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Ultrashort optical solitons in transparent nonlinear media with arbitrary dispersion

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Abstract

We consider the propagation of ultrashort optical pulses in nonlinear fibers and suggest a new theoretical framework for the description of pulse dynamics and exact characterization of solitary solutions. Our approach deals with a proper complex generalization of the nonlinear Maxwell equations and completely avoids the use of the slowly varying envelope approximation. The only essential restriction is that fiber dispersion does not favor both the so-called Cherenkov radiation, as well as the resonant generation of the third harmonics, as these effects destroy ultrashort solitons. Assuming that it is not the case, we derive a continuous family of solitary solutions connecting fundamental solitons to nearly single-cycle ultrashort ones for arbitrary anomalous dispersion and cubic nonlinearity.

1 Introduction

Optical solitons are stable localized wave packets resulting from interplay between nonlinearity and dispersion. They are important on their own [2], and provide simple "building units" for complicated chaotic states such as optical supercontinuum [10]. A fundamental optical soliton has universal $\cosh^{-1}(\tau/t_0)$ shape, the latter is imposed on a continuous carrier wave $e^{i(\beta_0 z-\omega_0 t)}$ where the wave vector $\beta_0 = \beta(\omega_0)$ is yielded by the carrier frequency ω_0 and by the dispersive propagation constant $\beta(\omega)$. The related retarded time $\tau = t - z/V$ is determined by the group velocity $V = 1/\beta'(\omega_0)$ at the carrier frequency. The free parameter t_0 may take different values, it determines the temporal width of the pulse. The universal $\cosh$-shape is observed for $\omega_0 t_0 \gg 1$, i.e., the soliton should contain many cycles of the carrier wave. Decreasing t_0 we come up against new effects which are typical for the few-cycle solitons:

- for a favorable dispersion $\beta(\omega)$ the soliton radiates energy in the form of dispersive waves. Strictly speaking, the soliton is then destroyed, in practice its living time still may be exponentially long [3]. Correct description of such radiation may involve non-polynomial representation of the dispersion profile [8, 6].
- effect of third harmonics generation may become important. It can destroy the soliton, except for very special dispersion profiles [11].
- soliton's shape deviates from the universal one. At some critical t_0 the shape function forms a cusp. Even in the absence of both the dispersive radiation and third harmonics generation the solitary solution just does not exist for smaller durations [9, 7, 4].

Both effects of the dispersive waves and the third harmonics owe to the fact that the fundamental solitons do not solve Maxwell equations directly. They only solve a simplified version of the

Maxwell equations in the form of the nonlinear Schrödinger equation (NLSE), derived for smooth pulse shapes at $\omega_0 t_0 \gg 1$ [2]. In the NLSE, in particular, $\beta(\omega)$ is replaced by a parabolic function, namely the quadratic Taylor expansion around $\omega = \omega_0$. On the contrary, cusp formation seems to be an universal feature of Maxwell equations: existence of the sub-cycle solitons is prohibited by Nature even in the absence of the higher harmonics and for the most favorable dispersion profile. The shortest pulse contains approximately one-and-half oscillations (full width at half maximum, FWHM).

Non-existence of the sub-cycle solitons was demonstrated previously using analytic solutions for the specially chosen linear [9, 7] and nonlinear [4] dispersions. Here we follow [5] and address this problem numerically for arbitrary dispersion. Specifically we suggest a modeling framework that allows separation and elimination of the higher harmonics without any reference to $\omega_0 t_0$ and to the slowly varying envelope approximation (SVEA). One can then investigate the generalized fundamental solitons while keeping many features of Maxwell equations that are not covered by NLSE, such as arbitrary nonlocal dispersion and bidirectional character of wave propagation. The fate of the fundamental soliton for $t_0 \to 0$ is then determined by dispersion: in all our calculations the ultrashort soliton was destroyed either by dispersive radiation or by formation of the singularity.

