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Numerical methods for generalized nonlinear Schrödinger equations

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Abstract

We present and analyze different splitting algorithms for numerical solution of the both classical and generalized nonlinear Schrödinger equations describing propagation of wave packets with special emphasis on applications to nonlinear fiber-optics. The considered generalizations take into account the higher-order corrections of the linear differential dispersion operator as well as the saturation of nonlinearity and the self-steepening of the field envelope function. For stabilization of the pseudo-spectral splitting schemes for generalized Schrödinger equations a regularization based on the approximation of the derivatives by the low number of Fourier modes is proposed. To illustrate the theoretically predicted performance of these schemes several numerical experiments have been done.

1 Introduction

Nonlinear Schrödinger equation (NLSE) is widely used to model pulse propagation in nonlinear dispersive media, e.g., along a micro-structured optical fiber [15, 2, 13]. Optical properties (dispersion) of the fiber are described by a relation between wave vector and frequency, $\beta = \beta(\omega)$, for a monochromatic linear wave in which the wave field is proportional to $e^{i(\beta z - \omega t)}$. The propagation coordinate z is directed along the fiber. The dispersion relation is typically encoded in a sequence of the so-called propagation constants $\beta_j = d^j\beta/d\omega^j$, $j = 0, 1, \ldots$, which are calculated or measured for some reference wave frequency ω_0. It is convenient to assume that ω_0 equals the pulse carrier frequency. In particular, β_0 is the carrier wave vector and $1/\beta_1$ describes the group velocity at the carrier frequency.

NLSE yields evolution of the complex pulse envelope, $\psi(z, \tau)$, along the fiber. Here the so-called retarded time $\tau = t - \beta_1 z$ corresponds to a coordinate system moving together with the pulse at the group velocity $1/\beta_1$. The latter generally differs from the phase velocity of the carrier wave ω_0/β_0. The corresponding electric field E is related to the field envelope ψ by a rapidly oscillating carrier wave $e^{i(\beta_0 z - \omega_0 t)}$:

$$E(z, t) = \mathrm{Re}\left[\psi(z, t - \beta_1 z)e^{i(\beta_0 z - \omega_0 t)}\right]. \tag{1}$$

An important precondition for the NLSE description is that the complex envelope ψ evolves slowly as compared to the carrier wave, such that $|\partial_\tau \psi| \ll |\omega_0 \psi|$. This slowly varying envelope approximation (SVEA) is used to derive a class of the NLSEs for $\psi(z, \tau)$.

The simplest, classical NLSE is given by the relation [3]

$$i\frac{\partial \psi}{\partial z} - \frac{\beta_2}{2}\frac{\partial^2 \psi}{\partial \tau^2} + N[\psi] = 0, \qquad N[\psi] = \gamma|\psi|^2\psi, \tag{2}$$

where $z \in [0, L]$ is the propagation coordinate, L indicates the total length of the fiber, and the retarded time τ plays role of the lateral coordinate of the considered computational domain. The nonlinear term $N[\psi]$ represents an instantaneous cubic (Kerr) nonlinearity, β_2 is referred to as the group-velocity dispersion parameter, and γ characterizes nonlinear properties of the fiber.

The initial conditions

$$\psi(z = 0, \tau) = \psi_0(\tau), \tag{3}$$

are defined by the incoming field envelope ψ_0, which can be expressed from the corresponding electric field $E(0, t)$ using the SVEA and the so-called sliding average [27]. The lateral boundary conditions are determined either by the natural assumption that the pulse is localized w.r.t. τ,

$$|\psi(z, \tau)| \to 0 \qquad \text{for} \qquad \tau \to \pm\infty, \tag{4}$$

or by the pulse periodicity assumption

$$\psi(z, \tau + T) \equiv \psi(z, \tau), \tag{5}$$

where $1/T$ is the pulse repetition rate.

The cubic NLSE (2) is just one representative of a huge family of the generalized nonlinear Schrödinger equations (GNLSE). The generalizations can be made by using more involved dispersion operator $\hat{\mathcal{D}}$,

$$i\frac{\partial\psi}{\partial z} + \hat{\mathcal{D}}\psi + N[\psi] = 0, \quad \text{where} \quad \hat{\mathcal{D}}\psi := \left[\sum_{m=2}^{M} \frac{\beta_m}{m!}\left(i\frac{\partial}{\partial\tau}\right)^m\right]\psi, \tag{6}$$

i.e., taking into account higher order propagation constants β_j, $j = 3, \ldots, M$, which can be important once operating close to the zero dispersion frequency (ZDF) at which $\beta_2 = \beta''(\omega_0)$ vanishes, or modeling ultra-short pulses determined on a large frequency range. Another type of generalizations occurs by assuming a more general form of nonlinearity $N[\psi]$, as, for example,

$$N[\psi] = f(|\psi|^2)\psi \quad \text{with} \quad \lim_{\xi\to 0}\frac{f(\xi)}{\xi} = \gamma \quad \text{and} \quad \inf_{\xi\geq 0} f(\xi) < +\infty;$$
$$N[\psi] = \left(\gamma + i\sigma\frac{\partial}{\partial\tau}\right)|\psi|^2\psi \quad \text{with} \quad \sigma = \gamma\omega_0^{-1};$$
$$N[\psi] = \left[\gamma_1|\psi|^2 + \gamma_2\int_0^\infty |\psi(z, \tau - \tau')|^2 h(\tau')d\tau'\right]\psi \quad \text{with} \quad \gamma_1 + \gamma_2 = \gamma.$$

Here, a real function f can describe a simple saturating nonlinearity e.g.,

$$f(\xi) = \gamma(\xi - s\xi^2) \quad \text{or} \quad f(\xi) = \frac{\gamma\xi}{1 + s\xi}, \quad \text{with} \quad s > 0. \tag{7}$$

An operator $i\sigma\frac{\partial}{\partial\tau}$ represents a self-steepening of the field envelope function and accounts for non-perfectness of the envelope description, whereas a convolution with the quickly decreasing real function $h(\tau')$ s.t. $\int_0^\infty h(\tau')d\tau' = 1$ represents a delayed medium response Raman

effect. Below in this paper we ignore a delayed medium response and consider the GNLSE (6) with the nonlinearity

$$N[\psi] = f(|\psi|^2)\psi + iR(\psi), \quad R(\psi) = \sigma_1|\psi|^2\frac{\partial}{\partial\tau}\psi + \sigma_2\psi\frac{\partial}{\partial\tau}|\psi|^2. \tag{8}$$

It is worth mentioning that the simplest Eq. (2) possess a very special property: it is completely integrable by means of the inverse-scattering method [29]. Usually this property is destroyed by generalizations, but in some cases the integrability is preserved. For example, the normalized GNLSE

$$i\frac{\partial\psi}{\partial z} + \frac{1}{2}\frac{\partial^2\psi}{\partial\tau^2} + |\psi|^2\psi + i\left[c_1\frac{\partial^3\psi}{\partial\tau^3} + c_2|\psi|^2\frac{\partial\psi}{\partial\tau} + c_3\psi\frac{\partial|\psi|^2}{\partial\tau}\right] = 0, \tag{9}$$

preserves integrability for special combinations of coefficients, such as $c_1 = 0$, $c_2 = 1$ and either $c_3 = 1$ or $c_3 = 0$ (different derivative Schrödinger equations), $c_1 = 1$, $c_2 = 6$, and $c_3 = 0$ (Hirota equation), or $c_1 = 1$, $c_2 = 6$, and $c_3 = 3$ (Sasa-Satsuma equation).

