the proof of Lemma 3.3 by combining (22) and (23). ☐

Hence, we cannot expect uniqueness of solutions to (19) for general bi-periodic Lipschitz interfaces separating the fluid and solid. It is also worth noting that uniqueness does not hold in general, even if Γ is the graph of some smooth bi-periodic function; see the non-uniqueness example below where Γ is a flat surface parallel to the ox_1x_2-plane.

Lemma 3.4. *Assume that $\Gamma=\Gamma_0:=\{x_3=0\}$ is a flat interface and the incident angle $\theta_2=0$. Suppose further that $k=k_p$ and $k=k\sin\theta_1+m_0$ for some $m_0\in\mathbb{Z}$. Then there exists at least one non-trivial radiation solution pair $(v^{sc},u)\in V$ to the homogeneous variational problem $A((v^{sc},u),(\varphi,\psi))=0$ for all $(\varphi,\psi)\in V$.*

Proof. Since the interface Γ_0 is invariant in x_2 and the incident direction $\hat\theta=(\sin\theta_1,0,-\cos\theta_1)$ is orthogonal to the x_2-axis, our scattering problem can be reduced to a problem in the ox_1x_3-plane. Therefore, we will look for radiating solutions v^{sc} and u of the special form

$$\begin{aligned}
v^{sc} &= \sum_{m\in\mathbb{Z}} v_m\, e^{i(\tilde\alpha_m x_1+\eta_m x_3)}, \quad x_3>0,\\
u &= \sum_{m\in\mathbb{Z}} \left(A_{p,m}\begin{pmatrix}\tilde\alpha_m\\0\\-\beta_m\end{pmatrix} e^{i(\tilde\alpha_m x_1-\beta_m x_3)} + A_{s,m}\begin{pmatrix}\gamma_m\\0\\\tilde\alpha_m\end{pmatrix} e^{i(\tilde\alpha_m x_1-\gamma_m x_3)}\right), \quad x_3<0,
\end{aligned}$$

with $v_m, A_{p,m}, A_{s,m} \in \mathbb{C}$, $\tilde{\alpha}_m := \alpha_1 + m = \alpha_n^{(1)}$ for $n = (m, 0)$. Here, $\alpha_1 = k \sin\theta_1$ due to the assumption that $\theta_2 = 0$. The parameters $\eta_m, \beta_m, \gamma_m$ for $m \in \mathbb{Z}$ are defined the same as $\eta_n, \beta_n, \gamma_n$ (see (6)) with $n = (m, 0)$ and $\alpha = (\alpha_1, 0)$. Note that the solution pair (v^{sc}, u) is independent of x_2.

Elementary calculations show that, using $\nu = (0, 0, -1)$ on Γ_0,

$$
\begin{aligned}
Tu|_{\Gamma_0} &= i \sum_{m\in\mathbb{Z}} \begin{pmatrix} 2\mu\tilde{\alpha}_m\beta_m & \omega^2\rho - 2\mu\tilde{\alpha}_m^2 \\ 2\mu\tilde{\alpha}_m^2 - \omega^2\rho & 2\mu\tilde{\alpha}_m\gamma_m \end{pmatrix} \begin{pmatrix} A_{p,m} \\ A_{s,m} \end{pmatrix} e^{i\tilde{\alpha}_m x_1}, \\
\nu \cdot u|_{\Gamma_0} &= \sum_{m\in\mathbb{Z}} \left(A_{p,m}\beta_m - A_{s,m}\tilde{\alpha}_m\right) e^{i\tilde{\alpha}_m x_1}, \\
\partial_\nu v^{sc}|_{\Gamma_0} &= \sum_{m\in\mathbb{Z}} -i\, v_m \eta_m e^{i\tilde{\alpha}_m x_1}.
\end{aligned}
$$

Hence, the coupling conditions between $v = v^{sc}$ and u on Γ_0 are equivalent to the algebraic equations

$$
\begin{pmatrix} 0 & 2\mu\tilde{\alpha}_m\beta_m & \omega^2\rho - 2\mu\tilde{\alpha}_m^2 \\ -1 & 2\mu\tilde{\alpha}_m^2 - \omega^2\rho & 2\mu\tilde{\alpha}_m\gamma_m \\ -\eta_m/(\rho_f\omega^2) & \beta_m & -\tilde{\alpha}_m \end{pmatrix} \begin{pmatrix} v_m \\ iA_{p,m} \\ iA_{s,m} \end{pmatrix} = 0. \tag{24}
$$

Denote by D_m the 3×3 matrix on the left hand side of (24). Its determinant is given by

$$
\text{Det}(D_m) = -\frac{\eta_m}{\rho_f\omega^2} \begin{vmatrix} 2\mu\tilde{\alpha}_m\beta_m & \omega^2\rho - 2\mu\tilde{\alpha}_m^2 \\ 2\mu\tilde{\alpha}_m^2 - \omega^2\rho & 2\mu\tilde{\alpha}_m\gamma_m \end{vmatrix} - \omega^2\rho\,\beta_m.
$$

Under the assumption that $k = k_p$ and $k = k\sin\theta_1 + m_0 = \tilde{\alpha}_{m_0}$ for some $m_0 \in \mathbb{Z}$, we have $\eta_{m_0} = \beta_{m_0} = 0$. Thus, the linear system (24) has the non-trivial solution $(v_{m_0}, A_{p,m_0}, A_{s,m_0})$ that satisfies the relation

$$
v_{m_0} + i\lambda k^2 A_{p,m_0} = 0, \quad A_{s,m_0} = 0.
$$

This implies that, one of the non-trivial solution (v^{sc}, u) is of the form

$$
v^{sc} = c\, e^{ikx_1} \quad \text{in} \quad x_3 > 0, \qquad u = -ic/(\lambda k^2)\,(0, 0, k)^\top e^{ikx_1} \qquad \text{in} \quad x_3 < 0,
$$

for any constant $c \in \mathbb{C}$. ☐

Although there is no uniqueness in general, we can verify the existence of solutions to (DP) at any frequency $\omega \in \mathbb{R}$ and the unique solvability for all frequencies excluding possibly a discrete set. The main results of this section are stated in the following theorem, where the number c_0 denotes the sound speed in the fluid.

