Institut für Angewandte Analysis und Stochastik

im Forschungsverbund Berlin e.V.

Quantization and measurability in gauge theory and gravity

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submitted: 22nd October 1992

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Preprint No. 18
Berlin 1992

Herausgegeben vom
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Abstract

Considering a mental experiment with a superposition of quasi-classical basic states for gauge theory and gravitation we obtain non-classical quantum observables. They can be interpreted as the difference of the gauge potentials of the basic states in gauge theory and a homeomorphism of the metrics of the basic states in general relativity. It is possible to consider gauge- and coordinate conditions (f.e. Lorentz gauge and harmonic coordinates) as new physical equations for these observables. For gravity we obtain in this way an interesting quasi-classical generalization of general relativity with two times — the time of general relativity and the harmonic time.

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0 Introduction

In classical gauge theory the gauge potential is not measurable. All observables of the gauge field depend only of integrals over closed curves of the gauge potential. This is connected with gauge symmetry. We cannot distinguish the potential A and the potential $T_g(A)$, obtained from A by a gauge transformation

$$A \to T_g(A) = g^{-1}Ag + g^{-1}dg.$$

Let's consider now quantum gauge theory and it's connection to classical theory. For every classical potential A there will be some coherent state or wave packet $C(A)$ in the correspondent quantum theory, and for every gauge transformation g some unitary operator T_g of the correspondent quantum symmetry group. There may be difficulties in this picture as several slightly different coherent states for the same potential and different composition rules for the T_g, but in the classical limit $h \to 0$ we will have

$$T_gC(A) \approx C(T_g(A)), \qquad\qquad T_gT_h \approx T_{gh}.$$

Gauge invariance of the quantum theory means that all observables O are gauge invariant ($T_gO = OT_g$). Then it is not possible to distinguish between th states $C(A)$ and $T_gC(A)$.

Now let's consider a superposition state $C(A) + C(A')$. It's clear that we cannot distinguish between this state and $T_gC(A) + T_gC(A')$. But what is with $C(A) + T_gC(A')$? These states are gauge-equivalent only in some degenerate cases like $T_g = 1$. Normally they are not gauge equivalent in the sense defined before. But on the other hand we have no classical observables to distinguish between these states. The measurement of F_{ij} gives the same result for these states — F_{ij} or F'_{ij} with probability $\frac{1}{2}$.

There are two possible solutions for this contradiction. The first is to have a quantum theory with $T_g = 1$ for all g. Then these states are all gauge-equivalent. The other possibility is that there are new non-classical observables which can distinguish between such states. We think that it is necessary to go the second way. There are new observables, namely the differences of the potentials $A_i(x) - A'_i(x)$. They are gauge-invariant if we use the same gauge-transformation for A and A'.

To show this we let's remember the Bohm-Aharonov-experiment. It has shown, that it is possible to get new observables in quantum mechanics. But in this experiment the state tested with a test particle was a classical state. We consider here a quantum state — a nontrivial superposition of quasi-classical states. We let it interact with some test particle and try to get some nontrivial interference picture. Considering the quasi-classical limit of the classical Schroedinger theory we obtain observables depending of the difference of the potentials $A_i(x) - A'_i(x)$.

If this relative gauge potential $A - A'$ is measurable, there must be a new evolution equation for it. The Maxwell equations cannot describe this evolution. It seams possible to find this equation considering the non-relativistic quasi-classical limit of the quantum field theory. That means, from this point of view field theories with different gauge-conditions are different physical theories.

For gravity we have a similar situation. We have to consider the group of all diffeomorphisms instead of the gauge group. The role of the difference of gauge potentials plays the relative position of the metrics. The result is the same — it is possible to measure this relative position by considering a superposition of gravitational fields. The experiment is the same, we use the non-relativistic limit of quantum gravity — the Schroedinger theory, but now with Newtonian potential. But here we have no accepted quantum field theory. We have to find a new physical equation to describe the new observables. As a natural candidate to complete the Einstein equations we consider here the harmonic coordinate condition.

Considering the Lorentz gauge as a physical equation, we change only the interpretation of some part of the theory. On the contrary, if we consider the harmonic condition as a physical equation, we obtain a lot of fundamental changes in gravitation. The topology becomes trivial, the scenario of the collapse to a black hole and the big bang changes. We get a new theory — a quasi-classical generalization of general relativity.