For a real-valued set of β_m, localized in τ pulses, and smooth enough initial function $\psi_0(\tau)$ the initial value problem (3), (4), (6), (8), can possesses some conservation laws. The classical cubic NLSE (2) has an infinite set of conservation laws [29]. For the general Eq. (6) with the vanishing generalized reaction term $R(\psi)$ (i.e., σ_1 and σ_2), at least three conservation laws can be found. One can easily derive them by writing (6) as a Hamiltonian equation

$$i\frac{\partial\psi}{\partial z} + \frac{\delta\mathcal{H}}{\delta\psi^*} = 0,$$

where ψ^* denotes a complex conjugate of ψ, functional $\mathcal{H}$ is defined below in (12), and $\frac{\delta\mathcal{H}}{\delta\psi^*}$ is a variational derivative. The first two integrals are derived from continuous symmetries of the Hamiltonian $\mathcal{H}$, namely

$$\mathcal{M}(\psi) := \int_{\mathbb{R}} |\psi(z,\tau)|^2\, d\tau = \text{const}, \tag{10}$$

and

$$\mathcal{S}(\psi) := \frac{i}{2}\int_{\mathbb{R}}\left(\psi^*\frac{\partial\psi}{\partial\tau} - \psi\frac{\partial\psi^*}{\partial\tau}\right) d\tau = \text{const}. \tag{11}$$

Finally, the Hamiltonian as such yields another conservation law,

$$\mathcal{H}(\psi) := \int_{\mathbb{R}}\left[\frac{\psi^*\hat{\mathcal{D}}\psi + \psi\hat{\mathcal{D}}\psi^*}{2} + F(|\psi|^2)\right] d\tau = \text{const}, \tag{12}$$

where $F(\xi) = \int_0^\xi f(s)\, ds$.

The same conservation laws (10), (11), and (12) with integration limits 0 and T are valid in the case of T-periodic boundary conditions (5). Generalizations of these integrals can be derived also for non-vanishing σ_1 and σ_2, see Ref. [7, 18, 21] and Section 3 of this paper.

2 Numerical approximation of NLSE

There are many numerical algorithms for the solution of nonlinear Schrödinger type problem. They are based on finite-difference schemes (see, e.g., [8, 11, 12, 22, 25]), finite-element and Galerkin approaches (see, [6, 8]), or spectral and pseudo-spectral methods (see the well-known pioneer paper [14], then these methods where developed in many papers [9, 10, 21, 26], see also references given therein). Various splitting techniques are used to construct efficient integrators of the obtained semi-discrete problems [17, 19, 26].

Our main goal is to compare efficiency of different solvers for the generalized NLSE. In this section we will construct numerical solvers and apply all of them for a standard nonlinear Schrödinger problem (2) with a cubic nonlinearity $N[\psi] = f(\psi) = |\psi|^2\psi$.

2.1 Numerical schemes

The considered domain $\bar{\Omega} := \Omega \cup \partial\Omega = [a, b]$ is covered by the following discrete uniform grid

$$\bar{\Omega}_h = \big\{\tau_j : \tau_j = a + jh, \quad j = 0, \dots, J, \quad h = (b-a)/J\big\}$$

with the grid points denoted by τ_j, the grid boundary $\partial\Omega_h = \{\tau_0 = a, \tau_J = b\}$, and the inner part of the grid $\Omega_h = \bar{\Omega}_h \setminus \partial\Omega_h$. Let ω_k be a uniform grid

$$\omega_k = \{z^n : z^n = nk,\, n = 0, \dots, N,\, Nk = L\},$$

where k is the discretization step. For simplicity, the step-size is taken constant. We consider numerical approximations U_j^n to the exact solution values $\psi_j^n = \psi(z^n, \tau_j)$ at the grid points $(z^n, \tau_j) \in \omega_k \times \bar{\Omega}_h$.

The following notation for difference and averaging operators is used

$$\begin{aligned}
&\partial_\tau U_j^n = (U_{j+1}^n - U_j^n)/h, \quad \partial_{\bar{\tau}} U_j^n = (U_j^n - U_{j-1}^n)/h,\\
&\partial_\tau^2 U_j^n := \partial_\tau \partial_{\bar{\tau}} U_j^n = \frac{U_{j+1}^n - 2U_j^n + U_{j-1}^n}{h^2},\\
&\partial_z U_j^n = (U_j^{n+1} - U_j^n)/k, \quad \widetilde{U}_j^{n+1/2} = (U_j^{n+1} + U_j^n)/2.
\end{aligned}$$

All numerical schemes for the problem (2), (3) presented below can be written as

$$\mathcal{B}\left(U_j^{n+1}, U_{j\pm1}^{n+1}, U_j^n, U_{j\pm1}^n\right) = 0, \quad z^n \in \omega_k, \;\; \tau_j \in \widetilde{\Omega}_h, \tag{13}$$

$$U_j^0 = \psi_0(\tau_j), \quad \tau_j \in \widetilde{\Omega}_h, \tag{14}$$

where the selection of the grid $\widetilde{\Omega}_h \subset \bar{\Omega}_h$ depends on the assumed conditions at the boundaries $\omega_k \times \partial\Omega_h$ of the computational domain. These conditions depend on the considered problem and/or preferred numerical scheme. For example, in the case of localized pulses satisfying (4) we can set

$$\widetilde{\Omega}_h = \Omega_h, \qquad U_j^n = \mu(z^n, \tau_j), \quad (z^n \times \tau_j) \in \omega_k \times \partial\Omega_h. \tag{15}$$

We use homogeneous Dirichlet condition $\mu = 0$ in (15) if the evolving envelope function does not reach lateral bounds of the computational domain, or some more sophisticated μ minimizing envelope back-reflection to the computational domain once the envelope function reaches the lateral bounds [8, 11]. For $(b-a)$-periodic sequence of envelope pulses satisfying (5) we can set

$$\widetilde{\Omega}_h = \tau_0 \cup \Omega_h, \qquad U_0^n = U_J^n, \quad U_{-1}^n = U_{J-1}^n, \quad z^n \in \omega_k. \tag{16}$$

