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Minimization of a fractional perimeter-Dirichlet integral functional

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ABSTRACT. We consider a minimization problem that combines the Dirichlet energy with the nonlocal perimeter of a level set, namely

$$\int_\Omega |\nabla u(x)|^2\,dx + \operatorname{Per}_\sigma\Big(\{u > 0\}, \Omega\Big),$$

with $\sigma \in (0,1)$. We obtain regularity results for the minimizers and for their free boundaries $\partial\{u > 0\}$ using blow-up analysis. We will also give related results about density estimates, monotonicity formulas, Euler-Lagrange equations and extension problems.

1. INTRODUCTION

Let Ω be a bounded domain $\mathbb{R}^n$ and $\sigma \in (0,1)$ a fixed parameter. In this paper we discuss regularity properties for minimizers of the energy functional

$$(1.1) \qquad J(u) := \int_\Omega |\nabla u|^2 dx + \operatorname{Per}_\sigma(E, \Omega), \qquad E = \{u > 0\} \text{ in } \Omega.$$

where $\operatorname{Per}_\sigma(E, \Omega)$ represents the σ-fractional perimeter of the set E in Ω.

Here the set E is fixed outside Ω and coincides with $\{u > 0\}$ in Ω, and we minimize J among all functions $u \in H^1(\Omega)$ with prescribed boundary data i.e. $u = \varphi$ on $\partial\Omega$ for some fixed $\varphi \in H^1(\Omega)$.

The fractional perimeter functional $\operatorname{Per}_\sigma(E, \Omega)$ was first introduced in [6] and it represents the Ω-contribution in the double integral of the norm $\|\chi_E\|_{H^{\sigma/2}}$. Precisely, for any measurable set $E \subseteq \mathbb{R}^n$

$$(1.2) \qquad \operatorname{Per}_\sigma(E, \Omega) := L(E \cap \Omega, E^c) + L(E \setminus \Omega, \Omega \setminus E),$$

where

$$L(A, B) := \int_{A \times B} \frac{dx\,dy}{|x - y|^{n+\sigma}}.$$

It is known (see [8, 3, 12, 11]) that up to multiplicative constants $\operatorname{Per}_\sigma(E, \mathbb{R}^n)$ converges to the classical perimeter functional as $\sigma \to 1$ and it converges to $|E|$, the Lebesgue measure of E, as $\sigma \to 0$. In this spirit, the functional in (1.1) formally interpolates between the two-phase free boundary problem treated in [1] (where the term $\operatorname{Per}_\sigma(E, \Omega)$ is replaced by the classical perimeter of E in Ω) and the Dirichlet-perimeter minimization functional treated in [4] (where $\operatorname{Per}_\sigma(E, \Omega)$ is replaced by the Lebesgue measure of E in Ω).

In fact, all previous models correspond to particular cases of the general nonlocal phase transition setting as discussed in [10] (see in particular Section 3.5 there): in our case, the square of the $H^{\sigma/2}$ norm of the function $\operatorname{sign} u$ is, in terms of [10], the double convolution of the "phase field parameter" ϕ with the corresponding fractional Laplacian kernel.

The existence of minimizers follows easily by the direct method in the calculus of variations, see Lemma 2.1 below. Our first regularity result deals with the Hölder regularity of solutions and density estimates for the free boundary ∂E.

Theorem 1.1. *Let (u, E) be a minimizer of J in B_1 with $0 \in \partial E$. Then u is $C^\alpha(B_1)$, with $\alpha := 1 - \frac{\sigma}{2}$) and*

$$(1.3) \qquad \|u\|_{C^\alpha(B_{r_0})} \leqslant C.$$

Moreover for any $r \leqslant r_0$

$$(1.4) \qquad \min\Big\{|B_r \cap E|,\ |B_r \cap E^c|\Big\} \geqslant cr^n.$$

The positive constants C, c above depend only on n and σ, and r_0 depends also on $\|u\|_{L^2(B_1)}$.

We remark that the Hölder exponent obtained in Theorem 1.1 is consistent with the natural scaling of the problem, namely

(1.5)
$$\text{if } u \text{ is a minimizer and } u_r(x) := r^{\frac{\sigma}{2}-1}u(rx),$$
$$\text{then } u_r \text{ is also a minimizer.}$$

A minimizer u is harmonic in its positive and negative sets and formally, at points x on the free boundary $\{u = 0\}$ it satisfies

(1.6)
$$\kappa_\sigma(x) := \int_{\mathbb{R}^n} \frac{\chi_{E^c} - \chi_E}{|x-y|^{n+\sigma}} \, dy = |\nabla u^+(x)|^2 - |\nabla u^-(x)|^2,$$

where $\kappa_\sigma(x)$ represents the σ-fractional curvature of ∂E at x (a precise statement will be given in Theorem 4.1).

Generically, we expect that the minimizer u is Lipschitz near the free boundary. Then the fractional curvature becomes the dominating term in the free boundary condition above and ∂E can be viewed as a perturbation of the σ-minimal surfaces which were treated in [6]. However, differently from the limiting cases $\sigma = 0$ and $\sigma = 1$, for $\sigma \in (0,1)$ it seems difficult to obtain the Lipschitz continuity of u at all points (see the discussion at the end of Section 5). For the regularity of the free boundary we use instead a monotonicity formula and study homogenous global minimizers. Following the strategy in [6] we obtain an improvement of flatness theorem for the free boundary ∂E. We also show in the spirit of [14, 15] that in dimension $n = 2$ all global minimizers are trivial and by the standard dimension reduction argument we obtain the following result.

Theorem 1.2. *Let (u, E) be a minimizer in B_1. Then ∂E is a $C^{1,\gamma}$-hypersurface and it satisfies the Euler-Lagrange equation* (1.6) *in the viscosity sense, outside a small singular set $\Sigma \subset \partial E$ of Haussdorff $(n-3)$-dimension.*

In particular in dimension $n = 2$ the free boundary is always a $C^{1,\gamma}$ curve. We remark that by using the strategy in [5] the $C^{1,\gamma}$ regularity of ∂E can be improved to C^∞ regularity.

The proofs of Theorem 1.1 and 1.2 require some additional results, that will be presented in the course of the paper, such as a monotonicity formula, a precise formulation of the Euler-Lagrange equation and an equivalent extension problem of local type.

The paper is organized as follows. In Section 2 we state various estimates for the change in the Dirichlet integral whenever we perturb the set E by $E \cup A$. We use these estimates throughout the paper and their proofs are postponed in the last section of the paper. We prove Theorem 1.1 in Section 3 and the improvement of flatness theorem in Section 4. The monotonicity formula and some of its consequences are presented in Section 5. Finally in Section 6 we prove Theorem 1.2 by showing the regularity of cones in dimension 2.

2. ESTIMATES FOR THE HARMONIC REPLACEMENT

In order to rigorously deal with the minimization concept of the functional in (1.1), we introduce some notation.

Let $\varphi \in H^1(\Omega)$ and $E_0 \subset \Omega^c$ be given. We want to minimize the energy

(2.1)
$$J_\Omega(u) := \int_\Omega |\nabla u|^2 dx + \text{Per}_\sigma(E, \Omega)$$

among all *admissible pairs* (u, E) that satisfy

$$u - \varphi \in H_0^1(\Omega), \qquad E \cap \Omega^c = E_0,$$

$$u \geqslant 0 \quad \text{a.e. in } E \cap \Omega, \qquad u \leqslant 0 \quad \text{a.e. in } E^c \cap \Omega.$$

We assume that there is an admissible pair with finite energy, say for simplicity $J(\phi, E_0 \cup \{\phi \geqslant 0\}) < \infty$. From the lower semicontinuity of J we easily obtain the existence of minimizers.

Lemma 2.1. *There exists a minimizing pair (u, E).*

Proof. Let (u_k, E_k) be a sequence of pairs along which J approaches its infimum. By compactness, after passing to a subsequence, we may assume that $u_k \rightharpoonup u$ in $H^1(\Omega)$, $u_k \to u$ in $L^2(\Omega)$ and $\chi_{E_k} \to \chi_E$ in $L^1(\Omega)$. Then (u, E) is admissible and by the lower semicontinuity of the fractional perimeter functional (i.e. Fatou's lemma) we obtain that (u, E) is a minimizing pair. $\square$

Notice that a minimizing pair in Ω is also a minimizing pair in any subdomain of Ω. We assume throughout, after possibly modifying E on a set of measure 0, that the topological boundary of E coincides with its essential boundary, that is

$$\partial E = \{x \in \mathbb{R}^n \text{ s.t. } 0 < |E \cap B_r(x)| < |B_r(x)| \quad \text{for all } r > 0\}.$$

We recall the notion of *harmonic replacement* from [4].

Definition 2.2. Let $\varphi \in H^1(\Omega)$ and $K \subset \Omega$ be a measurable set. Assume that the set

$$\mathcal{D} := \{v \text{ s.t. } v - \varphi \in H^1_0(\Omega) \quad \text{and } v = 0 \text{ a.e. in } K\}$$

is not empty. Then we denote by $\varphi_K \in \mathcal{D}$ the unique minimizer of

$$\min_{v \in \mathcal{D}} \int_\Omega |\nabla v|^2,$$

and say that φ_K is the harmonic replacement of φ that vanishes in K.

From the definition it follows that

$$\int_\Omega \nabla \varphi_K \cdot \nabla w = 0, \qquad \text{for all } w \in H^1_0(\Omega) \text{ with } w = 0 \text{ a.e. in } K.$$

Also, it is straightforward to check that if $\varphi \geqslant 0$ then φ_K is subharmonic. In this case we think that φ_K is defined pointwise as the limit of its solid averages.

Clearly if (u, E) is a minimizing pair then we obtain

$$u^+ = u^+_{E^c} \quad \text{and} \quad u^- = u^-_E.$$

Below we estimate the difference in the Dirichlet energies of the harmonic replacements in two different sets E and $E \setminus A$, in terms of the measure of the set $A \subset B_{3/4}$. These estimates depend on the geometry of E and A. We assume that $\varphi \in H^1(B_1) \cap L^\infty(B_1)$, $\varphi \geqslant 0$, and let

$$w := \varphi_{E^c}, \qquad v := \varphi_{E^c \cup A}.$$

The first lemma deals with the case when A is interior to a ball.

Lemma 2.3. *Assume v, w are as above and $A := B_\rho \cap E$ for some $\rho \in [\frac{1}{4}, \frac{3}{4}]$. Then*

$$\int_{B_1} |\nabla v|^2 - |\nabla w|^2 \, dx \leqslant C |A| \, \|w\|^2_{L^\infty(B_1)},$$

for some constant C depending only on n.

