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A compressible mixture model with phase transition

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Abstract

We introduce a new thermodynamically consistent diffuse interface model of Allen–Cahn/Navier–Stokes type for multi-component flows with phase transitions and chemical reactions. For the introduced diffuse interface model, we investigate physically admissible sharp interface limits by matched asymptotic techniques. We consider two scaling regimes, i.e. a non-dissipative and a dissipative regime, where we recover in the sharp interface limit a generalized Allen-Cahn/Euler system for mixtures with chemical reactions in the bulk phases equipped with admissible interfacial conditions. The interfacial conditions satify, for instance, a Young–Laplace and a Stefan type law.

1 Introduction

In this study, we propose a model for chemically reacting viscous fluid mixtures that may develop a transition between a liquid and a vapor phase. The mixture consists of N constituents and is described by N partial mass balance equations and a single equation of balance for the barycentric momentum. We exclusively consider isothermal evolutions. To describe the phase transition, we introduce an artificial phase field indicating the present phase by assigning the values 1 and -1 to the liquid and the vapor phase, respectively. Within the transition layer between two adjacent phases, the phase field smoothly changes between 1 and -1. However, usually the transition layers are very thin leading to steep gradients of the phase field.

This model belongs to the class of diffuse interface models. An alternative model class, that likewise represents phase transitions in fluid mixtures, contains sharp interface models. From the modelling point of view, sharp interface models have a simpler physical basis than diffuse interface models. For this reason, there arises always the non-trivial question if the sharp interface limits of a given diffuse model lead to admissible sharp interface models. The main concern of this paper is a careful discussion of this problem.

While diffuse interface models solve partial differential equations in the transition region, sharp interface models deal with jump conditions across the interface between the phases. Sometimes the jump conditions are mixed with geometric partial differential equations.

For two phases without chemical reactions, our compressible model reduces to an Allen–Cahn/Navier–Stokes type model, which is quite similar to the model derived by Blesgen [8]. Blesgen's model has been investigated analytically in [18, 14], where existence of strong local-in-time solutions and weak solutions has been shown.

We like to emphasize that the thermodynamical approaches of Blesgen's system and our derived model are different. For instance, the phase field variable χ in Blesgen's model, which satisfies the Allen-Cahn equation, is a physical quantity, namely the local mass fraction of one phase, whereas in our proposed model χ simply indicates the present phase. In addition, the function h that interpolates between the phases, which will be described later, is linear in Blesgen's model. This means that, in contrast to our proposed model, in general the equilibria depend on the chosen interpolation function h, which implies that Blesgen's system runs into different equilibria. Our approach with χ as an artificial phase field variable permits to obtain physical meaningful jump conditions at the interface agreeing with classical laws of thermodynamics.

A modified version of Blesgen's model can be found in [24]. In contrast to [8] and our introduced model, Witterstein [24] describes a mixture of two compressible fluids, which physically differ, exclusively by different Lamé coefficients which are assumed to depend on the phase field parameter and the mass density. Witterstein's model [24] also differs in the choice of the free energy, which contains two length scales. Moreover, the minima of the double well potential in the free energy have to be of different heights. This implies that the energy can only be controlled for transition regions with fixed width but not in the sharp interface limit.

Related to our work are diffuse interface models for incompressible and quasi-incompressible fluids. A diffuse interface model of Navier-Stokes-Cahn-Hilliard type for two incompressible, viscous Newtonian fluids, having the same densities, has been introduced by Hohenberg and Halperin in [17]. That model has been modified in several thermodynamically consistent ways such that different densities are allowed, see e.g. [16, 20, 4]. For existence results of strong local-in-time solutions and weak solutions, we refer to [1, 2, 3]. A diffuse interface model for two incompressible constituents which permits the transfer of mass between the phases due to diffusion and phase transitions has been proposed in [6, 5]. The densities of the fluids may be different, which leads to quasi–incompressibility of the mixture.

Our newly introduced diffuse interface model is given by the following system of PDEs for $(\rho, \rho_{\alpha=1,\dots,N-1}, \boldsymbol{v}, \chi)$ in $[0,T_f)\times\Omega$, $\Omega\subset\mathbb{R}^d$:

$$\begin{aligned}
\partial_t\rho + \operatorname{div}(\rho\boldsymbol{v}) &= 0,\\
\partial_t\rho_\alpha + \operatorname{div}(\rho_\alpha\boldsymbol{v}) - \operatorname{div}\Bigg(\sum_{\beta=1}^{N-1} M_{\alpha\beta}\nabla(\mu_\alpha-\mu_N)\Bigg) &= \sum_{i=1}^{N_R}\gamma_\alpha^i m_\alpha M_r^i\Bigg(1-\exp\left(\frac{A^i}{kT}\right)\Bigg),\\
\partial_t(\rho\boldsymbol{v}) + \operatorname{div}(\rho\boldsymbol{v}\otimes\boldsymbol{v}) + \nabla p + \operatorname{div}\left(\gamma\nabla\chi\otimes\nabla\chi - \boldsymbol{\sigma}_{NS}\right) &= 0,\\
\rho\partial_t\chi + \rho\boldsymbol{v}\cdot\nabla\chi &= -M_p\left(\frac{\partial\rho\psi}{\partial\chi} - \gamma\Delta\chi\right),
\end{aligned}$$

where p is the pressure, T the temperature, m_α the atomic mass of constituent α, k the Boltzmann constant,

$$\rho\psi \;=\; W(\chi) + \frac{\gamma}{2}|\nabla\chi|^2 + \rho f(\rho_1,\dots,\rho_N,\chi) \quad\text{and}\quad \mu_\alpha = \frac{\partial(\rho\psi)}{\partial\rho_\alpha}$$

with $\rho f(\rho_1,\dots,\rho_N,\chi) := h(\chi)\rho\psi_L(\rho_1,\dots,\rho_N) + (1-h(\chi))\rho\psi_V(\rho_1,\dots,\rho_N)$. In addition, γ_α^i are the stoichiometric coefficients of N_R possible chemical reactions, A^i the affinities and $M_{\alpha\beta}$, M_r^i and M_p the mobilities.

The work is organized as follows. In the upcoming section we derive the thermodynamically consistent model for multi-component flows with phase transitions and chemical reactions. The third section is devoted to the non-dimensionalization, the introduction of two interesting scaling regimes of the system and the setting of asymptotic analysis. Finally, in Sections 4 and 5, we determine the sharp interface limits for the two different scaling regimes introduced previously. We like to emphasize that Section 4.2 contains a conjecture on the incapability of viscous diffuse models to generate viscous sharp models.

2 The mixture model

2.1 Constituents and phases

We consider a fluid mixture consisting of N constituents $A_1, A_2, \dots, A_N$ indexed by $\alpha = 1,2,\dots,N$. The constituents may be subjected to chemical reactions. There are N_R reactions, indexed by $i = 1,2,\dots,N_R$, of the general type

$$a_1^iA_1 + a_2^iA_2 + \dots + a_N^iA_N \rightleftarrows b_1^iA_1 + b_2^iA_2 + \dots + b_N^iA_N. \tag{2.1}$$

Thus, there are forward(f) as well as backward(b) reactions. The constants $(a_\alpha^i)_{\alpha=1,2,\dots,N}$ and $(b_\alpha^i)_{\alpha=1,2,\dots,N}$ are positive integers and $\gamma_\alpha^i = b_\alpha^i - a_\alpha^i$ denotes the stoichiometric coefficient of constituent α in the reaction $i = 1,\dots,N_R$.

The fluid mixture may exist in the two phases liquid(L) and vapor(V). The two phases may coexist. In this paper, we describe the phases in the diffuse interface setting, where the interface between adjacent liquid and vapor phases is modelled by a thin layer. Within the layer, certain thermodynamic quantities smoothly change from values in one phase to different values in the adjacent phase. However, usually steep gradients occur.

2.2 Introduction of basic quantities and basic variables

Two phase mixtures can be modelled within three different model classes, i.e. Classes I-III. Class I considers as basic variables the mass densities $(\rho_\alpha)_{\alpha=1,2,\dots,N}$ of the constituents, the barycentric velocity $\boldsymbol{v}$, the temperature T and the phase field χ, which is used to indicate the present phase at $(t, \boldsymbol{x})$. It assumes values in the interval $[-1, 1]$ with $\chi = 1$ in the liquid and $\chi = -1$ in the vapor.. The basic variables of Class II are the mass densities $(\rho_\alpha)_{\alpha=1,2,\dots,N}$, the velocities $(\boldsymbol{v}_\alpha)_{\alpha=1,2,\dots,N}$ of the constituents, the temperature T and the phase field χ. Finally, in Class III we have the mass densities $(\rho_\alpha)_{\alpha=1,2,\dots,N}$, the velocities $(\boldsymbol{v}_\alpha)_{\alpha=1,2,\dots,N}$, the temperatures $(T_\alpha)_{\alpha=1,2,\dots,N}$ of the constituents and the phase field χ. In this study, we choose a description within Class I. The mixture occupies a region $\Omega \subset \mathbb{R}^d$. At any time $t \geq 0$, the thermodynamic state of Ω is described by N partial mass densities $(\rho_\alpha)_{\alpha=1,2,\dots,N}$, the barycentric velocity and by the temperature T of the mixture. These quantities may be functions of time $t \geq 0$ and space $\boldsymbol{x} = (x^i)_{i=1,\dots,d} = (x^1, \dots, x^d) \in \Omega$. However, we restrict ourselves to isothermal processes so that T appears only as a constant parameter in the equations.

Partial mass densities and partial velocities are used to define the total mass density ρ of the mixture and its barycentric velocity $\boldsymbol{v}$

$$\rho := \sum_{\alpha=1}^{N} \rho_\alpha, \qquad \boldsymbol{v} := \frac{1}{\rho} \sum_{\alpha=1}^{N} \rho_\alpha \boldsymbol{v}_\alpha. \tag{2.2}$$

The diffusion velocities $\boldsymbol{u}_\alpha$ and the corresponding diffusion fluxes $\boldsymbol{J}_\alpha$ are defined by

$$\boldsymbol{u}_\alpha := \boldsymbol{v}_\alpha - \boldsymbol{v}, \qquad \boldsymbol{J}_\alpha := \rho_\alpha \boldsymbol{u}_\alpha \quad \text{with} \quad \sum_{\alpha=1}^{N} \boldsymbol{J}_\alpha = 0\,. \tag{2.3}$$

Finally, we introduce the total number density of the mixture and the atomic fractions of the constituents

$$n := \sum_{\alpha=1}^{N} n_\alpha, \qquad y_\alpha := \frac{n_\alpha}{n} \quad \text{with} \quad \sum_{\alpha=1}^{N} y_\alpha = 1\,. \tag{2.4}$$

2.3 Equations of balance

The coupled system of PDEs for the basic variables of the Class I model relies on the equations of balance for the partial masses of the constituents, for the momentum of the mixture and for the phase field. Their generic structure reads

$$\partial_t \rho_\alpha + \operatorname{div}(\rho_\alpha \boldsymbol{v} + \boldsymbol{J}_\alpha) = r_\alpha, \qquad \alpha = 1, 2, \dots, N, \tag{2.5}$$

$$\partial_t(\rho \boldsymbol{v}) + \operatorname{div}(\rho \boldsymbol{v} \otimes \boldsymbol{v} - \boldsymbol{\sigma}) = \rho \boldsymbol{b}, \tag{2.6}$$

$$\partial_t(\rho \chi) + \operatorname{div}(\rho \chi \boldsymbol{v} + \boldsymbol{J}_\chi) = \xi_\chi. \tag{2.7}$$

The newly introduced quantities are: r_α - mass production rate of constituent α, $\boldsymbol{\sigma}$ - stress, $(\boldsymbol{J}_\chi, \xi_\chi)$ - flux and production rate of the phase field. The force density $\rho\boldsymbol{b}$ includes gravity and inertial forces, the latter only appear in case that the frame of reference is a non-inertial frame, i.e. a frame of reference that is undergoing acceleration with respect to an intertial frame.

The conservation law of mass for every single reaction $i = 1, 2, \dots, N_R$ reads

$$\sum_{\alpha=1}^{N} m_\alpha \gamma_\alpha^i = 0 \qquad \text{implying} \qquad \sum_{\alpha=1}^{N} r_\alpha = 0. \tag{2.8}$$

It is useful to decompose the N partial mass balances into the mass balance of the mixture and $N-1$ mass balances that serve as the basis for the diffusion equations, i.e.

$$\partial_t \rho + \operatorname{div}(\rho \boldsymbol{v}) = 0, \qquad \partial_t \rho_\alpha + \operatorname{div}(\rho_\alpha \boldsymbol{v} + \boldsymbol{J}_\alpha) = r_\alpha \quad \text{for} \quad \alpha = 1, 2, \dots, N-1. \tag{2.9}$$

2.4 General constitutive laws

2.4.1 Part 1: Basic assumptions

The variables ρ, $(\rho_\alpha)_{\alpha=1,2,\dots,N-1}$, $\boldsymbol{v}$ and χ are not the only quantities in the equations of balance. There are further quantities that must be given by thermodynamically consistent constitutive equations, such that the equations of balance become a PDE-system for the variables. Our constitutive model describes chemical reactions, diffusion, volume changes, viscosity and phase transitions including capillary effects.

The constitutive model for the mass production rates considers reactions with forward and backward path. The corresponding reaction rates R_f^i and R_b^i give the number of forward and backward reactions per volume and per time. Hence,

$$r_\alpha = \sum_{i=1}^{N_R} m_\alpha \gamma_\alpha^i (R_f^i - R_b^i). \tag{2.10}$$

The stress $\boldsymbol{\sigma}$ models volume changes, viscosity and capillarity. Due to these three phenomena we additively decompose the stress into three parts,

$$\boldsymbol{\sigma} = -p\mathbf{1} + \boldsymbol{\sigma}_{NS} + \boldsymbol{\sigma}_C, \tag{2.11}$$

where p denotes the pressure, $\boldsymbol{\sigma}_{NS}$ is the Navier-Stokes stress and $\boldsymbol{\sigma}_C$ is the so-called capillary stress.