With these periodic boundary conditions we avoid artificial envelope reflections from the lateral boundaries at large z, but feel an influence of the envelope pulses from the neighboring periods, instead. In many cases the usage of these conditions is presupposed by the fast Fourier transform driven split-step numerical schemes resolving the dispersion operator in the wave-vector domain. In these cases we assume that J is even and write the grid function as a discrete Fourier sum

$$U_j = \left[\mathcal{F}^{-1}(\widehat{U})\right]_j := \frac{1}{J}\sum_{\ell=-J/2}^{J/2-1} \widehat{U}_\ell \exp\left(\frac{2\pi i \ell j h}{b-a}\right),$$

where the Fourier coefficients are defined as

$$\widehat{U}_\ell = [\mathcal{F}(U)]_\ell := \sum_{j=-J/2}^{J/2-1} U_j \exp\left(-\frac{2\pi i \ell j h}{b-a}\right).$$

Here, $[\mathcal{F}^{-1}(\widehat{U})]_j$ and $[\mathcal{F}(U)]_\ell$ are corresponding components of the vectors $\mathcal{F}^{-1}(\widehat{U})$ and $\mathcal{F}(U)$ determined by the backward Fourier transform $\mathcal{F}^{-1}$ and the forward Fourier transform $\mathcal{F}$, respectively. The differential operators $\frac{\partial^s}{\partial \tau^s}$, $s = 1, 2, \ldots$ can be approximated by the exact derivative of the $J/2$-degree trigonometric interpolant D_j^s:

$$\begin{aligned} &\frac{\partial^s \psi(z,\tau_j)}{\partial \tau^s} \approx (D_J^s U)_j := \frac{1}{J}\sum_{\ell=-J/2}^{J/2-1} \omega_\ell^s \widehat{U}_\ell \exp\left(\frac{2\pi i \ell j h}{b-a}\right), \\ &\omega_\ell = \frac{2\pi\ell}{b-a}\, i, \quad \ell = -J/2, \ldots, J/2-1. \end{aligned} \tag{17}$$

Consequently, a general differential dispersion operator $\hat{\mathcal{D}}\psi$ defined in (6) is approximated as

$$\hat{\mathcal{D}}\psi(z,\tau_j) \approx \left(\hat{\mathcal{D}}_{h,J} U\right)_j := \frac{1}{J}\sum_{\ell=-J/2}^{J/2-1} \left[\sum_{m=2}^{M} \frac{(i\omega_\ell)^m \beta_m}{m!}\right] \widehat{U}_\ell \exp\left(\frac{2\pi i \ell j h}{b-a}\right). \tag{18}$$

We should take into account that on the uniform space grid we can only represent Fourier modes with $|\ell| \le J/2$. The energy of higher waves is moved by the discrete Fourier transform to the low number waves.

It is noteworthy, that both transformations can be computed very efficiently by the fast Fourier transform (FFT) algorithm, which reduces the number of arithmetic operations to $5J \log J$ multiplications.

In this paper we are also interested to check how accurately the discrete schemes constructed below preserve the conservation laws (10) and (12). In the case of standard NLSE (2) the discrete analogs of these laws are given by

$$\mathcal{M}_h(U^m) := \|U^m\|^2 = \|\psi_0\|^2, \tag{19}$$

$$\mathcal{H}_h(U^m) := \frac{\beta_2}{2}\|U^m\|_E^2 + (F(|U^m|^2), 1)_h = \frac{\beta_2}{2}\|\psi_0\|_E^2 + (F(|\psi_0|^2), 1)_h, \tag{20}$$

where $F(\xi) = \frac{\gamma\xi^2}{2}$ in our cubic nonlinearity case, and the discrete inner products and norms in the expressions above are defined as follows:

$$(V, W)_h = \sum_{j=1}^{J-1} V_j W_j^* h + \frac{V_0 W_0^* + V_J W_J^*}{2} h, \quad \|V\| = \sqrt{(V, V)_h},$$

$$(V, W)_2 = \sum_{j=0}^{J-1} V_j W_j^* h, \quad \|V\|_E^2 = \left(\partial_\tau V, \partial_\tau V\right)_2.$$

Crank-Nicolson scheme. The standard Crank-Nicolson scheme is given by

$$i\partial_z U_j^n - \frac{\beta_2}{2}\partial_\tau^2 \widetilde{U}_j^{n+1/2} + \frac{F(|U_j^{n+1}|^2) - F(|U_j^n|^2)}{|U_j^{n+1}|^2 - |U_j^n|^2}\,\widetilde{U}_j^{n+1/2} = 0, \quad \tau_j \in \widetilde{\Omega}_h. \tag{21}$$

For sufficiently smooth solutions it is easy to check that the approximation error of the discrete scheme (21), (14) with the corresponding boundary conditions (15) or (16) is $\mathcal{O}(k^2 + h^2)$. The convergence of a discrete solution of the Crank-Nicolson scheme (21) has been investigated in many papers, and the second-order accuracy is proved in various norms, see [6, 22] and references therein.

The implementation of the nonlinear discrete scheme (21), (14), (15) is done by using iterative algorithms, then a sequence of simple linear systems with tridiagonal matrices are solved. For periodic boundary conditions (16) the tridiagonal structure of the corresponding matrices is slightly violated, and approximately doubled number of the arithmetic operations is required for the effective implementation of the scheme.

It is easy to prove that the solution of Crank-Nicolson scheme (21), (14) supplemented by the boundary conditions (15) with $\mu = 0$ or (16) satisfies exactly discrete conservation laws (19) and (20).

Splitting algorithm. The second algorithm is based on splitting method. We split the diffraction and nonlinear interaction processes by using the symmetrical Strang splitting approach (see

[5, 16, 17, 23, 24, 26]):

$$
\begin{aligned}
&\frac{\partial \psi_l}{\partial z} = -i\frac{\beta_2}{2}\frac{\partial^2 \psi_l}{\partial \tau^2}, \quad z \in [z^n, z^{n+1}], \quad \psi_l(z^n, \tau) = \psi(z^n, \tau),\\
&\frac{\partial \psi_m}{\partial z} = if(|\psi_m|^2)\psi_m, \quad z \in [z^n, z^{n+1}], \quad \psi_m(z^n, \tau) = \psi_l(z^{n+1}, \tau),\\
&\psi(z^{n+1}, \tau) := \psi_m(z^{n+1}, \tau),\\
&\frac{\partial \psi_m}{\partial z} = if(|\psi_m|^2)\psi_m, \quad z \in [z^{n+1}, z^{n+2}], \quad \psi_m(z^{n+1}, \tau) = \psi(z^{n+1}, \tau),\\
&\frac{\partial \psi_l}{\partial z} = -i\frac{\beta_2}{2}\frac{\partial^2 \psi_l}{\partial \tau^2}, \quad z \in [z^{n+1}, z^{n+2}], \quad \psi_l(z^{n+1}, \tau) = \psi_m(z^{n+2}, \tau),\\
&\psi(z^{n+2}, \tau) := \psi_l(z^{n+2}, \tau).
\end{aligned}
\tag{22}
$$