The next lemma gives the same bound in the case when A is exterior to a ball under the additional hypothesis that A satisfies a density property.

Lemma 2.4. *Let v, w be as above and assume $E \cap B_{1/2} = \emptyset$. Let $A \subset B_{3/4} \setminus B_{1/2}$ be a closed set that satisfies the density property*

$$|A \cap B_r(x)| \geqslant \beta r^n \quad \text{for all } x \in \partial A \text{ and } B_r(x) \cap B_{1/2} = \emptyset,$$

for some $\beta > 0$. Then

$$\int_{B_1} |\nabla v|^2 - |\nabla w|^2 \ dx \leqslant C(\beta)|A| \, \|w\|_{L^\infty(B_1)}^2,$$

for some constant $C(\beta)$ depending only on n and β.

Finally we provide a more precise estimate in the case when ∂E is more regular.

Let $u \in H^1(B_1) \cap C(\overline{\Omega})$ be harmonic in the sets $E = \{u > 0\}$ and $\{u < 0\}$. Assume

$$0 \in \partial E \quad \text{and} \quad E = \{x_n > g(x')\}$$

is given by the subgraph in the e_n direction of a $C^{1,\gamma}$ function. For a sequence of $\varepsilon_k \to 0$ we consider sets

$$A_k := \{g(x') < x_n < f_k(x')\} \subset B_{\varepsilon_k},$$

for a sequence of functions f_k with bounded $C^{1,\gamma}$ norm. For each k we define $\bar{u}_k$ the perturbation of u for which the positive set is given by $E \cup A_k$, i.e.

$$\bar{u}_k^+ = u_{E^c \setminus A_\varepsilon}^+ \quad \bar{u}_k^- = u_{E \cup A}^-.$$

Lemma 2.5. *Then*

$$\lim_{k \to \infty} \frac{1}{|A_k|} \int_{B_1} |\nabla \bar{u}_k|^2 - |\nabla u|^2 \ dx = |\nabla u^-(0)|^2 - |\nabla u^+(0)|^2.$$

The proofs of Lemmas 2.3-2.5 will be completed in the last section.

3. PROOF OF THEOREM 1.1

In this section we obtain the Hölder continuity of minimizers and uniform density estimates for their free boundary. We adapt to our goals the strategy of [4], and we simplify some steps using Lemma 2.3. We start with a density estimate.

Lemma 3.1. *Let (u, E) be a minimizer in B_1 and assume*

$$0 \in \partial E \quad \text{and} \quad \|u^+\|_{L^\infty(B_1)} \leqslant M,$$

for some constant M. Then

$$|E \cap B_{1/2}| \geqslant \delta, \qquad \|u^-\|_{L^\infty(B_{1/2})} \leqslant K,$$

for some positive constant δ, K depending on n, σ and M.

Proof. First we prove the density estimate. For each $\rho \in [\frac{1}{4}, \frac{3}{4}]$, set

$$V_\rho = |E \cap B_\rho|, \quad a(\rho) = \mathcal{H}^{n-1}(E \cap \partial B_\rho).$$

and assume by contradiction that $V_{1/2} < \delta$ small.

For each such ρ we consider $\bar{u}$ the perturbation of u which has as positive set $E \setminus A$ with $A := E \cap B_\rho$, that is

$$\bar{u}^+ := u_{E^c \cup A}^+, \quad \bar{u}^- := u_{E \setminus A}^-.$$

From the minimality of (u, E) we find

(3.1)
$$\text{Per}_\sigma(E, B_1) - \text{Per}_\sigma(E \setminus A, B_1) \leqslant \int_{B_1} |\nabla \bar{u}|^2 - |\nabla u|^2 dx.$$

Since (see (7.2))

$$(3.2) \qquad \int_{B_1} |\nabla \bar{u}|^2 - |\nabla u|^2 dx = \int_{B_1} |\nabla \bar{u}^+|^2 - |\nabla u^+|^2 dx - \int_{B_1} |\nabla(\bar{u}^- - u^-)|^2 \, dx$$

$$\leqslant \int_{B_1} |\nabla \bar{u}^+|^2 - |\nabla u^+|^2 dx,$$

we use Lemma 2.3 and the definition of Per_σ (see (1.2)) and we conclude that

$$L(A, E^c) - L(A, E \setminus A) \leqslant CM^2 |A|.$$

Hence

$$(3.3) \qquad L(A, A^c) \leqslant 2L(A, E \setminus A) + CM^2|A| \leqslant 2L(A, B_\rho^c) + CM^2 V_\rho.$$

We estimate the left term by applying Sobolev inequality (see, e.g., Theorem 7 in [13]): we obtain that

$$V_\rho^{\frac{n-\sigma}{n}} = \|\chi_A\|^2_{L^{\frac{2n}{n-\sigma}}(\mathbb{R}^n)} \leqslant C\|\chi_A\|^2_{H^{\sigma/2}(\mathbb{R}^n)} = CL(A, A^c).$$

If $x \in B_\rho$ then

$$\int_{\mathcal{B}_\rho^c} \frac{1}{|x-y|^{n+\sigma}} dy \leqslant C \int_{\rho - |x|}^\infty \frac{1}{r^{n+\sigma}} r^{n-1} dr \leqslant C(\rho - |x|)^{-\sigma},$$

hence integrating in the set A we obtain

$$L(A, B_\rho^c) \leqslant C \int_0^\rho a(r)(\rho - r)^{-\sigma} \, dr.$$

We use these inequalities into (3.3) and the assumption that $V_\rho \leqslant \delta$ is sufficiently small to find

$$V_\rho^{\frac{n-\sigma}{n}} \leqslant C \int_0^\rho a(r)(\rho - r)^{-\sigma} \, dr.$$

Integrating the inequality above between $\frac{1}{4}$ and $t \in [\frac{1}{4}, \frac{1}{2}]$ gives

$$(3.4) \qquad \int_{1/4}^t V_\rho^{\frac{n-\sigma}{n}} \, d\rho \leqslant Ct^{1-\sigma} \int_0^t a(r) dr \leqslant CV_t.$$

The proof is now a standard De Giorgi iteration: let

$$t_k = \frac{1}{4} + \frac{1}{2^k}, \quad v_k = V_{t_k},$$

and notice that $t_2 = \frac{1}{2}$ and $t_\infty = \frac{1}{4}$. Equation (3.4) yields

$$2^{-(k+1)} v_{k+1}^{\frac{n-\sigma}{n}} \leqslant Cv_k.$$

Since $v_2 < \delta$, that is conveniently small, we obtain $v_k \to 0$ as $k \to \infty$. Thus $V_{1/4} = 0$ and we contradict that $0 \in \partial E$.

For the bound on u^- we write the energy inequality for $\rho = \frac{3}{4}$ and we estimate also the negative term in (3.2) by Poincare inequality

$$\int_{B_1} |\nabla(\bar{u}^- - u)|^2 dx \geqslant c \int_{B_1} |\bar{u}^- - u^-|^2 dx \geqslant c \int_{E \cap B_{1/2}} |\bar{u}^-|^2 dx \geqslant c\delta(\sup_{B_{1/2}} \bar{u}^-)^2,$$

where in the last inequality we used that u^- is harmonic in $B_{3/4}$.

We have

$$0 \leqslant L(A, E^c) \leqslant L(A, E \setminus A) + CM^2 V_\rho - c\delta(\sup_{B_{1/2}} \bar{u}^-)^2$$

and the desired conclusion follows since

$$L(A, E \setminus A) \leqslant L(B_\rho, B_\rho^c) \leqslant C, \quad V_\rho \leqslant C, \quad \text{and} \quad u^- \leqslant \bar{u}^-. \qquad \square$$

If (u, E) is a minimizing pair in B_r then the rescaled pair (u_r, E_r) is minimizing in B_1 with

(3.5)
$$u_r(x) := r^{\frac{\sigma}{2}-1} u(rx), \qquad E_r := r^{-1} E.$$

Let

$$\lambda_r^+ := \|u_r^+\|_{L^\infty(B_1)} = r^{\frac{\sigma}{2}-1} \|u^+\|_{L^\infty(B_r)},$$

and define λ_r^- similarly.

If either λ_r^+ or λ_r^- is less than 1 then, by Lemma 3.1 with $M = 1$,

$$\lambda_{r/2}^+ \leqslant C, \quad \lambda_{r/2}^- \leqslant C, \quad \text{and} \quad c \leqslant \frac{|E \cap B_r|}{|B_r|} \leqslant 1 - c,$$

with c, C constants depending on σ and n. Theorem 1.1 follows provided the inequalities above hold for all small r. Thus, in order to prove Theorem 1.1 it remains to show that for all $r \leqslant r_0$ either $\lambda_r^+ \leqslant 1$ or $\lambda_r^- \leqslant 1$. This follows from the next lemma which is a consequence of the Alt-Caffarelli-Friedman monotonicity formula in [2].

Lemma 3.2. *Let (u, E) be a minimizing pair in B_1, and assume $0 \in \partial E$. Then*

$$\lambda_r^+ \lambda_r^- \leqslant C r^\sigma \|u\|_{L^2(B_1)}^2, \qquad \forall r \in (0, 1/4],$$

with C depending only on n.

Proof. Similar arguments appear in Section 2 of [4]. We sketch the proof below.

First we prove that u^+ and u^- are continuous. For this we need to show that $u^+ = u^- = 0$ on ∂E. Assume by contradiction that, say for simplicity $u^-(0) > 0$. Since

$$\limsup_{x \to 0} u^-(x) = u^-(0),$$

we see that the density of E in B_r tends to 0 as $r \to 0$. Since $u^+ \geqslant 0$ is subharmonic and $u^+ = 0$ a.e. in E^c it follows that u^+ must vanish of infinite order at the origin. Then $\lambda_r^+ \leqslant 1$ for all small r and by the discussion above E has positive density in B_r for all small r and we reach a contradiction.

Since u^+ and u^- are continuous subharmonic functions with disjoint supports we can apply Alt-Caffarelli-Friedman monotonicity formula, according to which

$$\Psi(r) := \frac{1}{r^4} \int_{B_r} \frac{|\nabla u^+|^2}{|x|^{n-2}} \, dx \int_{B_r} \frac{|\nabla u^-|^2}{|x|^{n-2}} \, dx,$$

is increasing in r.