A new (physical and philosophical) interpretation of the terms of the theory seams necessary. It seams possible to interpret the harmonic time coordinate as the absolute, non-measurable time of classical quantum mechanics. From this point of view for quantization of gravity we have to use the scheme of classical quantum mechanics (with absolute time, states and probabilities) and not of relativistic field theory.

1 An Experiment

In this section we describe an easy mental experiment for gauge theory and for gravity. We consider a nontrivial superposition state of a source of the field — the first particle — an it's interaction with some test particle.

We assume some quasi-classical, nonrelativistic situation. The only quantum effect we are interested in is the superposition. Let $|\psi_l\rangle$ and $|\psi_r\rangle$ (left and right) be two quasi-classical states of a source of the field we are interested in. Let x_l and x_r be be their position. That means

$$\begin{aligned} X|\psi_{l/r}\rangle &\approx x_{l/r}|\psi_{l/r}\rangle \\ \psi_{l/r}(x) &\approx \delta(x - x_{l/r}) \end{aligned}$$

Now we need a possibility to construct a nontrivial superposition state of these states. That means, we need some observable O with eigenstates $|\psi_0\rangle = \frac{\sqrt{2}}{2}(|\psi_r\rangle + |\psi_l\rangle)$ and $|\psi_1\rangle = \frac{\sqrt{2}}{2}(|\psi_r\rangle - |\psi_l\rangle)$.

Then we consider a test particle in some initial state $|\varphi_0\rangle$. If it is a quasi-classical state we can describe the result of the interaction of $|\psi_{l/r}\rangle$ with $|\varphi_0\rangle$ (modulo a phase factor) using classical mechanics. The resulting two-particle state also can be considered as a quasi-classical state and will be nearly a tensor product.

For simplicity we assume that the state $|\psi_{l/r}\rangle$ of the first particle doesn't change. We can get this if the first particle is very heavy or if we have an external field holding the first particle in it's position.

Now we can describe our experiment. We start with the state $|\psi_0\rangle$ of the first particle and the state $|\varphi_0\rangle$ of the test particle. Then we let them interact. After this interaction we measure:

- our observable O of the first particle with eigenstates $|\psi_0\rangle$ and $|\psi_1\rangle$,
- the coordinate x of the test particle.

1.1 Case of Electric Field

Consider now the case of the electric field. At first we are not interested in relativistic or magnetic effects. That's why we use the classical Schroedinger two-particle theory for the description of the results. Using P, X, M, Q for momentum, coordinate, mass and charge of the first particle and p, x, m, q for the test particle we have the following Hamiltonian:

$$H = \frac{P^2}{2M} + V_{ext}(X) + \frac{p^2}{2m} - \frac{Qq}{4\pi\varepsilon_0|X - x|}.$$

V_{ext} describes some external potential holding the first particle at x_l and x_r (that means with sharp minima in $x_{l/r}$).

If the first particle is in one of the states $|\psi_{l/r}\rangle$ we can replace the X in the interaction term by $x_{l/r}$. Now our Hamiltonian is the sum of two one-particle operators. We have assumed that $|\psi_{l/r}\rangle$ is (nearly) stable. For the result of our one-particle problem for the test particle with the Hamiltonian

$$H_{l/r} = \frac{p^2}{2m} - \frac{Qq}{4\pi\varepsilon_0|x_{l/r} - x|}.$$

and the initial value $|\varphi_0\rangle$ we use the description $|\varphi_{l/r}\rangle$. Then the solution of the two-particle-problem is approximative

$$|\psi_{l/r}\rangle \otimes |\varphi_{l/r}\rangle.$$

Now we can solve the problem for the initial value $|\psi_0\rangle \otimes |\varphi_0\rangle$. We obtain the full solution

$$\tfrac{\sqrt{2}}{2}(|\psi_l\rangle \otimes |\varphi_l\rangle + |\psi_r\rangle \otimes |\varphi_r\rangle).$$

Using the designation $|\varphi_{0/1}\rangle = \frac{\sqrt{2}}{2}(|\varphi_l\rangle \pm |\varphi_r\rangle)$ we obtain the solution

$$\tfrac{\sqrt{2}}{2}(|\psi_0\rangle \otimes |\varphi_0\rangle + |\psi_1\rangle \otimes |\varphi_1\rangle).$$

Now we can describe the result of our measure. If the first particle is in the state $|\psi_0\rangle$, then the test particle is in the state $|\varphi_0\rangle$ and we obtain as the result of our measure of the coordinate $\rho_0(x) = |\varphi_0(x)|^2$. In the other case we obtain $\rho_1(x) = |\varphi_1(x)|^2$.