Assuming that n is even, we formulate two sequential steps of the corresponding finite difference scheme as follows:

$$
\begin{aligned}
&i\frac{U_j^{n+\frac{1}{2}} - U_j^n}{k} = \frac{\beta_2}{4}\partial_\tau^2\big(U_j^{n+\frac{1}{2}} + U_j^n\big), \quad \tau_j \in \widetilde{\Omega}_h,\\
&U_j^{n+1} = \exp\big[ikf\big(|U_j^{n+\frac{1}{2}}|^2\big)\big]U_j^{n+\frac{1}{2}}, \quad \tau_j \in \widetilde{\Omega}_h,\\
&U_j^{n+\frac{3}{2}} = \exp\big[ikf\big(|U_j^{n+1}|^2\big)\big]U_j^{n+1}, \quad \tau_j \in \widetilde{\Omega}_h,\\
&i\frac{U_j^{n+2} - U_j^{n+\frac{3}{2}}}{k} = \frac{\beta_2}{4}\partial_\tau^2\big(U_j^{n+2} + U_j^{n+\frac{3}{2}}\big), \quad \tau_j \in \widetilde{\Omega}_h.
\end{aligned}
\tag{23}
$$

The important property of the splitting algorithm is that the nonlinear part of the problem is solved exactly by the explicit formula. For sufficiently smooth solutions it is easy to check that the approximation error of this discrete scheme is $\mathcal{O}(k^2 + h^2)$. The splitting scheme satisfies the first discrete conservation law (19).

Split-step pseudo-spectral scheme. We use the same splitting algorithm (22) as a template for the third algorithm. Now the diffraction part is approximated by using the pseudo-spectral method [9, 10, 21, 26]. In general, this algorithm assumes periodicity of functions ψ (5), i.e., the related grid functions U satisfy the periodic boundary conditions (16).

According to the Fourier pseudo-spectral approach the second order dispersion operator $\frac{\partial^2}{\partial \tau^2}$ in (22) is approximated by the exact second derivative of the $J/2$-degree trigonometric interpolant D_J^2, defined in (17). Then the symmetrical splitting algorithm can be written as

$$
\begin{aligned}
&\widehat{U}_\ell^{n+\frac{1}{2}} = e^{-ik\beta_2\omega_\ell^2/2}\,\widehat{U}_\ell^n, \quad \ell = -J/2, \ldots, J/2-1,\\
&U_j^{n+1} = \exp\big[ikf\big(|U_j^{n+\frac{1}{2}}|^2\big)\big]U_j^{n+\frac{1}{2}}, \quad \tau_j \in \widetilde{\Omega}_h,\\
&U_j^{n+\frac{3}{2}} = \exp\big[ikf\big(|U_j^{n+1}|^2\big)\big]U_j^{n+1}, \quad \tau_j \in \widetilde{\Omega}_h,\\
&\widehat{U}_\ell^{n+2} = e^{-ik\beta_2\omega_\ell^2/2}\,\widehat{U}_\ell^{n+\frac{3}{2}}, \quad \ell = -J/2, \ldots, J/2-1,
\end{aligned}
\tag{24}
$$

where each intermediate subproblem within the given small z-coordinate step is solved exactly, and the scheme preserves the first discrete conservation law (19). If results at intermediate levels z^n are not required, we can use a trapezoidal symmetrical splitting scheme, which twice reduces the number of required operations. In this case this algorithm formally can be written as

$$U^N = e^{k\mathcal{L}} e^{2k\mathcal{N}} e^{2k\mathcal{L}} \cdots e^{2k\mathcal{N}} e^{k\mathcal{L}} U^0, \tag{25}$$

where $\mathcal{L}$ denotes the linear diffraction operator and $\mathcal{N}$ denotes the nonlinear interaction. Obviously a similar modification is valid for the splitting algorithm (23).

The pseudo-spectral algorithms are very efficient in the case when solutions are smooth functions. The estimate for the interpolation error of the $J/2$-degree trigonometric interpolant $I_J u$ is given by [10]

$$\|u - I_J u\|_{H^l} \le c J^{l-m} \left\| \frac{d^m u}{d\tau^m} \right\|_{L_2}.$$

Thus for functions $u \in C^\infty$ we have the exponential convergence rate.

Other pseudo-spectral algorithms. Next we formulate a more general pseudo-spectral algorithm for solving NLSE (2). The discrete Fourier transformation is applied for the NLSE in order to get a system of nonlinear ODEs for the Fourier spectral coefficients:

$$i\partial_z \widehat{U}_\ell(z) - \frac{\beta_2}{2}\omega_\ell^2 \widehat{U}_\ell(z) + \left[\mathcal{F}\big(f(|U|^2)U(z)\big)\right]_\ell = 0, \quad \ell = -J/2, \ldots, J/2-1.$$

This system can be rewritten in the operator form as

$$\partial_z \widehat{U}(z) = G\widehat{U} := iA\widehat{U} + i\mathcal{F}\big(f(|U|^2 U\big), \quad 0 < z \le L, \tag{26}$$

where $A = -\frac{\beta_2}{2}\,\mathrm{diag}(\omega^2_{-J/2}, \ldots, \omega^2_{J/2-1})$, coefficients ω_l are defined in (17), and $\widehat{U} = \left(\widehat{U}_{-J/2}, \ldots, \widehat{U}_{J/2-1}\right)^T$ is a column vector with T denoting a transpose.

In the following sections this algorithm will be easily extended to a large variety of NLSE-type problems.

2.2 Time integration

For time integration, the adaptive explicit Runge-Kutta-Fehlberg (RKF) method is used [16]. Here we should put two important remarks dealing with the computational efficiency of the pseudo-spectral algorithm (26).

First, in order to step forward from $\widehat{U}^n$ to new approximation $\widehat{U}^{n+1}$ by using the RKF algorithm we should compute six vectors $G(\widehat{U}^m)$, thus the forward and backward discrete Fourier transformations needed to evaluate the nonlinear function $f(|U|^2)U$ should be computed six times.