From the definition of the harmonic replacement it follows that (see Lemma 2.3 in [4] for example)

$$\triangle (u^+)^2 = 2|\nabla u^+|^2$$

and we find

$$c\|u^+\|_{L^\infty(B_{r/2})}^2 \leqslant c \fint_{B_r} (u^+)^2 dx \leqslant \int_{B_r} \frac{|\nabla u^+|^2}{|x|^{n-2}} \, dx \leqslant C \fint_{B_{2r}} (u^+)^2 dx.$$

We use these bounds in the monotonicity formula above and obtain the conclusion. $\qquad \square$

4. Improvement of flatness for the free boundary

In this section we obtain the Euler-Lagrange equation at points on the free boundary and also we show that if ∂E is sufficiently flat in some ball B_r then ∂E is a $C^{1,\gamma}$ graph in $B_{r/2}$. The proofs are similar to the corresponding proofs for nonlocal minimal surfaces in [6]. The difference is that when we perturb E by a set A, the change in the nonlocal perimeter is bounded by the change in the Dirichlet integrals (instead of 0), and by Section 2, this can be bounded in terms of $|A|$.

Our main theorem on this topic is the following.

Theorem 4.1. *Assume (u, E) is minimal in B_1 and that in B_1*

$$\{x_n > \varepsilon_0\} \subset E \subset \{x_n > -\varepsilon_0\}, \qquad \|u\|_{L^\infty} \leqslant 1,$$

for some $\varepsilon_0 > 0$ small depending on σ and n. Then $\partial E \cap B_{1/2}$ is a $C^{1,\gamma}$ graph in the e_n direction and it satisfies the Euler-Lagrange equation in the viscosity sense

$$(4.1) \qquad \int_{\mathbb{R}^n} \frac{\chi_{E^c} - \chi_E}{|y|^{n+\sigma}} \, dy = |\nabla u^+(x)|^2 - |\nabla u^-(x)|^2, \qquad x \in \partial E.$$

The constant γ above depends on n and σ. The Euler-Lagrange equation in the viscosity sense means that at any point x where ∂E has a tangent C^2 surface included in E (respectively E^c) we have $\geqslant$ (respectively $\leqslant$) in (4.1).

First we bound the σ-curvature of ∂E at points x that have a tangent ball from E^c.

Lemma 4.2. *Let (u, E) be a minimizing pair in B_1. Assume that $B_{1/4}(-e_n/4)$ is tangent from exterior to E at 0. Then*

$$\int_{\mathbb{R}^n} \frac{\chi_{E^c} - \chi_E}{|x|^{n+\sigma}} \, dx \leqslant C \|u^+\|_{L^\infty(B_1)}^2$$

with C depending on n and σ. If moreover ∂E is a $C^{1,\gamma}$ surface near 0 then

$$\int_{\mathbb{R}^n} \frac{\chi_{E^c} - \chi_E}{|x|^{n+\sigma}} \, dx \leqslant |\nabla u^+(0)|^2 - |\nabla u^-(0)|^2.$$

Proof. We follow closely the proof of Theorem 5.1 of [6].

After a dilation we may assume that E^c contains $B_2(-2e_n)$. Fix $\delta > 0$ small, and $\varepsilon \ll \delta$. Let T be the radial reflection with respect to the sphere $\partial B_{1+\varepsilon}(-e_n)$

We define the sets:

$$A^- := B_{1+\varepsilon}(-e_n) \cap E, \quad A^+ := T(A^-) \cap E, \quad A := A^- \cup A^+.$$

and let

$$F := T(B_\delta \cap (E \setminus A)).$$

It is easy to check that $F \subset E^c \cap B_\delta$.

Let $\bar{u}$ be the perturbation of u which has as positive set $E \setminus A$ as in the proof of Lemma 3.1. First we estimate the right hand side in the energy inequality (3.1). Let $\tilde{u}$ be the perturbation of u which has as positive set $E \setminus A^-$. We use Lemmata 2.3 and 2.4 and we obtain

$$\int_{B_1} |\nabla \bar{u}|^2 - |\nabla u|^2 dx = \int_{B_1} |\nabla \bar{u}|^2 - |\nabla \tilde{u}|^2 dx + \int_{B_1} |\nabla \tilde{u}|^2 - |\nabla u|^2 dx$$

$$\leqslant \int_{B_1} |\nabla \bar{u}^+|^2 - |\nabla \tilde{u}^+|^2 dx + \int_{B_1} |\nabla \tilde{u}^+|^2 - |\nabla u^+|^2 dx$$

$$\leqslant C|A| \, \|u^+\|_{L^\infty(B_1)}^2.$$

Notice that
$$\tilde{u}^+ = u^+_{E^c \cup A^-}, \quad \bar{u}^+ = u^+_{E^c \cup A^- \cup T(A^-)}$$
and, by Theorem 1.1, $T(A^-)$ satisfies the uniform density property of Lemma 2.4.

Now we consider the left hand side of the energy inequality (3.1):
$$\mathrm{Per}_\sigma(E, B_1) - \mathrm{Per}_\sigma(E \setminus A, B_1) = L(A, E^c) - L(A, \mathcal{C}(E \setminus A)) =$$
$$[L(A, E^c \setminus B_\delta) - L(A, E \setminus B_\delta)] + [L(A, F) - L(A, T(F))] + L(A, (E^c \cap B_\delta) \setminus F)$$
$$:= I_1 + I_2 + I_3 \geqslant I_1 + I_2.$$

We estimate I_1 and I_2 as in [6], and we conclude that
$$\left| \frac{1}{|A|} I_1 - \int_{\mathbb{R}^n \setminus B_\delta} \frac{\chi_{E^c} - \chi_E}{|x|^{n+s}} dx \right| \leqslant C\varepsilon^{1/2} \delta^{-1-s}$$

and
$$I_2 \geqslant -C\delta^{1-s}|A| - C\varepsilon L(A^-, F).$$

It remains to show that for all small ε

(4.2)
$$L(A^-, F) \leqslant CL(A^-, B^c_{1+\varepsilon}(-e_n))$$

since then, as in Lemma 5.2 of [6], there exists a sequence of $\varepsilon \to 0$ such that
$$\varepsilon L(A^-, F) \leqslant C\varepsilon^\eta |A^-|,$$

and our result follows.

We prove (4.2) by writing the energy inequality for $\tilde{u}$ defined above. We have $L(A^-, F) \leqslant L(A^-, E^c)$ and
$$L(A^-, E^c) \leqslant L(A^-, E \setminus A^-) + \int_{B_1} |\nabla \tilde{u}^+|^2 - |\nabla u^+|^2 dx$$
$$\leqslant L(A^-, B^c_{1+\varepsilon}(-e_n)) + C|A^-| \|u\|^2_{L^\infty(B_1)}$$
$$\leqslant 2L(A^-, B^c_{1+\varepsilon}(-e_n)).$$

where the last inequality holds for all small ε.

In the case when ∂E is a $C^{1,\gamma}$ surface near 0 we can estimate the change in the Dirichlet integral by Lemma 2.5 and obtain the second part of our conclusion. $\qquad \square$

With the results already obtained, Theorem 4.1 now follows easily from the improvement of flatness property of ∂E:

Proposition 4.3. *Assume (u, E) is a minimal pair in B_1 and fix $0 < \alpha < s$. There exists k_0 depending on s, n and α such that if*
$$0 \in \partial E, \qquad \|u\|_{L^\infty(B_1)} \leqslant 1, \quad \text{and for all balls } B_{2^{-k}} \text{ with } 0 \leqslant k \leqslant k_0 \text{ we have}$$

(4.3)
$$\{x \cdot e_k > 2^{-k(\alpha+1)}\} \subset E \subset \{x \cdot e_k > -2^{-k(\alpha+1)}\}, \qquad |e_k| = 1,$$

then there exist vectors e_k for all $k \in \mathbb{N}$ for which the inclusion above remains valid.

The proof now follows closely Theorem 6.8 in [6]. We sketch it below.

Assume (4.3) holds for some large $k \geqslant k_0$. Then by comparison principle we find that
$$u^\pm \leqslant Cr, \quad \text{in } B_r \text{ for all } r \geqslant 2^{-k}.$$

for some C depending on n and α.

Rescaling by a factor 2^k the pair (u, E), the situation above can be described as follows: if for all l with $0 \leqslant l \leqslant k$

$$\|u\|_{L^\infty(B_{2^l})} \leqslant 2^l 2^{-(\sigma k)/2},$$

$$\partial E \cap B_{2^l} \subset \{|x \cdot e_l| \leqslant 2^l 2^{\alpha(l-k)}\}, \quad |e_l| = 1$$

then the inclusion holds also for $l = -1$, i.e.

(4.4)
$$\partial E \cap B_{1/2} \subset \{|x \cdot e_{-1}| \leqslant 2^{-1} 2^{-\alpha(k+1)}\}.$$

For some fixed l we see that $\partial E \cap B_{2^l}$ has $C(l)2^{-\alpha k}$ flatness, and u is bounded by $C(l)2^{-(\sigma k)/2}$ in B_{2^l}.

First we give a rough Harnack inequality that provides compactness for a sequence of blow-ups.

Lemma 4.4. *Assume that for some large k, ($k > k_1$)*

$$\partial E \cap B_1 \subset \{|x_n| \leqslant a := 2^{-k\alpha}\}, \qquad \|u\|_{L^\infty(B_1)} \leqslant a^{\sigma/(2\alpha)}$$

and

$$\partial E \cap B_{2^l} \subset \{|x \cdot e_l| \leqslant a2^{l(1+\alpha)}\}, \quad l = 0, 1, \ldots, k.$$

Then either

$$\partial E \cap B_\delta \subset \left\{ \frac{x_n}{a} \leqslant 1 - \delta^2 \right\} \quad \text{or} \quad \partial E \cap B_\delta \subset \left\{ \frac{x_n}{a} \geqslant -1 + \delta^2 \right\},$$

for δ small, depending on σ, n, α, ($\alpha < \sigma$).

Proof. The proof is the same as Lemma 6.9 in [6]. The only difference is that at the contact point y between the paraboloid P and ∂E the quantity

(4.5)
$$\frac{1}{a} \int_{\mathbb{R}^n} \frac{\chi_{E^c} - \chi_E}{|x - y|^2} dx$$

is not bounded above by 0, instead by Lemma 4.2, it is bounded by

$$\frac{1}{a} C \|u\|^2_{L^\infty(B_1)} \leqslant C a^{(\sigma/\alpha)-1} \to 0 \quad \text{as } a \to 0,$$

and all the arguments apply as before. $\qquad \square$

Completion of the proof of Proposition 4.3. As k becomes much larger than k_1, we can apply Harnack inequality several times as in [6]. This gives compactness of the sets

$$\partial E^* := \left\{ (x', \frac{x_n}{a}) \text{ s.t. } x \in \partial E \right\},$$

as $a \to 0$. Precisely, we consider pairs (u, E) that are minimal in B_{2^k} with $0 \in \partial E$, for which

$$\partial E \cap B_1 \subset \{|x_n| \leqslant a := 2^{-k\alpha}\}, \qquad \|u\|_{L^\infty(B_1)} \leqslant a^{\sigma/(2\alpha)}.$$

and for all $0 \leqslant l \leqslant k$

$$\partial E \cap B_{2^l} \subset \{|x \cdot e_l| \leqslant a2^{l(1+\alpha)}\}, \qquad \|u\|_{L^\infty(B_{2^l})} \leqslant 2^l a^{\sigma/(2\alpha)}.$$

and we want to show that (4.4) holds.

