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Asymptotic minimaxity of chi–square tests

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Summary We show that the sequence of chi-square tests is asymptotically minimax if a number of cells increases with increasing sample size. The proof utilizes Theorem about asymptotic normality of chi-square test statistics obtained under new compact assumptions.

1 Introduction.

In recent years numerous papers have been devoted to studying the problems of nonparametric estimation. The problems of hypothesis testing versus nonparametric sets of alternatives have been also considered in many papers (see Mann and Wald (1942), Stein (1956), Ingster (1985),(1988), (1993),(1994), Ermakov (1988),(1990),(1994),(1995) and others). The especial attention has been paid on the goodness-of-fit testing as one of widespread problem of nonparametric statistics. The research in this domain is naturally separated into two parts: hypothesis testing about density and hypothesis testing about distribution function.

The setting of the first problem can be described as follows. Let $X_1, ..., X_n$ be i.i.d.r.v.'s with unknown density f. Let a priori information be given $f \in U$ where U is a set of functions satisfying some smoothness conditions. The problem is to test a hypothesis $f = f_0$ versus alternatives $f \in \Gamma_n = \{f : ||f - f_0|| > b_n > 0, f \in U\}$.. Here f_0 is a fixed density and $|| \cdot ||$ is a norm in Banach space containing the set U. For this setting Ingster (1986),(1993) has studied rates of convergence b_n, ($b_n \to 0$ as $n \to \infty$), allowing to distinquish the hypothesis and sets of alternatives Γ_n. In the case of L_2 norm the asymptotically minimax sequences of tests have been constructed in Ingster (1988) when densities f are infinitely differentiable and in Ermakov (1988) under weak assumptions on smoothness of density f.

The problem of hypothesis testing on distribution function is natural to study in the framework of statistical inference on a value of functional. Let $X_1, \ldots, X_n$ be i.i.d.r.v.'s with distribution function (c.d.f.) F, let $\hat{F}_n$ be the empirical df of $X_1, \ldots, X_n$ and let T be the functional on the set $\Im$ of all c.d.f.'s. Suppose the problem is to test a hypothesis $T(F) = 0$, $F \in \Im$ versus alternatives $F \in \Im(T, b_n) = \{F : T(F) > b_n, F \in \Im\}$. For this setting the asymptotic minimaxity of test statistics $T(\hat{F}_n)$ is a standard solution of the problem. In particular, the asymptotic minimaxity of Kolmogorov tests (see Ermakov (1990),(1995)) has been proved for the functional $T(F) = \max\{|F(x) - x|, x \in (0,1)\}$. The same results have been also obtained for the functionals T corresponding to omega-square and rank tests and have been announced for chi-square tests with an increasing number of cells (see Ermakov (1990),(1992),(1995)). Note that the asymptotic minimaxity of chi-square tests with fixed cells follows easily from Wald (1943) results (see also

Borovkov (1984)).

In practical applications we choose a number of cells depending on a sample size n. If a sample size n increases a number of cells $k = k(n)$ is also increases. Therefore it is of interest to prove asymptotic minimaxity of chi-square tests when a number of cells $k = k_n$ depends on a sample size n.

The exact setting of the problem is as follows. Let $X_1, \ldots, X_n$ be i.i.d.r.v.'s with df $F(x)$, $x \in [0,1]$. Let the interval $[0,1]$ be divided into $k = k_n$ subintervals $I_{jn} = [e_{jn}, e_{j+1,n})$, $p_{jn} = e_{j+1,n} - e_{jn} > 0$, $e_{0n} = 0$, $e_{kn} = 1$, $1 \le j \le k = k_n$. Fix the values g_{jn}, $0 < C_1 < G_{jn} < C_2 < \infty$, denote $r_{jn} = F(e_{jn}) - F(e_{j-1,n})$ for $1 \le j \le k_n$ and define the functional

$$T_n(F) = n \sum_{j=1}^{k} g_{jn}(r_{jn} - p_{jn})^2 p_{jn}^{-1}.$$

The functional T_n assigns the chi-square test statistic $T_n(\hat{F}_n)$. As we show in the paper these test statistics $T_n(\hat{F}_n)$ are asymptotically minimax in the problem of hypothesis testing $T_n(F) = 0$, $F \in \Im$ versus alternatives $F \in \Im(T_n, b_n)$.

Note that the setting under consideration can be also interpreted in terms of hypothesis testing about density. From this point one can say that the hypothesis testing based on chi-square tests has intermediate position between the problems of hypothesis testing about density and distribution function. The corresponding interpretation uses the representation of chi-square test statistic as L_2–distance between the hystogram and the approximation of density of hypothesis by the function $\bar{f}_n(x)$ such that $\bar{f}_n(x) = r_{jn}/p_{jn}$ for $x \in I_{jn}$, $1 \le j \le k$. Note that in the case of $F = \hat{F}_n$ the function $\bar{f}_n$ coincides with the usual hystogram. We call $\bar{f}_n(x)$ the hystogram function of c.d.f. F. For some choice of weights g_{jn} we have

$$T_n(F) = n \int_0^1 (\bar{f}(x) - \bar{f}_0(x))^2 dx. \tag{1.1}$$

Here $\bar{f}_0$ is the hystogram of uniform distribution, that is, $\bar{f}_0(x) = 1$ for all $x \in (0,1)$. Thus the following situation takes place. We cannot test the hypothesis about density in L_2-norm without any a priori information about its smoothness. At the same time we can replace the density by its simple approximation $\bar{f}$, to consider the problem of hypothesis testing in terms of functional $T_n(F)$ and the problem is solved.

The connection of chi-square tests with the problem of hypothesis testing about density is well-known. Using this connection Ingster (1986),(1993) has studied the problem of choice of number of cells for chi-square tests under a priori information on density smoothness. He considered the case of equal cells, $p_{jn} = 1/k$ and showed that, if a density f belongs to a ball in Sobolev space W_2^{2r}, then the optimal choice of number of cells is $k = O(n^{2/(4r+1)})$. Thus, if f belongs to balls in W_2^{2r} for all r then $k = o(n^{2-\epsilon})$ with arbitrary $\epsilon > 0$. The other aproaches to this problem can be found in Mann and Wald (1943), Greenwood and Nikulin (1987), and Kallenberg, Oosterhoff and Schribver (1985).

The results of the paper are as follows. If the cells have equal lengths $p_{jn} = 1/k$, $1 \le j \le k = k(n)$. the asymptotic minimaxity of chi–square tests is proved for $k_n = o(n^2)$. For the cells with different lengths the same assertion is shown when $n \min\{p_{in}, 1 \le i \le k(n)\} \to \infty$ as $n \to \infty$. Such an assumption implies that the number of random variables in each cell tends to infinity as $n \to \infty$. If this assumption does not satisfied and $k = o(n^2)$ then the asymptotic minimaxity is proved for some modification of chi-square test statistics.

The proofs of results are based on the following approach. We assign the functional T_n as a sum of squares of quasi-ortogonal functions. As a consequence the problem becomes similar to that of hypothesis testing about density (see Ermakov (1988),(1994), Ingster (1993),(1994)). The direct application of corresponding methods allows to show asymptotic minimaxity of chi–square test statistics only in the case $k = o(n)$. To weaken this assumption we modify essentially the method of the proof. In Ermakov (1988),(1994) and Ingster (1993),(1994) the proofs have been based on the analysis of expansion of logarithm of likelihood ratio. In this paper we analyze directly the expansion of likelihood ratio.

The upper bounds for type I and type II error probabilities of chi-square tests follow from Theorem on asymptotic normality of chi-square test statistics. This Theorem is given in the following version. If the type I and type II error probabilities of chi-square tests are detached from one and zero then the sequence of chi-square tests statistics is asymptotically normal both in the case of hypothesis and under the sequence of alternatives. The proof is based on the representation of chi-square test-statistics as U-statistics and the application of central limit Theorem for martingales (see Brown (1971)) in the spirit of Hall (1984) paper. The problem of asymptotic normality of chi-square test statistics has been considered in numerous publications. We should mention Tumanyan (1956), Steck (1957), Morris (1975), Medvedev (1977) and others.

The L_2 –distance (1.1) between hystogram and density is not unique measure of deviation of hystogram utilized in hypothesis testing. The functionals corresponding to Kullback-Leibler information

$$S_{1n}(F) = n \sum_{j=1}^{k} r_{jn} \log(r_{jn}/p_{jn})$$

and Hellinger distance

$$S_{2n}(F) = n\sum_{j=1}^{k}(r_{jn}^{1/2} - p_{jn}^{1/2})^2$$

are also widely applied in hypothesis testing. As wellknown, in many cases chi-square test statistics $T_n(\hat{F}_n$ and test statistics $S_{1n}(\hat{F}_n)$, $S_{2n}(\hat{F}_n)$ are asymptotically equivalent (has the same asymptotic behaviour). In the paper we indicate the conditions when such an equivalence takes place if a number of cells increases with increasing sample size. We also consider the problems of hypothesis testing $S_{1n}(F) = 0$ versus $F \in \Im(S_{1n}, b_n)$ and $S_{2n}(F) = 0$ versus $F \in \Im(S_{2n}, b_n)$. For these problems we prove asymptotic minimaxity of both chi-square test statistics $T_n(\hat{F}_n)$ and test statistics $S_{1n}(\hat{F}_n)$, $S_{2n}(\hat{F}_n)$. The last assertion follows easily from the asymptotic equivalence result.

