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Homogenization of the nonlinear bending theory for plates

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Abstract We carry out the spatially periodic homogenization of nonlinear bending theory for plates. The derivation is rigorous in the sense of Γ-convergence. In contrast to what one naturally would expect, our result shows that the limiting functional is not simply a quadratic functional of the second fundamental form of the deformed plate as it is the case in nonlinear plate theory. It turns out that the limiting functional discriminates between whether the deformed plate is locally shaped like a "cylinder" or not. For the derivation we investigate the oscillatory behavior of sequences of second fundamental forms associated with isometric immersions of class $W^{2,2}$, using two-scale convergence. This is a non-trivial task, since one has to treat two-scale convergence in connection with a nonlinear differential constraint.

Keywords homogenization · nonlinear bending theory · two-scale convergence · nonlinear differential constraint

1 Introduction

In this article we study the periodic homogenization of the nonlinear plate model introduced by Kirchhoff in 1850. In that model the elastic behavior of thin plates – undergoing bending only – are described as follows: The reference configuration of the plate in its undeformed, flat state is modeled by a bounded Lipschitz domain $S \subset \mathbb{R}^2$, while *bending deformations* are described by *isometric immersions* $u : S \to \mathbb{R}^3$ – differentiable maps that satisfy the isometry constraint

$$\partial_j u \cdot \partial_j u = \delta_{ij}, \tag{1}$$

where δ_{ij} denotes the Kronecker delta. The elastic *bending energy* of the deformed plate $u(S)$ is given by the variational integral

$$\int_S Q(\boldsymbol{I\!I}), \tag{2}$$

where $\boldsymbol{I\!I}$ is the second fundamental form associated with u (see (12) below), and Q is the quadratic energy density from linearized elasticity. We are interested in the minimizers of (2), since they are related to equilibrium shapes of thin elastic plates subject to external forces and boundary conditions. Indeed, Friesecke, James, Müller obtained in their celebrated work [FJM02] Kirchhoff's nonlinear plate model from nonlinear three-dimensional elasticity in the zero-thickness limit. The connection is rigorous in the sense of Γ-convergence, which roughly speaking means that (almost) minimizers to a large class of minimization problems from three-dimensional nonlinear elasticity converge to solutions to minimization problems associated with the bending energy (2).

The energy density Q encodes the elastic properties of the material and, when the material is heterogeneous, depends on $x \in S$ in addition. In the case of a periodic composite material with small period $\varepsilon \ll 1$, the energy density might be written in the form $Q(\frac{x}{\varepsilon}, F)$ where $Q(y, F)$ is periodic in y. For definiteness, let Q satisfy the following

Assumption 1 *Let* $Q : \mathbb{R}^2 \times \mathbb{R}^{2\times 2}_{\text{sym}} \to [0, \infty)$ *be*

(Q1) measurable and $[0, 1)^2$*-periodic in* $y \in \mathbb{R}^2$*,*

(Q2) convex and quadratic in $F \in \mathbb{R}^{2\times 2}$*,*

(Q3) bounded and non-degenerate in the sense of

$$\alpha|\text{sym}F|^2 \le Q(y, F) \le \frac{1}{\alpha}|\text{sym}F|^2 \tag{3}$$

for all $A \in \mathbb{R}^{2\times 2}$*, almost every* $y \in \mathbb{R}^2$ *and for some constant of ellipticity* $\alpha > 0$ *which is fixed from now on.*

We reformulate the bending energy (2) as the functional $\mathcal{E}^\varepsilon : L^2(\Omega, \mathbb{R}^3) \to [0, \infty]$ given by

$$\mathcal{E}^\varepsilon(u) := \begin{cases} \displaystyle\int_S Q\left(\frac{x}{\varepsilon}, \boldsymbol{I\!I}(x)\right) dx & \text{for } u \in W^{2,2}_{\text{iso}}(S), \\ \infty & \text{else,} \end{cases} \tag{4}$$

where $W^{2,2}_{\text{iso}}(S)$ denotes the subset of maps $u \in W^{2,2}(S, \mathbb{R}^3)$ that satisfy (1) almost everywhere in S.

Our goal is to understand the homogenization limit, $\varepsilon \downarrow 0$, in the spirit of Γ-convergence. For the description of the limit we need to classify the geometry of surfaces $u(S)$ with $u \in W^{2,2}_{\text{iso}}(S)$. For simplicity, let us first assume that u is a smooth isometric immersion. Since S is flat, the Gauss curvature of the surface $u(S)$ vanishes, and by a classical result from geometry we know that locally $u(S)$ is either *flat* (when u is affine), or a developable surface. In the latter case the surface has either the shape of a *cylinder* or a *cone*. (With slight abuse of the standard terminology, we refer to tangent developable surfaces as *cones*.) For the flat part of the surface $u(S)$ we introduce the notation

$$C_{\nabla u} = \{\, x \in S \,:\, u(S) \text{ is affine in a neighborhood of } u(x) \,\}.$$

By developability, for every point $x \in S \setminus C_{\nabla u}$, there exists a unit vector $N(x)$ such that ∇u is constant on the line segment through x with direction $N(x)$. If there exists a unit vector $\bar{N}$ such that the set $N^{-1}(\bar{N})$ has density 1 at x, we say that the surface has the shape of a *cylinder* there, and we call x a *cylindrical* point. Points $x \in S \setminus C_{\nabla u}$ where this does not hold true will be called *conical* points. (This dichotomy is only valid up to a null set, cf. Definition 1.) We write $Z_{\nabla u}$ and $K_{\nabla u}$ to denote the set of cylindrical and conical points, respectively. As we explain in Section 2.1 below, the assumption that u is smooth is unnecessary, and these notions extend to $W^{2,2}$-isometric immersions, see Definition 1.

For the definition of the limiting functional we require averaged and homogenized versions of Q. Since the second fundamental form almost surely belongs to the cone of symmetric 2×2-matrices with rank at most one, it suffices to define the relaxed versions of Q for such matrices: for a unit vector $T \in \mathbb{R}^2$ and $\mu \in \mathbb{R}$ set

$$Q_{\text{av}}(\mu T \otimes T) := \mu^2 \int_{(0,1)^2} Q(y, T \otimes T)\, dy, \tag{5}$$

$$Q_{\text{hom}}(\mu T \otimes T) := \mu^2 \min_{\alpha \in W^{1,2}_{T\text{-per}}(\mathbb{R})} \left\{ \int_{(0,1)^2} Q\Big(y, (1 + \alpha'(T \cdot y))\, T \otimes T\Big)\, dy \right\}; \tag{6}$$

here $W^{1,2}_{T\text{-per}}(\mathbb{R})$ denotes the closure w. r. t. the $W^{1,2}$-norm of the set of doubly periodic functions in $C^\infty(\mathbb{R})$ with periods $T \cdot e_1$ and $T \cdot e_2$, see Subsection 1.1 for details. Note that the expression for Q_{hom} differs from the usual formula used for the homogenization of convex integrands – in fact, as we will see in Subsection 1.1, it can be interpreted as mixture of a one-dimensional averaging and homogenization.

The $\Gamma(L^2)$-limit of $\mathcal{E}^\varepsilon$ is then given by the functional $\mathcal{E}_{\text{hom}} : L^2(\Omega, \mathbb{R}^3) \to [0, \infty]$,

$$\mathcal{E}^{\text{hom}}(u) := \begin{cases} \displaystyle\int_S (1 - \chi_{\nabla u}(x)) Q_{\text{av}}(\boldsymbol{I\!I}) + \chi_{\nabla u}(x) Q_{\text{hom}}(\boldsymbol{I\!I}) & \text{for } u \in W^{2,2}_{\text{iso}}(S), \\ \infty & \text{else,} \end{cases}$$

where $\chi_{\nabla u}$ denotes the indicator function of $Z_{\nabla u}$, see Definition 1 below.

We shall consider boundary conditions of the following form: Let $L_{BC} \neq \emptyset$ denote a line segment of the form $L_{BC} = \{\, x_0 + tN \,:\, t \in \mathbb{R} \,\} \cap S$ (for some $x_0 \in \mathbb{R}^2$ and some unit vector $N \in \mathbb{R}^2$). We assume that

$$u = \varphi_{BC} \text{ and } \nabla u = \nabla \varphi_{BC} \qquad \text{on } L_{BC}, \tag{BC}$$

where $\varphi_{BC} : \mathbb{R}^2 \to \mathbb{R}^3$ is a fixed rigid isometric immersion, i. e. $\nabla \varphi_{BC}$ is constant and satisfies (1).

We are now in position to state our main result.

Theorem 1 *Let $S \subset \mathbb{R}^2$ be a convex Lipschitz domain and let Q satisfy (Q1) – (Q3).*

(a) Consider $u^\varepsilon \in L^2(S, \mathbb{R}^3)$ with finite energy, i. e.

$$\limsup_{\varepsilon \downarrow 0} \mathcal{E}^\varepsilon(u^\varepsilon) < \infty.$$

Then there exists $u \in W^{2,2}_{iso}(S)$ such that $u^\varepsilon - ⨍_S u^\varepsilon \to u$ in $L^2(S, \mathbb{R}^3)$ as $\varepsilon \downarrow 0$ (after possibly passing to subsequences).

(b) Let u^ε converge to some u in $L^2(S, \mathbb{R}^3)$ as $\varepsilon \downarrow 0$. Then

$$\liminf_{\varepsilon \downarrow 0} \mathcal{E}^\varepsilon(u^\varepsilon) \geq \mathcal{E}^{hom}(u).$$

(c) *For every* $u \in L^2(S,\mathbb{R}^3)$ *there exists a sequence* $u^\varepsilon \in L^2(S,\mathbb{R}^3)$ *that converges to* u *and*

$$\lim_{\varepsilon\downarrow 0} \mathcal{E}^\varepsilon(u^\varepsilon) = \mathcal{E}^{hom}(u).$$

Moreover, if $u \in W^{2,2}_{iso}(S)$ *satisfies* (BC), *then* u^ε *can be chosen such that* $u^\varepsilon \in W^{2,2}_{iso}(S)$ *satisfies the boundary condition* (BC) *in addition.*

The limit $\mathcal{E}^{\mathrm{hom}}$ is not a standard Kirchhoff plate model. In particular, it is not possible to recast $\mathcal{E}^{\mathrm{hom}}$ into the form of (2). Still, it is a generalized Kirchhoff plate model in the sense that the energy locally is quadratic in the second fundamental form.

Remark 1 (1) The result also holds true for non-convex Lipschitz domains S with the property that there exists some $\Sigma \subset \partial S$ with $\mathcal{H}^1(\Sigma) = 0$, and the outer normal to S exists and is continuous on $\partial S \setminus \Sigma$. We limit ourselves to the convex case here for the sake of brevity. Our main point is the proof of part (b) of Theorem 1, which is completely independent of whether S is convex or not. The construction of a recovery sequence in part (c) however becomes somewhat more involved for non-convex domains. It is nevertheless possible by appealing to the results of [Hor11b] and [Hor11a].

(2) We have chosen to set the boundary conditions (BC) on a line segment in the interior of the domain. We have done so for the sake the simplicity. If the boundary of S contains a flat part, we could also set the boundary conditions there. It is possible to treat this case by enlarging the domain and extending the isometric immersion affinely – in this way, the boundary conditions on the flat part of the initial domain become boundary conditions on a line segment lying in the interior of the enlarged domain.

Let us comment on the proof of Theorem 1. Since $\mathcal{E}^\varepsilon$ is non-convex and singular with non-convex effective domain the derivation of the Γ-limit is subtle and standard tools, e. g. compactness and representation results for Γ-limits that rely on integral representations, are not applicable. To overcome these difficulties we take advantage of two observations: First, as a functional of the second fundamental form the mapping

$$\boldsymbol{I\!I} \mapsto \int_S Q(\tfrac{x}{\varepsilon}, \boldsymbol{I\!I})\, dx \tag{7}$$

is convex and quadratic, so that we can pass to the limit $\varepsilon \downarrow 0$ in (7) by classical homogenization techniques, in particular two-scale convergence. Secondly, the nonlinear isometry constraint yields a strong rigidity and allows the second fundamental form to oscillate only in a very restricted way.

This second observation is the heart of the matter and requires to describe the structure of two-scale limits of vector fields under a *nonlinear* differential constraint, cf. Remark 4 for more details. While the interplay of two-scale convergence and *linear* differential constraints is reasonably well understood, e.g. see [FK10], in the nonlinear case no systematic approach seems to be available. In fact, to our knowledge our result is the first attempt in that direction in the nonlinear case. Since the main focus of this paper is the derivation of the Γ-limit to $\mathcal{E}^\varepsilon$, we content ourselves with a partial identification of the two-scale limit which is yet strong enough to treat Theorem 1. To motivate this in more detail consider a sequence u^ε that weakly converges in $W^{2,2}_{\mathrm{iso}}(S)$ to some limit u. Let $\boldsymbol{I\!I}^\varepsilon$ denote the second fundamental form associated with u^ε. Since $\boldsymbol{I\!I}^\varepsilon$ is bounded in $L^2(S,\mathbb{R}^{2\times 2})$, we may pass to a weakly two-scale convergent sequence. Since $Q(y,F)$ is convex in F, standard results from two-scale convergence, cf. Lemma 6, yield the lower bound

$$\liminf_{\varepsilon\downarrow 0} \int_S Q(\frac{x}{\varepsilon}, \boldsymbol{I\!I}^\varepsilon(x))\, dx \geq \inf_{H(x,y)} \int_{S\times(0,1)^2} Q(y, H(x,y))\, dy dx,$$

where the infimum is taken over all weak two-scale limits $H(x,y)$ of arbitrary subsequences of $\boldsymbol{I\!I}^\varepsilon$. Seeking for a lower bound that only depends on the limit u, we need to identify the class of limits $H(x,y)$ that might emerge as weak two-scale limits of $\boldsymbol{I\!I}^\varepsilon$. This is done in Section 3. As we shall see in Proposition 2 only certain oscillations on scale ε are compatible with the nonlinear isometry constraint (1). Loosely speaking, we observe that on cylindrical regions of the limiting plate $u(S)$, only oscillations on scale ε parallel to the line of curvature are possible, while on conical regions all oscillations on scale ε are suppressed.

Theorem 1 is a homogenization result for a singular integral functional whose effective domain $\{\mathcal{E}^\varepsilon < \infty\}$ is *non-convex*. Questions regarding homogenization and relaxation of singular integral functionals related to hyperelasticity have been actively studied in the last years, e.g. [AHM11] and the references therein. Typically, these interesting works study integral functionals of the form $u \mapsto \int W(\frac{x}{\varepsilon}, \nabla u(x))\,dx$ where u denotes a deformation and W satisfies non-standard growth conditions allowing for attainment of the value $+\infty$. Compared to that, in our situation the singular behavior is of different nature. It is due to the non-convex differential constraint (1) and, thus, requires a completely different approach.

The paper is organized as follows: In Subsection 1.1 we discuss the homogenized quadratic form $Q_{\hom}$ in more detail. In Subsection 1.2 we put our limiting model $\mathcal{E}^{\hom}$ in relation with models derived from three-dimensional elasticity via simultaneous dimension reduction and homogenization. In Section 2 we recall some basic preliminaries from geometry and two-scale convergence. Section 3 is the core of the paper. There we analyze the structure of oscillations of the second fundamental form. Finally, in the last section we give the proof of Theorem 1.

1.1 Homogenization formula and homogenization effects

Theorem 1 states in particular that locally, there are no homogenization effects if the deformation u is not a cylindrical isometric immersion. On the cylindrical part non-trivial homogenization effects occur and the effective behavior is captured by $Q_{\hom}$ which is defined via (6). The formula involves the space $W^{1,2}_{T\text{-per}}(\mathbb{R})$ which is defined as follows: For any unit vector $T \in \mathbb{R}^2$ we set

$$C^1_{T\text{-per}}(\mathbb{R}) := \{\, \alpha \in C(\mathbb{R}) \,:\, \alpha(s + T\cdot k) = \alpha(s) \text{ for all } s \in \mathbb{R},\, k \in \mathbb{Z}^2 \,\},$$

and define $W^{1,2}_{T\text{-per}}(\mathbb{R})$ as the closure of $C^1_{T\text{-per}(\mathbb{R})}$ w. r. t. the norm

$$\|\alpha\|^2_{W^{1,2}_{T\text{-per}}(\mathbb{R})} := \int_{(0,\max\{T\cdot e_1, T\cdot e_2\})} \alpha^2(s) + |\alpha'(s)|^2\,ds.$$

The space $C^1_{T\text{-per}}(\mathbb{R})$ and thus $W^{1,2}_{T\text{-per}}(\mathbb{R})$ can be characterized as follows: Consider

$$\mathcal{S}^1_* := \{\, T \in \mathcal{S}^1 \,:\, T \in r\mathbb{Z}^2 \text{ for some } r \in \mathbb{R} \,\}$$

We will call $T \in \mathcal{S}^1_*$ a "rational" direction. We define

$$r(T) := \begin{cases} \sup\{\, r > 0 \,:\, T \in r\mathbb{Z}^d \,\} & \text{if } T \in \mathcal{S}^1_*, \\ 0 & \text{otherwise.} \end{cases} \tag{8}$$

If the ratio of the components of T, i. e. $T\cdot e_1$ and $T\cdot e_2$, is irrational, then $r(T) = 0$ and $C^1_{T\text{-per}}(\mathbb{R})$ only contains the constant functions. Otherwise $C^1_{T\text{-per}}(\mathbb{R})$ consists precisely of those functions in $C^1(\mathbb{R})$ that are periodic with period $r(T)$.

