Weierstraß-Institut
für Angewandte Analysis und Stochastik
Leibniz-Institut im Forschungsverbund Berlin e. V.

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A random cloud model for the Schrödinger equation

Wolfgang Wagner

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Weierstrass Institute
Mohrenstrasse 39
10117 Berlin, Germany
E-Mail: wolfgang.wagner@wias-berlin.de

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Weierstraß-Institut für Angewandte Analysis und Stochastik (WIAS)
Leibniz-Institut im Forschungsverbund Berlin e. V.
Mohrenstraße 39
10117 Berlin
Germany

Fax: +49 30 20372-303
E-Mail: preprint@wias-berlin.de
World Wide Web: http://www.wias-berlin.de/

Abstract

The paper is concerned with the construction of a stochastic model for the spatially discretized time-dependent Schrödinger equation. The model is based on a particle system with a Markov jump evolution. The particles are characterized by a sign (plus or minus), a position (discrete grid) and a type (real or imaginary). The jumps are determined by the creation of offsprings. The main result is the construction of a family of complex-valued random variables such that their expected values coincide with the solution of the Schrödinger equation.

Contents

1 Introduction

One of the basic equations in quantum mechanics was established by Erwin Schrödinger in 1926 [22]. The time-dependent form of the Schrödinger equation for a single electron is

$$i\,\hbar\,\frac{\partial}{\partial t}\,\Phi(t,x) \;\;=\;\; -\frac{\hbar^2}{2m}\,\Delta_x\Phi(t,x) - q\,V(x)\,\Phi(t,x)\,, \tag{1.1}$$

where m is the electron mass, q is the electron charge, V is the electric potential, $\hbar$ is Planck's constant divided by 2π , and i denotes the imaginary unit. Equation (1.1) describes the time evolution of the so-called wave-function Φ , which represents the quantum state of the electron.

This paper is concerned with the construction of a stochastic model for the Schrödinger equation. The probabilistic approach to quantum mechanics goes back to Feynman ([9], [10]). Developing Feynman's ideas, Kac introduced integration on the space of trajectories of the Wiener process [13]. In other areas, stochastic models for deterministic equations have been studied earlier. Some connections between random walks and difference equations were found in [5]. The basic equations for general drift-diffusion processes were established by Kolmogoroff [15]. Stochastic particle models for the Boltzmann equation were studied by Leontovich [16] and later by Kac [14] (see [21, Section 2.3.3] and [23] for more details).

The paper is organized as follows. The model and the results are presented in Section 2. The model is based on a particle system with a Markov jump evolution. The particles are characterized by a sign (plus or minus), a position (discrete grid) and a type (real or imaginary). The jumps are determined by the creation of offsprings. The main result is the construction of a family of complex-valued random variables such that their expected values coincide with the solution of the spatially discretized Schrödinger equation. The proofs are given in Section 3. Finally, Section 4 contains a sketch of possible generalizations as well as some comments concerning the physical interpretation of the model.

2 Model and results

The model is presented in its simplest form in order to make the basic idea as transparent as possible. Modifications and extensions will be discussed later.

We consider the discretized one-dimensional version of the Schrödinger equation (1.1),

$$i\,\hbar\,\frac{\partial}{\partial t}\,\Phi^{(\varepsilon)}(t,x) \;\;=\;\; -\frac{\hbar^2}{2m}\,\Delta_x^{(\varepsilon)}\Phi^{(\varepsilon)}(t,x) - q\,V(x)\,\Phi^{(\varepsilon)}(t,x)\,, \tag{2.1}$$

with initial condition

$$\Phi^{(\varepsilon)}(0,x) \;\;=\;\; \Phi_0(x)\,. \tag{2.2}$$

The discrete Laplacian

$$\Delta^{(\varepsilon)}f(x) \;\;=\;\; \frac{f(x+\varepsilon) - 2\,f(x) + f(x-\varepsilon)}{\varepsilon^2}\,, \qquad \varepsilon > 0\,, \tag{2.3}$$

is defined for functions f on the grid

$$\mathbb{R}_\varepsilon \;=\; \{\varepsilon\, j\,,\quad j=\ldots,-1,0,1,\ldots\}\,. \tag{2.4}$$

The main result is the construction of a family of complex-valued random variables

$$\xi^{(\varepsilon)}(t,x)\,,\qquad t\geq 0\,,\quad x\in\mathbb{R}_\varepsilon$$

such that their expected values coincide with the solution of equation (2.1).

2.1 The stochastic particle system

We introduce a particle system

$$\Big(u_j(t),x_j(t),y_j(t)\Big),\qquad j=1,\ldots,N(t)\,, \tag{2.5}$$

where $u_j=\pm 1$ is the sign, $x_j\in X\subset\mathbb{R}$ is the position and $y_j=1,2$ is the type (real or imaginary) of the particle. The system (2.5) evolves as a Markov jump processes. Independently of each other, the particles create "offsprings" that are added to the system. Let

$$Z=\{-1,1\}\times X\times\{1,2\} \tag{2.6}$$

denote the state space of an individual particle.

For any particle at state $z=(u,x,y)\in Z$, the waiting time τ until the next "creation event" is exponentially distributed,

$$\mathbb{P}(\tau\geq t)\;=\;\exp(-\lambda(z)\,t)\,,\qquad t\geq 0\,,$$

where

$$\lambda(z)\;=\;\frac{2\hbar}{m\,\varepsilon^2}+\frac{q}{\hbar}\,|V(x)|\,. \tag{2.7}$$

Note that the parameter (2.7) has the physical dimension "number per second", when ε has the dimension "meter". This is consistent with λ being a jump rate. Indeed, the physical dimensions of the various quantities are: $q\sim$ [C] (Coulomb); $V\sim$ [J/C], where J(Joule)=$\mathrm{kg\,m^2\,s^{-2}}$; $m\sim$ [kg]; $\hbar\sim$ [eV s], where $\mathrm{eV}\sim\mathrm{J}$.

The creation event has the form

$$z=(u,x,y)\;\;\Rightarrow\;\;\begin{cases}(u'(z),x-\varepsilon,y'(z))\,, & \text{with probability } \frac{c_1(\varepsilon)}{\lambda(z)}\,,\\ (u'(z),x+\varepsilon,y'(z))\,, & \text{with probability } \frac{c_1(\varepsilon)}{\lambda(z)}\,,\\ (-u'(z),x,y'(z))\,, & \text{with probability } \frac{2\,c_1(\varepsilon)}{\lambda(z)}\,,\\ (u'(z)\,\mathrm{sign}\,V(x),x,y'(z))\,, & \text{with probability } \frac{|c_2(x)|}{\lambda(z)}\,,\end{cases} \tag{2.8}$$

where

$$u'(z)\;=\;\begin{cases}u\;\;, & \text{if}\quad y=1\,,\\ -u\,, & \text{if}\quad y=2\,,\end{cases} \tag{2.9}$$

$$y'(z)\;=\;\begin{cases}2\,, & \text{if}\quad y=1\,,\\ 1\,, & \text{if}\quad y=2\,,\end{cases} \tag{2.10}$$

and

$$c_1(\varepsilon)=\frac{\hbar}{2m\,\varepsilon^2}\,,\qquad c_2(x)=\frac{q}{\hbar}\,V(x)\,. \tag{2.11}$$

2.2 Probabilistic representation

The main result is a probabilistic representation of the solution of equation (2.1) in terms of the particle system (2.5).