Theorem 3.5. **(i)** *For the incident plane wave v^{in} of the form (1), there always exists a solution $(v, u) \in V$ to the variational problem (19) and hence to (DP).*

(ii) *Assume $\sqrt{\lambda + 2\mu} \le c_0\rho$. There exists a small frequency $\omega_0 > 0$ such that uniqueness of solutions to (19) holds for all $\omega \in (0, \omega_0]$. Moreover, the variational problem (19) admits a uniqueness solution for all frequencies excluding a discrete set $\mathcal{D}$ with the only accumulation point at infinity.*

Proof. (i) The variational problem (19) can be formulated as the equivalent operator equation $\mathcal{A}(v,u) = \mathcal{F}_0$, where $\mathcal{F}_0 \in V'$ is defined as the right hand side of (19). By the Fredholm alternative and Lemma 3.3, this operator equation (19) is solvable provided $\mathcal{F}_0$ is orthogonal to all solutions $(\tilde{v},\tilde{u})$ of the homogeneous adjoint equation $\mathcal{A}^*(\tilde{v},\tilde{u}) = 0$, i.e., $\langle \mathcal{F}_0, (\tilde{v},\tilde{u})\rangle = 0$. Note that such $\tilde{v}$ can always be extended to a solution of the Helmholtz equation in the unbounded domain Ω^+ by setting

$$\tilde{v}(x) \;=\; \sum_{n\in\mathbb{Z}^2} \tilde{v}_n \exp(i\,\alpha_n\cdot\tilde{x}\,-i\overline{\eta}_n x_3), \quad x_3 > b,$$

where the Rayleigh coefficients $\tilde{v}_n$ are determined by the n-th Fourier coefficient of $(e^{-i\alpha\cdot\tilde{x}}\tilde{v})|_{\Gamma_b^+}$. On the other hand, by arguing as in the proof of Lemma 3.3, it can be derived from the imaginary part of the equation

$$0 = \langle \mathcal{A}^*(\tilde{v},\tilde{u}), (\varphi,\psi)\rangle = \langle (\tilde{v},\tilde{u}), \mathcal{A}(\varphi,\psi)\rangle = \overline{A\big((\varphi,\psi),(\tilde{v},\tilde{u})\big)}$$

with $(\varphi,\psi) = (\tilde{v},\tilde{u})$ that $\tilde{v}$ has vanishing Rayleigh coefficients of the incoming modes, i.e., $\tilde{v}_n = 0$ for $|\alpha_n| < k$. In particular, we have $\tilde{v}_0 = 0$ and hence

$$\langle \mathcal{F}_0, (\tilde{v},\tilde{u})\rangle = \int_{\Gamma_b^+} f_0\,\overline{\tilde{v}}ds = \int_{\Gamma_b^+} f_0\,\overline{\tilde{v}_0}\exp(-i\,\alpha_0\cdot\tilde{x}\,+i\eta_0 b)ds(\tilde{x}) = 0,$$

with f_0 given in (18). Applying the Fredholm alternative yields the existence of a solution to (DP).

(ii) We first prove uniqueness for small frequencies. The assumption $\sqrt{\lambda+2\mu} \le c_0\rho$ implies that $k \le k_p$. If $\mathcal{A}(v^{sc},u) = 0$ for some $(v^{sc},u) \in V$, we conclude from $k \le k_p$ and Lemma 3.3 that the zero-order Rayleigh coefficients of v^{sc} and u vanish, i.e., $v_0 = 0, A_{p,0} = 0$ and $\mathbf{A}_{s,0} = 0$. This together with the asymptotic behavior

$$|\eta_n| \ge C_0\,(1+|n|^2)^{1/2}, \quad |n| \ne 0, \qquad \text{as} \quad k = \omega/c_0 \to 0^+,$$

for some constant $C_0 > 0$, leads to the estimate (see (16))

$$\begin{aligned}
\operatorname{Re}\left\{-\int_{\Gamma_b^+} \overline{v}^{sc}\,\mathcal{T}^+ v^{sc} ds\right\} &= 4\pi^2 \sum_{|n|\ne 0} |\eta_n|^2\,|v_n\,e^{i\eta_n b}|^2 \\
&= 4\pi^2 \sum_{n\in\mathbb{Z}^2} |\eta_n|^2\,|v_n\,e^{i\eta_n b}|^2 \\
&\ge C_1\,||v^{sc}||^2_{H^{1/2}_\alpha(\Gamma_b^+)},
\end{aligned} \tag{25}$$

for some $C_1 > 0$ and $\omega \in (0,\omega_1]$ with $\omega_1 > 0$ being sufficiently small. In a completely similar manner, from asymptotic properties of the matrix W_n as $\omega \to 0^+$ (see [12, Lemma 2]) we obtain

$$\operatorname{Re}\left\{-\int_{\Gamma_b^-} \overline{u}\cdot\mathcal{T}^- u\,ds\right\} \ge C_2\,||u||^2_{H^{1/2}_\alpha(\Gamma_b^-)^3}. \tag{26}$$

Inserting (25) into (19) and setting $(\varphi,\psi) = (v^{sc},0)$, $v^{in} = 0$, we arrive at

$$\begin{aligned}
0 &= \operatorname{Re} A((v^{sc},u),(v^{sc},0)) \\
&\ge ||\nabla v^{sc}||^2_{L^2(\Omega_b^+)} + C_1\,||v^{sc}||^2_{H^{1/2}_\alpha(\Gamma_b^+)} - \omega^2/c_0^2\,||v^{sc}||^2_{L^2(\Omega_b^+)} - \omega^2\,\rho_f \int_\Gamma u\cdot\nu\overline{v^{sc}}\,ds.
\end{aligned}$$

Applying Friedrich's and Cauchy-Schwarz inequalities, it follows that

$$0 \geq C_3 ||v^{sc}||^2_{H^1_\alpha(\Omega_b^+)} - C_4\, \omega^2 ||u||^2_{L^2(\Gamma)^3}, \quad \omega \in (0, \omega_1], \tag{27}$$

for some constants $C_3, C_4 > 0$ uniformly in all $\omega \in (0, \omega_1]$. Similarly, inserting (26) into (19) with $(\varphi, \psi) = (0, u)$ and $f_0 = 0$ and applying Korn's inequality (see e.g., [20, Chap. 5.4] or [12]), we obtain

$$0 = \mathrm{Re}\, A\big((v^{sc}, u), (0, u)\big) \geq C_5 ||u||^2_{H^1_\alpha(\Omega_b^-)^3} - C_6\, ||v^{sc}||^2_{L^2(\Gamma)}, \quad \omega \in (0, \omega_1], \tag{28}$$

where $C_5, C_6 > 0$ are independent of $\omega \in (0, \omega_1]$. Combining (27), (28) and using the trace lemma yields $v^{sc} = 0, u = 0$ for all $\omega \in (0, \omega_0]$ with some small frequency $\omega_0 > 0$. Existence follows directly from uniqueness as the consequence of the Fredholm alternative.