Next, we obtain a more explicit formula for $Q_{\hom}(T \otimes T)$. If $r(T) = 0$, then we have $Q_{\hom}(T\otimes T) = Q_{\mathrm{av}}(T \otimes T)$. Otherwise, consider for $t \in [0, r(T))$ the finite union of line segments

$$L_t := \left\{ y \in [0,1)^2 : T\cdot y - t \in r(T)\mathbb{Z} \right\},$$

and define $q_{\mathrm{av},T} : [0, r(T)) \to \mathbb{R}$ by

$$q_{\mathrm{av},T}(t) = r(T) \int_{L_t} Q(y, T\otimes T)\, d\mathcal{H}^1(y), \tag{9}$$

which in fact is an average since $\mathcal{H}^1(L_t) = r(T)^{-1}$ for all $t \in [0, r(T))$. With this notation, we have by Fubini

$$Q_{\hom}(T\otimes T) = \min\left\{ \fint_0^{r(T)} q_{\mathrm{av},T}(t)(1+\alpha'(t))^2 dt : \alpha \in W^{1,2}_{T\text{-per}}(\mathbb{R}) \right\}.$$

The solution of this one-dimensional minimization problem which is obtained by integrating the associated Euler-Lagrange equation is well known. A minimizer α_* (whose dependency on T we suppress in the notation) is given by

$$\alpha_*(t) := \frac{1}{\fint_0^{r(T)} \frac{ds}{q_{\mathrm{av},T}(s)}} \int_0^t \frac{ds}{q_{\mathrm{av},T}(s)} \tag{10}$$

and we obtain

$$Q_{\mathrm{hom}}(T\otimes T)=\fint_0^{r(T)} q_{\mathrm{av},T}(t)(1+\alpha_*'(t))^2dt=\frac{1}{\fint_0^{r(T)}\frac{dt}{q_{\mathrm{av},T}(t)}}\,. \tag{11}$$

Thus we have averaging in the direction perpendicular to T (eq. (9)) and homogenization in the direction of T (eq. (11)). The averaging takes place over a set of $\mathcal{H}^1$-measure $r(T)^{-1}$, and the homogenization takes place over a set of $\mathcal{H}^1$-measure $r(T)$. The better T agrees with the periodic microstructure of the material (which by assumption (Q1) is aligned with the coordinate axes), the smaller is $r(T)$. Hence, the better T is chosen to match with the coordinate axes, the more room there is for homogenization effects to make the material softer with respect to bending in this direction.

1.2 Relation to 3d nonlinear elasticity

As mentioned in the introduction Kirchhoff's plate model can be rigorously derived from nonlinear 3d elasticity. In the following we compare the limit $\mathcal{E}^{\mathrm{hom}}$ from Theorem 1 to effective models obtained from 3d elasticity via simultaneous dimension reduction and homogenization. To that end we consider the energy functional

$$\mathcal{E}^{\varepsilon,h}(u):=\frac{1}{h^2}\fint_{\Omega_h} W(\tfrac{x_1}{\varepsilon},\tfrac{x_2}{\varepsilon},\nabla u(x))\,dx,$$

where $\Omega_h:=S\times(-\frac{h}{2},\frac{h}{2})$ models the reference domain of a thin, three-dimensional plate with thickness $h>0$, and $W:\mathbb{R}^2\times\mathbb{R}^{3\times3}\to[0,\infty]$ denotes a stored energy function of an elastic composite material. We assume that $W(y,F)$ is $[0,1)^2$-periodic in y, and frame-indifferent, non-degenerate, and C^2 in a neighborhood of the identity in F (see [FJM02] for details).

The energy $\mathcal{E}^{\varepsilon,h}$ models a hyperelastic material whose stress free reference state is the thin domain Ω_h. The described material is a composite that periodically varies in in-plane directions. Note that $\mathcal{E}^{\varepsilon,h}$ admits two small length scales: the thickness h and the material fine-scale ε. The limit $h\downarrow 0$ corresponds to dimension reduction, while $\varepsilon\downarrow 0$ amounts to homogenization. In [FJM02] it is shown that $\mathcal{E}^{\varepsilon,h}$ Γ-converges for $h\downarrow 0$ (and fixed $\varepsilon>0$) to the energy $\mathcal{E}^{\varepsilon}$, cf. (4), where Q is obtained from the quadratic form $G\mapsto\frac{\partial^2 W}{\partial F^2}(y,I)(G,G)$ by a relaxation formula, and (by the assumptions on W) automatically satisfies Assumption 1. Hence, in combination with Theorem 1 we deduce that $\mathcal{E}^{\mathrm{hom}}$ is the double-limit of the 3d-energy $\mathcal{E}^{\varepsilon,h}$ that correspond to "homogenization after dimension reduction"; i. e.

$$\mathcal{E}^{\mathrm{hom}}=\Gamma\text{-}\lim_{\varepsilon\downarrow0}\ \Gamma\text{-}\lim_{h\downarrow0}\mathcal{E}^{\varepsilon,h}.$$

We therefore expect $\mathcal{E}^{\mathrm{hom}}$ to be a good model for the three-dimensional plate in situations where $h\ll\varepsilon\ll 1$.

An alternative way to obtain an effective model from $\mathcal{E}^{\varepsilon,h}$ is to simultaneously pass to the limit $(\varepsilon,h)\to(0,0)$. This has been studied in the case of rods, plates and shells, see [HNV13], [HV12], [Neu10], [Neu12], [NV13], and [Vel12]. In particular, in [Neu12] the simpler situation of elastic rods has been analyzed in detail, i. e. when Ω_h is replaced by a thin rod-like domain of the form $(0,1)\times hB$ where B denotes the two-dimensional cross-section of the rod. As shown in [Neu12] the obtained Γ-limit depends on the relative scaling between ε and h. More precisely, under the assumption that the ratio $\frac{h}{\varepsilon}$ converges to a prescribed scaling factor $\gamma\in[0,+\infty]$, it is shown that the initial energy Γ-converges to a bending torsion model for inextensible rods, whose effective energy density continuously depends on the scaling factor γ. Moreover, it is shown that the model obtained in the case $\gamma=0$ (which corresponds to simultaneous dimension reduction and homogenization in the regime $h\ll\varepsilon\ll1$) is equivalent to the model obtained by the sequential limit "$\varepsilon\downarrow 0$ after $h\downarrow 0$".

For plates, as considered here, this suggests the following: For a given scaling factor $\gamma>0$ consider the limit $\mathcal{E}^{\gamma}=\Gamma\text{-}\lim_{h\downarrow0}\mathcal{E}^{\varepsilon(h),h}$ where we assume that $\frac{h}{\varepsilon(h)}\to\gamma$ as $h\downarrow 0$. This limit corresponds to a simultaneous dimension reduction and homogenization of $\mathcal{E}^{\varepsilon,h}$ in the case when the fine-scale ε and h do not separate. The analysis for rods described above suggests that $\mathcal{E}^{\mathrm{hom}}$ can be recovered from $\mathcal{E}^{\gamma}$ in the limit $\gamma\downarrow 0$. Surprisingly this is **not** the case for plates: As shown most recently by Hornung and Velčić and the first author in [HNV13, Theorem 2.4], for $\gamma\in(0,\infty)$ the limit $\mathcal{E}^{\gamma}$ takes the form of the plate model (4) with Q replaced by the relaxed and homogenized quadratic form Q_γ that depends on the scaling factor γ. A close look at the relaxation formula defining Q_γ shows that typically $\limsup_{\gamma\downarrow 0} Q_\gamma < Q_{\rm av}$. This implies that on the level of the associated energies $\mathcal{E}^\gamma$ and $\mathcal{E}^{\rm hom}$ we typically have $\limsup_{\gamma\downarrow 0}\mathcal{E}^\gamma(u) < \mathcal{E}^{\rm hom}(u)$ for *conical* deformations $u \in W^{2,2}_{\rm iso}(S)$, in contrast to the case of rods, where $\lim_{\gamma\downarrow 0}\mathcal{E}^\gamma = \mathcal{E}^0 = \mathcal{E}^{\rm hom}$.

2 Notation and preliminaries

Throughout this article we use the following notation:

- e_1, e_2 denotes the standard Euclidean basis of $\mathbb{R}^2$;
- we write $a \cdot b$ for the inner product in $\mathbb{R}^2$, $|\cdot|$ for the induced Euclidean norm, and denote the coefficients of $a \in \mathbb{R}^2$ by $a_i := a \cdot e_i$, $i = 1, 2$;
- for $a = (a_1, a_2) \in \mathbb{R}^2$ we set $a^\perp := (-a_2, a_1)$;
- $\mathcal{S}^1 := \{\, e \in \mathbb{R}^2 \,:\, |e| = 1\}$, and $\mathcal{S}^1_* := \{\, T \in \mathcal{S}^1 \,:\, T \in r\mathbb{Z}^2 \text{ for some } r \in \mathbb{R}\,\}$
- for $a = (a_1, a_2), b = (b_1, b_2) \in \mathbb{R}^2$ we denote by $a \otimes b$ the unique 2×2 matrix characterized by $e_i \cdot (a \otimes b)e_j = a_i b_j$;
- we denote the entries of $A \in \mathbb{R}^{2\times 2}$ by A_{ij} so that $A = \sum_{i,j=1}^2 A_{ij}(e_i \otimes e_j)$, and we write $A : B := \sum_{i,j=1}^d A_{ij}B_{ij}$ for the inner product in $\mathbb{R}^{2\times 2}$;
- A^t denotes the transposed of $A \in \mathbb{R}^{2\times 2}$;
- $B(x, R)$ denotes the open ball in $\mathbb{R}^2$ with center x and radius R;
- $a \times b$ denotes the vector product in $\mathbb{R}^3$.

2.1 Properties of $W^{2,2}$-isometric immersions

We denote by

$$W^{2,2}_{\rm iso}(S) := \{\, u \in W^{2,2}(S, \mathbb{R}^3) \,:\, u \text{ satisfies (1) a.e. in } S\,\}$$

the set of Sobolev isometries. The second fundamental form associated with $u \in W^{2,2}_{\rm iso}(S)$ is given by the matrix field $\boldsymbol{I\!I} : S \to \mathbb{R}^{2\times 2}$ with entries

$$\boldsymbol{I\!I}_{ij} := -\partial_i n \cdot \partial_j u, \tag{12}$$

where $n := \partial_1 u \times \partial_2 u$ denotes the normal field to the surface $u(S)$.

¿From classical geometry it is well known that a smooth surface in $\mathbb{R}^3$ that is isometric to a flat surface is developable — locally it is either flat, a cylinder or a cone. As shown by Kirchheim [Kir01] (see also [Pak04], [Hor11b] and [Hor11a]) $W^{2,2}$-isometries share this property. In the following we make this precise. Throughout the paper we use the notation $[x; N] := \{\, x + sN \,:\, s \in \mathbb{R}\,\}$ for the line through x parallel to N, and $[x; N]_S$ for the connected component of $[x; N] \cap S$ that contains x. We start our survey with a regularity result on the gradient of isometries:

Lemma 1 (see [MP05, Proposition 5]) *Let $S \subset \mathbb{R}^2$ be a Lipschitz domain. Then ∇u is continuous for all $u \in W^{2,2}_{iso}(S)$.*

In the following let S be a convex Lipschitz domain and $u \in W^{2,2}_{\rm iso}(S)$. We shall introduce some objects to describe the geometry of $u(S)$. We say $x \in S$ is a *flat point of* ∇u, if ∇u is constant in some neighborhood of x and introduce the (open) set

$$C_{\nabla u} := \{\, x \in S \,:\, x \text{ is a flat point of } \nabla u\,\}.$$

For our purpose it is convenient to describe the geometry of the *non-flat* part $S \setminus C_{\nabla u}$ by means of *asymptotic lines*. We say a unit vector $N \in \mathbb{R}^2$ is called an *asymptotic direction (for ∇u) at $x \in S$* if

$$\exists s_0 > 0 \text{ such that } \nabla u(x) = \nabla u(x + sN) \text{ for all } s \in (-s_0, s_0). \tag{13}$$

When u is a smooth isometry, then it is known from classical geometry that at every non-flat point x there exists an asymptotic direction $N(x)$ that is unique up to a sign. In fact, we know more: There exists a mapping $N : S \setminus C_{\nabla u} \to \mathcal{S}^1 := \{ N \in \mathbb{R}^2 : |N| = 1 \}$ such that for all $x, y \in S \setminus C_{\nabla u}$

$$\nabla u \text{ is constant on } [x; N(x)]_S, \tag{14a}$$

$$[x; N(x)]_S \cap [y; N(x)]_S \neq \emptyset \implies [x; N(x)] = [y; N(y)]. \tag{14b}$$

This observation extends to $W^{2,2}$-isometries:

Proposition 1 ([Pak04]) *Let $u \in W^{2,2}_{iso}(S, \mathbb{R}^3)$. Then there exists a locally Lipschitz continuous vector field $N : S \setminus C_{\nabla u} \to \mathcal{S}^1$ such that* (14a) *and* (14b) *is true for all $x, y \in S \setminus C_{\nabla u}$. Furthermore, the field $S \setminus C_{\nabla u} \ni x \mapsto N(x) \otimes N(x)$ is unique.*

For isometries of class C^2, Proposition 1 is contained in the more general result [HN59]. In the form above, the proposition has been proven in [Pak04], using ideas from [Kir01].

On $S \setminus C_{\nabla u}$ the second fundamental form $\boldsymbol{I\!I}$ is proportional to $N^\perp \otimes N^\perp$, which has the geometric meaning that $T := -N^\perp$ is the principal direction along which $u(S)$ is curved. This elementary observation is made precise in the following lemma which can be found in [FJM06] and [Hor11b]:

Lemma 2 *Let $S \subset \mathbb{R}^2$ be bounded and $u \in W^{2,2}_{iso}(S)$, then almost everywhere on S*

$$\partial_i \partial_j u \cdot n = \boldsymbol{I\!I}_{ij}, \tag{15}$$

$$\partial_2 \boldsymbol{I\!I}_{11} = \partial_1 \boldsymbol{I\!I}_{12}, \tag{16}$$

$$\partial_2 \boldsymbol{I\!I}_{21} = \partial_1 \boldsymbol{I\!I}_{22}, \tag{17}$$

and there exists $T : S \to \mathcal{S}^1$ with $T(x) = -N(x)^\perp$ for $x \in S \setminus C_{\nabla u}$ and $\mu \in L^2(S)$ such that

$$\boldsymbol{I\!I} = \mu T \otimes T \qquad \text{a.e. on } S. \tag{18}$$

For a given $u \in W^{2,2}_{\text{iso}}(S)$, we distinguish two subsets of the non-flat part of S, which we call the *cylindrical* and the *conical* part. To do so, we define for $T \in \mathcal{S}^1$,

$$\chi_{\nabla u, T}(x) := \begin{cases} 1 & \text{if } x \in S \setminus C_{\nabla u} \text{ and } N(x) \cdot T = 0, \\ 0 & \text{else,} \end{cases} \tag{19}$$

$$\chi^*_{\nabla u, T}(x) := \begin{cases} \lim_{r \downarrow 0} \fint_{B(x,r)} \chi_{\nabla u, T}(y) dy & \text{if the limit exists,} \\ 0 & \text{else.} \end{cases} \tag{20}$$

Definition 1 For $u \in W^{2,2}_{\text{iso}}(S)$. We say $x \in S \setminus C_{\nabla u}$ is

$$\begin{cases} \text{cylindrical, and write } x \in Z_{\nabla u}, & \text{if } \exists T \in \mathcal{S}^1 : \chi^*_{\nabla u, T}(x) = 1, \\ \text{conical, and write } x \in K_{\nabla u}, & \text{if } \forall T \in \mathcal{S}^1 : \chi^*_{\nabla u, T}(x) = 0. \end{cases}$$

We write $\chi_{\nabla u}$ for the indicator function of $Z_{\nabla u}$.

We conclude this section with some elementary properties of the introduced decomposition.

