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Uniqueness and nondegeneracy of positive solutions of $(-\Delta)^s u + u = u^p$ in $\mathbb{R}^N$ when s is close to 1

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Abstract

We consider the equation $(-\Delta)^s u + u = u^p$, with $s \in (0,1)$ in the subcritical range of p. We prove that if s is sufficiently close to 1 the equation possesses a unique minimizer, which is nondegenerate.

1 Introduction

The purpose of this paper is to provide some nondegeneracy and uniqueness result for solutions of an equation driven by a nonlocal operator. In striking contrast with the local case, extremely little is known about these topics in the nonlocal framework and a satisfactory analysis of the problem is still largely missing, in spite of some striking recent contributions in specific cases.

Our approach is to obtain some nondegeneracy and uniqueness results by compactness and bifurcation arguments from the local case, that is when the fractional parameter involved is sufficiently close to being an integer.

Let us introduce the setting in which the problem is posed. Let $N \geqslant 2$ be the dimension of the ambient space $\mathbb{R}^N$ and let $s \in (0,1]$ be our fractional parameter.

We consider the fractional exponent

$$2_s^* := \begin{cases} \dfrac{2N}{N-2s} & \text{if } N \geqslant 3, \text{ or } N=2 \text{ and } s \in (0,1), \\ +\infty & \text{if } N=2 \text{ and } s=1. \end{cases}$$

We recall that this exponent plays the role of the classical critical Sobolev exponent for the fractional Sobolev spaces (see, e.g., [9] for a gentle introduction to the topic, and notice that 2_s^* is increasing in s and coincides with the classical Sobolev exponent for $s=1$). We consider here the fractional Sobolev space

$$H^s(\mathbb{R}^N) := \Big\{ u \in L^2(\mathbb{R}^N) \,:\, \int_{\mathbb{R}^N} |\xi|^{2s} |\widehat{u}|^2 \, d\xi < \infty \Big\},$$

with norm

$$\|u\|_s^2 := \int_{\mathbb{R}^N} (1+|\xi|^{2s}) |\widehat{u}|^2 \, d\xi = \|u\|^2_{L^2(\mathbb{R}^N)} + \int_{\mathbb{R}^N} |\xi|^{2s} |\widehat{u}|^2 \, d\xi,$$

where, as usual, $\widehat{u}$ is the Fourier transform of the function u, namely

$$\widehat{u}(\xi) := \frac{1}{(2\pi)^{\frac{N}{2}}} \int_{\mathbb{R}^N} e^{-\imath \xi \cdot x} u(x) dx.$$

We also denote by $H^s_{rad}(\mathbb{R}^N)$ the space of the radially symmetric functions of $H^s(\mathbb{R}^N)$. We recall that 2^*_s provides a compactness threshold for such radial functions, since $L^q(\mathbb{R}^N)$ is compactly embedded in $H^s_{rad}(\mathbb{R}^N)$ for every $q \in (1, 2^*_s)$ (see Proposition 1.1 in [20]).

In this functional framework, we are concerned with the uniqueness and nondegeneracy properties of the positive functions solving the fractional elliptic semilinear problem

$$(-\Delta)^s u + u = u^p \quad \text{in } \mathbb{R}^N. \tag{1.1}$$

Here we take $p \in (1, 2^*_s - 1)$ (i.e., the exponent $p+1$ is subcritical with respect to the above mentioned embeddings). Problems of this type has received a great attention recently, both by themselves and in connection with solitary solutions of nonlinear dispersive wave equations (such as the Benjamin-Ono equation, the Benjamin-Bona-Mahony equation and the fractional Schrödinger equation, see e.g. [3, 4, 15, 21, 22, 31]).

In this framework, the classical, local Hamiltonian operator is replaced by a fractional, nonlocal one, and the classical diffusion induced by Brownian motions is replaced by a non-local diffusion driven by $2s$-stable Lévy processes.

These type of fractional operators are now becoming also very popular in real-world models (for instance in financial mathematics, nonlocal stochastic control, nonlocal electrostatics, denoising and image processing, oceanography, dislocation dynamics in crystals, etc.), see for instance [9] and references therein.

Since the fractional Laplacian of $\varphi \in C^\infty_c(\mathbb{R}^N)$ may be defined via Fourier transform as

$$\widehat{(-\Delta)^s}\varphi(\xi) := |\xi|^{2s}\widehat{\varphi}(\xi) \qquad \text{for } \xi \in \mathbb{R}^N, \tag{1.2}$$

we may apply Plancherel's formula and adopt a weak (or distributional) notion of solution $u \in H^s(\mathbb{R}^N)$ for problem (1.1) via the identity

$$\frac{1}{2}\int_{\mathbb{R}^N} |\xi|^{2s}(\widehat{u}\,\overline{\widehat{\varphi}} + \overline{\widehat{u}}\,\widehat{\varphi})\,d\xi = \int_{\mathbb{R}^N} |\xi|^{2s}\widehat{u}\,\overline{\widehat{\varphi}}\,d\xi = \int_{\mathbb{R}^N} (u^p - u)\varphi\,dx$$

for any $\varphi \in H^s(\mathbb{R}^N)$. This notion of solution may be reduced to the one in the viscosity sense (see [25, 28]) and therefore the fractional Laplace regularity theory applies (see [30]). It is known that problem (1.1) admits a positive radial solution (see [10, 14]). Such solution is called a ground state, since it is obtained (up to scaling) by a constrained minimization problem of the functional

$$J_s(u,\nu) := \frac{1}{2}\|u\|_s^2 - \frac{\nu}{p+1}\int_{\mathbb{R}^N} |u|^{p+1}\,dx,$$

namely it attains the following greatest lower bound:

$$\nu_s := \inf_{u \in H^s(\mathbb{R}^N)} \frac{\|u\|_s^2}{\left(\int_{\mathbb{R}^N} |u|^{p+1}\right)^{2/(p+1)}} = \inf_{\substack{u \in H^s(\mathbb{R}^N)\\ \|u\|_{L^{p+1}(\mathbb{R}^N)}=1}} \|u\|_s^2. \tag{1.3}$$

We observe that if u_s is such that $\|u_s\|_{L^{p+1}(\mathbb{R}^N)} = 1$ and $\nu_s = \|u\|_s^2$, than it is a solution of

$$(1.4) \qquad (-\Delta)^s u_s + u_s = \nu_s u_s^p$$

and so it solves (1.1) (up to scaling). Also its derivatives $\partial_i u_s$ are solution of the linearized equation

$$(1.5) \qquad (-\Delta)^s(\partial_i u_s) + \partial_i u_s = p\nu_s u_s^{p-1}\partial_i u_s$$

and therefore

(1.6) $\partial_i u_s$ belongs to the kernel of the operator $J_s''(u_s, \nu_s)$.

The first result of this paper is nondegeneracy, namely that these derivative and their linear combinations exhaust $Ker(J_s''(u_s, \nu_s))$ at least when[1] s is sufficiently close to 1.

Theorem 1.1 *There exists $s_0 \in (0,1)$ such that for every $s \in (s_0, 1)$ if u_s is a minimizer for ν_s then*

$$Ker(J_s''(u_s, \nu_s)) = span\{\partial_i u_s,\ i = 1, \dots, N\}.$$

Our next result is a uniqueness property.

Theorem 1.2 *There exists $s_0 \in (0,1)$ such that for every $s \in (s_0, 1)$, the minimizer for ν_s is unique, up to translations.*

In the local case $s = 1$, the results in Theorems 1.1 and 1.2 were obtained in [19, 23, 24] but the specific arguments used there are not directly applicable to the nonlocal case $s \in (0,1)$. Before this paper, the only results available in the nonlocal case were the ones obtained in [2] for $N = 1$, $s = 1/2$ and $p = 2$, and recently extended in [15] for $N = 1$ and all $s \in (0,1)$.

After this paper was completed, arxived in [12] and submitted, the striking paper [16] has appeared, showing that Theorems 1.1 and 1.2 hold for any $s \in (0,1)$.

We also point out that, soon after [12], some interesting nondegeneracy results have been obtained in [8] for a related, but different, fractional problem.

For other recent variational problems related to the fractional Laplacian see, for instance, [13,26,27,29] and references therein. The rest of the paper is organized as follows. In Section 2 we collect some preliminary material, likely well-known to the expert readers, concerning some uniform estimates on the minimizers, some related asymptotics and a (up to now classical) local realization of the fractional Laplacian. Then, in Section 3, we prove the nondegeneracy result of Theorem 1.1. The uniqueness result of Theorem 1.2 is proved in Sections 4 and 5, by combining a series of arguments related to the construction of a branch of pseudo-minimizers $U_1 + \omega_s$, with s varies near 1, which are uniquely determined by their perturbation ω_s. Uniqueness is then deduced by showing that radially symmetric minimizers belongs to such a branch.

[1] Of course, since we are interested here in the case s close to 1 with a fixed exponent p, we fix $S \in (0,1)$ and $p \in 2_S^* - 1$, and all the arguments we present assume implicitly that $s \in [S, 1]$.

