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**Spectrum and amplitude equations for scalar
delay-differential equations with large delay**

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Abstract

The subject of the paper are scalar delay-differential equations with large delay. Firstly, we describe the asymptotic properties of the spectrum of linear equations. Using these properties, we classify possible types of destabilization of steady states. In the limit of large delay, this classification is similar to the one for parabolic partial differential equations. We present a derivation and error estimates for amplitude equations, which describe universally the local behavior of scalar delay-differential equations close to the destabilization threshold.

1 Introduction

Delay differential equations (DDE) with large delay appear to be a very useful tool for modeling many technologically relevant applied systems. For example, in laser devices with optical feedback [38, 26, 2, 11] a large time delay appears due to the very fast internal time scale dynamics of the laser. As a result, the external feedback introduces a new time scale in the system, which can be modeled as the large time delay.

From the theoretical viewpoint, DDEs with large delay can be considered as singularly perturbed problems, which are challenging for analytical as well as numerical analysis. During the last decades, a number of important results have been obtained, which shed some light on the properties of these systems [29, 28, 18, 17, 16, 20, 22]. In particular, they studied the existence and asymptotic behavior of periodic orbits of a class of DDEs, and noticed, that their properties can be related to the properties of a reduced discrete map. For example, a flip bifurcation of the map implies square waves in the corresponding singularly perturbed system of DDEs. Another important milestone in the analysis of singularly perturbed DDEs was the derivation of amplitude equations [12, 23, 13, 24, 1, 35, 40], which describe the local dynamics of DDEs at the bifurcation threshold. These derivations have been performed by formal asymptotic methods. Recently, the structure of the spectrum of linear systems with large delay was described for the cases of constant [27, 41] and periodic [37] coefficients. Extensions for chaotic systems and systems with multiple delays are known as well [19, 7].

In this paper, we describe in details the asymptotic properties of the spectrum of a linear scalar DDE operator with large delay. Using these properties, we classify possible types of instabilities. Finally, for the case of vanishing quadratic nonlinearities, we present the derivation and an error estimate for the amplitude equations, which proves closeness of a solution of the amplitude equation and the corresponding solution of the original DDE with large delay. Even though the mathematical justification of amplitude equations has been established for certain classes of partial differential equations [39, 25, 36, 33, 32] or nonlinear oscillator chains [10, 15], similar

results for systems with time delay are still challenging because of the strong differences in the corresponding analysis for the associated linear evolution semigroups. In particular, a proof of the validity of the amplitude equations was not available previously.

We consider the following DDE

$$\frac{dy(t)}{dt} = g(y(t), y(t - \tau)) \quad (1.1)$$

with $y(t) \in \mathbb{R}$ and a smooth function $g : \mathbb{R}^2 \rightarrow \mathbb{R}$, which vanishes at zero $g(0, 0) = 0$. The delay $\tau > 0$ is assumed to be large. For convenience, we introduce the small parameter $\varepsilon = 1/\tau$.

Our main goal is to describe properties of system (1.1) for $\varepsilon \rightarrow 0$ in the vicinity of the equilibrium $y = 0$. To this end, we describe in Sec. 2 the spectral properties of the linearized system and determine possible types of destabilizing bifurcations. In Sec. 5, we show that, sufficiently close to the bifurcation point, the solutions can be described by an amplitude equation, i.e. by solutions of a partial differential equation (PDE) of the form

$$\frac{\partial u}{\partial \theta} = \alpha \frac{\partial^2 u}{\partial x^2} + \beta \frac{\partial u}{\partial x} + pu + \gamma u^3, \quad (1.2)$$

with some periodicity conditions and real coefficients, which will be specified later. To obtain this result, we assume that the function g has no quadratic nonlinearities (i.e. $D^2g(0, 0) = 0$) but a nontrivial cubic term (i.e. $D^3g(0, 0) \neq 0$). Finally, we present an example in Sec. 6.

2 Linear scalar DDEs with large delay

2.1 Asymptotic properties of a spectrum for autonomous linear systems

In this section, we consider the spectral properties of the linear DDE

$$y'(t) = ay(t) + by(t - \tau). \quad (2.1)$$

This equation corresponds to the linearization of (1.1) in $y = 0$ with $a = \partial_1 g(0, 0)$ and $b = \partial_2 g(0, 0)$. If not stated otherwise, we assume that $b \neq 0$ and $a \neq 0$. The spectrum of systems of linear DDEs with large delay have been discussed in [27]. In the present section we focus on the scalar case $y \in \mathbb{R}$ and treat it in more detail. The characteristic equation of (2.1) reads

$$\chi(\lambda, \varepsilon) = -\lambda + a + be^{-\lambda/\varepsilon} = 0. \quad (2.2)$$

Let us discuss the asymptotic behaviors of infinitely many roots $\{\lambda_j(\varepsilon), j \in \mathbb{Z}\}$ of (2.2) with respect to the small parameter ε . All possible situations can be covered by the following cases: (a) $\text{Re}[\lambda(\varepsilon)/\varepsilon] \rightarrow c = \text{const}$; (b) $\text{Re}[\lambda(\varepsilon)/\varepsilon] \rightarrow \infty$; see details in [27].

Definition 2.1. The set of roots, which satisfy condition (a) will be called *pseudo-continuous spectrum* (PCS). The roots, which have asymptotics (b) will be called *strongly unstable*.

(a) The pseudo-continuous spectrum

The PCS satisfies $\text{Re}[\lambda/\varepsilon] \rightarrow c = \text{const.}$ In order to find such solutions, we use the ansatz

$$\lambda_{PCS} = \mu\varepsilon + i\omega. \quad (2.3)$$

Substituting (2.3) into (2.2) and keeping the terms of order $\mathcal{O}(\varepsilon^0)$, we obtain the equation

$$-i\omega + a + be^{-\mu}e^{-i\omega/\varepsilon} = 0, \quad (2.4)$$

which can be solved with respect to μ and gives

$$\mu(\omega) = -\frac{1}{2} \ln \frac{\omega^2 + a^2}{b^2}. \quad (2.5)$$

Given the function (2.5), the curve

$$\lambda : \mathbb{R} \rightarrow \mathbb{C}, \quad \omega \mapsto \lambda(\omega) = \mu(\omega) + i\omega \quad (2.6)$$

contains all solutions of the equation (2.4), i.e., it approximates the PCS provided the real parts are rescaled by ε (compare (2.3) and (2.6)). The positions of the solutions on the curve (2.6) are determined by an additional equation for ω , which is obtained by insertion of (2.5) into (2.4)

$$\arg\left(\frac{i\omega - a}{b}\right) = \frac{\omega}{\varepsilon} \pmod{2\pi}. \quad (2.7)$$

Clearly, equation (2.7) has a countable number of solutions ω_k whose distances are proportional to ε . Consequently, the points $\mu(\omega_k) + i\omega_k$ cover the curve (2.6) densely as $\varepsilon \rightarrow 0$.

Definition 2.2. The curve (2.6) will be called *the asymptotic continuous spectrum (ACS)*, since the PCS approaches it asymptotically (see Lemma 2.3). Note that the PCS is a discrete set while the ACS is a continuous curve.

(b) The strongly unstable spectrum

The strongly unstable spectrum satisfies the asymptotic relation $\text{Re}[\lambda/\varepsilon] \rightarrow \infty$ as $\varepsilon \rightarrow 0$. In this case $be^{-\lambda/\varepsilon} \rightarrow 0$ and necessarily $\lambda \rightarrow a$, which is possible only if $a > 0$. Thus, the strongly unstable spectrum exists in the case $a > 0$ and satisfies

$$\lambda_{SU} \rightarrow a, \quad a > 0.$$

Using the implicit function theorem, it can be shown that the strongly unstable spectrum contains a single eigenvalue for $a > 0$ and is empty for $a < 0$ [27]. The presence of a strongly unstable spectrum indicates *strong instability* in the sense, that almost all solutions escape a vicinity of the origin after a time $\sim 1/a$, which is much less than the delay time $\tau = 1/\varepsilon$. On the other hand, the unstable PCS may introduce a *weak instability*, since $\text{Re} \lambda_{PCS} \sim \varepsilon$, and the time that solutions are staying in the vicinity of the origin is of the order of the delay τ .