Lemma 3 *The sets $Z_{\nabla u}$ and $K_{\nabla u}$ are measurable. Furthermore, there exists a null set $E \subset S$ such that*

$$S = C_{\nabla u} \cup Z_{\nabla u} \cup K_{\nabla u} \cup E. \tag{21}$$

There exists a countable set $\mathcal{S}_{\nabla u} \subset \mathcal{S}^1$ of pairwise non-parallel vectors, such that

$$Z_{\nabla u} = \bigcup_{T \in \mathcal{S}_{\nabla u}} \{\chi^*_{\nabla u, T} = 1\}, \tag{22}$$

$$\chi_{\nabla u} = \sum_{T \in \mathcal{S}_{\nabla u}} \chi_{\nabla u, T} \qquad \text{a.e. in } S. \tag{23}$$

Proof Consider the set

$$\mathcal{T} := \{ T \in \mathcal{S}^1 \,:\, \mathcal{L}^2(\{x \in S \setminus C_{\nabla u} \,:\, N(x) = T^\perp\}) > 0\},$$

which can be written as

$$\mathcal{T} = \bigcup_{k\in\mathbb{N}} \mathcal{T}_k, \qquad \mathcal{T}_k := \{ T \in \mathcal{S}^1 \,:\, \mathcal{L}^2(\{x \in S \setminus C_{\nabla u} \,:\, N(x) = T^\perp\}) > \tfrac{1}{k}\}.$$

Since $\mathcal{L}^2(S) < \infty$, and since the sets $\{x \in S \setminus C_{\nabla u} \,:\, N(x) = T^\perp\}$, $T \in \mathcal{S}^1$, are pairwise disjoint, each $\mathcal{T}_k$ only contains a finite number of elements, and thus $\mathcal{T}$ is countable. From the definition of $\chi^*_{\nabla u,T}$ it is clear that each $T \in \mathcal{S}^1$ with $\chi^*_{\nabla u,T}(x) = 1$ for some $x \in S$ must be an element of $\mathcal{T}$ or $-\mathcal{T}$. Hence, the set

$$\tilde{\mathcal{S}} := \{ T \in \mathcal{S}^1 \,:\, \exists x \in S \text{ s.t. } \chi^*_{\nabla u,T}(x) = 1 \}$$

is at most countable, and we get

$$Z_{\nabla u} = \bigcup_{T\in\tilde{\mathcal{S}}} \{\chi^*_{\nabla u,T} = 1\}. \tag{24}$$

Since this is a countable union of measurable sets, we deduce that $Z_{\nabla u}$ is measurable. By virtue of the invariance property $\chi^*_{\nabla u,T} = \chi^*_{\nabla u,-T}$, we may replace in (24) the set $\tilde{\mathcal{T}}$ by a suitable set $\mathcal{S}_{\nabla u} \subset \tilde{\mathcal{T}}$ of mutually non-parallel vectors. This proves (22).

By the Lebesgue Differentiation Theorem, we have $\chi_{\nabla u,T} = \chi^*_{\nabla u,T}$ almost everywhere in S. Hence, since $\mathcal{S}_{\nabla u}$ is countable, we can find a common null set $A \subset S$ such that $\chi_{\nabla u,T}(x) = \chi^*_{\nabla u,T}(x)$ for all $x \in S \setminus A$ and all $T \in \mathcal{S}_{\nabla u}$, and thus (23) follows.

We finally prove (21). Set

$$E := S \setminus (C_{\nabla u} \cup Z_{\nabla u} \cup K_{\nabla u}),$$

and let $x \in E$. Then there exists $T \in \mathcal{S}^1$ such that $0 < \chi^*_{\nabla u,T}(x) < 1$. By the same reasoning as above, we deduce that $T \in \mathcal{T}$. Since this is true for any $x \in E$, we get $E \subset \bigcup_{T\in\mathcal{T}}\{0 < \chi^*_{\nabla u,T} < 1\}$. Since indicator functions $\{0,1\}$-valued, the Lebesgue Differentiation Theorem implies that $\{0 < \chi^*_{\nabla u,T} < 1\}$ is a null set, and thus E is contained in a countable union of null sets, and thus a null set itself. □

Remark 2 We are grateful to an anonymous referee, who pointed out to us that neither $Z_{\nabla u}$ nor $K_{\nabla u}$ can be sensibly defined as open sets. Indeed, these sets (defined as above) could be of positive measure, but not contain any open ball – i.e., they could be of fat Cantor type.

2.2 Two-scale convergence.

Let $Y = [0,1)^2$ denote the unit cell in $\mathbb{R}^2$, and let $\mathcal{Y} := \mathbb{R}^2/\mathbb{Z}^2$ denote the unit torus. We denote by $C(\mathcal{Y})$ (resp. $C^\infty(\mathcal{Y})$) the space of continuous (resp. smooth) functions on the torus. We tacitly identify functions in $C(\mathcal{Y})$ (resp. $C^\infty(\mathcal{Y})$) with continuous (resp. smooth), Y-periodic, functions on $\mathbb{R}^2$. We denote by $L^2(\mathcal{Y})$ (resp. $W^{1,2}(\mathcal{Y})$) the closure of $C^\infty(\mathcal{Y})$ as a subspace of $L^2_{\mathrm{loc}}(\mathbb{R}^2)$ (resp. $W^{1,2}_{\mathrm{loc}}(\mathbb{R}^2)$). Note that $L^2(Y) \simeq L^2(\mathcal{Y})$, while $W^{1,2}(\mathcal{Y}) \neq W^{1,2}(Y)$. From [Ngu89] and [All92] we cite the definition of weak two-scale convergence:

Definition 2 A bounded sequence $w^\varepsilon \in L^2(S)$ *weakly two-scale converges* to $w \in L^2(S \times \mathcal{Y})$ if and only if

$$\lim_{\varepsilon\downarrow 0} \int_S w^\varepsilon(x)\psi(x, x/\varepsilon)dx = \int_{S\times Y} w(x,y)\psi(x,y)\,dx\,dy \quad \forall \psi \in C_0^\infty(S \times \mathcal{Y}).$$

Then we write $w^\varepsilon \stackrel{2}{\rightharpoonup} w$ in $L^2(S \times \mathcal{Y})$. If the sequence satisfies in addition

$$\lim_{\varepsilon\downarrow 0} \int_S |w^\varepsilon(x)|^2 dx = \int_{S\times Y} |w(x,y)|^2\,dx\,dy$$

then we say that w^ε is *strongly two-scale convergent* to w and write $w^\varepsilon \stackrel{2}{\to} w$. For vector valued functions we define weak and strong two-scale convergence component-wise.

The following result can be found in [All92]. It is an elementary but fundamental property of two-scale convergence and allows to pass to the limit in products of weakly convergent sequences.

Lemma 4 *Let $S \subset \mathbb{R}^2$ be open and bounded. Consider two sequences w^ε and ψ^ε that are bounded in $L^2(S)$, and suppose that $w^\varepsilon \xrightarrow{2} w$ strongly two-scale and $\psi^\varepsilon \xrightharpoonup{2} \psi$ weakly two-scale in $L^2(S \times \mathcal{Y})$. Then*

$$\int_S w^\varepsilon(x)\psi^\varepsilon(x)\,dx \to \int_{S\times Y} w(x,y)\psi(x,y)\,dydx.$$

The following lemma can be found as Proposition 2.12 in [Vis06] and is helpful for the computation of strong two-scale limits for products.

Lemma 5 *Let $p, q \geq 1$, and let v^ε, w^ε be sequences in $L^p(S)$, $L^q(\mathcal{Y})$ respectively, with $v^\varepsilon \to v$ in $L^p(S)$ and $w^\varepsilon \to w$ in $L^q(\mathcal{Y})$. Then*

$$v^\varepsilon(x)w^\varepsilon(x/\varepsilon) \xrightarrow{2} v(x)w(y) \quad \text{in } L^r(S\times\mathcal{Y}),$$

where $r^{-1} = p^{-1} + q^{-1}$.

Two-scale convergence allows to conveniently pass to limits in convex functionals with periodic coefficients. The following lemma is a special case of [Vis07, Proposition 1.3]

Lemma 6 *Let $A \subset \mathbb{R}^2$ be open and bounded, and let Q satisfy Assumption 1.*

(a) Suppose that $G^\varepsilon \in L^2(A, \mathbb{R}^{2\times 2})$ weakly two-scale converges to $G \in L^2(A\times\mathcal{Y}, \mathbb{R}^{2\times 2})$. Then

$$\liminf_{\varepsilon\downarrow 0} \int_A Q(\tfrac{x}{\varepsilon}, G^\varepsilon(x))\,dx \geq \int_{A\times Y} Q(y, G(x,y))\,dydx.$$

(b) Suppose that $G^\varepsilon \in L^2(A, \mathbb{R}^{2\times 2})$ strongly two-scale converges to $G \in L^2(A\times\mathcal{Y}, \mathbb{R}^{2\times 2})$. Then

$$\lim_{\varepsilon\downarrow 0} \int_A Q(\tfrac{x}{\varepsilon}, G^\varepsilon(x))\,dx = \int_{A\times Y} Q(y, G(x,y))\,dydx.$$

3 Two-scale limits of second fundamental forms

In this section we analyze the structure of two-scale limits of second fundamental forms. We consider the following generic situation:

(LB) Let u^ε be a sequence in $W^{2,2}_{\text{iso}}(S)$, let $u \in W^{2,2}_{\text{iso}}(S)$, and let $G \in L^2(S\times\mathcal{Y}, \mathbb{R}^{2\times 2})$. Suppose that

$$\begin{cases} u^\varepsilon \rightharpoonup u & \text{weakly in } W^{2,2}(S), \\ \boldsymbol{I\!I}^\varepsilon \xrightharpoonup{2} \boldsymbol{I\!I}(x) + G(x,y) & \text{weakly two-scale in } L^2(S\times\mathcal{Y}, \mathbb{R}^{2\times 2}), \end{cases} \tag{25}$$

as $\varepsilon \downarrow 0$.

(Note that (LB) is generic, since from every sequence $u^\varepsilon \in W^{2,2}_{\text{iso}}(S)$ that is bounded in $W^{2,2}(S)$ we may extract a subsequence that satisfies (LB)). The two-scale field G captures certain modes of oscillations of $\boldsymbol{I\!I}^\varepsilon$ that emerge in the limit $\varepsilon \downarrow 0$. Our goal is to understand and identify the structure of G.

Some information on G can easily be obtained by standard results of two-scale convergence: As a consequence of (16) and (17) we may represent the second fundamental form of an arbitrary isometry as the Hessian of a scalar field. In particular, $\boldsymbol{I\!I}^\varepsilon = \nabla^2\varphi^\varepsilon$ for some $\varphi^\varepsilon \in W^{2,2}(S)$. As an immediate consequence, we find that $G(x,y) = \nabla_y^2\psi(x,y)$ where $\psi \in L^2(S, H^2(\mathcal{Y}))$. However, this simple reasoning, which does not exploit the nonlinear constraint (1), is far from being optimal. In fact, below we show that oscillations of $\boldsymbol{I\!I}^\varepsilon$ on scale ε are suppressed in regions where the limiting isometric immersion u is neither cylindrical nor flat. Moreover, we prove that at points where u is cylindrical, oscillations on scale ε can only emerge perpendicular to asymptotic directions.

Our findings are summarized in the upcoming result, which is the main tool in proving the lower bound for the Γ-convergence result.

Proposition 2 *Suppose (LB). Then the following properties hold:*

(a) (conical case). $G = 0$ *almost everywhere in* $K_{\nabla u} \times \mathcal{Y}$.

(b) (cylindrical case). Let $\mathcal{S}_{\nabla u}$ *denote the set introduced in Lemma 3. Then for each* $T \in \mathcal{S}_{\nabla u} \cap \mathcal{S}^1_*$ *there exists a function* $\alpha_T \in L^2(S, W^{1,2}_{T\text{-}per}(\mathbb{R}))$ *such that*

$$\chi_{\nabla u}(x)G(x,y) = \sum_{T \in \mathcal{S}_{\nabla u} \cap \mathcal{S}^1_*} \chi_{\nabla u,T}(x)\partial_s \alpha_T(x, T \cdot y)\Big(T \otimes T\Big) \qquad \text{for a.e. } (x,y) \in S \times \mathcal{Y}. \tag{26}$$

Here $\partial_s \alpha_T$ *denotes the derivative of* α_T *w. r. t. its second component.*

(The proof is postponed to the end of this section.)

Remark 3 1. In the proof of Theorem 1 the preceding proposition is used to establish the lower-bound part of the Γ-convergence statement. The proposition yields a characterization of the possible two-scale limits of $\boldsymbol{I\!I}^\varepsilon$. The characterization on non-flat regions of u is optimal. Yet, regarding flat regions, Proposition 2 is partial, since there it does not yield any detailed information on the behavior of $G(x,y)$. Still, Proposition 2 is sufficient for identifying the Γ-limit in the proof of Theorem 1.

2. We would like to emphasize that on the right-hand side of (26) only directions $T \in \mathcal{S}^1$ in rational directions appear. In particular, (26) says that on the (possibly non-negligible) set

$$\Big\{ x \in S \,:\, \sum_{T \in \mathcal{S}_{\nabla u} \setminus \mathcal{S}^1_*} \chi_{\nabla u,T}(x) = 1 \Big\} \subset Z_{\nabla u}$$

the two-scale field G vanishes. This effect is due to the nature of two-scale convergence, which "resolves" only oscillations in rational directions and "filters out" oscillations in irrational directions. Let us remark that this behavior is beneficial for our purpose: Since the considered material is periodic, only oscillations adapted to the material's periodicity account for homogenization.

The crucial observation in the argument of Proposition 2 is that in the situation of (LB), the possible oscillations of $\boldsymbol{I\!I}^\varepsilon$ on the length scale ε are restricted to a very particular set, namely those parts of the domain where the asymptotic directions of the limit u agree with the direction of the oscillation. The following lemma expresses this fact on the level of G.

Lemma 7 *Suppose (LB), and let* $N : S \setminus C_{\nabla u} \to \mathcal{S}^1$ *denote the Lipschitz field associated with* u *via Proposition 1. Then for every* $k \in \mathbb{Z}^2 \setminus \{0\}$ *the function* $f_k : S \to \mathbb{R}$ *defined by*

$$f_k(x) := (1 - \tilde{\chi}_k(x)) \int_Y G(x,y) \exp(2\pi \mathrm{i} k \cdot y) dy,$$

$$\tilde{\chi}_k(x) := \begin{cases} 1 & \text{if } x \in C_{\nabla u} \text{ or if } x \in S \setminus C_{\nabla u} \text{ and } N(x) \cdot k = 0 \\ 0 & \text{else} \end{cases}$$

is identically 0 *almost everywhere.*

(The proof is postponed to the end of this section.)

The argument of this result makes use of several auxiliary lemmas, that we state next. First, we need to extend the field N of asymptotic directions, see Proposition 1, to the flat region. We only require a local extension to balls away from the boundary of S. This is the content of the upcoming Lemma 8, which – despite being elementary – plays a crucial role in our analysis.

Lemma 8 *Let* $u \in W^{2,2}_{iso}(S)$. *Consider a ball* B *with* $2B \subset S$. *Then there exists a Lipschitz continuous function* $N : B \to \mathcal{S}^1$ *such that for all* $x, y \in B$:

$$\nabla u \text{ is constant on } [x; N(x)]_B, \tag{27}$$

$$[x; N(x)]_{2B} \cap [y; N(x)]_{2B} \neq \emptyset \implies [x; N(x)] = [y; N(y)] \tag{28}$$

Moreover, we have

$$\mathrm{Lip}(N) \leq \frac{1}{\mathrm{radius}(B)}. \tag{29}$$

(The proof of Lemma 8 is postponed to the end of this section.)

Since the Lipschitz bound (29) only depends on the radius of B, and in particular not on the isometry u, we get the following compactness result:

Corollary 1 *Let B denote a ball with $2B \subset S$. Consider a sequence $u^\varepsilon \in W^{2,2}_{iso}(S)$ and let $N^\varepsilon : B \to \mathcal{S}^1$ denote the Lipschitz function associated with u^ε via Lemma 8. Then there exists a Lipschitz function $\tilde N : B \to \mathcal{S}^1$ and $\tilde\mu \in L^2(B \times \mathcal{Y})$ such that (up to subsequences)*

$$N^\varepsilon \otimes N^\varepsilon \to \tilde N \otimes \tilde N \qquad \text{uniformly in } B, \tag{30}$$

$$\boldsymbol{I\!I}^\varepsilon \overset{2}{\rightharpoonup} \tilde\mu(x,y)\left(\tilde N^\perp(x) \otimes \tilde N^\perp(x)\right) \qquad \text{two-scale in } L^2(B \times \mathcal{Y}). \tag{31}$$

Moreover, if $u^\varepsilon \rightharpoonup u$ weakly in $W^{2,2}(S,\mathbb{R}^3)$ and $N : B \to \mathcal{S}^1$ is associated with u via Lemma 8, then we have

$$\tilde N \otimes \tilde N = N \otimes N \qquad \text{in } B \setminus C_{\nabla u}. \tag{32}$$

(The proof of Corollary 1 is postponed to the end of this section.)

The following is a standard construction of the so-called line of curvature coordinates; see e. g. [Hor11b] and [Hor11a].