2 Preliminaries

2.1 Uniform estimates and asymptotics

By Lion's concentration compactness, minimizers for ν_s always exists (see, e.g. [10, 14] for details) and do not change sign. In this paper, we will consider only positive minimizers. They are radially symmetric by [14] (and, as usual, we take the center of symmetry to be the origin of $\mathbb{R}^N$). The minimizers attain the minimal value ν_s of the functional in (1.3) and they are normalized to have norm 1 in $L^p(\mathbb{R}^N)$. Also, thanks to Theorem 1.2 in [14], we have the decay estimate

$$u_s \leqslant C|x|^{-(N+2s)} \quad \text{in } \mathbb{R}^N.$$

We call $\mathcal{M}_s$ the the space of these positive, radially symmetric even minimizers u_s for ν_s normalized so that $\|u_s\|_{L^{p+1}(\mathbb{R}^N)} = 1$. Therefore if $u_s \in \mathcal{M}_s$ then

(2.1) $$\|u_s\|_{L^\infty(\mathbb{R}^N)} = |u_s(x_0^s)|,$$

for some $x_0^s \in \mathbb{R}^N$. Now we state a uniform bound on ν_s:

Lemma 2.1 *We have that* $\sup\limits_{s\in(0,1]} \nu_s < +\infty$.

Proof. Let $u_1 \in \mathcal{M}_1$. Notice that $|\xi|^{2s} \leqslant 1 + |\xi|^2$ and therefore

$$\|u_1\|_s \leqslant 2\|u_1\|_{L^2(\mathbb{R}^N)} + \int_{\mathbb{R}^N} |\xi|^2 |\widehat{u}|^2 \, d\xi \leqslant 2\|u_1\|_1 = 2\nu_1.$$

Since $\nu_s \leqslant \|u_1\|_s$, the desired result follows. □

The following result provides uniform bounds on the minimizers.

Lemma 2.2 *Given* $s_0 \in (0,1)$*, we have*

(2.2) $$0 < \gamma_{s_0} := \sup_{s\in(s_0,1)} \sup_{u_s\in\mathcal{M}_s} \|u_s\|_{L^\infty(\mathbb{R}^N)} < \infty.$$

Also, given $s_1 > 1/2$ *and* $\beta \in (0,1)$*,*

(2.3) $$\sup_{s\in(s_1,1)} \sup_{u_s\in\mathcal{M}_s} \|u_s\|_{C^{1,\beta}(\mathbb{R}^N)} < \infty.$$

Proof. The first inequality in (2.2) is obvious since

$$\gamma_{s_0} \geqslant \sup_{u_1\in\mathcal{M}_1} \|u_1\|_{L^\infty(\mathbb{R}^N)} > 0.$$

Now we prove the second inequality in (2.2). For this, we define

(2.4) $$\lambda_s := \|u_s\|_{L^\infty(\mathbb{R}^N)}$$

and we argue by contradiction: we suppose that $\lambda_s \to \infty$ for a sequence $s \to \bar\sigma \in [s_0, 1]$. We set

$$v_s(x) := \lambda_s^{-1} u_s(\lambda_s^{\frac{2}{2s-N}} x + x_0^s)$$

so that

$$\|v_s\|_{L^\infty(\mathbb{R}^N)} = 1 = v_s(0),$$
$$\widehat{v_s}(\xi) = \lambda_s^{-1+\frac{2N}{N-2s}} e^{\imath \xi \cdot x_0^s} \widehat{u_s}(\lambda_s^{\frac{2}{N-2s}} \xi)$$

and, by Lemma 2.1,

$$\int_{\mathbb{R}^N} |\xi|^{2s} |\widehat{v_s}|^2 \, d\xi = \int_{\mathbb{R}^N} |\xi|^{2s} |\widehat{u_s}|^2 \, d\xi \leqslant \nu_s \leqslant \text{Const.}$$

Therefore $v_s \rightharpoonup v$ in $H^t(\mathbb{R}^N)$ for every $t < \bar\sigma$ and

$$v_s \to v \text{ in } L^2_{loc}(\mathbb{R}^N).$$

Also, from (1.4),

$$(-\Delta)^s v_s(x) = -\lambda_s^{\frac{4s}{-N+2s}} v_s(x) + \lambda_s^{p-\frac{N+2s}{N-2s}} \nu_s v_s^p(x). \tag{2.5}$$

Now we recall Proposition 2.1.9 in [30], according to which we have that there is a constant $C(s, N, \alpha)$ such that

$$\|v_s\|_{C^{0,\alpha}(\mathbb{R}^N)} \leqslant C(s, N, \alpha) \left(\|(-\Delta)^s v_s\|_{L^\infty(\mathbb{R}^N)} + \|v_s\|_{L^\infty(\mathbb{R}^N)} \right), \tag{2.6}$$

where one can fix $\alpha < 2\bar\sigma$ for $2\bar\sigma < 1$ and $\alpha < 2\bar\sigma - 1$ for $2\bar\sigma > 1$ and the constant $C(s, N, \alpha)$ is bounded uniformly in $s \in [s_0, 1]$. From Lemma 2.1, (2.5) and (2.6), we see that $\|v_s\|_{C^{0,\alpha}(\mathbb{R}^N)}$ is bounded uniformly when $s \to \bar\sigma$. Accordingly, by the Ascoli theorem, we may suppose that v_s converges locally uniformly to v and passing to the limit in (2.5), we have that $v \equiv 0$. In particular

$$0 = \lim_{s \to \bar\sigma} |v_s(0)| = \lim_{s \to \bar\sigma} \lambda_s^{-1} |u_s(x_0^s)| = 1,$$

due to (2.1) and (2.4). This is a contradiction and so (2.2) is proved.

To prove (2.3) we use once again Proposition 2.1.9 in [30], see also [5], according to which, for any $s \in (s_1, 1]$,

$$\|u_s\|_{C^{1,\beta}(\mathbb{R}^N)} \leqslant C(s, N, \alpha) \left(\|(-\Delta)^s u_s\|_{L^\infty(\mathbb{R}^N)} + \|u_s\|_{L^\infty(\mathbb{R}^N)} \right),$$

where $C(s, N, \alpha)$ is uniformly bounded on $[s_1, 1]$. Then, the latter inequality implies (2.3), thanks to (1.4), (2.2) and Lemma 2.1. $\square$

Corollary 2.3 *Given $s_0 \in (0,1)$, we have*

$$\sup_{s \in (s_0,1)} \sup_{u_s \in \mathcal{M}_s} \|u_s\|_{2s} < \infty.$$

Proof. Let $s_0 \in (0,1)$, $u_s \in \mathcal{M}_s$ and $f_s(x) := \nu_s u_s^p(x) - u_s(x)$. Notice that

$$\int_{\mathbb{R}^N} |u_s|^{2p}\, dx \leqslant \|u_s\|_{L^\infty(\mathbb{R}^N)}^{2(p-1)} \int_{\mathbb{R}^N} |u_s|^2\, dx \leqslant C_1,$$

with $C_1 > 0$ independent of s and u_s, thanks to (2.2), Lemma 2.1 and the fact that $p > 1$. Moreover,

$$\|u_s\|_{L^2(\mathbb{R}^N)}^2 \leqslant \nu_s \leqslant C_2,$$

with $C_2 > 0$ independent of s and u_s, thanks to Lemma 2.1. As a consequence, and using Lemma 2.1 once more, we obtain that

$$\|f_s(x)\|_{L^2(\mathbb{R}^N)} \leqslant |\nu_s|\, \|u_s^p\|_{L^2(\mathbb{R}^N)} + \|u_s\|_{L^2(\mathbb{R}^N)} \leqslant C_3,$$

with $C_3 > 0$ independent of s and u_s. Also, from (1.4), $(-\Delta)^s u_s = f_s$, that is, recalling (1.2),

$$|\xi|^{2s}\widehat{u_s} = \widehat{f_s}$$

and so

$$\begin{aligned}\|u_s\|_{2s}^2 &= \|u_s\|_{L^2(\mathbb{R}^N)}^2 + \int_{\mathbb{R}^N} |\xi|^{4s}|\widehat{u}_s|^2\, d\xi \leqslant \nu_s + \int_{\mathbb{R}^N} |\widehat{f}_s|^2\, d\xi \\ &= \nu_s + \|f_s\|_{L^2(\mathbb{R}^N)}^2 \leqslant C_2 + C_3,\end{aligned}$$

and the desired result plainly follows. □

Next result is a general approximation argument on the fractional Laplacian:

Lemma 2.4 *Let s, $\bar\sigma \in (0,1]$ and*

(2.7) $$\delta > 2|\bar\sigma - s|.$$

Then, for any $\varphi \in H^{2(\bar\sigma+\delta)}(\mathbb{R}^N)$,

$$\|(-\Delta)^{\bar\sigma}\varphi - (-\Delta)^s\varphi\|_{L^2(\mathbb{R}^N)} \leqslant C_{\bar\sigma,\delta}|\bar\sigma - s|\, \|\varphi\|_{2(\bar\sigma+\delta)},$$

for a suitable $C_{\bar\sigma,\delta} > 0$.