If (u_m, E_m) is a sequence of pairs as above with $a_m \to 0$ there exists a subsequence m_k such that

$$\partial E^*_{m_k} \to (x', \omega(x'))$$

uniformly on compact sets, where $\omega : \mathbb{R}^{n-1} \to \mathbb{R}$ is Hölder continuous and

$$\omega(0) = 0, \quad |\omega| \leqslant C(1 + |x'|^{1+\alpha}).$$

Moreover, since the quantity in (4.5) tends to 0, the proof of Lemma 6.11 of [6] works as before, thus

$$\triangle^{\frac{\sigma+1}{2}} w = 0 \quad \text{in } \mathbb{R}^{n-1}.$$

This shows that ω is a linear function and therefore (4.4) holds for all large m. $\qquad\square$

5. A MONOTONICITY FORMULA

The goal of this section is to establish a Weiss-type monotonicity formula for minimizing pairs (u, E), that is different from the Alt-Caffarelli-Friedman monotonicity formula used in Lemma 3.2. For this scope, we first introduce the localized energy for the σ-perimeter by using the extension problem in one more dimension as in [6]. With a measurable set $E \subset \mathbb{R}^n$ we associate a function $U(x, z)$ defined in $\mathbb{R}^{n+1}_+$ as

$$U(\cdot, z) := (\chi_E - \chi_{E^c}) * P(\cdot, z), \quad \text{with} \quad P(x, z) := \tilde{c}_{n,\sigma} \frac{z^\sigma}{(|x|^2 + z^2)^{(n+\sigma)/2}},$$

where $\tilde{c}_{n,\sigma}$ is a normalizing constant depending on n and σ.

For a bounded Lipschitz domain $\Omega \subset \mathbb{R}^{n+1}$ we denote by

$$\Omega_0 := \Omega \cap \{z = 0\} \subset \mathbb{R}^n, \qquad \Omega_+ := \Omega \cap \{z > 0\},$$

and denote the extended variables as

$$X := (x, z) \in \mathbb{R}^{n+1}_+, \qquad \mathcal{B}^+_r := \{|X| < r\}.$$

The relation between the σ-perimeter and its extension is given by Lemma 7.2 in [6]. Precisely, let E be a set with $\text{Per}_\sigma(E, B_r) < \infty$ and U its extension, and let F be a set which coincides with E outside a compact set included in B_r. Then

$$\text{Per}_\sigma(F, B_r) - \text{Per}_\sigma(E, B_r) = c_{n,\sigma} \inf_{\Omega, V} \int_{\Omega^+} z^{1-\sigma}(|\nabla V|^2 - |\nabla U|^2) dX.$$

Here the infimum is taken over all bounded Lipschitz sets with $\Omega_0 \subset B_r$ and all functions V that agree with U near $\partial\Omega$ and whose trace on $\{z = 0\}$ is given by $\chi_F - \chi_{F^c}$. The constant $c_{n,\sigma} > 0$ above is a normalizing constant. As a consequence we obtain the following characterization of minimizing pairs (u, E) using the extension U of E.

Proposition 5.1. *The pair (u, E) is minimizing in B_r if and only if*

$$\int_{B_r} |\nabla u|^2 \, dx + c_{n,\sigma} \int_{\Omega^+} z^{1-\sigma} |\nabla U|^2 \, dX$$

$$\leqslant \int_{B_r} |\nabla v|^2 \, dx + c_{n,\sigma} \int_{\Omega^+} z^{1-\sigma} |\nabla V|^2 \, dX$$

for any bounded Lipschitz domain Ω with $\Omega_0 \subset B_r$ and any functions v, V that satisfy

1) $V = U$ in a neighborhood of $\partial\Omega$,

2) the trace of V on $\{z = 0\}$ is $\chi_F - \chi_{F^c}$ for some set $F \subset \mathbb{R}^n$,

3) $v = u$ near ∂B_r, and $v \geqslant 0$ a.e. in F, $v \leqslant 0$ a.e. in F^c.

Now we present a Weiss-type monotonicity formula for minimizing pairs (u, E).

Theorem 5.2. *Let (u, E) be a minimizing pair in B_ρ. Then*

$$\Phi_u(r) := r^{\sigma-n} \left(\int_{B_r} |\nabla u|^2 \, dx + c_{n,\sigma} \int_{\mathcal{B}^+_r} z^{1-\sigma} |\nabla U|^2 \, dX \right)$$

$$- \left(1 - \frac{\sigma}{2}\right) r^{\sigma-n-1} \int_{\partial B_r} u^2 \, d\mathcal{H}^{n-1}$$

is increasing in $r \in (0, \rho)$.

Moreover, Φ_u is constant if and only if u is homogeneous of degree $1 - \frac{\sigma}{2}$ and U is homogeneous of degree 0.

Proof. The proof is a suitable modification of the one of Theorem 8.1 in [6]. We notice that Φ_u possesses the natural scaling

$$\Phi_u(rs) = \Phi_{u_r}(s),$$

where (u_r, E_r) is the rescaling given in (3.5).

We prove that

$$\frac{d}{dr}\Phi(u, U, r) \geqslant 0 \text{ for a.e. } r.$$

By scaling it suffices to consider the case when $r = 1$ and r is a "regular"radius for $|\nabla u|^2 dx$, $z^{1-\sigma}|\nabla U|^2 dxdz$ and E. We use the short notation $\Phi(r)$ for $\Phi_u(r)$ and write

$$\Phi(r) = G(r) - H(r),$$

with

$$G(r) := r^{\sigma - n} \left(\int_{B_r} |\nabla u|^2 \, dx + c_{n,\sigma} \int_{B_r^+} z^{1-\sigma}|\nabla U|^2 \, dX \right)$$

$$H(r) := \left(1 - \frac{\sigma}{2}\right) r^{\sigma - n - 1} \int_{\partial B_r} u^2 \, d\mathcal{H}^{n-1}.$$

Below we use the minimality to obtain a bound for $G'(1)$. We denote as usual u_ν and u_τ for the normal and tangential gradient of u on ∂B_r. Let $\varepsilon > 0$ be small. We compute $G(1)$ by writing the integrals in $B_{1-\varepsilon}$ and $B_1 \setminus B_{1-\varepsilon}$:

$$G_u(1) = \int_{B_{1-\varepsilon}} |\nabla u|^2 \, dx + \varepsilon \int_{\partial B_1} |\nabla u|^2 \, d\mathcal{H}^{n-1}$$

$$+ c_{n,s} \left(\int_{\mathcal{B}_{1-\varepsilon}^+} z^{1-\sigma}|\nabla U|^2 \, dx \, dz + \varepsilon \int_{\partial \mathcal{B}_1^+} z^{1-\sigma}|\nabla U|^2 \, d\mathcal{H}^n \right) + o(\varepsilon)$$

$$= (1-\varepsilon)^{n-\sigma} G(1-\varepsilon) + \varepsilon \int_{\partial B_1} |u_\tau|^2 + |u_\nu|^2 \, d\mathcal{H}^{n-1}$$

$$+ \varepsilon \, c_{n,\sigma} \int_{\partial \mathcal{B}_1^+} z^{1-\sigma}(|U_\tau|^2 + |U_\nu|^2) \, d\mathcal{H}^n + o(\varepsilon).$$

We now consider a competitor $(u^\varepsilon, U^\varepsilon)$ for (u, U) defined as

$$u^\varepsilon(x) := \begin{cases} (1-\varepsilon)^{1-\frac{\sigma}{2}} \, u(\frac{x}{1-\varepsilon}) & \text{if } x \in B_{1-\varepsilon}, \\[2mm] |x|^{1-\frac{\sigma}{2})} \, u(\frac{x}{|x|}) & \text{if } x \in B_1 \setminus B_{1-\varepsilon}, \\[2mm] u(x) & \text{if } x \in B_1^c, \end{cases}$$

and

$$U^\varepsilon(X) := \begin{cases} U(\frac{X}{1-\varepsilon}) & \text{if } x \in \mathcal{B}_{1-\varepsilon}^+, \\[2mm] U(\frac{X}{|X|}) & \text{if } x \in \mathcal{B}_1^+ \setminus \mathcal{B}_{1-\varepsilon}^+, \\[2mm] U(X) & \text{if } |X| \geqslant 1. \end{cases}$$

From Proposition 5.1 we obtain

$$G_u(1) \leqslant G_{u^\varepsilon}(1).$$

We compute $G_{u^\varepsilon}(1)$ noticing that u^ε in $B_{1-\varepsilon}$ coincides with the rescaling $u_{1/(1-\varepsilon)}$ hence

$$G_{u^\varepsilon}(1) = (1-\varepsilon)^{n-\sigma} G_{u_{1-\varepsilon}}(1-\varepsilon) + \varepsilon \, c_{n,\sigma} \int_{\partial \mathcal{B}_1^+} |U_\tau|^2 d\mathcal{H}^n$$

$$+ \varepsilon \int_{\partial B_1} \left(|u_\tau|^2 + \left(1 - \frac{\sigma}{2}\right)^2 u^2 \right) d\mathcal{H}^{n-1} + o(\varepsilon).$$

By scaling, the first term in the sum above equals $(1-\varepsilon)^{n-\sigma} G_u(1)$. Plugging $G_u(1)$ and $G_{u^\varepsilon}(1)$ in the inequality above gives

$$G_u(1) \geqslant G_u(1-\varepsilon) + \varepsilon \int_{\partial B_1} |u_\nu|^2 - \left(1 - \frac{\sigma}{2}\right)^2 u^2 \, d\mathcal{H}^{n-1}$$

$$+ \varepsilon \, c_{n,\sigma} \int_{\partial \mathcal{B}_1^+} z^{1-\sigma} |U_\nu|^2 \, d\mathcal{H}^n + o(\varepsilon),$$

hence

$$G'(1) \geqslant \int_{\partial B_1} |u_\nu|^2 - \left(1 - \frac{\sigma}{2}\right)^2 u^2 \, d\mathcal{H}^{n-1} + c_{n,\sigma} \int_{\partial \mathcal{B}_1^+} z^{1-\sigma} |U_\nu|^2 \, d\mathcal{H}^n.$$

On the other hand,

$$H'(1) = \left(1 - \frac{\sigma}{2}\right) \int_{\partial B_1} 2uu_\nu + (\sigma - 2)u^2 \, d\mathcal{H}^{n-1}.$$

and we conclude that

$$\Phi'(1) \geqslant \int_{\partial B_1} \left(u_\nu - \left(1 - \frac{\sigma}{2}\right)u \right)^2 d\mathcal{H}^{n-1} + c_{n,\sigma} \int_{\partial \mathcal{B}_1^+} z^{1-\sigma} |U_\nu|^2 \, d\mathcal{H}^n,$$

and the conclusion follows. $\qquad\qquad\square$

The monotonicity formula allows us to characterize the blow-up limit of a sequence of rescalings (u_r, E_r). First we need to show that minimizing pairs remain closed under limits.