Lemma 9 *Let $u \in W^{2,2}_{iso}(S)$ and let $\boldsymbol{I\!I}$ denote its second fundamental form. Let $B = B(x_0, R)$ denote a ball with $2B \subset S$, and denote by $N : B \to \mathcal{S}^1$ the Lipschitz field associated with u according to Lemma 8.*

(i) There exists a function $\Gamma \in W^{2,\infty}([-R,R],B)$ with

$$\Gamma(0) = x_0, \qquad (\Gamma)'(t) = -(N^\varepsilon)^\perp(\Gamma(t)) \qquad \text{for all } t \in [-R,R],$$

and additionally

$$\max_{t\in[-R,R]} |\kappa(t)| \le \frac{1}{R} \tag{33}$$

where $\kappa(t) := \Gamma''(t) \cdot N(\Gamma(t))$.

(ii) For $(t,s) \in Q := (-\frac{R}{2}, \frac{R}{2})^2$ define $\Phi(t,s) := \Gamma(t) + sN(\Gamma(t))$. Then the map $\Phi : Q \to \Phi(Q)$ is one-to-one and Lipschitz continuous with

$$\mathrm{Lip}(\Phi) \le 2, \qquad \frac{1}{2} \le \det \nabla\Phi \le 2, \tag{34}$$

and satisfies

$$\tfrac{1}{4}B \subset \Phi(Q) \subset S. \tag{35}$$

Moreover, there exists $\kappa_n \in L^2((-\frac{R}{2}, -\frac{R}{2}))$ such that

$$\boldsymbol{I\!I}(\Phi(t,s)) = \frac{\kappa_n(t)}{1 - s\kappa(t)}\,\Gamma'(t) \otimes \Gamma'(t), \tag{36}$$

$$\boldsymbol{I\!I}(\Phi(t,s))|\det \nabla\Phi(t,s)| = \kappa_n(t)\,\Gamma'(t) \otimes \Gamma'(t), \tag{37}$$

almost everywhere in Q.

(The proof of Lemma 9 is postponed to the end of this section.)

After these preparations, we can start with the proofs of Lemma 7 and Proposition 2 in earnest.

Proof (of Lemma 7) Let $\tilde B$ be a ball such that $2\tilde B \subset S$. We will show that

$$f_k = 0 \qquad \text{a.e. in } \tilde B. \tag{38}$$

Since S can be covered by countably many of such balls, this proves the claim of the lemma.

We denote by $N^\varepsilon : \tilde B \to \mathcal{S}^1$ the Lipschitz function associated with u^ε according to Lemma 8. Thanks to Corollary 1 we may assume (by possibly passing to a subsequence) that there exists a Lipschitz field $N^0 : \tilde B \to \mathcal{S}^1$ such that $N^\varepsilon \otimes N^\varepsilon \to N^0 \otimes N^0$ uniformly in $\tilde B$ as $\varepsilon \downarrow 0$.

Step 1. Decomposition of the domain.

For $\varepsilon \geq 0$ and $\delta > 0$ define the sets

$$A_k^{\varepsilon,\delta} := \{\, x \in \tilde{B} \,:\, |N^\varepsilon(x) \cdot k| < \delta \,\},$$
$$A_k^{0,0} := \{\, x \in \tilde{B} \,:\, N^0(x) \cdot k = 0\}.$$

We write $\chi_k^{\varepsilon,\delta}$ for the characteristic function associated to $A_k^{\varepsilon,\delta}$. Note that

$$\begin{aligned} \chi_k^{\varepsilon,\delta} &\to \chi_k^{0,\delta} \text{ pointwise as } \varepsilon\downarrow 0\,, \\ \chi_k^{0,\delta} &\to \chi_k^{0,0} \text{ pointwise as } \delta\downarrow 0\,. \end{aligned} \tag{39}$$

The former is just a consequence of the uniform convergence $N^\varepsilon \otimes N^\varepsilon \to N^0 \otimes N^0$, and the latter is obvious from the definitions.

Recall that $N : S \setminus C_{\nabla u} \to \mathcal{S}^1$ denotes the vector field associated with u via Proposition 1. By (32) we have $N^0 || N$ on $\tilde{B} \setminus C_{\nabla u}$. Hence, in the definition of $\tilde{\chi}_k$ we may replace N by N^0, so that $A_k^{0,0} \subset \tilde{B} \cap \{\tilde{\chi}_k = 1\}$. Consequently, for (38), it suffices to show that the function

$$\tilde{f}_k(x) := (1 - \chi_k^{0,0}(x)) \int_Y G(x,y) \exp(2\pi \mathrm{i} k \cdot y) dy$$

is identically 0 almost everywhere in $\tilde{B}$. To show the latter, it is enough to prove $\int_B \tilde{f}_k(x)dx = 0$ for every ball B satisfying $4B \subset \tilde{B}$, since $\tilde{B}$ can be finely covered by such balls.
From now on let B be such a ball. As a consequence of (39) and Lemma 5, we have

$$\chi_k^{\varepsilon,\delta}(x) \exp\Big(\frac{2\pi \mathrm{i} k \cdot x}{\varepsilon}\Big) \overset{2}{\to} \chi_k^{0,\delta}(x) \exp(2\pi \mathrm{i} k \cdot y) \quad \text{in } L^2(B \times \mathcal{Y}) \quad \text{as } \varepsilon\downarrow 0\,. \tag{40}$$

In combination with Lemma 4, and since $\int_Y \exp(2\pi \mathrm{i} k \cdot y)\, dy = 0$ we get

$$\begin{aligned} &\lim_{\varepsilon\downarrow 0} \int_B \chi_k^{\varepsilon,\delta}(x) \boldsymbol{\Pi}^\varepsilon(x) \exp\Big(\frac{2\pi \mathrm{i} k \cdot x}{\varepsilon}\Big) dx \\ &= \int_{B\times Y} \chi_k^{0,\delta}(x)(\boldsymbol{\Pi}(x) + G(x,y)) \exp(2\pi \mathrm{i} k \cdot y) dx dy \\ &= \int_{B\times Y} \chi_k^{0,\delta}(x) G(x,y) \exp(2\pi \mathrm{i} k \cdot y) dx dy. \end{aligned}$$

Also, for any function $f \in L^1(S)$, we have by the continuity of the integral

$$\chi_k^{0,\delta} f \to \chi_k^{0,0} f \quad \text{in } L^1(S) \quad \text{as } \delta\downarrow 0\,.$$

Hence,

$$\begin{aligned} \int_B \tilde{f}_k &= \int_{B\times Y} (1 - \chi_k^{0,0}(x)) G(x,y) \exp(2\pi \mathrm{i} k \cdot y) dx dy \\ &= \lim_{\delta\downarrow 0} \lim_{\varepsilon\downarrow 0} \int_B (1 - \chi_k^{\varepsilon,\delta}(x)) \boldsymbol{\Pi}^\varepsilon(x) \exp\Big(\frac{2\pi \mathrm{i} k \cdot x}{\varepsilon}\Big) dx\,. \end{aligned} \tag{41}$$

Step 2. Conclusion.

In view of Step 1, in order to conclude the proof we only need to prove: for any $\delta > 0$ we have

$$\lim_{\varepsilon\downarrow 0} \int_B (1 - \chi_k^{\varepsilon,\delta}(x)) \boldsymbol{\Pi}^\varepsilon(x) \exp\Big(\frac{2\pi \mathrm{i} k \cdot x}{\varepsilon}\Big) dx = 0\,. \tag{42}$$

In the argument we make use of the line of curvature coordinates: An application of Lemma 9 to u^ε yields a chart

$$Q := (-2R, 2R), \qquad \Phi^\varepsilon : Q \to S, \qquad \Phi^\varepsilon(t,s) := \Gamma^\varepsilon(t) + sN^\varepsilon(\Gamma(t))$$

such that $B \subset \Phi^\varepsilon(Q) \subset S$. For brevity we set $N^\varepsilon(t) = N^\varepsilon(\Gamma^\varepsilon(t))$, $T^\varepsilon(t) = -N^\varepsilon(t)^\perp$, and write

$$\chi_B^\varepsilon(t,s) := \begin{cases} 1 & \Phi^\varepsilon(t,s) \in B, \\ 0 & \text{else,} \end{cases} \qquad \text{and} \qquad \rho^{\varepsilon,\delta}(t,s) := 1 - \chi_k^{\varepsilon,\delta}(\Phi^\varepsilon(t,s))$$

for the indicator functions of B and the complement of $A_k^{\varepsilon,\delta}$ in the new coordinates. With this notation the associated change of coordinates reads

$$\int_B (1-\chi_k^{\varepsilon,\delta})\boldsymbol{\Pi}^\varepsilon(x)\exp\Big(\frac{2\pi \mathrm{i}k\cdot x}{\varepsilon}\Big)dx \\ = \int_Q \chi_B^\varepsilon(t,s)\rho^{\varepsilon,\delta}(t,s)\boldsymbol{\Pi}^\varepsilon(\Phi^\varepsilon(s,t))\exp\Big(\frac{2\pi \mathrm{i}k\cdot \Phi^\varepsilon(t,s)}{\varepsilon}\Big)|\det\nabla\Phi^\varepsilon(s,t)|dsdt.$$

Using the definition of Φ^ε and (37) the right-hand side simplifies to

$$\int_Q \chi_B^\varepsilon(t,s)\rho^{\varepsilon,\delta}(t,s)\kappa_n^\varepsilon(t)T^\varepsilon(t)\otimes T^\varepsilon(t)\exp\left(\frac{2\pi ik\cdot\Gamma^\varepsilon(t)}{\varepsilon}\right)\exp\left(s\frac{2\pi ik\cdot N^\varepsilon(t)}{\varepsilon}\right)dsdt\,.$$

Since the field of asymptotic directions N^ε only depends on t (in the new coordinates), it follows from the definition of $A_k^{\varepsilon,\delta}$ that $\rho^{\varepsilon,\delta}(t,s)=\rho^{\varepsilon,\delta}(t)$ does not depend on s. Hence, we get

$$\int_B (1-\chi_k^{\varepsilon,\delta})\boldsymbol{\Pi}^\varepsilon(x)\exp\Big(\frac{2\pi \mathrm{i}k\cdot x}{\varepsilon}\Big)dx = \int_Q \chi_B^\varepsilon(t,s)f^\varepsilon(t)\partial_s G^\varepsilon(t,s)dsdt,$$

where

$$\begin{aligned} f_\varepsilon(t,s) =& \kappa_n^\varepsilon(t)T^\varepsilon(t)\otimes T^\varepsilon(t)\exp\left(\frac{2\pi ik\cdot\Gamma^\varepsilon(t)}{\varepsilon}\right),\\ G_\varepsilon(s,t) =& \rho^{\varepsilon,\delta}(t)\frac{\varepsilon}{2\pi \mathrm{i}N^\varepsilon(t)\cdot k}\exp\left(s\frac{2\pi ik\cdot N^\varepsilon(t)}{\varepsilon}\right). \end{aligned}$$

Note that G^ε is well-defined, since $|N^\varepsilon\cdot k|^{-1}\le\delta^{-1}$, whenever $\rho^{\varepsilon,\delta}$ is non-zero. Clearly, for (42) it suffices to prove

$$\lim_{\varepsilon\downarrow 0}\int_Q \chi_B^\varepsilon(t,s)f^\varepsilon(t)\partial_s G^\varepsilon(t,s)dsdt = 0. \tag{43}$$

To that end, we first claim that for all $t\in(-2R,2R)$:

$$\left|\int_{-2R}^{2R}\chi_B^\varepsilon(t,s)\partial_s G^\varepsilon(t,s)\,ds\right|\le 4\frac{\varepsilon}{\delta}. \tag{44}$$

Indeed, since B is convex, and $s\mapsto\Phi^\varepsilon(t,s)$ is linear, we deduce that $s\mapsto\chi_B^\varepsilon(t,s)$ is the indicator function of an open (possibly empty) interval, say $(s_1^\varepsilon(t),s_2^\varepsilon(t))\subset(-2R,2R)$. Hence, an integration by parts yields

$$\left|\int_{-2R}^{2R}\chi_B^\varepsilon(t,s)\partial_s G^\varepsilon(t,s)\,ds\right| = \left|\int_{s_1^\varepsilon(t)}^{s_2^\varepsilon(t)}\partial_s G^\varepsilon(t,s)\,ds\right|\le 2\|G^\varepsilon\|_{L^\infty(Q)}\le 4\frac{\varepsilon}{\delta},$$

which proves (44). By Fubini's theorem and the triangle inequality, we have

$$|\int_Q \chi_B^\varepsilon(t,s)f^\varepsilon(t)\partial_s G^\varepsilon(t,s)dsdt|\le\int_{-2R}^{2R}|f^\varepsilon(t)|\left|\int_{-2R}^{2R}\chi_B^\varepsilon(t,s)\partial_s G^\varepsilon(t,s)\,ds\right|\,dt.$$

To complete the proof it remains to argue that $\int_{-2R}^{2R}|f^\varepsilon(t)|\,dt$ is uniformly bounded in ε. Here comes the argument:

$$\begin{aligned} \int_{-2R}^{2R}|f^\varepsilon(t)|\,dt &= \frac{1}{4R}\int_Q|\kappa_n^\varepsilon(t)T^\varepsilon(t)\otimes T^\varepsilon(t)|\,dtds\\ &\overset{(37)}{=} \frac{1}{4R}\int_Q|\boldsymbol{\Pi}^\varepsilon(\Phi^\varepsilon(t,s))||\det\nabla\Phi^\varepsilon(t,s)|\,dtds\\ &= \frac{1}{4R}\int_{\Phi^\varepsilon(Q)}|\boldsymbol{\Pi}^\varepsilon(x)|dx \overset{\Phi^\varepsilon(Q)\subset S}{\le} \frac{1}{4R}\int_S|\boldsymbol{\Pi}^\varepsilon(x)|dx. \end{aligned}$$

Since $\boldsymbol{\Pi}^\varepsilon$ weakly converges in $L^2(S)$ as $\varepsilon\downarrow 0$, we deduce that the right-hand side is uniformly bounded in ε. □

Remark 4 As a consequence of (16) and (17) we may represent the second fundamental form $\boldsymbol{II}$ of an arbitrary $W^{2,2}$-isometry as $\boldsymbol{II} = \nabla^2\varphi$ where $\varphi \in W^{2,2}(S)$ is a scalar function that solves the degenerate Monge-Ampère equation

$$\det \nabla^2\varphi = 0. \tag{45}$$

Above $\nabla^2\varphi$ denotes the Hessian of φ. As in [Pak04], Proposition 1 can be reformulated for scalar functions that belong to the non-convex space

$$\mathcal{A} := \{\, \varphi \in W^{2,2}(S) \,:\, \det \nabla^2\varphi = 0 \,\}.$$

Without much effort we recover the result of Lemma 7 on the level of the functions $\varphi \in \mathcal{A}$; i. e. the following statement: Consider a sequence $\varphi^\varepsilon \in W^{2,2}(S)$ of solutions to (45) and assume that φ^ε weakly converges to some φ in $W^{2,2}(S)$, and $\nabla^2\varphi^\varepsilon$ converges weakly two-scale to $\nabla^2\varphi + G$ in $L^2(S \times \mathcal{Y})$. If the limit φ is locally equal to an affine function, i. e. for some open set $O \ni x_0$, $A \in \mathbb{R}^2$ and $a \in \mathbb{R}$ we have

$$\int_O |\varphi(x) - (A \cdot x + a)|^2 \, dx > 0 \,,$$

we write $x_0 \in C_{\nabla\varphi}$. For $k \in \mathbb{Z}^2 \setminus \{0\}$, we define

$$A_k := \{x \in S \setminus C_{\nabla\varphi} \,:\, \nabla^2\varphi(x) : k^\perp \otimes k^\perp = 0\} \,,$$

write χ_k for the associated characteristic function, and set $\tilde{\chi}_k = \chi_k + \chi_{C_{\nabla\varphi}}$. Then for every $k \in \mathbb{Z}^2 \setminus \{0\}$, the function

$$x \mapsto (1 - \tilde{\chi}_k(x)) \int_Y G(x, y) \exp(2\pi i k \cdot y) dy$$

is 0 almost everywhere.
Rephrased in that form, it is apparent that Lemma 7 entails a characterization of two-scale limits under the *nonlinear differential constraint* (45). Note that the interplay of two-scale convergence and *linear* differential constraints is reasonably well understood, see e.g. [FK10] for general results in that direction. In contrast, to our knowledge our result is the first treatment of a *nonlinear* differential constraint.

We are now ready for the proof of Proposition 2.

Proof (of Proposition 2)
Step 1. Argument for (a).

Since $\boldsymbol{II}^\varepsilon \rightharpoonup \boldsymbol{II}$ in $L^2(S)$, we have

$$\int_Y G(x, y) dy = 0 \text{ for a.e. } x \in S \,. \tag{46}$$

Recalling the definition of $\tilde{\chi}_k$ from Lemma 7, we have $\tilde{\chi}_k(x) = 0$ for all $k \in \mathbb{Z}^2 \setminus \{0\}$ and almost every $x \in K_{\nabla u}$. Hence, by the conclusion of that lemma and (46), we have $\int_Y G(x, y) \exp(2\pi i k \cdot y) dy = 0$ for almost every $x \in K_{\nabla u}$ and every $k \in \mathbb{Z}^2$. This implies $y \mapsto G(x, y)$ is identical to 0 in $L^2(\mathcal{Y})$ for almost every $x \in K_{\nabla u}$, which yields the claim.