There are various possibilities to characterize the phase transition by different choices of the flux $\boldsymbol{J}_\chi$ and the production rate ξ_χ. For example, the choice $\boldsymbol{J}_\chi \neq 0$ and $\xi_\chi = 0$ describes a transformation of phases due to diffusion and under the constraint of constant total phase fractions. The evolution equation for χ becomes the Cahn-Hilliard equation. In this study, we choose a different case, namely

$$\boldsymbol{J}_\chi = 0 \qquad \text{and} \qquad \xi_\chi \neq 0, \tag{2.12}$$

describing a situation where the phases exclusively change by a mechanism similar to a chemical reaction without a constraint to their total mass fractions. In this case the evolution equation for χ becomes the Allen-Cahn equation.

2.4.2 Part 2: Consequences of the 2$^{\text{nd}}$ law of thermodynamics

The considered mixture needs constitutive functions for the reaction rates R_f^i, R_b^i the diffusion fluxes $\boldsymbol{J}_\alpha$, the pressure p, the Navier-Stokes stress $\boldsymbol{\sigma}_{NS}$, the capillary stress $\boldsymbol{\sigma}_C$ and the production rate ξ_χ of the phases.

The thermodynamically consistent constitutive model relies on a free energy density of the general form

$$\rho\psi = \rho\tilde{\psi}(T, \rho_1, \rho_2, \dots, \rho_N, \chi, \nabla\chi). \tag{2.13}$$

In this paper we do not state and explicitly exploit the five axioms representing the 2$^{\text{nd}}$ law of thermodynamics. We refer the reader to the review article [9] and to [11]. In the following, we only list the constitutive relations and the representation of the entropy production.

1. Chemical potentials and pressure:

$$\mu_\alpha := \frac{\partial \rho\psi}{\partial \rho_\alpha}, \quad \alpha = 1, \dots, N, \qquad p = -\rho\psi + \sum_{\alpha=1}^{N} \mu_\alpha \rho_\alpha. \tag{2.14}$$

Note that μ_α is a definition and the representation of the pressure is then a consequence of the 2$^{\text{nd}}$ law and is called Gibbs-Duhem equation.

2. Representation of the stresses:

$$\boldsymbol{\sigma}_{NS} = \left(\lambda_1 + \frac{2}{d}\lambda_2\right) \operatorname{div}(\boldsymbol{v})\mathbf{1} + \lambda_2 \left(\nabla \boldsymbol{v} + (\nabla \boldsymbol{v})^T - \frac{2}{d}\operatorname{div}(\boldsymbol{v})\mathbf{1}\right), \ \boldsymbol{\sigma}_C = -\frac{\partial \rho\psi}{\partial \nabla\chi} \otimes \nabla\chi. \tag{2.15}$$

Bulk and shear viscosity satisfy the inequalities $\lambda_1 + \frac{2}{d}\lambda_2 \geq 0$ and $\lambda_2 \geq 0$, where d is the dimension of the considered space.

3. Diffusion laws and reaction rates:

$$\boldsymbol{J}_\alpha = -\sum_{\beta=1}^{N-1} M_{\alpha\beta} \nabla(\mu_\beta - \mu_N), \ \alpha = 1, \dots, N-1, \qquad R_b^i = R_f^i \exp\left(\frac{A^i}{kT}\right), \ i = 1, \dots, N_R, \tag{2.16}$$

where A^i is given by the law of actions, i.e.

$$A^i = \sum_{\alpha=1}^{N} m_\alpha \gamma_\alpha^i \mu_\alpha, \tag{2.17}$$

and is called the chemical affinity. The $(N-1)\times(N-1)$ matrix $M_{\alpha\beta}$ of diffusion mobilities is symmetric and positive definite. In this study, the mobilities are assumed to be constant. In equation(2.16), the Boltzmann constant k is introduced such that the argument of the exp-function becomes dimensionless.

The 2$^{\text{nd}}$ law prescribes the ratio of the reaction rates to be as in (2.16). Thus, either the forward or the backward rate can be modelled. We set $R_f^i = M_r^i$ and choose the reaction mobility $M_r^i > 0$ as constant.

4. Production rate of phases:

$$\xi_\chi = -M_p \left(\frac{\partial \rho\psi}{\partial \chi} - \operatorname{div}\left(\frac{\partial \rho\psi}{\partial \nabla\chi}\right)\right). \tag{2.18}$$

The quantity $M_p > 0$ denotes the phase mobility.

5. Representation and sign of the entropy production:

$$\xi_s = \frac{1}{T}\Bigg(-\sum_{\alpha=1}^{N} \boldsymbol{J}_\alpha \cdot \nabla\mu_\alpha + \boldsymbol{\sigma}_{NS} : \nabla \boldsymbol{v} - \xi_\chi \left(\frac{\partial \rho\psi}{\partial \chi} - \operatorname{div}\left(\frac{\partial \rho\psi}{\partial \nabla\chi}\right)\right) - \sum_{i=1}^{N_R} \left(\sum_{\alpha=1}^{N} m_\alpha \gamma_\alpha^i \mu_\alpha\right) (R_f^i - R_b^i) \Bigg) \geq 0\,. \tag{2.19}$$

The entropy production must be non negative for every solution of the balance equations, i.e. $\xi_s \geq 0$. Equilibrium is a solution of the balance equations with $\xi_s = 0$.

2.5 Special free energy densities for fluid mixtures capable of liquid-vapor phase transitions

In order to make the generic free energy function in (2.13) explicit we decompose $\rho\psi = \rho\tilde{\psi}(\rho_1, \dots, \rho_N, \chi, \nabla\chi)$ according to

$$\rho\psi = W(\chi) + \frac{\gamma}{2}|\nabla\chi|^2 + h(\chi)\rho\psi_L(\rho_1, \dots, \rho_N) + (1 - h(\chi))\rho\psi_V(\rho_1, \dots, \rho_N), \tag{2.20}$$

where $W(\chi) = (\chi - 1)^2(\chi + 1)^2$ and $h : \mathbb{R} \to [0, 1]$ is a smooth interpolation function satisfying

$$h(z) = \begin{cases} 1 & \text{for} \quad z \geq 1 \\ 0 & \text{for} \quad z \leq -1 \end{cases}, \quad \text{in particular,} \quad h'(z) = 0 \text{ for all } |z| \geq 1 \,. \tag{2.21}$$

The double well function W has its minima in the pure phases and controls the phase transition. The gradient term in (2.20) models capillarity effects and the coefficient $\gamma > 0$ is related to the surface tension between two adjacent phases. The two functions $\rho\psi_L, \rho\psi_V : (0, \infty)^N \longrightarrow [0, \infty)$ are the free energy density functions of the pure phases which we assume to be given by a combination of isotropic elastic response and entropy of mixing, i.e.

$$\rho\psi_{L/V} = \sum_{\alpha=1}^{N} \rho_\alpha \psi_\alpha^R + (K_{L/V} - p^R)\left(1 - \frac{n}{n^R}\right) + K_{L/V}\frac{n}{n^R}\ln\left(\frac{n}{n^R}\right) + kT\sum_{\alpha=1}^{N} n_\alpha \ln\left(\frac{n_\alpha}{n}\right),$$

where $K_{L/V}$ are the bulk moduli, and the superscript R indicates a reference value of the corresponding quantity.

Remark 2.1. In terms of the asymptotic analysis performed later in this paper, the crucial property of the energy densities chosen here is that they are convex, such that the map $(\rho_\alpha)_{\alpha=1,\dots,N} \mapsto (\mu_\alpha)_{\alpha=1,\dots,N}$ is a diffeomorphism for any fixed $\chi \in [-1, 1]$.

2.6 Summary

The resulting system of equations can be written as

$$\begin{aligned} \partial_t\rho + \operatorname{div}(\rho\boldsymbol{v}) &= 0, && (2.22)\\ \partial_t\rho_\alpha + \operatorname{div}(\rho_\alpha\boldsymbol{v}) - \operatorname{div}\Big(\sum_{\beta=1}^{N-1} M_{\alpha\beta}\nabla(\mu_\alpha - \mu_N)\Big) &= \sum_{i=1}^{N_R} \gamma_\alpha^i m_\alpha M_r^i \Big(1 - \exp\left(\frac{A^i}{kT}\right)\Big), && (2.23)\\ \partial_t(\rho\boldsymbol{v}) + \operatorname{div}(\rho\boldsymbol{v}\otimes\boldsymbol{v}) + \nabla p + \operatorname{div}(\gamma\nabla\chi\otimes\nabla\chi - \boldsymbol{\sigma}_{NS}) &= 0, && (2.24)\\ \rho\partial_t\chi + \rho\boldsymbol{v}\cdot\nabla\chi &= -M_p\left(\frac{\partial\rho\psi}{\partial\chi} - \gamma\Delta\chi\right). && (2.25) \end{aligned}$$

The pressure p and the affinities A^i are taken from $(2.14)_2$ and (2.17). Furthermore, we have used the following abbreviations

$$\begin{aligned} \rho\psi &= W(\chi) + \frac{\gamma}{2}|\nabla\chi|^2 + \rho f(\rho_1, \dots, \rho_N, \chi), && (2.26)\\ \mu_\alpha &= g_\alpha^R + \frac{K(\chi)}{m_\alpha n^R}\ln\left(\frac{n}{n^R}\right) + \frac{kT}{m_\alpha}\ln\left(\frac{n_\alpha}{n}\right) && (2.27) \end{aligned}$$

with

$$\begin{aligned} \rho f(\rho_1, \dots, \rho_N, \chi) &:= h(\chi)\rho\psi_L(\rho_1, \dots, \rho_N) + (1 - h(\chi))\rho\psi_V(\rho_1, \dots, \rho_N), && (2.28)\\ K(\chi) &:= h(\chi)K_L + (1 - h(\chi))K_V, && (2.29) \end{aligned}$$

where g_α^R, n^R are reference quantities.

2.7 Energy inequality

To address the issue of stability of (2.22)-(2.25), we prove an energy inequality. Let $\Omega \subset \mathbb{R}^d$ be some open and bounded domain with C^1–boundary and $T_f > 0$ some time up to which we assume that classical solutions of (2.22)-(2.25) exist.

Lemma 2.2 (Energy inequality)**.** *Let $((\rho_\alpha)_\alpha, \boldsymbol{v}, \chi)$ be a classical solution of* (2.22)-(2.25) *in $(0, T_f) \times \Omega$, then the following inequality is fulfilled for all $t \in (0, T_f)$:*

$$\frac{d}{dt}\int_\Omega \Big(W(\chi) + \frac{\gamma}{2}|\nabla\chi|^2 + (\rho f)((\rho_\alpha)_\alpha, \chi) + \frac{\rho}{2}|\boldsymbol{v}|^2\Big)\, d\boldsymbol{x} + \sum_{\alpha=1}^{N}\int_{\partial\Omega} \boldsymbol{n}\cdot(\rho_\alpha\mu_\alpha\boldsymbol{v} + \mu_\alpha\boldsymbol{J}_\alpha)\, d\boldsymbol{\sigma}$$
$$- \int_{\partial\Omega} \boldsymbol{n}\cdot\Big(\gamma\chi_t\nabla\chi - \frac{\rho}{2}\boldsymbol{v}|\boldsymbol{v}|^2 + \boldsymbol{\sigma}_{NS}\boldsymbol{v}\Big)\, d\boldsymbol{\sigma} = -D_1 - D_2 - D_3 - D_4 \le 0, \quad (2.30)$$

where $\boldsymbol{n}$ denotes the outer unit normal to $\partial\Omega$, and

$$D_1 := \int_\Omega \frac{M_p}{\rho}\Big(W'(\chi) - \gamma\Delta\chi + \frac{\partial(\rho f)}{\partial\chi}\Big)^2\, d\boldsymbol{x}, \quad (2.31)$$

$$D_2 := \int_\Omega \sum_{\alpha,\beta=1}^{N-1} M_{\alpha\beta}\nabla(\mu_\alpha - \mu_N)\cdot\nabla(\mu_\beta - \mu_N)\, d\boldsymbol{x}, \quad (2.32)$$

$$D_3 := -\int_\Omega \sum_{i=1}^{N_R} M_r^i A^i\left(1 - \exp\left(\frac{A^i}{kT}\right)\right)\, d\boldsymbol{x} = kT\int_\Omega \sum_{i=1}^{N_R} \log\Big(\frac{R_b^i}{R_f^i}\Big)(R_b^i - R_f^i)\, d\boldsymbol{x}, \quad (2.33)$$

$$D_4 := \int_\Omega \boldsymbol{\sigma}_{NS} : (\nabla\boldsymbol{v})\, d\boldsymbol{x}. \quad (2.34)$$

Corollary 2.3. *Let $((\rho_\alpha)_\alpha, \boldsymbol{v}, \chi)$ be a classical solution of* (2.22)-(2.25) *in $(0, T_f) \times \Omega$ satisfying the boundary conditions $\nabla\chi\cdot\boldsymbol{n} = 0$, $\boldsymbol{J}_\alpha\cdot\boldsymbol{n} = 0$, $\alpha = 1, \dots, N-1$, and $\boldsymbol{v} = 0$ on $(0, T_f)\times\partial\Omega$, then for all $t \in (0, T_f)$:*

$$\frac{d}{dt}\int_\Omega \Big(W(\chi) + \frac{\gamma}{2}|\nabla\chi|^2 + (\rho f)((\rho_\alpha)_\alpha, \chi) + \frac{\rho}{2}|\boldsymbol{v}|^2\Big)\, d\boldsymbol{x} \le 0. \quad (2.35)$$