Second, the specific stability properties of the explicit Runge-Kutta methods should be taken into account. Let us consider the standard test problem

$$i\frac{du}{dz} - \lambda u = 0, \quad 0 < z < L, \quad u(0) = 1. \tag{27}$$

Here the parameter λ defines the the spectral coefficient and it satisfies the asymptotic relation

$$\lambda = (cj)^p, \quad p \geq 2,$$

where p is the corresponding order of the derivative in the dispersion term, j defines the jth spectral mode. By introducing $z' = \lambda z$, we put equation (27) into non-dimensional form

$$i\frac{du}{dz'} - u = 0, \quad 0 < z' < L', \quad L' = \lambda L.$$

We solve this problem by using the classical explicit fourth-order Runge-Kutta method or by the Runge-Kutta-Fehlberg method [16]. Let us denote the error of the obtained discrete solution $U(z)$ by $E(z) = u(z) - U(z)$. It satisfies the estimate

$$|E(L')| \leq CL'(k')^4,$$

where $k' = \lambda k$ and k is the discretization step. The linear dependence of the error on L' follows from the fact that the exact solution is a periodical function and the error depends linearly on the number of full cycles of the solution. Returning back to the physical parameters we get the error estimate

$$|E(L)| \leq CL\lambda^5 k^4. \tag{28}$$

Thus in order to guarantee some specified accuracy ε, the discretization step k must be related to λ as

$$k \leq C\frac{(\varepsilon/L)^{1/4}}{\lambda^{5/4}}. \tag{29}$$

It is well-known that the A-stability region of the classical fourth-order Runge-Kutta (RK4) method includes an interval in the imaginary axis $\left[-i2\sqrt{2}, i2\sqrt{2}\right]$ [16]. Thus, due to the stability requirements, we get the restriction on the integration step

$$k \leq \frac{2\sqrt{2}}{\lambda}, \tag{30}$$

which guarantees the boundedness of the solution. Comparison of estimates (29) and (30) shows that the A-stability requirement is slightly weaker than the approximation requirement (29). But here we should take into account that the approximation estimate must be guaranteed only for spectral modes having a sufficient amount of energy, while the stability estimate must be fulfilled for all modes. Thus the stability estimate will be the most restrictive requirement if the number of modes is large. The last remark is important for the adaptive Runge-Kutta-Fehlberg method, which is only ρ-stable, i.e. the imaginary axis is not included into the A-stability region of this method.

For illustration of the obtained theoretical results we present results of computational experiments. The test problem (27) is solved in domain $0 < z \leq 5$ with the dispersion parameters

$$\lambda_j = \frac{1}{2}(\pi j)^2, \quad j = 1, 2, 4, 8.$$

We have listed in Table 1 the numbers of the discretization points N^1_{RK}, N^1_{RKF}, which are sufficient to get a bounded solution $|U(z)| \leq 2.1$ to solve the test problem with the classical fourth-order Runge-Kutta and the adaptive Runge-Kutta-Fehlberg methods respectively. We also have listed the numbers of the discretization points N^2_{RK}, N^2_{RKF}, which are sufficient to solve the test problem with the accuracy $1.22 \cdot 10^{-4}$.

Table 1: Results of computational experiments for RK and RKF solvers

	N^1_{RK}	N^1_{RKF}	N^2_{RK}	N^2_{RKF}
$j = 1$	9	11	160	100
$j = 2$	35	66	900	565
$j = 4$	140	400	5120	3200
$j = 8$	560	2200	28800	17920

The results of computational experiments have confirmed the theoretical conclusion that the classical RK4 solver guarantees the boundedness of the discrete solution for weaker requirements on the discrete time step k, but the adaptive RKF solver gives a more accurate approximation for the same fixed values of k. Fully in agreement with the theoretical conclusions, the asymptotics of N^1_{RK} follows the estimate (30), while N^1_{RKF}, N^2_{RK} and N^2_{RKF} satisfy the estimate (29).

2.3 Numerical Experiments

In this section we solve the normalized nonlinear Schrödinger equation

$$i\frac{\partial \psi}{\partial z} - \frac{\partial^2 \psi}{\partial \tau^2} - |\psi|^2 \psi = 0, \quad z > 0. \tag{31}$$

The single soliton solution of (31) is given by [16]

$$\psi(z,\tau) = e^{-i\left(\frac{1}{2}c\tau - \left(\frac{1}{4}c^2 - \frac{1}{2}\right)z\right)} \operatorname{sech}\left(\sqrt{1/2}(\tau - cz)\right).$$

For the numerical tests we consider the superposition of two solitons. The initial profile is given

$$\psi(0,\tau) = e^{-i\frac{1}{2}c_1\tau} \operatorname{sech}\left(\sqrt{1/2}\,\tau\right) + e^{-i\frac{1}{2}c_2(\tau-\delta)} \operatorname{sech}\left(\sqrt{1/2}\,(\tau-\delta)\right),$$

with constants $c_1 = 1$, $c_2 = 1/10$ and $\delta = 25$. We solve the problem in the domain $-20 \leq \tau \leq 80$ for $0 \leq z \leq 40$. For these values of z the solution is negligibly small outside the τ-domain, thus the choice of boundary conditions is not important for this test problem. The most difficult part of the test is to simulate accurately the collision of the solitons, when the faster soliton catches the slower and passes through it. The shape and velocity of solitons remains unchanged after the collision.

In all computations the errors have been estimated by comparing the results with a numerical reference solution U_{ref} calculated by the pseudo-spectral splitting algorithm (24) on a fine grid with $J = 8192$, $N = 200000$.

First, we have investigated the error introduced mainly by finite difference approximation of the diffraction operator. We have listed in Table 2 the errors $Z_{CN}(L;N,J) = U_{CN} - U_{ref}$ in the maximum norm of the discrete solution U_{CN} of Crank-Nicolson scheme (21) at the final point $L = 40$, when the faster soliton have passed through the slower one. The integration was done with a very small step k in order to make the influence of this part of error negligibly small.

Table 2: Errors of the solution of scheme (21) in the maximum norm for the fixed small k and various $h = 100/J$ at $L = 40$.

	$J = 512$	$J = 1024$	$J = 2048$	$J = 4096$	$J = 8192$
$\|Z_{CN}(L;J)\|_\infty$	0.3153	0.0772	0.0190	0.0046	0.0011

The presented results show that the error changes as $\mathcal{O}(h^2)$ in accordance with the theoretical results.

Next we have compared the full approximation Crank-Nicolson scheme (21) with the splitting scheme (23) on a fixed uniform transversal grid Ω_h with $J = 8192$. Our aim is to study the influence of splitting errors to the accuracy of the discrete solution. Results are presented in Table 3.

Table 3: Convergence analysis of the discrete solutions of the full approximation scheme (21) and the splitting scheme (23) with respect to the truncation errors of the integration method. Errors are computed in the maximum norm for fixed h = 100/8192 and various $k = L/N$ at $L = 40$. The last row of each block defines the errors of both schemes for a balanced relation of discrete steps $J = CN$.