Theorem 2.1 *Consider the particle system (2.5) with $X = \mathbb{R}_\varepsilon$ (cf. (2.4)) and $N(0) = 1$. Assume*

$$|V(x)| \;\leq\; V_{\max} \qquad \forall\, x \in X \tag{2.12}$$

and

$$\|\Phi_0\| := \sum_{x\in X} \Big[|\Phi_{0,1}(x)| + |\Phi_{0,2}(x)| \Big] < \infty\,, \tag{2.13}$$

where $\Phi_0 = \Phi_{0,1} + i\,\Phi_{0,2}$ (cf. (2.2)). Define the random variables

$$\xi^{(\varepsilon)}(t,x) = \|\Phi_0\| \left\{ \eta^{(\varepsilon)}_{1,1}(t,x) - \eta^{(\varepsilon)}_{-1,1}(t,x) + i \left[\eta^{(\varepsilon)}_{1,2}(t,x) - \eta^{(\varepsilon)}_{-1,2}(t,x) \right] \right\}, \tag{2.14}$$

where

$$\eta^{(\varepsilon)}_{u,y}(t,x) = \#\left\{ j = 1,\dots,N(t) \;:\; \Big(u_j(t), x_j(t), y_j(t)\Big) = (u,x,y) \right\} \tag{2.15}$$

are the occupation numbers. If

$$\mathbb{E}\,\xi^{(\varepsilon)}(0,x) \;=\; \Phi_0(x) \qquad \forall\, x \in X\,, \tag{2.16}$$

then

$$\mathbb{E}\,\xi^{(\varepsilon)}(t,x) \;=\; \Phi^{(\varepsilon)}(t,x) \qquad \forall\, t > 0\,, \quad x \in X\,, \tag{2.17}$$

where $\mathbb{E}$ denotes mathematical expectation.

Remark 2.2 *Condition (2.16) is fulfilled, when the initial particle is generated as follows. Choose the position $x_1(0)$ according to the density*

$$\frac{|\Phi_{0,1}(x)| + |\Phi_{0,2}(x)|}{\|\Phi_0\|} \qquad x \in X\,.$$

Given $x_1(0) = x$, choose sign and type

$$(u_1(0), y_1(0)) = \begin{cases} (1,1)\,, & \textit{with probability}\;\; \Phi^+_{0,1}(x)/(|\Phi_{0,1}(x)| + |\Phi_{0,2}(x)|)\,, \\ (-1,1)\,, & \textit{with probability}\;\; \Phi^-_{0,1}(x)/(|\Phi_{0,1}(x)| + |\Phi_{0,2}(x)|)\,, \\ (1,2)\,, & \textit{with probability}\;\; \Phi^+_{0,2}(x)/(|\Phi_{0,1}(x)| + |\Phi_{0,2}(x)|)\,, \\ (-1,2)\,, & \textit{with probability}\;\; \Phi^-_{0,2}(x)/(|\Phi_{0,1}(x)| + |\Phi_{0,2}(x)|)\,, \end{cases}$$

where the superscripts "+" and "-" denote the positive and negative parts of a function. Indeed, (2.14) and (2.15) imply

$$\begin{aligned} &\frac{1}{\|\Phi_0\|}\,\mathbb{E}\,\xi^{(\varepsilon)}(0,x) = \\ &\qquad \mathbb{P}\Big((u_1(0),x_1(0),y_1(0)) = (1,x,1)\Big) - \mathbb{P}\Big((u_1(0),x_1(0),y_1(0)) = (-1,x,1)\Big) + \\ &\qquad i\left[\mathbb{P}\Big((u_1(0),x_1(0),y_1(0)) = (1,x,2)\Big) - \mathbb{P}\Big((u_1(0),x_1(0),y_1(0)) = (-1,x,2)\Big)\right] \\ &= \;\; \frac{1}{\|\Phi_0\|}\left\{\Phi^+_{0,1}(x) - \Phi^-_{0,1}(x) + i\left[\Phi^+_{0,2}(x) - \Phi^-_{0,2}(x)\right]\right\} = \frac{1}{\|\Phi_0\|}\,\Phi_0(x)\,. \end{aligned}$$

Finally, some modifications of the stochastic model are introduced, for which the result (2.17) also holds.

model with creation of more than one particle

In the basic model each jump consists in the creation of exactly one particle. However, also groups of particles can be created.

For example, creation of pairs of particles is obtained, when (2.7) and (2.8) are replaced by

$$\lambda(z) \;\; = \;\; 2\,c_1(\varepsilon) + |c_2(x)| \tag{2.18}$$

and

$$z \;\; \Rightarrow \;\; \begin{cases} (u'(z), x-\varepsilon, y'(z)), (-u'(z), x, y'(z))\,, & \text{with probability } \frac{c_1(\varepsilon)}{\lambda(z)}\,, \\ (u'(z), x+\varepsilon, y'(z)), (-u'(z), x, y'(z))\,, & \text{with probability } \frac{c_1(\varepsilon)}{\lambda(z)}\,, \\ (u'(z)\,\mathrm{sign}\,V(x), x, y'(z))\,, & \text{with probability } \frac{|c_2(x)|}{\lambda(z)}\,, \end{cases} \tag{2.19}$$

respectively.

Creation of four particles is obtained, when (2.7) and (2.8) are replaced by

$$\lambda(z) \;\; = \;\; c_1(\varepsilon) + |c_2(x)| \tag{2.20}$$

and

$$z \;\; \Rightarrow \;\; \begin{cases} (u'(z), x-\varepsilon, y'(z)), 2\,(-u'(z), x, y'(z))\,, & \\ (u'(z), x+\varepsilon, y'(z))\,, & \text{with probability } \frac{c_1(\varepsilon)}{\lambda(z)}\,, \\ (u'(z)\,\mathrm{sign}\,V(x), x, y'(z))\,, & \text{with probability } \frac{|c_2(x)|}{\lambda(z)}\,, \end{cases} \tag{2.21}$$

respectively.

model with cancellation of pairs of particles

Another modification is obtained, when pairs of particles of the form

$$z_j = (1, x, y)\,, \qquad z_j' = (-1, x, y) \tag{2.22}$$

are immediately removed from the system. Clearly this transformation does not change the random variables (2.14). In the modifications with creation of more than one particle (cf. (2.19), (2.21)) there can be cancellation of more than one pair of particles at the same time.