2 Model equation

Although the mathematical approach described in this section is very general, for the sake of concreteness we focus on one-dimensional optical pulse with a single-component electric field $E = E(z,t)$ propagating along the fiber. z denotes the propagation coordinate. In the linear case the pulse field is governed by a linear wave equation with real coefficients. To find a real-valued $E(z,t)$, one may look first for a complex-valued solution, $\mathcal{E}(z,t)$, *of the same equation*. Such solutions appear in pairs, $\mathcal{E}(z,t)$ and $\mathcal{E}^*(z,t)$, and using linearity one can combine them to get a real-valued field

$$E(z,t) = \frac{\mathcal{E}(z,t) + \mathcal{E}^*(z,t)}{2} = \mathrm{Re}[\mathcal{E}(z,t)]. \tag{1}$$

A common opinion is that such trick works only for linear equations. In fact, one can generalize the approach also for the nonlinear case. The only difference is that in the nonlinear case the real $E(z,t)$ and complex $\mathcal{E}(z,t)$ fields are governed by *different equations*. To illustrate this fact we now consider pulse propagation in a nonlinear fiber as yielded by the nonlinear wave equation

$$\partial_z^2 E - \frac{1}{c^2}\partial_t^2(\hat{\epsilon}E + \chi^{(3)}E^3) = 0, \tag{2}$$

where c is the speed of light, $\chi^{(3)}$ is the nonlinear susceptibility of the third order, and the operator $\hat{\epsilon}$ describes linear dispersion, i.e., in the frequency domain

$$E(z,t) = \sum_\omega E_\omega(z) e^{-i\omega t} \quad \Rightarrow \quad \hat{\epsilon}E = \sum_\omega \epsilon(\omega) E_\omega(z) e^{-i\omega t}. \tag{3}$$

The effective dispersion function, $\epsilon(\omega)$, describes bulk medium dispersion, and, if necessary, contains contributions from the waveguide geometry. Equation (2) applies both to the bulk propagation and to a single-mode nonlinear fiber. It is further natural to consider the transparency band, i.e., assume that $\mathrm{Im}[\epsilon(\omega)] \approx 0$. For simplicity we also assume that $\chi^{(3)} \approx$ const for the frequency range of interest. Now, consider the following complementary equation

$$\partial_z^2 \mathcal{E} - \frac{1}{c^2}\partial_t^2 \left(\hat{\epsilon}\mathcal{E} + \frac{3}{4}\chi^{(3)}|\mathcal{E}|^2\mathcal{E} + \frac{1}{4}\chi^{(3)}\mathcal{E}^3 \right) = 0, \tag{4}$$

for the complex field $\mathcal{E}(z,t)$, where by definition $(\hat{\epsilon}\mathcal{E})_\omega = \epsilon(\omega)\mathcal{E}_\omega$ as motivated by Eq. (3). The standard property

$$\epsilon(-\omega) = \epsilon^*(\omega) \quad \Rightarrow \quad \hat{\epsilon}\left(\mathcal{E}^*\right) = \left(\hat{\epsilon}\mathcal{E}\right)^*$$

ensures that complex solutions of the Eq. (4) also appear in pairs, $\mathcal{E}(z,t)$ and $\mathcal{E}^*(z,t)$. Moreover, one can directly check that the linear combination (1) yields a real-valued solution of Eq. (2). Equation (4) can be considered as one of many possible complexifications of Eq. (2).

In the linear case we benefit from the complexification because one can immediately substitute $e^{i(\beta z-\omega t)}$ for the wave field. The reasoning is different in the nonlinear case. Note, that nonlinear terms in Eq. (4) explicitly distinguish between the self-phase modulation (SPM) and THG. If the latter can be ignored, one is left with a kind of “bidirectional” NLSE

$$\partial_z^2 \mathcal{E} - \frac{1}{c^2}\partial_t^2 \left(\hat{\epsilon}\mathcal{E} + \frac{3\chi^{(3)}}{4}|\mathcal{E}|^2\mathcal{E} \right) = 0, \tag{5}$$