	$N = 800$	$N = 1600$	$N = 3200$	$N = 6400$
$\|Z_{CN}(L;N,8192)\|_\infty$	0.01641	0.00305	0.00038	0.00094
$\|\widetilde{Z}_{CN}(L;N)\|_\infty$	0.01751	0.00415	0.00148	0.00016
$\|Z_{CN}\left(L;N,\frac{8192\,N}{3200}\right)\|_\infty$	0.00675	0.00154	0.00038	0.00010
$\|Z_{SS}(L;N,8192)\|_\infty$	0.07760	0.02030	0.00431	0.00045
$\|\widetilde{Z}_{SS}(L;N)\|_\infty$	0.07870	0.02141	0.00541	0.00155
$\|Z_{SS}\left(L;N,\frac{8192\,N}{6400}\right)\|_\infty$	0.02200	0.00669	0.00176	0.00045

The first row of the table lists the global errors $\|Z_{CN}(L;N,J)\|_\infty$ of the discrete solutions U_{CN} of the full approximation scheme (21) in the maximum norm. The behavior of the error is quite intricate, these results can be explained assuming that the errors due to the z-stepping algorithm and the time approximation can cancel out each other and we predict that the global error can be represented in the following form:

$$Z_{CN}(L;N,J) \approx \widetilde{Z}_{CN}(L;N) - C_2h^2, \tag{32}$$

where $\widetilde{Z}_{CN}(L;N)$ describes the z-stepping error. By taking from Table 2 that $C_2h^2 = 0.0011$ for $J = 8192$ and solving the equation (32), we compute the modified errors $\|\widetilde{Z}_{CN}(L;N)\|_\infty$.

They are presented in the second row of Table 3. The indicated interaction of two types of errors can be used to minimize the global error of the discrete solution. We keep the linear relation for discrete steps $k = Ch$, in order to maximize the cancellation effect of two different sources of approximation error. In the third row of Table 3 we present errors $\|Z_{CN}(L; N, J)\|_\infty$ of the Crank-Nicolson scheme when $J = 8192N/3200$. The results of numerical experiments show that the obtained errors are smaller even though the computational grid is more sparse.

The maximum norms of errors Z_{SS} and $\widetilde{Z}_{SS}$ induced by the symmetrical splitting scheme (23) are given in the last three rows of Table 3. Once estimating Z_{SS} in the last row of the table we have used $J = 8192N/6400$. Comparing the obtained results we note that in order to achieve the same accuracy the splitting algorithm requires twice smaller integration step than the full approximation algorithm (21), but this increase of computations is compensated by the non-iterative origin of the splitting algorithm.

Next we consider the pseudo-spectral splitting scheme (24). The accuracy of spectral approximation of the solitons is very high even for a quite small number of modes. The exponential convergence rate can be illustrated by the following results (they are obtained for the same problem of interacting solitons):

$$\|Z_{PSSS}(L; N, 256)\|_\infty = 4.12 \times 10^{-4}, \quad \|Z_{PSSS}(L; N, 512)\|_\infty = 6.36 \times 10^{-7},$$

where $N = 200000$ is used in numerical experiments.

The behavior of the splitting error is very similar to that one obtained for the finite difference splitting scheme (23) The results are presented in Table 4.

Table 4: Convergence analysis of the discrete solutions of the pseudo-spectral splitting scheme (24) with respect to the integration step k. Errors are computed in the maximum norm for fixed $h = 100/512$ and various $k = L/N$ at $L = 40$.

	$N=400$	$N=800$	$N=1600$	$N=3200$	$N=6400$
$\|Z_{PSSS}(L; 512, N)\|_\infty$	0.6466	0.1934	0.0506	0.0128	0.00321

It follows from the presented results that errors of of the discrete solution of the pseudo-spectral splitting scheme (24) are approximately two times larger than errors of the finite difference splitting scheme (23) (see the errors presented in the fifth line of Table 3)

In the last experiment we have investigated the stability and approximation accuracy properties of the explicit solver (26) based on the pseudo-spectral approximation and the adaptive RKF algorithm. We note, that being ρ-stable, this adaptive integration algorithm should automatically guarantee not only the approximation requirement but also the boundedness of the solution by selecting a sufficiently small integration step k. In Table 5 we list the number of integration steps N for different numbers of spectral modes J.

The presented results have confirmed the theoretically established results, that for smaller J the more restrictive is the requirement (29), which can be written as $k \leq CJ^{-2.5}$, while for larger numbers of spectral modes the stability requirement (30) starts to be a dominant one.

Table 5: Stability analysis of the discrete solution of the pseudo-spectral scheme (26). N is the number of integration steps of the explicit adaptive RKF solver for different numbers of spectral modes J.

	$J = 256$	$J = 512$	$J = 1024$
N	3347	19004	86427

Our main conclusion of this section is that for given benchmark problems the most efficient is the symmetrical splitting scheme (23). The high-order approximation accuracy of the pseudo-spectral scheme (26) is not sufficient to compensate two drawbacks – the need to compute six forward and backward FFT per one step and to keep integration step sufficiently small due to the stability requirements of the explicit RKF solver.

3 Generalized NLSE

In this section we consider splitting algorithms for a generalized nonlinear Schrödinger equation (6) with a nonlinear term defined in (8). We note, that for simple cubic nonlinear function $f = \gamma|\psi|^2\psi$, vanishing higher order dispersion terms, β_j, $j \geq 4$, and $\beta_2\gamma < 1$ this problem is equivalent to Eq. (9).

It is known that for certain values of the coefficients and certain initial conditions, solutions of this problem experience finite-time blow up [20]. Therefore, the development of robust numerical schemes for such problems is important and challenging task.

3.1 Numerical algorithms

Different forms of the GNLS problem (6), (8) are solved numerically in papers [18, 21], see also references therein. Pathria and Morris [21] have investigated spectral split step methods, they have restricted to the first order splitting techniques. The splitting algorithms treat separately the linear diffraction $\mathcal{L}$, the local nonlinear interaction $\mathcal{N}$ and the generalized reaction $\mathcal{Q}$. All algorithms differ only in the way how the subproblem $\mathcal{Q}$ is treated. Four different algorithms are proposed to solve the subproblem $\mathcal{Q}$. The high-order spectral splitting algorithms are investigated in [18], where only the linear diffraction $\mathcal{L}$ and nonlinear reaction $\mathcal{N} + \mathcal{Q}$ processes are split in the schemes.