2.3 Classical case

Here we study the analog of the stochastic model (2.5) in the "classical" case, namely for the discretized heat equation (cf. (2.3))

$$\frac{\partial}{\partial t}\,\Phi^{(\varepsilon)}(t,x) \;\; = \;\; a\,\Delta_x^{(\varepsilon)}\,\Phi^{(\varepsilon)}(t,x) + V(x)\,\Phi^{(\varepsilon)}(t,x)\,, \tag{2.23}$$

where $a \geq 0$. We introduce

$$\Big(u_j(t), x_j(t)\Big), \qquad j = 1, \ldots, N(t)\,, \tag{2.24}$$

$$Z = \{-1, 1\} \times X_\varepsilon$$

and

$$\lambda(z) \;\; = \;\; 4\, c_1(\varepsilon) + |c_2(x)|\,, \qquad z = (u, x) \in Z\,,$$

where

$$c_1(\varepsilon) = \frac{a}{\varepsilon^2}\,, \qquad c_2(x) = V(x)\,.$$

The creation event has the form (cf. (2.8))

$$z = (u, x) \;\; \Rightarrow \;\; \begin{cases} (u, x - \varepsilon)\,, & \text{with probability } \frac{c_1(\varepsilon)}{\lambda(z)}\,, \\ (u, x + \varepsilon)\,, & \text{with probability } \frac{c_1(\varepsilon)}{\lambda(z)}\,, \\ (-u, x)\,, & \text{with probability } \frac{2\, c_1(\varepsilon)}{\lambda(z)}\,, \\ (u \operatorname{sign} V(x), x)\,, & \text{with probability } \frac{|c_2(x)|}{\lambda(z)}\,. \end{cases} \tag{2.25}$$

The modification with creation of pairs of particles takes the form (cf. (2.19))

$$z = (u, x) \;\; \Rightarrow \;\; \begin{cases} (u, x - \varepsilon), (-u, x)\,, & \text{with probability } \frac{c_1(\varepsilon)}{\lambda(z)}\,, \\ (u, x + \varepsilon), (-u, x)\,, & \text{with probability } \frac{c_1(\varepsilon)}{\lambda(z)}\,, \\ (u \operatorname{sign} V(x), x)\,, & \text{with probability } \frac{|c_2(x)|}{\lambda(z)}\,, \end{cases} \tag{2.26}$$

where λ is given in (2.18).

Remark 2.3 *In the modification with creation of pairs (cf. (2.26)) and cancellation (cf. (2.22)), the particles either jump to a new position or create offsprings at the same position.*

Assume $\Phi_0 \geq 0$ and $\|\Phi_0\| = 1$. Then the initial particle has a positive sign and a position distributed according to Φ_0 . The random variable (2.14) takes the form

$$\xi^{(\varepsilon)}(t, x) = \eta_1^{(\varepsilon)}(t, x) - \eta_{-1}^{(\varepsilon)}(t, x)\,, \tag{2.27}$$

where

$$\eta_u^{(\varepsilon)}(t, x) = \#\left\{j = 1, \ldots, N(t) \;:\; \Big(u_j(t), x_j(t)\Big) = (u, x)\right\}. \tag{2.28}$$

Next we consider two special cases and provide some explicit solutions.

pure transport

If $V = 0$, then the random cloud (2.24) reduces to one particle. This particle jumps to one of the neighboring positions with equal probability keeping its sign obtained at the start (cf. (2.26)). The expectation of the random variable (2.27) equals the probability that the particle is located at time t at position x.

The limit $\varepsilon \to 0$ gives the Wiener process. For $a > 0$, equation

$$\frac{\partial}{\partial t}\,\Phi(t,x) = a\,\Delta_x\,\Phi(t,x) \tag{2.29}$$

has the solution

$$\Phi(t,x) = \int_{\mathbb{R}} dx_0\,\Phi_0(x_0)\,p_W(t,x_0,x)\,, \tag{2.30}$$

where

$$p_W(t,x_0,x) = \frac{1}{\sqrt{4\pi\,a\,t}}\,\exp\left(-\frac{(x-x_0)^2}{4\,a\,t}\right). \tag{2.31}$$

Note that

$$\lim_{t\to 0}\,p_W(t,x_0,x) = \delta(x-x_0)\,.$$

The solution (2.30) represents the distribution of the Wiener process at time t, when it starts according to Φ_0. Another interpretation of (2.30) is

$$\Phi(t,x) = \mathbb{E}\,\Phi_0(W_x(t))\,, \tag{2.32}$$

where $W_x(0) = x$. But the "backward" representation (2.32) is based on symmetry properties of (2.31), which do not hold in more general situations.

no transport

If $a = 0$, then equation (2.23) has the solution

$$\Phi^{(\varepsilon)}(t,x) = \Phi_0(x)\,\exp(t\,V(x))\,, \tag{2.33}$$

which does not depend on ε.

In the stochastic model each particle (u,x) creates offsprings of the form $(u\,\mathrm{sign}V(x),x)$. The random variables (2.27) are independent for different x.

Example 2.4 *If the initial particle has position x and $V(x) > 0$, then all created particles are the same. According to (2.27), (2.28), one obtains $\xi^{(\varepsilon)}(t,x) = N(t)$, where $N(t)$ is a birth process with generator*

$$Af(N) = (f(N+1) - f(N))\,V(x)\,N\,.$$

Dynkin's formula

$$\frac{d}{dt}\,\mathbb{E}\,N(t) = V(x)\,\mathbb{E}\,N(t)$$

gives

$$\mathbb{E}\,N(t) = \exp(t\,V(x))\,.$$

Example 2.5 *If the initial particle has position x and $V(x) < 0$, then the first new particle cancels the original one, and the process stops. One obtains $\xi^{(\varepsilon)}(t,x) = N(t)$, where $N(t) = 1$ or $N(t) = 0$, with*

$$\mathbb{P}(N(t) = 1) = \exp(-t\,|V(x)|)\,.$$

The effect of cancellation is illustrated by finding the explicit solution of the original model (2.25). Let $N_+(t)$ and $N_-(t)$ denote the numbers of positive and negative particles so that $\xi^{(\varepsilon)}(t,x) = N_+(t) - N_-(t)$. There are jumps $N_+ \to N_+ + 1$, with rate $|V(x)|\,N_-$, and $N_- \to N_- + 1$, with rate $|V(x)|\,N_+$. The generator is

$$Af(N_+,N_-) = \\ [f(N_+ + 1, N_-) - f(N_+,N_-)]\,|V(x)|\,N_- + [f(N_+, N_- + 1) - f(N_+,N_-)]\,|V(x)|\,N_+\,.$$

Dynkin's formula gives a system of two differential equations

$$\frac{d}{dt}\,\mathbb{E}\,N_+(t) = |V(x)|\,\mathbb{E}\,N_-(t)\,, \qquad \frac{d}{dt}\,\mathbb{E}\,N_-(t) = |V(x)|\,\mathbb{E}\,N_+(t)\,.$$

Using a vector notation, one obtains

$$\frac{d}{dt}\,\mathbb{E}\,\bar{N}(t) = |V(x)|\,B\,\bar{N}(t)\,, \qquad \textit{where} \quad B = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix},$$

and

$$\mathbb{E}\,\bar{N}(t) = \exp(|V(x)|\,B\,t)\,\mathbb{E}\,\bar{N}(0)\,, \qquad \textit{where} \quad \mathbb{E}\,\bar{N}(0) = \begin{pmatrix} 1 \\ 0 \end{pmatrix}.$$