which will be referred to as biNLSE. The biNLSE is more general than the standard NLSE and possess a rich set of solitary solutions. However, in order to ignore $\mathcal{E}^3$ self-consistently, one first has to relate $\mathcal{E}(z,t)$ to the analytic signal. As explained in the next section, such a relation exists if fiber dispersion does not favor THG. In other words, we shall assume that the corresponding phase-matching conditions

$$\begin{aligned} \beta(\omega_1) + \beta(\omega_2) + \beta(\omega_3) &= \beta(\omega_4), \\ \omega_1 + \omega_2 + \omega_3 = \omega_4, &\qquad \omega_i > 0, \end{aligned} \tag{6}$$

are *not satisfied for any four positive frequencies* from the transparency region.

3 Field representation by analytic signal

The analytic signal relates the real physical field $E(z,t)$ to some naturally defined complex one $\mathcal{E}^{\text{as}}(z,t)$, such that Eq. (1) is automatically satisfied. $\mathcal{E}^{\text{as}}(z,t)$ is defined simply by keeping only positive frequencies in the spectral representation (3)

$$\mathcal{E}^{\text{as}}(z,t) = 2\sum_{\omega>0} E_\omega(z) e^{-i\omega t}, \qquad E(z,t) = \mathrm{Re}[\mathcal{E}^{\text{as}}(z,t)]. \tag{7}$$

Why should one believe that $\mathcal{E}^{\text{as}}(z,t)$ is provided by solutions of Eq. (5)? Of course, we can use the input field $[E(z,t)]_{z=0}$ and $[\partial_z E(z,t)]_{z=0}$ to obtain $[\mathcal{E}^{\text{as}}(z,t)]_{z=0}$ and $[\partial_z\mathcal{E}^{\text{as}}(z,t)]_{z=0}$,

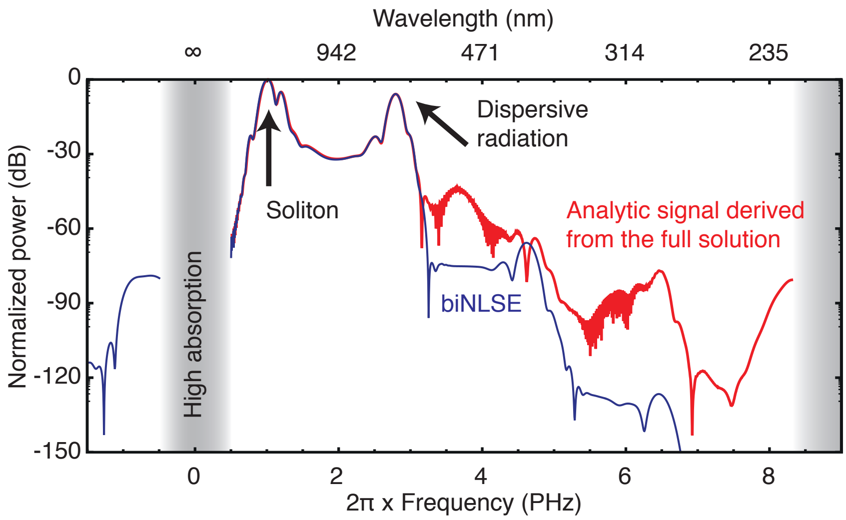

Figure 1: Blue line: power spectrum $|\mathcal{E}_\omega(z)|^2$ at $z = 3\,$mm for a three-cycle soliton (soliton number $= 2$) at initial carrier wavelength $\lambda_0 = 1.6\mu$m obtained from biNLSE (5) for the bulk fused silica dispersion. Red line: power spectrum $|\mathcal{E}^{\text{as}}_\omega(z)|^2$ of the exact analytic signal derived from the full Eq. (2). The reduced model properly describes both the soliton and the dispersive radiation. Still $\mathcal{E}(z,t)$ contains small negative-frequency part and is slightly inaccurate for $\omega \gtrsim 3\omega_0$. Two models also show excellent agreement in the time domain [5]. Therefore, to a good approximation, $\mathcal{E}(z,t)$ is determined by SPM and yields the analytic signal.