Step 2. Argument for (b).

Since rational directions $T \in \mathcal{S}^1_*$ play a special role in our argument, set $\mathcal{S}_{\nabla u,*} := \mathcal{S}_{\nabla u} \cap \mathcal{S}^1_*$. Let B denote a ball with $2B \subset S$. Since S can be covered by countably many of such balls, it suffices to prove identity (26) for almost every $(x, y) \in B \times \mathcal{Y}$. Furthermore, thanks to (23), it suffices to show that for all $T \in \mathcal{S}_{\nabla u,*}$ there exists $\alpha_T \in L^2(B, W^{1,2}_{T\text{-per}}(\mathbb{R}))$ such that

$$\chi_{\nabla u}(x) G(x, y) = \sum_{T \in \mathcal{S}_{\nabla u,*}} \chi_{\nabla u,T}(x) \partial_s \alpha_T(x, T \cdot y)(T \otimes T) \qquad \text{for a.e. } (x, y) \in B \times \mathcal{Y}. \tag{47}$$

From now on all identities hold for almost every $(x, y) \in B \times \mathcal{Y}$ or for almost every $x \in B$, respectively.
We start our argument for (47) with an application of Corollary 1: By (31) and (32) there exists $\tilde{\mu} \in L^2(B \times \mathcal{Y})$ such that

$$\chi_{\nabla u}(x) \Big(\boldsymbol{II}(x) + G(x, y) \Big) = \chi_{\nabla u}(x) \tilde{\mu}(x, y) (N^\perp(x) \otimes N^\perp(x)).$$

Due to the definition of $\chi_{\nabla u,T}$ and by (23) we find that

$$\chi_{\nabla u}(x)G(x,y) = \sum_{T\in\mathcal{S}_{\nabla u}} \chi_{\nabla u,T}(x)\mu(x,y)(T\otimes T),$$

where $\mu(x,y) := \tilde{\mu}(x,y) - \int_Y \tilde{\mu}(x,y)\,dy$. Hence, in order to deduce (47) we only need to show that

$$\chi_{\nabla u,T}(x)\mu(x,y) = \begin{cases} \chi_{\nabla u,T}(x)\partial_s\alpha_T(x,T\cdot y) & \text{if } T\in\mathcal{S}_{\nabla u,*}, \\ 0 & \text{if } T\in\mathcal{S}_{\nabla u}\setminus\mathcal{S}_{\nabla u,*}. \end{cases} \tag{48}$$

Here comes the argument. First, we represent $\mu(x,y)$ via a Fourier-series w. r. t. y:

$$\mu(x,y) = \sum_{k\in\mathbb{Z}^2} a_k(x)\exp(2\pi i k\cdot y) \qquad \text{for some } a\in L^2(B,\ell^2(\mathbb{Z}^2)). \tag{49}$$

From $\int_Y \mu(x,y)\,dy = 0$ we deduce that $a_0 = 0$. Now recall the definition of $\tilde{\chi}_k$ from Lemma 7 and note that for all

$$k\in\mathbb{Z}^2\setminus\{0\} \text{ with } k^\perp\cdot T\neq 0, \tag{50}$$

we have

$$\chi_{\nabla u,T} = (1-\tilde{\chi}_k) \qquad \text{a.e. in } B. \tag{51}$$

Hence, an application of Lemma 7 shows that for all k satisfying (50) we have

$$\chi_{\nabla u,T}(x)\int_Y \mu(x,y)\exp(2\pi i k\cdot y)\,dy = 0, \tag{52}$$

and thus $\chi_{\nabla u,T}a_{-k} = 0$. If $T\in\mathcal{S}_{\nabla u}\setminus\mathcal{S}_{\nabla u,*}$, then (50) is satisfied for every $k\in\mathbb{Z}^2\setminus\{0\}$ and (48) follows. It remains to consider the case $T\in\mathcal{S}_{\nabla u,*}$. From (49) – (52) we learn that

$$\begin{aligned}\chi_{\nabla u,T}(x)\mu(x,y) &= \chi_{\nabla u,T}(x)\sum_{\substack{k\in\mathbb{Z}^2\setminus\{0\}\\ k||T}} a_k(x)\exp(2\pi i k\cdot y)\\ &= \chi_{\nabla u,T}(x)\partial_s\alpha_T(x,T\cdot y),\end{aligned}$$

where $\alpha_T\in L^2(B,W^{1,2}(S))$ is given explicitly by

$$\alpha_T(x,s) := \chi_{\nabla u,T}(x)\sum_{\substack{k\in\mathbb{Z}^2\setminus\{0\}\\ k||T}} \frac{a_k(x)}{2\pi i(k\cdot T)}\exp(2\pi i(k\cdot T)s).$$

Thanks to the elementary identity

$$(k\cdot T)(s+k'\cdot T) = (k\cdot T)s + k\cdot k' \in (k\cdot T)s+\mathbb{Z},$$

which holds for all $s\in\mathbb{R}$, $k\in\mathbb{Z}^2\setminus\{0\}$ with $k||T$, and $k'\in\mathbb{Z}^2$, we deduce that $\alpha_T(x,s)$ satisfies the required periodicity property in s, i.e. $\alpha_T\in L^2(B,W^{1,2}_{T\text{-per}}(\mathbb{R}))$. This completes the argument for (48), and the proof of the proposition. □

Finally, we present the proofs of the auxiliary results, Lemma 8 and Corollary 1.

Proof (of Lemma 8) **Step 1.** We claim that it suffices to construct a vector field $\tilde{N}: B\to\mathcal{S}^1$ that satisfies (27) and (28) (with N replaced by $\tilde{N}$) such that $F: B\to\mathbb{R}^{2\times 2}$, $F(x) := (\tilde{N}(x)\otimes\tilde{N}(x))$ is continuous. Here comes the argument: Since B is simply connected, there exists a continuous vector field $N: B\to\mathcal{S}^1$ with $F = N\otimes N$. Hence, it remains to check that N satisfies (29). To that end let $x,y\in B$. We need to show that

$$|N(x)-N(y)| \leq \frac{1}{\mathrm{radius}(B)}|x-y|. \tag{53}$$

We distinguish the following cases:

– If either $[x, N(x)] = [y, N(y)]$ or $[x, N(x)] \cap [y, N(y)] = \emptyset$, then $N(x)$ and $N(y)$ must be parallel. We argue that $N(x) = N(y)$, which means that (53) is trivially fulfilled. Indeed, if this were not the case, then $N(x)$ and $N(y)$ would be in different connected components of $\mathcal{S}^1 \setminus \{\pm(x-y)/|x-y|\}$. By the continuity of N and the fact that $[x, y]$ – the line segment connecting x and y – is contained in B, there would have to exist $z \in [x, y] \setminus \{x, y\}$ such that $N(z) \in \{\pm(x-y)/|x-y|\}$, and thus $[z; N(z)]_B \cap [x, N(x)]_B = \{x\} \neq \emptyset$ in contradiction to eq. (28).
– If $[x, N(x)] \neq [y, N(y)]$ and $[x, N(x)] \cap [y, N(y)] \neq \emptyset$, then the lines intersect in some point $A \in \mathbb{R}^2$. By elementary geometry and by appealing to the continuity of N as in the argument above, we deduce that

$$\text{either } N(x) = \tfrac{x-A}{|x-A|}, \quad N(y) = \tfrac{y-A}{|y-A|}$$
$$\text{or } N(x) = -\tfrac{x-A}{|x-A|}, \ N(y) = -\tfrac{y-A}{|y-A|}.$$

By (28) we necessarily have $A \notin 2B$, so that (assuming without loss of generality that $|x-A| \leq |y-A|$)

$$\begin{aligned}|N(x) - N(y)| \leq & \left| N(x) - \frac{|y-A|}{|x-A|} N(y) \right| \\ \leq & \frac{1}{|x-A|} |x - A - y + A| \\ \leq & \frac{1}{\text{radius}(B)} |x-y|.\end{aligned}$$

Step 2. Structure of the connected components of $C_{\nabla u} \cap B$.

Let U be a connected component of $C_{\nabla u} \cap B$. We claim that the boundary of U in B can be written as the union of at most 2 disjoint line segments, and the corresponding lines do not intersect in $2B$, that is: there exists $k \in \{0, 1, 2\}$, and $x_i \in B$, $N_i \in \mathcal{S}^1$ for $1 \leq i \leq k$, such that

$$\partial U \cap B = \bigcup_{i=1}^{k} [x_i; N_i]_B, \tag{54}$$
$$[x_i; N_i]_{2B} \cap [x_j; N_j]_{2B} = \emptyset \text{ for } i \neq j. \tag{55}$$

We first define some notation that we are going to use in the argument. For distinct points $A, C \in \mathbb{R}^2$, let $\overline{AC}$ denote the line $\{A + t(C - A) : t \in \mathbb{R}\}$ and let $\overrightarrow{AC}$ denote the half line $\{A + t(C - A) : t \in [0, \infty)\}$. For pairwise distinct points $A, C, D \in \mathbb{R}^2$, let $\angle ACD$ denote the smaller angle enclosed by the half lines $\overrightarrow{CA}$ and $\overrightarrow{CD}$. We adopt the convention that all such angles are positive. Let the center of B be denoted by O.

Now, notice that the boundary of U in B has to be the union of open disjoint line segments since this is true for the boundary of $C_{\nabla u}$ in B by Proposition 1. Furthermore, the corresponding lines do not intersect in $2B$. This proves eqs. (54) and (55) for some $k \in \mathbb{N}$, and it remains to show that $k \leq 2$. Assume the contrary. Then there exist three lines L_1, L_2, L_3 such that (cf. Figure 1)

– $L_i \cap L_j \cap 2B = \emptyset$ for $i \neq j$
– $L_i \cap B \neq \emptyset$ for $i = 1, 2, 3$
– $\bigcup_{i=1}^{3} L_i \cap B \subset \partial U$

Let m_i, $i \in \{1, 2, 3\}$ be the midpoints of $L_i \cap B$. Since U is connected, either the L_i, $i = 1, 2, 3$ enclose a triangle $\triangle \subset \mathbb{R}^2$ or two of the lines are parallel and the third is not.

In the first case, let A_i be the corner of the triangle that is opposite to the side containing m_i, see Figure 2. Let $i, j \in \{1, 2, 3\}$, $i \neq j$. Since m_j is the midpoint of $L_j \cap B$, the line $\overline{Om_j}$ is orthogonal to L_j, and (see Figure 3)

$$\sin(\angle OA_i m_j) = |m_j - O|/|A_i - O| < 1/2.$$

The latter estimate holds since $m_j \in B$ and $A_i \notin 2B$ by assumption. Hence the enclosed angle is less than $\pi/6$. This is true for all pairs $i \neq j$. If (i, j, k) is some permutation of $(1, 2, 3)$, then

$$\angle m_i A_j m_k \leq \angle OA_j m_i + \angle OA_j m_k.$$

(Inequality occurs if O is outside $\triangle$.) The contradiction is obtained by using the fact that the sum of the angles in $\triangle$ is equal to π,

$$\pi = \angle m_1 A_2 m_3 + \angle m_2 A_1 m_3 + \angle m_3 A_2 m_1 < \pi.$$

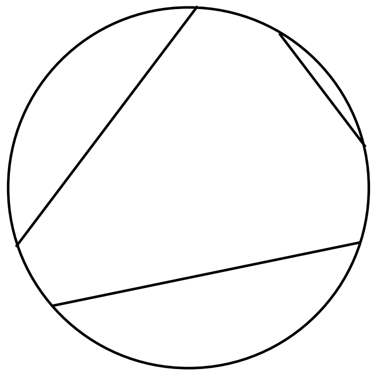

Fig. 1 Three line segments contained in ∂U.

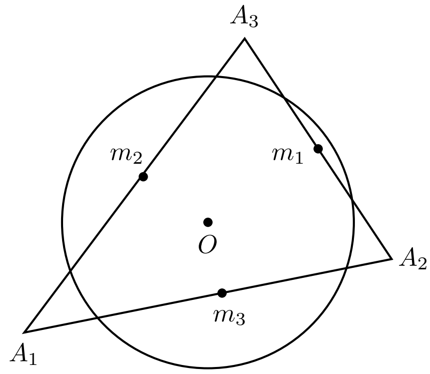

Fig. 2 The triangle $\triangle$ containing the line segments, and the ball B.

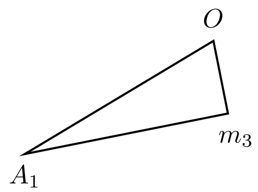

Fig. 3 The Sine of the angle enclosed by $\overrightarrow{A_1O}$ and $\overrightarrow{A_1m_3}$ is given by $|m_3 - O|/|A_1 - O|$. This ratio is smaller than 1/2 since $m_3 \in B$ and $A_1 \notin 2B$. Thus the angle is smaller than $\pi/6$.

In the case that two lines, say L_1 and L_2, are parallel, let m_i, $i \in \{1,2,3\}$ be as before, A_1 the point where L_2 and L_3 intersect, and A_2 the point where L_1 and L_3 intersect, see Figure 4.

With the same reasoning as before, the angles $\angle m_1 A_2 m_3$, $\angle m_3 A_1 m_2$ are both smaller than $\pi/3$. Since L_1 and L_2 are parallel, the sum of these angles has to be π, which produces the contradiction, and finishes the proof of (54) and (55) with $k \le 2$.

Step 3. Conclusion: Construction of $\tilde{N}$.

By Step 1, to complete the proof we only need to construct a vector field $\tilde{N} : B \to \mathcal{S}^1$ that satisfies (27) and (28) such that $F = \tilde{N} \otimes \tilde{N}$ is continuous on B. In the trivial case $C_{\nabla u} = B$ we simply set $\tilde{N} = e_1$. Suppose now that $C_{\nabla u} \neq B$. We define $\tilde{N}$ on $B \setminus C_{\nabla u}$ via Proposition 1. The thus defined $F = \tilde{N} \otimes \tilde{N}$ is continuous on $B \setminus C_{\nabla u}$ and $\tilde{N}$ satisfies (27) and (28) for $x, y \in B \setminus C_{\nabla u}$. On the remainder $B \cap C_{\nabla u}$ we define $\tilde{N}$ on each connected component U separately as described next. Note that on U (27) is trivially fulfilled. Since $U \neq B$, by Step 2 the boundary $\partial U \cap B$ consists of one or two connected components. If $\partial U \cap B = [x_1; N_1]_B$ for some $x_1 \in B$ and $N_1 \in \mathcal{S}^1$, we set $\tilde{N} = N_1$ on U. If $\partial U \cap B = [x_1; N_1]_B \cup [x_2; N_2]_B$ for some $x_1, x_2 \in B$ and $N_1, N_2 \in \mathcal{S}^1$, we distinguish two cases:

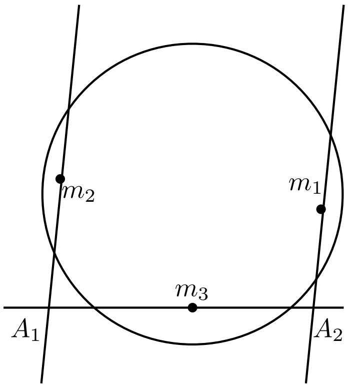

Fig. 4 The case of parallel line segments contained in ∂U.

- if N_1 and N_2 are not parallel, then there exists a unique $A \in [x_1; N_1] \cap [x_2; N_2]$ and we set $\tilde{N}(y) := (A - y)/|A - y|$ for $y \in U$;
- if N_1 and N_2 are parallel, then we set $\tilde{N} = N_1$.

The thus defined vector field $\tilde{N} : B \to \mathcal{S}^1$ satisfies (27) and (28) by construction. By step 1, it remains to show that $F = \tilde{N} \otimes \tilde{N}$ is continuous. We parallel the argument from step 1: If either $[x, \tilde{N}(x)] = [y, \tilde{N}(y)]$ or $[x, \tilde{N}(x)] \cap [y, \tilde{N}(y)] = \emptyset$ then $F(x) = F(y)$. Otherwise, the lines $[x, \tilde{N}(x)]$, $[y, \tilde{N}(y)]$ intersect in exactly one point $A \in \mathbb{R}^2$, and

$$\begin{aligned} F(x) &= \frac{x - A}{|x - A|} \otimes \frac{x - A}{|x - A|} \\ F(y) &= \frac{y - A}{|y - A|} \otimes \frac{y - A}{|y - A|}. \end{aligned}$$

By $A \notin 2B$, $|F(x) - F(y)| \leq \frac{2}{\text{radius}\, B}|x - y|$, which proves the continuity of F. □

Proof (of Corollary 1) **Step 1.** Argument for (30).