Let's compute the result in the quasi-classical case. To compute $\varphi_{l/r}$ for some point x we have to compute the classical trajectory $\gamma_{l/r}$ from the start point x_0 to x for the Hamiltonian $H_{l/r}$. Then we can compute a phase factor $f_{l/r}$ as the exponent of the integral of the action over $\gamma_{l/r}$. The module of the sum $f = f_l \pm f_r$ nearly defines the interference picture we get.

Using the description

$$A_0^{l/r}(x) = \frac{Q}{4\pi\varepsilon_0 |x_{l/r} - x|}$$

and f_0 for the value of f without electric field we obtain

$$f_{l/r} = f_0 \exp -iq \int A_0^{l/r}(\gamma_{l/r}(t))dt.$$

That means, the result of the measure depends on their quotient

$$\exp -iq \int A_0^l(\gamma_l(t)) - A_0^r(\gamma_r(t))dt.$$

It's clear how to write this in a relativistic invariant form:

$$\exp -iq \int A_i^l(\gamma_l(t)) - A_i^r(\gamma_r(t))dx^i.$$

But this is a dependence of the difference of two gauge potentials and obviously depends on the gauge condition we use. You can consider this as a dependence of an integral of gauge fields over a closed path (γ_l and in other direction γ_r), but with different gauge potentials on different parts of the path. It is gauge invariant only if we use the same gauge transformation for all fields.

1.2 Case of Gravity

To obtain an analogical result for gravity is simpler as for the electric field. It is not necessary to compute the phase factor. To see this we have to consider the analog of the difference of the gauge potentials in general relativity.

The symmetry group — the analog of the group of gauge transformations in general relativity — is the group of coordinate transformations. Consider now many gravitational fields in some coordinate system. What we have to find are things which are invariant if we use the same coordinate transformation for all field but change if we use different coordinate transformations for every field.

It's easy to find such invariants. We can define some equivalence between events in different metrics. Two events are equivalent if they have the same coordinates. If we use the same coordinate transformation for all fields, this equivalence relation doesn't change. From mathematical point of view these are matched diffeomorphisms of different metric spaces or different metrics on the same manifold.

Now let's compute the results of our experiment and search for measurables depending on this equivalence. For the description of the experiment we use the Schroedinger theory, but now with the Newtonian potential:

$$H = \frac{P^2}{2M} + V_{ext}(X) + \frac{p^2}{2m} - G\frac{Mm}{|x-X|}.$$

We get an analogical solution $\rho_0(x) = |\varphi_0(x)|^2$ and $\rho_1(x) = |\varphi_1(x)|^2$. And we see that these functions $\rho - 0/1$ depend on our equivalence relation described before! To compute $\varphi_{0/1}(x)$ we have to add functions defined for different gravitational fields. That means, in the language of general relativity they are events in different metrics. Obviously we need a correspondence between these events to define the sum or the difference of functions on different metrics.

Consider now the probability of transition from $|\psi_0\rangle$ to $|\psi_1\rangle$. We get

$$p_{0\to 1} = \int \rho_1(x)dx = \tfrac{1}{2}(1 - \Re\langle\varphi_r|\varphi_l\rangle).$$

It also depends on our equivalence relation. This result is interesting because we don't need a measure of coordinates to measure it. We need only our observable O of the first particle.