In the following, we assume that $a < 0$, since otherwise the stationary state is strongly unstable. In this case, the destabilization of the equilibrium can only be mediated by the PCS. The following lemma summarizes the main properties of the PCS.

Lemma 2.3. *Let $b \neq 0$ and $a \neq 0$. Then, the PCS approaches the ACS (2.6) as $\varepsilon \rightarrow 0$. More specifically, for any $\Omega > 0$ there exist $\varepsilon_0 > 0$ and $M > 0$ such that for all ε , $0 < \varepsilon < \varepsilon_0$, all roots λ of (2.2) which lie in the set $Q(M, \Omega, \varepsilon) = \{z \in \mathbb{C} : |\operatorname{Re} z| \leq M\varepsilon, |\operatorname{Im} z| \leq \Omega\}$ have the form*

$$\lambda = \lambda(\omega_k) = i\omega_k + \varepsilon\lambda_1(\omega_k) \left(1 + \varepsilon\lambda_2(\omega_k) + \varepsilon^2\lambda_3(\omega_k) + \varepsilon^3h(\varepsilon, \omega_k)\right) \quad (2.8)$$

$$\omega_k = 2\pi k\varepsilon, \quad k \in \mathbb{Z}, \quad (2.9)$$

where $h : [-\Omega, \Omega] \times [0, \varepsilon_0] \rightarrow \mathbb{C}$ is an analytic function,

$$\lambda_1(\omega) = \mu(\omega) - i\psi(\omega), \quad (2.10)$$

$$\lambda_2(\omega) = \frac{1}{a - i\omega}, \quad \text{and, } \lambda_3(\omega) = \frac{2 + \lambda_1(\omega)}{2(a - i\omega)^2} \quad (2.11)$$

with $\mu(\omega)$ given by (2.5) and $\psi(\omega) := \arg\left(\frac{i\omega - a}{b}\right) \in (-\pi, \pi]$.

Proof. We only consider the case $ab < 0$, the case $ab > 0$ can be treated analogously. First observe that an arbitrary root $\lambda \in Q(M, \Omega, \varepsilon)$ of (2.2) can be written as

$$\lambda = i\omega_k + \varepsilon g, \quad (2.12)$$

with $\omega_k = 2\pi k\varepsilon$, $k \in \mathbb{Z}$, such that $g \in \mathcal{A}(M) := \{z \in \mathbb{C} : |\Im(z)| < \pi/2, |\Re(z)| < M\}$. Inserting (2.12) into (2.2) we find that g has to solve

$$\mathcal{M}(\omega, \varepsilon, g) = i\omega + \varepsilon g - a - be^{-g} = 0, \quad (2.13)$$

for $\omega = \omega_k$. The equation (2.13) locally defines an implicit function $g = G(\omega, \varepsilon)$ since

$$\partial_g \mathcal{M}(\omega, \varepsilon, g) = \varepsilon + be^{-g} \neq 0$$

for sufficiently small ε . Because \mathcal{M} is analytic, so is $G(\omega, \varepsilon)$.

Now we show that there exists an $\varepsilon_0 > 0$ such that (2.13) exhibits exactly one solution $g \in \mathcal{A}(M)$ for $|\omega| \leq \Omega$ and $\varepsilon \leq \varepsilon_0$. This will guarantee, that there are no other solutions except those, which are extensions of $\varepsilon \mapsto G(\omega, \varepsilon)$ for $\varepsilon \in [0, \varepsilon_0]$. Existence of a unique solution g follows from the contraction mapping principle applied to the function

$$F(g) := -\ln \left| \frac{i\omega - a + \varepsilon g}{b} \right| - i \arg \left(\frac{i\omega - a + \varepsilon g}{b} \right),$$

whose fixed points correspond to solutions of (2.13). Let us show that, for sufficiently large $M > 0$ and sufficiently small $\varepsilon_0 > 0$, F maps $\mathcal{A}(M)$ into itself and is a contraction for arbitrary $-\Omega \leq \omega \leq \Omega$ and $\varepsilon \leq \varepsilon_0$. First, choose an arbitrary $M > \max_{|\omega| \leq \Omega} \left| \ln \left| \frac{i\omega - a}{b} \right| \right| = \max\{\ln |a/b|, \left| \ln \left| \frac{i\Omega - a}{b} \right| \right|\}$. Since the real part of $F(g)$ depends continuously on ε uniformly in $|\omega| \leq \Omega$, its modulus is smaller than M for sufficiently small ε . Furthermore, observe that the modulus of the imaginary part of $F(g)$ is strictly smaller than $\pi/2$ for small ε (here we used the assumption $ab < 0$). Therefore, F maps $\mathcal{A}(M)$ into itself for $\varepsilon \leq \varepsilon_0$ and sufficiently small

$\varepsilon_0 > 0$. Moreover, F is a contraction, since $|F'(g)| = \mathcal{O}(\varepsilon_0)$ uniformly for all $|\omega| \leq \Omega$, $g \in \mathcal{A}(M)$, and $\varepsilon \leq \varepsilon_0$.

Let us now consider again $\omega = \omega_k = 2\pi k\varepsilon$. This case corresponds to a characteristic root $\lambda = i\omega_k + \varepsilon G(\omega_k, \varepsilon)$. As a solution to (2.13) the value $G(\omega_k, 0)$ fulfills

$$\mathcal{M}(\omega_k, 0, G(\omega_k, 0)) = i\omega_k - a - be^{-g_0} = 0,$$

which implies

$$G(\omega_k, 0) = \mu(\omega_k) + i \arg\left(\frac{i\omega_k - a}{b}\right) + i2\pi m = \lambda_1(\omega_k) + i2\pi m$$

for some $m \in \mathbb{Z}$. Since $G \in \mathcal{A}(M)$ we have $m = 0$. Therefore we conclude $G(\omega_k, 0) = \lambda_1(\omega_k)$ and

$$G(\omega_k, \varepsilon) = \lambda_1(\omega_k) + \varepsilon \partial_\varepsilon g(\omega_k, 0) + \frac{\varepsilon^2}{2} \partial_\varepsilon^2 g(\omega_k, 0) + \varepsilon^3 h(\omega_k, \varepsilon), \quad (2.14)$$

where $h : [-\Omega, \Omega] \times [0, \varepsilon_0]$ is analytic. The higher terms of (2.14) are readily computed as

$$\lambda_2(\omega) = \frac{\partial_\varepsilon G(\omega, 0)}{\lambda_1(\omega)} = \frac{1}{a - i\omega} \text{ and } \lambda_3(\omega) = \frac{\partial_\varepsilon^2 G(\omega, 0)}{2\lambda_1(\omega)} = \frac{2 + \lambda_1(\omega)}{2(a - i\omega)^2}.$$

□

Remark 2.4. It is easy to check that all solutions of the characteristic equation with $|\operatorname{Im} \lambda| > \Omega$ satisfy

$$\operatorname{Re} \lambda < -\varepsilon \ln \frac{\Omega}{|b|}.$$

This means that for large enough Ω , they all have negative real parts with sufficiently large magnitude. Thus, the critical spectrum is indeed described by Lemma 2.3.

The real part $\mu(\omega)$ of the ACS (2.6) is a smooth unimodal function, which has maximum at $\mu(0) = -\ln |a/b|$ [see Fig. 2.1]. Lemma 2.3 states that in rescaled coordinates $(\operatorname{Re} \lambda / \varepsilon, \operatorname{Im} \lambda)$, the critical spectrum of the DDE approaches the curve of the ACS, which is independent of ε . As a result, the stability of the equilibrium is determined by the ACS and is independent of ε if it is sufficiently small.