Proof of Lemma 2.2. We directly compute

$$\begin{aligned}
&\frac{d}{dt}\int_\Omega \Big(W(\chi) + \frac{\gamma}{2}|\nabla\chi|^2 + (\rho f)((\rho_\alpha)_\alpha, \chi) + \frac{\rho}{2}|\boldsymbol{v}|^2\Big)\, d\boldsymbol{x} \\
&\quad = \int_\Omega \Big(W'(\chi)\chi_t + \gamma\nabla\chi\cdot\nabla\chi_t + \sum_\alpha \rho_{\alpha,t}\mu_\alpha + \frac{\partial(\rho f)}{\partial\chi}\chi_t - \frac{1}{2}\rho_t|\boldsymbol{v}|^2 + \boldsymbol{v}\cdot(\rho\boldsymbol{v})_t\Big) d\boldsymbol{x} \\
&\quad = \int_\Omega \Big(\chi_t\Big(W'(\chi) - \gamma\Delta\chi + \frac{\partial(\rho f)}{\partial\chi}\Big) + \sum_\alpha \rho_{\alpha,t}\mu_\alpha - \frac{1}{2}\rho_t|\boldsymbol{v}|^2 + \boldsymbol{v}\cdot(\rho\boldsymbol{v})_t\Big)\, d\boldsymbol{x} \\
&\qquad + \int_{\partial\Omega} \gamma\chi_t\boldsymbol{n}\cdot\nabla\chi\, d\boldsymbol{\sigma}.
\end{aligned} \quad (2.36)$$

We insert the evolution equations (2.22)-(2.25) into (2.36) to eliminate the time derivatives. Several terms cancel out such that we obtain

$$
\begin{aligned}
&\frac{d}{dt}\int_\Omega \Big(W(\chi)+\frac{\gamma}{2}|\nabla\chi|^2+\rho f((\rho_\alpha)_\alpha,\chi)+\frac{\rho}{2}|\boldsymbol{v}|^2\Big)\,d\boldsymbol{x}\\
&\quad=-\int_\Omega\left(\sum_{\alpha=1}^N(\operatorname{div}(\rho_\alpha\boldsymbol{v}+\boldsymbol{J}_\alpha)-r_\alpha)\mu_\alpha-\frac{1}{2}\operatorname{div}(\rho\boldsymbol{v})|\boldsymbol{v}|^2+\operatorname{div}(\rho\boldsymbol{v}\otimes\boldsymbol{v})\cdot\boldsymbol{v}\right)\,d\boldsymbol{x}\\
&\qquad-\int_\Omega\left(\boldsymbol{v}\cdot\sum_{\alpha=1}^N\rho_\alpha\nabla\mu_\alpha-\operatorname{div}(\boldsymbol{\sigma}_{NS})\cdot\boldsymbol{v}\right)\,d\boldsymbol{x}-D_1+\int_{\partial\Omega}\gamma\chi_t\boldsymbol{n}\cdot\nabla\chi\,d\boldsymbol{\sigma},
\end{aligned}
\tag{2.37}
$$

where $\boldsymbol{J}_\alpha$ and r_α are given by (2.10) and (2.16). Using integration by parts in (2.37), we get

$$
\begin{aligned}
&\frac{d}{dt}\int_\Omega \Big(W(\chi)+\frac{\gamma}{2}|\nabla\chi|^2+\rho f((\rho_\alpha)_\alpha,\chi)+\frac{\rho}{2}|\boldsymbol{v}|^2\Big)\,d\boldsymbol{x}\\
&=\int_\Omega\sum_{\alpha=1}^N(\boldsymbol{J}_\alpha\cdot\nabla\mu_\alpha+r_\alpha\mu_\alpha)\,d\boldsymbol{x}-D_1-D_4\\
&\quad+\int_{\partial\Omega}\boldsymbol{n}\cdot\Big(\gamma\chi_t\nabla\chi-\sum_{\alpha=1}^N\mu_\alpha(\rho_\alpha\boldsymbol{v}+\boldsymbol{J}_\alpha)-\frac{\rho}{2}\boldsymbol{v}|\boldsymbol{v}|^2+\boldsymbol{\sigma}_{NS}\boldsymbol{v}\Big)\,d\boldsymbol{\sigma}\\
&=-\sum_{j=1}^4 D_j+\int_{\partial\Omega}\boldsymbol{n}\cdot\Big(\gamma\chi_t\nabla\chi-\sum_{\alpha=1}^N\mu_\alpha(\rho_\alpha\boldsymbol{v}+\boldsymbol{J}_\alpha)-\frac{\rho}{2}\boldsymbol{v}|\boldsymbol{v}|^2+\boldsymbol{\sigma}_{NS}\boldsymbol{v}\Big)\,d\boldsymbol{\sigma}.
\end{aligned}
\tag{2.38}
$$

□

3 Asymptotic analysis

To avoid physically impossible scalings, we first nondimensionalize the system (2.22)-(2.25).

3.1 Non-dimensionalization

We introduce reference quantities, denoted by superscript ^{c}, and non-dimensional quantities, denoted by *, i.e.

$$
\begin{aligned}
&\boldsymbol{x}=x^c\boldsymbol{x}^*,\ t=t^ct^*,\ \rho_\alpha=\rho^c\rho_\alpha^*,\ \boldsymbol{v}=v^c\boldsymbol{v}^*,\ \mu_\beta=\mu^c\mu_\beta^*,\ M_{\alpha\beta}=M^cM_{\alpha\beta}^*,\ M_r^i=M_r^c(M_r^i)^*,\\
&\gamma_\beta=\gamma_r^c\gamma_\beta^*,\ m_\beta=m^cm_\beta^*,\ \lambda_{1,2}=\lambda^c\lambda_{1,2}^*,\ M_p=M_p^cM_p^*,\ W=W^cW^*,\ (\rho f)=(\rho f)^c(\rho f)^*,\\
&\gamma=\gamma^c\gamma^*,\ A^i=A^c(A^i)^*.
\end{aligned}
$$

Note that χ and the interpolation function h are already nondimensionalized and $\rho = \rho^c \rho^*$ with $\rho^* = \sum_\alpha \rho^*_\alpha$. Thereby, we get

$$
\begin{aligned}
\frac{\rho^c}{t^c}\partial_{t^*}\rho^* + \frac{\rho^c v^c}{x^c}\mathrm{div}^*(\rho^*\boldsymbol{v}^*) = 0,\\
\frac{\rho^c}{t^c}\partial_{t^*}\rho^*_\alpha + \frac{\rho^c v^c}{x^c}\mathrm{div}^*(\rho^*_\alpha\boldsymbol{v}^*) - \frac{M^c\mu^c}{(x^c)^2}\mathrm{div}^*\Big(\sum_{\beta=1}^{N-1} M^*_{\alpha\beta}\nabla^*(\mu^*_\beta - \mu^*_N)\Big)\\
-M^c_r\gamma^c_r m^c\sum_{i=1}^{N_R}(\gamma^i_\alpha)^* m^*_\alpha (M^i_r)^*\left(1-\exp\left(\frac{A^c(A^i)^*}{kT}\right)\right) = 0,\\
\frac{\rho^c v^c}{t^c}\partial_{t^*}(\rho^*\boldsymbol{v}^*) + \frac{\rho^c(v^c)^2}{x^c}\mathrm{div}^*(\rho^*\boldsymbol{v}^*\otimes\boldsymbol{v}^*) + \frac{1}{x^c}\nabla^*\left(\sum_{\alpha=1}^{N}\rho^c\mu^c\rho^*_\alpha\mu^*_\alpha - (\rho f)^c(\rho f)^*\right)\\
-\frac{1}{x^c}\nabla^*\Big(W^cW^* + \frac{\gamma^c\gamma^*}{2(x^c)^2}|\nabla^*\chi|^2\Big) + \frac{\gamma^c\gamma^*}{(x^c)^3}\mathrm{div}^*(\nabla^*\chi\otimes\nabla^*\chi) - \frac{\lambda^c v^c}{(x^c)^2}\mathrm{div}^*(\boldsymbol{\sigma}^*_{NS}) = 0,\\
\frac{1}{t^c}\partial_{t^*}\chi + \frac{v^c}{x^c}\boldsymbol{v}^*\cdot\nabla^*\chi + \frac{M^c_pM^*_p}{\rho^c\rho^*}\left(W^cW^{*\prime} - \frac{\gamma^c\gamma^*}{(x^c)^2}\Delta^*\chi + (\rho f)^c\frac{\partial(\rho f)^*}{\partial\chi}\right) = 0.
\end{aligned}
\tag{3.1}
$$

As we are interested in hyperbolic scalings we set $x^c = v^c t^c$, $(\rho f)^c = \rho^c\mu^c$ and define the following Mach and Reynolds numbers

$$
\mathrm{M}_W := v^c\sqrt{\frac{\rho^c}{W^c}},\quad \mathrm{M}_{\rho f} := v^c\sqrt{\frac{\rho^c}{(\rho f)^c}},\quad \mathrm{Re} := \frac{\rho^c v^c x^c}{\lambda^c}. \tag{3.2}
$$

Moreover, we choose $A^c = kT$ and define additionally nondimensional quantities related to the reaction and diffusion mobilities

$$
\bar{M}_d = \frac{M^c\mu^c}{v^c x^c\rho^c},\quad \bar{M}_r = \frac{M^c_r\gamma^c_r m^c t^c}{\rho^c},\quad \bar{M}_p = \frac{M^c_p t^c W^c}{\rho^c}. \tag{3.3}
$$

We assume that the small parameter

$$
\varepsilon := \sqrt{\frac{\gamma^c}{(x^c)^2 W^c}} \tag{3.4}
$$

is proportional to the width of the interfacial layer. This can be justified by Γ-Limit methods, cf. [23, 21, 22, 13]. Using these nondimensional parameters and suppressing * in the notation, we get

$$
\begin{aligned}
\partial_t\rho + \mathrm{div}(\rho\boldsymbol{v}) = 0,\\
\partial_t\rho_\alpha + \mathrm{div}(\rho_\alpha\boldsymbol{v}) - \bar{M}_d\,\mathrm{div}\Big(\sum_{\beta=1}^{N-1}M_{\alpha\beta}\nabla(\mu_\beta-\mu_N)\Big) - \bar{M}_r\sum_{i=1}^{N_R}\gamma^i_\alpha m_\alpha M^i_r(1-\exp(A^i)) = 0,\\
\partial_t(\rho\boldsymbol{v}) + \mathrm{div}(\rho\boldsymbol{v}\otimes\boldsymbol{v}) + \frac{1}{\mathrm{M}^2_{\rho f}}\nabla\left(\sum_{\alpha=1}^{N}\rho_\alpha\mu_\alpha - \rho f\right) - \frac{1}{\mathrm{M}^2_W}\nabla(W + \varepsilon^2\frac{\gamma}{2}|\nabla\chi|^2)\\
+\frac{\varepsilon^2}{\mathrm{M}^2_W}\gamma\mathrm{div}(\nabla\chi\otimes\nabla\chi) - \frac{1}{\mathrm{Re}}\mathrm{div}(\boldsymbol{\sigma}_{NS}) = 0,\\
\partial_t\chi + \boldsymbol{v}\cdot\nabla\chi + \bar{M}_p\frac{M_p}{\rho}\left(W' - \varepsilon^2\gamma\Delta\chi + \frac{\mathrm{M}^2_W}{\mathrm{M}^2_{\rho f}}\frac{\partial\rho f}{\partial\chi}\right) = 0.
\end{aligned}
\tag{3.5}
$$

In the sequel, we will consider two scaling regimes. For both of them we choose

$$
\bar{M}_d = \bar{M}_r = \mathrm{M}_{\rho f} = 1,\quad \mathrm{M}_W = \sqrt{\varepsilon},\quad \frac{1}{\mathrm{Re}} = \varepsilon^2. \tag{3.6}
$$

The scalings only differ in $\bar{M}_p$. Taking

$$\bar{M}_p = \frac{1}{\varepsilon^3} \text{ leads to a non-dissipative regime and } \bar{M}_p = \frac{1}{\varepsilon^2} \text{ to a dissipative regime.} \tag{3.7}$$

Remark 3.1. We are aware of the fact that it would be favorable also to obtain the full Navier-Stokes equations in the bulk in the leading order. This would correspond to the scaling $\mathrm{Re} = 1$. We will show in Subsection 4.2 that such a scaling immediately rules out mass fluxes across the interface. As we are interested in situations in which phase change occurs, we do not pursue this scaling.

3.2 Assumptions and definitions for formal asymptotics

To keep this paper self-contained, we state the necessary assumptions and definitions of formal asymptotic expansions.

3.2.1 Outer setting

We define the two bulk phases for $t \in [0, T_f)$ by

$$\Omega^-(t;\varepsilon) := \{\boldsymbol{x} \in \Omega :\ \chi_\varepsilon(t,\boldsymbol{x}) < 0\} \quad \text{and} \quad \Omega^+(t;\varepsilon) := \{\boldsymbol{x} \in \Omega :\ \chi_\varepsilon(t,\boldsymbol{x}) > 0\}.$$

We assume that the solutions $((\rho_{\alpha,\varepsilon})_\alpha, \boldsymbol{v}_\varepsilon, \chi_\varepsilon)$ of the considered scalings of (2.22)-(2.25) have expansions in ε in the outer regions $\Omega^\pm(t;\varepsilon)$, (in fact, we only need expansions up to the first two summands of each series):

$$\chi_\varepsilon(t,\boldsymbol{x}) = \sum_{i=0}^{\infty} \varepsilon^i \chi_i(t,\boldsymbol{x}), \quad \boldsymbol{v}_\varepsilon(t,\boldsymbol{x}) = \sum_{i=0}^{\infty} \varepsilon^i \boldsymbol{v}_i(t,\boldsymbol{x}) \text{ and } \rho_{\alpha,\varepsilon}(t,\boldsymbol{x}) = \sum_{i=0}^{\infty} \varepsilon^i \rho_{\alpha,i}(t,\boldsymbol{x}). \tag{3.8}$$

Hence, we may expand W into its Taylor series.