Proof. We start with some elementary inequalities. First of all, if $\tau \in [0,1)$ then $(1+\tau^{2\bar\sigma+\delta})\tau^{2|\bar\sigma-s|} \leqslant 2 \cdot 1$. On the other hand, if $\tau \geqslant 1$ then $(1+\tau^{2\bar\sigma+\delta})\tau^{2|\bar\sigma-s|} \leqslant (2\cdot\tau^{2\bar\sigma+\delta})\tau^{\delta}$, thanks to (2.7). All in all, we obtain that, for any $\tau \geqslant 0$,

(2.8) $$(1+\tau^{2\bar\sigma+\delta})\tau^{2|\bar\sigma-s|} \leqslant 2(1+\tau^{2(\bar\sigma+\delta)}).$$

Moreover, for any $t \in \mathbb{R}$,

(2.9) $$|e^t - 1| \leqslant \sum_{k=1}^{+\infty} \frac{|t|^k}{k!} \leqslant \sum_{k=1}^{+\infty} \frac{|t|^k}{(k-1)!} = |t| e^{|t|}.$$

Furthermore, the map $(0,1) \ni \tau \mapsto \tau^{2\bar\sigma} \log \tau$ is minimized at $\tau = e^{-1/2\bar\sigma}$ and therefore

(2.10) $$|\tau^{2\bar\sigma} \log \tau| \leqslant (2\bar\sigma e)^{-1} \qquad \text{for any } \tau \in (0,1).$$

Similarly, the map $[1,\infty) \ni \tau \mapsto \tau^{-\delta} \log \tau$ is maximized at $\tau = e^{1/\delta}$ and so

(2.11) $$|\tau^{-\delta} \log \tau| \leqslant (\delta e)^{-1} \qquad \text{for any } \tau \in [1,\infty).$$

By combining (2.10) and (2.11), we obtain that, for any $\tau > 0$,

(2.12) $$|\tau^{2\bar\sigma} \log \tau| \leqslant C_{\bar\sigma,\delta} \left(1 + \tau^{2\bar\sigma+\delta}\right)$$

where

(2.13) $$C_{\bar\sigma,\delta} := (2\bar\sigma e)^{-1} + (\delta e)^{-1}.$$

Thus, using (2.8), (2.9) and (2.12), we obtain that, for any $\xi \in \mathbb{R}^N \setminus \{0\}$,

(2.14) $$\begin{aligned} ||\xi|^{2s} - |\xi|^{2\bar\sigma}| &= |\xi|^{2\bar\sigma} |\xi^{2(s-\bar\sigma)} - 1| = |\xi^{2\bar\sigma}|\, |e^{2(s-\bar\sigma)\log|\xi|} - 1| \\ &\leqslant |\xi|^{2\bar\sigma} \left|2(\bar\sigma - s)\log|\xi|\right| e^{2|\bar\sigma - s|\,|\log|\xi||} = |\xi|^{2\bar\sigma}\, 2|\bar\sigma - s| \left|\log|\xi|\right| |\xi|^{2|\bar\sigma - s|} \\ &\leqslant 2C_{\bar\sigma,\delta}|\bar\sigma - s|(1 + |\xi|^{2\bar\sigma+\delta})\, |\xi|^{2|\bar\sigma - s|} \leqslant 4C_{\bar\sigma,\delta}|\bar\sigma - s|(1 + |\xi|^{2(\bar\sigma+\delta)}). \end{aligned}$$

As a consequence

$$\begin{aligned} \int_{\mathbb{R}^N} |[(-\Delta)^s\varphi - (-\Delta)^{\bar\sigma}\varphi]|^2 &= \int_{\mathbb{R}^N} ||\xi|^{2s} - |\xi|^{2\bar\sigma}|^2 |\widehat{\varphi}|^2 \\ &\leqslant \text{Const}\, C_{\bar\sigma,\delta}^2 (\bar\sigma - s)^2 \int_{\mathbb{R}^N} (1 + |\xi|^{4(\bar\sigma+\delta)}) |\widehat{\varphi}|^2 \, d\xi \leqslant \text{Const}\, C_{\bar\sigma,\delta} \, \|\varphi\|_{2(\bar\sigma+\delta)}^2, \end{aligned}$$

as desired. □

Corollary 2.5 *Fix $\sigma \in (0,1]$. Then $\lim_{s\to\sigma} \nu_s = \nu_\sigma$.*

Proof. Let $s_0 \in (0,1)$. Let s, $s' \in (s_0, 1]$, that will be taken one close to the other, namely such that

(2.15) $$s > 2|s - s'|.$$

Let $u_s \in \mathcal{M}_s$. Since $\|u_s\|_{L^{p+1}(\mathbb{R}^N)} = 1$, we obtain that $\nu_{s'} \leqslant \|u_s\|_{s'}^2$. Hence, recalling (2.13) and (2.14) (used here with $\bar\sigma := s'$ and $\delta := s$, and notice that (2.7) is warranted by (2.15)), we conclude that

$$\begin{aligned} \nu_{s'} - \nu_s &\leqslant \|u_s\|_{s'}^2 - \|u_s\|_s^2 \\ &= \int_{\mathbb{R}^N} \left(|\xi|^{2s'} - |\xi|^{2s}\right) |\widehat{u_s}|^2 \leqslant \text{Const}\, |s' - s| \int_{\mathbb{R}^N} \left(1 + |\xi|^{4s}\right) |\widehat{u_s}|^2 \\ &= \text{Const}\, |s' - s| \, \|u_s\|_{2s} \end{aligned}$$

The constants here above only depend on the fixed s_0, but not on s and s'. Since the roles of s and s' may be interchanged, and recalling Corollary 2.3, we obtain that

$$|\nu_{s'} - \nu_s| \leqslant \text{Const}\,|s' - s|$$

and the desired result plainly follows. $\square$

From now on, we will use the uniqueness and nondegeneracy results for the local case. Namely, we recall that there exists a unique radial minimizer $U_1(x) = \bar{U}_1(|x|)$ for ν_1, such that

$$Ker(J_1''(U_1, \nu_1)) = \text{span}\{\partial_j U_1,\ j = 1, \dots, N\}, \tag{2.16}$$

see, e.g. [19, 23, 24].

Lemma 2.6 *Fix $\bar{\sigma} \in (0,1]$. Let $s_n \in (0,1)$ be such that $s_n \to \bar{\sigma}$. Let $u_{s_n} \in \mathcal{M}_{s_n}$. Then there exist $\bar{u} \in \mathcal{M}_{\bar{\sigma}}$ and a subsequence (still denoted by s_n) such that if*

$$\omega_{s_n}(x) := u_{s_n}(x) - \bar{u}, \tag{2.17}$$

we have that

$$\|\omega_{s_n}\|_{2s_n} \to 0 \text{ as } n \to \infty.$$

Moreover, if $\bar{\sigma} = 1$ then

$$\|\omega_{s_n}\|_{2} \to 0 \text{ as } n \to \infty.$$

Proof. To alleviate the notation, we write s instead of s_n. From Corollary 2.3 we have that u_s is bounded in $H^t(\mathbb{R}^N)$ for every $t < \bar{\sigma}$. Therefore, by compactness (see Proposition 1.1 in [20]), we obtain that there exists $\bar{u}$ such that

$$u_s \to \bar{u} \text{ in } L^q(\mathbb{R}^N) \text{ for every } q \in (2, 2^*_{\bar{\sigma}}).$$

Since we have uniform decay bounds at infinity and uniform L^∞ bounds (recall Lemma 2.2), this and the interpolation inequality implies that the convergence also holds for $q \in (1, 2]$, hence

$$u_s \to \bar{u} \text{ in } L^q(\mathbb{R}^N) \text{ for every } q \in (1, 2^*_{\bar{\sigma}}). \tag{2.18}$$

In particular, $\|\bar{u}_{\bar{\sigma}}\|_{L^{p+1}(\mathbb{R}^N)} = 1$ and $\bar{u}$ is radially symmetric. What is more, by Fatou lemma, it follows that $\bar{u} \in H^{\bar{\sigma}}(\mathbb{R}^N)$ because $\int_{\mathbb{R}^N} |\xi|^{2s} |\widehat{u_s}|^2 d\xi \leqslant \nu_s \leqslant Conts$. Also, by (1.4),

$$\int_{\mathbb{R}^N} u_s (-\Delta)^s \varphi + \int_{\mathbb{R}^N} u_s \varphi = \nu_s \int_{\mathbb{R}^N} u_s^p \varphi \quad \forall \varphi \in C_c^\infty(\mathbb{R}^N). \tag{2.19}$$

Using Lemma 2.4,

$$\int_{\mathbb{R}^N} |[(-\Delta)^s \varphi - (-\Delta)^{\bar{\sigma}} \varphi]|^2 \, d\xi \leqslant \text{Const}\,(\bar{\sigma} - s)^2 \int_{\mathbb{R}^N} (1 + |\xi|^4) |\widehat{\varphi}|^2 \, d\xi \leqslant \text{Const} \|\varphi\|_2^2.$$

Hence we can pass to the limit in (2.19) and conclude that $\bar{u}$ is a distributional solution to the equation

$$(-\Delta)^{\bar{\sigma}}\bar{u} + \bar{u} = \nu_{\bar{\sigma}}\bar{u}^p \tag{2.20}$$

that belongs to $H^{\bar{\sigma}}(\mathbb{R}^N)$.

So, by testing the equation against u itself, we see that $\|u\|_{\bar{\sigma}}^2 = \nu_{\bar{\sigma}}\|u\|_{L^{p+1}(\mathbb{R}^N)}^{p+1} = \nu_{\bar{\sigma}}$, hence $\bar{u}$ is a minimizer for $\nu_{\bar{\sigma}}$.