In view of the analytic Fredholm theory (see e.g. [8, Theorem 8.26] or [18, Theorem I. 5. 1]) and the unique solvability of (DP) at small frequencies, we obtain uniqueness and existence for all frequencies $\omega \in \mathbb{R}^+ \backslash \mathcal{D}$, where $\mathcal{D}$ is a discrete set including the set $\mathcal{D}_0$ of the Rayleigh frequencies given by

$$\mathcal{D}_0 := \{\omega : \text{there exists some } n \in \mathbb{Z}^2 \text{ such that } \eta_n(\omega) = 0, \beta_n(\omega) = 0 \text{ or } \gamma_n(\omega) = 0\}.$$

Note that the DtN maps $\mathcal{T}^\pm$ are not analytic with respect to $\omega \in \mathcal{D}_0$. Moreover, we conclude from the arguments in [12, Theorem 6] or [16, Theorem 3.3] that $\mathcal{D}$ cannot have a finite accumulation point. The proof is complete. □

Remark 3.6. *Theorem 3.5 (i) remains valid for a broad class of incident waves of the form*

$$v^{in} = \sum_{n \in \mathbb{Z}^2 : |\alpha_n| < k} q_n \exp(i\alpha_n \cdot \tilde{x} - i\eta_n x_3), \quad q_n \in \mathbb{C}.$$

4 Factorization method for inverse problem

4.1 The admissible set of incident acoustic waves

In contrast to the inverse scattering from bounded obstacles, the incident angle θ_1 has to be restricted to $[0, \pi/2)$ in order to identify the scattering surface from above. However, it seems not suitable to employ incident waves with distinct angles $\theta_1 \in [0, \pi/2), \theta_2 \in [0, 2\pi)$, since the quasi-periodicity parameters α_1 and α_2 of the scattered field vary with the direction of incidence. To define our admissible set of incident wave, we first recall the free space Green function $\Phi(x, y)$ for the Helmholtz equation $(\Delta + k^2)u = 0$ in $\mathbb{R}^3$:

$$\Phi(x, y) = \frac{e^{ik|x-y|}}{4\pi|x-y|}, \qquad x = (\tilde{x}, x_3), \quad y = (\tilde{y}, y_3) \in \mathbb{R}^3,\ x \neq y,$$

and the following free space α-quasiperiodic Green function $G(x, y)$ for the Helmholtz equation $(\Delta + k^2)u = 0$:

$$G(x, y) = \frac{i}{8\pi^2} \sum_{n \in \mathbb{Z}^2} \frac{1}{\eta_n} \exp(i\alpha_n \cdot (\tilde{x} - \tilde{y}) + i\eta_n |x_3 - y_3|), \quad x - y \neq (2n\pi, 0),\ n \in \mathbb{Z}^2, \tag{29}$$

with α_n, η_n defined as in (6). In the subsequent sections, we suppose that $\eta_n(\omega) \neq 0$ for all $n \in \mathbb{Z}^2$, so that the above series (29) is well-defined. It is well-known (see e.g. [2, 34]) that the difference of Φ and G is an analytic function. Moreover, the function $x \to G(x,y) - G(x,y') =: G_D(x,y)$ is the α-quasiperiodic Green's function of the Helmholtz equation in the half space $x_3 > 0$ satisfying the Dirichlet boundary condition on $x_3 = 0$. For $x_3 > y_3$, we have an alternative expression of $G_D(x,y)$ given by

$$G_D(x,y) = \sum_{n \in \mathbb{Z}^2} g_n(y)\, \exp(i\alpha_n \cdot \tilde{x} + i\eta_n x_3), \tag{30}$$

with

$$g_n(y) := \frac{i}{8\pi^2 \eta_n} \big(\exp(-i\alpha_n \cdot \tilde{y} - i\eta_n y_3) - \exp(i\alpha_n \cdot \tilde{y} + i\eta_n y_3) \big), \quad n \in \mathbb{Z}^2. \tag{31}$$

The expression (30) states that, for $x_3 > y_3$, the function $x \to G_D(x,y)$ is actually an up-ward going radiating solution with the Rayleigh coefficients $g_n(y)$. Inspired by the existing factorization methods for diffraction gratings [3] as well as for bounded obstacle scattering in a half-space [25, Chapter 2.6], we define the admissible set $\mathcal{I}$ of incident acoustic waves as follows:

$$\mathcal{I} := \left\{ v_n^{in}(x) = \frac{i}{8\pi^2 \eta_n} \big(\exp(i\alpha_n \cdot \tilde{x} - i\eta_n x_3) - \exp(i\alpha_n \cdot \tilde{x} + i\eta_n x_3) \big) : |n| < M \right\}, \ M \in \mathbb{N}^+,$$

where v_n^{in}'s have the same quasiperiodicity parameter α for each $n \in \mathbb{Z}^2$. By the definition of η_n (see (6)), one can observe further that the incident wave $v_n^{in}(y)$ coincides with the conjugate of the n-th Rayleigh coefficient of the function $x \to G_D(x,y)$ for $x_3 > y_3$, i.e., $v_n^{in}(y) = \overline{g_n(y)}$ for all $n \in \mathbb{Z}^2$.

For the inverse problem (**IP**), we suppose there is *a priori* knowledge that the unknown interface Γ lies between the planes $\Gamma_b^+ = \{x_3 = b\}$ and $\Gamma_0 = \{x_3 = 0\}$ for some $b > \Gamma^+$. In the rest of the paper, we will investigate the following equivalent issue to (**IP**):

(IP)': Recover the scattering interface Γ from the set $\{v_n^{(j)} : |j|, |n| < M\}$ for some $M \in \mathbb{N}^+$, where $v_n^{(j)}$ $(n \in \mathbb{Z}^2)$ denotes the n-th Rayleigh coefficient of the scattered acoustic near field $v_j^{sc}(\tilde{x}, b)$ generated by the incident wave $v_j^{in} \in \mathcal{I}$.

Note that Rayleigh coefficients $v_n^{(j)}$ can be directly computed from the near-field data $v_j^{sc}(\tilde{x}, b)$ via the integral over Γ_b^+.

Remark 4.1. *The incident wave $v_n^{in} \in \mathcal{I}$ consists of two parts: the first part is a downward propagating wave mode, whereas the second part is an upward mode which is not physically meaningful. The scattered field corresponding to the upward mode is just itself, thus the scattered field v_n^{sc} can be produced by linear supposition. When $|\alpha_n| > k$, the downward evanescent mode $\exp(i\alpha_n \cdot \tilde{x} - i\eta_n x_3) = \exp(i\alpha_n \cdot \tilde{x} + |\eta_n| x_3)$ may be physically generated at the prism face by total internal reflection, which has already been in practical use in near-field optics; see e.g., [6, 9, 17] and references therein.*

The assertion of Theorem 3.5 (ii) is applicable to v_n^{in} for all $n \in \mathbb{Z}^2$. Throughout the rest sections the frequency of incidence is assumed to be such that problem (DP) is always uniquely solvable.