We call a family $(\hat{\Gamma}(t))_{t\in[0,T_f)}$ an oriented $C^{1,2}$-family of hypersurfaces if for each point $(t_0, \boldsymbol{x}_0) \in (0, T_f) \times \mathbb{R}^d$ with $\boldsymbol{x}_0 \in \hat{\Gamma}(t_0)$ the following properties are satisfied:

(i) There exist an open subset $O \subset \mathbb{R}^d$ containing $\boldsymbol{x}_0$, $\delta > 0$ and a function $u \in C^{1,2}((t_0 - \delta, t_0 + \delta) \times O)$ such that

$$O \cap \hat{\Gamma}(t) = \{\boldsymbol{x} \in O \mid u(t,\boldsymbol{x}) = 0\} \quad \text{and}$$
$$\nabla u(t,\boldsymbol{x}) \neq 0 \quad \text{for } \boldsymbol{x} \in O \cap \hat{\Gamma}(t), t \in (t_0 - \delta, t_0 + \delta).$$

(ii) There exists a unit normal field $\boldsymbol{\nu}$ for $\hat{\Gamma}$ such that $\boldsymbol{\nu} \in C^0\left(\bigcup_{0<t<T_f}(\{t\} \times \hat{\Gamma}(t)), \mathbb{R}^d\right)$ and $\boldsymbol{\nu}(t,\cdot) \in C^1(\hat{\Gamma}(t), \mathbb{R}^d)$.

We assume that for $\varepsilon > 0$ small enough,

$$\Gamma_\varepsilon := \{(t,\boldsymbol{x}) \in [0,T_f) \times \Omega \ :\ \chi_\varepsilon(t,\boldsymbol{x}) = 0\}$$

is a set of smoothly evolving oriented $C^{1,2}$-hypersurfaces in $[0, T_f) \times \mathbb{R}^d$. In addition, we assume the existence of a limiting $C^{1,2}$-family of oriented hypersurfaces Γ for ε going to zero. The curve Γ is the zeroth order of the interface. We denote the limiting bulk regions by $\Omega^+(t)$ and $\Omega^-(t)$. Further orders of Γ_ε are not required here. They would be needed if we considered higher order jump conditions, see [12].

3.2.2 Inner setting

In a neighborhood of Γ, we introduce a new coordinate system. To this end, let $\boldsymbol{\varrho}$ be a local parameterization of Γ:

$$\boldsymbol{\varrho} : [0, T_f) \times U \to \mathbb{R}^d,$$

where $[0, T_f) \subset \mathbb{R}$ and $U \subset \mathbb{R}^{d-1}$ are the time interval and the spatial parameter domain, respectively.

Next, we parameterize a neighborhood of $\boldsymbol{\varrho}(t, U)$ in $\mathbb{R}^d$ as follows:

$$(t, \boldsymbol{x}) = (t, \boldsymbol{\varrho}(t, \boldsymbol{s}) + \varepsilon z \boldsymbol{\nu}(t, \boldsymbol{s})) \tag{3.9}$$

with $0 < \varepsilon \le \varepsilon_0$ for some $\varepsilon_0 > 0$ and $z \in \mathbb{R}$. The normal and tangential velocity of the interface Γ are related to the parameterization via

$$\boldsymbol{w}_{\boldsymbol{\nu}} := w_{\boldsymbol{\nu}} \boldsymbol{\nu} := (\partial_t \boldsymbol{\varrho} \cdot \boldsymbol{\nu})\boldsymbol{\nu} \quad \text{and} \quad \boldsymbol{w}_{\boldsymbol{\tau}} := \partial_t \boldsymbol{\varrho} - (\partial_t \boldsymbol{\varrho} \cdot \boldsymbol{\nu})\boldsymbol{\nu}. \tag{3.10}$$

For a generic function, depending on outer variables f we denote the corresponding function in inner variables by capital F, i.e.

$$F(t, \boldsymbol{s}, z) = f(t, \boldsymbol{x}).$$

The partial derivatives of these functions transform as follows:

$$\begin{pmatrix} \nabla f \\ \partial_t f \end{pmatrix} = \begin{pmatrix} \frac{1}{|\boldsymbol{T}|^2}\boldsymbol{T} & \varepsilon^{-1}\boldsymbol{\nu} & 0 \\ -\boldsymbol{w}_{\boldsymbol{\tau}} & -\varepsilon^{-1} w_{\boldsymbol{\nu}} & 1 \end{pmatrix} \begin{pmatrix} \nabla_\Gamma F \\ \partial_z F \\ \partial_t F \end{pmatrix} + \mathcal{O}(\varepsilon)$$

where $\boldsymbol{T}$ is a $d \times (d-1)$-matrix whose columns are given by a basis of tangent vectors on Γ. Moreover, we have

$$\begin{aligned} &\operatorname{div} \boldsymbol{f} = \tfrac{1}{\varepsilon}\partial_z \boldsymbol{F} \cdot \boldsymbol{\nu} + \operatorname{div}_\Gamma \boldsymbol{F} + \mathcal{O}(\varepsilon), \\ &\Delta f = \tfrac{1}{\varepsilon^2}\partial_{zz} F - \tfrac{1}{\varepsilon}\kappa \partial_z F - z|\kappa|^2 \partial_z F + \Delta_\Gamma F + \mathcal{O}(\varepsilon), \end{aligned}$$

where $\nabla_\Gamma, \operatorname{div}_\Gamma, \Delta_\Gamma$ are the surface gradient, the surface divergence, and the surface Laplacian on Γ, and κ is the mean curvature, respectively.

For the inner counterpart $((R_{\alpha,\varepsilon})_\alpha, \boldsymbol{V}_\varepsilon, X_\varepsilon)$ of the outer functions $((\rho_{\alpha,\varepsilon})_\alpha, \boldsymbol{v}_\varepsilon, \chi_\varepsilon)$, we assume:

$$R_{\alpha,\varepsilon}(t, \boldsymbol{s}, z) = \sum_{i=0}^{\infty} \varepsilon^i R_{\alpha,i}(t, \boldsymbol{s}, z), \ \boldsymbol{V}_\varepsilon(t, \boldsymbol{s}, z) = \sum_{i=0}^{\infty} \varepsilon^i \boldsymbol{V}_i(t, \boldsymbol{s}, z), \ X_\varepsilon(t, \boldsymbol{s}, z) = \sum_{i=0}^{\infty} \varepsilon^i X_i(t, \boldsymbol{s}, z) \tag{3.11}$$

Remark 3.2. Due to our definitions of Γ_ε and Γ we cannot expect $X_0(t, \boldsymbol{s}, 0) = 0$ but there will be a translational quantity depending on t and $\boldsymbol{s}$. We could expand the interface position in ε, which would ensure $X_0(t, \boldsymbol{s}, 0) = 0$. However, we prefer the definition of Γ as in the orders studied here the translational constant causes no problems as no interfacial mass density appears. This is in contrast to the situation in [12].

3.3 Matching relations

In matched asymptotic techniques, inner and outer quantities are linked via certain matching conditions, see e.g. [10]. We impose the following asymptotic behavior for a generic quantity f as $z \to \pm\infty$:

$$F_0(t, \boldsymbol{s}, z) - f_0^{\pm} = o(1/|z|), \tag{3.12}$$

$$F_1(t,\boldsymbol{s},z) - f_1^{\pm} - ((\nabla f_0)^{\pm}\cdot\boldsymbol{\nu}(t,\boldsymbol{s}))z = o(1/|z|), \tag{3.13}$$

where the superscript $\pm$ denotes $\lim_{\varepsilon\searrow 0} f(t,\boldsymbol{\varrho}(t,\boldsymbol{s})\pm\varepsilon\boldsymbol{\nu}(t,\boldsymbol{s}))$. Moreover, we have

$$\partial_z F_0(t,\boldsymbol{s},z) = o(1/|z|), \tag{3.14}$$

$$\partial_{zz} F_0(t,\boldsymbol{s},z) = o(1/|z|), \tag{3.15}$$

$$\partial_z F_1(t,\boldsymbol{s},z) - (\nabla f_0)^{\pm}\cdot\boldsymbol{\nu}(t,\boldsymbol{s}) = o(1/|z|), \tag{3.16}$$

$$\nabla_\Gamma F_0(t,\boldsymbol{s},z) - (\nabla f_0)^{\pm} + ((\nabla f_0)^{\pm}\cdot\boldsymbol{\nu}(t,\boldsymbol{s}))\boldsymbol{\nu}(t,\boldsymbol{s}) = o(1/|z|). \tag{3.17}$$

The idea behind this matching method is that the large z–behavior (for small ε) of the inner quantities coincides with the traces of the outer quantities, see e.g. [19]. To this end, a formal term-by-term matching of the ε-expansion of the inner quantities to the Taylor polynomials of the outer ones is made, see [10, 15].

4 Sharp interface limit of the dissipative regime

4.1 Low viscosity case

We start by defining outer solutions in the bulk phases. They are obtained by inserting (3.8) into the scaled equations and comparing the terms order by order.

Definition 4.1. A tuple $((\rho_{\alpha,0})_{\alpha=1,\dots,N},\boldsymbol{v}_0,\chi_0,\chi_1)$ with

$$\begin{aligned}
\rho_{\alpha,0} &\in C^0([0,T_f),C^2(\Omega^{\pm},\mathbb{R}_+))\cap C^1([0,T_f),C^0(\Omega^{\pm},\mathbb{R}_+)),\\
\boldsymbol{v}_0 &\in C^0([0,T_f),C^1(\Omega^{\pm},\mathbb{R}^d))\cap C^1([0,T_f),C^0(\Omega^{\pm},\mathbb{R}^d)),\\
\chi_0 &\in C^0([0,T_f),C^2(\Omega^{\pm},\mathbb{R})),\\
\chi_1 &\in C^0([0,T_f),C^1(\Omega^{\pm},\mathbb{R})),
\end{aligned} \tag{4.1}$$

where $\mathbb{R}_+ := \{x\in\mathbb{R} : x>0\}$, is called an outer solution of the *dissipative regime* provided

$$\partial_t\rho_0 + \operatorname{div}(\rho_0\boldsymbol{v}_0) = 0, \tag{4.2}$$

$$\partial_t\rho_{\alpha,0} + \operatorname{div}(\rho_{\alpha,0}\boldsymbol{v}_0) - \operatorname{div}\left(\sum_{\beta=1}^{N-1} M_{\alpha\beta}\nabla(\mu_{\beta,0}-\mu_{N,0})\right) - \sum_{i=1}^{N_R}\gamma_\alpha^i m_\alpha M_r^i\left(1-\exp\left(A_0^i\right)\right) = 0, \tag{4.3}$$

$$W'(\chi_0) = 0,\ \text{in particular, } \nabla(W(\chi_0)) = 0, \tag{4.4}$$

$$\partial_t(\rho_0\boldsymbol{v}_0) + \operatorname{div}(\rho_0\boldsymbol{v}_0\otimes\boldsymbol{v}_0) + \nabla\left(\sum_{\alpha=1}^{N}\rho_{\alpha,0}\mu_{\alpha,0} - (\rho f_0)\right) - \nabla(W'(\chi_0)\chi_1) = 0, \tag{4.5}$$

$$W''(\chi_0)\chi_1 + \frac{\partial(\rho f)}{\partial\chi}(\rho_{1,0},\dots,\rho_{N,0},\chi_0) = 0, \tag{4.6}$$

where we use the following abbreviations

$$\mu_{\alpha,0} := \mu_\alpha(\rho_{1,0},\dots,\rho_{N,0},\chi_0),\quad \rho f_0 := \rho f(\rho_{1,0},\dots,\rho_{N,0},\chi_0),\quad A_0^i = \sum_{\alpha=1}^{N} m_\alpha\gamma_\alpha^i\mu_{\alpha,0}. \tag{4.7}$$

Next, we define inner solutions. They are obtained from the scaled system by changing coordinates via (3.9) and inserting (3.11).

Definition 4.2. A tuple $((R_{\alpha,0})_{\alpha=1,\dots,N}, (R_{\alpha,1})_{\alpha=1,\dots,N}, \boldsymbol{V}_0, X_0, X_1)$ with $X_0 \not\equiv 0$ and

$$\begin{aligned}
R_{\alpha,0} &\in C^0([0,T_f), C^0(U, C^2(\mathbb{R}_+))), \\
R_{\alpha,1} &\in C^0([0,T_f), C^0(U, C^2(\mathbb{R}))), \\
\boldsymbol{V}_0 &\in C^0([0,T_f), C^0(U, C^1(\mathbb{R}^d))), \\
X_0 &\in C^0([0,T_f), C^1(U, C^0(\mathbb{R}))) \cap C^0([0,T_f), C^0(U, C^2(\mathbb{R}))), \\
X_1 &\in C^0([0,T_f), C^0(U, C^2(\mathbb{R}))),
\end{aligned} \tag{4.8}$$

is called an *inner solution of the dissipative regime* with normal velocity $w_{\boldsymbol{\nu}}$ provided

$$\Big(\sum_{\beta=1}^{N-1} M_{\alpha\beta}(\mathcal{M}_{\beta,0} - \mathcal{M}_{N,0})_z\Big)_z = 0, \tag{4.9}$$

$$W'(X_0) - \gamma X_{0,zz}, \text{ in particular, } 0 = \boldsymbol{\nu}(-W(X_0))_z + \gamma \boldsymbol{\nu} X_{0,z} X_{0,zz} = 0, \tag{4.10}$$

$$(R_{\alpha,0}(\boldsymbol{V}_0 \cdot \boldsymbol{\nu} - w_{\boldsymbol{\nu}}))_z - \sum_{\beta=1}^{N-1} M_{\alpha\beta}\Big((\mathcal{M}_{\beta,1} - \mathcal{M}_{N,1})_z - \kappa(\mathcal{M}_{\beta,0} - \mathcal{M}_{N,0})\Big)_z = 0, \tag{4.11}$$

$$(R_0(\boldsymbol{V}_0 \cdot \boldsymbol{\nu} - w_{\boldsymbol{\nu}}))_z = (j_0)_z = 0, \tag{4.12}$$

$$\begin{aligned}
&j_0 \boldsymbol{V}_{0,z} + \boldsymbol{\nu}\Big(\sum_{\alpha=1}^{N} R_{\alpha,0}\mathcal{M}_{\alpha,0} - (RF_0) - W'(X_0)X_1\Big)_z \\
&\quad + \gamma\boldsymbol{\nu}(X_{0,z}X_{1,zz} + X_{0,zz}X_{1,z} - \kappa X_{0,z}^2) - \nabla_\Gamma W(X_0) + \gamma X_{0,zz}\nabla_\Gamma(X_0) = 0,
\end{aligned} \tag{4.13}$$

$$\frac{j_0}{M_p} X_{0,z} + W''(X_0)X_1 - \gamma X_{1,zz} + \gamma\kappa X_{0,z} + \frac{\partial(\rho f)}{\partial \chi}(R_{1,0}, \dots, R_{N,0}, X_0) = 0, \tag{4.14}$$

where (4.9) and (4.11) hold for $\alpha = 1, \dots, N-1$ and we use

$$\begin{aligned}
&j_0 := R_0((\boldsymbol{V}_0 \cdot \boldsymbol{\nu}) - w_{\boldsymbol{\nu}}), \quad \mathcal{M}_{\beta,0} := \mu_\beta(R_{1,0}, \dots, R_{N,0}, X_0), \\
&RF_0 := (\rho f)(R_{1,0}, \dots, R_{N,0}, X_0), \\
&\mathcal{M}_{\beta,1} := \sum_{\alpha=1}^{N} \frac{\partial \mu_\beta}{\partial \rho_\alpha}(R_{1,0}, \dots, R_{N,0}, X_0)R_{\alpha,1} + \frac{\partial \mu_\beta}{\partial \chi}(R_{1,0}, \dots, R_{N,0}, X_0)X_1.
\end{aligned} \tag{4.15}$$

Finally, we need to define matching solutions which consist of compatible outer and inner solutions.