Since $B^2 = I$, it follows that

$$\begin{aligned} &\exp(|V(x)|\,B\,t) = \\ &\qquad I + (|V(x)|\,t)\,B + \frac{(|V(x)|\,t)^2}{2}\,I + \frac{(|V(x)|\,t)^3}{3!}\,B + \frac{(|V(x)|\,t)^4}{4!}\,I + \ldots \\ &= \quad I\left[1 + \frac{(|V(x)|\,t)^2}{2} + \ldots\right] + B\left[(|V(x)|\,t) + \frac{(|V(x)|\,t)^3}{3!} + \ldots\right]. \end{aligned}$$

One obtains

$$\mathbb{E}\,N_+(t) = 1 + \frac{(|V(x)|\,t)^2}{2} + \ldots$$

and

$$\mathbb{E}\,N_-(t) = (|V(x)|\,t) + \frac{(|V(x)|\,t)^3}{3!} + \ldots$$

so that

$$\mathbb{E}\,N_+(t) - \mathbb{E}\,N_-(t) = \exp(-|V(x)|\,t)\,.$$

3 Proofs

Theorem 2.1 will be proved as a special case of a more general setup. This approach covers the modifications of the basic model introduced in Section 2 as well as some of the extensions that will be discussed in Section 4.

3.1 General approach

Consider a particle system

$$\Big(z_j(t)\Big), \qquad j = 1, \dots, N(t)\,, \qquad t \geq 0\,, \tag{3.1}$$

where $z_j(t)$ belongs to a locally compact separable metric space Z and $N(t)$ is the number of particles. The time evolution of the system is determined by a Markov jump mechanism. The transition kernel has the form

$$Q(\bar{z}, d\bar{z}_1) = \sum_{j=1}^{N} \int_{\mathcal{Z}_K} q(z_j, dz'_1, \dots, dz'_k)\, \delta_{J(\bar{z};z'_1,\dots,z'_k)}(d\bar{z}_1)\,, \tag{3.2}$$

where

$$\bar{z} = (z_1, \dots, z_N) \in \mathcal{Z} = \cup_{N=1}^{\infty} Z^N\,. \tag{3.3}$$

The kernel q governs the generation of at most K "offsprings" belonging to the space

$$\mathcal{Z}_K = \cup_{k=1}^{K} Z^k\,. \tag{3.4}$$

The jump transformation

$$J(\bar{z}; z'_1, \dots, z'_k) = (z_1, \dots, z_N, z'_1, \dots, z'_k) \tag{3.5}$$

adds the new particles to the system.

Theorem 3.1 *Assume*

$$N(0) = n \qquad \textit{for some} \quad n = 1, 2, \dots \tag{3.6}$$

and

$$q(z, \mathcal{Z}_K) \;\leq\; C_q < \infty \qquad \forall\, z \in Z\,. \tag{3.7}$$

Consider the random measures

$$\mu(t, dz) = \sum_{j=1}^{N(t)} \delta_{z_j(t)}(dz)\,. \tag{3.8}$$

Then

$$\nu(t, dz) = \mathbb{E}\,\mu(t, dz) \tag{3.9}$$

satisfies

$$\int_Z \psi(z)\,\nu(t,dz) = \int_Z \psi(z)\,\nu(0,dz) + \int_0^t \left(\int_Z \nu(s,dz) \int_{\mathcal{Z}_K} q(z, dz'_1, \ldots, dz'_k) \Big[\psi(z'_1) + \ldots + \psi(z'_k) \Big] \right) ds\,, \tag{3.10}$$

for any bounded measurable function ψ .

Proof. First we check regularity of the process (3.1). The jump rate is unbounded, but satisfies

$$\lambda(\bar{z}) := Q(\bar{z}, \mathcal{Z}) \;\; \leq \;\; C_q\, N \qquad \forall\, \bar{z} = (z_1, \ldots, z_N) \in \mathcal{Z}\,,$$

according to assumption (3.7). One obtains (cf. (3.6))

$$\lambda(\zeta_l) \;\; \leq \;\; C_q\,(n + l\,K)\,, \qquad l = 0, 1, 2, \ldots\,,$$

where $\zeta_l \in \mathcal{Z}$ denotes the embedded Markov chain of the jump process (3.1). Thus, regularity follows from the criterion (cf., e.g., [3, p.337])

$$\mathbb{P}\left(\sum_{l=0}^{\infty} \frac{1}{\lambda(\zeta_l)} = \infty \right) \;\; = \;\; 1\,.$$

The generator of the process (3.1) takes the form (cf. (3.2))

$$\begin{aligned} (A\Psi)(\bar{z}) \;\; &= \;\; \int_{\mathcal{Z}} [\Psi(\bar{z}_1) - \Psi(\bar{z})]\, Q(\bar{z}, \bar{z}_1) \\ &= \;\; \sum_{j=1}^{N} \int_{\mathcal{Z}_K} q(z_j, dz'_1, \ldots, dz'_k) \Big[\Psi(J(\bar{z}; z'_1, \ldots, z'_k)) - \Psi(\bar{z}) \Big]. \end{aligned} \tag{3.11}$$

Dynkin's formula (cf. [4, p.380]) states

$$\mathbb{E}\,[\Psi(\bar{Z}(t))] \;\; = \;\; \mathbb{E}\,[\Psi(\bar{Z}(0))] + \mathbb{E}\left[\int_0^t (A\Psi)(\bar{Z}(s))\, ds \right], \tag{3.12}$$

where $\bar{Z}(t)$ denotes the process (3.1). With

$$\Psi(\bar{z}) = \psi(z_1) + \ldots + \psi(z_N) \tag{3.13}$$

one obtains (cf. (3.5))

$$(A\Psi)(\bar{z}) \;\; = \;\; \sum_{j=1}^{N} \int_{\mathcal{Z}_K} q(z_j, dz'_1, \ldots, dz'_k) \Big[\psi(z'_1) + \ldots + \psi(z'_k) \Big].$$

Thus, (3.12) takes the form

$$\mathbb{E}\left[\sum_{j=1}^{N(t)} \psi(z_j(t)) \right] = \mathbb{E}\left[\sum_{j=1}^{N(t)} \psi(z_j(0)) \right] + \mathbb{E}\left[\int_0^t \left(\sum_{j=1}^{N(s)} \int_{\mathcal{Z}_K} q(z_j(s), dz'_1, \ldots, dz'_k) \Big[\psi(z'_1) + \ldots + \psi(z'_k) \Big] \right) ds \right], \tag{3.14}$$

which implies (3.10).