Pseudo-spectral split-step algorithm. We are using the symmetrical split-step approximation where N is even and, like in [21], all three processes are resolved separately:

$$U^{n+1} = e^{k\mathcal{L}}e^{k\mathcal{Q}}e^{k\mathcal{N}}U^n, \;\; U^{n+2} = e^{k\mathcal{N}}e^{k\mathcal{Q}}e^{k\mathcal{L}}U^{n+1}, \;\; n = 0, 2, \ldots, N-2. \qquad (33)$$

The solver for the local nonlinear $\mathcal{N}$ term is defined in the second line of the pseudo-spectral splitting algorithm (24). The first line of the same algorithm defines also the solver for the linear

term $\mathcal{L}$ once the dispersion operator is determined by β_2 alone. In general case, the linear operator $\mathcal{L}$ is determined by

$$\widehat{U}_\ell^{n+\frac{1}{2}} = \exp\left(ik \sum_{m=2}^{M} \frac{(i\omega_\ell)^m \beta_m}{m!} \right) \widehat{U}_\ell^n, \quad \ell = -J/2, \ldots, J/2-1. \tag{34}$$

The subproblem $\mathcal{Q}$ is solved in a different way than in [21]. We define the following system for the vector $\widehat{U}$ of spectral coefficients

$$-\partial_z \widehat{U}(z) = Q(\widehat{U}) := \mathcal{F}\big((\sigma_1+\sigma_2)|U|^2 D_{J_0}^1 U + \sigma_1 U^2 D_{J_0}^1 U^*\big), \;\; 0 < z \le L, \tag{35}$$

where the first order derivative is approximated by the exact derivative of the $J_0/2$-degree trigonometric interpolant $D_{J_0}^1$ defined in (17). In order to control the possible ill-posedness of the numerical differentiation, we introduce the regularization by restricting the degree of the trigonometric interpolant $J_0 < J$. The obtained system of ordinary differential equations is solved by using the Runge-Kutta-Fehlberg adaptive algorithm.

Generalized pseudo-spectral algorithm. We have also solved the GNLS problem (6), (8) using the following generalization of the pseudo-spectral algorithm (26) discussed in Section 2:

$$\partial_z \widehat{U}(z) + i\mathcal{F}\Big(\hat{\mathcal{D}}_{h,J_1} U - f(|U|^2)U\Big) + Q(\widehat{U}) = 0, \quad 0 < z \le L, \tag{36}$$

where $Q(\widehat{U})$ is defined in (35). The advantage of this approach is that the approximation error can be controlled automatically by the robust adaptive RKF integration algorithm. But we should take into account that due to conditional stability the restriction on the integration step depends linearly on the spectrum of the discrete diffraction operator. In order to control the stiffness of the matrix, we introduce into the scheme the additional regularization: the differential dispersion operator $\hat{\mathcal{D}}$ is approximated by the exact differential operator of the reduced $J_1/2$-degree trigonometric interpolant $\hat{\mathcal{D}}_{h,J_1}$ from (18) with $J_1 < J$.

3.2 NLSE with the generalized nonlinearity

In this section we consider a generalized nonlinear Schrödinger problem (6), (8) with a nonlinear function $f(|\psi|^2) = \gamma_1|\psi|^2 + \gamma_2|\psi|^4$, an additional higher-order (e.g., saturating) term in the nonlinear part, and a standard second-order dispersion operator $\hat{\mathcal{D}} = -\frac{\beta_2}{2}\frac{\partial^2}{\partial\tau^2}$. Assuming that the solution ψ and all it derivatives converge to zero as $\tau \to \pm\infty$, one can show that the solution of this problem conserves the following three quantities [20], which can be used as convenient aposteriori measures to test the accuracy of the discrete solution [18, 21]:

$$\mathcal{M} = \int_{-\infty}^{\infty} |\psi|^2 \, d\tau, \qquad \mathcal{S} = \int_{-\infty}^{\infty} \left(\beta_2 \operatorname{Im}\left(\psi \frac{\partial \psi^*}{\partial \tau}\right) + \sigma_1 |\psi|^4 \right) d\tau,$$

$$\mathcal{H} = \int_{-\infty}^{\infty} \left[\frac{\beta_2}{2} \left| \frac{\partial \psi}{\partial \tau} \right|^2 + F(|\psi|^2) + \frac{2\sigma_1+\sigma_2}{2} |\psi|^2 \left(\operatorname{Im}\left(\psi \frac{\partial \psi^*}{\partial \tau}\right) + \frac{2\sigma_1}{3\beta_2}|\psi|^4 \right) \right] d\tau.$$

It is noted in [21] that the quantities $\mathcal{M}$, $\mathcal{S}$ and $\mathcal{H}$ provide aposteriori check on the numerical results and that the energy integral $\mathcal{H}$ is the most sensitive of the three in indicating numerical difficulties.

To test the performance of the numerical schemes for GNLSE with a standard second order dispersion operator we have solved numerically the test problem from [18, 21]

$$i\frac{\partial\psi}{\partial z}+\frac{\partial^2\psi}{\partial\tau^2}+\frac{1}{2}|\psi|^2\psi-\frac{7}{4}|\psi|^4\psi-i\frac{\partial|\psi|^2}{\partial\tau}\psi-2i|\psi|^2\frac{\partial\psi}{\partial\tau}=0.$$

This equation has a traveling solitary wave solution:

$$\psi(z,\tau)=\left(\frac{4}{4+3\sinh^2(\tau-2z-15)}\right)^{1/2}\exp\left(i\Phi(z,\tau)\right)$$
$$\Phi(z,\tau)=2\tanh^{-1}\left(0.5\tanh(\tau-2z-15)\right)+\tau-15.$$

The purpose of the numerical experiments is to verify numerically (i) that the proposed split-step and full approximation schemes exhibit the second-order and fourth-order convergence in space, respectively, (ii) to compare the stability regions of both schemes and (iii) to check the conservativity of proposed algorithms.

In all computations the regularization parameter $J_0=60$ is used to compute the approximation of the derivative $D^1_{J_0}U$. In Table 6 we presents errors of the solution of pseudo-spectral splitting scheme (33) with respect to the space integration step. The errors are computed at $L=2$, the computations are done in the transversal domain $[0,35]$.

Table 6: Convergence analysis of the discrete solutions of the pseudo-spectral splitting scheme (33) with respect to the space integration step. Errors are computed in the maximum norm for two different $h=35/J$ and various $k=L/N$ at $L=2$.

	$N=100$	$N=200$	$N=400$	$N=800$	$N=1600$
$J=256$	1.077e-3	2.672e-4	6.667e-5	1.665e-5	4.150e-6
$J=1024$	1.082e-3	2.686e-4	6.702e-5	1.676e-5	4.197e-6

The presented results show that the error changes as $\mathcal{O}(k^2)$ in accordance with the theoretical results. The regularized algorithm is stable with respect to the number of modes J. In the case, when the derivative of the solution is computed as $D^1_J U$, i.e. no regularization is done, the scheme becomes unstable for $J\geq 512$ and $N\geq 300$. Such a behavior of the error shows the boundary effects of high order modes amplified due to numerical differentiation.