Let's now consider the quasi-classical case — if the states of the test particle $|\varphi_{l/r}\rangle$ can be considered as wave packets near classical trajectories $\gamma_{l/r}$. If these trajectories are so different, that there is de facto no intersection of the wave packets, we have $p_{0\to 1} = \frac{1}{2}$, and we obtain two equal peaks as the result of our measurement. A more interesting result we obtain, if we have an intersection (in coordinate space, not in phase space) of these trajectories. Then we obtain an interesting interference picture, different for $|\psi_0\rangle$ and $|\psi_1\rangle$. Considering only this qualitative picture we can define the points of intersection of γ_r and γ_l. This can be used for some kind of direct measure of equivalent events.

We have found a lot of interesting observables for the case of gravity:

- The transition probability $p_{0\to 1}$, defined by the scalar product of φ_r with φ_l.

- The probability distributions $\rho_{0/1}(x)$, defined by the sum and the difference of φ_r and φ_l.

- The qualitative picture, defining the points of intersection of the quasi-classical trajectories γ_r and γ_l.

All they are depending on some equivalence relation between events of different gravitational fields — our analog of the difference of gauge potentials for gravity.

2 Consequences

The main consequence of our results is that every quantum field theory has to define in it's quasi-classical, non-relativistic limit the behaviour of these measurables. This limit has to coincide with the results of the Schroedinger theory.

The problem is that this behaviour is fully undefined by the correspondent classical theory — gauge theory or general relativity. That means, there must be a new equation in the theory. In non-relativistic situation the behaviour is well-defined by the Schroedinger theory. But if we try to use the same quasi-classical language to compute some first order relativistic correction we will see that it is impossible to get any result because of the arbitrariness of gauge potentials and coordinate system. To repeat the classical result or to obtain a first order relativistic correction we need some new equation describing our new observables.

Let's consider possibilities to do this. At first we have to find variables to describe our new observables. For gauge fields it seams natural to define a new (now measurable) gauge potential A as the difference of the gauge potential of the current field and the gauge potential of the vacuum. For gravity we can use the coordinates of the equivalent event of the trivial (vacuum) field as the new observable. Now the new equation we need has the form of a gauge or coordinate condition. But it is only the form — from our point of view we have a new physical equation and it is possible to prove it by an experiment.

That means, we need some gauge or coordinate condition that looks like a physical equation. For this problem we have some standard candidates — the Lorentz gauge

$$\partial_i A_i(x) = 0$$

in gauge theory and the harmonic coordinates

$$\partial_i (g^{ij}\sqrt{-g}) = 0,$$

$$\Box x^i = 0$$

in general relativity. They fulfil the following necessary conditions:

- they define the evolution of our new observables if they are given for some initial state.
- they define a correct causality for the dependence of the new observables from initial values.
- they give the correct classical limit.
- they looks very nice — like relativistic physical equations.

But there are also other possibilities for such equations. For example, we can use the equations

$$\Box A_i(x) = 0$$

and assume that the initial values fulfil the Lorentz condition.

2.1 Quantum Gauge Theories

Using one of these equations we can define a quasi-classical theory which enables us to compute relativistic corrections for our experiment.

But for the electric field and other gauge fields we have already a good quantum field theory. It's possible to compute the results of our experiment using this theory. What will we get? I think we get approximately the same result as using the gauge condition used in the field theory a a new physical equation. I haven't proved it exactly, because our states are not asymptotically free.

We get a new interpretation of the role of the gauge condition in the quantum theory. It is necessary to use some gauge condition and different gauge conditions define different quantum theories. In principle it is possible to distinguish between these theories by experiment. It doesn't mean that there is no gauge symmetry in the theory — but there is only one gauge freedom for all fields, and that's why the gauge symmetry becomes trivial.

This contradicts to the usual point of view, that the results of the quantum field theories are independent of the gauge used. May be there is an error in the proof of the independence, may be it is a consequence of the restriction of the set of states to asymptotically free states only. From our point of view there are only two alternatives:

- the theory is independent of the gauge condition used. Then the Schroedinger theory is not the non-relativistic limit of the correspondent field theory.
- the theory depends on the gauge used, and it is in principle possible to verify it by experiment.