Furthermore, by (1.4), (2.17) and (2.20),

$$\begin{aligned}(-\Delta)^s\omega_s + \omega_s &= \nu_s[(\bar{u}+\omega_s)^p - \bar{u}^p] \\ &\quad +[(-\Delta)^{\bar{\sigma}}\bar{u} - (-\Delta)^s\bar{u}] + (\nu_s - \nu_{\bar{\sigma}})\bar{u}^p.\end{aligned} \tag{2.21}$$

Also, from the fundamental theorem of calculus

$$\begin{aligned}(\bar{u}+\omega_s)^p - \bar{u}^p &= \int_0^1 \frac{d}{dt}(\bar{u}+t\omega_s)^p dt \\ &= p\omega_s\int_0^1(\bar{u}+t\omega_s)^{p-1}dt,\end{aligned}$$

so that using (2.17) and (2.2)

$$\begin{aligned}&|(\bar{u}+\omega_s)^p - \bar{u}^p| \leqslant p|\omega_s|(\|u_s\|_{L^\infty(\mathbb{R}^N)} + 2\|\bar{u}\|_{L^\infty(\mathbb{R}^N)})^{p-1} \\ &\qquad \leqslant \text{Const } |\omega_s|.\end{aligned} \tag{2.22}$$

Next we observe that, since $\bar{u}, u_s \in C^2(\mathbb{R}^N)$, (2.21) holds pointwise and thus, by (2.22), we obtain

$$\begin{aligned}&\|(-\Delta)^s\omega_s\|_{L^2(\mathbb{R}^N)}^2 \\ &\quad \leqslant \|\omega_s\|_{L^2(\mathbb{R}^N)}^2 + \text{Const }(|\bar{\sigma}-s|^2 + |\nu_{\bar{\sigma}}-\nu_s|^2 + \|\omega_s\|_{L^2(\mathbb{R}^N)}^2) \to 0\end{aligned} \tag{2.23}$$

as $s \to \bar{\sigma}$. This and (2.18) imply that $\|\omega_s\|_{2s} \to 0$ as $s \nearrow \bar{\sigma}$, as desired.

Next we consider the case $\bar{\sigma} = 1$. By (1.5) and (2.2) we have that for every s close to 1

$$\|\partial_j u_s\|_{2s} \leqslant \text{Const.}$$

From this, (2.20) and (2.17), we deduce that

$$\|\partial_j \omega_s\|_{2s} \leqslant \text{Const.}$$

In particular $\|\omega_s\|_{2s+1}$ is uniformly bounded. We let f_s be the right hand side of (2.21) so that

$$(-\Delta)^s\omega_s + \omega_s = f_s$$

and so

$$-\Delta\omega_s + \omega_s = f_s + [-\Delta\omega_s - (-\Delta)^s\omega_s].$$

Using Lemma 2.4, we conclude that, for every $\delta \in (0, 1/4)$,

$$\int_{\mathbb{R}^N}[-\Delta\omega_s - (-\Delta)^s\omega_s]^2 \leqslant C_{N,\delta}(1-s)\|\omega_s\|_{2+\delta} \leqslant \|\omega_s\|_{2s+1} \leqslant (1-s)\text{Const},$$

provided s is close to 1. Also, by recalling (2.23) and (2.18), we obtain that $\|f_s\|_{L^2(\mathbb{R}^N)} \to 0$ as $s \nearrow 1$, and therefore $\|\omega_s\|_2 \to 0$. □

2.2 Local realization of $(-\Delta)^s$ for $s \in (0,1)$

Following [6], we recall here an extension property that provides a local realization of the fractional Laplacian by means of a divergence operator in a higher dimension halfspace. Namely, given $u \in H^s(\mathbb{R}^N)$, there exists a unique $\mathcal{H}(u) \in H^1(\mathbb{R}^{N+1}_+; t^{1-2s})$ such that

$$\begin{cases} \operatorname{div}(t^{1-2s}\nabla\mathcal{H}(u)) = 0 \quad \text{in } \mathbb{R}^{N+1}_+, \\ \mathcal{H}(u) = u \quad \text{in } \mathbb{R}^N, \\ \lim_{t\searrow 0} t^{1-2s}\mathcal{H}(u)_t := t^{1-2s}\mathcal{H}(u)_t = \kappa_s(-\Delta)^s u \quad \text{on } \mathbb{R}^N, \end{cases} \tag{2.24}$$

where κ_s is a positive normalization constant. Equivalently for every $\Psi \in H^1(\mathbb{R}^{N+1}_+; t^{1-2s})$

$$\int_{\mathbb{R}^{N+1}_+} \nabla\mathcal{H}(u)\cdot\nabla\Psi\, t^{1-2s} dt\, dx = \kappa_s \int_{\mathbb{R}^N} |\xi|^{2s}\widehat{u}\widehat{\Psi} d\xi, \tag{2.25}$$

where here and hereafter we denote the trace of a function with the same letter. From now on, $\mathcal{H}$ will denote the s-harmonic operator. Moreover, the trace property holds, i.e. for any $\Phi \in H^1(\mathbb{R}^{N+1}_+; t^{1-2s})$, the trace Φ on $\mathbb{R}^N$ belongs to $H^s(\mathbb{R}^N)$. As $\mathcal{H}(tr(\Phi)) := \mathcal{H}(\Phi)$ has minimal Dirichlet energy, it follows that

$$\int_{\mathbb{R}^{N+1}_+} |\nabla\Phi|^2 t^{1-2s} dt\, dx \geqslant \int_{\mathbb{R}^{N+1}_+} |\nabla\mathcal{H}(\Phi)|^2 t^{1-2s} dt\, dx = \kappa_s \int_{\mathbb{R}^N} |\xi|^{2s}|\widehat{\Phi}|^2 d\xi.$$

Hence $\mathcal{H}(u_s)$ is radially symmetric with respect to the x variable and it is a minimizer for

$$\nu_s = \inf_{U\in H^1(\mathbb{R}^{N+1}_+; t^{1-2s})} \frac{\kappa_s^{-1}\displaystyle\int_{\mathbb{R}^{N+1}_+} |\nabla U|^2 t^{1-2s} dt\, dx + \int_{\mathbb{R}^N} |U|^2 dx}{\left(\displaystyle\int_{\mathbb{R}^N} |U|^{p+1} dx\right)^{2/(p+1)}} \tag{2.26}$$

and, by (2.24),

$$\begin{cases} \operatorname{div}(t^{1-2s}\nabla\mathcal{H}(u_s)) = 0 \quad \text{in } \mathbb{R}^{N+1}_+ \\ \kappa_s^{-1} t^{1-2s}\mathcal{H}(u)_t + \mathcal{H}(u) = \nu_s\mathcal{H}(u)^p \quad \text{on } \mathbb{R}^N. \end{cases} \tag{2.27}$$

In this setting, we define

$$\mathcal{J}_s(U,\nu) := \frac{1}{2}\int_{\mathbb{R}^{N+1}_+} |\nabla U|^2 t^{1-2s} dt\, dx + \frac{\kappa_s}{2}\int_{\mathbb{R}^N} U^2 dx - \frac{\nu\kappa_s}{p+1}\int_{\mathbb{R}^N} |U|^p dx.$$

3 Nondegenracy

3.1 Preliminary observations

In this section, we assume that $u_s \in \mathcal{M}_s$ and we prove that it is nondegenerate for s sufficiently close to 1. For this, we denote by $\perp_s$ the orthogonality relation in $H^s(\mathbb{R}^N)$ and we start by estimating the second variation of the functional.

Lemma 3.1 *For every $\varphi \perp_s u_s$ we have that*

$$0 \leqslant J_s''(u_s, \nu_s)[\varphi, \varphi] = \|\varphi\|_s^2 - p\nu_s \int_{\mathbb{R}^N} u_s^{p-1}\varphi^2 \, dx. \tag{3.1}$$

Proof. Let $\varepsilon > 0$. Since $\varphi \perp_s u_s$, we have

$$\|\varepsilon\varphi + u_s\|_s^2 = \varepsilon^2\|\varphi\|_s^2 + \|u_s\|_s^2. \tag{3.2}$$

Also, by a Taylor expansion we obtain

$$\begin{aligned} &\int_{\mathbb{R}^N} |\varepsilon\varphi + u_s|^{p+1} \\ &\quad = \int_{\mathbb{R}^N} |u_s|^{p+1} + \varepsilon(p+1)\int_{\mathbb{R}^N} u_s^p\varphi + \frac{\varepsilon^2 p(p+1)}{2}\int_{\mathbb{R}^N} u_s^{p-1}\varphi^2 + O(\varepsilon^3). \end{aligned} \tag{3.3}$$

Furthermore, by testing (1.4) against φ and using again that $\varphi \perp_s u_s$, we conclude that

$$\int_{\mathbb{R}^N} u_s^p\varphi = 0,$$

hence the first order in ε in (3.3) vanishes. Consequently, recalling also that functions in $\mathcal{M}_s$ are normalized with $\|u\|_{L^{p+1}(\mathbb{R}^N)} = 1$, we write (3.3) as

$$\int_{\mathbb{R}^N} |\varepsilon\varphi + u_s|^{p+1} = 1 + \frac{\varepsilon^2 p(p+1)}{2}\int_{\mathbb{R}^N} u_s^{p-1}\varphi^2 + O(\varepsilon^3). \tag{3.4}$$

Now we recall the Taylor expansion

$$\frac{1}{(1+x)^{2/(p+1)}} = 1 - \frac{2}{p+1}x + O(x^2) \tag{3.5}$$

for small x. Thus, by inserting (3.4) into (3.5), we obtain

$$\frac{1}{\left(\displaystyle\int_{\mathbb{R}^N} |\varepsilon\varphi + u_s|^{p+1}\right)^{2/(p+1)}} = 1 - \varepsilon^2 p\int_{\mathbb{R}^N} u_s^{p-1}\varphi^2 + O(\varepsilon^3).$$

From this and (3.2) we obtain

$$\begin{aligned} &\frac{\|\varepsilon\varphi + u_s\|_s^2}{\left(\displaystyle\int_{\mathbb{R}^N} |\varepsilon\varphi + u_s|^{p+1}\right)^{2/(p+1)}} \\ &\quad = \left(1 - \varepsilon^2 p\int_{\mathbb{R}^N} u_s^{p-1}\varphi^2 + O(\varepsilon^3)\right)\left(\varepsilon^2\|\varphi\|_s^2 + \|u_s\|_s^2\right) \\ &\quad = \|u_s\|_s^2 + \varepsilon^2\left(\|\varphi\|_s^2 - p\|u_s\|_s^2\int_{\mathbb{R}^N} u_s^{p-1}\varphi^2\right) + O(\varepsilon^3). \end{aligned}$$