the latter two quantities naturally serve as initial conditions for the biNLSE (5). Therefore, by construction $\mathcal{E} \equiv \mathcal{E}^{\text{as}}$ at the beginning of the fiber. Still the difference $\|\mathcal{E} - \mathcal{E}^{\text{as}}\| \neq 0$ for $z > 0$, just because the negative frequency components are permanently generated by the $|\mathcal{E}|^2\mathcal{E}$ term in Eq. (5). However, a presupposed failure of the phase-matching conditions (6) guaranties that the negative-frequency part remains small and $\mathcal{E} \approx \mathcal{E}^{\text{as}}$. Indeed, Eqs. (6) directly indicate that the sum-frequency generated from any three positive frequencies by virtue of the $\mathcal{E}^3$ term is non-resonant and therefore $\mathcal{E}^3$ has only minor backward effect on solutions of Eq. (4). Moreover, assume that a negative frequency component $\omega_* < 0$ is generated by three waves with positive frequencies ω_1, ω_4, ω_3 by virtue of the $|\mathcal{E}|^2\mathcal{E}$ term. The excitation would be resonant if

$$\beta(\omega_1) - \beta(\omega_4) + \beta(\omega_3) = \beta(\omega_*), \qquad \omega_1 - \omega_4 + \omega_3 = \omega_*.$$

Replacing the only negative frequency ω_* with $-\omega_2$ we see that such three-wave interactions are again non-resonant due to condition (6). Therefore, the initially small negative-frequency part of $\mathcal{E}$, while being present, is not accumulated in the course of pulse propagation.

We conclude that for favorable dispersion $\beta(\omega)$ the complex field $\mathcal{E}$ yielded by Eq. (5) should be a good approximation to the analytic signal $\mathcal{E}^{\text{as}}$ yielded by the full wave equation Eq. (2) and definition (7). This conclusion is confirmed by numerical examples, as illustrated in Fig. 1. We now turn to solitary solutions of Eq. (5).

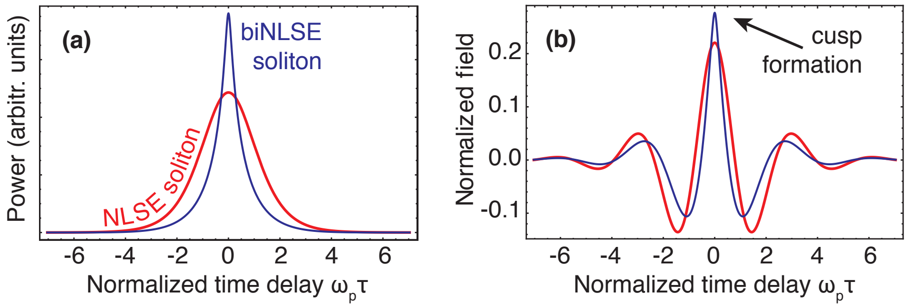

Figure 2: Ultrashort solitary solution of the biNLSE (5) (blue thin line) and the standard fundamental soliton which approximates pulse electric field as good as possible (red thick line). Calculations are made for the Drude dispersion, $\epsilon(\omega) = 1 - \omega_p^2/\omega^2$, pulse carrier frequency is twice the plasma frequency, $\omega_0 = 2\omega_p$. (a) power, (b) electric field. Further decrease of the pulse duration leads to cusp formation.