Proposition 5.3. *Assume (u_m, E_m) are minimizing pairs in B_2 and*

$$u_m \to u \quad \text{in } L^2(B_2), \quad \text{and} \quad E_m \to E \quad \text{in } L^1_{\text{loc}}(\mathbb{R}^n).$$

Then (u, E) is a minimizing pair in B_1 and $u_m \to u$ in $H^1(B_1)$ and

$$\text{Per}_\sigma(E_m, B_1) \to \text{Per}_\sigma(E, B_1).$$

Proof. First we show that $u_m \to u$ in $H^1(B_1)$. Since $\nabla u_m \rightharpoonup \nabla u$ weakly in L^2 it suffices to show that

$$\int_{B_1} |\nabla u_m|^2 dx \to \int_{B_1} |\nabla u|^2 dx.$$

Indeed, since u_m and u are continuous functions which are harmonic in their positive and negative sets we have

$$\triangle u^2 = 2|\nabla u|^2, \quad \triangle u_m^2 = 2|\nabla u_m|^2,$$

and the limit above follows since $u_m^2 \to u^2$ in L^1.

Let (v, F) be a compact perturbation for (u, E) in B_1. Precisely, assume $F = E$ and $v = u$ outside a compact set of B_1, and $v \geqslant 0$ a.e. in F, $v \leqslant 0$ a.e. in F^c. Let

$$w_m^+ = \min\{u_m^+, u^+\}$$

and define v_m^+ such that $v_m^+ = v^+$ in $B_{1-2\varepsilon}$, $v_m^+ = w_m^+$ in the annulus $B_{1+\varepsilon} \setminus B_{1-\varepsilon}$ and $v_m^+ = u_m^+$ outside $B_{1+2\varepsilon}$. In $B_1 \setminus B_{1-2\varepsilon}$ we define v_m^+ as an interpolation between v^+ and w_m^+ i.e.

$$v_m^+ = \eta v^+ + (1-\eta)w_m^+,$$

with η a cutoff function with $\eta = 1$ in $B_{1-2\varepsilon}$ and $\eta = 0$ outside $B_{1-\varepsilon}$. Similarly, in $B_{1+2\varepsilon} \setminus B_1$ we let v_m^+ to be an interpolation between u_m^+ and w_m^+.

We define v_m^- similarly. We have

$$v_m \geqslant 0 \quad \text{a.e. in } F_m, \quad v_m \leqslant 0 \quad \text{a.e. in } F_m^c, \text{ with}$$

$$F_m := (F \cap B_1) \cup (E_n \setminus B_1),$$

thus (v_m, F_m) is a compact perturbation of (u_m, E_m). From the minimality of (u_m, E_m) (see (2.1)) we find

$$J_{B_2}(u_m) \leqslant J_{B_2}(v_m).$$

By construction,

$$\int_{B_2} |\nabla v_m|^2 - |\nabla u_m|^2 dx \leqslant \int_{B_1} |\nabla v|^2 - |\nabla u_m|^2 dx + c_m(\varepsilon),$$

with

$$c_m(\varepsilon) := C\varepsilon^{-2} \int_{B_2} (u_m - u)^2 dx + C \int_{B_{1+2\varepsilon} - B_{1-2\varepsilon}} |\nabla u|^2 + |\nabla u_m|^2 dx.$$

Notice also that

$$\mathrm{Per}_\sigma(F_m, B_2) - \mathrm{Per}_\sigma(E_m, B_2) \leqslant \mathrm{Per}_\sigma(F, B_1) - \mathrm{Per}_\sigma(E_m, B_1) + b_m,$$

with

$$b_m := L(B_1, (E_m \Delta E) \setminus B_1).$$

Since $E_m \to E$ in $L^1_{\mathrm{loc}}(\mathbb{R}^n)$ it follows easily that $b_m \to 0$ (see Theorem 3.3 in [6]). Using the last two inequalities in the energy inequality and letting first $m \to \infty$ and then $\varepsilon \to 0$ we find

$$\limsup J_{B_1}(u_m) \leqslant J_{B_1}(v).$$

On the other hand from the lower semicontinuity of J we have

$$\liminf J_{B_1}(u_m) \geqslant J_{B_1}(u).$$

This shows that (u, E) is a minimizing pair and that $J_{B_1}(u_m) \to J_{B_1}(u)$ and our conclusion follows. $\qquad \square$

Next we consider the limit of a sequence of rescalings u_r, E_r, U_r as $r \to 0$,

$$u_r(x) = r^{\frac{\sigma}{2}-1} u(rx), \quad E_r = r^{-1}E, \quad U_r(X) = U(rX).$$

Proposition 5.4 (Tangent cone). *Assume (u, E) is a minimizing pair in B_1, and $0 \in \partial E$. There exists a sequence of $r = r_k \to 0$ such that*

$$u_r \to \bar{u} \quad \text{in } L^2_{\mathrm{loc}}(\mathbb{R}^n), \quad E_r \to \bar{E} \quad \text{in } L^1_{\mathrm{loc}}(\mathbb{R}^n), \quad U_r \to \bar{U} \quad \text{in } L^2_{\mathrm{loc}}(\mathbb{R}^n, z^{1-\sigma}dX)$$

with $\bar{u}$ homogeneous of degree $1 - \frac{\sigma}{2}$, $\bar{U}$ homogeneous of degree 0 and $(\bar{u}, \bar{E})$ a minimizing pair in $\mathbb{R}^n$.

We refer to a minimizing homogeneous pair $(\bar{u}, \bar{E})$ as a *minimizing cone*. From Theorem 1.1 we see that on compact sets $u_r \to u$ uniformly and $E_r \to \bar{E}$ in Hausdorff distance.

Proof. By compactness we can find a sequence such that $u_r \to \bar{u}$ and $E_r \to \bar{E}$ as above. From Proposition 5.3 we have $\mathrm{Per}_\sigma(E_r) \to \mathrm{Per}_\sigma(\bar{E})$ and, as in Proposition 9.1 in [6], this implies the convergence above of U_r to U, and

$$\Phi_{u_r}(t) \to \Phi_{\bar{u}}(t) \quad \text{as } r \to 0.$$

Then $\Phi_{\bar{u}}(t) = \Phi_u(0+)$ and the conclusion follows from Theorem 5.2. Notice from the definition of Φ that $\Phi(0+)$ is bounded since $u \in C^\alpha(B_1)$, with $\alpha = 1 - \frac{\sigma}{2}$, thanks to Theorem 1.1. $\qquad \square$

Let $(\bar{u}, \bar{E})$ be a minimizing cone. We define its energy as $\Phi_{\bar{u}}$ which is a constant (recall Theorem 5.2). From the homogeneity of $\bar{u}$ it follows that

$$\Phi_{\bar{u}} = c_{n,\sigma} \int_{B_1} |\nabla \bar{U}|^2 dX,$$

hence the energy depends only on $\bar{E}$.

Since $\bar{u}^{\pm}$ are complementary homogeneous harmonic functions in $\bar{E}$ respectively $\bar{E}^c$, at least one of them, say $\bar{u}^-$, has homogeneity greater or equal to 1, thus $\bar{u}^- = 0$. Then $\bar{u}^+$ is homogeneous of degree $1 - \frac{\sigma}{2}$ and

$$\int_{\mathbb{R}^n} \frac{\chi_{\bar{E}^c} - \chi_{\bar{E}}}{|y - x|^{n+\sigma}} \, dy = |\nabla \bar{u}^+(x)|^2, \qquad \forall x \in \partial \bar{E},$$

holds in the viscosity sense. Notice that both terms are homogeneous of degree $-\sigma$.

If $\bar{u}^+ \equiv 0$ then the study of minimizing cones reduces to the study of σ-minimal surfaces. This is the case when $\sigma = 1$ which was treated in [4]. Indeed, the homogeneity of a positive harmonic function in a mean-convex cone E which vanishes on ∂E cannot be less than 1. This follows since a multiple of the distance function to ∂E is superharmonic and is an upper barrier for $\bar{u}^+$. When $\sigma < 1$ it is not clear whether or not there exist minimizing cones with $\bar{u} \neq 0$ and it seems difficult to relate the σ-curvature of ∂E with the homogeneity of $\bar{u}^+$.

When $\bar{E} = \Pi$ is a half-space then $\bar{u} \equiv 0$ and we call $(0, \Pi)$ a trivial cone. If the blow-up limit $(\bar{u}, \bar{E})$ of a minimizing pair (u, E) is trivial then we say that $0 \in \partial E$ is a regular point of the free boundary. By Theorem 4.1, ∂E is a $C^{1,\gamma}$ surface in a neighborhood of its regular points.

We remark that if E admits an exterior tangent ball at $0 \in \partial E$ then $\bar{E} \subset \Pi$ and $\bar{u}^+ = 0$. Then, we use the Euler-Lagrange equation (Lemma 4.2) and obtain $E = \Pi$. Thus any point on ∂E which admits a tangent ball from E or E^c is a regular point. Therefore the set of regular points is dense in ∂E. We summarize these results below.

Proposition 5.5. *Let (u, E) be a minimal pair, $0 \in \partial E$, and let $(\bar{u}, \bar{E})$ be its tangent cone as in Proposition 5.4. If $\bar{E}$ is a half-space (i.e. if 0 is a regular point) then ∂E is a $C^{1,\gamma}$ surface and the free boundary equation* (4.1) *holds. Moreover, all points on ∂E which have a tangent ball from either E or E^c are regular points.*

By a standard argument (see Theorem 9.6 in [6]), we also obtain that the trivial cone has the least energy amongst all minimizing cones. Precisely if $(\bar{u}, \bar{E})$ is a minimizing cone then

$$\Phi_{\bar{u}} \geqslant \Phi_{\Pi},$$

and if $\bar{E}$ is not a half-space then

$$\Phi_{\bar{u}} \geqslant \Phi_{\Pi} + \delta_0$$

for some $\delta_0 > 0$ depending only on n, σ.