Then the desired result follows since u_s attains the minimal value $\nu_s = \|u_s\|_s^2$. □

Lemma 3.2 *Let $\Phi \in H^1(\mathbb{R}^{N+1}_+; t^{1-2s})$ be such that*

$$\kappa_s^{-1} \int_{\mathbb{R}^{N+1}_+} \nabla \Phi \cdot \nabla \mathcal{H}(u_s) t^{1-2s} dt\, dx + \int_{\mathbb{R}^N} \Phi \mathcal{H}(u_s) dx = 0. \tag{3.6}$$

Then

$$\mathcal{J}_s''(\mathcal{H}(u_s))[\Phi, \Phi] = \kappa_s^{-1} \int_{\mathbb{R}^{N+1}_+} |\nabla \Phi|^2 t^{1-2s} dz + \int_{\mathbb{R}^N} \Phi^2 dx - p\nu_s \int_{\mathbb{R}^N} u_s^{p-1} \Phi^2 dx \geqslant 0. \tag{3.7}$$

In particular for any $g \in H^1(\mathbb{R}^2_{++}; t^{1-2s} r^{N-1})$

$$\begin{aligned} A_1(g,g) \quad := \quad & \int_{\mathbb{R}^2_+} g_t^2 t^{1-2s} r^{N-1} dt dr + \int_{\mathbb{R}^2_{++}} g_r^2 t^{1-2s} r^{N-1} dt dr \\ & + (N-1) \int_{\mathbb{R}^2_{++}} g^2 t^{1-2s} r^{N-3} dt dr + \kappa_s \int_{\mathbb{R}_+} g^2 r^{N-1} dr \\ & - p\nu_s \kappa_s \int_{\mathbb{R}_+} u_s^{p-1} g^2 r^{N-1} dr \geqslant 0. \end{aligned} \tag{3.8}$$

Proof. The proof of (3.7) is similar to the proof of Lemma 3.1, since $\mathcal{H}(u_s)$ minimizes (2.26). Next, let $g \in H^1(\mathbb{R}^2_{++}; t^{1-2s} r^{N-1})$ and define $\Phi(x) := g(t, |x|) \frac{x^i}{|x|}$. Since $\mathcal{H}(u_s)$ is radial in the x variable, Φ satisfies (3.6) by odd symmetry. Then (3.6), (3.7) and the use of polar coordinates yield (3.8). □

Lemma 3.3 *Let $w \in Ker J_s''(u_s, \nu_s)$. Then*

$$w = w_0(|x|) + \sum_{i=0}^{N} c^i \partial_i u_s,$$

where

$$w_0(r) = \int_{S^{N-1}} w(r\theta) d\sigma(\theta)$$

and $c^i \in \mathbb{R}$.

Proof. Let $w \in Ker(J_s''(u_s, \nu_s))$ which means

$$(-\Delta)^s w + w - p\nu_s u_s^{p-1} w = 0 \quad \text{in } \mathbb{R}^N.$$

Let $\mathcal{H}(w) \in H^1(\mathbb{R}^{N+1}_+; t^{1-2s})$ be the s-harmonic extension of w which satisfies

$$\kappa_s^{-1} \int_{\mathbb{R}^{N+1}_+} \nabla \mathcal{H}(w) \cdot \nabla \Psi\, t^{1-2s} dt\, dx + \int_{\mathbb{R}^N} \mathcal{H}(w) \Psi\, dx - p\nu_s \int_{\mathbb{R}^N} u_s^{p-1} \mathcal{H}(w) \Psi dx = 0, \tag{3.9}$$

for all $\Psi \in H^1(\mathbb{R}^{N+1}_+; t^{1-2s})$. Now we consider the spherical harmonics on $\mathbb{R}^N$ for $N \geqslant 2$, i.e. the solution of the classical eigenvalue problem

$$-\Delta_{S^{N-1}} Y_k^i = \lambda_k Y_k^i \text{ on } S^{N-1}.$$

We let n_k be the multiplicity of λ_k. It is known that $n_0 = 1$ and $n_1 = N$ (see e.g. formulae (3.1.11) and (3.1.12) in [17]). In addition $\lambda_0 = 0$, $\lambda_1 = N - 1$ and $\lambda_k > N - 1$ for $k \geqslant 2$. Also Y_0 is constant, while

$$Y_1^i = \frac{x^i}{|x|} \text{ for } i = 1, \dots, N.$$

With this setting, we decompose $\mathcal{H}(w)$ in the spherical harmonics and we obtain

$$\mathcal{H}(w)(t,x) = \sum_{k \in \mathbb{N}} \sum_{i=1}^{n_k} f_i^k(t,|x|) Y_k^i \left(\frac{x}{|x|} \right), \tag{3.10}$$

where $f_i^k \in H^1(\mathbb{R}^2_+; t^{1-2s} r^{N-1})$. By testing (3.9) against the function $\Psi = h(t,|x|) Y_k^i$ and using polar coordinates, we obtain that, for any $h \in H^1(\mathbb{R}^2_+; t^{1-2s} r^{N-1})$, any $k \in \mathbb{N}$ and any $i \in [1, n_k]$,

$$\begin{aligned} A_k(f_i^k, h) \;\; := & \int_{\mathbb{R}^2_{++}} (f_i^k)_t h_t t^{1-2s} r^{N-1} dtdr + \int_{\mathbb{R}^2_{++}} (f_i^k)_r h_r t^{1-2s} r^{N-1} dtdr \\ & + \lambda_k \int_{\mathbb{R}^2_{++}} f_i^k h t^{1-2s} r^{N-3} dtdr + \kappa_s \int_{\mathbb{R}_+} f_i^k h r^{N-1} dr \\ & - p \nu_s \kappa_s \int_{\mathbb{R}_+} u_s^{p-1} f_i^k h r^{N-1} dr = 0. \end{aligned}$$

Now we observe that

$$A_k(f_i^k, f_i^k) = A_1(f_i^k, f_i^k) + (\lambda_k - (N-1)) \int_{\mathbb{R}^2_{++}} \int_{S^{N-1}} (f_i^k)^2 t^{1-2s} r^{N-3} dtdr.$$

By Lemma 3.2 and the fact that $\lambda_k > N - 1$ for $k \geqslant 2$, we obtain from the identities above that

$$\begin{aligned} 0 = A_k(f_i^k, f_i^k) &= A_1(f_i^k, f_i^k) + (\lambda_k - (N-1)) \int_{\mathbb{R}^2_{++}} \int_{S^{N-1}} (f_i^k)^2 t^{1-2s} r^{N-3} dtdr \\ &\geqslant (\lambda_k - (N-1)) \int_{\mathbb{R}^2_{++}} \int_{S^{N-1}} (f_i^k)^2 t^{1-2s} r^{N-3} dtdr \geqslant 0. \end{aligned}$$

As a consequence, $f_i^k = 0$ for every $k \geqslant 2$. Accordingly, (3.10) becomes

$$\mathcal{H}(w)(t,x) = \sum_{i=1}^{N} f_i^1(t,|x|) Y_k^i \left(\frac{x}{|x|} \right).$$

To complete the proof we need to characterize f_i^1. For this, we notice that, for $i = 1, \dots, N$, the function

$$f_i^1(t,r) = \int_{S^{N-1}} \mathcal{H}(w)(t, r\theta) \theta^i d\sigma(\theta)$$

satisfies $f_i^1(t,0) = 0$ and

$$\begin{aligned} A_1(f_i^1, h) \;\; = & \int_{\mathbb{R}^2_{++}} (f_i^1)_t h_t t^{1-2s} r^{N-1} dtdr + \int_{\mathbb{R}^2_{++}} (f_i^1)_r h_r t^{1-2s} r^{N-1} dtdr \\ & + (N-1) \int_{\mathbb{R}^2_{++}} f_i^1 h t^{1-2s} r^{N-3} dtdr + \kappa_s \int_{\mathbb{R}_{++}} f_i^1 h r^{N-1} dr \\ & - p \nu_s \kappa_s \int_{\mathbb{R}_{++}} u_s^{p-1} f_i^1 h r^{N-1} dr = 0, \end{aligned} \tag{3.11}$$

for every $h \in H^1(\mathbb{R}^2_+; t^{1-2s}r^{N-1})$, due to (3.9).

Now we define $\bar{U}(t,|x|) = \mathcal{H}(u_s)(t,x)$. Then we have

$$\begin{cases} \operatorname{div}(t^{1-2s}r^{N-1}\nabla \bar{U}) = 0 \quad \text{in } \mathbb{R}^2_{++} \\ \\ \lim_{t\searrow 0} -t^{1-2s}r^{N-1}\bar{U}_t + \kappa_s r^{N-1}\bar{U} = \kappa_s r^{N-1}\bar{U}^p \quad \text{on } \mathbb{R}_+. \\ \\ \lim_{r\searrow 0} r^{N-1}\bar{U}_r(t,0) = 0. \end{cases}$$

We set $V := \bar{U}_r$ and we differentiating the above equation with respect to r. We obtain

$$\begin{cases} -\operatorname{div}(t^{1-2s}r^{N-1}\nabla V) + (N-1)t^{1-2s}r^{N-3}V = 0 \quad \text{in } \mathbb{R}^2_{++} \\ \\ \lim_{t\searrow 0} -t^{1-2s}r^{N-1}V_t + \kappa_s r^{N-1}V = \kappa_s p r^{N-1}\bar{U}^{p-1}V \quad \text{on } \mathbb{R}_+ \\ \\ \lim_{r\searrow 0} r^{N-1}V(t,0) = 0. \end{cases} \tag{3.12}$$

Since $\bar{U}_r$ does not change sign, we may assume that $V < 0$ on $\mathbb{R}^2_{++}$.