4.2 Auxiliary boundary value problems

In this subsection we introduce several auxiliary boundary value problems that will be used later in establishing the factorization method. For $h \in H_\alpha^{1/2}(\Gamma)$, consider the boundary problem of finding the α-quasiperiodic solution $v \in H_\alpha^1(\Omega_0^-)$ such that

$$\text{(I)} \begin{cases} \Delta v + k^2 v = 0, & \text{in} \quad \Omega_0^- := \{x \in \Omega_b^- : x_3 > 0\}, \\ v = 0 \quad \text{on} \quad \Gamma_0, & v = h \quad \text{on} \quad \Gamma. \end{cases}$$

Clearly, the region Ω_0^- denotes the domain between Γ_0 and the scattering interface. The above problem (I) is uniquely solvable for each $h \in H_\alpha^{1/2}(\Gamma)$ provided $k \in \mathbb{R}^+ \backslash \tilde{\mathcal{D}}_1$, where $\tilde{\mathcal{D}}_1$ consists of α-quasiperiodic Dirichlet eigenvalues of the negative Laplacian over the periodic layer Ω_0^-. It was shown in [19] that $\tilde{\mathcal{D}}_1$ has only a finite number of eigenvalues in the interval $(-N, N)$ for any $N > 0$. Now we assume $k \notin \tilde{\mathcal{D}}_1$. Then, the normal derivative of v on Γ defines another Dirichlet to Neumann operator $\mathcal{T}_\alpha : H_\alpha^{1/2}(\Gamma) \to H_\alpha^{-1/2}(\Gamma)$ by

$$\mathcal{T}_\alpha(h) := (\partial_\nu v)|_\Gamma, \quad h \in H_\alpha^{1/2}(\Gamma),$$

where v is the unique solution to problem (I). In particular, for each incident wave $v_j^{in} \in \mathcal{I}$, it holds that

$$\mathcal{T}_\alpha(v_j^{in}|_\Gamma) = (\partial_\nu v_j^{in})|_\Gamma.$$

With the definition of $\mathcal{T}_\alpha$, we introduce the second auxiliary boundary value problem:

$$\text{(II)} \begin{cases} \Delta v^{sc} + k^2 v^{sc} = 0, & \text{in} \quad \Omega^+, \\ \Delta^* u + \rho\omega^2 u = 0, & \text{in} \quad \Omega^-, \\ \eta\, u \cdot \nu - \partial_\nu v^{sc} = \mathcal{T}_\alpha \varphi, & \text{on} \quad \Gamma, \qquad \eta = \rho_f \omega^2 > 0, \\ Tu + v^{sc}\,\nu = -\varphi\nu, & \text{on} \quad \Gamma, \end{cases}$$

where $v^{sc} \in H_\alpha^1(\Omega_b^+)$ resp. $u \in H_\alpha^1(\Omega_b^-)^3$ is required to satisfy the upward resp. downward Rayleigh expansions (5) resp. (9). By Theorem 3.5, it is readily seen that problem (II) admits a unique solution (v^{sc}, u) for every $\varphi \in H_\alpha^{1/2}(\Gamma)$ if $\omega \notin \mathcal{D} \cup \mathcal{D}_1$, where $\mathcal{D}$ is the discrete set involving in Theorem 3.5 (ii) and $\mathcal{D}_1 := \{\omega : \omega/c_0 \in \tilde{\mathcal{D}}_1\}$.

Analogously to the data-to-pattern operator for bounded obstacle scattering problem, we define the operator $G : H_\alpha^{1/2}(\Gamma) \to l^2$ as

$$G(\varphi) := \{\exp(i\eta_n b)\, v_n : n \in \mathbb{Z}^2\}, \quad \varphi \in H_\alpha^{1/2}(\Gamma), \tag{32}$$

where v_n denotes the n-th Rayleigh coefficient of v^{sc} solving problem (II), and $b > \Gamma^+$ represents the measurement position of our scattered acoustic field. Note that we have $G(\varphi) \in l^2$ due to the fact that $v^{sc}(\tilde{x}, b) \in L^2(\Gamma_b^+)$. Obviously, our scattering problem (DP) with $v^{in} = v_j^{in} \in \mathcal{I}$ can be reformulated as the boundary value problem (II) by setting $\varphi = v_j^{in} \in \mathcal{I}$. This implies that

$$G(v_j^{in}|_\Gamma) = \{\exp(i\eta_n b)\, v_n^{(j)} : n \in \mathbb{Z}^2\}, \quad \text{for all} \quad v_j^{in} \in \mathcal{I},\ j \in \mathbb{Z}^2.$$

To verify the factorization method for the interaction problem, we still need the following interior boundary value problem: Find $w \in H^1_\alpha(\Omega_0^-), u \in H^1_\alpha(\Omega_0^-)^3$ such that

$$
\text{(III)} \quad \begin{cases} \Delta w + k^2 w = 0 & \text{in} \quad \Omega_0^-, \\ \Delta^* u + \rho\omega^2 u = 0 & \text{in} \quad \Omega_0^-, \\ \eta\, u \cdot \nu - \partial_\nu w = f & \text{on} \quad \Gamma, \\ Tu + w\,\nu = g & \text{on} \quad \Gamma, \\ Tu = \mathcal{T}_\alpha^- u & \text{on} \quad \Gamma_0, \\ w = 0 & \text{on} \quad \Gamma_0, \end{cases}
$$

with $f \in H_\alpha^{-1/2}(\Gamma), g \in H_\alpha^{-1/2}(\Gamma)^3$. Problem (III) plays the same role of the interior eigenvalue problem for the FSI problem with a bounded elastic body; see [26]. Note that the condition $Tu = \mathcal{T}_\alpha^- u$ is equivalent to the downward Rayleigh expansion of u. Following the argument in the proof of Theorem 3.5, one can verify the existence and uniqueness of solutions to (III) for all $\omega \in \mathbb{R}^+\backslash\mathcal{D}_2$ with some discrete set $\mathcal{D}_2$. Further, one can observe that, if v is a solution to (I) for some $h \in H_\alpha^{1/2}(\Gamma)$, then $(w, u) = (v, 0)$ is one solution pair to (III) with $f = -\partial_\nu v$, $g = v\,\nu$.

In the subsequent sections we assume $\omega \notin \mathcal{D} \cup \mathcal{D}_1 \cup \mathcal{D}_2$, so that the above problems (I), (II), (III) and (**DP**) are always uniquely solvable. In particular, the mapping $(f, g) \to (w, u)$ in problem (III) is bounded from $H_\alpha^{-1/2}(\Gamma) \times H_\alpha^{-1/2}(\Gamma)^3$ into $H^1_\alpha(\Omega_0^-) \times H^1_\alpha(\Omega_0^-)^3$.

4.3 Properties of G

This subsection concerns properties of the operator G. We first show that the range $\mathrm{Range}(G)$ of G can be used to characterize the domain Ω^- beneath Γ, and then prove the denseness of $\mathrm{Range}(G)$ in l^2.

Lemma 4.2. *Let $g_n(y)$ be given as in (31). The sequence $\{\exp(i\eta_n b)g_n(y) : n \in \mathbb{Z}^2\}$ lies in the range of G if and only if $y \in \Omega^-$.*

Proof. Assume $y \in \Omega^-$. Let (w, u) be the unique solution to problem (III) with

$$
f = (\partial_\nu G_D(\cdot, y))|_\Gamma \in H_\alpha^{-1/2}(\Gamma), \quad g = -\nu G_D(\cdot, y) \in H_\alpha^{-1/2}(\Gamma)^3.
$$

By the definition of $\mathcal{T}$, we see $\mathcal{T}_\alpha(w|_\Gamma) = (\partial_\nu w)|_\Gamma$. Hence, the solution $(v^{sc}, u) = (G_D(\cdot, y), u)$ solves problem (II) with $\phi = w|_\Gamma$. Together with the definition of G, this implies that

$$
G(w|_\Gamma) = \{\exp(i\eta_n b)g_n(y) : n \in \mathbb{Z}^2\},
$$

where the sequence $g_n(y)$, defined by (31), is the n-th Rayleigh coefficient of the function $x \to G_D(x, y)$.