Dynkin's formula holds for functions such that

$$\mathbb{E}\,|\Psi(\bar{Z}(t))| < \infty \qquad \forall\, t \geq 0\,. \tag{3.15}$$

Property (3.15) is satisfied for the function (3.13) provided that ψ is measurable and bounded. Indeed, one obtains

$$\mathbb{E}\,|\Psi(\bar{Z}(t))| \leq \|\psi\|\,\mathbb{E}\,N(t)$$

and (cf. (3.6), (3.7))

$$\mathbb{E}\,N(t) \leq n + K\,C_q \int_0^t \mathbb{E}\,N(s)\,ds$$

so that

$$\mathbb{E}\,N(t) \leq n\,\exp(K\,C_q\,t)\,,$$

according to Gronwall's inequality. ■

3.2 Theorem 2.1

Theorem 2.1 is a consequence of Theorem 3.1. Consider $K = 1$ so that $\mathcal{Z}_K = Z$ (cf. (3.4)), where Z is defined in (2.6). Below we use the notations $U = \{-1, 1\}$ and $Y = \{1, 2\}$. According to (2.7)–(2.11), one obtains

$$\begin{aligned} q(z, dz') = |c_2|\,\delta_{(u'(z)\,\mathrm{sign}\,V(x),x,y'(z))}(dz') + \\ c_1\left[\delta_{(u'(z),x-\varepsilon,y'(z))}(dz') + \delta_{(u'(z),x+\varepsilon,y'(z))}(dz') + 2\,\delta_{(-u'(z),x,y'(z))}(dz')\right], \end{aligned} \tag{3.16}$$

where $c_1 = c_1(\varepsilon)$ and $c_2 = c_2(x)$. Note that (cf. (3.10))

$$\begin{aligned} \int_Z \nu(t, dz) \int_{\mathcal{Z}_K} q(z, dz'_1, \ldots, dz'_k)\Big[\psi(z'_1) + \ldots + \psi(z'_k)\Big] = \\ \int_Z \nu(t, du, dx, dy)\Big[|c_2|\,\psi(u'(z)\,\mathrm{sign}\,V(x), x, y'(z)) + \\ c_1\Big(\psi(u'(z), x-\varepsilon, y'(z)) + \psi(u'(z), x+\varepsilon, y'(z)) + 2\,\psi(-u'(z), x, y'(z))\Big)\Big]. \end{aligned} \tag{3.17}$$

With

$$\psi(z) = u\,\varphi(x, y) \tag{3.18}$$

and

$$f(t, dx, dy) = \int_U u\,\nu(t, du, dx, dy)\,, \tag{3.19}$$

one obtains

$$\int_Z \psi(z)\,\nu(t,dz) = \int_{X\times Y} \varphi(x,y)\,f(t,dx,dy) \tag{3.20}$$

and (cf. (2.11))

$$\begin{aligned}
\int_Z \nu(t,dz) \int_{\mathcal{Z}_K} q(z,dz'_1,\ldots,dz'_k)\Big[\psi(z'_1)+\ldots+\psi(z'_k)\Big] = \qquad (3.21)\\
\int_{U\times X} \nu(t,du,dx,1)\Big[|c_2|\,\psi(u\,\mathrm{sign}\,V(x),x,2)+\\
c_1\Big(\psi(u,x-\varepsilon,2)+\psi(u,x+\varepsilon,2)+2\,\psi(-u,x,2)\Big)\Big]+\\
\int_{U\times X} \nu(t,du,dx,2)\Big[|c_2|\,\psi(-u\,\mathrm{sign}\,V(x),x,1)+\\
c_1\Big(\psi(-u,x-\varepsilon,1)+\psi(-u,x+\varepsilon,1)+2\,\psi(u,x,1)\Big)\Big]\\
= \int_X f(t,dx,1)\Big[c_2\,\varphi(x,2)+c_1\Big(\varphi(x-\varepsilon,2)+\varphi(x+\varepsilon,2)-2\,\varphi(x,2)\Big)\Big]-\\
\int_X f(t,dx,2)\Big[c_2\,\varphi(x,1)+c_1\Big(\varphi(x-\varepsilon,1)+\varphi(x+\varepsilon,1)-2\,\varphi(x,1)\Big)\Big].
\end{aligned}$$

According to (2.12), one obtains (cf. (3.16))

$$q(z,Z) \;\le\; \frac{2\hbar}{m\,\varepsilon^2}+\frac{q}{\hbar}\,V_{\max}$$

so that (3.7) is satisfied. With $\varphi(x,2)=0$ and $\varphi(x,1)=0$, respectively, it follows from (3.10), (3.20) and (3.21) that

$$\begin{aligned}
\int_X \varphi(x,1)\,f(t,dx,1) = \int_X \varphi(x,1)\,f(0,dx,1)-\\
\int_0^t ds \int_X f(s,dx,2)\Big[c_2\,\varphi(x,1)+c_1\Big(\varphi(x-\varepsilon,1)+\varphi(x+\varepsilon,1)-2\,\varphi(x,1)\Big)\Big]
\end{aligned}$$

and

$$\begin{aligned}
\int_X \varphi(x,2)\,f(t,dx,2) = \int_X \varphi(x,2)\,f(0,dx,2)+\\
\int_0^t ds \int_X f(s,dx,1)\Big[c_2\,\varphi(x,2)+c_1\Big(\varphi(x-\varepsilon,2)+\varphi(x+\varepsilon,2)-2\,\varphi(x,2)\Big)\Big].
\end{aligned}$$

When taking into account that f has a density (discrete position space), then, by removing test functions, one obtains

$$\begin{aligned}
\frac{\partial}{\partial t}\,f(t,x,1) &= -c_2\,f(t,x,2)-c_1\Big[f(t,x+\varepsilon,2)+f(t,x-\varepsilon,2)-2\,f(t,x,2)\Big]\\
\frac{\partial}{\partial t}\,f(t,x,2) &= c_2\,f(t,x,1)+c_1\Big[f(t,x+\varepsilon,1)+f(t,x-\varepsilon,1)-2\,f(t,x,1)\Big]. \qquad (3.22)
\end{aligned}$$

Note that (cf. (2.15), (3.8))

$$\eta^{(\varepsilon)}_{u,y}(t,x) \;=\; \mu(t,u,x,y)$$

and (cf. (2.14), (3.9))

$$\mathbb{E}\,\xi^{(\varepsilon)}(t,x) = \|\Phi_0\|\left\{\nu(t,1,x,1)-\nu(t,-1,x,1)+i\left[\nu(t,1,x,2)-\nu(t,-1,x,2)\right]\right\} \tag{3.23}$$

Since (cf. (3.19))

$$f(t,x,y)=\int_U u\,\nu(t,du,x,y)=\nu(t,1,x,y)-\nu(t,-1,x,y)\,,$$

it follows from (3.23) that

$$\mathbb{E}\,\xi^{(\varepsilon)}(t,x) \;=\; \|\Phi_0\|\Big[f(t,x,1)+i\,f(t,x,2)\Big].$$

On the other hand, with

$$\Phi^{(\varepsilon)}(t,x) \;=\; \Phi^{(\varepsilon)}_1(t,x)+i\,\Phi^{(\varepsilon)}_2(t,x)\,,$$

equation (2.1) takes the form (cf. (2.11))

$$\begin{aligned}
\frac{\partial}{\partial t}\,\Phi^{(\varepsilon)}_1(t,x) &= -c_1\Big[\Phi^{(\varepsilon)}_2(t,x+\varepsilon)-2\,\Phi^{(\varepsilon)}_2(t,x)+\Phi^{(\varepsilon)}_2(t,x-\varepsilon)\Big]-c_2\,\Phi^{(\varepsilon)}_2(t,x)\\
\frac{\partial}{\partial t}\,\Phi^{(\varepsilon)}_2(t,x) &= c_1\Big[\Phi^{(\varepsilon)}_1(t,x+\varepsilon)-2\,\Phi^{(\varepsilon)}_1(t,x)+\Phi^{(\varepsilon)}_1(t,x-\varepsilon)\Big]+c_2\,\Phi^{(\varepsilon)}_1(t,x)\,,
\end{aligned}$$

which is (3.22). Thus, $\mathbb{E}\,\xi^{(\varepsilon)}(t,x)$ and $\Phi^{(\varepsilon)}(t,x)$ satisfy the same system of equations. The initial conditions coincide, according to (2.16). Uniqueness holds, according to (2.12) and (2.13), so that (2.17) follows.