Definition 4.3. A tuple $((\rho_{\alpha,0})_{\alpha=1,\dots,N}, \boldsymbol{v}_0, \chi_0, \chi_1, (R_{\alpha,0})_{\alpha=1,\dots,N}, (R_{\alpha,1})_{\alpha=1,\dots,N}, \boldsymbol{V}_0, X_0, X_1)$ is called a *matching solution of the dissipative regime* provided $((\rho_{\alpha,0})_{\alpha=1,\dots,N}, \boldsymbol{v}_0, \chi_0, \chi_1)$ is an outer solution and the tuple $((R_{\alpha,0})_{\alpha=1,\dots,N}, (R_{\alpha,1})_{\alpha=1,\dots,N}, \boldsymbol{V}_0, X_0, X_1)$ is an inner solution and both are linked by the matching conditions, see Subsection 3.3.

Theorem 4.1. *Let $((\rho_{\alpha,0})_{\alpha=1,\dots,N}, \boldsymbol{v}_0, \chi_0, \chi_1, (R_{\alpha,0})_{\alpha=1,\dots,N}, (R_{\alpha,1})_{\alpha=1,\dots,N}, \boldsymbol{V}_0, X_0, X_1)$ be a matching solution of the dissipative regime, then the following equations are satisfied in the bulk regions $\Omega^\pm$:*

$$\chi_0 = \pm 1, \chi_1 = 0, \tag{4.16}$$

$$\partial_t \rho_0 + \operatorname{div}(\rho_0 \boldsymbol{v}_0) = 0, \tag{4.17}$$

$$\partial_t \rho_{\alpha,0} + \operatorname{div}(\rho_{\alpha,0}\boldsymbol{v}_0) - \operatorname{div}\Big(\sum_{\beta=1}^{N-1} M_{\alpha\beta}\nabla(\mu_{\beta,0} - \mu_{N,0})\Big) - \sum_{i=1}^{N_R} \gamma_\alpha^i m_\alpha M_r^i \left(1 - \exp\left(A_0^i\right)\right) = 0, \tag{4.18}$$

$$\partial_t(\rho_0 \boldsymbol{v}_0) + \operatorname{div}(\rho_0 \boldsymbol{v}_0 \otimes \boldsymbol{v}_0) + \nabla\bigg(\sum_{\alpha=1}^{N} \rho_{\alpha,0}\mu_{\alpha,0} - \rho f_0\bigg) = 0. \tag{4.19}$$

Moreover, the following conditions are fulfilled at the interface:

$$[\![\mu_{\alpha,0} - \mu_{N,0}]\!] = 0 \quad \textit{for all } \alpha = 1, \dots, N-1, \tag{4.20}$$

$$[\![\rho_0(\boldsymbol{v}_0 \cdot \boldsymbol{\nu} - w_{\boldsymbol{\nu}})]\!] = 0, \tag{4.21}$$

$$[\![\rho_{\alpha,0}(\boldsymbol{v}_0 \cdot \boldsymbol{\nu} - w_{\boldsymbol{\nu}})]\!] = [\![\sum_{\beta=1}^{N-1} M_{\alpha\beta}\nabla(\mu_{\beta,0} - \mu_{N,0}) \cdot \boldsymbol{\nu}]\!] \quad \textit{for all } \alpha = 1, \dots, N-1, \tag{4.22}$$

$$[\![j_0 \boldsymbol{v}_0 + \Big(\sum_{\alpha=1}^{N} \rho_{\alpha,0}\mu_{\alpha,0} - \rho f_0\Big)\boldsymbol{\nu}]\!] = \gamma\kappa\boldsymbol{\nu}\int_{-\infty}^{\infty}(X_{0,z})^2\,dz, \tag{4.23}$$

$$[\![\frac{j_0^2}{2\rho_0^2} + \mu_{N,0}]\!] = -\frac{j_0}{M_p}\int_{-\infty}^{\infty}\frac{1}{R_0}(X_{0,z})^2\,dz, \tag{4.24}$$

where $j_0 = \rho_0^{\pm}(\boldsymbol{v}_0^{\pm} \cdot \boldsymbol{\nu} - w_{\boldsymbol{\nu}})$.

Remark 4.4. In view of (4.10), the surface tension coefficient can be rewritten as follows

$$\gamma\int_{-\infty}^{\infty}(X_{0,z})^2\,dz = \sqrt{2\gamma}\int_{-1}^{1}\sqrt{W(X)}\,dX.$$

We will decompose the proof of Theorem 4.1 into several lemmata. Our first lemma ascertains that we have pure phases in the bulk.

Lemma 4.5. *Let* χ_0, χ_1 *be given as in Definition 4.3, then*

$$\chi_0 \in \{-1, 1\} \quad \textit{and} \quad \chi_1 = 0.$$

Furthermore, all solutions $\Psi \in C^2(\mathbb{R})$ *of the ordinary differential equation*

$$W'(\Psi) - \gamma\partial_{zz}\Psi = 0 \tag{4.25}$$

with $\partial_z\Psi \to 0, \Psi \to \pm 1$ *as* $z \to \pm\infty$ *are given by the one parameter family*

$$\Psi(z) = \bar{\Psi}(z - \bar{z}), \quad \bar{z} \in \mathbb{R}, \tag{4.26}$$

where $\bar{\Psi}$ *is the unique strictly monotonically increasing solution of* (4.25) *satisfying* $\bar{\Psi}(0) = 0$. *In particular, all* X_0 *as in Definition 4.3 are given by the one parameter family*

$$X_0(t, \boldsymbol{s}, \cdot) = \bar{\Psi}(\cdot - \bar{z}(t, \boldsymbol{s})), \quad \bar{z} \in \mathbb{R}.$$

Proof. From (4.4) we know $\chi_0 \in \{\pm 1, 0\}$. Thus, by continuity, χ_0 is constant in $\Omega^{\pm}$. A phase portrait analysis which can be found in [7] shows that (4.25) with $\Psi \to \pm 1$ as $z \to \pm\infty$ implies (4.26) and $\chi_0^{\pm} = \pm 1$. Hence, $\chi_0 = \pm 1$ in $\Omega^{\pm}$ and, therefore, $\frac{\partial \rho f}{\partial \chi}(\rho_{1,0}, \dots, \rho_{N,0}, \chi_0) = 0$ because of (2.21). Thus, $\chi_1 = 0$ because of $W''(\pm 1) \neq 0$ and equation (4.6). □

Our next step is to use the continuity of the mass flux across the interface to eliminate the normal velocity from the equations.

Remark 4.6. As $R_0 > 0$ equation (4.12) holds if and only if $\boldsymbol{V}_0 \cdot \boldsymbol{\nu} = \frac{j_0}{R_0} + w_{\boldsymbol{\nu}}$ for some j_0 independent of z.

In order to determine the functions $R_{\alpha,0}$, we reformulate several of the inner equations such that we obtain equations, which do not contain X_1.

Lemma 4.7. *Equations* (4.9)*,* (4.13)*,* (4.14) *are equivalent to* (4.9) *and*

$$\frac{j_0}{R_0}\left(\frac{j_0}{R_0}\right)_z + (\mathcal{M}_{N,0})_z + \frac{j_0}{M_p}\frac{(X_{0,z})^2}{R_0} = 0, \tag{4.27}$$

$$\left(\frac{j_0^2}{R_0}\right)_z + \left(\sum_{\alpha=1}^{N} R_{\alpha,0}\mathcal{M}_{\alpha,0} - RF_0 - W'(X_0)X_1\right)_z + \gamma((X_{0,z}X_{1,z})_z - \kappa X_{0,z}^2) = 0, \tag{4.28}$$

$$j_0 \boldsymbol{V}_0 \cdot \boldsymbol{t} = 0 \tag{4.29}$$

for any tangent vector $\boldsymbol{t}$ to Γ.

Proof. The tangent part of (4.13) simplifies to (4.29) because of (4.10). Combining the normal part of (4.13) and (4.14) gives

$$\begin{aligned}
j_0 \boldsymbol{V}_{0,z} \cdot \boldsymbol{\nu} &+ \left(\sum_{\alpha=1}^{N} R_{\alpha,0}\mathcal{M}_{\alpha,0} - RF_0 - W'(X_0)X_1\right)_z + \gamma X_{0,zz}X_{1,z} \\
&= -\gamma X_{0,z}(X_{1,zz} - \kappa X_{0,z}) \\
&= -\frac{j_0}{M_p}(X_{0,z})^2 - X_{0,z}W''(X_0)X_1 - (h(X_0))_z(\rho\psi_L(\rho_1, \dots, \rho_N) - \rho\psi_V(\rho_1, \dots, \rho_N)).
\end{aligned} \tag{4.30}$$

The first equality in (4.30) is (4.28). The combination of the first and third line of (4.30) can be simplified by using (4.10) such that

$$\begin{aligned}
j_0 \boldsymbol{V}_{0,z} \cdot \boldsymbol{\nu} &+ \left(\sum_{\alpha=1}^{N} R_{\alpha,0}\mathcal{M}_{\alpha,0} - RF_0\right)_z \\
&= -\frac{j_0}{M_p}(X_{0,z})^2 - (h(X_0))_z(\rho\psi_L(\rho_1, \dots, \rho_N) - \rho\psi_V(\rho_1, \dots, \rho_N)).
\end{aligned} \tag{4.31}$$

By definition (2.28), equation (4.31) is equivalent to

$$j_0\left(\frac{j_0}{R_0}\right)_z + \frac{j_0}{M_p}(X_{0,z})^2 + \sum_{\alpha=1}^{N} R_{\alpha,0}\left(\mu_\alpha(R_{1,0}, \dots, R_{N,0}, X_0)\right)_z = 0. \tag{4.32}$$

Because of (4.9), the positive definiteness of $M_{\alpha\beta}$ and $R_0 \neq 0$ equation (4.32) implies (4.27). □

Before we can derive necessary and sufficient conditions for the existence of $R_{1,0}, \dots, R_{N,0}$, we need several technical prerequisites.

Lemma 4.8. *Let the functions $\hat{n}, \hat{y}_\alpha : \mathbb{R}^{N+2} \to (0, \infty)$ defined by*

$$\hat{n}(c_1, \dots, c_N, a, z) := n^R\left(\sum_{\alpha=1}^{N} \exp\left(m_\alpha \frac{(c_\alpha - c_N + a - g_\alpha^R)}{kT}\right)\right)^{\frac{n^R kT}{K(X_0(z))}}, \tag{4.33}$$

$$\hat{y}_\alpha(c_1, \dots, c_N, a, z) := \left(\exp\left(m_\alpha(c_\alpha - c_N + a - g_\alpha^R)\right) \left(\frac{n^R}{n}\right)^{\frac{K(X_0(z))}{n^R}} \right)^{\frac{1}{kT}}. \tag{4.34}$$

Then,

$$R_{\alpha,0} = m_\alpha y_\alpha n \in C^1(\mathbb{R}), \quad \alpha = 1, \dots, N,$$

with

$$n(z) = \hat{n}(c_1, \dots, c_N, f(z), z) \quad \textit{and} \quad y_\alpha(z) = \hat{y}_\alpha(c_1, \dots, c_N, f(z), z) \tag{4.35}$$

is a solution of the auxiliary problem

$$\begin{aligned} \mu_{\alpha,0}(R_{1,0}, \dots, R_{N,0}, X_0) - \mu_{N,0}(R_{1,0}, \dots, R_{N,0}, X_0) = c_\alpha - c_N, \quad \alpha = 1, \dots, N-1, \\ \mu_{N,0}(R_{1,0}, \dots, R_{N,0}, X_0) = f(z) \end{aligned} \tag{4.36}$$

for $c_1, \dots, c_N \in \mathbb{R}$ *and* $f \in C^1(\mathbb{R})$.