We shall use the following notations. Denote by letters C, c arbitrary constants, $\chi(A)$ the indicator of event A, $\sharp\{i, i \in B\}$ the number of elements of set B, $[z]$ the whole part of $z \in R^1$ and $\Phi(x) = (2\pi)^{-1/2}\int_{-\infty}^{x}\exp\{-x^2/2\}dx$ the standard normal distribution function. If the context is clear the index n will be omitted in notation. For example $p_j = p_{jn}$, and $g_j = g_{jn}$.

2 Main Results.

For any test $L_n = L_n(X_1, ..., X_n)$ denote $\alpha(L_n)$ its level and $\beta(L_n, F)$ its type II error probability under alternative $F \in \Im(T_n, b_n)$. Put $\beta(L_n, b_n) = \sup\{\beta(L_n, F) : F \in \Im(T_n, b_n)\}$

Denote

$$d_n = E_0T_n(X_n) = \sum_{j=1}^{k} g_j - \sum_{j=1}^{k} g_j p_j,$$

$$\sigma_n^2 = VarT_n(X_n) = 2k^{-1}\sum_{j=1}^{k} g_j^2.$$

Introduce the sequence of chi-square tests $K_n = \chi\{k^{-1/2}\sigma_n^{-1}(T_n(X^{(n)}) - d_n) > x_{\alpha_n}(1 + o(1))\}$ with x_{α_n} defined by the level $\alpha_n = \alpha(K_n)$.

Suppose that at least one from the following Assumptions holds.

A. $k = k_n = o(n^2)$, $k_n \to \infty$ as $n \to \infty$ and $p_{jn} = 1/k$ for all $1 \le j \le k$.

B. $n\min\{p_{jn}, 1 \le j \le k\} \to \infty$, $\max\{p_{jn}, 1 \le j \le k\} \to 0$ as $n \to \infty$.

C. $nk^{1/2}\min\{p_{jn}, 1 \le j \le k\} \to \infty$, $\max\{p_{jn}, 1 \le j \le k\} \to 0$ as $n \to \infty$.

If $p_{jn} \asymp 1/k_n$ for all $1 \le j \le k(n)$ then Assumption C implies $k_n = o(n^2)$ and $p_{jn} = o(1)$ as $n \to \infty$.

We call a sequence of tests K_n asymptotically minimax if for any sequence of tests L_n, $\alpha(L_n) < \alpha(K_n)$ it holds

$$\limsup_{n\to\infty} \beta(K_n, b_n) - \beta(L_n, b_n) \le 0.$$

Theorem 2.1. *Assume* A *or* B. *Let*

$$0 < \liminf_{n\to\infty} b_n k^{-1/2} \le \limsup_{n\to\infty} b_n k^{-1/2} < \infty. \tag{2.1}$$

Then the sequence of chi-square tests K_n is asymptotically minimax.

Let $0 < \gamma_1 \le \alpha_n = \alpha(K_n) \le \gamma_2 < 1$ Then x_{α_n} can be set by equation $\alpha_n = 1 - \Phi(x_{\alpha_n})$ and

$$\beta(K_n, b_n) = \Phi(x_{\alpha_n} - b_n k^{-1/2})(1 + o(1)) \tag{2.2}$$

as $n \to \infty$.

Remark. It follows from Theorem 2.1 that if $\alpha(K_n) > \alpha > 0$ and $b_n k_n^{-1/2} \to 0$ then $\beta(K_n, b_n) = 1 + o(1)$ as $n \to \infty$. If $\alpha(K_n) > \alpha > 0$ and $b_n k_n^{-1/2} \to \infty$ as $n \to \infty$ then $\beta(K_n, b_n) = o(1)$ as $n \to \infty$.

The proof of Theorem 2.1 is based on the following representation of functionals T_n in terms of quasiortogonal system of functions $\phi_{jn}(x) = \chi(x \in I_{jn}) - p_{jn}$, $x \in (0,1)$, $1 \le j \le k(n)$

$$T_n(F) = n \sum_{j=1}^{k} g_{jn} \Big(\int_0^1 \phi_{jn}(x) dF(x)\Big)^2 p_{jn}^{-1}. \tag{2.3}$$

Such an assignment of functional $T_n(F)$ allows to utilize the technique developed in the problems of asymptotically minimax hypothesis testing about density (see Ermakov (1988),(1994) and Ingster (1993),(1994)). For this problem the corresponding functional $\bar{T}_n$ is L_2–norm of deviation of alternative, that is,

$$\bar{T}_n(F) = \sum_{j=1}^{\infty} \Big(\int_0^1 \xi_j(x) d(F - F_0)(x)\Big)^2$$

where ξ_j, $1 \le j < \infty$, is ortonormal system of functions and F_0 is a distribution function of hypothesis. The functions ϕ_{jn}, $1 \le j \le k$, are not ortogonal and this cause the essential differences in the proof.

In more general situation of Assumption C we can prove only asymptotic minimaxity of some modification of chi-square test statistics

$$T_{1n}(\hat{F}_n) = T_n(\hat{F}_n) - \sum_{j=1}^{k} g_{jn} \lambda_{jn} \frac{1 - 2p_{jn}}{p_{jn} n} - \sum_{j=1}^{k} g_{jn} p_{jn}$$

with $\lambda_{jn} = \hat{F}_n(e_{jn}) - \hat{F}_n(e_{j-1,n})$.

Such an assignment of test statistics T_{1n} can be explained as follows. We have

$$T_n(\hat{F}_n) = n^{-1} \sum_{j=1}^{k} g_{jn} (\sum_{s_1=1}^{n} \phi_j(X_s))^2 p_{jn}^{-1}, \tag{2.4}$$

$$T_{1n}(\hat{F}_n) = T_n(\hat{F}_n) - N_n(\hat{F}_n) =$$

$$2n^{-1} \sum_{j=1}^{k} g_{jn} \sum_{1 \le s_1 < s_2 \le n} \phi_j(X_{s_1}) \phi_j(X_{s_2}) p_{jn}^{-1} \tag{2.5}$$

where

$$N_n(\hat{F}_n) = n^{-1} \sum_{j=1}^{k} g_{jn} \sum_{s=1}^{n} \phi_j^2(X_s) p_{jn}^{-1}.$$

If Assumption A fullfils then $N_n(\hat{F}_n) = d_n = \text{const}$.

If Assumption B is valid then

$$E_F N_n(\hat{F}_n) - E_0 N_n(\hat{F}_n) = o(E_F T_n(\hat{F}_n) - E_0 T_n(\hat{F}_n)), \tag{2.6}$$

$$Var_F N_n(\hat{F}_n) = o(Var_F T_n(\hat{F}_n)) \tag{2.7}$$

for all $F \in \Im(T_n, b_n)$.

Thus the influence of addendums N_n is neglected in these cases.

If Assumption C is valid then (2.6) holds and in many cases

$$Var_F T_{1n}(\hat{F}_n) = o(Var_F T_n(\hat{F}_n)). \tag{2.8}$$

This implies that the influence of N_n is essential. For this reason we have been obliged to introduce the test statistics $T_{1n}(\hat{F}_n)$. Note that a similar modification of asymptotically minimax tests statistics has been also considered in the problems of hypothesis testing on density (see Ermakov (1988), Ingster (1994)).

Let $g_{jn} = 1$, $p_{jn} = 1/k$ for all $1 \le j \le k$. Then the test statistics

$$T_{1n}(X^{(n)}) = 2n^{-1}(\sum_{j=1}^{k} \sum_{1 \le s_1 < s_2 \le n} \chi(X_{s_1} \in I_{jn}) \chi(X_{s_2} \in I_{jn}) - \frac{n(n-1)}{2k}) \tag{2.9}$$

can be interpreted as a normalized sum of pairs of observations X_{s_1},X_{s_2} containing in the cells. Thus the hypothesis and the sets of alternatives are distinquishable by test statistics $T_{1n}(X^{(n)})$ if a number of cells containing at least two observations tends to infinity with increasing sample size.

Introduce a sequence of tests $K_{1n}(X^{(n)}) = \chi\{k^{-1/2} \sigma_n^{-1} T_{1n}(\hat{F}_n) > x_{\alpha_n}(1 + o(1))\}$ with x_{α_n} defined by the levels $\alpha_n = \alpha(K_{1n})$.

Theorem 2.2. *Assume* C *and* (2.1). *Then the sequence of tests* K_{1n} *is asymptotically minimax. If* $0 < \gamma_1 < \alpha_n < \gamma_2 < 1$ *then*

$$\beta(K_{1n}, b_n) = \Phi(x_{\alpha_n} - b_n k^{-1/2})(1 + o(1)) \tag{2.10}$$

with x_{α_n} *defined by equation* $\alpha_n = 1 - \Phi(x_{\alpha_n})$.

Remark. Assume A and (2.1). Then the sets of alternatives are defined by the equation

$$T_n(F) = n \int_0^1 (\bar{f}(x) - 1)^2 dx \geq b_n \asymp k^{1/2}.$$

Thus the assumption $k_n = o(n^2)$ allows to cover all possible deviations of L_2 − norm of density.

Remark. It follows from Theorems 2.1, 2.2 and (1.1) that we have to take into account two factors in the choice of chi-square test statistics. The first one is L_2-distance b_n between hystogram of hypothesis and alternatives. The second factor is deviation of density from its hystogram constructed by k_n cells. Thus, from this point of view the statistical procedure has to have the following character. We indicate a desired value b_n and find values k_n such that by a priori information the distance between density and its hystogram has the same order b_n. After that a hypothesis is tested. In such a way a choice of number of cells k_n is defined by L_2-distance b_n and density smoothness.