4 Solitons and spectral renormalisation

Equation (5) is well suited for the description of solitons, because the soliton spectrum is concentrated near its carrier frequency and the main precondition (6) can be easily verified. To derive solitary solutions we substitute the expression

$$\mathcal{E}(z,t) = f(\tau)e^{i(\kappa+\beta_0)z - i\omega_0 t}, \qquad \tau = t - \beta_1 z \tag{8}$$

into Eq. (5). Here, $f(\tau)$ denotes the unknown shape function, we use a standard notation $\beta_m = \beta^{(m)}(\omega_0)$, and solitons are parametrized by the nonlinear correction κ to the carrier wavenumber β_0. The soliton duration parameter t_0 is calculated self-consistently, for the fundamental soliton $f(\tau)$ is proportional to $\cosh^{-1}(\tau/t_0)$ and $t_0 = \sqrt{|\beta_2|/(2\kappa)}$, see, e.g., [2]. The resulting equation in the frequency domain reads

$$\left[\kappa - \tilde{\beta}(\Omega)\right] f_\Omega = \frac{\omega_0 + \Omega}{c} \frac{3\chi^{(3)}/8}{n(\omega_0+\Omega) + \eta} \left(|f|^2 f\right)_\Omega, \tag{9}$$

$$\tilde{\beta}(\Omega) = \beta(\omega_0 + \Omega) - \beta_0 - \beta_1\Omega, \qquad \eta = \frac{\kappa - \tilde{\beta}(\Omega)}{\omega_0 + \Omega}\frac{c}{2}. \tag{10}$$

where f_Ω denotes a Furrier component of $f(\tau)$. We define operator $\hat{\mathcal{N}}$ such that Eq. (9) becomes $f = \hat{\mathcal{N}}[f]$ the latter is solved using successive iterations in a full analogy with the spectral renormalisation method for NLSE [1]. Note, that each iteration is rescaled to improve convergence, first $f_{n+1/2} = \hat{\mathcal{N}}[f_n]$ and then $f_{n+1} = s_n f_{n+1/2}$, where s_n yields $\langle f_{n+1}|f_{n+1}\rangle = \langle f_n|f_n\rangle$. The final solution is given by $f(\tau) = \sqrt{s_\infty} f_\infty(\tau)$.

In general, the universal $\cosh^{-1}$ soliton shape always appears for $\kappa \to 0$. As κ increases the soliton duration decreases and its shape becomes more and more sharp. Our approach allows to trace solitons up to a nearly single-cycle duration. The further decrease of the pulse width appears to be limited by two other effects. Most often too short solitons are destroyed by Cherenkov radiation [3, 10]. Up to small nonlinear correction, Cherenkov resonant frequency ω_r is determined by the condition $\beta(\omega_r) = \beta_0 - \beta_1(\omega_r - \omega_0)$ indicating that operator $\hat{\mathcal{N}}$ in Eq. (9)

is singular. On the other hand, for a favorable, i.e., convex with $\beta''(\omega) < 0$, dispersion function the Cherenkov resonance does not appear. The solitons are then destroyed by singularity, the envelope forms a cusp in accord with analytical predictions [7, 4]. An exemplary shortest ultrashort solution calculated for a Drude dispersion function is shown in Fig 2.

To conclude we note that familiar spectrally-narrow, $\Omega \ll \omega_0$, solitons appear from Eq. (9) under the following four simplifications:

(1)	no higher-order dispersion	$\tilde{\beta}(\Omega) \to \beta_2\Omega^2/2$, where $\beta_2 < 0$
(2)	shock term is ignored	$(\omega_0 + \Omega)/c \to \omega_0/c$
(3)	bidirectionality is ignored	$\eta \to 0$
(4)	nonlinear dispersion is ignored	$\frac{3}{8}\chi^{(3)}/n(\omega_0 + \Omega) \to n_2 = \frac{3}{8}\chi^{(3)}/n(\omega_0)$

One can now use the full Eq. (9) and investigate the relative role of these effects, which are all neglected in the simplest integrable NLSE. The most important one is proven to be the higher-order dispersion followed by the shock term. Bidirectionality and dispersion of the nonlinear term are less important and can be ignored up to single-cycle regime. Even in the latter case their contribution to the soliton shape is only several percents, as compared to the higher-order dispersion and self-steepening [5].

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