Since N^ε is a vector field of unit vectors, and since $\mathrm{Lip}(N^\varepsilon)$ is bounded uniformly in $\varepsilon > 0$, the sequence N^ε is bounded in $W^{1,\infty}(B, \mathbb{R}^2)$. Hence, $N^\varepsilon \overset{*}{\rightharpoonup} \tilde{N}$ weakly-star in $W^{1,\infty}$, up to a subsequence (that we do not relabel), and $\tilde{N} \in W^{1,\infty}(B, \mathbb{R}^2)$. Since $W^{1,\infty}(B, \mathbb{R}^2)$ is compactly embedded into the Hölder spaces $C^{0,\alpha}(B, \mathbb{R}^2)$, $0 \leq \alpha < 1$, the convergence holds uniformly and we deduce that $\tilde{N}(x) \in \mathcal{S}^1$ almost everywhere.

Step 2. Argument for (31).

Set $T^\varepsilon(x) := -(N^\varepsilon(x))^\perp$. By (18) we have

$$\boldsymbol{I\!I}^\varepsilon(x) = \mu^\varepsilon(x)\, T^\varepsilon(x) \otimes T^\varepsilon(x) \qquad \text{for some } \mu^\varepsilon \in L^2(B). \tag{56}$$

The sequence μ^ε is bounded in $L^2(B)$. Hence, we can pass (to a further) subsequence with $\mu^\varepsilon \overset{2}{\rightharpoonup} \tilde{\mu}(x, y)$ two-scale in $L^2(B \times \mathcal{Y})$. Combined with the uniform convergence $N^\varepsilon \to \tilde{N}$, (31) follows via Lemma 5.

Step 3. Argument for (32).

For convenience set $T := -N^\perp$. Note that (56) remains valid when the superscript ε is dropped. By assumption we have $u^\varepsilon \rightharpoonup u$ in $W^{2,2}$, and thus $\boldsymbol{I\!I}^\varepsilon \rightharpoonup \boldsymbol{I\!I}$ weakly in $L^2(S, \mathbb{R}^{2\times 2})$. Since $N^\varepsilon \otimes N^\varepsilon \to \tilde{N} \otimes \tilde{N}$ uniformly in B we obtain

$$\int_B \big(\boldsymbol{I\!I} : (\tilde{N} \otimes \tilde{N})\big)\varphi\, dx = \lim_{\varepsilon \downarrow 0} \int_B \big(\boldsymbol{I\!I}^\varepsilon : (N^\varepsilon \otimes N^\varepsilon)\big)\varphi\, dx,$$

for all $\varphi \in L^2(B)$. By orthogonality, the right-hand side vanishes, and thus

$$0 = \int_B \big(\boldsymbol{I\!I} : (\tilde{N} \otimes \tilde{N})\big)\varphi\, dx. \tag{57}$$

The combination of identity (56) (with the superscript ε dropped) and (57) (with $\varphi = \mu$) yields

$$0 = \int_B |\mu|^2 (T \otimes T) : (\tilde{N} \otimes \tilde{N})\, dx = \int_B |\mu|^2 |T \cdot \tilde{N}|^2\, dx.$$

Since $|\mu|^2 > 0$ almost everywhere in $B \setminus C_{\nabla u}$, the previous identity implies that $\tilde{N}$ and T are orthogonal in that region, and thus, by the continuity of $\tilde{N}$ and $N = T^{\perp}$, we obtain (32). □

Proof (of Lemma 9) (i) We will write $N(t) := N(\Gamma(t))$. The existence and regularity of the curve Γ follows from a standard fix point argument. Since $\Gamma'(t) = -N^{\perp}(\Gamma(t))$ is a unit vector, we deduce that $\Gamma''(t)$ is orthogonal to $\Gamma'(t)$ and thus parallel to $N(t)$. Hence, there exists an L^2 function $\kappa(t)$ such that $\Gamma''(t) = \kappa(t)N(t)$. We have for almost every t

$$|\kappa(t)| = |\Gamma''(t)| = |\nabla N(\Gamma(t))\Gamma'(t)| \le |\nabla N(\Gamma(t))| \le \mathrm{Lip}(N).$$

The estimate $\mathrm{Lip}(N) \le \frac{1}{R}$ (cf. (29)) completes the argument.
(ii) Let $(t, s), (t', s') \in Q$. Then

$$\begin{aligned}
|\Phi(t, s) - \Phi(t', s')| \le& |\Phi(t, s) - \Phi(t', s)| + |\Phi(t', s) - \Phi(t', s')| \\
\le& |\Gamma(t) - \Gamma(t')| + |N(t) - N(t')||s| + |N(t')||s - s'| \\
\le& |t - t'| + R^{-1}|t - t'|R/2 + |s - s'| \\
\le& 2|(t, s) - (t', s')|\,.
\end{aligned}$$

This proves the first estimate in (34). Hence, (28) implies that Φ is one-to-one. A direct calculation yields

$$\nabla\Phi = \Big(\Gamma'(t),\ N(t)\Big)\Big(\mathrm{Id} - s\kappa(t) e_1 \otimes e_1\Big),$$

Since (Γ', N) is a rotation, and $|s\kappa(t)| \le \frac{1}{2}$ by eq. (33), we get

$$\frac{1}{2} \le \det \nabla\Phi = 1 - s\kappa(t) \le 2\,. \tag{58}$$

This completes the proof of eq. (34).

A proof of the inclusion (35) can be found in [Hor11b, Remark 5]. For (36), see e. g. [Hor11a, Proposition 1]. Identity (37) follows from (36) combined with $|\det \nabla\Phi| = 1 - s\kappa(t)$. □

4 Proof of Theorem 1

4.1 Proof of Theorem 1 (a) & (b) – compactness and lower bound

Proof (of statement (a) – compactness) In view of the coercivity assumption (Q3) and Poincaré's inequality, any sequence u^ε with finite energy and mean zero is bounded in $W^{2,2}(S, \mathbb{R}^3)$. Hence, the statement follows from the observation that $W^{2,2}_{\mathrm{iso}}(S)$ is closed under weak convergence in $W^{2,2}(S, \mathbb{R}^3)$. □

Proof (of statement (b) – lower bound) By the compactness statement (a), we may assume without loss of generality that $u^\varepsilon, u \in W^{2,2}_{\mathrm{iso}}(S)$ and

$$u^\varepsilon \rightharpoonup u \qquad \text{weakly in } W^{2,2}(S, \mathbb{R}^3), \tag{59}$$

$$\boldsymbol{I\!I}^\varepsilon \rightharpoonup \boldsymbol{I\!I} \qquad \text{weakly in } L^2(S, \mathbb{R}^{2\times 2}), \tag{60}$$

$$\boldsymbol{I\!I}^\varepsilon \overset{2}{\rightharpoonup} \boldsymbol{I\!I} + G \qquad \text{weakly two-scale in } L^2(S \times \mathcal{Y}, \mathbb{R}^{2\times 2}), \tag{61}$$

where $\boldsymbol{I\!I}^\varepsilon$ and $\boldsymbol{I\!I}$ denote the second fundamental forms associated with u^ε and u, and $G(x, y)$ is a function in $L^2(S \times \mathcal{Y}, \mathbb{R}^{2\times 2})$. By Lemma 6 (a) we have

$$\liminf_{\varepsilon \downarrow 0} \int_S Q\big(\frac{x}{\varepsilon}, \boldsymbol{I\!I}^\varepsilon(x)\big)\, dx \ge \int_{S\times Y} Q(y, \boldsymbol{I\!I}(x) + G(x, y))\, dy dx.$$

Hence, it suffices to show that

$$\int_{S\times Y}(1-\chi_{\nabla u}(x))Q(y,\boldsymbol{I\!I}(x)+G(x,y))\,dydx \geq \int_S(1-\chi_{\nabla u}(x))Q_{\rm av}(\boldsymbol{I\!I}(x))\,dx, \tag{62}$$

$$\int_{S\times Y}\chi_{\nabla u}(x)Q(y,\boldsymbol{I\!I}(x)+G(x,y))\,dydx \geq \int_S\chi_{\nabla u}(x)Q_{\rm hom}(\boldsymbol{I\!I}(x))\,dx. \tag{63}$$

We start with (62). By (21) we have $\{\chi_{\nabla u}=0\}\subset C_{\nabla u}\cup K_{\nabla u}\cup E$ for some null set E. An application of Proposition 2 shows that $G=0$ almost everywhere on $K_{\nabla u}\times\mathcal{Y}$, so that

$$\begin{aligned}[\text{LHS of (62)}] &\geq \int_{S\setminus C_{\nabla u}}(1-\chi_{\nabla u}(x))\left(\int_Y Q(y,\boldsymbol{I\!I}(x))\,dy\right)\,dx\\ &= \int_S(1-\chi_{\nabla u}(x))Q_{\rm av}(\boldsymbol{I\!I}(x))\,dx.\end{aligned}$$

For the last identity we used that $Q_{\rm av}(\boldsymbol{I\!I}(x))=0$ almost everywhere in $C_{\nabla u}$.

It remains to prove (63). Let $\mathcal{S}_{\nabla u}\subset\mathcal{S}^1$ denote the set from Lemma 3, and recall that $\mathcal{S}_{\nabla u}$ is at most countable. From (23), Fubini's theorem, and the fact that the functions $\chi_{\nabla u,T}$ are $\{0,1\}$-valued, we deduce that

$$\begin{aligned}&\int_{S\times Y}\chi_{\nabla u}(x)Q\Big(y,\boldsymbol{I\!I}(x)+G(x,y)\Big)\,dydx\\ &=\sum_{T\in\mathcal{S}_{\nabla u}}\int_S\chi_{\nabla u,T}(x)\left(\int_Y Q\Big(y,\boldsymbol{I\!I}(x)+G(x,y)\Big)\,dy\right)\,dx.\end{aligned} \tag{64}$$

From (18) and Proposition 2 (b), we deduce that there exists $\mu\in L^2(S)$ and for all $T\in\mathcal{S}_{\nabla u}\cap\mathcal{S}^1_*$ a function $\alpha_T\in L^2(S,W^{1,2}_{T\text{-per}}(\mathbb{R}))$ such that for almost every $(x,y)\in S\times\mathcal{Y}$:

$$\chi_{\nabla u,T}(x)(\boldsymbol{I\!I}(x)+G(x,y))=\chi_{\nabla u,T}(x)\begin{cases}\mu(x)(T\otimes T) & \text{if } T\in\mathcal{S}_{\nabla u}\setminus\mathcal{S}^1_*,\\ (\mu(x)+\partial_s\alpha_T(x,T\cdot y))(T\otimes T) & \text{if } T\in\mathcal{S}_{\nabla u}\cap\mathcal{S}^1_*.\end{cases}$$

Hence, in view of the definition of $Q_{\rm hom}(T\otimes T)$, see (6), we have for all $T\in\mathcal{S}_{\nabla u}\cap\mathcal{S}^1_*$ and almost every $x\in S$:

$$\begin{aligned}\chi_{\nabla u,T}(x)\int_Y Q\Big(y,(\boldsymbol{I\!I}(x)+G(x,y))\Big)\,dy &\geq \chi_{\nabla u,T}(x)\mu^2(x)Q_{\rm hom}(T\otimes T)\\ &= \chi_{\nabla u,T}(x)Q_{\rm hom}(\mu(x)T\otimes T)\\ &= \chi_{\nabla u,T}(x)Q_{\rm hom}(\boldsymbol{I\!I}(x)),\end{aligned}$$

and similarly, for all $T\in\mathcal{S}_{\nabla u}\setminus\mathcal{S}^1_*$ and almost every $x\in S$:

$$\begin{aligned}\chi_{\nabla u,T}(x)\int_Y Q\Big(y,(\boldsymbol{I\!I}(x)+G(x,y))\Big)\,dy &\geq \chi_{\nabla u,T}(x)Q_{\rm av}(\boldsymbol{I\!I}(x))\\ &\overset{T\notin\mathcal{S}^1_*}{=} \chi_{\nabla u,T}(x)Q_{\rm hom}(\boldsymbol{I\!I}(x)).\end{aligned}$$

Combined with (64), the claimed inequality (63) follows. □

4.2 Proof of Theorem 1 (c) - construction of recovery sequences

The construction of the recovery sequence consists of two parts. In the first part, which is the heart of the matter, we locally modify u in order to recover the oscillatory effects of homogenization. This is done on what we call "patches", i.e. "regular" subdomains on which u can conveniently be described by line of curvature coordinates, see Definition 3. In a second part we apply an approximation scheme due to [Pak04], [Hor11b] and [Hor11a]. In these works the approximation of Sobolev isometries by smooth isometries is discussed, and as a central step it is shown that any Sobolev isometry can be approximated by isometries whose gradients are *finitely developable*, see below for the precise definition.

For the definition of a "patch", we introduce (as in [Pak04]) for $u \in W^{2,2}_{\text{iso}}(S)$ the set $\hat{C}_{\nabla u} \subset C_{\nabla u}$ as the union of all connected components $U \subset C_{\nabla u}$ with the property that $\partial U \cap S$ consists of more than two connected components. In [Pak04] it is shown that the field of asymptotic directions N can be extended to $S \setminus \hat{C}_{\nabla u}$.

This will not quite be enough for our purposes, since we wish to consider affine boundary conditions posed on a line segment. In order to treat the boundary condition (BC), we need the following variant of this statement:

Lemma 10 *Let $u \in W^{2,2}_{iso}(S)$. Then there exists a locally Lipschitz continuous vector field $N : S \setminus \hat{C}_{\nabla u} \to \mathcal{S}^1$ such that* (14a) *and* (14b) *hold for all $x, y \in S \setminus \hat{C}_{\nabla u}$. Moreover, if u satisfies* (BC)*, then we can chose N such that*

$$\begin{cases} \text{either } L_{BC} \subset \hat{C}_{\nabla u}, \\ \text{or } L_{BC} = [x, N(x)]_S \text{ for some } x \in S \setminus \hat{C}_{\nabla u}. \end{cases} \tag{65}$$

The proof of this and the following lemmas is postponed to the end of this section.

Remark 5 As stated in Remark 2 under Proposition 1 of [Hor11b], the choice of the vector field $N : S \setminus \hat{C}_{\nabla u} \to \mathcal{S}^1$ is non-unique. The lemma above makes a particular choice. The results of [Hor11b] do not depend on the choice of this vector field, cf. again the remark just mentioned. In particular, in the statement of Theorem 2 below, we may assume that N is the vector field constructed in Lemma 10.

Definition 3 We call an open set $V \subset S \setminus \hat{C}_{\nabla u}$ a **patch** for (u, N), if it can be parametrized by a single line of curvature chart $\Phi : M \to V$ in the following sense:

(a) there exist $\Gamma \in W^{2,\infty}([0,\ell], S \setminus \hat{C}_{\nabla u})$ with $\ell > 0$ such that

$$\Gamma'(t) = -N^{\perp}(\Gamma(t)), \qquad \Gamma'(t) \cdot \Gamma'(t') > 0$$

for all $t, t' \in [0, \ell]$.

(b) $V = \Phi(M)$ where

$$\begin{aligned} M &:= \{\, (t,s) \in (0,\ell) \times \mathbb{R} \,:\, \Gamma(t) + sN(\Gamma(t)) \in S \,\}, \\ \Phi : M \to V, &\qquad \Phi(t,s) := \Gamma(t) + sN(\Gamma(t)). \end{aligned}$$

The approximation of $u \in W^{2,2}_{\text{iso}}(S)$ mentioned above is carried out with the help of two theorems below, which we quote from [Hor11b, Hor11a]. They deliver the desired approximation in two steps: First, we approximate $u \in W^{2,2}_{\text{iso}}(S)$ by $u^{\delta} \in W^{2,2}_{\text{iso}}(S)$ such that ∇u^{δ} is **finitely developable**. This means that $\hat{C}_{\nabla u^{\delta}}$ consists of finitely many connected components, and each connected component $U \subset \hat{C}_{\nabla u^{\delta}}$ has the property that $\partial U \cap S$ consists of finitely many connected components.
In the second approximation step, $u \in W^{2,2}_{\text{iso}}(S)$ with finitely developable gradient is approximated by a map $u^{\delta'} \in W^{2,2}_{\text{iso}}(S)$, with the property that it can be parametrized by finitely many patches.

Proposition 3 ([Hor11a], Proposition 5) *Let $u \in W^{2,2}_{iso}(S)$. Then for every $\delta > 0$ there exists $u^{\delta} \in W^{2,2}_{iso}(S)$ with the following properties:*

(i) The gradient ∇u^{δ} is finitely developable.
(ii) $u^{\delta} = u$ on the set

$$\begin{aligned} S_{\delta} := \bigcup \Big\{ & [x; N(x)]_S \,:\, x \in E_{\delta} \setminus \hat{C}_{\nabla u} \Big\} \cup \\ & \bigcup \Big\{ U \,:\, U \text{ is a connected component of } \hat{C}_{\nabla u} \text{ with } U \cap E_{\delta} \neq \emptyset \Big\}, \end{aligned}$$

where $E_{\delta} := \{\, x \in S \,:\, \operatorname{dist}(x, \partial S) > \delta \,\}$. Moreover, u^{δ} is affine on every connected component of $S \setminus \bar{S}_{\delta}$.
(iii) $u^{\delta} \to u$ strongly in $W^{2,2}(S; \mathbb{R}^3)$ as $\delta \downarrow 0$.