The fact, that the spectrum of the DDE with large delay asymptotically approaches a continuous spectrum, allows us to classify possible destabilization scenarios similarly to the case of spatially extended systems [41]. In the case of scalar DDEs a possible scenario is shown in the figure 2.1. For $|b| < |a|$, the PCS is stable and, if additionally $a < 0$, there is no strongly unstable spectrum. As a result the equilibrium is asymptotically stable for sufficiently small ε . The destabilization occurs at $|a| = |b|$ and $a < 0$, where the PCS crosses the imaginary axis. After the crossing, for $|b| > |a|$, the PCS is unstable and there are always unstable eigenvalues provided ε is sufficiently small. The number of such unstable eigenvalues grows linearly with $\tau \rightarrow \infty$ and cover the whole curve of the ACS. The figure 2.1 shows also numerically computed eigenvalues for $\tau = 70$, for comparison. As we have proved before, the rescaled real part

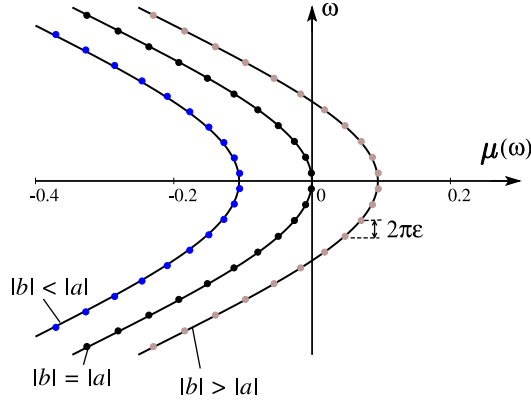


Figure 2.1: Illustration of the ACS $\mu(\omega)$. Parameter values are $b = 0.9$, $b = 1.0$, and $b = 1.1$, respectively. For all curves, $a = 1$ is chosen. The PCS is asymptotically located along the curves in the rescaled coordinates $(\text{Re } \lambda/\varepsilon, \text{Im } \lambda)$. The points show numerically computed eigenvalues for the same parameter values and $\tau = 70$.

of the ACS, $\mu(\omega)$, has the form of a unimodal function with maximum at $\omega = 0$. Hence, the leading eigenvalues (those with the maximal real parts) correspond to small ω values. Note that the strongly unstable spectrum is absent if $a < 0$. The following lemma provides the third-order expansion for the leading eigenvalues.

Lemma 2.5. *Assume $a < 0$ and $b \neq 0$. Then the leading eigenvalues of the scalar DDE (2.1) satisfy the following estimates. For the case $ab < 0$:*

$$\begin{aligned} \lambda_{crit}(k) = (q + i2\pi k) \varepsilon + \frac{1}{a} (q + i2\pi k) \varepsilon^2 \\ + \frac{1}{a^2} \left(q + \frac{q^2}{2} - 2\pi^2 k^2 + i2\pi k(1 + q) \right) \varepsilon^3 + \mathcal{O}(\varepsilon^4), \end{aligned} \quad (2.15)$$

and for the case $ab > 0$:

$$\begin{aligned} \lambda_{crit}(k) = (q + i2\pi k - i\pi) \varepsilon + \frac{1}{a} (q + i2\pi k - i\pi) \varepsilon^2 \\ + \frac{1}{a^2} \left(q + \frac{q^2 - \pi^2}{2} - 2\pi^2 k(k - 1) + i\pi(2k - 1)(1 + q) \right) \varepsilon^3 + \mathcal{O}(\varepsilon^4). \end{aligned} \quad (2.16)$$

Here $k = 0, \pm 1, \pm 2, \dots$ and $q = -\ln \left| \frac{a}{b} \right| = \mu(0)$ is the maximal value of the ACS.

Proof. In order to obtain the expressions (2.15) and (2.16), we expand Eq. (2.9) in $\omega_k = 2\pi k\varepsilon$ for $k \ll \frac{1}{\varepsilon}$, i.e. $\omega_k = \mathcal{O}(\varepsilon)$:

$$\begin{aligned} (2.9) = i\omega_k + \varepsilon \left(\lambda_1(0) + \lambda_1'(0) \omega_k + \lambda_1''(0) \frac{\omega_k^2}{2} \right) \\ \times \left(1 + \varepsilon (\lambda_2(0) + \lambda_2'(0) \omega_k) + \varepsilon^2 \lambda_3(0) \right) + \mathcal{O}(\varepsilon^4). \end{aligned} \quad (2.17)$$

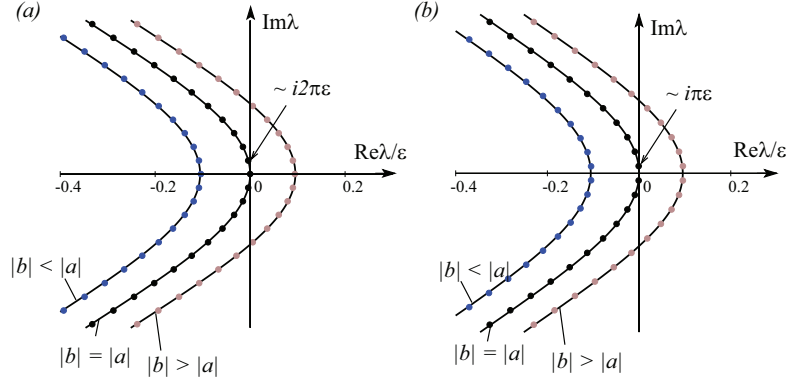


Figure 2.2: Two possible types of bifurcations in the scalar DDEs. The case (a) $a = -b$ corresponds to a transcritical or pitchfork bifurcation of the reduced map as well as for (2.1). In this case, there is a leading real eigenvalue. The case (b) $a = b$ corresponds to a flip bifurcation of the reduced discrete map [eq. (2.19)], resp. Hopf bifurcation for (2.1). The leading pair of eigenvalues has imaginary parts $\sim \pm i\pi\varepsilon$.

From (2.10) and (2.11) we obtain

$$\begin{aligned}
 \lambda_1(0) &= q - i\kappa\pi \\
 \lambda_2(0) &= \frac{1}{a} \\
 \lambda_3(0) &= \frac{2 + q - \kappa i\pi}{2a^2} \\
 \lambda_1'(0) &= \frac{i}{a} \\
 \lambda_1''(0) &= -\frac{1}{a^2} \\
 \lambda_2'(0) &= \frac{i}{a^2},
 \end{aligned}$$

with $\kappa = \frac{1 - \text{sgn}(b)}{2}$ ($\kappa = 0$ if $ab < 0$ and $\kappa = 1$ if $ab > 0$). Insertion into (2.17) gives

$$\begin{aligned}
 \lambda_{crit}(k) &= (q + i(2\pi k - \kappa\pi))\varepsilon + \frac{1}{a}(q + i(2\pi k - \kappa\pi))\varepsilon^2 \\
 &+ \frac{1}{a^2} \left(q + \frac{1}{2}q^2 - \left(2k^2 - \kappa \left(2k - \frac{1}{2} \right) \right) \pi^2 + i(2\pi k - \kappa\pi)(1 + q) \right) \varepsilon^3 + \mathcal{O}(\varepsilon^4)
 \end{aligned}$$

which corresponds to either (2.15) (if $\kappa = 0$), or (2.16) (if $\kappa = 1$). \square

Lemma 2.5 implies, that there are two cases, for which the location of leading eigenvalues is qualitatively different. These cases differ by the sign of the product ab . In the case when $ab < 0$, there is a real eigenvalue and the imaginary parts of the other leading eigenvalues are given asymptotically as $2\pi k\varepsilon$. In the second case, when $ab > 0$, there is no real eigenvalue and the imaginary parts of the leading eigenvalues are given asymptotically as $i\pi(1 + 2k)\varepsilon$. Both cases are illustrated in Fig. 2.2(a) and (b).