6. PROOF OF THEOREM 1.2

In this section we prove Theorem 1.2 using the dimension reduction argument of Federer. As in Section 10 in [6], in order to obtain Theorem 1.2 it suffices to prove the following two propositions.

Proposition 6.1. *The pair (u, E) is minimizing in $\mathbb{R}^n$ if and only if $(u(x), E \times \mathbb{R})$ is minimizing in $\mathbb{R}^{n+1}$.*

Proposition 6.2. *In dimension $n = 2$, all minimizing cones are the trivial.*

Proof of Proposition 6.1. The proof is similar to the one of Theorem 10.1 in [6]. We just sketch the main difference. The only issue that needs to be discussed is the existence of a perturbation which is admissible when we prove that (u, E) is minimizing in $\mathbb{R}^n$ if $(u(x), E \times \mathbb{R})$ is minimizing in $\mathbb{R}^{n+1}$.

Precisely let $v(x)$, $V(x, z)$ be admissible functions which coincide with u, respectively U say outside $\mathcal{B}_{1/2}^+$. It suffices to construct an admissible pair $w(x, x_{n+1})$ and $W(x, x_{n+1}, z)$ in one dimension higher i.e. in $\mathcal{B}_1 \times [0, 1]$ such that on the n dimensional slice $x_{n+1} = 0$, (w, W) coincides with (u, U), and on the slice $x_{n+1} = 1$, (w, W) coincides with (v, V).

For $x_{n+1} \in [0, 1/4]$ we define

$$W(x, x_{n+1}, z) = U(x, z), \quad \text{and} \quad w(x, x_{n+1}) := (1 - \varphi + \varphi\eta(x))u(x)$$

with $\varphi = \varphi(x_{n+1})$ a smooth function vanishing for $x_{n+1} \leqslant 0$ and which equals 1 for $x_{n+1} \geqslant 1/4$. The function η above is a cutoff function which vanishes in $B_{1/2}$ and equals 1 outside $B_{3/4}$.

Similarly we construct W and w for $x_{n+1} \in [3/4, 1]$, by using the pair (v, V).

In the interval $x_{n+1} \in [1/4, 3/4]$ we extend w to be constant in the x_{n+1} variable. We also extend W to be constant in the annulus $\mathcal{B}_1^+ \setminus \mathcal{B}_{1/2}^+$. It remains to construct W in the inner cylinder $\mathcal{B}_{1/2} \times [1/4, 3/4]$. Since $w = 0$ on the "bottomöf this cylinder, any choice for W with trace ± 1 on $\{x_{n+1} = 0\}$ makes the pair (w, W) admissible. Now we can argue precisely as in the proof of the σ-minimal surfaces, and the construction for the interpolating W is given in Lemma 10.2 in [6]. $\qquad\square$

Proof of Proposition 6.2. We follow the methods in [14, 15] where the same result was proved for σ-minimal surfaces. We remark that the assumption that $n = 2$ is only necessary at the end of the proof. We define

$$\mathcal{E}_r(v, V) := \int_{B_\rho} |\nabla v|^2 \, dx + c_{n,\sigma} \int_{\mathcal{B}_r^+} z^{1-\sigma} |\nabla V(X)|^2 \, dX.$$

By Proposition 5.1, we know that (u, U) minimizes $\mathcal{E}$ under domain variations. We consider a diffeomorphism on $\mathbb{R}^{n+1}$ given, for any $X \in \mathbb{R}_+^{n+1}$ by

(6.1) $$X \mapsto Y := X + \varphi(|X|/R)e_1,$$

where $\varphi \in C^\infty(\mathbb{R})$, $\varphi = 1$ in $[-1/2, 1/2]$ and $\varphi = 0$ outside $(-3/4, 3/4)$, and R is a large parameter. We define $U_R^+(Y) := U(X)$ and similarly, if we change e_1 into $-e_1$ in (6.1), we may define U_R^-. The diffeomorphism in (6.1) restricts to a diffeomorphism in $\mathbb{R}^n$ just by considering points of the type $X = (x, 0)$, i.e.

$$y := x + \varphi(|x|/R)e_1.$$

and we set $u_R^+(y) := u(x)$, and similarly we define u_R^-. We claim that

(6.2) $$\mathcal{E}_R(u_R^+, U_R^+) + \mathcal{E}_R(u_R^-, U_R^-) - 2\mathcal{E}_R(u, U) \leqslant CR^{n-2-\sigma},$$

for some C independent of R. By Proposition 5.1, the minimality of (u, U) gives

$$\mathcal{E}_R(u, U) \leqslant \mathcal{E}_R(u_R^-, U_R^-),$$

and the last two inequalities imply

(6.3) $$\mathcal{E}_R(u_R^+, U_R^+) \leqslant \mathcal{E}_R(u, U) + CR^{n-2-\sigma}.$$

To prove (6.2), by direct calculations (or see formula (11) in [14]) we obtain

$$\left(|\nabla u_R^+|^2 + |\nabla u_R^-|^2\right) dy = 2(1 + O(1/R^2)\chi_{B_R \setminus B_{R/2}})|\nabla u|^2 \, dx$$

$$z^{1-\sigma}\left(|\nabla U_R^+|^2 + |\nabla U_R^-|^2\right) dY = 2z^{1-\sigma}(1 + O(1/R^2)\chi_{B_R^+ \setminus B_{R/2}^+})|\nabla U|^2 \, dX.$$

We use that $|\nabla u(x)|^2$ and $z^{1-\sigma}|\nabla U(X)|^2$ are homogeneous of degree $-\sigma$ respectively $-1-\sigma$ and obtain

$$\int_{B_R} \left(|\nabla u_R^+|^2 + |\nabla u_R^-|^2 \right) dy - 2 \int_{B_R} |\nabla u|^2 \, dx$$
$$\leqslant CR^{-2} \int_{B_R \setminus B_{R/2}} |\nabla u|^2 \, dx \leqslant CR^{-2} \cdot R^{n-\sigma}$$

and

$$\int_{B_R^+} z^{1-\sigma} \left(|\nabla U_R^+|^2 + |\nabla U_R^-|^2 \right) dY - 2 \int_{B_R^+} 2z^{1-\sigma}|\nabla U|^2 \, dX$$
$$\leqslant CR^{-2} \int_{B_R^+ \setminus B_{R/2}^+} z^{1-\sigma}|\nabla U|^2 \, dX \leqslant CR^{-2} \cdot R^{n-\sigma}$$

and so the proof of (6.2) is complete.

Next we perform an argument similar to the one of Theorem 1 of [14] (the main difference here is that two functions are involved in the minimization procedure instead of a single one). For this, we assume now that $n = 2$, we argue by contradiction and we suppose that E is not a halfplane. Thus, there exist $M > 0$ and $p \in B_M$, say on the e_2-axis, such that p lies in the interior of E, and $p + e_1$ and $p - e_1$ lie in E^c. Therefore, if R is sufficiently large we have that

(6.4)
$$u_R^+(x) = u(x - e_1), \text{ for all } x \in B_{2M}$$
$$U_R^+(X) = U(X - e_1), \text{ for all } X \in B_{2M}^+,$$
$$u_R^+(x) = U(x) \text{ for all } x \in \mathbb{R}^2 \setminus B_R, \text{ and}$$
$$U_R^+(X) = U(Y) \text{ for all } X \in \mathbb{R}_+^3 \setminus B_R^+.$$

We define

$$v_R(x) := \min\{u(x), u_R^+(x)\}, \quad w_R(x) := \max\{u(x), u_R^+(x)\},$$
$$V_R(X) := \min\{U(X), U_R^+(X)\} \quad \text{and} \quad W_R(X) := \max\{U(X), U_R^+(X)\}$$

and $P := (p, 0) \in \mathbb{R}^3$. From (6.4) and the trace property of U we have that

(6.5) $$U_R^+ < W_R = U \text{ in a neighborhood of } P, \text{ and}$$

(6.6) $$U < W_R = U_R^+ \text{ in a neighborhood of } P + e_1.$$

Moreover

$$\mathcal{E}_R(u, U) \leqslant \mathcal{E}_R(v_R, V_R)$$

and

$$\mathcal{E}_R(v_R, V_R) + \mathcal{E}_R(w_R, W_R) = \mathcal{E}_R(u, U) + \mathcal{E}_R(u_R^+, U_R^+),$$

therefore

(6.7) $$\mathcal{E}_R(w_R, W_R) \leqslant \mathcal{E}_R(u_R^+, U_R^+).$$

Now we observe that

$$(w_R, W_R) \text{ is not a minimizer for } \mathcal{E}_{2M}$$

with respect to compact perturbations in $B_{2M} \times \mathcal{B}_{2M}^+$. Otherwise W_R would be a minimizer too: then the fact that $U \leqslant W_R$, (6.5) and the strong maximum principle would give that $U = W_R$ in $\mathcal{B}_{2M}^+$, but this would be in contradiction with (6.6). Thus there exists $\delta > 0$ and a competitor

$$(u_*, U_*) \text{ that coincides with } (w_R, W_R) \text{ outside } B_{2M} \times \mathcal{B}_{2M}^+$$

(with $u_* = w_R$) and such that

$$\mathcal{E}_{2M}(u_*, U_*) + \delta \leqslant \mathcal{E}_{2M}(w_R, W_R).$$

Here $\delta > 0$ is independent of R since (w_R, W_R) does not depend on R when restricted to $B_{2M} \times \mathcal{B}_{2M}^+$ (recall (6.4)). We conclude that

$$\mathcal{E}_R(u_*, U_*) + \delta \leqslant \mathcal{E}_R(w_R, W_R).$$

Combining this with (6.3) and (6.7) we obtain

$$\mathcal{E}_R(u_*, U_*) + \delta \leqslant \mathcal{E}_R(w_R, W_R) \leqslant \mathcal{E}_R(u_R^+, U_R^+) \leqslant \mathcal{E}_R(u, U) + CR^{-\sigma}.$$

If R is large enough we obtain that $\mathcal{E}_R(u_*, U^*) < \mathcal{E}_R(u, U)$, which contradicts the minimality of (u, U) and completes the proof of Proposition 6.2. $\qquad\square$

7. PROOFS OF LEMMAS 2.3 - 2.5

In this section we estimate the difference in the Dirichlet energies of the harmonic replacements in two different sets E and $E \setminus A$, with $A \subset B_{3/4}$. We assume that $\varphi \in H^1(B_1) \cap L^\infty(B_1)$, $\varphi \geqslant 0$, and let

$$w := \varphi_{E^c}, \qquad v := \varphi_{E^c \cup A}.$$

Here above, we used the notation for the harmonic replacements of φ that vanish in E^c and $E^c \cup A$, as introduced in Definition 2.2. We remark that the existence of v follows from the existence of w. Indeed, given w we can easily find an explicit test function with finite energy which vanishes in $E^c \cup B_{3/4}$, for example a function of the form $w(1 - \eta)$ with η a cutoff function.