Theorem 2 ([Hor11b], Theorem 2) *Let $u \in W^{2,2}_{iso}(S)$ with finitely developable gradient, let $V_1, \dots, V_m$ be the connected components of $\hat{C}_{\nabla u}$ and let $N : S \setminus \hat{C}_{\nabla u} \to \mathcal{S}^1$ be the vector field associated to u via Lemma 10. Then for all $\delta > 0$ there exists $n \in \mathbb{N}$ with $n \geq m$ and curves $\Gamma^{(k)} \in W^{2,\infty}([0, T_k]; S \setminus \hat{C}_{\nabla u})$ for $k = m+1, \dots, n$, such that, with*

$$V_k = S \cap \{\Gamma^{(k)}(t) + sN(\Gamma^{(k)}(t)) : t \in (0, T_k), s \in \mathbb{R}\}, \quad k = m+1, \dots, n\,,$$

the following holds true:

(i) $N(\Gamma^{(k)}(t)) \cdot (\Gamma^{(k)})'(t) = 0$ for $k = m+1, \dots, n$, $t \in [0, T_k]$.

(ii) We have

$$E_\delta := \{\, x \in S \,:\, \operatorname{dist}(x, \partial S) > \delta \,\} \subset int\left(\cup_{k=1}^n \bar{V}_k\right)\,.$$

(iii) Whenever $j, k \in \{1, \dots, n\}$ with $j \neq k$, then

$$V_j \cap V_k = \emptyset\,.$$

After having collected these results from the literature, we come to the heart of the recovery sequence construction – the construction on a single patch.

Lemma 11 (Construction on a single patch) *Let V be a patch for (u, N). Then there exists a sequence $u^\varepsilon \in W^{2,2}_{iso}(S)$ such that*

$$\lim_{\varepsilon \downarrow 0} \int_V Q(\tfrac{x}{\varepsilon}, \boldsymbol{I\!I}^\varepsilon(x))\,dx \tag{66a}$$

$$= \int_V (1 - \chi_{\nabla u}(x)) Q_{av}(\boldsymbol{I\!I}(x)) + \chi_{\nabla u}(x) Q_{\hom}(\boldsymbol{I\!I}(x))\,dx,$$

$$\boldsymbol{I\!I}^\varepsilon \rightharpoonup \boldsymbol{I\!I} \text{ weakly in } L^2(V), \tag{66b}$$

$$u^\varepsilon \text{ is affine on each line segment } [x; N(x)]_S,\ x \in \overline{V} \cap S. \tag{66c}$$

As already announced, the preceding lemma will be combined with Theorem 2 for the construction of the recovery sequence for the case of $u \in W^{2,2}_{\text{iso}}(S)$ with finitely developable gradient:

Lemma 12 (Construction in the finitely developable case) *Let $u \in W^{2,2}_{iso}(S)$ such that ∇u is finitely developable. Then there exists a sequence $u^\varepsilon \in W^{2,2}_{iso}(S)$ such that*

$$\lim_{\varepsilon \downarrow 0} \|u^\varepsilon - u\|_{L^2(S)} = 0, \tag{67a}$$

$$\lim_{\varepsilon \downarrow 0} \mathcal{E}^\varepsilon(u^\varepsilon) = \mathcal{E}^0(u), \tag{67b}$$

$$\text{if } u \text{ satisfies (BC), then } u^\varepsilon \text{ satisfies (BC).} \tag{67c}$$

The construction of the recovery sequence for arbitrary $u \in W^{2,2}_{\text{iso}}(S)$ satisfying the boundary condition (BC) is then achieved by combining Lemma 12 with Proposition 3. This is what we will do next; the proof of the theorem is followed by the proofs of the auxiliary results above.

Proof (of Theorem 1 (c)) We only need to consider the case with prescribed boundary conditions, since otherwise we might artificially introduce boundary conditions by introducing a line segment L_{BC} on which u is affine. Let $N : S \setminus \hat{C}_{\nabla u} \to \mathcal{S}^1$ be as in Lemma 10. We use Proposition 3 to approximate u by $u^\delta \in W^{2,2}_{\text{iso}}(S)$ with finitely developable gradient. We also adapt the definitions of E_δ, S_δ from the statement of that proposition. For the treatment of the boundary conditions, we shall always assume that $\delta > 0$ is so small that $L_{BC} \cap E_\delta \neq \emptyset$. Note that

$$E_\delta \subset S_\delta \qquad \text{and} \qquad L_{BC} \subset S_\delta. \tag{68}$$

The first inclusion directly follows from the definition of S_δ. The argument for the second inclusion is postponed to the end of this proof.
By Proposition 3, we have $\lim_{\delta \downarrow 0} \|u^\delta - u\|_{L^2(S)} = 0$,

$$u^\delta = u \text{ in } S_\delta, \tag{69}$$

and u^δ is affine on each connected component of $S \setminus \bar{S}_\delta$. Note that the latter implies that

$$|\boldsymbol{I\!I}^\delta(x)| \le |\boldsymbol{I\!I}(x)| \qquad \text{a.e. in } S. \tag{70}$$

Since u satisfies (BC), it follows from the second inclusion in (68) and (69) that u^δ satisfies (BC). Furthermore, (69) implies that $\chi_{\nabla u^\delta} = \chi_{\nabla u}$ and $\boldsymbol{I\!I}^\delta = \boldsymbol{I\!I}$ almost everywhere on S_δ. Hence,

$$\begin{aligned} &\mathcal{E}^0(u^\delta) - \mathcal{E}^0(u) \\ = &\int_{S\setminus S_\delta} (1-\chi_{\nabla u^\delta}(x))Q_{\rm av}(\boldsymbol{I\!I}^\delta(x)) + \chi_{\nabla u^\delta}(x)Q_{\rm hom}(\boldsymbol{I\!I}^\delta(x))\,dx \\ &- \int_{S\setminus S_\delta} (1-\chi_{\nabla u}(x))Q_{\rm av}(\boldsymbol{I\!I}(x)) + \chi_{\nabla u}(x)Q_{\rm hom}(\boldsymbol{I\!I}(x))\,dx. \end{aligned}$$

Because of $S \setminus S_\delta \subset S \setminus E_\delta$ (cf. (68)), $0 \le Q_{\rm hom}(F) \le Q_{\rm av}(F) \le \frac{1}{\alpha}|F|^2$ (cf. (3)), and (70), we estimate

$$|\mathcal{E}^0(u^\delta) - \mathcal{E}^0(u)| \le \frac{2}{\alpha}\int_{S\setminus E_\delta} |\boldsymbol{I\!I}(x)|^2\,dx,$$

and thus conclude

$$\lim_{\delta\downarrow 0}\big(\|u^\delta - u\|_{L^2(S)} + \big|\mathcal{E}^0(u^\delta) - \mathcal{E}^0(u)\big|\big) = 0. \tag{71}$$

Next, we apply Lemma 12: For each $\delta > 0$ there exists a sequence $u^{\delta,\varepsilon} \in W^{2,2}_{\rm iso}(S)$ such that each $u^{\delta,\varepsilon}$ satisfies (BC), and

$$\lim_{\varepsilon\downarrow 0}\big(\|u^{\delta,\varepsilon} - u^\delta\|_{L^2(S)} + \big|\mathcal{E}^\varepsilon(u^{\delta,\varepsilon}) - \mathcal{E}^0(u^\delta)\big|\big) = 0. \tag{72}$$

Combined with (71) we get

$$\lim_{\delta\downarrow 0}\lim_{\varepsilon\downarrow 0}\big(\|u^{\delta,\varepsilon} - u\|_{L^2(S)} + \big|\mathcal{E}^\varepsilon(u^{\delta,\varepsilon}) - \mathcal{E}^0(u)\big|\big) = 0.$$

By a standard diagonalization argument due to Attouch (see [Att84, Corollary 1.16]), there exists a map $\varepsilon \mapsto \delta(\varepsilon) \in \mathbb{N}$ such that

$$\lim_{\delta\downarrow 0}\lim_{\varepsilon\downarrow 0}\big(\|u^\varepsilon - u\|_{L^2(S)} + \big|\mathcal{E}^\varepsilon(u^\varepsilon) - \mathcal{E}^0(u)\big|\big) = 0.$$

Moreover, since each $u^{\delta,\varepsilon}$ satisfies (BC), the diagonal sequence u^ε satisfies (BC) as well.

To complete the proof, it remains to prove the second inclusion in (68), i.e. $L_{BC} \subset S_\delta$. If $L_{BC} \subset \hat{C}_{\nabla u}$, then there exists a connected component $U \subset \hat{C}_{\nabla u}$ that contains L_{BC}, and from $\emptyset \ne L_{BC} \cap E_\delta \subset U \cap E_\delta$ we deduce that $S_\delta \supset U \supset L_{BC}$ as claimed. Likewise, if $L_{BC} \not\subset \hat{C}_{\nabla u}$, then there exists $x \in L_{BC} \cap (S \setminus \hat{C}_{\nabla u})$. From Lemma 10 we infer that $L_{BC} = [x, N(x)]_S$. Since $L_{BC} \cap E_\delta \ne \emptyset$ we deduce $L_{BC} \subset S_\delta$ from the definition of S_δ.

□

In the remainder of this section we present the proofs of Lemma 10 – Lemma 12.

Proof (of Lemma 10) This is very similar to step 3 in the proof of Lemma 8, and we are going to be brief. We need to construct N on every connected component of $C_{\nabla u} \setminus \hat{C}_{\nabla u}$. Let U be such a connected component.

By (BC) we have either $L_{BC} \cap U = \emptyset$ or $L_{BC} \subset U$. First suppose that $L_{BC} \cap U = \emptyset$. Let L_1, L_2 be the two connected components of $\partial U \cap S$. Since $L_1, L_2 \subset S \setminus C_{\nabla u}$, N is defined there, and takes values N_1, N_2 respectively.

- if N_1 and N_2 are not parallel, then there exists a unique $A \in [x_1; N_1] \cap [x_2; N_2]$ and we set $N(y) := (A-y)/|A-y|$ for $y \in U$;
- if N_1 and N_2 are parallel, then we set $N(y) = N_1$ for $y \in U$.

Now suppose $L_{BC} \subset U$. Choose $x, \bar{N}$ such that $L_{BC} = [x; \bar{N}]_S$, and set $N(x) = \bar{N}$ on L_{BC}. Then subdivide U into the two connected components of $U \setminus L_{BC}$, and carry out the construction from the previous case.

In this way, we obtain a vector field $N : S \setminus \hat{C}_{\nabla u} \to \mathcal{S}$ with the property that $N \otimes N$ is locally Lipschitz (cf. the proof of Lemma 8). Since every connected component U of $S \setminus \hat{C}_{\nabla u}$ is simply connected, there exists a continuous lifting $\tilde{N} : U \to \mathcal{S}^1$. This defines the wished for vector field. □

Proof (of Lemma 11) Let $\Gamma \in W^{2,\infty}([0,\ell], S \setminus \hat{C}_{\nabla u})$ and $\Phi : M \to V$ be associated with V according to Definition 3. Set $L_1 := [\Gamma(0), N(\Gamma(0)]_S$ and note that L_1 is one of the two connected components of $\partial V \cap S$. To simplify the presentation, we say that an isometry $v \in W^{2,2}_{\text{iso}}(V)$ satisfies property (A), if

$$\begin{cases} v \text{ is affine on each line segment } [x, N(x)]_S \text{ for all } x \in \overline{V} \cap S, \\ v = u \text{ and } \nabla v = \nabla u \text{ on } L_1 \end{cases} \tag{A}$$

By virtue of the definition of N, see Lemma 10, u itself satisfies property (A).

Step 1. A reduction step.

We claim that it suffices to prove the following statement:

(S) For arbitrary $J \in \mathbb{N}$, mutually non-parallel vectors $T_1, \ldots, T_J \in \mathcal{S}^1$, and functions $\alpha_j \in C^\infty_{T_j\text{-per}}(\mathbb{R})$, $j = 1, \ldots, J$, there exists a sequence $v^\varepsilon \in W^{2,2}_{\text{iso}}(V)$ such that v^ε satisfies property (A) and the associated fundamental form satisfies

$$\boldsymbol{II}^\varepsilon \stackrel{2}{\to} \Big(1 + \sum_{j=1}^{J} \chi_{\nabla u, T_j}(x) \alpha_j'(T_j \cdot y)\Big) \boldsymbol{II}(x). \tag{73}$$

Here comes the argument. Recall the definition of $\mathcal{S}_{\nabla u}$ from Lemma 3. Let $T_1, T_2, \ldots$ be an enumeration of $\mathcal{S}_{\nabla u}$. By definition we have $\chi^*_{\nabla u}(x) = \sum_{j=1}^{\infty} \chi_{\nabla u, T_j}(x)$ for almost every $x \in S$, and thus $\lim_{J\uparrow\infty} \int_S |(\sum_{j=1}^{J} \chi_{\nabla u, T_j}) - \chi^*_{\nabla u}| Q_{\text{hom}}(\boldsymbol{II})\, dx = 0$. Therefore, for every $\delta > 0$ we can find $J^\delta > 0$ and functions $\alpha_{\delta,j} \in C^\infty_{T_j\text{-per}}(\mathbb{R})$, $j = 1, \ldots, J^\delta$, such that

$$\begin{aligned} &\int_V \Big|\chi_{\nabla u}(x) - \Big(\sum_{j=1}^{J^\delta} \chi_{\nabla u, T_j}(x)\Big)\Big| Q_{\text{av}}(\boldsymbol{II}(x))\, dx \le \delta, \\ &\left| \int_V \chi_{\nabla u}(x) Q_{\text{hom}}(\boldsymbol{II}(x)) - \sum_{j=1}^{J^\delta} \chi_{\nabla u, T_j}(x) \int_Y Q(y, \alpha'_{\delta,j}(T_j \cdot y) \boldsymbol{II}(x))\, dy dx \right| < \delta. \end{aligned} \tag{74}$$

By assumption (S), there exists a sequence $v^{\delta,\varepsilon} \in W^{2,2}_{\text{iso}}(S)$ with

$$v^{\delta,\varepsilon} = u \text{ and } \nabla v^{\delta,\varepsilon} = \nabla u \text{ on } L_1, \tag{75}$$

and

$$\boldsymbol{II}^{\delta,\varepsilon} \stackrel{2}{\to} \Big(1 + \sum_{j=1}^{J^\delta} \chi_{\nabla u, T_j}(x) \alpha'_{\delta,j}(T_j \cdot y)\Big) \boldsymbol{II}(x) \qquad \text{as } \varepsilon \downarrow 0. \tag{76}$$

We finally claim that the sought sequence u^ε can be obtained as a diagonal sequence of $v^{\delta,\varepsilon}$. To that end set

$$\begin{aligned} e^{\delta,\varepsilon} &:= \int_V Q(\tfrac{x}{\varepsilon}, \boldsymbol{II}^{\delta,\varepsilon}(x))\, dx, \\ e^0 &:= \int_V (1 - \chi_{\nabla u}(x)) Q_{\text{av}}(\boldsymbol{II}(x)) + \chi_{\nabla u}(x) Q_{\text{hom}}(\boldsymbol{II}(x))\, dx, \end{aligned}$$

and consider

$$c(\delta, \varepsilon) := \|v^{\delta,\varepsilon} - u\|_{L^2(S)} + |e^{\delta,\varepsilon} - e^0|.$$

We shall prove that

$$\limsup_{\delta\downarrow 0} \limsup_{\varepsilon\downarrow 0} c(\delta, \varepsilon) = 0. \tag{77}$$

Indeed, (76) implies that $\boldsymbol{I\!I}^{\delta,\varepsilon} \rightharpoonup \boldsymbol{I\!I}$ weakly in $L^2(V)$ as $\varepsilon \downarrow 0$. Combined with(75) we deduce that $v^{\delta,\varepsilon} \to u$ strongly in $L^2(V)$ as $\varepsilon \downarrow 0$. It remains to show $\lim_{\delta\downarrow 0}\lim_{\varepsilon\downarrow 0} e^{\delta,\varepsilon} = e^0$. From Lemma 6 (b) and (76) we get

$$\begin{aligned}\lim_{\varepsilon\downarrow 0} e^{\delta,\varepsilon} &= \int_{V\times Y} Q\Big(y, \Big(1+\sum_{j=1}^{J^\delta}\chi_{\nabla u,T_j}(x)\alpha'_{\delta,j}(T_j\cdot y)\Big)\boldsymbol{I\!I}(x)\Big)\,dydx \\ &= \int_{V\times Y}\Big(1-(\sum_{j=1}^{J^\delta}\chi_{\nabla u,T_j}(x))\Big)Q\Big(y,\boldsymbol{I\!I}(x)\Big)\,dydx \\ &\quad + \int_{V\times Y}\sum_{j=1}^{J^\delta}\chi_{\nabla u,T_j}(x)Q\Big(y,\Big(1+\alpha'_{\delta,j}(T_j\cdot y)\Big)\boldsymbol{I\!I}(x)\Big)\,dydx.\end{aligned}$$

Combined with (74), (77) follows.