The two above mentioned cases correspond to qualitatively different dynamics in the vicinity of equilibrium. The reasons for such difference can be understood by inspecting the properties of the corresponding reduced map. Observe that equation (1.1) is equivalent to the following singularly perturbed equation

$$\varepsilon \frac{d\bar{y}(\bar{t})}{d\bar{t}} = g(\bar{y}(\bar{t}), \bar{y}(\bar{t} - 1)) \quad (2.18)$$

after the transformations $\bar{t} = t/\tau$ and $\bar{y}(\bar{t}) = y(t\tau)$. The reduced map can be obtained by substituting formally $\varepsilon = 0$ into (2.18)

$$g(\bar{y}(\bar{t}), \bar{y}(\bar{t} - 1)) = 0.$$

Introducing the discrete variable $z_n = \bar{y}(\bar{t} - 1)$, we obtain the one-dimensional discrete map $z_n \rightarrow z_{n+1}$ determined by

$$g(z_{n+1}, z_n) = 0. \quad (2.19)$$

In general, the properties of the map (2.19) have much in common with the properties of DDE [29, 21, 22, 18]. For example, the asymptotic stability of an equilibrium for the map (2.19) implies the asymptotic stability of the corresponding equilibrium for the DDE with sufficiently large delay. Indeed, denoting $a = \partial_1 g(0, 0)$ and $b = \partial_2 g(0, 0)$, the linearization of (2.19) around the origin is

$$z_{n+1} = -\frac{b}{a}z_n,$$

which is asymptotically stable if $|b| < |a|$. The bifurcation of the map occurs when

$$\left| \frac{a}{b} \right| = 1 \quad (2.20)$$

as well as the bifurcation of the ACS (2.6) for the DDE (1.1), since $\mu(0) = 0$ when (2.20) is fulfilled.

Returning now to the two qualitatively different cases for the spectrum configuration shown in Fig. 2.2, we notice that they correspond to different bifurcations of the reduced map occurring at the moment $|a| = |b|$:

1. When $a = -b$, the map (2.19) has the multiplier 1, which may correspond to a transcritical or pitchfork bifurcation. The corresponding bifurcation of the DDE (1.1) is mediated by the critical eigenvalues with imaginary parts $i2\pi k\varepsilon$, in particular by the leading eigenvalue $\lambda_{crit}(0) = 0$, cf. Fig. 2.2(b).

2. When $a = b$, the map (2.19) undergoes a flip bifurcation. In this case, the critical eigenvalues of DDE have imaginary parts $i\pi(1 + 2k)\varepsilon$, cf. Fig. 2.2(a). In particular, the most unstable pair of eigenvalues produces a Hopf bifurcation with the period $2\pi/\pi\varepsilon = 2\tau$. The appearance of square waves with the period 2τ in this situation has been studied in [28, 29, 4, 3, 17].

In the both above mentioned cases, the primary destabilization bifurcation is followed by subharmonic Hopf bifurcations with frequencies $\omega_H = 2\pi k\varepsilon$ or $\omega_H = \pi\varepsilon(1 + 2k)$, respectively, as $\left| \frac{a}{b} \right|$ increases.

3 Estimates for evolution operators

In this section we derive estimates for evolution operators of linear, nonautonomous DDEs. The following Lemma 3.1 shows that a DDE of the form

$$y'(t) = a(t)y(t) + b(t)y(t - \tau), \quad \tau = 1/\varepsilon, \quad (3.1)$$

can be at most weakly unstable, if it has a stable instantaneous part. This result appears in various physical applications, such as stability of synchronization in delay-coupled systems [9, 19]. More precisely, the following result holds true

Lemma 3.1. *Let the linear ordinary differential equation*

$$x' = a(t)x \quad (3.2)$$

be exponentially stable, i.e.

$$|x(t; t_0, x_0)| \leq M e^{-\eta(t-t_0)} |x_0|, \quad t \geq t_0$$

with some $M > 0, \eta > 0$. Here $x(t; t_0, y_0)$ denotes a solution of (3.2) with an initial condition $x(t_0; t_0, x_0) = x_0$. Let also $b(t)$ be bounded on some interval $t \in [t_0, T]$ with $|b(t)| \leq B_0$.

Then for any smooth initial function $\varphi(s), s \in [-\tau, 0]$, the solution of the initial value problem for the DDE (3.1) with initial condition $y(t_0 + s; t_0, \varphi) = \varphi(s), s \in [-\tau, 0]$ satisfies

$$|y(t; t_0, \varphi)| \leq L e^{\varepsilon \ln(L)(t-t_0)} |\varphi|_C, \quad t \in [t_0, T], \quad (3.3)$$

with $L = M(1 + B_0/\eta)$.

Proof. In order to estimate the solution of the DDE (3.1) on the time interval $[0, \tau]$ let us use the variation of constants formula

$$y(t; t_0, \varphi) = \Phi(t, t_0)\varphi(0) + \int_{t_0}^t \Phi(t, s)b(s)\varphi(s - \tau)ds,$$

where $\Phi(t, s) = e^{\int_s^t a(s)ds}$. The following estimate holds true for all $t_0 \leq t \leq t_0 + \tau$:

$$\begin{aligned} |y(t; t_0, \varphi)| &\leq |\Phi(t, t_0)\varphi(0)| + \int_{t_0}^t |\Phi(t, s)b(s)\varphi(s - \tau)| ds \\ &\leq M e^{-\eta(t-t_0)} |\varphi|_C + M B_0 |\varphi|_C \int_{t_0}^t e^{-\eta(t-s)} ds \\ &= M |\varphi|_C \left(\frac{B_0}{\eta} + \left(1 - \frac{B_0}{\eta}\right) e^{-\eta(t-t_0)} \right) \leq L |\varphi|_C, \end{aligned}$$

where $L = M(1 + B_0/\eta)$. Similarly, we show that for $t \in [t_0 + \tau, t_0 + 2\tau]$

$$|y(t; t_0, \varphi)| \leq L^2 |\varphi|_C,$$

and for $t \in [t_0 + (j-1)\tau, t_0 + j\tau]$ we find

$$|y(t; t_0, \varphi)| \leq L^j |\varphi|_C. \quad (3.4)$$

Hence,

$$|y(t; t_0, \varphi)| \leq |\varphi|_C L e^{\ln(L)(j-1)} = |\varphi|_C L e^{\varepsilon \ln(L)(j-1)\tau} \leq |\varphi|_C L e^{\varepsilon \ln(L)(t-t_0)}.$$

□

We remark that Lemma 3.1 is also valid for systems of DDE's. The proof is literally the same. The following Lemma gives an exponential estimate for the DDE with a nonautonomous perturbation

$$y'(t) = ay(t) + by(t-\tau) + \varepsilon^n a_1(t)y(t) + \varepsilon^n b_1(t)y(t-\tau), \quad \tau = \frac{1}{\varepsilon}, \quad n \in \mathbb{N}. \quad (3.5)$$

Lemma 3.2. *Let $a < 0$ and $|a| = |b|$. Assume also that $a_1(t)$ and $b_1(t)$ are continuous and uniformly bounded, i.e. there exists a constant $M > 0$ such that $|a_1(t)| \leq M$ and $|b_1(t)| \leq M$ for all $0 \leq t \leq T$. Then, for small enough $\varepsilon = 1/\tau > 0$, the solution of the initial value problem for (3.5) with an initial condition $\varphi(s)$, $s \in [-\tau, 0]$, satisfies the following estimate*

$$|y(t; \varphi)| \leq (1 + \varepsilon^n c) e^{c\varepsilon^{n+1}t} |\varphi|_C, \quad 0 \leq t \leq T,$$

where $c = 3M/|b|$.