Results of computational experiments with full approximation scheme (36) have confirmed the theoretical results. Due to conditional stability of the explicit scheme the restriction on integration step k leads to the need to integrate the system with very small steps, much smaller than required by the approximation accuracy. For the given test problem the scheme is unstable if $N\leq 300$ for $J=256$, $N\leq 3600$ for $J=512$ and $N\leq 18000$ for $J=1024$. In

order to improve the stability of the pseudo-spectral scheme (36) it is recommended to include the regularization technique for approximation of the second derivative $D_{J_1}^2 U$. If parameter $J_1 = 60$ is used, than the full approximation scheme (36) computes in stable way for any integration steps and the accuracy of order $\mathcal{O}(10^{-6})$ is achieved for $N = 100$ (due to the forth order approximation accuracy of RKF algorithm).

3.3 GNLSE with the third-order dispersion

In the rest of this paper we consider a generalized NLSE with an additional third-order dispersion (TOD) term (see [4, 28])

$$i\frac{\partial\psi}{\partial z} + \frac{1}{2}\frac{\partial^2\psi}{\partial\tau^2} - i\varepsilon\frac{\partial^3\psi}{\partial\tau^3} + |\psi|^2\psi = 0, \tag{37}$$

where the TOD operator is included as a perturbation of the classical nonlinear Schrödinger equation (2). It is well-known that for the unperturbed ($\varepsilon = 0$) equation (37) the fundamental soliton solution reads

$$\psi^0(z,\tau) = A\,\mathrm{sech}(A\tau)\exp(ikz), \tag{38}$$

where $k = A^2/2$ is the soliton wave number. To explain the TOD effect, let us use the Ansatz $\psi(z,\tau) = A\,\varphi(s)\exp(iA^2z/2)$, $s = A\tau$ motivated by Eq. (38) with the yet unknown function $\varphi(s)$. The latter should be derived from the equation

$$\hat{\mathcal{T}}\varphi = |\varphi|^2\varphi, \quad \text{where} \quad \hat{\mathcal{T}}\varphi = \left(\frac{1}{2} - \frac{1}{2}\frac{\partial^2}{\partial s^2} + iA\epsilon\frac{\partial^3}{\partial s^3}\right)\varphi. \tag{39}$$

Now, for $\epsilon = 0$ the operator $\hat{\mathcal{T}}$ is positive-defined and Eq. (39) yields the solitary solution (38). The soliton can be derived even for a more general nonlinearity (8) by properly rescaled successive iterations [1]. For $\epsilon \neq 0$ this is not the case, because $\hat{\mathcal{T}}$ becomes singular. In terms of pulse propagation that means that the initial pulse (38) emits radiation. Two different cases should be distinguished here. If $A\varepsilon < 0.04$ the emitted radiation is extremely weak [28]. To first order of ε we have a quasi-solitary solution with the shape

$$\psi^\varepsilon(z,\tau) = A\,\mathrm{sech}(A\tau^\epsilon)\exp\left(ikz + i\varepsilon[2A^2\tau - 3\tanh(A\tau^\epsilon)]\right), \tag{40}$$

where $\tau^\epsilon = \tau - \varepsilon A^2 z$. If $A\varepsilon > 0.04$, the radiation is strong and the carrier wavelength of the pulse is quickly shifted into the red. The underlying physical mechanism is similar to that of Cherenkov radiation created by superluminal objects in dispersive media [4].

In this section our main aim is to analyze the influence of the TOD to the stability and accuracy of the split step scheme (33) and the full approximation pseudo-spectral algorithm (36). It follows from the computational experiments that for the accurate approximation of the soliton (40) it is sufficient to use $J = 128$ spectral modes. Thus if we use more spectral modes, then it is important to check how different algorithms are resolving these excessive modes.

As it was proved in the previous section the RKF method used for integration of generalized pseudospectral algorithm (36) is only conditionally ρ-stable, thus in the case of TOD the integration step k automatically will be restricted as $k \leq Ch^3$. This conclusion is confirmed by results

of computational experiments. The problem (37) was solved in the domain $-20 \leq \tau \leq 20$ for $0 \leq z \leq 100$, and $A = 1$, $\varepsilon = 0.02$. In Table 7 we present the total CPU times T in seconds and the number of integration steps N for different numbers of spectral modes J, the accuracy of RKF algorithm was fixed to 10^{-7}. Computations were performed on VGTU cluster of computers "Vilkas", consisting of nodes with Intel®Core$^{\mathrm{TM}}$ processor i7-860 @ 2.80 GHz and 4 GB RAM.

Table 7: Results of numerical experiments: problem (37) is solved by the pseudo-spectral scheme (36). T is the the total CPU time is seconds and N is the number of integration steps of the explicit adaptive RKF solver for different numbers of spectral modes J.

	$J = 128$	$J = 256$	$J = 512$
N	11963	74511	701209
T	1.7	20.4	395

In the case of the split step scheme (33) the unconditional stability of the discrete scheme guarantees a more flexible control of the integration step k, it is sufficient to resolve accurately the active spectral modes and the remaining modes will be kept bounded. This theoretical conclusion is confirmed by results of computational experiments. The number of integration steps is fixed to $N = 50000$. The accuracy of discrete solutions in the maximum norm is obtained of order 10^{-4} for all numbers of spectral modes. Finally, we present the CPU times $T(J)$ of these experiments:

$$T(128) = 1.8s, \quad T(256) = 3.3s, \quad T(512) = 7s.$$

As interesting remark we note, that our computations have confirmed the theoretical accuracy of the corrected form of the soliton (40). For this reason we have integrated numerically the TOD problem for $A = 1$ and different values of the small parameter ε. The errors

$$Z^{\epsilon}(z; J) = U - \psi^{\varepsilon}(z, t)$$

(U is the numerical solution of the problem obtained by the scheme (36)) in the maximum norm estimated at $z = 500$ are given in Table 8.

Table 8: Results of numerical experiments: problem (37) is solved by the scheme (36) until $z = L = 500$ using $J = 512$, and different ε.

	$\varepsilon = 0.02$	$\varepsilon = 0.01$	$\varepsilon = 0.005$
$\|Z^{\varepsilon}(500; J)\|_{\infty}$	0.0619	0.0164	0.0048

The presented results agree well with the theoretical prediction that these errors should be of order $O(\varepsilon^2)$.

4 Conclusions

We have considered several splitting algorithms for numerical integration of standard NLSE and GNLSE. These algorithms are based on separate treatment of local and/or nonlocal nonlinearities and linear dispersion processes. We have shown, that the numerical schemes for GNLSE obtained by the straightforward generalization of the related pseudo-spectral schemes of NLSE can be only conditionally stable. The stability of these schemes can be improved by neglecting higher-order Fourier modes in the approximation of the derivative terms of GNLSE.

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