Finally, we deduce from (77), by appealing to a standard diagonalization argument (see [Att84]), that there exists a map $\varepsilon \mapsto \delta(\varepsilon)$ such that $c(\delta(\varepsilon),\varepsilon) \to 0$ as $\varepsilon \downarrow 0$. Hence, the diagonal sequence $u^\varepsilon := v^{\delta(\varepsilon),\varepsilon}$ strongly converges in $L^2(V)$ to u, and its energy satisfies $\lim_{\varepsilon\downarrow 0}\int_V Q(\frac{x}{\varepsilon},\boldsymbol{I\!I}^\varepsilon(x))\,dx = e^0$. Since this especially implies that the associated sequence of fundamental forms $\boldsymbol{I\!I}^\varepsilon$ is bounded in $L^2(V)$, we can upgrade the convergence of u^ε and deduce that $u^\varepsilon \rightharpoonup u$ weakly in $W^{2,2}(V)$ as claimed. This in particular implies that $\boldsymbol{I\!I}^\varepsilon \rightharpoonup \boldsymbol{I\!I}$ in weakly $L^2(V)$. Moreover, since each $u^{\delta(\varepsilon),\varepsilon}$ satisfies property (A), the same is true for u^ε.

The rest of the proof is devoted to show statement (S) in Step 1.

Step 2. Line of curvature parametrisation of $u|_V$.

Recall that

$$\Phi(t,s) := \Gamma(t) + sN(t), \qquad N(t) := N(\Gamma(t)), \qquad T(t) := -N^\perp(\Gamma(t)).$$

Following [Hor11b] we introduce the framed curve $(\gamma, R) : [0,\ell] \to \mathbb{R}^3 \times SO(3)$

$$\begin{aligned}&\gamma(t) := u(\Gamma(t)), \qquad \nu(t) := (\nabla u(\Gamma(t))N(t), \qquad n := \gamma'(t)\wedge\nu(t),\\ &R(t) := (\gamma'(t),\nu(t),n(t))^t.\end{aligned}$$

Then a direct computation shows that (e.g. see [Hor11b, Proposition 1])

$$\begin{aligned}u(\Phi(t,s)) &= \gamma(t) + s\nu(t),\\ \nabla u(\Phi(t,s)) &= \gamma'(t)\otimes T(t) + \nu(t)\otimes N(t),\\ \boldsymbol{I\!I}(\Phi(t,s)) &= \frac{\kappa_n(t)}{1-s\kappa(t)}(T(t)\otimes T(t)),\end{aligned}$$

with scalar curvatures

$$\kappa(t) := \Gamma''\cdot N, \qquad \kappa_n(t) := \gamma''(t)\cdot n(t),$$

and the frame R is the unique solution in $W^{1,2}((0,\ell),SO(3))$ to the system

$$R' = \begin{pmatrix} 0 & \kappa & \kappa_n\\ -\kappa & 0 & 0\\ -\kappa_n & 0 & 0\end{pmatrix} R, \qquad R(0) = (\gamma'(0),\nu(0),n(0))^t.$$

Step 3. Manipulation of κ_n.

We claim that for any $\theta \in L^\infty([0,\ell])$ there exists $u_\theta \in W^{2,2}_{\mathrm{iso}}(V)$ satisfying property (A), and

$$\boldsymbol{I\!I}_\theta(\Phi(t,s)) = \frac{(1+\theta(t))\kappa_n(t)}{1-s\kappa(t)}(T(t)\otimes T(t)). \tag{78}$$

Indeed, this follows from [Hor11b, Proposition 2]. For the convenience of the reader we briefly recall the construction: Let $R_\theta \in W^{1,2}((0,\ell),SO(3))$ be the unique solution to

$$R'_\theta = \begin{pmatrix} 0 & \kappa & (1+\theta)\kappa_n\\ -\kappa & 0 & 0\\ -(1+\theta)\kappa_n & 0 & 0\end{pmatrix} R_\theta, \qquad R_\theta(0) = (\gamma'(0),\nu(0),n(0))^t,$$

and define γ_θ, ν_θ, n_θ via

$$R_\theta = (\gamma'_\theta, \nu_\theta, n_\theta), \qquad \gamma_\theta(0) = \gamma(0).$$

Now the isometry $u_\theta : V \to \mathbb{R}^3$ is given by

$$u_\theta(\Phi(t,s)) = \gamma_\theta(t) + s\nu_\theta(t)$$

and its fundamental form satisfies (78). By construction u_θ satisfies property (A).

Step 4. Proof of statement (S).

Let $J \in \mathbb{N}$, $T_1, \ldots, T_J$ and α_j as in statement (S). For $\varepsilon > 0$ we define the function

$$\theta^\varepsilon(x) := \sum_{j=1}^{J} \chi_{\nabla u, T_j}(x) \alpha'_j(\tfrac{T_j \cdot x}{\varepsilon}).$$

Note that we have $\theta^\varepsilon \in L^\infty$, since the α_j's are smooth and the sum is finite. Since $N(\Phi(t,s))$ is independent of s, the function

$$\tilde{\theta}^\varepsilon(t,s) := \theta^\varepsilon(\Phi(t,s)), \qquad (t,s) \in M$$

is independent of s. Hence, an application of Step 3 shows that there exists an isometry $u^\varepsilon = u_{\theta^\varepsilon}$ satisfying property (A) and

$$\boldsymbol{II}^\varepsilon(\Phi(t,s)) = \frac{(1 + \tilde{\theta}^\varepsilon(t))\kappa_n(t))}{1 - s\kappa(t)} T(t) \otimes T(t).$$

With $\Phi(t,s) = x$, this can be rewritten as

$$\boldsymbol{II}^\varepsilon(x) = (1 + \theta^\varepsilon(x))\boldsymbol{II}(x). \tag{79}$$

Since $y \mapsto \alpha'_j(T_j \cdot y)$ is a $\mathcal{Y}$-periodic function, we have

$$\chi_{\nabla u, T_j}(x) \alpha'_j(\tfrac{T_j \cdot x}{\varepsilon}) \boldsymbol{II}(x) \overset{2}{\to} \chi_{\nabla u, T_j}(x) \alpha'_j(T_j \cdot y) \boldsymbol{II}(x)$$

strongly two-scale in $L^2(V \times \mathcal{Y})$ for $j = 1, \ldots, J$. Hence, (73) follows by superposition.

□

Proof (of Lemma 12) We only need to consider the case with prescribed boundary conditions. Let $N : S \setminus \hat{C}_{\nabla u} \to \mathcal{S}^1$ be as in Lemma 10. Here and below we assume that $\delta > 0$ is so small that $L_{BC} \cap E_\delta \neq \emptyset$, where $E_\delta := \{\, x \in S : \operatorname{dist}(x, \partial S) > \delta \,\}$.

By appealing to a diagonalization argument similar to the one in the proof of Theorem 1 (c), we only need to prove the following statement: For all $\delta > 0$ there exists a sequence $u^{\delta,\varepsilon} \in W^{2,2}_{\text{iso}}(S)$ such that $u^{\delta,\varepsilon}$ satisfies (BC) and

$$\limsup_{\delta \downarrow 0} \limsup_{\varepsilon \downarrow 0} \left(\|u^{\delta,\varepsilon} - u\|_{L^2(S)} + |\mathcal{E}^\varepsilon(u^{\delta,\varepsilon}) - \mathcal{E}^0(u)| \right) = 0\,. \tag{80}$$

Let us explain the construction of $u^{\delta,\varepsilon}$. By assumption, ∇u is finitely developable, and we may apply Theorem 2. Hence, there exists a finite number of mutually disjoint patches $V_1^{(\delta)}, \ldots, V_{m(\delta)}^{(\delta)}$ such that

$$E_\delta \setminus \hat{C}_{\nabla u} \subset \bigcup_{k=1}^{m(\delta)} \overline{V_k^{(\delta)}} =: V_\delta\,. \tag{81}$$

In view of Definition 3, the boundary $\partial V_k^{(\delta)} \cap S$ of each patch $V_k^{(\delta)}$ consists of two connected components. They are line segments of the form $[x, N(x)]_S$. Define

$$\mathcal{L}^{(\delta)} := \left\{\, L : L \text{ is a connected component of } \partial V_k^{(\delta)} \cap S \text{ for some } 1 \leq k \leq m(\delta) \,\right\} \cup \{L_{BC}\},$$

and note that u is affine on each $L \in \mathcal{L}^{(\delta)}$. We divide the rest of the argument into two steps.

Step 1. In this step δ is fixed. Hence we write $m(\delta) = m$, $V_k^{(\delta)} = V_k$, $\mathcal{L}^{(\delta)} = \mathcal{L}$. Also, the objects we introduce here will depend on δ, but we are going to suppress the superscript δ to alleviate the notation. Set $V_0 := \emptyset$. We claim that for $k = 0, \dots, m$ there exists a sequence $u_k^\varepsilon \in W^{2,2}_{\text{iso}}(S)$ such that

$$u_k^\varepsilon \text{ is affine on each } L \in \mathcal{L}, \tag{82a}$$

$$\boldsymbol{I\!I}_k^\varepsilon = \boldsymbol{I\!I}_{k-1}^\varepsilon \text{ a.e. on } S \setminus V_k \quad (\text{for } k > 0) \tag{82b}$$

for all $\varepsilon > 0$, and

$$\boldsymbol{I\!I}_k^\varepsilon \rightharpoonup \boldsymbol{I\!I} \text{ weakly in } L^2(S) \text{ as } \varepsilon \downarrow 0, \tag{82c}$$

$$\begin{aligned}&\lim_{\varepsilon\downarrow 0} \int_{V_k} Q(\tfrac{x}{\varepsilon}, \boldsymbol{I\!I}_k^\varepsilon(x))\,dx \\ &\quad = \int_{V_k} (1 - \chi_{\nabla u}(x)) Q_{\text{av}}(\boldsymbol{I\!I}(x)) + \chi_{\nabla u}(x) Q_{\text{hom}}(\boldsymbol{I\!I}(x))\,dx.\end{aligned} \tag{82d}$$

We construct u_k^ε inductively. The trivial sequence $u_0^\varepsilon := u$ clearly satisfies (82a) – (82d) for $k = 0$. Now assume that these properties are satisfied for some fixed index $0 \le k < m$ and a sequence u_k^ε. We apply Lemma 11 to the patch V_{k+1} and obtain a sequence $v^\varepsilon \in W^{2,2}_{\text{iso}}(V_{k+1})$ satisfying (66a) – (66c).

In the following we define u_{k+1}^ε by "merging" v^ε and u_k^ε. To that end let $\varepsilon > 0$ be fixed for a moment. We claim that there exists $\tilde{v} \in W^{2,2}_{\text{iso}}(S)$ that coincides with v^ε on V_{k+1}, and is equal (up to a rigid motion) with u_k^ε on each connected component of $S \setminus V_{k+1}$. Indeed, since V_{k+1} is a patch, its boundary $\partial V_{k+1} \cap S$ consists of two line-segments $L_1, L_2 \in \mathcal{L}$. Furthermore, due to the convexity of S, the set $S \setminus \overline{V}_{k+1}$ consists of two connected components U_1 and U_2. By (66c) and (82a) the functions v^ε and u_k^ε are affine on L_1 and L_2. Hence, there exist rigid motions $\varphi_1, \varphi_2 : \mathbb{R}^3 \to \mathbb{R}^3$ such that

$$u_{k+1}^\varepsilon(x) := \begin{cases} \varphi_1 \circ u_k^\varepsilon(x) & \text{if } x \in \overline{U}_1 \cap S \\ v^\varepsilon(x) & \text{if } x \in V_{k+1} \\ \varphi_2 \circ u_k^\varepsilon(x) & \text{if } x \in \overline{U}_2 \cap S \end{cases}$$

defines a function in $W^{2,2}_{\text{iso}}(S)$. We claim that for each $L \in \mathcal{L}$

$$u_{k+1}^\varepsilon \text{ is affine on } L. \tag{83}$$

For the argument we distinguish the two cases $L \cap V_{k+1} \neq \emptyset$ and $L \cap V_{k+1} = \emptyset$. In the latter case, the claim directly follows from property (82a) and the fact that affine maps remain affine under composition with a rigid motion. Since the patches are mutually disjoint, and lines in $\mathcal{L}$ do not intersect, the case $L \cap V_{k+1} \neq \emptyset$ is only possible, if $L = L_{BC}$. Hence, there exists $x_0 \in V_{k+1} \cap L_{BC}$. Since V_{k+1} is a patch, x_0 necessarily belongs to $S \setminus \hat{C}_{\nabla u}$, and thus $L_{BC} = [x_0, N(x_0)]_S$ due to the construction of N (see Lemma 10). Now the claim follows from (66c).

It remains to check that u_{k+1}^ε satisfies properties (82b) – (82d). Since the composition with a rigid motion does not change the second fundamental form, u_{k+1}^ε satisfies (82b), and properties (82c) and (82d) are inherited from properties (66a) and (66b) satisfied by v^ε.

Step 2. Construction of $u^{\delta,\varepsilon}$.

We set $u^{\delta,\varepsilon} := \varphi^{\delta,\varepsilon} \circ u^{\delta,\varepsilon}_{m(\delta)}$, where $u^{\delta,\varepsilon}_{m(\delta)} \equiv u^\varepsilon_m$ is the isometry constructed in Step 1, and $\varphi^{\delta,\varepsilon}$ is a rigid motion, which is chosen in such a way that $u^{\delta,\varepsilon}$ satisfies (BC). (Note that this is possible, since $u^{\delta,\varepsilon}_{m(\delta)}$ is affine on L_{BC} by (82a)). Recall the definition of V_δ, see (81). From (82a) – (82d) we learn that

$$\boldsymbol{I\!I}^{\delta,\varepsilon} = \boldsymbol{I\!I} \qquad \text{on } S \setminus V_\delta, \tag{84}$$

and as $\varepsilon \downarrow 0$:

$$\boldsymbol{I\!I}^{\delta,\varepsilon} \rightharpoonup \boldsymbol{I\!I} \qquad \text{weakly in } L^2(S) \text{ as } \varepsilon \downarrow 0, \tag{85}$$

$$\begin{aligned}&\lim_{\varepsilon\downarrow 0} \int_{V_\delta} Q(\tfrac{x}{\varepsilon}, \boldsymbol{I\!I}^{\delta,\varepsilon}(x))\,dx \\ &\quad = \int_{V_\delta} (1 - \chi_{\nabla u}(x)) Q_{\text{av}}(\boldsymbol{I\!I}(x)) + \chi_{\nabla u}(x) Q_{\text{hom}}(\boldsymbol{I\!I}(x))\,dx.\end{aligned} \tag{86}$$

Since $u^{\delta,\varepsilon}$ satisfies (BC), we deduce from (85) that

$$\|u^{\delta,\varepsilon} - u\|_{L^2(S)} \to 0 \quad \text{as } \varepsilon \downarrow 0\,. \tag{87}$$

Next we estimate the difference $\mathcal{E}^\varepsilon(u^{\delta,\varepsilon}) - \mathcal{E}^0(u)$. From (84) and (86) we deduce that

$$\begin{aligned}
&\lim_{\varepsilon\downarrow 0} \mathcal{E}^\varepsilon(u^{\delta,\varepsilon})\\
&= \int_{V_\delta} (1-\chi_{\nabla u}(x))Q_{\mathrm{av}}(\boldsymbol{I\!I}(x)) + \chi_{\nabla u}(x)Q_{\mathrm{hom}}(\boldsymbol{I\!I}(x))\,dx\\
&\quad + \int_{S\setminus V_\delta} Q_{\mathrm{av}}(\boldsymbol{I\!I}(x))\,dx\\
&= \int_S (1-\chi_{\nabla u}(x))Q_{\mathrm{av}}(\boldsymbol{I\!I}(x)) + \chi_{\nabla u}(x)Q_{\mathrm{hom}}(\boldsymbol{I\!I}(x))\,dx\\
&\quad - \int_{S\setminus V_\delta} (1-\chi_{\nabla u}(x))Q_{\mathrm{av}}(\boldsymbol{I\!I}(x)) + \chi_{\nabla u}(x)Q_{\mathrm{hom}}(\boldsymbol{I\!I}(x))\,dx\\
&\quad + \int_{S\setminus V_\delta} Q_{\mathrm{av}}(\boldsymbol{I\!I}(x))\,dx
\end{aligned}$$

Since $Q_{\mathrm{hom}}(F) \le Q_{\mathrm{av}}(F) \le \frac{1}{\alpha}|\mathrm{sym}F|^2$, where α is the constant of ellipticity (cf. (3)), and because $S\setminus V_\delta \subset S\setminus E_\delta$, we finally get

$$\lim_{\varepsilon\downarrow 0} \left|\mathcal{E}^\varepsilon(u^{\delta,\varepsilon}) - \mathcal{E}^0(u)\right| \le \frac{2}{\alpha}\int_{S\setminus E_\delta} |\boldsymbol{I\!I}(x)|^2\,dx. \tag{88}$$

In combination with (87), this proves (80) and thus completes the proof of the lemma. □

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