3.3 Modifications

creation of more than one particle

For the modification (2.18), (2.19), one obtains $K=2$ and

$$\begin{aligned}
&q(z,dz') = |c_2|\,\delta_{(u'(z)\,\mathrm{sign}\,V(x),x,y'(z))}(dz')+\\
&\quad c_1\Big[\delta_{(u'(z),x-\varepsilon,y'(z)),(-u'(z),x,y'(z))}(dz'_1,dz'_2)+\delta_{(u'(z),x+\varepsilon,y'(z)),(-u'(z),x,y'(z))}(dz'_1,dz'_2)\Big],
\end{aligned}$$

For the modification (2.20), (2.21), one obtains $K=4$ and

$$\begin{aligned}
&q(z,dz') = |c_2|\,\delta_{(u'(z)\,\mathrm{sign}\,V(x),x,y'(z))}(dz')+\\
&\quad c_1\,\delta_{(u'(z),x-\varepsilon,y'(z)),(-u'(z),x,y'(z)),(-u'(z),x,y'(z)),(u'(z),x+\varepsilon,y'(z))}(dz'_1,dz'_2,dz'_3,dz'_4)\,.
\end{aligned}$$

In both cases (3.17) holds, which leads to the same equation.

cancellation of pairs of particles

Instead of J given in (3.5) one uses

$$\tilde{J}(\bar{z}; z'_1, \ldots, z'_K) = T(J(\bar{z}; z'_1, \ldots, z'_K)) = T(z_1, \ldots, z_N, z'_1, \ldots, z'_K)\,,$$

where the transformation T removes pairs of the form (2.22). The state space (3.3) is extended by including $N = 0$ so that the empty system is covered. Let $\tilde{Q}$ and $\tilde{A}$ be defined in analogy with (3.2) and (3.11), respectively, with J replaced by $\tilde{J}$. The processes corresponding to A and $\tilde{A}$ are different (cf. Example 2.5), but the random variables (2.14) are the same. This is due to the fact that $A\Psi = \tilde{A}\Psi$ for the functions Ψ defined in (3.13).

4 Extensions and comments

First several generalizations of the random cloud model (2.5) are sketched. Working out the details is left to future studies. The paper will be finished by providing comments concerning the physical interpretation of the random cloud model.

4.1 Single particle state space

The extension of the single particle state space (2.6) to

$$Z = U \times X \times Y\,, \tag{4.1}$$

for more general sets U, X and Y, needs some additional technical effort. Details will be discussed separately for each of the three components of the particle state.

weight

The set $U \subset \mathbb{R}$ in (4.1) generalizes the sign of a particle to an arbitrary scalar "weight". The random variables (2.14) are adapted by summing up the weights of particles with the same type and position. Partial cancellation is obtained by adding positive and negative weights.

Weights may be generated by taking into account the influence of the potential V not via individual creation events as in (2.8), but via accumulating these events over a certain time interval. This idea is best explained in the classical case. According to Remark 2.3, a particle (u, x) creates offsprings with rate $|V(x)|$, while waiting for the next jump to a new position. These contributions are expressed via (2.33) and lead to a weight transformation $u \exp(V(x)\,\tau)$, where τ is the jump time. The random cloud (2.24) reduces to just one particle, which performs a random walk and changes its weight according to the values of the potential V along the trajectory. The weight reaches the value

$$\exp\left(\sum_k V(x_k)\,\tau_k\right) \tag{4.2}$$

where (x_k, τ_k) is the sequence of positions and jump times. In the limit $\varepsilon \to 0$, expression (4.2) leads to the Feynman-Kac formula. In the quantum case the weight transfer idea can be implemented in the discrete model by using the weight factor $\exp(i\, V(x)\, \tau)$ and adjusting the weights corresponding to particles with real and imaginary type at a given position. But taking the limit $\varepsilon \to 0$ is much more difficult (if at all possible).

position

The position space X in (4.1) may be multi-dimensional. This covers the case of the Schrödinger equation for many interacting quantum objects in three dimensions. Different masses of the quantum objects are taken into account in the intensity (2.7). A non-discrete position space $X = \mathbb{R}^d$ is possible, when keeping the discretization parameter ε in the Laplacian. This would need some technical assumptions assuring the existence of densities. However, a discrete position space is convenient for performing cancellation. Finally, a bounded position space can be considered. The treatment of boundary conditions is typically straightforward in stochastic particle models.

type

The type space Y in (4.1) may consist of more than two elements. This covers arbitrary systems of equations. In the classical case (heat equation) the type space consists of one element.

The Schrödinger equation (1.1) takes the form

$$\begin{aligned} \frac{\partial}{\partial t}\, \Phi_1(t,x) &= -\frac{\hbar}{2m}\, \Delta_x \Phi_2(t,x) - \frac{q}{\hbar}\, V(x)\, \Phi_2(t,x) \\ \frac{\partial}{\partial t}\, \Phi_2(t,x) &= \frac{\hbar}{2m}\, \Delta_x \Phi_1(t,x) + \frac{q}{\hbar}\, V(x)\, \Phi_1(t,x)\,, \end{aligned} \tag{4.3}$$

or

$$\frac{\partial}{\partial t}\, \bar{\Phi}(t,x) = \left[\frac{\hbar}{2m}\, \Delta_x + \frac{q}{\hbar}\, V(x)\right] B\, \bar{\Phi}(t,x)\,, \tag{4.4}$$

where

$$\bar{\Phi}(t,x) = \begin{pmatrix} \Phi_1(t,x) \\ \Phi_2(t,x) \end{pmatrix} \quad \text{and} \quad B = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}.$$

The method works for rather general systems of equations of the form

$$\frac{\partial}{\partial t}\, \Psi(t,x) = L_x\, \Psi(t,x)\,,$$

where L is a partial differential operator generalizing the operator in (4.4). Discretization of the partial derivatives leads to a linear system of first order ordinary differential equations. In the context of quantum theory, it might be interesting to study the random cloud model for the Dirac equation (actually, a system of four equations).