Given $g \in C^\infty_c(\mathbb{R}^2_{++} \cup \{t=0\})$, we define

$$\psi := \frac{g}{V} \in H^1(\mathbb{R}^2_{++}; t^{1-2s}r^{N-1}).$$

Simple computations show that

$$|\nabla g|^2 = |V\nabla\psi|^2 + \nabla V \cdot \nabla(V\psi^2).$$

Hence we have

$$\int_{\mathbb{R}^2_+} |\nabla g|^2 t^{1-2s}r^{N-1}dtdr$$
$$= \int_{\mathbb{R}^2_+} |V\nabla\psi|^2 t^{1-2s}r^{N-1}dtdr + \int_{\mathbb{R}^2_{++}} \nabla(V\psi^2)\cdot(t^{1-2s}r^{N-1}\nabla V)dtdr.$$

Integrating by parts, by using the above identities and (3.12), we get

$$\int_{\mathbb{R}^2_{++}} |\nabla g|^2 t^{1-2s}r^{N-1}dtdr + (N-1)\int_{\mathbb{R}^2_{++}} g^2 t^{1-2s}r^{N-3}dtdr + \kappa_s\int_{\mathbb{R}_+} g^2 r^{N-1}dr$$
$$-\kappa_s p\int_{\mathbb{R}_+} u_s^{p-1}g^2 r^{N-1}dr = \int_{\mathbb{R}^2_{++}} |V\nabla\psi|^2 t^{1-2s}r^{N-1}dtdr.$$

In particular, by density and recalling (3.11), we have that, for every $i = 1, \dots, N$,

$$A_1(f_i^1, f_i^1) = 0 \geqslant \int_{\mathbb{R}^2_{++}} \left|V\nabla\left(f_i^1 V^{-1}\right)\right|^2 t^{1-2s}r^{N-1}dtdr.$$

This implies that the last term vanishes and therefore

$$\frac{f_i^1}{V} \equiv c^i$$

for some constant $c^i \in \mathbb{R}$. We then conclude that $f_i^1(0, |x|) = c^i \bar{U}_s'(|x|)$ for all $x \in \mathbb{R}^N$.

Thus, we have proved that for any $w \in Ker(J_s''(u_s, \nu_s))$

$$\mathcal{H}(w)(0, x) = w(x) = f_1^0(0, |x|) + \sum_{i=1}^{N} f_i^1(0, |x|) \frac{x^i}{|x|} = f_1^0(0, |x|) + \sum_{i=0}^{1} c^i \partial_k u_s(x),$$

as desired. □

Now we are ready to prove our nondegeneracy result for s close to 1.

3.2 Completion of the proof of Theorem 1.1

Let $v_s \in Ker(J''(u_s, \nu_s))$ be a radial function.

Claim: If s is close to 1, we have $v_s \equiv 0$.
Assume by contradiction that there exists a sequence s_n – still denoted by s – with $s \nearrow 1$ and such that $v_s \neq 0$. Up to normalization, we can assume that $\|v_s\|_{L^{p+1}(\mathbb{R}^N)} = 1$. By Corollary 2.5, we know that $\nu_s \to \nu_1$ and $u_s \to U_1(\cdot - a)$ in L^{p+1}, for some $a \in \mathbb{R}^N$. Since u_s is symmetric with respect to the origin, $a = 0$. By Hölder inequality

$$\int_{\mathbb{R}^N} |\xi|^{2s} |\widehat{v_s}|^2 d\xi \leqslant \|v_s\|_s^2 \leqslant p\nu_s \|u_s\|_{L^{p+1}}^{p-1} \|v_s\|_{L^{p+1}(\mathbb{R}^N)}^2 = p\nu_s \leqslant \text{Const.,} \tag{3.13}$$

by Lemma 2.1. Since v_s is a radial sequence and bounded in $H^t(\mathbb{R}^N)$ for every $t \in (0, 1)$, by compactness (see [20]) $v_s \to v$ in $L^q(\mathbb{R}^N)$ for every $q \in (2, 2_1^*)$, and then also for $q = 2$ (by repeating the argument above (2.18)). In particular $\|v\|_{L^{p+1}(\mathbb{R}^N)} = 1$. Next we observe that v_s is a solution of the linearized equation and therefore for any $\varphi \in C_c^\infty(\mathbb{R}^N)$

$$\int_{\mathbb{R}^N} v_s(-\Delta)^s \varphi + \int_{\mathbb{R}^N} v_s \varphi - p\nu_s \int_{\mathbb{R}^N} u_s^{p-1} v_s \varphi = 0$$

so by (2.2) and the fact that $(-\Delta)^s \varphi \to -\Delta \varphi$ in $L^2(\mathbb{R}^N)$ thanks to Lemma 2.4, we infer that

$$\int_{\mathbb{R}^N} v(-\Delta)\varphi + \int_{\mathbb{R}^N} v\varphi - p\nu_1 \int_{\mathbb{R}^N} U_1^{p-1} v\varphi = 0.$$

Applying Fatou lemma to (3.13), we get $v \in H^1(\mathbb{R}^N)$. We then conclude that v is radial, nontrivial and belongs to $Ker(J''(U_1, \nu_1))$. This is clearly a contradiction and the claim is proved. □

4 Uniqueness (preliminary observations)

4.1 Preliminary observations

Now we prove Theorem 1.2. The first part of the proof of the following result is quite standard but the last part requires a more delicate analysis on radial functions.

Lemma 4.1 *Let* $\Lambda_s := (Ker(J_s''(u_s,\nu_s)) \oplus \mathbb{R}u_s)^{\perp_s}$.

1 *We have*

(4.1)
$$J_s''(u_s,\nu_s)[u_s,u_s] = (1-p)\|u_s\|_s^2.$$

2 *There exists* $s_0 \in (0,1)$ *such that for every* $s \in (s_0,1)$ *and every minimizer* u_s *for* ν_s

(4.2)
$$K(s,u_s) := \inf_{\varphi\in\Lambda_s\setminus\{0\}} \frac{J_s''(u_s,\nu_s)[\varphi,\varphi]}{\|\varphi\|_s^2} > 0.$$

3 *Let*

$$\Lambda_s^r := \{\varphi \in H^s_{rad}(\mathbb{R}^N), \varphi \perp_s u_s\}$$

and

$$K_r(s,u_s) := \inf_{\Lambda_s^r\setminus\{0\}} \frac{J_s''(u_s,\nu_s)[\varphi,\varphi]}{\|\varphi\|_s^2}.$$

Then there exits $s_0 \in (0,1)$ *such that*

(4.3)
$$\inf_{s\in(s_0,1]} \inf_{u\in\mathcal{M}_s} K_r(s,u) > 0.$$

Proof. The statement in (4.1) is immediate from (3.1).

Now we prove (4.2). We first show that for any $\varphi \in \Lambda_s$

(4.4)
$$J_s''(u_s,\nu_s)[\varphi,\varphi] = 0 \Longrightarrow \varphi \equiv 0.$$

That is to say that $J_s''(u_s,\nu_s)$ defines a scalar product on Λ_s by Lemma 3.1. For this, assume that $\varphi \in \Lambda_s$ and

$$J_s''(u_s,\nu_s)[\varphi,\varphi] = 0.$$

Pick $\psi \in H^s(\mathbb{R}^N)$ such that $\psi \perp_s u_s$. By Lemma 3.1

$$J_s''(u_s,\nu_s)[\varphi+\varepsilon\psi,\varphi+\varepsilon\psi] \geqslant 0.$$

Hence

$$\begin{aligned} 0 &\leqslant J_s''(u_s,\nu_s)[\varphi,\varphi] + 2\varepsilon J_s''(u_s,\nu_s)[\varphi,\psi] + \varepsilon^2 J_s''(u_s,\nu_s)[\psi,\psi] \\ &= 2\varepsilon J_s''(u_s,\nu_s)[\varphi,\psi] + \varepsilon^2 J_s''(u_s,\nu_s)[\psi,\psi]. \end{aligned}$$

Then we conclude that

$$J_s''(u_s, \nu_s)[\varphi, \psi] = 0 \text{ for any } \psi \perp_s u_s. \tag{4.5}$$

Now we observe that, since $\varphi \perp_s u_s$, we deduce from (1.4) that

$$0 = \langle \varphi, u_s \rangle_s = \nu_s \int_{\mathbb{R}^N} u_s^p \varphi$$

and so

$$J_s''(u_s, \nu_s)[\varphi, u_s] = \langle \varphi, u_s \rangle_s - p\nu_s \int_{\mathbb{R}^N} u_s^p \varphi = 0.$$

This and (4.5) yield that $\varphi \in Ker(J_s''(u_s, \nu_s))$. Since also $\varphi \perp_s Ker(J_s''(u_s, \nu_s))$ it follows that $\varphi = 0$, and (4.4) is proved.