Proof. Let us first consider the reduced ordinary differential equation

$$y' = (a + \varepsilon^n a_1(t)) y. \quad (3.6)$$

The fundamental solution of (3.6) is $\Phi(t, s) = e^{a(t-s) + \varepsilon^n \int_s^t a_1(s) ds}$. Since $a < 0$, the function $|\Phi(t, s)| \leq e^{(a+\varepsilon^n M)(t-s)}$ is exponentially decaying for all $\varepsilon < \varepsilon_1$, where

$$\varepsilon_1 = (|a|/M)^{1/n}. \quad (3.7)$$

Considering the initial value problem for the DDE (3.5) on the interval $0 \leq t \leq \tau$, we substitute $y(t-\tau) = \varphi(t-\tau)$ and apply the variation of constants formula to obtain

$$y(t; \varphi) = \Phi(t, 0)\varphi(0) + \int_0^t \Phi(t, s) (b + \varepsilon^n b_1(s)) \varphi(s-\tau) ds.$$

Hence

$$\begin{aligned} |y(t; \varphi)| &\leq e^{(a+\varepsilon^n M)t} |\varphi(0)| + (|b| + \varepsilon^n M) \int_0^t e^{(a+\varepsilon^n M)(t-s)} |\varphi(s-\tau)| ds \\ &\leq e^{(a+\varepsilon^n M)t} |\varphi|_C + e^{(a+\varepsilon^n M)t} (|b| + \varepsilon^n M) |\varphi|_C \int_0^t e^{-(a+\varepsilon^n M)s} ds \\ &\leq e^{(a+\varepsilon^n M)t} |\varphi|_C + \frac{|b| + \varepsilon^n M}{-a - \varepsilon^n M} (1 - e^{(a+\varepsilon^n M)t}) |\varphi|_C \\ &\leq |\varphi|_C \frac{|b| + \varepsilon^n M}{|b| - \varepsilon^n M} + |\varphi|_C e^{(a+\varepsilon^n M)t} \left(1 - \frac{|b| + \varepsilon^n M}{|b| - \varepsilon^n M} \right). \end{aligned}$$

Let us notice that the second term is negative, since

$$\frac{|b| + \varepsilon^n M}{|b| - \varepsilon^n M} > 1.$$

For $\varepsilon \leq \varepsilon_0 := (|b|/(3M))^{1/n}$ we have

$$\frac{|b| + \varepsilon^n M}{|b| - \varepsilon^n M} = \frac{1 + \varepsilon^n M/|b|}{1 - \varepsilon^n M/|b|} \leq 1 + c\varepsilon^n$$

with $c = 3M/|b|$. Therefore, we obtain

$$|y(t; \varphi)| \leq \frac{|b| + \varepsilon^n M}{|b| - \varepsilon^n M} |\varphi|_C \leq (1 + \varepsilon^n c) |\varphi|_C, \text{ for } t \in [0, \tau]. \quad (3.8)$$

By induction we find

$$|y(t; \varphi)| \leq |\varphi|_C (1 + \varepsilon^n c)^j, \text{ for } t \in [(j-1)\tau, j\tau]. \quad (3.9)$$

Equation (3.9) and $\tau = 1/\varepsilon$ imply that

$$\begin{aligned} |y(t; \varphi)| &\leq |\varphi|_C (1 + \varepsilon^n c) e^{\ln(1+\varepsilon^n c)(j-1)} = |\varphi|_C (1 + \varepsilon^n c) e^{\varepsilon \ln(1+\varepsilon^n c)(j-1)\tau} \\ &\leq |\varphi|_C (1 + \varepsilon^n c) e^{\varepsilon \ln(1+\varepsilon^n c)t} \leq (1 + \varepsilon^n c) |\varphi|_C e^{c\varepsilon^{n+1}t}, \end{aligned}$$

which is the desired result. \square

Remark 3.3. If the nonautonomous perturbations in (3.5) are absent and $a = -b$, then all solutions are bounded. This follows from the fact that $\frac{d}{dt}y^2(t) \leq 0$ at timepoints t with $|y(t)| \geq |y(s)|$ for all earlier times $s \leq t$. For the case $a = b < 0$, the zero solution is asymptotically stable, since all eigenvalues have negative real parts.

4 Estimations for the weakly nonlinear equation

In this section we derive a technical estimate, which will enable us to prove the main result, Theorem 5.1, of this paper. Consider the following linear equation, which is weakly and nonlinearly perturbed

$$\begin{aligned} y'(t) &= (a + \varepsilon^n a_1(t)) y(t) + (b + \varepsilon^n b_1(t)) y(t - \tau) \\ &\quad + \varepsilon^n g_1(\varepsilon, t) + \varepsilon^{n+1} g_2(\varepsilon, t, y(t), y(t - \tau)), \end{aligned} \quad (4.1)$$

where $\tau = 1/\varepsilon$. The following result gives useful a priori bounds.

Lemma 4.1. *Let $y(t) = y(t; \varphi, \varepsilon)$ be a solution of (4.1) with initial function $\varphi \in C([- \tau, 0], \mathbb{R})$ and $\varepsilon \in [0, r]$, $r > 0$. Let $|b| = |a|$ and $a < 0$, and let the following assumptions be fulfilled:*

(i) *The functions $a_1(t)$, $b_1(t)$, and $g_1(\varepsilon, t)$ are continuous and bounded with some $M > 0$, i.e. $|b_1(t)|, |a_1(t)|, |g_1(\varepsilon, t)| \leq M$, for $t \in [0, \infty)$ and $\varepsilon \in [0, r]$.*

(ii) The function $(y_1, y_2) \mapsto g_2(\varepsilon, t, y_1, y_2)$ is continuous and uniformly bounded, i.e. for all $R > 0$ there exists $G_2(R) > 0$ such that $|g_2(\varepsilon, t, y_1, y_2)| < G_2(R)$ for all $(\varepsilon, t) \in [0, r] \times \mathbb{R}$ and $|y_1| \leq R, |y_2| \leq R$.

Then, for each $T_0 > 0$ and $d_0 > 0$, there exist $\varepsilon_0 > 0, C > 0$ such that for all $\varepsilon \in [0, \varepsilon_0]$ the following statement holds: If $|\varphi|_C < d_0$, then $y(t)$ is defined for $t \in I_\varepsilon = [-\tau, T_0/\varepsilon^{n+1}]$ and the estimate

$$|y(t)| < C \quad (4.2)$$

is satisfied for all $t \in I_\varepsilon$.

Proof. Similarly to the proof of the previous lemmas, we consider the integral equation

$$y(t) = \Phi(t, 0) \varphi(0) + \int_0^t \Phi(t, s) \{ (b + \varepsilon^n b_1(s)) y(s - \tau) + \varepsilon^n g_1(\varepsilon, s) + \varepsilon^{n+1} g_2(\varepsilon, s, y(s), y(s - \tau)) \} ds \quad (4.3)$$

with $\Phi(t, s) = \exp\left(\int_s^t (a + \varepsilon^n a_1(u)) du\right)$, which is equivalent to (4.1) provided they both are equipped with the initial condition $\varphi(s)$ on $[-\tau, 0]$.

The remaining proof runs as follows. In step (i) we derive a function $H(C, \varepsilon_0)$ which bounds $y(t)$ as long as $|y(t)| < C, t \in I_\varepsilon$, and $\varepsilon \leq \varepsilon_0$. Then we show in step (ii) that there exist $\bar{C} (\bar{C} > d_0)$ and ε_0 such that $H(\bar{C}, \varepsilon_0) < \bar{C}$. Therefore, $|y(t)| < \bar{C}$ for all $t \in I_\varepsilon$, if $|\varphi|_C < d_0$. Indeed, if a solution passes $|y(t^*)| = \bar{C}$ at some time $t = t^*$ and $|y(t)| < \bar{C}$ for all $t < t^*$, the above implies $t^* \notin I_\varepsilon$.