In this paper we do not consider adaptive methods of choice of a number of cells. The study of such procedures requires a special technique which is out of scope of the paper. Recently such interesting procedures has been proposed in Bogdan (1995), Fan (1995) and Kallenberg and Ledwina (1995).

The results on asymptotic normality of tests statistics $T_n(X^{(n)})$ and $T_{1n}(X^{(n)})$ will be given in terms of parameters $\theta_n = \theta_n(F) = \{\theta_{jn}\}_1^k$, $\theta_{jn} = (r_{jn} - p_{jn})/p_{jn}$. It is clear that the distributions of tests statistics $T_n(X^{(n)})$, $T_{1n}(X^{(n)})$ are uniquely defined by these parameters.

Denote

$$M_n(\theta) = (n-1) \sum_{j=1}^{k} \theta_{jn}^2 p_{jn}.$$

Theorem 2.3. *Assume* A *or* B. *Then*

$$E_\theta T_n(\hat{F}_n) - d_n = M_n(\theta_n)(1 + o(1)), \tag{2.11}$$

$$Var_\theta T_n(\hat{F}_n) = k\sigma_n^2 + o(M^2(\theta_n) + k). \tag{2.12}$$

Assume C. *Then*

$$E_\theta T_{1n}(\hat{F}_n) = 0, \tag{2.13}$$

$$Var_\theta T_{1n}(\hat{F}_n) = k\sigma_n^2 + o(M_n^2(\theta_n) + k). \tag{2.14}$$

By Chebyshov inequality it follows from (2.11)–(2.14) that if $k_n^{-1/2}M_n(\theta_n) \to \infty$ then $\beta(K_n,\theta_n) \to 0$, $\beta(K_{1n},\theta_n) \to 0$ as $n \to \infty$. Hence it suffices to study the asymptotic behaviour of $\beta(K_n,F)$, $\beta(K_{1n},F)$ under the assumption $\theta_n(F) \in \Gamma_n(C) = \{\theta(F) : M_n(\theta(F)) < Ck^{1/2}\}$.

Theorem 2.4. *Assume* A *or* B. *Then* P_θ *distributions* $k^{-1/2}\sigma_n^{-1}(T_n(\hat{F}_n) - k)$ *converge to normal uniformly in* $\theta = \theta_n(F) \in \Gamma_n(C)$.

Assume C. *Then* P_θ *distributions* $k_n^{-1/2}\sigma_n^{-1}T_{1n}(\hat{F}_n)$ *converge to normal uniformly in* $\theta = \theta_n(F) \in \Gamma_n(C)$.

Remark. In Theorem 2.4 the distribution function F does not assumed fixed as opposed to the previous results (see Tumanian (1956), Morris (1975), Medvedev (1977) and others).

As mentioned we also consider the problems of hypothesis testing for the sets of alternatives $F \in \Im(S_{1n},b_n)$ and $F \in \Im(S_{2n},b_n)$. The corresponding results are obtained under essentially more strong assumptions.

C1. $C_1/k_n < p_{jn} < C_2/k_n$ for all $1 \le j \le k_n$.
C2. $k_n = o(n^{2/3})$ as $n \to \infty$.

For any test L_n denote $\beta_i(L_n,b_n) = \sup\{\beta(L_n,F) : F \in \Im(S_{in},b_n)\}$, $i = 1,2$.

Theorem 2.5. *Assume* C1,C2 *and* (2.1). *Then, for the problem of testing a hypothesis* $S_{in}(F) = 0$ *versus* $F \in \Im(S_{in}, i^{-2}b_n)$,$i = 1,2$, *the sequence of chi-square tests* K_{1n} *is asymptotically minimax. If* $0 < \gamma_1 < \alpha_n = \alpha(K_{1n}) < \gamma_2 < 1$ *then*

$$\beta_i(K_{1n},b_n) = \Phi(x_{\alpha_n} - b_n k_n^{-1/2})(1 + o(1)). \tag{2.15}$$

with x_{α_n} *defined by the equation* $\alpha_n = 1 - \Phi(x_{\alpha_n})$

Make the following assumption.

C3. $k_n = o(n^{1/2}/\log n)$ as $n \to \infty$.

Define the sets of c.d.f.'s $\Re(S_{in},C,c_n) = \Gamma_n(C) \cap \{F : F(x)$ has a density $f(x) = dF(x)/dx$ and $\sup|f(x) - 1| < c_n\}$.

Theorem 2.6. *Assume* C1,C3 *and let* $c_n \to 0$ *as* $n \to \infty$. *Then, for all* $\epsilon > 0$

$$P(k_n^{-1/2}|T_n(\hat{F}_n) - S_{1n}(\hat{F}_n)| > \epsilon) = o(1), \tag{2.16}$$

uniformly in $F \in \Re(S_{1n},C,c_n)$ *as* $n \to \infty$ *and*

$$P(k_n^{-1/2}|T_n(\hat{F}_n) - 4S_{2n}(\hat{F}_n)| > \epsilon) = o(1) \tag{2.17}$$

uniformly in $F \in \Re(S_{2n},C,c_n)$ *as* $n \to \infty$.

Remark. A slight modification of technique developed in the paper allows to obtain similar results for Neyman test and to strengthen results on asymptotically minimax hypothesis testing about density see Ermakov (1988, 1994). These problems will be considered in another paper.

3 Proof of Theorems 2.1 and 2.2. Lower bounds. Main Lemmas.

The proof of lower bounds is based on some asymptotic analogue of Lemma on the least favourable distribution (see Lehmann (1986)). We assign a sequence of Bayes distributions and show that Bayes likelihood ratio can be presented as a function of test statistics $T_{1n}(X^{(n)})$. Then we prove that Bayes type II error probabilities of $K_{1n}(X^{(n)})$ equal $\beta(K_{1n}, b_n)(1+o(1))$. By asymptotic analogue of Lemma about the least favourable distribution and Theorems 2.3,2.4 this implies Theorem 2.2. By (2.6),(2.7) if A or B is satisfied then the difference of test statistics $T_n(X^{(n)})$ and $T_{1n}(X^{(n)})$ is unessential. Thus Theorem 2.1 follows from Theorem 2.2.

Denote by U_n the set of all c.d.f.'s G_τ with the densities

$$g_\tau(x) = 1 + \sum_{j=1}^{k} \tau_j \phi_j(x); \qquad \tau = \{\tau_j\}_1^k, \qquad \tau = \tau(G_\tau) \in R^k.$$

Define the sets

$$Q_n = \{\tau : n \sum_{j=1}^{k} g_j p_j (\tau_j - \sum_{t=1}^{k} \tau_t p_t)^2 > b_n, \tau = \{\tau_j\}_1^k,\ G_\tau \in U_n\}.$$

For any c.d.f. $F \in \Im(T_n, b_n)$ there exists c.d.f. G_τ, $\tau \in Q_n$ such that $\theta(F) = \tau(G_\tau)$ and $T_n(F) = T_n(G_\tau) = n\sum_{j=1}^k g_j p_j(\tau_j - \sum_{t=1}^k \tau_t p_t)^2$. Thus it suffices to prove Theorem 2.2 for the parametric sets of alternatives Q_n instead of $\Im(T_n, b_n)$.

Lemma 3.1. *For any j, t, $1 \le j < t \le k$ it holds*

$$E_\tau \phi_j(X_1) = p_j(\tau_j - \sum_{i=1}^{k} \tau_i p_i), \tag{3.1}$$

$$E_\tau \phi_j^2(X_1) = p_j(1 - p_j - \sum_{i=1}^{k} \tau_i p_i - \tau_j p_j(1-2p_j)), \tag{3.2}$$

$$E_\tau \phi_j(X_1)\phi_t(X_1) = -p_j p_t(1 - 2\sum_{i=1}^{k} \tau_i p_i + \tau_j + \tau_t). \tag{3.3}$$

Let $1 \le j_1 < j_2 < ... < j_l \le k$ and $\tau_{j_i} = \tau_{j_i}^{(n)} = o(1)$ as $n \to \infty$ for all i, $1 \le i \le l$. Then for any integers $t_1, ..., t_l$

$$E_\tau \phi_{j_1}^{t_1}(X_1)...\phi_{j_l}^{t_l}(X_1) = p_{j_1}...p_{j_l}(1+o(1)). \tag{3.4}$$

Denote

$$\rho_{n\delta} = k(1+\delta)(\sum_{j=1}^{k} g_j^2)^{-1}.$$

Put $\kappa_j^2 = \kappa_{jn\delta}^2 = n^{-1}k^{-1}g_j\rho_{n\delta}b_np_j^{-1}$ for all j, $1 \le j \le k$. Define a random vector $\eta^{(n)} = \eta_\delta^{(n)} = \{\eta_j\}_1^k$ with independent components η_j having Binomial distribution $P(\eta_j = \kappa_j) = P(\eta_j = -\kappa_j) = 1/2$. Denote μ_n the probability measure of random vector $\eta^{(n)}$. Then Bayes a priori probability measure ν_n is the conditional measure of μ_n under the condition $\eta^{(n)} \in Q_n$.

Note that the choice of Binomial distribution as the least favourable distribution does not play essential role in the proof. One can take arbitrary independent bounded random variables $\zeta_j^{(n)}$ with the same first and second moments as $\eta_j^{(n)}$, $1 \le j \le k$.

Define the sets $W_n = \{\theta : \theta_j = \pm\kappa_j, \theta = \{\theta_j\}_1^k, 1 \le j \le k\}$ $W_{1n} = \{\theta : \theta \in W_n, T_n(G_\theta) > b_n\}$ and $W_{2n} = W_n \setminus W_{1n}$.