Now we end the proof of statement 2. Assume by contradiction that there exits a sequence $\varphi_n \in \Lambda_s$ such that $\|\varphi_n\|_s = 1$ and

$$J_s''(u_s, \nu_s)[\varphi_n, \varphi_n] \to 0 \text{ as } n \to \infty. \tag{4.6}$$

Let φ be the weak limit of φ_n in $H^s(\mathbb{R}^N)$. Then, by Lemma 3.1, we have that

$$0 \leqslant J_s''(u_s, \nu_s)[\varphi, \varphi] \leqslant \liminf J_s''(u_s, \nu_s)[\varphi_n, \varphi_n] = 0.$$

We deduce from this and (4.4) that $\varphi = 0$, that is

$$\varphi_n \text{ converges to } 0 \text{ weakly in } H^s(\mathbb{R}^N). \tag{4.7}$$

Now, since $u_s^{p-1} \in L^{\frac{p+1}{p-1}}(\mathbb{R}^N)$, given $\varepsilon > 0$ there exists $w_\varepsilon \in C_c^\infty(\mathbb{R}^N)$ such that

$$\|u_s^{p-1} - w_\varepsilon\|_{L^{\frac{p+1}{p-1}}(\mathbb{R}^N)} < \varepsilon. \tag{4.8}$$

Now we use (4.7) and the compactness results in fractional Sobolev spaces (see, e.g., Theorem 7.1 in [9]): we obtain that φ_n converges to 0 in $L^2_{loc}(\mathbb{R}^n)$ and therefore

$$\left| \int_{\mathbb{R}^N} w_\varepsilon \varphi_n^2 \right| \leqslant \|w_\varepsilon\|_{L^\infty(\mathbb{R}^N)} \|\varphi_n\|^2_{L^2(\operatorname{Supp} w_\varepsilon)} \to 0 \tag{4.9}$$

as $n \to \infty$. Also, by Hölder inequality

$$\begin{aligned} \left| \int_{\mathbb{R}^N} u_s^{p-1} \varphi_n^2 \right| &\leqslant \|u_s^{p-1} - w_\varepsilon\|^{p-1}_{L^{\frac{p+1}{p-1}}(\mathbb{R}^N)} \|\varphi_n\|^2_{L^{p+1}(\mathbb{R}^N)} + \int_{\mathbb{R}^N} w_\varepsilon \varphi_n^2 \\ &\leqslant \varepsilon^{p-1} \nu_s^{-1} + \int_{\operatorname{Supp} w_\varepsilon} w_\varepsilon \varphi_n^2. \end{aligned}$$

This, (4.8) and (4.9) imply that

$$\int_{\mathbb{R}^N} u_s^{p-1} \varphi_n^2 = o(1) \text{ as } n \to \infty.$$

Hence, recalling Lemma 2.1, we obtain

$$J_s''(u_s,\nu_s)[\varphi_n,\varphi_n] = \|\varphi_n\|_s^2 - p\nu_s \int_{\mathbb{R}^N} u_s^{p-1}\varphi_n^2 = 1 + o(1).$$

But this is in contradiction with (4.6) and the proof of (4.2) is complete.

Now we prove (4.3). Assume by contradiction that for every $s_0 \in (0,1)$

$$\inf_{s\in(s_0,1]} \inf_{u_s\in\mathcal{M}_s} K_r(s,u_s) = 0.$$

Then there exits a sequence $s_n \nearrow 1$ and radial minimizers u_{s_n} for ν_{s_n} such that

$$K_r(s_n,u_{s_n}) \to 0 \text{ as } n \to \infty. \tag{4.10}$$

Up to a subsequence, and recalling Corollary 2.5, we may assume that $\nu_{s_n} \to \nu_1$ and, by Lemma 2.6, that

$$\|u_{s_n} - U_1\|_2 \to 0 \quad \text{as } n \to \infty. \tag{4.11}$$

For fixed $n \in \mathbb{N}$, by the Eckeland variational principle (see [11]) together with the Riesz representation theorem, we obtain that there exist $f_{n,m} \in \Lambda_s^r$ and a minimizing sequence $\psi_{n,m} \in \Lambda_s^r$ for $K_r(s_n,u_{s_n})$ such that

$$\|\psi_{n,m}\|_{s_n} = 1, \quad \forall m \in \mathbb{N}$$

and

$$J_s''(u_{s_n},\nu_{s_n})[\psi_{n,m},v] - K_r(s_n,u_{s_n})\langle\psi_{n,m},v\rangle_{s_n} = \langle f_{n,m},v\rangle_{s_n}, \quad \forall v \in \Lambda_{s_n}^r, \tag{4.12}$$

where $\|f_{n,m}\|_{s_n} \to 0$ as $m \to \infty$. Then there exists a sequence of sub-indices m_n such that $\|f_{n,m_n}\|_{s_n} \to 0$ as $n \to \infty$. In particular, from (4.12) we have

$$J_s''(u_{s_n},\nu_{s_n})[\psi_{n,m_n},v] - K_r(s_n,u_{s_n})\langle\psi_{n,m_n},v\rangle_{s_n} = \langle f_{n,m_n},v\rangle_{s_n}. \tag{4.13}$$

Let $w \in C_c^\infty(\mathbb{R}^N) \cap \Lambda_1^r$. Then, from (2.14) and (4.11) we have $\langle w,u_{s_n}\rangle_{s_n} = o(1)\|w\|_{2+1}$ and

$$\int_{\mathbb{R}^N} (1+|\xi|^{2s_n})(\imath\xi^j)u_{s_n}w = o(1)\|w\|_{2+2} \quad \forall j = 1,\dots,N.$$

We define

$$v_n = w - \frac{\langle w,u_{s_n}\rangle_{s_n}}{\|u_{s_n}\|_{s_n}^2}u_{s_n}.$$

By construction $v_n \in \Lambda_{s_n}^r$. Using it as test function in (4.13) and recalling that $\psi_{n,m_n} \in \Lambda_{s_n}^r$, we get

$$J_{s_n}''(u_{s_n},\nu_{s_n})[\psi_{n,m_n},w] - K_r(s_n,u_{s_n})\langle\psi_{n,m_n},w\rangle_{s_n} = o(1). \tag{4.14}$$

Since $\|\psi_{n,m_n}\|_{s_n} = 1$, we may assume that up to a subsequence $\psi_{n,m_n} \rightharpoonup \psi$ in $H^t(\mathbb{R}^N)$ for every fixed $t \in (0,1)$. Passing to the limit in (4.14) and recalling (4.10), we get

$$J_1''(U_1,\nu_1)[\psi,w] = 0 \qquad \forall w \in C_c^\infty(\mathbb{R}^N) \cap \Lambda_1^r.$$

Since, by Fatou's lemma, $\psi \in H^1(\mathbb{R}^N)$, the latter identity implies that $\psi = 0$, because the case $s = 1$ is nondegenerate and $\psi \in \Lambda_1^r$.

That is, $\psi_{n,m_n} \rightharpoonup \psi = 0$ in $H^t(\mathbb{R}^N)$ for every fixed $t \in (0,1)$ and so, by compactness,

$$\psi_{n,m_n} \to 0 \text{ in } L^{p+1}(\mathbb{R}^N).$$

Also, by (4.12), we have

$$J_s''(u_{s_n}, \nu_{s_n})[\psi_{n,m_n}, \psi_{n,m_n}] - K_r(s_n, u_{s_n}) \|\psi_{n,m_n}\|_{s_n}^2 = o(1)$$

and, by Hölder inequality

$$\begin{aligned} \left| \int_{\mathbb{R}^N} u_{s_n}^{p-1} \psi_{n,m_n}^2 \right| &\leqslant \left(\int_{\mathbb{R}^N} u_{s_n}^{p+1} \right)^{\frac{p-1}{p+1}} \left(\int_{\mathbb{R}^N} \psi_{n,m_n}^{p+1} \right)^{\frac{2}{p+1}} \\ &= \|\psi_{n,m_n}\|_{L^{p+1}(\mathbb{R}^N)} = o(1) \end{aligned} \tag{4.15}$$

as $n \to +\infty$. Therefore

$$1 - p \int_{\mathbb{R}^N} u_{s_n}^{p-1} \psi_{n,m_n}^2 = \|\psi_{n,m_n}\|_{s_n}^2 - p \int_{\mathbb{R}^N} u_{s_n}^{p-1} \psi_{n,m_n}^2 = o(1).$$

Hence, passing to the limit and using (4.15), we get $1 - 0 = 0$, that is a contradiction. $\square$

5 Uniqueness (construction of pseudo-minimizers and completion of the proof)

5.1 Construction of pseudo-minimizers

Pick u_s a radially symmetric even minimizer for ν_s. Define the mapping

$$\Phi_s : H^s_{rad}(\mathbb{R}^N) \to H^s_{rad}(\mathbb{R}^N) \tag{5.1}$$

by

$$\Phi_s(\omega) = J_s'\left(U_1 + \omega, \nu_s\right). \tag{5.2}$$

As customary, by (5.2), we mean: for all $w \in H^s_{rad}(\mathbb{R}^N)$

$$\langle \Phi_s(\omega), w \rangle_s = J_s'\left(U_1 + \omega, \nu_s\right)[w]. \tag{5.3}$$

Lemma 5.1 *For every $f \in H^s_{rad}(\mathbb{R}^N)$, there exists a unique $\bar{w}^s \in H^s_{rad}(\mathbb{R}^N)$ such that*

$$\langle \Phi_s'(0)[\bar{w}^s], w \rangle_s = \langle f, w \rangle_s \quad \forall w \in H^s_{rad}(\mathbb{R}^N). \tag{5.4}$$