Assume $\{\exp(i\eta_n b)g_n(y) : n \in \mathbb{Z}^2\} = G(\varphi)$ for some $\varphi \in H_\alpha^{1/2}(\Gamma)$, and let (v^{sc}, u) be the solution to (II) with the same φ. Then, it holds that $G_D(\cdot, y) = v^{sc}$ on Γ_b^+. Furthermore, we have $G_D(\cdot, y) = v^{sc}$ in $\Omega^+\backslash\{y\}$, due to the uniqueness to the Dirichlet boundary value problem of the quasiperiodic Helmholtz equation in the half space $x_3 > b$ and the unique continuation of solutions to the Helmholtz equation. If $y \in \Omega^+$, the boundedness of $\lim_{x\to y} v^{sc}(x)$ contradicts the singularity of $G_D(x, y)$ at $x = y$. If $y \in \Gamma$, we can always find an open bounded cone C with vertex at y such that $C \subset \Omega^+$. Clearly, $v^{sc} \in H^1(C)$ but $G_D(\cdot, y) \notin H^1(C)$. The latter contradiction implies that $y \in \Omega^-$. □

Lemma 4.3. *The operator $G : H_\alpha^{1/2}(\Gamma) \to l^2$ is compact and has dense range.*

Proof. For $\varphi \in H_\alpha^{1/2}(\Gamma)$, let (v^{sc}, u) be the unique solution to problem (II). Obviously, there holds the estimate

$$||G(\varphi)||_{l^2} = ||v^{sc}||_{L^2_\alpha(\Gamma_b^+)} \leq ||v^{sc}||_{H_\alpha^{1/2}(\Gamma_b^+)} \leq C\,||\varphi||_{H_\alpha^{1/2}(\Gamma)}.$$

The compactness of G follows immediately from the boundedness of the map $G_1(\varphi) = v^{sc} : H_\alpha^{1/2}(\Gamma) \to H_\alpha^{1/2}(\Gamma_b^+)$ and the compactness of $G_2 : H_\alpha^{1/2}(\Gamma_b^+) \to L^2_\alpha(\Gamma_b^+)$.

To prove the denseness of G, it suffices to verify the injectivity of its adjoint operator $G^* : l^2 \to H_\alpha^{-1/2}(\Gamma)$. To this end, we need an explicit expression of G^* which will be derived as follows.

For $\mathbf{d} = \{d_n : n \in \mathbb{Z}^2\} \in l^2$, define the Rayleigh series

$$w(x) = \sum_{n\in\mathbb{Z}^2} d_n \exp(i\alpha_n \cdot \tilde{x} + i\eta_n(x_3 - b)), \quad x_3 \geq b,$$

which converges absolutely in any compact support of $x_3 > b$. For $\varphi \in H_\alpha^{1/2}(\Gamma)$, denote by (v^{sc}, u) the unique solution to problem (II). By the definitions of G (see (32)), we see

$$\left\langle G\varphi, \mathbf{d}\right\rangle_{l^2} = \frac{1}{4\pi^2}\int_{\Gamma_b^+} v^{sc}(x)\,\overline{w}(x)\,ds, \tag{33}$$

where $\left\langle \cdot,\cdot\right\rangle_{l^2}$ denotes the inner product in l^2. Using the periodic version of Green's representation formula, it holds

$$v^{sc}(x) = \int_\Gamma \left[G_D(x,y)\partial_\nu v^{sc}(y) - v^{sc}(y)\partial_\nu G_D(x,y)\right]\,ds(y), \quad x \in \Omega^+.$$

Inserting the above expression into (33) and changing the order of integration yields

$$\left\langle G\varphi, \mathbf{d}\right\rangle_{l^2} = \frac{1}{4\pi^2}\int_\Gamma \left[q(y)\partial_\nu v^{sc}(y) - v^{sc}(y)\partial_\nu q(y)\right]\,ds(y)$$

where

$$q(y) := \int_{\Gamma_b^+} G_D(x,y)\overline{w}(x)ds(x), \quad y_3 < b,$$

is -α-quasiperiodic with respect to $\tilde{y}$. Since $q = 0$ on Γ_0, we have $\partial_\nu q = \mathcal{T}_{-\alpha}(q)$ on Γ and thus

$$\left\langle G\varphi, \mathbf{d}\right\rangle_{l^2} = \frac{1}{4\pi^2}\int_\Gamma \left[q(y)\partial_\nu v^{sc}(y) - v^{sc}(y)\,(\mathcal{T}_{-\alpha}q)(y)\right]\,ds(y). \tag{34}$$

Now, let $(\tilde{v}^{sc}, \tilde{u})$ be the unique -α-quasiperiodic solution pair to (II) with $\varphi = q|_\Gamma$. Using the coupling conditions

$$\mathcal{T}_{-\alpha}(q) = \eta\tilde{u}\cdot\nu - \partial_\nu\tilde{v}^{sc}, \quad \nu\, q = -(T\tilde{u} + \nu\tilde{v}^{sc}) \quad \text{on } \Gamma, \tag{35}$$

it follows from (34) that

$$\left\langle G\varphi, \mathbf{d}\right\rangle_{l^2} = -\frac{1}{4\pi^2}\int_\Gamma \left[T(\tilde{u})\cdot\nu\;\partial_\nu v^{sc} + \eta\, v^{sc}\,\tilde{u}\cdot\nu\right]\,ds + \frac{1}{4\pi^2}\int_\Gamma \left[v^{sc}\,\partial_\nu\tilde{v}^{sc} - \tilde{v}^{sc}\,\partial_\nu v^{sc}\right]\,ds. \tag{36}$$

Applying Green's formula and using the upward Rayleigh expansions of v^{sc} and $\tilde{v}^{sc}$, one can straightforwardly verify that

$$\int_\Gamma [v^{sc}\,\partial_\nu \tilde{v}^{sc} - \tilde{v}^{sc}\,\partial_\nu v^{sc}]\;ds = \lim_{b\to\infty}\int_{\Gamma_b^+} [v^{sc}\,\partial_\nu \tilde{v}^{sc} - \tilde{v}^{sc}\,\partial_\nu v^{sc}]\;ds = 0. \tag{37}$$

In view of the coupling boundary conditions:

$$\partial_\nu v^{sc} = \eta\, u\cdot\nu - \mathcal{T}_{-\alpha}(\varphi), \quad \nu v^{sc} = -(\varphi\nu + Tu), \quad \text{on} \quad \Gamma.$$