Step (i). We derive the bounding function $H(C, \varepsilon_0)$. Assume a solution $y(t) = y(t; \varphi)$ of (4.3) with $|y(t)| < C$, where $|\varphi|_C < d_0$ and $0 < \varepsilon \leq \varepsilon_0 < r$ with ε_0 so small that $\varepsilon_0^n M < |b| = -a$. For $t \in [0, \tau]$ we have

$$\begin{aligned} |y(t)| &\leq \Phi(t, 0) |\varphi(0)| + \int_0^t \Phi(t, s) \{ |(b + \varepsilon^n b_1(s)) \varphi_0(s - \tau)| \\ &\quad + \varepsilon^n |g_1(\varepsilon, s)| + \varepsilon^{n+1} |g_2(\varepsilon, s, y(s), y(s - \tau))| \} ds \\ &< e^{(a + \varepsilon^n M)t} d_0 + \int_0^t e^{(a + \varepsilon^n M)(t-s)} ((|b| + M\varepsilon^n) d_0 + \varepsilon^n M + \varepsilon^{n+1} G_2(C)) ds \\ &\leq \frac{|b| + \varepsilon^n M}{|b| - \varepsilon^n M} d_0 + \frac{\varepsilon^n M + \varepsilon^{n+1} G_2(C)}{|b| - \varepsilon^n M} \\ &\leq (1 + c_1 \varepsilon^n) d_0 + \varepsilon^n c_2 =: d_1 \end{aligned}$$

with $c_1 = \frac{M}{|b| - \varepsilon_0^n M}$ and $c_2 = \frac{G_1 + \varepsilon_0 G_2(C)}{|b| - \varepsilon_0^n M}$. By the same reasoning, we obtain for $t \in [\tau, 2\tau]$

$$\begin{aligned} |y(t)| &< (1 + c_1 \varepsilon^n) d_1 + \varepsilon^n c_2 \\ &= (1 + c_1 \varepsilon^n)^2 d_0 + \varepsilon^n (1 + c_1 \varepsilon^n) c_2 + \varepsilon^n c_2 =: d_2. \end{aligned}$$

By induction we obtain for $t \in [(j-1)\tau, j\tau]$

$$\begin{aligned} |y(t)| < d_j &= \varepsilon^n c_2 \sum_{k=0}^{j-1} (1 + c_1 \varepsilon^n)^k + (1 + c_1 \varepsilon^n)^j d_0 \\ &= \frac{(c_2 + c_1 d_0) (1 + c_1 \varepsilon^n)^j - c_2}{c_1} \\ &\leq \frac{c_2 + c_1 d_0}{c_1} e^{j c_1 \varepsilon^n} = \frac{c_2 + c_1 d_0}{c_1} e^{j \tau c_1 \varepsilon^{n+1}}. \end{aligned}$$

This implies for all $t \in I_\varepsilon$:

$$|y(t)| < \frac{c_2 + c_1 d_0}{c_1} e^{(t+\tau)c_1 \varepsilon^{n+1}} \leq \frac{c_2 + c_1 d_0}{c_1} e^{c_1(T_0 + \varepsilon_0^n)} := H(C, \varepsilon_0). \quad (4.4)$$

Hence, the function $H(C, \varepsilon_0)$ is obtained. Note the dependence of c_2 on C , where C occurs only in the term $\varepsilon_0 G_2(C)$. Hence, for $\varepsilon_0 = 0$, the function $C \mapsto H(C, 0)$ is constant.

Step (ii). Next, we prove the existence of \bar{C} and $\varepsilon_0 > 0$ such that the inequality $H(\bar{C}, \varepsilon_0) < \bar{C}$ holds. Substituting c_1 and c_2 in (4.4) we obtain

$$H(C, \varepsilon_0) = \frac{G_1 + \varepsilon_0 G_2(C) + M d_0}{M} \exp \left[\frac{M(T_0 + \varepsilon_0^n)}{|b| - \varepsilon_0^n M} \right]. \quad (4.5)$$

Now choose

$$\bar{C} := 2 \frac{G_1 + M d_0}{M} \exp \left[\frac{M T_0}{|b|} \right] = 2H(\bar{C}, 0) = 2H(0, 0) > 2d_0. \quad (4.6)$$

Due to continuity of $H(C, \varepsilon_0)$ in ε_0 , there exists $\varepsilon_0 > 0$ which satisfies $H(\bar{C}, \varepsilon_0) < \bar{C}$. This proves the Lemma. \square

5 Amplitude equations

In this section we derive a PDE-approximation for the DDE

$$\frac{dy(t)}{dt} = g(y(t), y(t-\tau), \varepsilon^2), \quad g(0, 0, \varepsilon^2) = 0, \quad \tau = \varepsilon^{-1}. \quad (5.1)$$

In the vicinity of the stationary state $y = 0$, solutions of (5.1) can be approximated by solutions $u = u(\theta, x)$ of a corresponding parabolic PDE of the Ginzburg-Landau type, namely

$$\frac{\partial u}{\partial \theta} = \alpha \frac{\partial^2 u}{\partial x^2} + \beta \frac{\partial u}{\partial x} + pu + \gamma u^3, \quad x \in \mathbb{R}, \quad 0 \leq \theta \leq T_0 \quad (5.2)$$

with appropriate periodicity conditions in $x \in \mathbb{R}$. The equation (5.2) is called amplitude equation. An important feature of (5.2) is that it is no longer singularly perturbed. Moreover, it provides a universal description of the systems behavior close to a bifurcation threshold allowing

a universal description of different dynamical systems of various origin. For example, Ginzburg-Landau type equations are already well established normal forms for analyzing destabilization of reaction-diffusion systems or oscillator chains [6, 34, 31, 30, 25, 14].

In what follows, we assume that $g \in C^4(\mathbb{R}^3)$. We adopt the notation $g_1 := \partial_1 g(0, 0, 0)$ and $g_2 := \partial_2 g(0, 0, 0)$ for the derivatives with respect to the instantaneous and the delayed state variable respectively. As follows from the linear theory for large τ given in Sec. 2, the system admits two types of bifurcations for the ground state $y = 0$, which are governed by the pseudo-continuous spectrum (PCS) and occur when the conditions $g_1 = g_2$ in the flip case or $g_1 = -g_2$ in the pitchfork/transcritical case are satisfied.

Based on our analysis of the linear equation in Sec. 2, it is not difficult to check that if the system is ε^2 -close to the bifurcation, then the number of unstable eigenvalues (belonging to PCS) stays asymptotically bounded with $\varepsilon \rightarrow 0$. This is the reason why we assume the perturbation parameter in the right hand side of (5.1) to be proportional to ε^2 . If the perturbation were of order $\mathcal{O}(\varepsilon)$, the number of unstable eigenvalues would grow with $\varepsilon \rightarrow 0$ and we expect some singularly perturbed problem in the limit. For convenience, let us rewrite Eq. (5.1) by explicitly separating linear parts as

$$y'(t) = (a + a_1 \varepsilon^2) y(t) + (a\kappa + b_1 \varepsilon^2) y(t - \tau) + f(y(t), y(t - \tau), \varepsilon^2), \quad (5.3)$$

where $\kappa = \frac{a}{b} = -\text{sgn}(b)$. This means, $\kappa = 1$ stands for the flip case and $\kappa = -1$ for the pitchfork/transcritical case. The function $f \in C^4(\mathbb{R}^3)$ represents the nonlinearities of g , i.e. $f_1 = f_2 = 0$. Since we consider the case of cubic nonlinearities, we have $f_{11} = f_{12} = f_{22} = 0$. Here again, the corresponding derivatives are denoted by indices of the function f , e.g. $f_{12} = \partial_2 \partial_1 f(0, 0, 0)$ and so on.