Since $\kappa_n = \max\{\kappa_{jn\delta}, 1 \le j \le k\} = o(1)$ then $g_w(x) \ge 0$ for all $x \in (0,1)$ and $w \in W_n$. Thus g_w is a density for all $w \in W_n$.

Lemma 3.2.

$$\lim_{n\to\infty} P_\eta(T_n(G_\eta) \ge b_n) = 1. \tag{3.5}$$

The proofs of Lemma 3.2 and next Lemmas 3.3–3.5 will be given in section 4. Lemma 3.7 will be proved in section 5.

Take $\gamma_n = \mu_n$ or $\gamma_n = \nu_n$ and for any tests L_n denote

$$\beta(L_n, \gamma_n) \doteq \int_{W_n} \beta(L_n, \theta) d\gamma_n. \tag{3.6}$$

By Lemma 3.2, $\beta(L_n, \mu_n) = \beta(L_n, \nu_n)(1 + o(1))$ for any sequence of tests L_n such that $\beta(L_n, \mu_n) > c > 0$. Thus Theorem 2.2 will follow from

$$\lim_{\delta\to 0}\lim_{n\to\infty} \beta(K_{1n}, \mu_n) = \lim_{\delta\to 0}\lim_{n\to\infty} \beta(K_{1n}, b_n) =$$

$$\lim_{\delta\to 0}\lim_{n\to\infty} \beta(S_n, \mu_n). \tag{3.7}$$

Here $S_n = S_n(X^{(n)})$ is Bayes likelihood ratio test with the same level as K_n

$$S_n(X^{(n)}) = \chi\{E_\eta \prod_{s=1}^n g_\eta(X_s) > c_{\alpha_n}\}. \tag{3.8}$$

In the remaining part of section we show that Bayes likelihood ratio can be presented as a function $f_n(T_{1n}(X^{(n)}))(1 + o_P(1))$. This will be imply that the left and right-hand sides of (3.7) coincide.

We have

$$\prod_{s=1}^n (1 + \sum_{j=1}^k \eta_j\phi_j(X_s)) = 1 + \sum_{l=1}^n 1/l! R_l. \tag{3.9}$$

Here R_l is the sum of all addendums

$$\eta_{j_1}...\eta_{j_l}\phi_{j_1}(X_{s_1})...\phi_{j_l}(X_{s_l}) \tag{3.10}$$

such that the indices $s_1, ..., s_l$ are different, that is, $R_l = R_{1l} - R_{2l}$ where

$$R_{1l} = (\sum_{s=1}^{n} \sum_{j=1}^{k} \eta_j \phi_j(X_s))^l,$$

and R_{2l} contains all addends of R_{1l} having factors $\phi_{j_1}(X_{s_1})\phi_{j_2}(X_{s_2})$ with $s_1 = s_2$. The multiplier $1/l!$ arises in (3.9) since each addend of (3.9) is contained in R_l $l!$-times.

It is clear that $E_\eta R_l = 0$ for the add l.

If $l = 2$ then $E_\eta R_2 = T_{1n}(X^{(n)})$.

Put

$$J_l = J_l(X^n) = (2r-1)!! \sum{}' \kappa_{j1}^2 .. \kappa_{jr}^2 \phi_{j1}(X_{s1})\phi_{j1}(X_{s2}) \times$$
$$\phi_{j2}(X_{s3})\phi_{j2}(X_{s4})...\phi_{jr}(X_{s_{l-1}})\phi_{jr}(X_{sl}). \tag{3.11}$$

Here $\sum'$ denotes summation over all different indices $1 \leq j_1, ..., j_r \leq k$ and $1 \leq s_1, ..., s_l \leq k$ that is $j_{i_1} \neq j_{i_2}$ and $s_{t_1} \neq s_{t_2}$ for $1 \leq i_1 < i_2 \leq r$ and $1 \leq t_1 < t_2 \leq l$ respectively.

Lemma 3.3 *Let $l = 2r$, $r \geq 1$. Then*

$$E_\eta R_l = J_l + o_P(1). \tag{3.12}$$

Thus it remains to analyze the structure of J_l. Denote

$$\Lambda_n(X^{(n)}) = 2\sum_{j=1}^{k} \sum_{1 \leq s_1 < s_2 \leq n} \kappa_j^4 \phi_j^2(X_{s_1})\phi_j^2(X_{s_2}),$$

$$A_n = E\Lambda_n(X^{(n)}) = n^2 \sum_{j+1}^{k} \kappa_j^4 p_j^2(1 + o(1)).$$

Put $e = [r/2]$. For any $A > 0$ denote

$$H_l(A, X^{(n)}) = \frac{1}{2^r r!} T_{1n}^r(X^{(n)}) - \frac{1}{2^r e!} A^e - \frac{1}{2^{r-1}(e-1)!4!} H_2(A, X^{(n)}) A^{e-1} - ...$$

$$-\frac{1}{2^{r-i}(e-i)!(4i)!} A^{l-i} H_{2i}(A, X^{(n)}) - ... - \frac{1}{2^{e+1}(r-2)!} H_{l-2}(A, X^{(n)}) A$$

for even r and

$$H_l(A, X^{(n)}) = \frac{1}{2^e r!} T_n^r(X^{(n)}) - \frac{1}{2^r e!} A^e H_2(A, X^{(n)}) -$$

$$\frac{1}{2^{r-1}(e-1)!4!} A^{e-1} H_6(A, X^{(n)}) - ... - \frac{r!}{2^{e+1}(r-2)!} H_{l-2}(A, X^{(n)}) A$$

for add r.

Lemma 3.4. *Assume* A3. *Then*

$$J_l(X^{(n)}) = H_l(A_n, X^{(n)}) + o_P(1) = H_l(\Lambda_n(X^{(n)}), X^{(n)}) + o_P(1). \tag{3.13}$$

Lemma 3.5. *Assume A3. Then*

$$E(\sum_{i=l}^{\infty} \frac{1}{2^i i!} J_i(X^{(n)}))^2 = o(1) \tag{3.14}$$

as $l \to \infty$ uniformly in n.

Lemma 3.6. *Assume* A3. *Then*

$$\beta(K_{1n}, \mu_n) = \beta(S_n, \mu_n)(1 + o(1)). \tag{3.15}$$

Proof. It follows directly from (3.9) and Lemmas 3.3–3.5 that Bayes likelihood ratio can be presented as $f_n(T_{1n}(X^{(n)})) + o_P(1)$. This implies (3.15).

Lemma 3.7. *Assume* C. *Then, both in the case of hypothesis and Bayes alternatives the distribution of chi-square test statistics $k^{-1/2}\sigma_n^{-1/2}T_{1n}(X^{(n)})$ is asymptotically normal. It holds*

$$E_{\nu_n} T_{1n}(X^{(n)}) = n \sum_{j=1}^{k} g_j \kappa_j^2 p_j (1 + o(1)), \tag{3.16}$$

$$Var_{\nu_n} T_{1n}(X^{(n)}) = k\sigma_n^2(1 + o(1)) \tag{3.17}$$

as $n \to \infty$.

Now Theorems 2.1 and 2.2 follows from Lemmas 3.6,3.7 and Theorems 2.3,2.4.

4 Proof of Lemmas 3.2–3.5.

Proof of Lemma 3.2. Denote $\epsilon_j = \eta_j/|\eta_j|$, $1 \le j \le k$. We have

$$T_n(G_\eta) = b_n + b_n(\delta_n - 2\rho_n k^{-1} \sum_{t=1}^{k} \epsilon_t g_t^{3/2} p_t^{1/2} g_t^{1/2} +$$

$$\rho_n k^{-1} \sum_{t=1}^{k} g_t^2 p_t (\sum_{t=1}^{k} \epsilon_t p_t^{1/2} g_t^{1/2})^2). \tag{4.1}$$

By Chebyshov inequality we obtain

$$P(\sum_{t=1}^{k} \epsilon_t p_t^{1/2} g_t^{1/2} > C\delta_n^{1/2} k) = o(1), \tag{4.2}$$

$$P(\sum_{t=1}^{k} \epsilon_t p_t^{1/2} g_t^{3/2} > C\delta_n^{1/2} k) = o(1). \tag{4.3}$$

Now (4.1)–(4.3) imply (3.5).

Proof of Lemma 3.3. Denote $J_{0l} = E_\eta R_l - J_l$. We have

$$J_{0l} = \sum_{u=1}^{r} \sum^{*} V(t_1, \dots, t_u) \tag{4.4}$$

with

$$V(t_1, ..., t_u) = \frac{l!}{t_1! \dots t_u! m_1! \dots m_v!} \sum{}' E_\eta \eta_{j_1}^{t_1} \dots \eta_{j_u}^{t_u} \times$$

$$\phi_{j_1}(X_{s_1}) \dots \phi_{j_1}(X_{s_{t_1}}) \phi_{j_2}(X_{s_{t_1}+1}) \dots \phi_{j_2}(X_{s_{t_1+t_2}}) \phi_{j_3}(X_{s_{t_1+t_2+1}}) \dots \phi_{j_u}(X_{s_l}). \tag{4.5}$$

Here $\sum^*$ is taken over all integers $t_1, \dots, t_u$, with $t_1 + ... + t_u = 1$ such that at least one value $t_i > 2$ for $1 \le i \le u$. The sum $\sum'$ is taken over all different indices $1 \le j_1, \dots, j_u \le k$, and $1 \le s_1, \dots, s_l \le k$. The values $m_1, ..., m_v$ are defined by the relations $m_1 = \sharp\{t_i : t_i = d_1, d_1 = \min\{t_s, 1 \le s \le k\}\}$, $m_2 = \sharp\{t_i : t_i = d_2, d_2 = \min\{t_s, 1 \le s \le u, t_s > d_1\}\}$, ..., $m_v = \sharp\{t_i, t_i = d_v, d_v = \min\{t_s, 1 \le s \le u, t_s > d_{v-1}\}$.