4.2 Challenging problems

While the generalization of the single particle state space is straightforward, other extensions of the basic model are more challenging.

unbounded potential

An interesting problem is to weaken assumption (2.12) about the boundedness of the potential. This issue is mainly related to finding appropriate conditions on V assuring regularity of the stochastic process. On the other hand, modifications of the model based on the introduction of weights (cf. (4.2)) may work under less restrictive assumptions on V .

steady state

An important question is to understand under which conditions the random cloud stabilizes in time and approximates the steady state solution of the Schrödinger equation (1.1). A steady state solution $\Phi(x)$ satisfies

$$0 \;=\; -\frac{\hbar^2}{2m}\,\Delta_x\Phi(x) - q\,V(x)\,\Phi(x)\,. \tag{4.5}$$

More generally, a steady state is constructed using the equation

$$\left[\frac{\hbar^2}{2m}\,\Delta_x + q\,V(x)\right]\Phi(x) \;=\; E\,\Phi(x)\,, \tag{4.6}$$

where E has the physical dimension of energy. If (4.6) holds for some E , then (4.5) is fulfilled with $q\,V$ replaced by $q\,V - E$. The steady states are characterized by the values of E . The potential V admits different steady states, if such E exist.

There are various modifications of the random cloud model. Obviously many of them will not reach a steady state, i.e., a distribution that does not change in time. The cancellation mechanism is the basic ingredient for reducing the growth of the cloud. Modifications with appropriate combinations of creation events (happening at the same position) may help to stabilize the cloud. Also modifications with weights might be useful.

nonlinear equation

The main result can be reformulated as follows. Let the system (2.5) start with $n = 1, 2, \ldots$ independent copies of the particle generated according to Remark 2.2. Consider the normalized occupation numbers

$$\xi^{(n,\varepsilon)}(t,x) = \frac{\|\Phi_0\|}{n}\left\{\eta^{(\varepsilon)}_{1,1}(t,x) - \eta^{(\varepsilon)}_{-1,1}(t,x) + i\left[\eta^{(\varepsilon)}_{1,2}(t,x) - \eta^{(\varepsilon)}_{-1,2}(t,x)\right]\right\}, \tag{4.7}$$

which are the discrete version of empirical measures. According to the law of large numbers, assumption (2.16) and Theorem 2.1 imply

$$\lim_{n\to\infty}\xi^{(n,\varepsilon)}(t,x) \;=\; \Phi^{(\varepsilon)}(t,x) \qquad \forall\, x\in X \qquad \text{in probability.} \tag{4.8}$$

This results also holds for the modification with cancellation, since the random variables (4.7) do not change. However, in this case the clouds of offsprings of different initial particles are not independent.

A slightly more general version of the main result is the following. If

$$\lim_{n\to\infty} \xi^{(n,\varepsilon)}(0,x) \;=\; \Phi_0(x) \qquad \forall\, x \in X \qquad \text{in probability,} \tag{4.9}$$

then (4.8) holds. The advantage of this version is that it can be extended to equations with nonlinearities. The proof might be based on the general theory about the convergence of interacting particle systems to solutions of nonlinear kinetic equations (see, e.g., [7]).

vanishing grid size

If the solution of the Schrödinger equation (1.1) is sufficiently smooth, then it is approximated by the stochastic model in the limit of vanishing grid size,

$$\Phi(t,x) \;=\; \lim_{\varepsilon\to 0} \mathbb{E}\, \xi^{(\varepsilon)}(t,x)\,.$$

If the processes depend both on the grid size ε and the initial number of particles n, then one obtains (cf. (4.8))

$$\Phi(t,x) \;=\; \lim_{\varepsilon\to 0} \lim_{n\to\infty} \xi^{(n,\varepsilon)}(t,x)\,.$$

A fundamental open problem is to understand what happens to the stochastic model when the grid size goes to zero, i.e., whether

$$\lim_{\varepsilon\to 0} \xi^{(\varepsilon)}(t,x) \tag{4.10}$$

makes sense.

In the "classical" case the limit $\varepsilon \to 0$ is related to the Wiener process and the Feynman-Kac formula (cf. (4.2)). In the "quantum" case, all creation events change the type and some of them change the sign of the particle (cf. (2.8)). If the limit (4.10) exists, it will be a rather irregular object.

4.3 Physical interpretation

I am not a specialist in quantum mechanics, but it is tempting to speculate about the physical interpretation of the random cloud model. Before doing this, I briefly comment on the literature, mainly for readers not too familiar with the subject and its history.

literature

The traditional interpretation of quantum mechanics is formulated as follows (cf. [8, p.454]): "A physical system is completely described by a state function Ψ, which ... gives information only to the extent of specifying the probabilities of the results of various observations which can be made *on* the system *by* external observers. There are two fundamentally different ways in which the state function can change:

1. The discontinuous change brought about by the observation of a quantity with eigenstates $\Phi_1, \Phi_2, \ldots$, in which the state Ψ will be changed to the state Φ_j ; with probability $|(\Psi, \Phi_j)|^2$.

2. The continuous, deterministic change of state of an isolated system with time according to a wave equation $\partial\Psi/\partial t = A\,\Psi$, where A is a linear operator."

This interpretation is called "conventional" or "external observation" formulation of quantum mechanics. Another name is "Copenhagen interpretation", after the living place of Niels Bohr. A humorous characterization of this point of view was given in [17]: "If I were forced to sum up in one sentence what the Copenhagen interpretation says to me, it would be "Shut up and calculate!"." In other words, the traditional interpretation is pragmatic (and successful in predicting outcomes of experiments), but not completely satisfactory from a conceptual point of view. Many attempts have been made to provide other (better) interpretations of quantum mechanics.

Some approaches are mainly related to the "measurement problem" trying to explain or to avoid the "collapse" of the wave function during measurement.

- Bohm's interpretation in terms of "hidden" variables [1]

  According to [6, p.159], this approach tries "to construct explicit mechanisms for bringing about the collapse, but the conventionalists claim that it does not matter how the state vector is collapsed. To them the state vector does not represent reality but only an algorithm for making statistical predictions." In [2, p.454] it is noted that "the theory of the process of measurement involves a great many unclear features and unresolved problems, arising mainly because the role of the measuring instrument in the phenomenon of the "collapse" of the wave packet in a quantum mechanical measurement process is obscure." In this approach, a deterministic particle trajectory is constructed, which is governed by the solution of the Schrödinger equation. Other names are "pilot-wave theory" or "deBroglie-Bohm mechanics".

- Everett's "relative state" formulation [8]

  According to [6, p.155], this is a proposal "that pictures the universe as continually splitting into a multiplicity of mutually unobservable but equally real worlds, in each one of which a measurement does give a definite result." Later this approach became known as the "many worlds" interpretation.

Other approaches are directed towards explaining the quantum mechanical meaning of probability.

- Feynman's approach via path integrals [9]

  This probabilistic concept is based on random trajectories. In [9, Section 11] it is noted that "The formulation ... requires an unnatural and cumbersome subdivision of the time interval ..." and "There are very interesting mathematical problems involved in the attempt to avoid the subdivision and limiting processes. Some sort of complex measure is being associated with the space of functions ...". The concept of integration on function spaces was developed in [13] leading to the notion "Feynman-Kac formula".