Since w minimizes the Dirichlet energy among all functions which are fixed in E^c and have prescribed values on ∂B_1 we find

$$(7.1) \qquad \int_{B_1} \nabla w \cdot \nabla \psi \, dx = 0, \qquad \forall \psi \in H_0^1(B_1) \quad \text{with } \psi = 0 \text{ a.e. in } E^c,$$

and therefore

$$(7.2) \qquad \int_{B_1} |\nabla(w - \psi)|^2 - |\nabla w|^2 \, dx = \int_{B_1} |\nabla \psi|^2 \, dx.$$

By definition, v minimizes the Dirichlet energy among all functions which equal w on ∂B_1, and are 0 a.e. in $E^c \cup A$. We may relax this last condition to functions that are equal to 0 a.e. in E^c and are nonpositive in A, since then we can truncate them wherever they are negative. This and (7.2) show that

$$(7.3) \qquad \int_{B_1} |\nabla v|^2 - |\nabla w|^2 \, dx = \inf_{\psi \in \mathcal{A}} \int_{B_1} |\nabla \psi|^2 dx$$

where

$$\mathcal{A} := \{\psi \in H_0^1(B_1), \quad \psi = 0 \quad \text{a.e. in } E^c, \quad \psi \geqslant w \quad \text{a.e. in } A\}.$$

We use this characterization and show that the difference between the energies of v and w depends monotonically on φ, E and A. Precisely, for $i = \{1, 2\}$ let w_i, v_i be the corresponding functions for φ_i, E_i, A_i.

Lemma 7.1. *Assume*

$$\varphi_1 \leqslant \varphi_2, \quad E_1 \subset E_2, \quad A_1 \subset A_2.$$

Then

$$\int_{B_1} |\nabla v_1|^2 - |\nabla w_1|^2 \, dx \leqslant \int_{B_1} |\nabla v_2|^2 - |\nabla w_2|^2 \, dx.$$

Proof. Let $\bar{v}_2$ minimize the Dirichlet integral in B_1 among all the functions that equal v_2 a.e. in E_1^c and $\bar{v}_2 - v_2 \in H_0^1(B_1)$. Notice that $\bar{v}_2$ is well defined since v_2 is a test function with finite energy, so the minimizer exists by direct methods. As in (7.1) and (7.2) above, we find

$$\int_{B_1} |\nabla v_2|^2 - |\nabla \bar{v}_2|^2 \, dx = \int_{B_1} |\nabla(\bar{v}_2 - v_2)|^2 \, dx.$$

Since $\bar{v}_2 = v_2 = 0$ a.e. in $E_2^c \subset E_1^c$, and $\bar{v}_2 = w_2$ on ∂B_1 we find from the definition of w_2 that

$$\int_B |\nabla w_2|^2 \, dx \leqslant \int_B |\nabla \bar{v}_2|^2 \, dx.$$

hence

$$\int_{B_1} |\nabla(\bar{v}_2 - v_2)|^2 \, dx = \int_{B_1} |\nabla v_2|^2 - |\nabla \bar{v}_2|^2 \, dx \leqslant \int_{B_1} |\nabla v_2|^2 - |\nabla w_2|^2 \, dx.$$

Using the characterization in (7.3) for v_1, w_1 it suffices to show that $\bar{v}_2 - v_2 \in \mathcal{A}_1$. By construction $\bar{v}_2 - v_2 \in H_0^1(B_1)$, $\bar{v}_2 - v_2 = 0$ a.e. in E_1^c and $\bar{v}_2 - v_2 = \bar{v}_2$ a.e. in $A_1 \subset A_2$. It remains to check that $\bar{v}_2 \geqslant w_1$ which follows by maximum principle.

Indeed, let $h := (w_1 - \bar{v}_2)^+$. We have $h = 0$ a.e. in E_1^c and also $h \in H_0^1(B_1)$ since $\varphi_1 \leqslant \varphi_2$. From the definitions of w_1, $\bar{v}_2$ (see (7.1)) we obtain

$$\int_{B_1} \nabla w_1 \cdot \nabla h \, dx = 0, \qquad \int_{B_1} \nabla \bar{v}_2 \cdot \nabla h \, dx = 0.$$

Then

$$\int_{B_1} |\nabla(w_1 - \bar{v}_2)^+|^2 \, dx = \int_{B_1} \nabla(w_1 - \bar{v}_2) \cdot \nabla h \, dx = 0,$$

and the desired inequality $w_1 \leqslant \bar{v}_2$ is proved. $\qquad\square$

Proof of Lemma 2.3. After dividing w and v by an appropriate constant, we may assume that $\|w\|_{L^\infty(B_1)} = 1$. Then by Lemma 7.1 it suffices to prove our bound in the case when $\varphi = 1$, $B_1 \setminus B_\rho \subset E$ and $A = B_\rho \cap E$. In this case

$$v = c(\rho^{2-n} - |x|^{2-n})^+$$

for an appropriate c, and using symmetric rearrangement we see that the Dirichlet integral of w is minimized whenever w and the set A are radial. Therefore we need to prove the lemma only in the case when $E = B_r^c$, $A = B_\rho \setminus B_r$, for some $r \leqslant \rho$. We have

$$\int_{B_1} |\nabla v|^2 - |\nabla w|^2 \, dx = \int_{B_1} |\nabla(w - v)|^2 dx = \int_{B_1 \setminus B_r} (w - v)\triangle(v - w).$$

Using that in $B_\rho \setminus \overline{B}_r$

$$\triangle(v - w) = \triangle v = v_\nu d\mathcal{H}^{n-1}|_{\partial B_\rho},$$

and that $w - v = w \leqslant Cr$ on ∂B_ρ we find

$$\int_{B_1} |\nabla v|^2 - |\nabla w|^2 \, dx \leqslant Cr \leqslant C|A|,$$

and the lemma is proved. $\qquad\square$

Proof of Lemma 2.4. Assume that $\|w\|_{L^\infty(B_1)} = 1$ and as before, by Lemma 7.1, it suffices to obtain the bound in the case when $\varphi = 1$ and $E = B_{1/2}$. Then

$$w := c(2^{n-2} - |x|^{2-n})^+$$

for an appropriate c, and let

$$\bar{v} := \min\{w, C_0 d_A\},$$

where d_A represents the distance to the closed set A, and C_0 is a large constant depending only on n. Notice that by construction $\bar{v} - \varphi \in H_0^1(B_1)$, $\bar{v} = 0$ in A and $\bar{v}$ has bounded Lipschitz norm. Then

$$\int_{B_1} |\nabla v|^2 - |\nabla w|^2 \, dx \leqslant \int_{B_1} |\nabla \bar{v}|^2 - |\nabla w|^2 \, dx \leqslant C|S|,$$

where $S := \{\bar{v} < w\}$. It remains to show that $|S| \leqslant C(\beta)|A|$ which follows the uniform density property of A.

By choosing C_0 sufficiently large we have

$$S \subset \{C_0 d_A < w\} \subset \{6 d_A < d_{\partial B_{1/2}}\}.$$

Thus if $x \in S$ and $y \in \partial A$ is the closest point to x then it easily follows that

$$x \in B_{d_y/5}(y) \qquad \text{with} \quad d_y := d_{\partial B_{1/2}}(y).$$

Hence by Vitali's lemma we can find a collection of disjoint balls $B_{d_{y_i}/5}(y_i)$ such that

$$S \subset \bigcup_i B_{d_{y_i}}(y_i).$$

Thus, by adding the inequalities

$$|A \cap B_{d_{y_i}/5}(y_i)| \geqslant c(\beta)|B_{d_{y_i}}(y_i)|$$

we obtain that $|A| \geqslant c(\beta)|S|$. $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\square$

For the proof of Lemma 2.5 we first need a regularization result for the maximum of two $C^{1,\gamma}$ functions, $\gamma \in (0,1)$. In the next lemma we smooth out the "cornersöf the graph of the positive part of a $C^{1,\gamma}$ function without increasing its area too much.

Lemma 7.2. *Assume $h : \overline{\Omega} \to \mathbb{R}^+$ is a $C^{1,\gamma}$ function that satisfies $\{h > 0\} = \Omega$, $h = 0$ on $\partial\Omega$, and for any $z \in \overline{\Omega}$ there exists a linear function l_z (its tangent plane) such that*

$$|h - l_z| \leqslant \varepsilon |x - z|^{1+\gamma}, \qquad \forall x \in \overline{\Omega},$$

for some $\varepsilon > 0$ small. Let

$$K := \{z \in \overline{\Omega} \text{ s.t. } l_z + |x - z|^{1+\gamma} \geqslant 0 \quad \text{in } \mathbb{R}^n\}$$

and denote by

$$h^*(x) := \inf_{z \in K} \left(l_z + |x - z|^{1+\gamma} \right).$$

Then

$$\int_{\overline{\Omega}} h^* \, dx \leqslant (1 + \varepsilon^\sigma) \int_K h \, dx$$

with $\sigma > 0$ depending on n and γ.

Clearly if we replace $|x - z|^{1+\gamma}$ by $m|x - z|^{1+\gamma}$ the conclusion still holds since the problem remains invariant under multiplication by a constant m. The function h^* can be thought as a $C^{1,\gamma}$ upper envelope of norm $\|\nabla h\|_{C^\gamma}/\varepsilon$ of the function h (extended by 0 in the whole $\mathbb{R}^n$).

By construction $h^* \geqslant h$ in $\overline{\Omega}$, $h = h^*$ in K, and at any point $z \in K$ the graph of h is tangent by below to the $C^{1,\gamma}$ function $l_z + |x - z|^{1+\gamma} \geqslant 0$.

Proof. Notice that

$$z \in K \quad \Leftrightarrow \quad h(z) \geqslant c_0 |\nabla h(z)|^{\frac{\gamma+1}{\gamma}}, \qquad \text{with} \quad c_0 := \gamma(\gamma+1)^{-\frac{\gamma+1}{\gamma}}.$$

We show that for any $y \in \overline{\Omega} \setminus K$ there exists $d_y > 0$ such that

$$(7.4) \qquad \int_{(\overline{\Omega} \setminus K) \cap B_{d_y}(y)} h^* \, dx \leqslant \varepsilon^\sigma \int_{B_{d_y/5}(y) \cap K} h \, dx.$$

Then, by Vitali lemma, we cover $\overline{\Omega} \setminus K$ with a collection of balls $B_{d_{y_i}}(y_i)$ with $B_{d_{y_i}/5}(y_i)$ disjoint and we obtain the desired claim by summing (7.4) for all y_i.