In addition there exists a constant $C_1 > 0$ such that

$$\|(\Phi_s'(0))^{-1}\| \leqslant C_1 \quad \forall s \in (s_0, 1). \tag{5.5}$$

Proof. We observe that

$$\langle \Phi_s'(0)[w'], w\rangle_s = (J_s''\left(U_1, \nu_s\right)[w', w]).$$

Hence solving the equation

$$\langle \Phi_s'(0)[\bar{w}], w\rangle_s = \langle f, w\rangle_s \quad \forall w \in H^s_{rad}(\mathbb{R}^N)$$

is equivalent to find a solution $\bar{w}$ to the equation

$$J_s''\left(U_1, \nu_s\right)[\bar{w}, w] = \langle f, w\rangle_s, \tag{5.6}$$

for any $w \in H^s_{rad}(\mathbb{R}^N)$. To this scope, we observe that, for every $w \in H^s_{rad}(\mathbb{R}^N)$,

$$\begin{aligned} |(J_s''(U_1,\nu_s) - J_s''(u_s,\nu_s))[w,w]| &= \nu_s p \left| \int_{\mathbb{R}^N} (u_s^{p-1} - U_1^{p-1}) w^2 \right| \\ &\leqslant \nu_s p \|u_s^{p-1} - U_1^{p-1}\|_{L^{\frac{p+1}{p-1}}(\mathbb{R}^N)} \|w\|^2_{L^{p+1}(\mathbb{R}^N)}. \end{aligned} \tag{5.7}$$

From Lemma 2.6 and Corollary 2.5 we know that $\|u_s - U_1\|_s \to 0$ and $\nu_s \to \nu_1$ as $s \nearrow 1$. This implies that $u_s \to U_1$ in $L^{p+1}(\mathbb{R}^N)$ and thus we have

$$u_s^{p-1} \to U_1^{p-1} \quad \text{in } L^{\frac{p+1}{p-1}}(\mathbb{R}^N).$$

Therefore, from (5.7),

$$|(J_s''(U_1,\nu_s) - J_s''(u_s,\nu_s))[w,w]| = o(1)\|w\|^2_{L^{p+1}(\mathbb{R}^N)}. \tag{5.8}$$

This together with (4.3) and (4.1) in Lemma 4.1 implies that there exist $C, s_0 > 0$ such that for all $s \in (s_0, 1)$

$$|J_s''(U_1,\nu_s)[v,v]| \geqslant C\|v\|_s^2 \quad \forall v \in H^s_{rad}(\mathbb{R}^N). \tag{5.9}$$

Hence, by the Lax-Milgram theorem, there exits a unique $\bar{w}^s \in H^s_{rad}(\mathbb{R}^N)$ such that

$$J_s''(U_1,\nu_s)[\bar{w}^s] = f$$

and by (5.9)

$$\|\bar{w}^s\|_s \leqslant C\|f\|_s,$$

which gives the desired result. $\square$

Proposition 5.2 *For $r > 0$ and $s > 0$, we set*

$$\mathcal{B}_{r,s} = \Big\{\omega \in H^s_{rad}(\mathbb{R}^N) \,:\, \|\omega\|_s \leqslant r \max\{1-s, |\nu_1 - \nu_s|\}\Big\}.$$

Then there exist $s_0 \in (0,1)$, $r_0 > 0$ such that for any $s \in (s_0, 1)$, there exists a unique function $\omega^s \in \mathcal{B}_{r_0,s_0}$ such that

$$\Phi_s(\omega^s) = 0.$$

Proof. We transform the equation $\Phi_s(\omega) = 0$ to a fixed point equation:

$$\omega = -(\Phi_s'(0))^{-1} \{\Phi_s(0) + Q_s(\omega)\}, \tag{5.10}$$

where

$$Q_s(\omega) := \Phi_s(\omega) - \Phi_s(0) - \Phi_s'(0)[\omega].$$

Notice that the definition above is well-posed thanks to (5.5). We observe that if $\omega \in H^s_{rad}(\mathbb{R}^N)$ then the mapping $\omega \mapsto (\Phi_s'(0))^{-1} \{\Phi_s(0) + Q_s(\omega)\}$ is radial too, since U_1 is radial.

For very $\bar{\omega} \in H^s_{rad}(\mathbb{R}^N)$, we set

$$\begin{aligned} \mathcal{N}_s(\omega)[\bar{\omega}] &:= J_s'(U_1 + \omega, \nu_s)[\bar{\omega}] - J_s'(U_1, \nu_s)[\bar{\omega}] - J_s''(U_1, \nu_s)[\omega, \bar{\omega}] \\ &= \nu_s \left(- \int_{\mathbb{R}^N} |U_1 + \omega|^p \bar{\omega} dx + \int_{\mathbb{R}^N} U_1^p \bar{\omega} dx + \int_{\mathbb{R}^N} U_1^{p-1} \omega \bar{\omega} dx \right). \end{aligned}$$

Notice that

$$Q_s(\omega) = \mathcal{N}_s(\omega). \tag{5.11}$$

Also, referring to page 128 in [1], we obtain

$$|\mathcal{N}_s(\omega)[\bar{\omega}]| \leqslant C(\|\omega\|_s^2 + \|\omega\|_s^p) \|\bar{\omega}\|_s$$

and

$$\|\mathcal{N}_s(\omega_1) - \mathcal{N}_s(\omega_2)\| \leqslant C(\|\omega_1\|_s + \|\omega_1\|_s^{p-1} + \|\omega_2\|_s + \|\omega_2\|_s^{p-1}) \|\omega_1 - \omega_2\|_s.$$

This, together with (5.11), implies that for every $\|\omega_1\|_s, \|\omega_2\|_s < 1$,

$$\|Q_s(\omega_1)\| \leqslant C_3 \|\omega_1\|_s^{\min(2,p)} \tag{5.12}$$

and

$$\|Q_s(\omega_1) - Q_s(\omega_1)\| \leqslant C_3 \|\omega_1 - \omega_2\|_s, \tag{5.13}$$

where C_3 is independent on $s \in (s_0, 1)$.

Now we claim that there exists a constant $C_2 > 0$ independent on $s \in (s_0, 1)$ such that

$$\|\Phi_s(0)\| \leqslant C_2 \max\{1 - s, |\nu_1 - \nu_s|). \tag{5.14}$$

By (2.14) we conclude that

$$|J_s'(U_1, \nu_s)[v] - J_1'(U_1, \nu_1)[v]| \leqslant (1 - s) C_{\delta,N} \|U_1\|_{2-s+\delta} \|v\|_s + |\nu_1 - \nu_s| \|v\|_s.$$

Since, from (1.4), $J_1'(U_1, \nu_1) = 0$, we get (5.14).

Now we finish the proof of Proposition 5.2. We shall solve the fixed point equation (5.10) in a ball of the form

$$\mathcal{B}_{r,s} = \{\omega \in H^s_{rad}(\mathbb{R}^N) \, : \, \|\omega\|_s \leqslant r\alpha_s\},$$

where $\alpha_s = \max\{1-s, |\nu_1 - \nu_s|\}$ and $r > 0$ will be fixed in a minute. Indeed for $\omega \in \mathcal{B}_{r,s}$, we exploit (5.5), (5.14) and (5.12) to deduce that

$$\|(\Phi_s'(0))^{-1}\{\Phi_s(0) + Q_s(\omega)\}\|_s \leqslant C_1\left(C_2\alpha_s + C_3 r^{\min(2,p)}\alpha_s^{\min(2,p)}\right).$$

There exists $r_0 > 0$ large and $s_0 \in (0,1)$ (possibly depending on r_0) such that for any $s \in (s_0, 1)$ we have

$$\begin{aligned} r_0\alpha_{s_0} &\geqslant C_1\left(C_2\alpha_{s_0} + C_3 r_0^{\min(2,p)}\alpha_{s_0}^{\min(2,p)}\right) \\ &> C_1\left(C_2\alpha_s + C_3 r_0^{\min(2,p)}\alpha_{s_0}^{\min(2,p)}\right), \end{aligned}$$

since α_s is small as $s \nearrow 1$. It follows that for every $s \in (s_0, 1)$, the mapping

$$\omega \mapsto -(\Phi_s'(0))^{-1}\{\Phi_s(0) + Q_s(\omega)\}$$

maps $\mathcal{B}_{r_0,s_0}$ into itself. Increasing s_0 if necessary, this map is a contraction on $\mathcal{B}_{r_0,s_0}$ by (5.13). Hence by the Banach fixed point theorem, for every $s \in (s_0, 1)$, there exists a unique function $\omega^s \in \mathcal{B}_{r_0,s_0}$ solving the fixed point equation (5.10). $\square$

The set of pseudo-minimizers is given by $\{U_1 + \omega_s \,:\, \Phi_s(\omega_s) = 0,\, s \in (s_0, 1)\}$. We now prove uniqueness, up to translations, of the minimizers for ν_s when s is close to 1 by showing that minimizers belong to such a set.

5.2 Completion of the proof of Theorem 1.2

Let u_s^1 and u_s^2 be two minimizers for ν_s. We know that they are symmetric under rotation, so we may and do assume that they are both symmetric with respect to the origin of $\mathbb{R}^N$. Our aim is to show that $u_s^1 = u_s^2$ provided s is close to 1 (no confusion should arise between the superscripts 1 and 2 and some exponents that shall occur in the course of the proof).

By Lemma 2.6, we know that $u_s^i = U_1 + \omega_s^i$ with $\|\omega_s^i\|_s \to 0$ as $s \nearrow 1$, for $i = 1, 2$ and ω^i is symmetric with respect to the origin for $i = 1, 2$. Then we have $\Phi_s(\omega_s^i) = 0$ for s close to 1 and thus by uniqueness (Proposition 5.2) we conclude that $\omega_s^1 = \omega_s^2$.

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