For some $\varepsilon_0 > 0$ and $0 < \varepsilon \leq \varepsilon_0$, consider a small solution $y(t)$ of the DDE (5.1). We adopt the multiscale representation

$$y(t) = \varepsilon u(\xi(t, \varepsilon)) + \varepsilon^4 \varrho(\xi(t, \varepsilon)), \quad \text{with } \xi(t, \varepsilon) = (\varepsilon t, \varepsilon^2 t, \varepsilon^3 t), \quad (5.4)$$

a multiscalar function u and a residuum ϱ . For a moment, let us assume that these functions are smooth and bounded and formally derive some requirements for u based upon this assumption. We define a 1-shift in the first coordinate by $\sigma(\xi_1, \xi_2, \xi_3) := (\xi_1 - 1, \xi_2, \xi_3)$. Then $\xi(t - \tau, \varepsilon) = \sigma(\xi(t, \varepsilon)) - (0, \varepsilon, \varepsilon^2)$ and

$$\begin{aligned} y(t - \tau) &= \varepsilon u(\xi(t - \tau, \varepsilon)) + \varepsilon^4 \varrho(\xi(t - \tau, \varepsilon)) \\ &= \varepsilon (u(\sigma(\xi(t, \varepsilon)) - (0, \varepsilon, \varepsilon^2))) + \varepsilon^4 \varrho(\xi(t - \tau, \varepsilon)) \\ &= \varepsilon u(\sigma(\xi(t, \varepsilon))) - \varepsilon^2 \partial_2 u(\sigma(\xi(t, \varepsilon))) + \frac{\varepsilon^3}{2} \partial_2^2 u(\sigma(\xi(t, \varepsilon))) \\ &\quad - \varepsilon^3 \partial_3 u(\sigma(\xi(t, \varepsilon))) + \varepsilon^4 \eta(\xi(t, \varepsilon), u) + \varepsilon^4 \varrho(\xi(t - \tau, \varepsilon)), \end{aligned}$$

where $\eta(\xi, u)$ depends linearly on derivatives of u up to the third order and allows for an estimation as $|\eta| \leq C_1 |u|_{C^4}$ with some constant C_1 . We insert (5.5) and (5.4) into (5.3) and compare terms at different orders of ε . Terms of order $\mathcal{O}(\varepsilon)$ give

$$u(\xi) = -\kappa u(\sigma(\xi)). \quad (5.5)$$

This means that the function u must be anti-periodic with respect to the first argument in the case of period doubling and periodic in the pitchfork/transcritical case. Comparison of terms of order $\mathcal{O}(\varepsilon^2)$ yields

$$\partial_1 u(\xi) = -a\kappa\partial_2 u(\sigma(\xi)) = a\partial_2 u(\xi). \quad (5.6)$$

This implies, that u can be written as a function of two arguments $x = \xi_1 + \frac{1}{a}\xi_2$ and $\theta = \xi_3$ which we denote by $A(x, \theta)$, i.e., $u(\xi_1, \xi_2, \xi_3) = A(\xi_1 + \frac{1}{a}\xi_2, \xi_3)$. Following from (5.5), we have

$$A(x, \theta) = -\kappa A(x - 1, \theta). \quad (5.7)$$

The $\mathcal{O}(\varepsilon^3)$ -terms give

$$\partial_\theta A(x, \theta) = \frac{1}{2a^2}\partial_x^2 A(x, \theta) + \frac{1}{a^2}\partial_x A(x, \theta) - \frac{(a_1 - \kappa b_1)}{a}A(x, \theta) - \frac{\varsigma}{a}A(x, \theta)^3 \quad (5.8)$$

where $\varsigma = \frac{1}{6}(f_{111} - 3\kappa f_{112} + 3f_{122} - \kappa f_{222})$. For any bounded solution A of (5.8), we define

$$y_A(t, \varepsilon) := \varepsilon A\left(\varepsilon t + \frac{1}{a}\varepsilon^2 t, \varepsilon^3 t\right), \quad (5.9)$$

which we call a "formal approximation". It was shown in [5, Lemma 3.1] that, if $\varsigma < 0$, then solutions of (5.8) with $x \in \mathbb{R}$ and initial data

$$A_0 \in C_b^4(\mathbb{R}) = \{v \in C_b(\mathbb{R}) \mid \partial_x^k v \in C_b(\mathbb{R}) \text{ for } k = 0, \dots, 4\}$$

remain in $C_b^4(\mathbb{R})$ for all times $t \geq 0$ and are unique and uniformly bounded. This means that $M := \sup_{(x, \theta) \in \mathbb{R} \times \mathbb{R}_+} |A(x, \theta)| < \infty$ is finite and, accordingly, $\sup_{t \in \mathbb{R}} |y_A(t, \varepsilon)| \leq \varepsilon M \leq \varepsilon_0 M =: m$. By the above reasoning, any C^4 -smooth and bounded A defines a C^4 -smooth and bounded $y_A(t, \varepsilon)$, which fulfills

$$\begin{aligned} y_A'(t, \varepsilon) &= (a + a_1\varepsilon^2)y_A(t, \varepsilon) + (a\kappa + b_1\varepsilon^2)y_A(t - \tau, \varepsilon) \\ &\quad + f(y_A(t, \varepsilon), y_A(t - \tau, \varepsilon), \varepsilon^2) + \varepsilon^4 r(y_A(t, \varepsilon), y_A(t - \tau, \varepsilon), \varepsilon^2), \end{aligned} \quad (5.10)$$

where the remainder $r(y_A(t, \varepsilon), y_A(t - \tau, \varepsilon), \varepsilon^2)$ is bounded by a constant which is, for a given $\varepsilon_0 > 0$, independent of $\varepsilon \in [0, \varepsilon_0]$ and only depends on derivatives of f up to fourth order evaluated within $[-m, m]^2 \times [0, \varepsilon_0]$. By uniqueness the (anti-)periodicity property (5.7) for A_0 persists under the evolution of (5.8).

Theorem 5.1. *Let $A \in C^1([0, T_A], C_b^4(\mathbb{R}))$ be a solution of system (5.2) with initial data A_0 , which satisfies*

$$A_0(x) = -\kappa A_0(x + 1). \quad (5.11)$$

and coefficients as in (5.8)

$$\alpha = 1/2a^2, \quad \beta = 2\alpha, \quad p = -\frac{a_1 - \kappa b_1}{a}, \quad \gamma = -\frac{1}{6a}(f_{111} - \kappa f_{222} + 3f_{122} - 3\kappa f_{112}).$$

Let y_A be the formal approximation

$$y_A(t, \varepsilon) = \varepsilon A\left(\varepsilon t + \frac{1}{a}\varepsilon^2 t, \varepsilon^3 t\right). \quad (5.12)$$

Then, for each $T_0 \in (0, T_A]$ and $d_0 > 0$, there exist $C > 0$ and $\varepsilon_0 > 0$ such that the following statement holds for all $\varepsilon \in [0, \varepsilon_0]$. If $y = y(t; \varphi_0)$ is a solution of the DDE (5.3) with an initial value $y(s; \varphi) = \varphi_0(s)$, $s \in [0, \tau]$, which satisfies

$$\left| \varphi_0(s) - \varepsilon A_0 \left(\varepsilon s + \frac{1}{a} \varepsilon^2 s \right) \right| \leq d_0 \varepsilon^2, \quad s \in [0, \tau],$$

then $y(t, \varphi_0)$ is defined for $t \in I_\varepsilon = [0, T_0/\varepsilon^3]$ and

$$\sup_{t \in I_\varepsilon} |y(t; \varphi_0) - y_A(t, \varepsilon)| \leq C \varepsilon^2$$

Proof. For the sake of brevity, we write $y(t) = y(t; \varphi)$ and $y_A(t) = y_A(t, \varepsilon)$. Consider the rescaled error

$$R(t) = \varepsilon^{-2} (y(t) - y_A(t)). \quad (5.13)$$

Differentiating (5.13), taking into account that $y(t)$ satisfies (5.3) and $y_A(t)$ satisfies (5.10), we obtain the equation for the error

$$\begin{aligned} \varepsilon^2 R'(t) &= (a + a_1 \varepsilon^2) \varepsilon^2 R(t) + (a\kappa + b_1 \varepsilon^2) \varepsilon^2 R(t - \tau) + f(y(t), y(t - \tau), \varepsilon^2) \\ &\quad - f(y_A(t), y_A(t - \tau), \varepsilon^2) - \varepsilon^4 r(y_A(t), y_A(t - \tau), \varepsilon^2). \end{aligned}$$