We deduce from (36) and (37) that

$$\begin{aligned} \big\langle G\varphi, \mathbf{d}\big\rangle_{l^2} &= \frac{1}{4\pi^2}\int_\Gamma [\eta\,\tilde{u}\cdot\nu\,\varphi + \mathcal{T}_{-\alpha}(\varphi)\;\nu\cdot T\tilde{u}]\;ds - \frac{\eta}{4\pi^2}\int_\Gamma [T\tilde{u}\cdot u - \tilde{u}\cdot Tu]\;ds \\ &= \frac{1}{4\pi^2}\int_\Gamma \varphi\;[\eta\,\tilde{u}\cdot\nu\; + \mathcal{T}_{-\alpha}\,(\nu\cdot T\tilde{u})]\;ds, \end{aligned}$$

where the last equality follows from the fact that $\nu\cdot T\tilde{u}\in H^{1/2}_{-\alpha}(\Gamma)$ and

$$\begin{aligned} \int_\Gamma (\mathcal{T}_{-\alpha} g)\cdot h\;ds &= \int_\Gamma (\mathcal{T}_{-\alpha} h)\cdot g\;ds, \quad \text{for all } h, g\in H^{1/2}_{-\alpha}(\Gamma), \\ \int_\Gamma [T\tilde{u}\cdot u - \tilde{u}\cdot Tu]\;ds &= \lim_{b\to\infty}\int_{\Gamma_b^-} [T\tilde{u}\cdot u - \tilde{u}\cdot Tu]\;ds = 0. \end{aligned}$$

Therefore, we obtain the expression of the adjoint operator G^*:

$$G^*(\mathbf{d}) = \frac{\eta}{4\pi^2}\,\overline{[\eta\,\tilde{u}\cdot\nu + \mathcal{T}_{-\alpha}(\nu\cdot T(\tilde{u}))]}, \quad \mathbf{d}\in l^2.$$

To prove the injectivity of G^*, we need to prove that $G^*(\mathbf{d}) = 0$ implies $\mathbf{d} = 0$ in l^2. Indeed, by the expression of G^* we see $\eta\,\tilde{u}\cdot\nu = -\mathcal{T}_{-\alpha}(\nu\cdot T\tilde{u})$, and thus by (35),

$$\partial_\nu \tilde{v}^{sc} = -\mathcal{T}_{-\alpha}(q|_\Gamma + \nu\cdot T\tilde{u}), \quad \tilde{v}^{sc} = -(q|_\Gamma + \nu\cdot T\tilde{u}) \quad \text{on} \quad \Gamma. \tag{38}$$

Define the -α-quasiperiodic function

$$Q := \begin{cases} \tilde{v}^{sc} & \text{in} \quad \Omega^+, \\ \tilde{Q} & \text{in} \quad \Omega_0^-, \end{cases}$$

where $\tilde{Q}$ is the unique $-\alpha$-quasiperiodic solution to problem (I) with $h = -(q|_\Gamma + \nu\cdot T\tilde{u})$. The equations in (38) combined with the definition of $\mathcal{T}_{-\alpha}$ imply that

$$Q^+ = Q^-, \quad \partial_\nu Q^+ = \partial_\nu Q^- \quad \text{on} \quad \Gamma,$$

where the superscripts $'+'$ and $'-'$ denote respectively the limits from above and below. Thus, Q satisfies the Helmholtz equation in the half-space $x_3 > 0$, with the Dirichlet boundary condition $Q = 0$ on Γ_0. By uniqueness to the exterior boundary problem we obtain $Q = 0$ in $x_3 > 0$ and in particular, $\tilde{v}^{sc} = 0$ in Ω^+. Consequently, from (38) we see

$$q = -T\tilde{u}\cdot\nu, \qquad \partial_\nu q = \mathcal{T}_{-\alpha}(q) = -\mathcal{T}_{-\alpha}(\nu\cdot T\tilde{u}) = \eta\,\tilde{u}\cdot\nu \quad \text{on} \quad \Gamma.$$

Hence, the solution $q\in H^1_{-\alpha}(\Omega_0^-), \tilde{u}\in H^1_{-\alpha}(\Omega_0^-)$ is the unique $-\alpha$-quasiperiodic solution to problem (III) with $f = 0, g = 0$. By uniqueness it holds that $q = 0$ in Ω_0^-, and by unique continuation, $q = 0$ in Ω_b^+. Consequently, we get $q = 0$ on Γ_b^+ and $q = 0$ in $x_3 > b$ due to the uniqueness of α-quasiperiodic solutions to the exterior boundary value problem in the half space $x_3 > b$. Finally, according to the jump relation $0 = \partial_\nu q^- - \partial_\nu q^+ = \overline{w}$ on Γ_b^+ and the definition of w, we obtain $d_n = 0$ for all $n\in\mathbb{Z}^2$, i.e., $\mathbf{d} = 0$. The proof is complete.

□

4.4 Near-field operator and its factorization

Recall that $\Gamma_b^+ = \{x : x_3 = b, 0 < x_1, x_2 < 2\pi\}$ with $b > \Gamma^+$ denotes the measure position of the scattered acoustic field in the fluid. Introduce the number

$$\tau_n := \overline{\exp(i\eta_n b)} = \begin{cases} \exp(-i\eta_n b), & \text{if} \quad |\alpha_n| < k, \\ \exp(-|\eta_n| b), & \text{if} \quad |\alpha_n| > k. \end{cases} \tag{39}$$

The operator $H : l^2 \to H_\alpha^{1/2}(\Gamma)$ is defined as

$$H(\mathbf{a})(x) = \sum_{n\in\mathbb{Z}^2} a_n\, \tau_n v_n^{in}(x), \quad x \in \Gamma, \qquad \mathbf{a} = \{a_n : n \in \mathbb{Z}^2\}.$$

The operator H is restriction of the supposition of quasiperiodic incident waves v_n^{in} with the weight $a_n\tau_n$ to Γ. Its adjoint operator $H^* : H_\alpha^{-1/2}(\Gamma) \to l^2$ is given by

$$H^*(\phi) = \{d_n : n \in \mathbb{Z}^2\}, \quad d_n := \int_\Gamma \exp(i\eta_n b)\, g_n(y)\, \phi(y)\, ds(y), \quad \phi \in H_\alpha^{-1/2}(\Gamma), \tag{40}$$

with $g_n = \overline{v}_n^{in}$ given in (31). The near-field operator $N : l^2 \to l^2$ is defined as

$$N(\mathbf{a}) = G\, H(\mathbf{a}), \quad \mathbf{a} \in l^2. \tag{41}$$

It maps a supposition of the admissible incident waves to the set of Fourier coefficients of the corresponding scattered field on $x_3 = b$.