Our hypotheses and (7.4) remain invariant under the scaling

$$h_\lambda(x) = \lambda^{1+\gamma} h(x/\lambda),$$

thus we may assume for simplicity that $y = 0$ and $\nabla h(0) = e_n$. Since $0 \notin K$ we have $h(0) \in [0, c_0)$, and by our hypothesis

$$|h(x) - (h(0) + x_n)| \leqslant \varepsilon |x|^{1+\gamma},$$

hence

$$|h(x) - (h(0) + x_n)| \leqslant \varepsilon^{1/2} \quad \text{if} \quad |x| \leqslant 2d_0 := \varepsilon^{-\frac{1}{2(\gamma+1)}}.$$

This implies that for some C_0 sufficiently large,

$$\overline{\Omega} \cap B_{d_0} \subset \{x_n \geqslant -C_0\},$$

$$|\nabla h| \leqslant 2, \quad h \geqslant c_0 2^{\frac{\gamma+1}{\gamma}} \quad \text{in the set} \quad B_{d_0} \cap \{x_n \geqslant C_0\}.$$

We obtain

$$B_{d_0} \cap \{x_n \geqslant C_0\} \subset K, \quad \text{and} \quad h^* \leqslant C \quad \text{in} \quad B_{d_0} \cap \{|x_n| \leqslant C_0\}$$

hence

$$\int_{(\overline{\Omega} \setminus K) \cap B_{d_0}} h^* \, dx \leqslant C d_0^{n-1}, \quad \int_{K \cap B_{d_0/5}} h \, dx \geqslant c d_0^{n+1},$$

and (7.4) follows. $\qquad \square$

Assume for simplicity that E is a set

$$E := \{x_n \geqslant g(x')\},$$

where g is a $C^{1,\gamma}$ function and $g(0) = 0$, $\nabla_{x'} g(0) = 0$.

Let $u \in H^1(E \cap \overline{B}_1)$, be positive and harmonic in the interior with $u = 0$ on ∂E. First we state a consequence of $C^{1,\gamma}$ estimates for harmonic functions.

Lemma 7.3. *Let $F = \{x_n \geqslant f(x')\}$ be a compact perturbation of E in $B_{1/2}$ and denote by v the harmonic function in $F \cap B_1$ which vanishes on $\partial F \cap B_1$ and equals u on ∂B_1. Assume that f, g are $C^{1,\gamma}$ functions with norm bounded by a constant M, $\|u\|_{L^2} \leqslant M$ and also that $|f - g| \leqslant \varepsilon$. Then*

$$\|\nabla u - \nabla v\|_{L^\infty(E \cap F \cap B_{1/2})} \leqslant C \varepsilon^{\frac{\gamma}{1+\gamma}}.$$

for some constant C depending on n, γ and M.

Proof. By boundary $C^{1,\gamma}$ estimates

$$\|v\|_{C^{1,\gamma}(B_{3/4} \cap F)} \leqslant C \quad \Rightarrow \quad |u - v| \leqslant C\varepsilon \quad \text{on} \quad \partial(E \cap F \cap B_1).$$

By maximum principle, the last inequality holds also in the interior of the domain and the conclusion follows since $u - v$ has bounded $C^{1,\gamma}$ norm in $B_{3/4} \cap E \cap F$. $\qquad \square$

Completion of the proof of Lemma 2.5. We estimate the change in the Dirichlet integral for the harmonic replacement of u whenever we perturb E by a small $C^{1,\gamma}$ set $A \subset B_\varepsilon$. We distinguish two cases, when A is interior to E and when A is exterior to E. Assume for simplicity that $|\nabla u(0)| = 1$.

Case 1: The set A is interior to E,

(7.5) $$A = \{g(x') \leqslant x_n < f(x')\} \subset B_\varepsilon,$$

for some function f with $C^{1,\gamma}$ norm bounded by a constant M. We let $\bar{u} := u_{E^c \cup A}$ and we want to show that

(7.6) $$\lim_{\varepsilon \to 0} \frac{1}{|A|} \int_{B_1} (|\nabla \bar{u}|^2 - |\nabla u|^2)\, dx = 1.$$

After modifying f in the set $B_{2\varepsilon} \setminus B_\varepsilon$ we may assume that $f = g$ outside $B_{2\varepsilon}$ and f has bounded $C^{1,\gamma}$ norm. From (7.5) we also obtain that

(7.7) $$\|g\|_{C^{1,\frac{\gamma}{2}}(B'_{2\varepsilon})}, \quad \|f\|_{C^{1,\frac{\gamma}{2}}(B'_{2\varepsilon})} \quad \text{are bounded by } C\varepsilon^{\frac{\gamma}{2}}.$$

We have

$$\int_{B_1} |\nabla \bar{u}|^2 - |\nabla u|^2\, dx = \int_{B_1} \nabla(\bar{u} - u) \cdot \nabla(\bar{u} + u)\, dx.$$

After integrating by parts in the sets $E \setminus A$ and A we find

(7.8) $$\int_{B_1} |\nabla \bar{u}|^2 - |\nabla u|^2\, dx = \int_{\partial A} u\, \bar{u}_\nu\, d\mathcal{H}^{n-1},$$

with ν the exterior normal to A. We need to estimate

$$\int_\Gamma u\, \bar{u}_\nu\, d\mathcal{H}^{n-1} \quad \text{with} \quad \Gamma := \{(x', f(x')) \text{ s.t. } f(x') > g(x')\}.$$

Let $T \subset \Gamma$ be a measurable set and denote by $T' \subset \mathbb{R}^{n-1}$ its projection along e_n direction. Since in B_ε, $u_n = 1 + o(1)$ with $o(1) \to 0$ as $\varepsilon \to 0$, we use (7.7) and we see that

$$(1 + o(1)) \inf_T \bar{u}_\nu \int_{T'} h\, dx' \leqslant \int_T u\, \bar{u}_\nu\, d\mathcal{H}^{n-1} \leqslant (1 + o(1)) \sup_T \bar{u}_\nu \int_{T'} h\, dx',$$

with

$$h := f - g.$$

For the upper bound we use that $\bar{u} \leqslant v$ with v defined in Lemma 7.3. Then $\bar{u}_\nu \leqslant v_\nu = 1 + o(1)$ in Γ and we find that

(7.9) $$\int_\Gamma u\, \bar{u}_\nu\, d\mathcal{H}^{n-1} \leqslant (1 + o(1))|A|.$$

For the lower bound we use Lemma 7.2 for h^+ and consider its $C^{1,\gamma/2}$ envelope of norm $\varepsilon^{\gamma/4} \gg \varepsilon^{\gamma/2}$. Denote by $K' \subset \mathbb{R}^{n-1}$ the contact set between h^+ and its envelope and let $K \subset \Gamma$ be the corresponding set that projects onto K'.

At any point $z \in K$ there is a $C^{1,\gamma/2}$ graph

$$G_z := \{x_n = f_z(x')\} \qquad f_z := g + l_z + \varepsilon^{\frac{\gamma}{4}}|x' - z'|^{1+\frac{\gamma}{2}},$$

and G_z is tangent by above to A and is included in $E \setminus A$. Moreover after using a cutoff function we may assume h_z has small $C^{1,\gamma/2}$ norm in a neighborhood of 0 and coincides with g outside this

neighborhood. Let v_z denote the corresponding harmonic function for h_z as in Lemma 7.3. Then $\bar{u} \geqslant v_z$, or $\bar{u}_\nu(z) \geqslant 1 + o(1)$ and we obtain

$$(7.10) \qquad \int_K \bar{u}_\nu\, u\, d\mathcal{H}^{n-1} \geqslant (1 + o(1)) \int_{K'} h\, dx' \geqslant (1 + o(1)) \int_{\Gamma'} h\, dx',$$

where in the last inequality we used Lemma 7.2. Then (7.6) follows from (7.9) and (7.10).

Case 2: The set A is exterior to E,

$$A = \{f(x') < x_n \leqslant g(x')\} \subset B_\varepsilon,$$

for some function f with $C^{1,\gamma}$ norm bounded. We let $\bar{u} := u_{E^c \setminus A}$ and we want to show that

$$(7.11) \qquad \lim_{\varepsilon \to 0} \frac{1}{|A|} \int_{B_1} (|\nabla u|^2 - |\nabla \bar{u}|^2) dx = 1.$$

As before we may assume that $h = g$ outside $B_{2\varepsilon}$ and (7.7) holds. Since

$$(7.12) \qquad \int_{B_1} |\nabla \bar{u}|^2 - |\nabla u|^2\, dx = \int_{\partial A} \bar{u}\, u_\nu\, d\mathcal{H}^{n-1}$$

and

$$(7.13) \qquad u_\nu = 1 + o(1)$$

we need to estimate

$$\int_\Gamma \bar{u}\, d\mathcal{H}^{n-1} \qquad \text{with} \quad \Gamma := \{(x', g(x')) \text{ s.t. } g(x') > f(x')\}.$$

The function v defined in Lemma 7.3 is a lower barrier for $\bar{u}$ and since $v_n = 1 + o(1)$ we obtain

$$(7.14) \qquad \int_\Gamma \bar{u}\, d\mathcal{H}^{n-1} \geqslant (1 + o(1)) \int_{\Gamma'} h\, dx', \qquad \text{with } h := (g - f)^+.$$

For the upper bound we apply Lemma 7.2 for the function h as in case 1 above. For any $z = (z', f(z'))$, $z' \in \Gamma'$ we define the graph G_z of the function

$$G_z := \{x_n = f_z(x')\}, \qquad f_z := g - l_z - \varepsilon^{\frac{\gamma}{4}}|x' - z'|^{1 + \frac{\gamma}{2}},$$

which is included in E^c and it is tangent to A by below at z. Since $\bar{u} \leqslant v_z$ and $\partial_n v_z = 1 + o(1)$ we obtain

$$\bar{u} \leqslant (1 + o(1))(x_n - f_z(x')).$$

After taking the infimum over all $z \in \Gamma$ we find

$$\bar{u}(x', g(x_n)) \leqslant (1 + o(1))h^*(x') \qquad \forall x' \in \Gamma'.$$

By Lemma 7.2 we find

$$(7.15) \qquad \int_\Gamma \bar{u}\, d\mathcal{H}^{n-1} \leqslant (1 + o(1)) \int_{\Gamma'} h^*\, dx' \leqslant (1 + o(1)) \int_{\Gamma'} h\, dx'.$$

Now, (7.11) is a consequence of (7.12), (7.13), (7.14) and (7.15), and this ends the proof of Lemma 2.5. $\qquad\square$

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