Proof. The assertion can be verified by standard calculations. However, for convenience, we give a short sketch. We consider the problem

$$\mu_{\alpha,0}(R_{1,0}, \dots, R_{N,0}, X_0) = c_\alpha - c_N + a, \quad \alpha = 1, \dots, N.$$

Then, by (2.27) and (2.4),

$$\frac{K_0}{n^R m_\alpha} \ln\left(\frac{n}{n^R}\right) + \frac{kT}{m_\alpha} \ln y_\alpha = c_\alpha - c_N + a - g_\alpha^R$$

where $K_0 = K(X_0)$. Hence, we obtain

$$\frac{K_0}{n^R kT} \ln\left(\frac{n}{n^R}\right) + \ln y_\alpha = \frac{(c_\alpha - c_N + a - g_\alpha^R) m_\alpha}{kT}. \tag{4.37}$$

Because of $\sum_{\alpha=1}^N y_\alpha = 1$, we obtain

$$\sum_{\alpha=1}^N \exp^{\frac{K_0}{n^R kT} \ln\left(\frac{n}{n^R}\right)} \exp^{\ln y_\alpha} = \exp^{\frac{K_0}{n^R kT} \ln\left(\frac{n}{n^R}\right)} = \sum_{\alpha=1}^N \exp^{\frac{(c_\alpha - c_N + a - g_\alpha^R) m_\alpha}{kT}}.$$

In consequence,

$$n = n^R \left(\sum_{\alpha=1}^N \exp^{\frac{(c_\alpha - c_N + a - g_\alpha^R) m_\alpha}{kT}} \right)^{\frac{n^R kT}{K_0}}. \tag{4.38}$$

Now, solving (4.37) for y_α yields

$$y_\alpha = \left(\exp\left(m_\alpha(c_\alpha - c_N + a - g_\alpha^R)\right) \left(\frac{n^R}{n}\right)^{\frac{K_0}{n^R}} \right)^{\frac{1}{kT}}.$$

Since, by definition, $R_\alpha(z) = m_\alpha y_\alpha(z) n(z)$, we obtain the claim. □

For $c_1, \dots, c_N, j \in \mathbb{R}$ and $b > 0$ let

$$\hat{R}_\alpha[(c_\alpha)_\alpha, b, j], \quad \alpha = 1, \dots, N, \tag{4.39}$$

denote the solution of (4.36) with

$$f(z)=\hat{f}(z):=-\frac{j^2}{2b^2}-\frac{j}{bM_p}\int_{-\infty}^{z}(X_{0,z})^2\,d\tilde{z}. \tag{4.40}$$

Then, for f given by (4.40) there exist constants $\underline{f},\bar{f}$ with $\underline{f}(b,j)<\bar{f}(b,j)$ such that

$$\underline{f}(b,j)<f(z)<\bar{f}(b,j)\quad\text{for all } z\in\mathbb{R}, \tag{4.41}$$

and, therefore, by the continuity properties of $\hat{n},\hat{y}_\alpha$, there are constants $\underline{R},\overline{R}$ with $0<\underline{R}((c_\alpha)_\alpha,b,j)<\overline{R}((c_\alpha)_\alpha,b,j)$ such that

$$0<\underline{R}((c_\alpha)_\alpha,b,j)<\sum_{\alpha=1}^{N}\hat{R}_\alpha[(c_\alpha)_\alpha,b,j](z)<\overline{R}((c_\alpha)_\alpha,b,j)\quad\text{for all } z\in\mathbb{R}. \tag{4.42}$$

In view of our existence result for $R_{\alpha,0}$, $\alpha=1,\dots,N$, we define the following constants

$$\tilde{L}((c_\alpha)_\alpha,b,j):=\frac{8j^2}{\underline{R}^3((c_\alpha)_\alpha,b,j)}+\frac{4|j|}{M_p\underline{R}^2((c_\alpha)_\alpha,b,j)}\int_{-\infty}^{\infty}(X_{0,z})^2\,dz, \tag{4.43}$$

$$L((c_\alpha)_\alpha,b,j):=\max_{\alpha}\max_{\underline{a}<a<\bar{a}}\max_{z\in\mathbb{R}}\frac{\partial(\hat{n}\hat{y}_\alpha)}{\partial a}((c_\alpha)_\alpha,a,z), \tag{4.44}$$

where

$$\bar{a}((c_\alpha)_\alpha,b,j):=\frac{2|j|}{M_p\underline{R}((c_\alpha)_\alpha,b,j)}\int_{-\infty}^{\infty}(X_{0,z})^2\,dz,$$
$$\underline{a}((c_\alpha)_\alpha,b,j):=-\frac{2j^2}{\underline{R}^2((c_\alpha)_\alpha,b,j)}-\bar{a}((c_\alpha)_\alpha,b,j).$$

Finally, we need the following technical lemma.

Lemma 4.9. *Let $\eta>0$ and $F:(-\eta,\eta)\times(0,\infty)^N\to\mathbb{R}^N$ be a differentiable mapping such that $F_0(\cdot):=F(0,\cdot)$ is a diffeomorphism of $(0,\infty)^N$ onto $\mathbb{R}^N$. Then, for every connected and compact set $K\subset\mathbb{R}^N$ there exists some $\delta>0$ such that for $|j|\le\delta$ there exists a neighborhood V of K and an inverse $G(j,\cdot)$ of $F(j,\cdot)$ defined on V which is differentiable.*

Proof. Firstly, we choose some compact and connected $U\subset(0,\infty)^N$ such that ∂U is a closed and connected hypersurface of $(0,\infty)^N$ and

$$K\subset F_0(U)\quad\text{with}\quad \operatorname{dist}(K,F_0(\partial U))>1.$$

Due to continuity arguments there exists a $\tilde{\delta}>0$ such that for all $|j|\le\tilde{\delta}$ and $\boldsymbol{x}\in U$:

$$\det(DF_j(\boldsymbol{x}))>0 \tag{4.45}$$

Let us define

$$C_1:=\max_{\boldsymbol{y}\in F_0(U)}\|(DF_0)^{-1}(\boldsymbol{y})\|, \tag{4.46}$$

$$C_2 := \max_{|j|\le\tilde\delta, \boldsymbol{x}\in U} \|\partial_j(\nabla F)(j,\boldsymbol{x})\|, \tag{4.47}$$

$$C_3 := \max_{|j|\le\tilde\delta, \boldsymbol{x}\in U} \|\partial_j F(j,\boldsymbol{x})\|. \tag{4.48}$$

Now, for $|j| < \bar\delta := \min\{\tilde\delta, \frac{1}{C_1 C_2}\}$, $\boldsymbol{x} \neq \tilde{\boldsymbol{x}}$ and $\boldsymbol{x}, \tilde{\boldsymbol{x}} \in U$ we get

$$\begin{aligned} \|F(j,\boldsymbol{x}) - F(j,\tilde{\boldsymbol{x}})\| &\ge \|F_0(\boldsymbol{x}) - F_0(\tilde{\boldsymbol{x}})\| - \|F(j,\tilde{\boldsymbol{x}}) - F(0,\tilde{\boldsymbol{x}}) - F(j,\boldsymbol{x}) + F(0,\boldsymbol{x})\| \\ &\ge \frac{1}{C_1}\|\boldsymbol{x} - \tilde{\boldsymbol{x}}\| - C_2\,|j|\,\|\tilde{\boldsymbol{x}} - \boldsymbol{x}\| > 0, \end{aligned} \tag{4.49}$$

since

$$\|\boldsymbol{x} - \tilde{\boldsymbol{x}}\| = ||F_0^{-1}(F_0(\boldsymbol{x})) - F_0^{-1}(F_0(\tilde{\boldsymbol{x}}))|| \le C_1 ||F_0(\boldsymbol{x}) - F_0(\tilde{\boldsymbol{x}})||. \tag{4.50}$$

Thus, for $|j|$ small enough the map $F(j,\cdot)|_U$ is injective and a diffeomorphism on its image.

It remains to show $K \subset (F_j(U))^\circ$ for $|j|$ sufficiently small. For $|j| < \frac{C_3}{2}$ we have

$$\mathrm{dist}(F_j(\bar{\boldsymbol{x}}), F_0(\bar{\boldsymbol{x}})) < \frac{1}{2} \quad \text{for any } \bar{\boldsymbol{x}} \in \partial U.$$

Hence, we obtain

$$\mathrm{dist}(K, F_j(\bar{\boldsymbol{x}})) \ge \mathrm{dist}(K, F_0(\bar{\boldsymbol{x}})) - \mathrm{dist}(F_0(\bar{\boldsymbol{x}}), F_j(\bar{\boldsymbol{x}})) > \frac{1}{2} \quad \text{for any } \bar{\boldsymbol{x}} \in \partial U.$$

This implies $\mathrm{dist}(K, F_j(\partial U)) > 1/2$ and, therefore, $B_{1/2}(K) \subset F_j(U)$, $B_{1/2}(K) = \{\boldsymbol{x} \in \mathbb{R}^n : \mathrm{dist}(\boldsymbol{x}, K) < 1/2\}$, which is a consequence of the following consideration: The hypersurface $F_j(\partial U)$ decomposes $\mathbb{R}^N$ into two connected components. One of those is $F_j(U)$. As $F_j(F_0^{-1}(K)) \subset B_{1/2}(K)$ for $|j| < \frac{C_3}{2}$ we can conclude $B_{1/2}(K) \subset F_j(U)$. □

Now we can prove the following existence result for $R_{\alpha,0}$, $\alpha = 1, \dots, N$.

Lemma 4.10. *Let $\rho_1^-, \dots, \rho_N^- > 0$ be given. Further, let $j_0 \in \mathbb{R}$ with $|j_0|$ small enough, satisfying,*

$$\big(\max_{\beta=1,\dots,N} m_\beta\big) L((\mu_\alpha^-)_\alpha, \rho^-, j_0)\, \tilde{L}((\mu_\alpha^-)_\alpha, \rho^-, j_0)\, (\overline{R}((\mu_\alpha^-)_\alpha, \rho^-, j_0) + \rho^-) < \frac{\underline{R}((\mu_\alpha^-)_\alpha, \rho^-, j_0)}{4N}, \tag{4.51}$$

$$\big(\max_{\beta=1,\dots,N} m_\beta\big) L((\mu_\alpha^-)_\alpha, \rho^-, j_0)\, \tilde{L}((\mu_\alpha^-)_\alpha, \rho^-, j_0) < \frac{1}{2N}, \tag{4.52}$$

where $\rho^- := \sum_\alpha \rho_\alpha^-$ and $\mu_\alpha^- := \mu_\alpha(\rho_1^-, \dots, \rho_N^-, -1)$, $\alpha = 1, \dots, N$. Then there exist functions $R_{1,0}, \dots, R_{N,0} \in C^0(\mathbb{R}, \mathbb{R}_+)$ and $\rho_1^+, \dots, \rho_N^+ > 0$ such that

$$\mu_\alpha(R_{1,0}(z), \dots, R_{N,0}(z), X_0(z)) - \mu_N(R_{1,0}(z), \dots, R_{N,0}(z), X_0(z)) - \mu_\alpha^- + \mu_N^- = 0, \tag{4.53}$$

$$\mu_N(R_{1,0}(z), \dots, R_{N,0}(z), X_0(z)) + \frac{j_0^2}{2(\sum_\alpha R_{\alpha,0}(z))^2} + \frac{j_0}{M_p}\int_{-\infty}^z \frac{(X_{0,z}(\tilde z))^2}{\sum_\alpha R_{\alpha,0}(\tilde z)}\, d\tilde z = 0, \tag{4.54}$$

$$\lim_{z\to\pm\infty} R_{\alpha,0}(z) - \rho_\alpha^\pm = 0, \tag{4.55}$$

and (4.20) *and* (4.24) *are satisfied. In particular, the functions $R_{1,0}, \dots, R_{N,0}$ solve* (4.9) *and* (4.27).

Proof. The proof is based on a fixed point argument. Let us define

$$F[R_{1,0},\dots,R_{N,0},X_0](z) := -\frac{1}{2}\frac{j_0^2}{\left(\sum_\alpha R_{\alpha,0}(z)\right)^2} - \frac{j_0}{M_p}\int_{-\infty}^{z}\frac{(X_{0,z}(\tilde z))^2}{\sum_\alpha R_{\alpha,0}(\tilde z)}\,d\tilde z \tag{4.56}$$

and

$$\begin{aligned} A := \Big\{ & f\in C(\mathbb{R},\mathbb{R}^N) \,|\, f_\alpha(z)\ge 0 \text{ for } \alpha=1,\dots,N, \text{all } z\in\mathbb{R} \text{ and} \\ & \frac{1}{2}\underline{R}((\mu_\alpha^-)_\alpha,\rho^-,j_0) \le \sum_\alpha f_\alpha(z) \le 2\overline{R}((\mu_\alpha^-)_\alpha,\rho^-,j_0) \text{ for all } z\in\mathbb{R}\Big\}. \end{aligned} \tag{4.57}$$

For given $(R^n_{\alpha,0})_\alpha\in A$, we define $(R^{n+1}_{\alpha,0})_\alpha$ to be the solution of (4.36) with $c_\alpha=\mu_\alpha^-$, $\alpha=1,\dots,N$, and

$$f(z) = F[R^n_{1,0},\dots,R^n_{N,0},X_0](z).$$

Next we show that Banach's fixed point theorem applies to the map $(R^n_{\alpha,0})_\alpha \mapsto (R^{n+1}_{\alpha,0})_\alpha$. We first verify that A is mapped to A. For $(R^n_{\alpha,0})_\alpha\in A$ we have

$$\|\rho_\alpha^- - R^n_{\alpha,0}\|_\infty \le \rho^- + 2\overline{R}((\mu_\alpha^-)_\alpha,\rho^-,j_0)$$

and, therefore,

$$\begin{aligned} \|F[\rho_1^-,\dots,\rho_N^-,X_0] - F[R^n_{1,0},\dots,R^n_{N,0},X_0]\|_\infty &\le N\tilde L((\mu_\alpha^-)_\alpha,\rho^-,j_0)\,\|\rho_\alpha^- - R^n_{\alpha,0}\|_\infty \\ &\le 2N\tilde L((\mu_\alpha^-)_\alpha,\rho^-,j_0)\left(\rho^- + \overline{R}((\mu_\alpha^-)_\alpha,\rho^-,j_0)\right). \end{aligned} \tag{4.58}$$

We have, see (4.39) for the definition of $\hat R$,

$$\begin{aligned} \hat R_\alpha[(\mu_\alpha^-)_\alpha,\rho^-,j_0](z) &= m_\alpha\,\hat n\big((\mu_\alpha^-)_\alpha, F[\rho_1^-,\dots,\rho_N^-,X_0],z\big)\;\hat y_\alpha\big((\mu_\alpha^-)_\alpha, F[\rho_1^-,\dots,\rho_N^-,X_0],z\big), \\ R^{n+1}_{\alpha,0}(z) &= m_\alpha\,\hat n\big((\mu_\alpha^-)_\alpha, F[R^n_{1,0},\dots,R^n_{N,0},X_0],z\big)\;\hat y_\alpha\big((\mu_\alpha^-)_\alpha, F[R^n_{1,0},\dots,R^n_{N,0},X_0],z\big), \end{aligned} \tag{4.59}$$

which automatically ensures $R^{n+1}_{\alpha,0}(z)\ge 0$. Using (4.44), (4.58) and (4.59), we obtain for $\alpha=1,\dots,N$,

$$\|\hat R_\alpha[(\mu_\alpha^-)_\alpha,\rho^-,j_0] - R^{n+1}_{\alpha,0}\|_\infty \le 2NL\big((\mu_\alpha^-)_\alpha,\rho^-,j_0\big)\;m_\alpha\;\tilde L\big((\mu_\alpha^-)_\alpha,\rho^-,j_0\big)\;\left(\rho^- + \overline{R}((\mu_\alpha^-)_\alpha,\rho^-,j_0)\right) \tag{4.60}$$

and, thus, by (4.51), we get

$$\|\hat R_\alpha[(\mu_\alpha^-)_\alpha,\rho^-,j_0] - R^{n+1}_{\alpha,0}\|_\infty \le \frac{1}{2}\underline{R}((\mu_\alpha^-)_\alpha,\rho^-,j_0). \tag{4.61}$$

Equations (4.42) and (4.61) imply $(R^{n+1}_{\alpha,0})_\alpha\in A$.