- Nelson's stochastic mechanics [18]

  This approach is based on Markov processes given by stochastic differential equations (drift-diffusion processes). Successes and failures of stochastic mechanics have been discussed recently in [19].

In his famous lectures (published in 1965) Feynman considered the right-hand side of the Schrödinger equation "for a particle moving freely in space with no forces, no disturbances" and noted [12, Section 16-5]: "Where did we get that from? Nowhere. It's not possible to derive it from anything you know. It came out of the mind of Schrödinger, invented in his struggle to find an understanding of the experimental observations of the real world." In 1982 Feynman said [11, p.471]: "... we always have had a great deal of difficulty in understanding the world view that quantum mechanics represents. At least I do, ...". Even now, in 2013, the views among physicists on this topic are still highly controversial [20].

random cloud model

The derivation of the random cloud model is not based on any kind of physical intuition. It starts from the Schrödinger equation and uses the machinery of Markov jump processes (for rather general particle systems). There is a specific branching mechanism related to the creation of offsprings. But the connection with the Schrödinger equation is established in a direct way, without applying the general theory of branching processes.

The "quantum particles" (elements of the random cloud) do not move on a "trajectory". Instead of taking one position after the other, they create offsprings, thus spreading some "information" about their presence to the environment. Quantum particles can be deleted by the process of "pair cancellation". Even a single object (delta-function in the Schrödinger equation) immediately becomes a cloud. The speed of propagation of the cloud is inversely proportional to the grid size so that it becomes infinite with vanishing grid size. Clouds created by different particles are independent, but cancellation introduces an interaction.

The "classical" case (heat equation, Schrödinger equation "with imaginary time") is recovered for one of the modifications of the random cloud model (pair generation plus cancellation). Here the particle creates two offsprings, but one of them immediately cancels with the original particle. Thus, the cloud reduces to just one particle. This particle "moves" as a Wiener trajectory accumulating (or loosing) "weight" according to the potential V .

The derivation of the random cloud model is technically easy. Once the main idea is there, everything looks almost trivial. The basic ingredients of the derivation are related to the different parts of the single particle state:

- sign

  In the terminology of kinetic transport equations, the Schrödinger equation has a general "gain term" and no specific "loss term". This leads to the generation of signed particles instead of ordinary particle jumps.

- position

  The discrete position space allows one to catch the much simpler pre-limit (with respect to vanishing grid size) behaviour of the model.

- type

  The writing of the Schrödinger equation in terms of complex-valued functions might be useful in certain respects, but for the purpose of guessing a probabilistic representation it is misleading. Considering the Schrödinger equation as a coupled system of two equations for real-valued functions (cf. (4.3)) is more helpful in this context.

The random cloud model provides a probabilistic representation of the solution of the Schrödinger equation. The probability measure is determined on a space of measure-valued trajectories, while usually trajectories have values in the state space of a single particle. Both the sign (positive/negative) and the type (real/imaginary) are treated via trajectories and do not influence the notion of the probability measure.

According to the random cloud model, a quantum object is not a “particle”, but a “cloud of particles”. Correspondingly, the solution of the Schrödinger equation is related to a functional of the cloud. Its interpretation in terms of the occupation probability of a single particle (“probability amplitude”) assumes classical properties of the quantum object, in analogy with the Brownian motion.

In the quantum case any single particle creates a cloud so that the “one particle picture” is not consistent. However, the “many particle picture” is similar to classical statistical mechanics, where a cloud of particles is used to approximate, e.g., the solution of the Boltzmann equation. The “empirical measure approach” has many advantages compared to the “single particle approach” via densities and factorization. In particular, it leads to a unified treatment of linear and nonlinear equations.

physical reality

Once the random cloud model reproduces the evolution determined by the Schrödinger equation, it should reflect some aspects of physical reality. The model is technically simple, compared to other approaches, which makes it rather appealing. For a serious discussion of its physical relevance, profound knowledge is necessary of how quantum mechanical experiments and measurements are performed. One should be able to distinguish between the “real” observations and their interpretations based on a given model. Since I do not have that knowledge, I will only collect thoughts related to some “commonly known” parts of quantum mechanics.

- double slit experiment

  This is one of the basic experiments in quantum mechanics. It is observed that the single slit pictures do not sum up to the picture obtained when both slits are open. In [10, p.533] this observation was interpreted as “... the discovery that in nature the laws of combining probabilities were *not* those of the classical probability theory of Laplace.” In the report on [10] by B. O. Koopman (MR0047541) this conclusion is described as “a fundamental error (common, unfortunately, to some others who have treated the subject)”, adding the comment “The fallacy is that total probability applies only when the events are mutually exclusive. In the present case this is only true if the classical picture is used, according to which the electron occupies a definite position at each time and moves through such positions ... But in quantum mechanics it is in principle meaningless to say that an electron

really has some position ... when no position-observation is made. Much less is it meaningless to say that it really moves, in the sense of occupying a succession of positions in the course of time, when no appropriate observations are made."

According to the random cloud model, a quantum object passes through both slits, which is consistent with the above observations.

- wave-particle duality

  This term reflects the fact that observations suggest both a particle and a wave nature of quantum objects. Referring to the double slit experiment, it is noted in [2, p.456] that "the electron acts like an individual particle in producing a track on one frame of the film". On the other hand, interference properties are observed which are typical for waves.

  The random cloud model resolves this "contradiction". It produces wave-like behaviour, since it is linked to the Schrödinger equation. At the same time, individual members of the cloud produce tracks on the measuring device.

- uncertainty principle

  This relationship states that certain quantum mechanical quantities (e.g., position and momentum of an electron) cannot be measured simultaneously with arbitrary precision. The issue is strongly related to the question of how quantum mechanical quantities are measured (observed). According to [8, p.454], "Much of the discussion of öbserversïn quantum mechanics has to do with photoelectric cells, photographic plates, and similar devices."

  It seems to me that in the double slit experiment the measuring device can be considered as part of the boundary of a finite domain so that the potential inside the domain is not changed by the measurement. But in general the measuring procedure influences the potential.

  The purpose of a measurement is to derive a macroscopically observable quantity so that the quantum object obtains some classical feature. It is plausible that this can not be achieved for all quantities simultaneously. I guess that in fact a measurement takes some averages over time and/or space. Time averages need some stability of the quantum object, which is related to the "steady state" issue discussed earlier (cf. (4.6)).

  Measurements of a quantum object are performed via interactions with other entities. So it should be quite difficult to measure the position (or the momentum) of a single quantum object. I doubt that anybody has observed the trajectory of an electron.

  The random cloud model suggests the point of view that classical quantities like position and momentum do not exists for a quantum object, which is a cloud of particles. If the cloud is measured at different times, a certain change in the position space is observed. But the single element of the cloud does not move at all, it is either still in its original position or has been removed by cancellation. This jump-like transport mechanism does not seem to be just a consequence of the fact that the random cloud model is restricted to a discrete position space. Perhaps the quantum world is discrete (and discontinuous) in its nature, while continuity and differentiability are concepts of the macroscopic world.

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