In the rest of the paper we shall use the single layer potential

$$(S\psi)(x) := \int_\Gamma G_D(x,y)\psi(y)\, ds(y), \quad x \in \mathbb{R}^3, \quad \psi \in H_\alpha^{-1/2}(\Gamma), \tag{42}$$

whose kernel is the Dirichlet Green's function in the half-space $x_3 > 0$. Since $(S\psi)|_{\Gamma_0} = 0$, by the definition of $\mathcal{T}_\alpha$ we have $\mathcal{T}_\alpha((S\psi)|_\Gamma) = \partial_\nu(S\psi)$ on Γ. The jump relation for the normal derivatives of $S\psi$ on Γ yields

$$\partial_\nu(S\psi)^+ - \partial_\nu(S\psi)^- = \psi, \quad \text{on} \quad \Gamma. \tag{43}$$

For $\psi \in H_\alpha^{-1/2}(\Gamma)$, let (w,u) be the unique solution to problem (III) with $f = \psi, g = 0$, and define

$$(J\psi)(x) := [w(x) - (S\psi)(x)]|_\Gamma, \quad x \in \Gamma. \tag{44}$$

Then it is easy to check that on Γ,

$$\begin{aligned} \eta\, u\cdot\nu - \partial_\nu(S\psi)^+ &= \partial_\nu w + \psi - \partial_\nu(S\psi)^+ = \partial_\nu w - \partial_\nu(S\psi)^- = \mathcal{T}_\alpha(w - S\psi), \\ T(u) + (S\psi)\nu &= -w\nu + (S\psi)\nu = -(w - S\psi)\nu. \end{aligned}$$

This implies that the solution pair $(S\psi, u)$ solves problem (II) with $\varphi = (w - S\psi)|_\Gamma = J\psi$. In view that $g_n(y)$ is the n-th Rayleigh coefficient of $x \to G_D(x,y)$ for $x_3 > y_3$, we deduce from (40) and the definition of G that

$$GJ(\psi) = H^*(\psi) \qquad \text{for all } \psi \in H_\alpha^{-1/2}(\Gamma), \tag{45}$$

from which it follows $H = J^*G^*$. Hence, by (41) we obtain the factorization of the near-field operator

$$N = GJ^*G^*.$$

In the following we shall show properties of the middle operator J. Let $(\cdot,\cdot)$ denote the duality between $H_\alpha^{1/2}(\Gamma)$ and $H_\alpha^{-1/2}(\Gamma)$ extending the inner product of $L^2_\alpha(\Gamma)$.

Lemma 4.4. **(i)** $\mathrm{Im}\,(\psi, J\psi) \geq 0$ *for all* $\psi \in H_\alpha^{1/2}(\Gamma)$.

(ii) *The operator* $J : H_\alpha^{-1/2}(\Gamma) \to H_\alpha^{1/2}(\Gamma)$ *is injective.*

(iii) *There exists a selfadjoint and coercive operator* $J_0 : H_\alpha^{-1/2}(\Gamma) \to H_\alpha^{1/2}(\Gamma)$ *such that* $J - J_0 : H_\alpha^{-1/2}(\Gamma) \to H_\alpha^{1/2}(\Gamma)$ *is compact. Here the coercivity of* J_0 *means that*

$$(\psi, J_0\,\psi) \geq c||\psi||^2_{H_\alpha^{-1/2}(\Gamma)} \quad \text{for all} \quad \psi \in H_\alpha^{-1/2}(\Gamma),$$

where $c > 0$ *is a positive constant independent of* ψ.

Proof. For $\psi \in H_\alpha^{-1/2}(\Gamma)$, let (w,u) be the unique solution to problem (III) with $f = \psi, g = 0$, and define $S\psi$, $J\psi$ as in (42), (44), respectively.

(i) Using the jump relation (43) and the coupling conditions between u and w, we see

$$(\psi, J\psi) = (\psi, w - S\psi) = -(\partial_\nu w, w) - \eta\,(u, Tu) - (\partial_\nu(S\psi)^+ - \partial_\nu(S\psi)^-, S\psi). \tag{46}$$

Making use of the boundary condition $w = 0$ on Γ_0 and applying Green's formula to w and Betti's formula to u gives

$$-(\partial_\nu w, w) \;=\; \int_{\Omega_0^-} [|\nabla w|^2 - k^2|w|^2]\,dx, \tag{47}$$

$$-(u, Tu) \;=\; \int_{\Omega_0^-} [\mathcal{E}(u,\overline{u}) - \omega^2\rho\,|u|^2]\,dx - \int_{\Gamma_0} u\cdot T\overline{u}\,ds. \tag{48}$$

Similarly, from the boundary condition $S\psi = 0$ on Γ_0 and Green's formula for $S\psi$ applied to Ω_b^+ and Ω_0^-, it follows that

$$(\partial_\nu(S\psi)^+ - \partial_\nu(S\psi)^-, S\psi) = \int_{\Omega_{b,0}} |\nabla(S\psi)|^2 - k^2|S\psi|^2\,dx - \int_{\Gamma_b^+} \partial_\nu(S\psi)\,\overline{(S\psi)}\,ds, \tag{49}$$

where the integral over $\Omega_{b,0} := \{x : 0 < x_3 < b, 0 < x_1, x_2 < 2\pi\}$ is understood as the sum of the integrals over Ω_0^- and Ω_b^+. Inserting (47)-(49) into (46) and taking the imaginary part of the resulting expression, we get

$$\begin{aligned}\mathrm{Im}\,(\psi, J\psi) \;&=\; -\eta\,\mathrm{Im}\int_{\Gamma_0} u\cdot T\overline{u}\,ds + \mathrm{Im}\int_{\Gamma_b^+} \partial_\nu(S\psi)\,\overline{(S\psi)}\,ds \\ &=\; \eta\,\mathrm{Im}\int_{\Gamma_0} \overline{u}\cdot Tu\,ds + \mathrm{Im}\int_{\Gamma_b^+} \partial_\nu(S\psi)\,\overline{(S\psi)}\,ds.\end{aligned}$$

Since $S\psi$ satisfies the Rayleigh expansion in $x_3 > \Gamma^+$, we deduce from (23) and the previous identity that $\mathrm{Im}\,(\psi, J\psi) \geq 0$ for all $\psi \in H_\alpha^{1/2}(\Gamma)$.

(ii) Assuming $J\psi = (w - S\psi)|_\Gamma = 0$ for some $\psi \in H_\alpha^{-1/2}(\Gamma)$, we shall prove $\psi = 0$. Obviously, the quasiperiodic function $q(x) := w(x) - S\psi(x)$ for $x \in \Omega_0^-$, satisfies the Helmholtz equation in Ω_0^- and vanishes on both Γ_0 and Γ. Since k^2 is not a Dirichlet eigenvalue of the negative Laplacian over the periodic layer Ω_0^-, it holds that $q \equiv 0$ and hence $w(x) = S\psi(x)$ for $x \in \Omega_0^-$. Consequently, we have $\partial_\nu w = \partial_\nu S\psi^-$ on Γ. Moreover, from the coupling conditions between u and w, we see

$$\begin{aligned} \eta\, u \cdot \nu - \partial_\nu S\psi^+ &= (\eta\, u \cdot \nu - \partial_\nu w) - (\partial_\nu S\psi^+ - \partial_\nu S\psi^-) = \psi - \psi = 0, \\ T(u) + (S\psi)\nu &= T(u) + w\nu = 0. \end{aligned}$$

Therefore, the solution pair $(S\psi, u)$ solves problem (**II**) with $\varphi = 0$, implying that $S\psi = 0$ in Ω_b^+ and $u = 0$ in Ω_0^-. In particular we get $S\psi = 0$ on Γ. Recalling that $S\psi$ also vanishes on Γ_0 and $\omega \notin \mathcal{D}_1$, we obtain $S\psi = 0$ in Ω_0^-. Finally, the jump relation (43) yields $\psi = 0$, i.e., J is injective.