Our next aim is to show that $(R^n_{\alpha,0})_\alpha\mapsto(R^{n+1}_{\alpha,0})_\alpha$ defines a contraction on A with respect to the $\|\cdot\|_\infty$-norm. To prove this, we consider apart from $(R^n_{\alpha,0})_\alpha$ a further arbitrary $(\tilde R^n_{\alpha,0})_\alpha\in A$ and, correspondingly to

$$R^{n+1}_{\alpha,0}(z) = m_\alpha\,\hat n\big((\mu_\alpha^-)_\alpha, F[R^n_{1,0},\dots,R^n_{N,0},X_0],z\big)\;\hat y_\alpha\big((\mu_\alpha^-)_\alpha, F[R^n_{1,0},\dots,R^n_{N,0},X_0],z\big),$$

we define

$$\tilde R^{n+1}_{\alpha,0}(z) = m_\alpha\,\hat n\big((\mu_\alpha^-)_\alpha, F[\tilde R^n_{1,0},\dots,\tilde R^n_{N,0},X_0],z\big)\;\hat y_\alpha\big((\mu_\alpha^-)_\alpha, F[\tilde R^n_{1,0},\dots,\tilde R^n_{N,0},X_0],z\big).$$

Note that $(R^n_{\alpha,0})_\alpha, (\tilde{R}^n_{\alpha,0})_\alpha \in A$ implies

$$\underline{a}((\mu^-_\alpha)_\alpha, \rho^-, j_0) \leq F[\tilde{R}^n_{1,0}, \dots, \tilde{R}^n_{N,0}, X_0],\ \ F[R^n_{1,0}, \dots, R^n_{N,0}, X_0] \leq \bar{a}((\mu^-_\alpha)_\alpha, \rho^-, j_0) \tag{4.62}$$

such that for $z \in \mathbb{R}$

$$|R^{n+1}_{\alpha,0}(z) - \tilde{R}^{n+1}_{\alpha,0}(z)| \leq m_\alpha\, L((\mu^-_\alpha)_\alpha, \rho^-, j_0)\ \left| F[\tilde{R}^n_{1,0}, \dots, \tilde{R}^n_{N,0}, X_0](z) - F[R^n_{1,0}, \dots, R^n_{N,0}, X_0](z) \right|. \tag{4.63}$$

It holds

$$\begin{aligned} &|F[\tilde{R}^n_{1,0}, \dots, \tilde{R}^n_{N,0}, X_0](z) - F[R^n_{1,0}, \dots, R^n_{N,0}, X_0](z)| \\ &\quad = \left| -\frac{j_0^2}{2}\left(\frac{1}{(\sum_\alpha R^n_{\alpha,0})^2} - \frac{1}{(\sum_\alpha \tilde{R}^n_{\alpha,0})^2}\right) - \frac{j_0}{M_p}\int_{-\infty}^{z}\left(\frac{1}{\sum_\alpha R^n_{\alpha,0}} - \frac{1}{\sum_\alpha \tilde{R}^n_{\alpha,0}}\right) X^2_{0,z}\, d\tilde{z} \right| \\ &\quad \leq 2N\tilde{L}((\mu^-_\alpha)_\alpha, \rho^-, j_0) \max_\beta \|R^n_{\beta,0} - \tilde{R}^n_{\beta,0}\|_\infty. \end{aligned} \tag{4.64}$$

We combine (4.63) and (4.64) to get

$$\max_\alpha \|R^{n+1}_{\alpha,0} - \tilde{R}^{n+1}_{\alpha,0}\|_\infty \leq 2N(\max_\beta m_\beta)\tilde{L}((\mu^-_\alpha)_\alpha, \rho^-, j_0)L((\mu^-_\alpha)_\alpha, \rho^-, j_0) \max_\beta \|R^n_{\beta,0} - \tilde{R}^n_{\beta,0}\|_\infty. \tag{4.65}$$

Because of (4.52), this shows the contraction property. Hence, the assumptions of the Banach fixed point theorem are satisfied and we obtain a unique fixed-point $R_{\alpha,0}$, which is given by $R_{\alpha,0}(z) = \lim_{n\to\infty} R^n_{\alpha,0}(z)$, $\alpha = 1, \dots, N$, for $z \in \mathbb{R}$.

Note that, because of (4.62), (4.53) and (4.54), we have

$$((R^{n+1}_{\alpha,0}(z))_\alpha, X_0(z)) \in \bigcap_{\eta=1}^{N} \mu_\eta^{-1}\big(\big[\mu^-_\eta - \mu^-_N + \underline{a}\big((\mu^-_\beta)_\beta, \rho^-, j_0\big), \mu^-_\eta - \mu^-_N + \overline{a}\big((\mu^-_\beta)_\beta, \rho^-, j_0)\big)\big]\big) \tag{4.66}$$

for $z \in \mathbb{R}$ and any $n \in \mathbb{N}$. As the set in (4.66) is independent of n and a compact subset of $(0,\infty)^N$ we can deduce $R_\alpha(z) > 0$ for all $z \in \mathbb{R}$.

It remains to check the interface conditions. Let $K \subset \mathbb{R}^N$ be a connected and compact set such that $B_\delta(K_{\hat{j}_0}) \subset K$, where $K_{\hat{j}_0} := \big[\mu^-_\alpha - \mu^-_N - \max_{j\in[-\hat{j}_0,\hat{j}_0]} \overline{a}\big((\mu^-_\alpha)_\alpha, \rho^-, j\big), \mu^-_\alpha - \mu^-_N + \max_{j\in[-\hat{j}_0,\hat{j}_0]} \overline{a}\big((\mu^-_\alpha)_\alpha, \rho^-, j\big)\big]$ and $B_\delta(K) = \{\boldsymbol{x} \in \mathbb{R}^n : \text{dist}(\boldsymbol{x}, K) < \delta\}$ for sufficiently small $\hat{j}_0$ and $\delta > 0$.

From Lemma 4.9 we know that the function

$$G_{j_0}(\rho_1, \dots, \rho_N) := \left(\mu_1((\rho_\alpha)_\alpha, 1) + \frac{j_0^2}{2(\sum_\beta \rho_\beta)^2}, \dots, \mu_N((\rho_\alpha)_\alpha, 1) + \frac{j_0^2}{2(\sum_\beta \rho_\beta)^2}\right)^T$$

has a local inverse J_{j_0} in a neighborhood of

$$(\hat{b}_{j_0,\alpha})_\alpha := \left(\mu^-_\alpha - \mu^-_N - \frac{j_0}{M_p}\int_{-\infty}^{\infty} \frac{1}{\sum_\beta R_{\beta,0}(\tilde{z})}\big(X_{0,z}(\tilde{z})\big)^2\, d\tilde{z}\right)_\alpha \in K_{\hat{j}_0},$$

for $j_0 < \hat{j}_0$ small enough. By construction, we have

$$\begin{aligned} &\mu_\alpha(R_{1,0}(z), \dots, R_{N,0}(z), X_0(z)) + \frac{j_0^2}{2(\sum_\beta R_{\beta,0}(z))^2} \\ &= \mu^-_\alpha - \mu^-_N - \frac{j_0}{M_p}\int_{-\infty}^{z} \frac{1}{\sum_\beta R_{\beta,0}(\tilde{z})}\big(X_{0,z}(\tilde{z})\big)^2\, d\tilde{z} \\ &=: b_{j_0,\alpha}(z) \end{aligned} \tag{4.67}$$

for $\alpha = 1, \dots, N$. As $\lim_{z\to+\infty} b_{j_0,\alpha}(z) = \hat{b}_{j_0,\alpha}$ we find

$$\lim_{z\to+\infty} R_{\alpha,0}(z) = \lim_{z\to+\infty} (J_{j_0})_\alpha(b_{j_0,1}(z), \dots, b_{j_0,N}(z)) = (J_{j_0})_\alpha(\hat{b}_{j_0,1}, \dots, \hat{b}_{j_0,N}).$$

By a similar argument, we obtain the existence of $\lim_{z\to-\infty} R_{\alpha,0}(z)$. This proves the existence of the limits $\rho_\alpha^\pm$ and the validity of the boundary conditions. □

Our next step is to study (4.28). For this, it is important to keep in mind that we already have determined X_0, $(R_{\alpha,0})_\alpha$ in a way which is independent of X_1.

Lemma 4.11. *Let*

$$\mathcal{L} : W^{2,1}(\mathbb{R}) \to L^1(\mathbb{R}), \quad \Psi \mapsto (W'(X_0)\Psi - \gamma X_{0,z}\Psi_z)_z. \tag{4.68}$$

Then equation (4.28) *can be expressed as*

$$\mathcal{L}X_1 = \left(\frac{j_0^2}{\sum_{\alpha=1}^N R_{\alpha,0}}\right)_z + \left(\sum_{\alpha=1}^N R_{\alpha,0}\mathcal{M}_{\alpha,0} - (RF_0)\right)_z - \gamma\kappa X_{0,z}^2. \tag{4.69}$$

In particular, equation (4.69) *has a solution if and only if* (4.23) *holds.*

Proof. We will determine the solutions in $L^\infty(\mathbb{R})$ of the homogeneous adjoint problem to (4.69), which is given by

$$W'(X_0)\Xi_z + \gamma(X_{0,z}\Xi_z)_z = 0. \tag{4.70}$$

Once we have determined these solutions the solvability conditions for (4.69) follow from Fredholm's theorem. As the operator $\mathcal{L}$ is linear and of second order it has two linearly independent solutions in $C^2(\mathbb{R})$. We need to determine linearly independent solutions in $C^2(\mathbb{R}) \cap L^\infty(\mathbb{R})$. Equation (4.70) has the trivial solution $\Xi_1 = \text{const}$ which is obviously contained in $L^\infty(\mathbb{R})$. To determine a linearly independent solution Ξ_2, we define $Z = (\Xi_2)_z$. Inserting this into (4.70) and using (4.10) gives

$$2X_{0,zz}Z + X_{0,z}Z_z = 0. \tag{4.71}$$

As $X_{0,z} \neq 0$ for $z \in \mathbb{R}$ we infer

$$Z(z) = \frac{k}{\left(X_{0,z}(z)\right)^2}, \qquad z \in \mathbb{R},$$

for some $k \in \mathbb{R}$. For $k \neq 0$ this implies $Z(z) \to \infty$ for $z \to \infty$. Hence, the solutions of (4.70) in $L^\infty(\mathbb{R}) \cap C^2(\mathbb{R})$ are given by the one parameter family $\Xi = c$ for any $c \in \mathbb{R}$. Thus, the only solvability condition for (4.69) is

$$0 = \int_{-\infty}^{\infty} \left(\frac{j_0^2}{\sum_{\alpha=1}^N R_{\alpha,0}}\right)_z + \left(\sum_{\alpha=1}^N R_{\alpha,0}\mathcal{M}_{\alpha,0} - (RF)_0\right)_z - \gamma\kappa X_{0,z}^2 \, dz,$$

which is (4.23). □

We have determined all inner quantities up to a translational constant, except $(R_{\alpha,1})_{\alpha=1,\dots,N}$. The only remaining inner equation is (4.11) for $\alpha = 1, \dots, N$.

Lemma 4.12. *For given* $((R_{\alpha,0})_{\alpha=1,\dots,N}, \boldsymbol{V}_0, X_0, X_1)$*, there exist* $(R_{\alpha,1})_{\alpha=1,\dots,N}$ *satisfying* (4.11) *if and only if* (4.22) *is fulfilled.*

Proof. Let us note that due to Lemma 4.8 the map

$$(0,\infty)^N \to \mathbb{R}^N, \qquad (\rho_\alpha)_{\alpha=1,\dots,N} \mapsto (\mu_1((\rho_\alpha)_{\alpha=1,\dots,N},\chi),\dots,\mu_N((\rho_\alpha)_{\alpha=1,\dots,N},\chi))^T$$

is a diffeomorphism for any fixed $\chi \in [-1,1]$. Thus, for fixed $\chi \in [-1,1]$, the matrix

$$\Big(\frac{\partial \mu_\beta}{\partial \rho_\gamma}((\rho_\alpha)_{\alpha=1,\dots,N},\chi)\Big)_{\beta,\gamma=1,\dots,N}$$

is invertible for any $\rho_1,\dots,\rho_N > 0$.