Since the nonlinearity f contains terms at least cubic in $y(t)$ and $y(t - \tau)$, we have

$$\begin{aligned} f(y(t), y(t - \tau), \varepsilon^2) - f(y_A(t), y_A(t - \tau), \varepsilon^2) &= \varepsilon^2 \partial_1 f(y_A(t), y_A(t - \tau), 0) R(t) \\ &\quad + \varepsilon^2 \partial_2 f(y_A(t), y_A(t - \tau), 0) R(t - \tau) + H(y_A(t), y_A(t - \tau), R(t), R(t - \tau), \varepsilon). \end{aligned} \quad (5.14)$$

Defining the functions a_2, b_2 and N via

$$\begin{aligned} \partial_1 f(y_A(t), y_A(t - \tau), 0) &= \varepsilon^2 a_2(t), \quad \partial_2 f(y_A(t), y_A(t - \tau), 0) = \varepsilon^2 b_2(t), \\ H(y_A(t), y_A(t - \tau), R(t), R(t - \tau), \varepsilon) &= \varepsilon^5 N(\varepsilon, t, R(t), R(t - \tau)), \end{aligned}$$

we see that $a_2(t)$ and $b_2(t)$ are bounded for $t \in I_\varepsilon$ uniformly in ε . Moreover, the nonlinear function N satisfies the assumption (ii) of Lemma 4.1. The equation for the error becomes

$$\begin{aligned} R'(t) &= (a + \varepsilon^2 (a_1 + a_2(t))) R(t) + (a\kappa + \varepsilon^2 (b_1 + b_2(t))) R(t - \tau) \\ &\quad - \varepsilon^2 r(y_A(t), y_A(t - \tau), \varepsilon^2) + \varepsilon^3 N(\varepsilon, t, R(t), R(t - \tau)), \end{aligned}$$

with initial condition

$$R(s) = R_0(s) = \varepsilon^{-2} (\varphi(s) - y_A(s)), \quad s \in [0, \tau].$$

Applying Lemma 4.1 for the case $n = 2$, $g_1 = -r$, $g_2 = N$, we obtain that for each $T_0 > 0$ and $d_0 > 0$ that there exist $\varepsilon_0, C > 0$ such that for all $\varepsilon \in [0, \varepsilon_0]$ the following statement holds: If $|R_0|_C \leq d_0$, i.e., $\sup_{s \in [0, \tau]} |\varphi(s) - y_A(s)| \leq d_0 \varepsilon^2$, then

$$|R(t; R_0)| \leq C, \quad \text{for all } t \in I_\varepsilon.$$

Thus, $y(t) = y_A(t) + \varepsilon^2 R(t)$ exists for $t \in I_\varepsilon$ and

$$|y(t) - y_A(t)| = \varepsilon^2 |R(t)| \leq C \varepsilon^2, \quad \text{for all } t \in I_\varepsilon.$$

This proves the theorem. \square

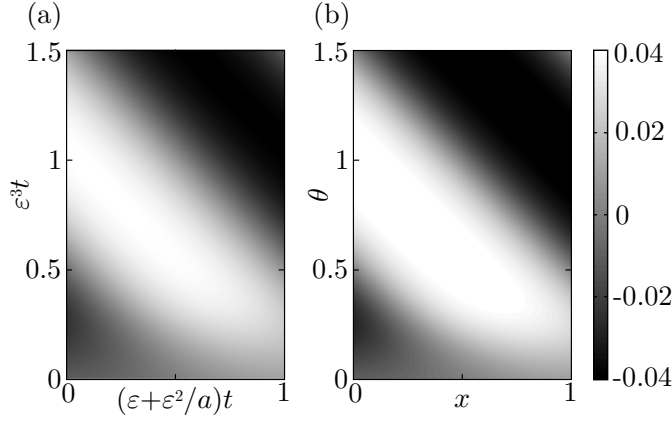


Figure 6.1: Chart (a) shows evolution for DDE (6.1) in the rescaled coordinates $x = (\varepsilon + \varepsilon^2/a)t$, $\theta = \varepsilon^3t$. Chart (b) shows the evolution of the amplitude equation (6.2)–(6.3). The magnitude of the solution is indicated by the different shades of gray.

6 Example

In order to illustrate the obtained results, let us consider the following DDE

$$y'(t) = -0.999y(t) - y(t - \tau) - 0.5y^3(t), \quad \tau = 100. \quad (6.1)$$

In terms of the previous sections, this means,

$$\varepsilon = \frac{1}{\tau} = 0.01, \quad a = b = -1, \quad \kappa = 1, \quad a_1 = 10, \quad b_1 = 0, \quad f(y_1, y_2, \varepsilon^2) = -0.5y_1^3.$$

The corresponding amplitude equation is

$$\frac{\partial A}{\partial \theta} = \frac{1}{2} \frac{\partial^2 A}{\partial x^2} + \frac{\partial A}{\partial x} + 10A - \frac{1}{2}A^3, \quad (6.2)$$

with the antiperiodicity condition

$$A_0(x) = -A_0(x + 1). \quad (6.3)$$

For the following initial condition

$$A_0(x) = -\cos(\pi x),$$

the solution of (6.2)–(6.3) is shown in Fig 6.1(b) as a gray-scaled plot. The corresponding solution for the DDE (6.1) in the coordinates $x = \varepsilon t + \frac{1}{a}\varepsilon^2t$, $\theta = \varepsilon^3t$ is shown in Fig 6.1(a). We note, that both solutions show the same behavior as predicted by the theory. For more numerical examples, we refer to the papers [13, 35, 1, 40].

7 Discussion

Let us mention some possible generalization of our results. Suppose that the DDE has a small and slow nonautonomous perturbation of the form $\varepsilon^3 h(\varepsilon t + \varepsilon^2 t/a, \varepsilon^3 t)$, i.e.

$$y'(t) = g(y(t), y(t - \tau)) + \varepsilon^3 h(\varepsilon t + \varepsilon^2 t/a, \varepsilon^3 t), \quad (7.1)$$

with $h(x, \theta) = -\kappa h(x + 1, \theta)$. Then the same procedure is applicable for (7.1) and, as a result, the nonautonomous and nonhomogeneous partial differential equation

$$\frac{\partial u}{\partial \theta} = \alpha \frac{\partial^2 u}{\partial x^2} + \beta \frac{\partial u}{\partial x} + pu + \gamma u^3 + h(x, \theta) \quad (7.2)$$

appears as the amplitude equation. If h is independent of θ , then the dynamics of (7.2) is that of a traveling gradient flow. Indeed, defining $\tilde{u}(x, \theta) = u(\theta, x + \beta\theta)$, we find

$$\frac{\partial \tilde{u}}{\partial \theta} = \alpha \frac{\partial^2 \tilde{u}}{\partial x^2} + p\tilde{u} + \gamma \tilde{u}^3 + h(x), \quad (7.3)$$

which is the L^2 -gradient flow for the energy functional

$$\mathcal{E}(\tilde{u}) = \int_0^1 \left(\frac{\alpha}{2} \left| \frac{\partial \tilde{u}}{\partial x} \right|^2 - \frac{p}{2} \tilde{u}^2 - \frac{\gamma}{4} \tilde{u}^4 - h\tilde{u} \right) dx.$$

Hence, for $\gamma < 0$ the global attractor for (7.3) contains spatially (anti-)periodic traveling waves $u(x, \theta) = U(x + \beta\theta)$ and heteroclinic connections between them, cf. [8]. Of course, considering a forcing $h(x, \theta)$ with nontrivial θ -dependence opens up the possibility of a much richer dynamical behavior and to implement suitable control mechanisms to stabilize desirable patterns.

Another possible extension of the presented results is the case of systems of DDE (i.e. non-scalar DDE). Systems possess some interesting features, which are not present in scalar DDEs. For example, the Turing mechanism of destabilization, which is not possible in the scalar case, can appear for systems [40].

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