It is clear that $EV(t_1, \dots, t_v) = 0$ and

$$\sigma^2(t_1, ..., t_v) = DV(t_1, ..., t_u) = \frac{(l!)^2}{(t_1! \dots t_u! m_1! ... m_v!)^2} \times$$

$$\sum{}' \kappa_{j_1}^{2t_1} \dots \kappa_{j_u}^{2t_u} E\phi_{j_1}^2(X_{s_1}) \dots \phi_{j_1}^2(X_{s_{t_1}}) \phi_{j_2}^2(X_{s_{t_1+1}}) \dots \phi_{j_u}^2(X_{s_l})(1 + o(1)) <$$

$$\frac{Cn^l (l!)^2}{(t_1! \dots t_u! m_1! \dots m_v!)^2} \sum_{j_1, \dots, j_u = 1}^{k} \kappa_{j_1}^{2t_1} \dots \kappa_{j_u}^{2t_u} p_{j_1}^{t_1} ... p_{j_u}^{t_u} \le$$

$$Ck^{-(l-u)} (l!)^2 (t_1! \dots t_u! m_1! \dots m_v!)^{-2}. \tag{4.6}$$

Note that

$$\omega(l, u) = \sum_{t_1 + ... + t_u = l} \frac{l!}{t_1! \dots t_u! m_1! \dots m_v!}$$

is Stirling number of second kind having the following generating function

$$\frac{1}{u!}(e^z - 1)^u = \sum_{t=u}^{\infty} \omega(t, u) \frac{z^t}{t!}.$$

Putting $z = 1$ we obtain

$$\omega(l, u) < \frac{l! \exp\{u\}}{u!} \tag{4.7}$$

By (4.6),(4.7), applying Stirling formular we obtain

$$(l!)^{-2} Var J_{0l} \leq (\sum_{u=1}^{r} \sum_{t_1+\ldots+t_u=l} \sigma(t_1,\ldots,t_u))^2 <$$

$$(l!)^{-2}(\sum_{u+1}^{r} \frac{l! \exp\{u\}}{u!} k^{-(l-u)/2})^2 = o(1). \tag{4.8}$$

Proof of Lemma 3.4. First of all, note that

$$J_l(X^{(n)}) = H_l(\Lambda_n(X^{(n)}, X^{(n)}) + J_{1l}(X^{(n)}) \tag{4.9}$$

where J_{1l} contains all addendums of J_l that can not be presented in the form $\Lambda_n^{l-i}(X^{(n)})H_{4i}$ for even r and $\Lambda_n^{l-i}(X^{(n)})H_{4i+2}$ for add r (the exact assignment of addendums J_{1l} is given below in (4.15))

Prove that

$$H_l(\Lambda_n(X^{(n)}), X^{(n)}) - H_l(A_n, X^{(n)}) = o_P(1). \tag{4.10}$$

For the proof of (4.10) it suffices to show that

$$E\Lambda_n^m(X^{(n)}) - (E\Lambda_n(X^{(n)}))^m = E\Lambda_n^m(X^{(n)}) - A_n^m + o(1) = o(1) \tag{4.11}$$

for all even m. The left-hand side of (4.11) can be written as the sum of following addends

$$Z_m = \sum_{j_1,\ldots,j_u} \sum_{s_1,\ldots,s_v=1}^{n} \kappa_{j_1}^{4t_1} \ldots \kappa_{j_u}^{4t_u} E\phi_{j_1}^2(X_{s_1}) \ldots \phi_{j_u}^2(X_{s_v}). \tag{4.12}$$

Here $t_1 + \ldots + t_u = m$, each function $\phi_{j_i}^2$,$1 \leq i \leq u$, enters in the product exactly $2t_i$- times and $2u < v < 2t$.

By direct calculations we obtain that

$$EZ_m < Cn^{-2m+v} k^{-m+u} (\min p_j)^{-2(m-u)} = o(1). \tag{4.13}$$

Lemma 4.1 *Assume* A1. *Then*

$$J_{1l}(X^{(n)}) = J_l(X^{(n)}) - H_l(\Lambda_n(X^{(n)}), X^{(n)}) = o_P(1) \tag{4.14}$$

as $n \to \infty$.

Proof. The left handside of (4.14) is the sum of following addends

$$B_n = \sum{}' \kappa_{j_1}^{2t_1} \ldots \kappa_{j_u}^{2t_u} \phi_{j_1}(X_{s_1}) \ldots \phi_{j_u}(X_{s_v}) \tag{4.15}$$

where the sum $\sum'$ is taken over all integers $t_1, ..., t_u$ such that $t_1 + \ldots + t_u = r$ and the product $\phi_{j_1}(X_{s_1} \ldots \phi_{j_u}(X_{s_v})$ has $2r$-factors. The sum B_n does not have to be equal $C\Lambda_n^{l-i}(X^{(n)})J_{2i}$ for some i, $1 \le i \le r$.

Estimate EB_n. Write the product in (4.15) as

$$\kappa_{j_1}\phi_{j_1}(X_{s_1})\kappa_{j_1}\phi_{j_1}(X_{s_2}) \ldots \kappa_{j_u}\phi_{j_u}(X_{s_{v-1}})\kappa_{j_u}\phi_{j_u}(X_{s_v}). \tag{4.16}$$

Let (4.16) contain elements $\kappa_{j_\alpha}\phi_{j_\alpha}(X_{s_t})\kappa_{j_\alpha}\phi_{j_\alpha}(X_{s_t})$ and the index s_t does not appear in the other factors. Then we have

$$E \sum_{s_t=1}^{n} \kappa_{j_\alpha}^2 \phi_{j_\alpha}^2(X_{s_t}) < Ck^{-1/2}. \tag{4.17}$$

Let $t_1 = 1$. Extract the factors $\kappa_{j_1}\phi_{j_1}(X_{s_1})$, $\kappa_{j_1}\phi_{j_1}(X_{s_2})$ and all factors with random variables X_{s_1},X_{s_2}. It is clear that we can estimate the corresponding sum

$$Y_n = 2\sum_{j_1}^{k} E \sum_{1 \le s_1 < s_2 \le n} \kappa_{j_1}\phi_{j_1}(X_{s_1})\kappa_{j_{i_1}}^{a_1}\phi_{j_{i_1}}^{a_1}(X_{s_1}) \ldots \kappa_{j_{i_t}}^{a_t}\phi_{j_{i_t}}^{a_t}(X_{s_1}) \times$$

$$\kappa_{j_1}\phi_{j_1}^{b_1}(X_{s_2}) \ldots \kappa_{j_{r_u}}^{b_u}\phi_{j_{r_u}}^{b_u}(X_{s_2}) \tag{4.18}$$

separately from remaining sums in (4.15).

Denote $\gamma_j(a) = p_j^{1/2}k^{-1/4}n^{-1/2}$ if $a = 1$ and $\gamma_j(a) = n^{-a/2}p_j^{a/2}k^{-a/4} = O(n^{-1}k^{-1/2})$ if $a > 1$.

Then, by direct calculation, we obtain

$$Y_n = O(\gamma_{j_{i_1}}(a_1) \ldots \gamma_{j_{i_t}}(a_t)\gamma_{j_{r_1}}(b_1) \ldots \gamma_{j_{r_u}}(b_u)). \tag{4.19}$$

Let $t_1 > 1$ and let $\kappa_{j_1}^{2t_1}\phi_{j_1}^{d_1}(X_{s_1}) \ldots \phi_{j_1}^{d_w}(X_{s_w})$ be all factors with index j_1. Suppliment to this product all elements $\kappa_{j_{i_1}}^{a_1}\phi_{j_{i_1}}^{a_1}(X_{s_1}) \ldots \kappa_{j_t}^{a_t}\phi_{j_t}^{a_t}(X_{s_w})$ containing all random variables X_{s_m}, $1 \le m \le w$. Then we obtain

$$E\sum_{j_1}^{k} \kappa_{j_1}^{2t_1}\phi_{j_1}^{d_1}(X_{s_1}) \ldots \phi_{j_1}^{d_w}(X_{s_w})\kappa_{j_{i_1}}^{a_1}\phi_{j_{i_1}}^{a_1}(X_{s_1}) \ldots \kappa_{j_t}^{a_t}\phi_{j_t}^{a_t}(X_{s_w}) <$$

$$E\sum_{j_1=1}^{k} \kappa_{j_1}^{2t_1}\phi_{j_1}^{d_1}(X_{s_1}) \ldots \phi_{j_{i_1}}^{d_w}(X_{s_w})\gamma_{j_{i_1}}(a_1) \ldots \gamma_{j_{i_t}}(a_t) <$$

$$D(t_1, w)\gamma_{j_{i_1}}(a_1) \ldots \gamma_{j_{i_t}}(a_t) \tag{4.20}$$

where

$$D(t_1, w)\sum_{j_1=1}^{k} n^{w-t_1}k^{-t_1/2}p_{j_1}^{w-t_1} = o(k^{-w/2+1})$$

for $w \le t_1$ and

$$D(t_1, w) = o(n^{w-t_1} k^{-t_1/2})$$

for $w > t_1$. Note that in the last case $2(w - t_1)$ does not exceed a number of r.v.'s X_{s_i} entering in $\phi_{j_1}^{d_1}(X_{s_1}) \dots \phi_{j_1}^{d_w}(X_{s_w})$ only one time. Each X_{s_i} enters in the product in (4.15) at least twice. Thus taking into accaunt the assignment of $\gamma_j(a)$ we obtain that the factor n^{w-t_1} for $w > t_1$ is reduced by other multipliers.