Therefore, in order to determine solvability conditions for (4.11), we may search for conditions determining whether there exist functions $\mathcal{M}_{\alpha,1} : \mathbb{R} \to \mathbb{R}$, $\alpha = 1,\dots,N$ satisfying (4.11). As soon as we have ensured the existence of the $(\mathcal{M}_{\alpha,1})_{\alpha=1,\dots,N}$ we can use (4.15) to compute corresponding $(R_{\alpha,1})_{\alpha=1,\dots,N}$. Conversely, if no $(\mathcal{M}_{\alpha,1}(z))_{\alpha=1,\dots,N}$ solving (4.11) exist there are also no solutions in terms of $(R_{\alpha,1})_{\alpha=1,\dots,N}$. Note that the matching condition (3.14) and (4.9) imply

$$(\mathcal{M}_{\beta,0} - \mathcal{M}_{N,0})_z = 0 \quad \text{for all } \beta = 1,\dots,N.$$

Thus, (4.11) reads

$$0 = (R_{\alpha,0}(\boldsymbol{V}_0 \cdot \boldsymbol{\nu} - w_{\boldsymbol{\nu}}))_z - \sum_{\beta=1}^{N-1} M_{\alpha\beta}\Big((\mathcal{M}_{\beta,1} - \mathcal{M}_{N,1})_z\Big)_z. \tag{4.72}$$

Obviously only the differences $\psi_\beta := \mathcal{M}_{\beta,1} - \mathcal{M}_{N,1}$ for $\beta = 1,\dots,N-1$ are of interest in (4.72). Our strategy is to use the Fredholm alternative theorem to determine the solvability conditions for the $(\psi_\beta)_{\beta=1,\dots,N-1}$. To this end, we introduce (arbitrary) auxiliary functions $\Xi_\beta \in C^\infty(\mathbb{R})$ for $\beta = 1,\dots,N-1$ such that

$$\Xi_\beta(z) = \begin{cases} (\nabla(\mu_{\beta,0} - \mu_{N,0}))^+ \cdot \boldsymbol{\nu} z + (\mu^+_{\beta,1} - \mu^+_{N,1}) & \text{for} \quad z > 1 \\ (\nabla(\mu_{\beta,0} - \mu_{N,0}))^- \cdot \boldsymbol{\nu} z + (\mu^-_{\beta,1} - \mu^-_{N,1}) & \text{for} \quad z < -1. \end{cases}$$

Defining $\Psi_\beta = \psi_\beta - \Xi_\beta$, we are interested in the following auxiliary problem: Find $(\Psi_\beta)_{\beta=1,\dots,N-1} \in L^1(\mathbb{R})$ such that

$$\sum_{\beta=1}^{N-1}(M_{\alpha\beta}(\Psi_\beta)_z)_z = (R_{\alpha,0}(\boldsymbol{V}_0 \cdot \boldsymbol{\nu} - w_{\boldsymbol{\nu}}))_z - \sum_{\beta=1}^{N-1}\Big(M_{\alpha\beta}(\Xi_\beta)_z\Big)_z. \tag{4.73}$$

To determine the solvability conditions for (4.73) we need to find all linearly independent solutions of the homogenous adjoint system of equations in $L^\infty(\mathbb{R})$. The homogenous adjoint system of equations is given by

$$\sum_{\beta=1}^{N-1}(M_{\alpha\beta}(Z_\beta)_z)_z = 0 \quad \text{for } \alpha = 1,\dots,N-1, \quad (Z_\beta)_{\beta=1,\dots,N-1}. \tag{4.74}$$

As the matrix $M_{\alpha\beta}$ is constant in z and positive definite the solutions of (4.74) are given by the $2N-2$ parameter family

$$Z_\beta(z) = a_\beta z + b_\beta \quad \text{with } a_\beta, b_\beta \in \mathbb{R}, \qquad \beta = 1,\dots,N-1.$$

The solutions are elements of $L^\infty(\mathbb{R})$ if $a_\beta = 0$. Thus, we obtain $N-1$ linearly independent solutions of (4.74), i.e. $\alpha = 1,\dots,N-1$, which can be chosen as

$$Z_\beta(z) = \delta_{\alpha\beta}, \qquad z \in \mathbb{R}.$$

Thus, by the Fredholm alternative theorem (4.73) is solvable if and only if

$$\int_{\mathbb{R}} (R_{\alpha,0}(\boldsymbol{V}_0 \cdot \boldsymbol{\nu} - w_{\boldsymbol{\nu}}))_z - \sum_{\beta=1}^{N-1}\Big(M_{\alpha\beta}(\Xi_\beta)_z\Big)_z \, dz = 0 \tag{4.75}$$

for $\alpha = 1,\dots,N-1$. Integrating (4.75) gives (4.22). □

Proof of Theorem 4.1. The bulk equations follow from the outer equations using the information on χ_0 and χ_1 given in Lemma 4.5. The interface conditions follow by combining the previous lemmata. $\square$

4.2 Viscosity of order 1

Here, we consider the same scaling as above with the only modification that we set $\mathrm{Re} = 1$. Further, we will exclude the case $\lambda_1 = \lambda_2 = 0$ since otherwise we have no Navier-Stokes stress. The conditions on the bulk and shear viscosity imply $\lambda_1 + 2\lambda_2 > 0$. Then, the leading orders of the mass, the momentum and the Allen-Cahn equation read

$$(R_0(\boldsymbol{V}_0 \cdot \boldsymbol{\nu} - w_{\boldsymbol{\nu}}))_z = 0, \tag{4.76}$$

$$\boldsymbol{\nu}(-W(X_0))_z + \gamma \boldsymbol{\nu} X_{0,z} X_{0,z} = (\lambda_1 + 2\lambda_2)(\boldsymbol{V}_0 \cdot \boldsymbol{\nu})\boldsymbol{\nu} + \lambda_2(\boldsymbol{V}_0 - (\boldsymbol{V}_0 \cdot \boldsymbol{\nu})\boldsymbol{\nu}), \tag{4.77}$$

$$W'(X_0) - \gamma X_{0,zz} = 0. \tag{4.78}$$

Inserting (4.78) into (4.77) implies

$$(\lambda_1 + 2\lambda_2)(\boldsymbol{V}_0 \cdot \boldsymbol{\nu})\boldsymbol{\nu} + \lambda_2(\boldsymbol{V}_0 - (\boldsymbol{V}_0 \cdot \boldsymbol{\nu})\boldsymbol{\nu}) = 0.$$

Due to the matching conditions and the assumption on λ_1, λ_2, this gives $[\![\boldsymbol{v}_0 \cdot \boldsymbol{\nu}]\!] = 0$, i.e., $\boldsymbol{v}_0^+ \cdot \boldsymbol{\nu} = \boldsymbol{v}_0^- \cdot \boldsymbol{\nu}$. Integrating (4.76), we find

$$[\![\rho_0]\!](\boldsymbol{v}_0^\pm \cdot \boldsymbol{\nu} - w_{\boldsymbol{\nu}}) = 0. \tag{4.79}$$

As $[\![\rho_0]\!] \neq 0$ at a proper phase boundary we get

$$\boldsymbol{v}_0^+ \cdot \boldsymbol{\nu} = \boldsymbol{v}_0^- \cdot \boldsymbol{\nu} = w_{\boldsymbol{\nu}}$$

which excludes mass fluxes across the interface.

Hence, the given diffuse interface model prevents a phase transition in the sharp limit if the viscous part of the stress survives in the leading order jump condition for the barycentric momentum. The same observation has been made for other diffuse models in the literature which fulfill the total mass balance. In fact, we were lead to the conjecture: A viscous diffuse model with $\mathrm{Re} = \mathcal{O}(1)$ satisfying the total mass balance is not capable to generate a thermodynamically consistent viscous sharp interface model.

5 Sharp interface limit of the non-dissipative regime

As the non-dissipative regime is rather similar to the dissipative regime treated in the last section we only outline the differences. The equations satisfied by outer solutions are obtained by inserting (3.8) into the scaled equations and comparing the terms order by order.

Definition 5.1. A tuple $((\rho_{\alpha,0})_{\alpha=1,\dots,N}, \boldsymbol{v}_0, \chi_0, \chi_1)$ is called an outer solution of the *non-dissipative regime* provided it is an outer solution of the dissipative regime.

Inner solutions are determined from the scaled system by changing coordinates via (3.9) and inserting (3.11).

Definition 5.2. A tuple $((R_{\alpha,0})_{\alpha=1,\dots,N}, (R_{\alpha,1})_{\alpha=1,\dots,N}, \boldsymbol{V}_0, X_0, X_1)$ with $X_0 \not\equiv 0$ and

$$\begin{aligned} R_{\alpha,0} &\in C^0([0,T_f), C^0(U, C^2(\mathbb{R}_+))), \\ R_{\alpha,1} &\in C^0([0,T_f), C^0(U, C^2(\mathbb{R}))), \\ \boldsymbol{V}_0 &\in C^0([0,T_f), C^0(U, C^1(\mathbb{R}^d))), \\ X_0 &\in C^0([0,T_f), C^1(U, C^0(\mathbb{R}))) \cap C^0([0,T_f), C^0(U, C^2(\mathbb{R}))), \\ X_1 &\in C^0([0,T_f), C^0(U, C^2(\mathbb{R}))), \end{aligned} \tag{5.1}$$

is called an *inner solution of the non-dissipative regime* with normal velocity $w_{\boldsymbol{\nu}}$ provided (4.9)–(4.13) and

$$0 = W''(X_0)X_1 - \gamma X_{1,zz} + \gamma\kappa X_{0,z} + \frac{\partial \rho f}{\partial \chi}(R_{1,0}, \dots, R_{N,0}, X_0) \tag{5.2}$$

are satisfied.

Matching solutions are combinations of compatible outer and inner solutions.

Definition 5.3. A tuple $((\rho_{\alpha,0})_{\alpha=1,\dots,N}, \boldsymbol{v}_0, \chi_0, \chi_1, (R_{\alpha,0})_{\alpha=1,\dots,N}, (R_{\alpha,1})_{\alpha=1,\dots,N}, \boldsymbol{V}_0, X_0, X_1)$ is called a *matching solution of the non-dissipative regime* provided $((\rho_{\alpha,0})_{\alpha=1,\dots,N}, \boldsymbol{v}_0, \chi_0, \chi_1)$ is an outer solution and $((R_{\alpha,0})_{\alpha=1,\dots,N}, (R_{\alpha,1})_{\alpha=1,\dots,N}, \boldsymbol{V}_0, X_0, X_1)$ is an inner solution and both are linked by the matching conditions.

Theorem 5.1. *Let* $((\rho_{\alpha,0})_{\alpha=1,\dots,N}, \boldsymbol{v}_0, \chi_0, \chi_1, (R_{\alpha,0})_{\alpha=1,\dots,N}, (R_{\alpha,1})_{\alpha=1,\dots,N}, \boldsymbol{V}_0, X_0, X_1)$ *be a matching solution of the non-dissipative regime, then the following equations are satisfied in the bulk:*

$$\chi_0 = \pm 1, \chi_1 = 0, \tag{5.3}$$

$$\partial_t \rho_0 + \operatorname{div}(\rho_0 \boldsymbol{v}_0) = 0, \tag{5.4}$$

$$\partial_t \rho_{\alpha,0} + \operatorname{div}(\rho_{\alpha,0}\boldsymbol{v}_0) - \operatorname{div}\Big(\sum_{\beta=1}^{N-1} M_{\alpha\beta}\nabla(\mu_{\beta,0} - \mu_{N,0})\Big) - \sum_{i=1}^{N_R} \gamma_\alpha^i m_\alpha M_r^i (1 - \exp(A_0^i)) = 0, \tag{5.5}$$

$$\partial_t(\rho_0 \boldsymbol{v}_0) + \operatorname{div}(\rho_0 \boldsymbol{v}_0 \otimes \boldsymbol{v}_0) + \nabla\Big(\sum_{\alpha=1}^{N} \rho_{\alpha,0}\mu_{\alpha,0} - \rho f_0\Big) = 0. \tag{5.6}$$

Moreover, the following conditions are fulfilled at the interface:

$$[\![\mu_{\alpha,0} - \mu_{N,0}]\!] = 0 \quad \textit{for all } \alpha = 1, \dots, N-1, \tag{5.7}$$

$$[\![\rho_0(\boldsymbol{v}_0 \cdot \boldsymbol{\nu} - w_{\boldsymbol{\nu}})]\!] = 0, \tag{5.8}$$

$$[\![\rho_{\alpha,0}(\boldsymbol{v}_0 \cdot \boldsymbol{\nu} - w_{\boldsymbol{\nu}})]\!] = [\![\sum_{\beta=1}^{N-1} M_{\alpha\beta}\nabla(\mu_{\beta,0} - \mu_{N,0}) \cdot \boldsymbol{\nu}]\!] \quad \textit{for all } \alpha = 1, \dots, N-1, \tag{5.9}$$

$$[\![j_0 \boldsymbol{v}_0 + \Big(\sum_{\alpha=1}^{N} \rho_{\alpha,0}\mu_{\alpha,0} - \rho f_0\Big)\boldsymbol{\nu}]\!] = \gamma\kappa\boldsymbol{\nu}\int_{-\infty}^{\infty} (X_{0,z})^2\, dz, \tag{5.10}$$

$$[\![\frac{j_0^2}{2\rho_0^2} + \mu_{N,0}]\!] = 0. \tag{5.11}$$

Proof. The proof is an analogue simplified version of the proof of Theorem 4.1 as no fixed point argument is needed to construct $R_{\alpha,0}$, $\alpha = 1, \dots, N$. □

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