(iii) The third assertion can be treated in the same manner as in the proof of [26, Theorem 2.4] for the fluid-solid interaction problem with a bounded elastic body. Note that the bounded obstacle and its exterior in [26] correspond respectively to our periodic layer Ω_0^- and the region Ω^+. In the present situation, the scalar function w and the vector function u are additionally required to satisfy the Dirichlet boundary condition $u = 0$ and the transparent boundary condition $Tu = \mathcal{T}^- u$ on Γ_0. The upward Rayleigh expansion will be in place of the Sommerfeld radiation condition imposed on the scattered acoustic field. Our single layer potential (42) will play the same role of the one in [26]. For brevity we omit the proof here.

□

4.5 Inversion algorithm

In this subsection we report the inversion algorithm for finding the bi-periodic interface separating the fluid and solid in one-periodic cell. By Lemma 4.2, the sequence $\{\exp(i\eta_n b) g_n(y) : n \in \mathbb{Z}^2\}$ can be used to characterize the domain Ω^- beneath Γ through the range $\mathrm{Range}(G)$ of G. In order to identify Γ, we still need to bridge the connection between $\mathrm{Range}(G)$ and $\mathrm{Range}(N)$, since the near-field operator N can be straightforwardly computed from knowledge of the Rayleigh coefficients due to the admissible incident waves. For this purpose, we shall apply the following range identity (see [25, Theorem 2.5.1]) to the factorization of the near-field operator established in (45).

Lemma 4.5 (**Range Identity**). *Let $X \subset Y \subset X^*$ be a Gelfand triple with Hilbert space Y and reflexive Banach space X such that the embedding is dense. Furthermore, let Y be a second Hilbert space and $F : Y \to Y$, $G : X \to Y$ and $T : X^* \to X$ be linear and bounded operators with $F = GTG^*$. Suppose further that*

(a) *G is compact and has dense range.*

(b) *There exists $t \in [0, 2\pi]$ such that $\mathrm{Re}\,[\exp(it)T]$ has the form $\mathrm{Re}\,[\exp(it)T] = T_0 + T_1$ with some compact operator T_1 and some coercive operator $T_0 : X^* \to X$, i.e. there exists $c > 0$ with*

$$(\varphi, T_0\varphi) \geq c\|\varphi\|^2 \quad \textit{for all} \quad \varphi \in X^*. \tag{50}$$

(c) *$\mathrm{Im}\, T$ is non-negative on $\mathcal{R}(G^*) \subset X^*$, i.e., $(\varphi, \mathrm{Im}\, T\varphi) \geq 0$ for all $\varphi \in \mathcal{R}(G^*)$.*

(d) *$\mathrm{Re}\,[\exp(it)T]$ is one-to-one or $\mathrm{Im}\, T$ is strictly positive on the closure $\overline{\mathcal{R}(G^*)}$ of $\mathcal{R}(G^*)$, i.e., for all $\varphi \in \overline{\mathcal{R}(G^*)}$ with $\varphi \neq 0$ it holds $(\varphi, \mathrm{Im}\, T\varphi) > 0$.*

Then the operator $F_\sharp := |\mathrm{Re}\,\exp(it)F| + |\mathrm{Im}\,F|$ is positive definite and the ranges of $G : X \to Y$ and $F_\sharp^{1/2} : Y \to Y$ coincide.

The above range identity plays a crucial role in various versions of the factorization method for wave scattering from impenetrable and penetrable scatterers. To apply Lemma 4.5, we set

$$t = 0,\ F = N,\ G = G,\ T = J^*,\ T_0 = J_0,\ T_1 = \mathrm{Re}\,(J - J_0),$$
$$Y = l^2,\ \ X = H_\alpha^{1/2}(\Gamma).$$

In our settings, all the conditions in Lemma 4.5 are satisfied. In fact, conditions (a) and (b) follow from Lemma 4.3 and Lemma 4.4 (iii), respectively. Conditions (c) and (d) are guaranteed by Lemma 4.4 (i) and (ii). Combining Lemmas 4.2 and 4.5, we get

Theorem 4.6. *Assume $\omega \notin \mathcal{D} \cup \mathcal{D}_1 \cup \mathcal{D}_2$. Set $N^\sharp := |\mathrm{Re}\,N| + |\mathrm{Im}\,N|$, and let $\eta_n, g_n(y)$ be given as in (6) and (31), respectively. Then,*

(i) *The sequence $\{\exp(i\eta_n b)g_n(y) : n \in \mathbb{Z}^2\}$ belongs to $\mathrm{Range}(N_\sharp^{1/2})$ if and only if $y \in \Omega_0^-$.*

(ii) *The near-field data $v_j^{sc}(\tilde{x}, b)$ for all $\tilde{x} \in (0, 2\pi) \times (0, 2\pi)$, $j \in \mathbb{Z}^2$ and some $b > \Gamma^+$ uniquely determine the interface Γ.*

Note that the uniqueness described in Theorem 4.6 (ii) is only a corollary of the first assertion. By Picard's theorem (see e.g, [8, Theorem 4.8]), the region Ω_0^- below Γ can be characterized through the eigensystem of the near-field operator as follows.

Corollary 4.7. *Suppose the assumptions in Theorem 4.6 hold. Let (λ_j, ψ_j) be an eigensystem of the (positive) operator $N_\sharp$. We have the following characterization of Ω_0^-:*

$$y \in \Omega_0^- \quad \Longleftrightarrow \quad \sum_{j=1}^{\infty} \frac{|\langle \phi_y, \psi_j \rangle_{l^2}|^2}{\lambda_j} < \infty,$$

with $\phi_y := \{\exp(i\eta_n b)g_n(y) : n \in \mathbb{Z}^2\} \in l^2$, or equivalently,

$$y \in \Omega_0^- \quad \Longleftrightarrow \quad W(y) := \sum_{j=1}^{\infty} \left[\frac{|\langle \phi_y, \psi_j \rangle_{l^2}|^2}{\lambda_j} \right]^{-1} > 0. \tag{51}$$

Thus, the interface Γ can be identified by first selecting sampling points from the set $\{(\tilde{y}, y_3) \in \mathbb{R}^3 : 0 < y_3 < b\}$ and then computing the value of the indicator function $W(y)$. The values $W(y)$ for y lying below the Γ will be relatively larger than those above Γ which are actually zero.

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