Now (4.17)–(4.20) together implies $E(J_l - H_l) = o(1)$.

Estimate EB_n^2. Note that B_n^2 can not be presented as a product of several factors $\Lambda_n(X^{(n)})$ and H_{2i} type and admits the same assignment as B_n in (4.15). This implies $EB_n^2 = o(1)$ and completes the proof of Lemma 3.4.

Proof of Lemma 3.5. We have

$$EJ_l^2 = \frac{1}{2^{2l}(l!)^2} \sum_{u=0}^{l} \sum_{v=0}^{u} \sum_{j_1,\dots,j_{2l-u}}^{\prime} \sum_{s_1,\dots s_{2l-2v}}^{\prime} \kappa_{j_1}^4 \dots \kappa_{j_u}^4 \times$$

$$\kappa_{j_{u+1}}^2 \dots \kappa_{j_{2l-u}}^2 \phi_{j_1}^2(X_{s_1}) \phi_{j_1}^2(X_{s_2}) \phi_{j_2}^2(X_{s_3}) \phi_{j_2}^2(X_{s_4}) \dots \phi_{j_v}^2(X_{s_{2v-1}}) \phi_{j_v}^2(X_{s_{2v}}) \times$$
$$\phi_{j_{v+1}}(X_{s_{2v+1}}) \phi_{j_{v+1}}(X_{s_{2v+2}}) \phi_{j_{2l-u}}(X_{s_{2l-2v-1}}) \phi_{j_{2l-u}}(X_{s_{2l-2v}}). \tag{4.21}$$

Here the indices $j_1, \dots, j_{2l-u}, s_1, \dots, s_{2l-2v}$ take over different values $1 \le j_1, \dots, j_{2l-u} \le k$, $1 \le s_1, \dots, s_{2l-2v} \le n$, that is, $j_{i_1} \ne j_{i_2}$, $s_{t_1} \ne s_{t_2}$ for $i_1 \ne i_2$, $1 \le i_1, i_2 \le 2l - u$, and $t_1 \ne t_2$, $1 \le t_1, t_2 \le 2l - 2v$ respectively. Each random variable X_{s_i}, $2v \le i \le 2l - 2v$ enters twice in the product $\phi_{j_{v+1}}(X_{s_{2v+1}}) \dots \phi_{j_{2l-u}}(X_{s_{2l-2v}})$ and each function ϕ_{j_i}, $v \le i \le 2l - u$, is also presented twice in the product.

By direct calculation we obtain

$$EJ_l^2 \le \frac{(2l)!}{2^{2l}(l!)^2(2l-2u)!u!2^u} \frac{u!}{(u-v)!v!}(2l-2v)!k^{v-l}. \tag{4.22}$$

Denote $x = (l-u)/l$, $y = v/l$. Applying Stirling formular we have

$$\log EJ_l^2 \le -(2l+u)\log 2 - 2l\log l + 2l\log(2l) - (2l-2u)\log(2l-2u) +$$

$$(2v-u) - (u-v)\log(u-v) - v\log v + (2l-2v)\log(2l-2v) - (l-v)\log k. \tag{4.23}$$

The extremum of the right hand side of (4.23) is achieved in $8x^2 = (1-x-y)/l^2$ and $y = 1/2(1-(1-k/l^2)^{1/2})$ if $k/l^2 < 1$ or $y = 0$, $y = 1 - x$ if $k/l^2 > 1$.

Let $x = 0$, $y = 1$. Then

$$EJ_l^2 < \frac{(2l)!}{2^{3l}(l!)^3} = o(1). \tag{4.24}$$

Let $y = 0$. Then $x = (8l)^{-1/2}$ and

$$EJ_l^2 \le C(\frac{2l}{ek})^l = o(1). \tag{4.25}$$

Let $k/l^2 < 1$. Then

$$EJ_l^2 \le \exp\{l(1-x-2y)\}(ly/k)^l \le$$
$$C\{1/2\exp\{(1-k/l^2)^{1/2}\}(1-(1-k/l^2)^{1/2}\}^l = o(1). \tag{4.26}$$

Since $e^z(1-z) < 2$ for all $0 \le z \le 1$.

Now (4.21)–(4.26) together implies (3.14).

5 Proofs of Theorems 2.3 and 2.4.

Proof of Theorem 2.3. Write $T_n(X^{(n)}) = I_{1n} + I_{2n} + I_{3n} + I_{4n}$ where

$$I_{1n} = 2 \sum_{1 \le i_1 < i_2 \le n} U_n(X_{i_1}, X_{i_2}),$$

$$U_n(x,y) = \sum_{j=1}^{k} g_j(\varphi_j(x) - E_\theta\varphi_j(X_i))(\varphi_j(y) - E_\theta\varphi_j(X_i))/(np_j),$$

$$I_{2n} = \sum_{j=1}^{k}\sum_{s=1}^{n} \phi_j^2(X_S)/(np_j),$$

$$I_{3n} = (n-1)\sum_{j=1}^{k} g_j(np_j)^{-1}\sum_{s=1}^{n}(\phi_j(X_s) - E_\theta\phi_j(X_s))E_\theta\phi_j(X_s),$$

$$I_{4n} = (n-1)\sum_{j=1}^{k} g_j p_j^{-1}(E_\theta\phi_j(X_i))^2.$$

We have

$$E_\theta I_{1n} = E_\theta I_{3n} = 0, \tag{5.1}$$

$$E_\theta I_{2n} = \sum_{j=1}^{k} g_j(1 - p_j + \theta_j p_j(1-2p_j)), \tag{5.2}$$

$$I_{4n} = (n-1)\sum_{j=1}^{k} g_j p_j \theta_j^2. \tag{5.3}$$

By Shwartz inequality we obtain

$$\left|\sum_{j=1}^{k} g_j p_j \theta_j\right| \le \left(\sum_{j=1}^{k} g_j p_j\right)^{1/2} \left(n^{-1}M(\theta)\right)^{1/2} = o\left(M(\theta)n^{-1/2}\right). \tag{5.4}$$

Now (5.1) - (5.4) imply (2.11), (2.13). Prove (2.14). We have

$$Var_\theta I_{1n} = 2\sum_{j=1}^{k} g_j^2 p_j^{-2} Var_\theta^2 \phi_j(X_i)$$

$$+2 \sum_{1 \le j_1 < j_2 \le k} g_{j_1} g_{j_2} p_{j_1}^{-1} p_{j_2}^{-1} (Cov_\theta(\phi_{j_1}(X_i)\phi_{j_2}(X_i)))^2$$

$$= 2 \sum_{j=1}^{k} g_j^2 (1 - p_j + \theta_j p_j (1 - 2p_j))^2 + 4 \sum_{1 \le j_1 < j_2 \le k} g_{j_1} g_{j_2} p_j p_{j_2} (1 + \theta_j)^2 (1 + \theta_{j_2})^2 .$$

By (5.4) we obtain

$$Var_\theta I_{1n} = k\sigma_n^2 + o(n^{-1} M(\theta) + n^{-2} M^2(\theta)). \tag{5.5}$$

We have

$$Var_\theta I_{3n} = I_{3\,1n} + I_{3\,2n} \tag{5.6}$$

where

$$I_{3\,1n} = 2n^{-1}(n-1)^2 \sum_{1 \le j_1 < j_2 \le k} g_{j_1} g_{j_2}\, Cov_\theta(\phi_{j_1}(X_i)\phi_{j_2}(X_i))$$

$$\times E_\theta \phi_{j_1}(X_i)\, E_\theta \phi_{j_2}(X_i)\, p_{j_1}^{-1} p_{j_2}^{-1}$$

$$= 2n^{-1}(n-1)^2 \sum_{1 \le j_1 < j_2 \le k} g_{j_1} g_{j_2}\, p_{j_1} p_{j_2} (1 + \theta_{j_1})(1 + \theta_{j_2})\, \theta_{j_1}\, \theta_{j_2} . \tag{5.7}$$

$$I_{3\,2n} = n^{-1}(n-1)^2 \sum_{j=1}^{k} g_j^2 Var_\theta \phi_j(X_i)\, E_\theta^2 \phi_j(X_i)\, p_j^{-2}$$

$$= n^{-1}(n-1)^2 \sum_{j=1}^{k} g_j^2 p_j (1 - p_j + \theta_j (1 - 2p_j) - p_j \theta_j^2) \theta_j^2 . \tag{5.8}$$

We estimate now a part of addendums in $I_{3\,1n}$, $I_{3\,2n}$. The estimates of other addendums are trivial and are omitted.

Using Shwartz inequality, Assumption C and (5.4) we have

$$n \sum_{j=1}^{k} g_j^2 p_j \theta_j^2 \le C\, M(\theta) \max\{|\theta_j|,\ 1 \le j \le k\}$$

$$\le C\, n^{-1/2} M^{3/2}(\theta) \max\{p_j^{-1/2} :\ 1 \le j \le k\} = o(k^{1/4} M^{3/2}(\theta)), \tag{5.9}$$

$$n \sum_{1 \le j_1 < j_2 \le k} g_j g_{j_2}\, p_{j_1} p_{j_2}\, \theta_{j_1}^2 \theta_{j_2} \le n^{-1/2} M^{3/2}(\theta), \tag{5.10}$$

$$n \sum_{1 \le j_1 < j_2 \le k} g_j g_{j_2}\, p_{j_1} p_{j_2}\, \theta_j^2 \theta_{j_2}^2 < n^{-1} M^2(\theta), \tag{5.11}$$

$$n \sum_{j=1}^{k} g_j^2\, p_j^2\, \theta_j^4 < n^{-1} M^2(\theta). \tag{5.12}$$

Therefore

$$Var_\theta I_{3n} = o(k + M_n^2(\theta)\,). \tag{5.13}$$

This completes the proof of (2.14).

Assume B. Then

$$Var_\theta\, I_{2n} = n^{-1} \sum_{j=1}^{k} g_j^2\, p_j^{-2}\, Var_\theta\, \phi_j^2(X_i)$$

$$< n^{-1} \sum_{j=1}^{k} g_j^2\, p_j^{-1}\, (1 + |\theta_j|)\, (1 + o(1)) = o(k) \qquad (5.14)$$

since

$$\sum_{j=1}^{k} (np_j)^{-3/2}\, (np_j)^{1/2}\, |\theta_j| < C \left(\sum_{j=1}^{k} (n^3 p_j^3)^{-1} \right)^{1/2} M_n^{1/2}(\theta) = o(k).$$

Assume C. Then

$$I_{2n} = k/n \sum_{j=1}^{k} \sum_{s=1}^{n} (\chi(X_s \in (e_{j-1}, e_j)) - p_j)^2 = k + k \sum_{j=1}^{k} \sum_{s=1}^{n} p_j^2 - D_n \qquad (5.15)$$

where

$$D_n = 2\, k/n \sum_{j=1}^{k} \sum_{s=1}^{n} p_j\, \chi\, (X_s \in (e_{j-1}, e_j)).$$

We have

$$E_\theta D_n = 2\, k \sum_{j=1}^{k} \sum_{s=1}^{n} p_j^2 (1 + \theta_j), \qquad (5.16)$$

$$Var_\theta D_n = 4\, k^2 n^{-2} \sum_{j=1}^{k} \sum_{s=1}^{n} p_j^3 (1 + \theta_{js}) - 1/n\, E_\theta^2\, N \le C\, n^{-1} + C\, n^{-1} k^{-1} \sum_{j=1}^{k} |\theta_j|$$

$$\le C\, n^{-1} + C\, n^{-1} k^{-1/2} \left(\sum_{j=1}^{k} \theta_j^2 \right)^{1/2} \le C\, n^{-1} + C\, n^{-1/2} M^{1/2}(\theta). \qquad (5.17)$$

We have

$$Cov_\theta\, I_{1n}\, I_{2n} \le D_\theta^{1/2} I_{1n}\, D_\theta^{1/2} I_{2n}, \qquad (5.18)$$

$$Cov_\theta\, I_{2n}\, I_{3n} \le D_\theta^{1/2} I_{2n}\, D_\theta^{1/2} I_{3n}, \qquad (5.19)$$

$$Cov_\theta\, I_{1n}\, I_{3n} \le D_\theta^{1/2} I_{1n}\, D_\theta^{1/2} I_{3n}. \qquad (5.20)$$

Now (5.6), (5.7), (5.13) – (5.20) together imply (2.12). This completes the proof of Theorem 2.3.

Proof of Theorem 2.4. By (5.6), (5.13) – (5.20) it suffices to prove asymptotic normality of I_{1n}.

Denote

$$Y_{nj} = \sum_{1<i<j} U_n(X_i, X_j), \quad Z_{nj} = \sum_{i=1}^{j} Y_{ni},$$

$$V_{nj} = E_\theta(Z_{nj}^2\,|X_1,\ldots,X_{j-1}),$$
$$V_n = \sum_{i=1}^n V_{ni}$$

where $1 \le j \le n.$

The sequence Z_{nj} is a martingale. Therefore we may apply the following variant of central limit theorem for martingales.

For any $\varepsilon > 0$ let

$$P_\theta(|\,V_n^2 - \sigma_n^2\,k|\,\sigma_n^{-2}k^{-1} > \varepsilon) = o(1), \tag{5.21}$$

$$k^{-1}\,\sigma_n^2 \sum_{i=1}^n E_\theta Y_{nj}^2\,\chi\,(|\,Y_{nj}\,| > \varepsilon\,\sigma_n\,k^{1/2}) = o(1) \tag{5.22}$$

uniformly in $\theta \in \Gamma_n$ as $n \to \infty$.

Then P_θ distributions $(k\,\sigma_n^2)^{-1/2}Z_{nn}$ converge to normal uniformly in $\theta \in \Gamma_n$ as $n \to \infty$.

The proof of this statement is similar to that of Central Limit Theorem for martigales given in Brown (1971) and is omitted.

Put $q_n(x,y) = E_\theta\,U_n(x,X_i)\,U_n(X_i,y)$ for any $x,y \in [0,1]$. By direct calculations we have

$$E_\theta(V_n^2 - k\,\sigma_n^2)^2 < C(n^4\,E_\theta q_n^2(X_1,\,X_2) + n^3\,E_\theta q_n^2(X_1,\,X_1)), \tag{5.23}$$

$$n^4\,E_\theta q_n^2(X_1,\,X_2) < \sum_{j=1}^k g_j^4 + o(M_n^2(\theta)) = o(k^2\,\sigma_n^4 + M_n^2(\theta)), \tag{5.24}$$

$$\begin{aligned} n^3\,E_\theta q_n^2(X_1,\,X_2) &= n^3\,E_0\,q_n^2(X_1,\,X_2) + o(M_n^2(\theta)) \\ &= \sum_{j=1}^k g_j^4/(np_j) + o(M_n^2(\theta)) = o(k^{3/2}\sigma_n^2) \end{aligned} \tag{5.25}$$

uniformly in $\theta \in \Gamma_n$ as $n \to \infty$.

Combining (5.23–5.25) together we obtain (5.21). For the proof of (5.22) we estimate $E_\theta\,Y_{nj}^4$. Write

$$E_\theta\,Y_{ns}^4 = \sum_{r=1}^k p_r(1+\theta_r)E_\theta\,(Y_{ns}^4 \mid X_s \in (e_{r-1}e_r)) = B_{n1} + B_{n2} \tag{5.26}$$

where

$$B_{1n} = 3\sum_{r=1}^k p_r(1+\theta_r) \sum_{j_1,j_2,j_3,j_4=1}^k g_{j_1}g_{j_2}g_{j_3}g_{j_4}$$
$$\times(n^4 p_{j_1}p_{j_2}p_{j_3}p_{j_4})^{-1}\,\Gamma^{(1)}_{j_1,j_2,j_3,j_4}\Pi_{j_1,j_2,j_3,j_4}\,,$$

$$\Gamma^{(1)}_{j_1,j_2,j_3,j_4} = (s-1)(s-2)E_\theta\,(\,(\phi_{j_1}(X_1) - p_{j_1}\theta_{j_1})(\phi_{j_2}(X_1) - p_{j_2}\theta_{j_2})$$

$$\times(\phi_{j_3}(X_2) - p_{j_3}\theta_{j_3})(\phi_{j_4}(X_2) - p_{j_4}\theta_{j_4}),$$

$$\Pi_{j_1,j_2,j_3,j_4} = E_\theta\Big(\Pi_{i=1}^4(\phi_{j_i}(X_1) - p_{j_i}\theta_{j_i}) \mid X_{s_1} \in (e_{r-1}e_r)\Big).$$

The assignment of sum B_{2n} is similar to B_{1n}. The unique difference is that $\Gamma^{(1)}_{j_1,j_2,j_3,j_4}$ is replaced by

$$\Gamma^{(2)}_{j_1,j_2,j_3,j_4} = (s-1)\, E_\theta\, \Pi_{i=1}^4\,(\phi_{j_i}(X_1) - p_{j_i}\theta_{j_i}).$$

For any j_1, j_2, j_3, j_4 , $j_1 \neq j_2$, $j_3 \neq j_4$, $1 \le j_1, j_2, j_3, j_4 \le k$ we have

$$\Gamma^{(1)}_{j_1,j_2,j_3,j_4} = p_{j_1}p_{j_2}p_{j_3}p_{j_4}(1+\theta_{j_1})(1+\theta_{j_2})(1+\theta_{j_3})(1+\theta_{j_4}),$$

$$\Gamma^{(1)}_{j_1,j_1,j_3,j_4} = p_{j_1}p_{j_3}p_{j_4}(1 - p_{j_1} + p_{j_1}\theta_{j_1}(1-2p_{j_1}) - p_{j_1}\theta_{j_1}^2)$$
$$\times(1+\theta_{j_3})(1+\theta_{j_4}),$$

$$\Gamma^{(1)}_{j_1,j_1,j_2,j_2} = p_{j_1}p_{j_2}(1 - p_{j_1} + p_{j_1}\theta_{j_1}(1-2p_{j_1}) - p_{j_1}\theta_{j_1}^2)$$
$$\times(1 - p_{j_2} + p_{j_2}\theta_{j_2}(1-2p_{j_2}) - p_{j_2}\theta_{j_2}^2).$$

If $j_1 \neq j_3$, $j_1 \neq j_4$, $j_2 \neq j_3$, $j_2 \neq j_4$, also then

$$\Gamma^{(2)}_{j_1,j_2,j_3,j_4} = -3\Pi_{i=1}^4\, p_{j_i}(1+\theta_{j_i})\left(1 - \sum_{i=1}^4 p_{j_i}\theta_{j_i}\right),$$

$$\Gamma^{(2)}_{j_1,j_1,j_2,j_3} = p_{j_1}p_{j_2}p_{j_3}(1+\theta_{j_1})(1+\theta_{j_2})(1+\theta_{j_3})$$
$$\times\Big(1 - (p_{j_1}(1+\theta_{j_1}) + p_{j_2}(1+\theta_{j_2}) + p_{j_3}(1+\theta_{j_3}))p_{j_1}(1+\theta_{j_1})\Big),$$

$$\Gamma^{(2)}_{j_1,j_1,j_2,j_2} = p_{j_1}p_{j_2}(1+\theta_{j_1})(1+\theta_{j_2})\Big(p_{j_1}(1+\theta_{j_1}) + p_{j_2}(1+\theta_{j_2})$$
$$-3p_{j_1}p_{j_2}(1+\theta_{j_1})(1+\theta_{j_2})\,(1 + p_{j_1}(1+\theta_{j_1}) + p_{j_2}(1+\theta_{j_2}))\Big),$$

$$\Gamma^{(2)}_{j_1,j_1,j_1,j_2} = -p_{j_1}p_{j_2}(1+\theta_{j_1})(1+\theta_{j_2})\Big((1 - p_{j_1} - p_{j_1}\theta_{j_1})^3$$
$$-(1 - p_{j_1}(1+\theta_{j_1}) - 2p_{j_2}(1+\theta_{j_2}))\,p_{j_1}^3(1+\theta_{j_1})\Big)\,p_{j_2}(1+\theta_{j_2}),$$

$$\Gamma^{(2)}_{j_1,j_2,j_3,j_4} = p_{j_1}(1+\theta_{j_1})\Big(1 - p_{j_1}(1+\theta_{j_1})\,(\,(1+\theta_{j_1})^3\,p_{j_1}^3 + (1 - p_{j_1}(1+\theta_{j_1}))^3\Big).$$

For any $j_1, j_2, j_3, j_4 \neq r$ we have

$$\Pi_{j_1,j_2,j_3,j_4} = p_{j_1}p_{j_2}p_{j_3}p_{j_4}(1+\theta_{j_1})(1+\theta_{j_2})(1+\theta_{j_3})(1+\theta_{j_4}) \tag{5.27}$$

and if $j_\ell = r$, $1 \le \ell \le 4$, then the corresponding multiplier $p_{j_\ell}(1+\theta_{j_\ell})$ in (5.27) is replaced by $(-1)(1 - p_r - p_r\theta_r)$.

It follows from $M(\theta) < C\,k^{1/2}$ that

$$\sum_{j=1}^k \theta_j \le k^{1/2}(\sum_{j=1}^k \theta_j^2)^{1/2} < C\,k, \tag{5.28}$$

$$\sum_{j=1}^{k} p_j \, \theta_j \le (\sum_{j=1}^{k} p_j \theta_j^2)^{1/2} < C\, n^{-1/2} k^{1/4}. \tag{5.29}$$

Using (5.28), (5.29) for the estimation of B_{1n}, B_{2n}, it is eary to see that the largest order of estimates is attained on the following addendums

$$3 \sum_{r=1}^{k} p_r (1+\theta_r)\,(g_r/(np_r)\,)^4\, \Gamma^{(1)}_{rrrr}\, \Pi_{rrrr}$$

$$< C \sum_{r=1}^{k} (1+\theta_r) n^{-4} p_r^{-1} s^2 < C\, n^{-1} k^{3/2} \tag{5.30}$$

$$\sum_{r=1}^{k} p_r (1+\theta_r)\,(g_r/(np_r)\,)^4\, \Gamma^{(2)}_{rrrr}\, \Pi_{rrrr}$$

$$< C(s-1) \sum_{r=1}^{k} p_r (1+|\,\theta_r\,|)\,(n^4 p_r^2)^{-1} = o(k^2/n)\,. \tag{5.31}$$

We have

$$E_\theta\, Y_{ns}^2\, \chi(|Y_{ns}| > \varepsilon\, k^{1/2}) < \varepsilon^{-2} k^{-1} E_\theta\, Y_{ns}^4\, \chi(|Y_{ns}| > \varepsilon\, k^{1/2}). \tag{5.32}$$

Now (5.26), (5.30)-(5.32) imply (2.22). This completes the proof of Theorem 2.4.

6 Proofs of Theorems 2.5 and 2.6.

Proof of Theorem 2.5. First the proof of lower bounds will be given. In this proof a priori Bayes distributions has the same assignment as in the proof of Theorems 2.1 and 2.2. Naturally we put here $g_{jn} = 1$ for all $1 \le j \le k_n$. Thus, a priori Bayes measure is a probability measure of random vector $\eta = \{\eta_j\}_1^k$ where $\eta_1, ..., \eta_k$ are i.i.d.r.v.'s having Binomial distribution $P(\eta_j = \pm\kappa_j) = 1/2$, $\kappa_j^2 = (1+\delta) n^{-1} k {-} 1 b_n p_j^{-1/2}$. Therefore, for the proof of Theorem 2.5 it suffices to show only that

$$S_{in}(G_\eta) = i^{-2} n \sum_{j=1}^{k} \eta_j^2 p_j (1+\zeta_n) \tag{6.1}$$

where $\zeta_n \rightarrow_P 0$ as $n \to \infty$.

We have

$$S_{1n}(F_\eta) = n \sum_{j=1}^{k} p_j ((1+\eta_j - \sum_{t=1}^{k} \eta_t p_t)^{1/2} - 1)^2, \tag{6.2}$$

$$S_{2n}(F_\eta) = 1/4n\sum_{j=1}^{k} p_j((1+\eta_j - \sum_{t=1}^{k}\eta_t p_t)\log((1+\eta_j - \sum_{t=1}^{k}\eta_t p_t) \tag{6.3}$$

By Bernstein inequality, for any $\epsilon > 0$ and all $n > n_0(\epsilon)$

$$P(|\sum_{t=1}^{k} p_t\eta_t|\epsilon) < \exp\{-\epsilon^2 nk/(2b_n)\} \tag{6.4}$$

Therefore (6.1) follows from (6.2)-(6.4) by Taylor formular.

Similarly to the proof of Theorems 2.1 and 2.2 the upper bounds can be proved under the assumptions $\sum_{i=1}^{k}\theta_i p_i = 0$ and $\theta \in \Gamma_n(C)$. This implies that

$$M_n(\theta) = n\sum_{j=1}^{k}\theta_{jn}^2 p_j < Ck^{1/2}. \tag{6.5}$$

By C1,C2 and (6.5) we get

$$\sum_{j=1}^{k}\theta_{jn}^2 < Ck^{3/2}n^{-1} = o(1)$$

and $\max\{|\theta_{jn}| : 1 \le j \le k_n\} = o(1)$ as $n \to \infty$.

Hence, by Taylor formular, we get

$$T_n(G_\theta) = i^{-2}S_{in}(G_\theta(1+o(1)) == i^{-2}nM_n(\theta)(1+o(1))$$

as $n \to \infty$.

The last relation shows that all arguments in the proof of Theorem 2.4 are also valid for this setting. This completes the proof of Theorem 2.5.

*Proof of Theorem 2.6.*Denote $\bar\phi_j(X_s) = \chi(X_s \in I_j) - p_j(1+\theta_j(1-p_j))$.

We have

$$S_{1n} = n\sum_{j=1}^{k} p_j((1+\theta_j(1-p_j)+(np_j)^{-1}\sum_{s=1}^{n}\bar\phi_j(X_s))\log(1+\theta_j(1-p_j)+(np_j)^{-1}\sum_{s=1}^{n}\bar\phi_j(X_s))$$

$$S_{2n} = n\sum_{j=1}^{k} p_j((1+\theta_j(1-p_j)+(np_j)^{-1}\sum_{s=1}^{n}\bar\phi_j(X_s))^{1/2}-1)^2$$

By Bernstein inequality we get

$$P(\max_{1\le j\le k}(np_j)^{-1/2}|\sum_{s=1}^{n}\bar\phi_j(X_s)| > \epsilon_n\log n) <$$

$$\sum_{j=1}^{k} P((np_j)^{-1/2}|\sum_{s=1}^{n}\bar\phi_j(X_s)| > \epsilon_n\log n)$$

$$< k\exp\{-\epsilon_n^2\log^2 n\} = kn^{-\epsilon_n^2\log n} = o(1) \tag{6.6}$$

with $\epsilon_n = o((\log n)^{-1/2})$. Since $\sup\{|f(x)-1|, x \in (0,1)\} < c_n$ then

$$\max\{|\theta_{jn}|, 1 \le j \le k\} \to 0 \tag{6.7}$$

as $n \to \infty$.

By (6.6),(6.7), expanding in Taylor series, we get

$$|T_n(\hat{F}_n) - i^{-2}S_{in}(\hat{F}_n)| < D_1 + D_2 \tag{6.8}$$

where

$$D_1 = Cn\sum_{j=1}^{k} p_j|\theta_j|^3, \tag{6.9}$$

$$D_2 = C\sum_{j=1}^{k}(np_j)^{-2}|\sum_{s=1}^{n}\bar{\phi}_j(X_s)|^3. \tag{6.10}$$

By (6.6), (6.7), for any $\delta > 0$ we have

$$D_1 = o(M(\theta)), \qquad P(D_2 > \delta k^{1/2}) = o(1). \tag{6.11}$$

Now (6.8)-(6.11) together imply (2.16),(2.17).

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173. Reiner Lauterbach, Stanislaus Maier–Paape: Heteroclinic cycles for reaction diffusion systems by forced symmetry–breaking.

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