3 Quasi-Classical Gravity

But what about gravity? We have no quantum field theory for gravity. We can consider new, quasi-classical theories of gravitation including new equations for our new observables into the equations of general relativity. For example, we obtain new, interesting effects, if we consider the harmonic coordinate condition as a physical equation. This was done by Logunov [1] [4]. We obtain interesting fundamental differences to general relativity:

- All solutions are defined in $\mathbb{R}^4$. There are no nontrivial topologies in the theory.
- The part of the black hole behind the horizon is not a part of the "full" solution, because we have another definition of a full solution in the theory.

[1] I cannot agree with his point of view that general relativity cannot describe the results of experiments in the classical situation. Here I agree with [5]. But the idea to consider the harmonic coordinate condition as a physical equation is very nice.

- Only the flat Friedmann metric is isotropic and homogeneous, since we have another definition of a homogeneous solution. The big bang is not a part of the solution (it is at $x^0 \to -\infty$), but we obtain a singularity for $\tau \to \infty$ and finite x^0.
- On the other hand experimental result are the same as in general relativity, because the equations of general relativity are a part of the new theory. For example, the post-Newton parameters are the same as for general relativity, and the results about black holes for observers outside of the black hole also coincide with general relativity.
- We have a new symmetry group — the group of affine transformations.

This is a new theory, since we have new predictions about the behaviour of quasi-classical superposition states. Let's call this theory here quasi-classical gravity. But it is not an alternative theory, because there is no contradiction to general relativity and general relativity is their "classical limit" [2].

4 Interpretation

We have some new terms in our theory. We need a physical and philosophical interpretation of these new terms. We give here an interpretation which revives the classical absolute time. The idea is to consider the new equations we have as equations for an absolute time and for absolute space coordinates:

$$\Box t = 0; \qquad \Box x^i = 0, i = 1, 2, 3;$$

Now we have two time scales in our theory — the other is the path-dependent time

$$\tau = \int_\gamma \sqrt{g_{ij} \frac{dx^i}{dt} \frac{dx^j}{dt}} dt.$$

defined by metric tensor of general relativity. It will be interpreted (as in general relativity) as the time defined by clocks. It depends on the path of the clock and of the gravitational field. In quantum mechanics this time τ

[2] Logunov considers his theory as an alternative theory to general relativity

will be uncertain because of the uncertainty of the path and the gravitational field.

This interpretation reduces the symmetry group of the theory from the full affine group of the flat space-time $\mathbb{R}^4$ to some affine variant of the Galilei group [3]. This seams like a step back to the last century. But there are a lot of arguments for introducing an absolute time:

- Why absolute time was thrown away from physics? There was a symmetry group (the Lorentz group) which was considered as the global, independent and fundamental symmetry group of nature. Now we know that this symmetry is only local, depends on the gravitational field and this field is in reality a quantum, uncertain field. I think only the non-measurability is not a sufficient reason to throw away absolute time.

- It seams natural to distinguish between two notions of time — a measurable time defined by clocks and a notion of time describing past, presence, future and causality. Consider, for example, the definition of time given by Newton [1]:

  > *Absolute, true, and mathematical time, in itself, and from its own nature, flows equally, without relation to any thing external; and by other name called Duration. Relative, apparent, and vulgar time, is some sensible and external measure of duration by motion, whether accurate or unequable, which is commonly used instead of true time; as an hour, a day, a month, a year. It may be, that there is no equable motion, whereby time may be accurately measured. All motions may be accelerated and retarded, but the flowing of absolute time is liable to no change.*

- Obviously the measurable aspect of time was developed by special and general relativity. But there is also a big difference between the notion of time in classical mechanics and in quantum mechanics. It is a development of the other, absolute aspect of time:

[3] Here we have another difference of our theory with the "relativistic theory of gravity" of Logunov [4]. In the theory of Logunov there is a special-relativistic metric γ^{ij} which plays a central role in the interpretation, and the affine group reduces to the Poincare group.

- there is no standard self-adjoint operator for time measurement. For every time measurement we have to built some operator A depending of the Hamiltonian: $[A, H] \approx i\hbar$.
- It's difficult to built such a time measurement operator, since the spectrum of the Hamiltonian is usually positive and it is inpossible to define an A with $[A, H] = i\hbar$. That means, time measurement is uncertain.
- superposition makes quantum theory global in space — for the description of a state at some moment we have to describe the full space.
- there is no superposition of states at different time. That means, the absolute time is defined exactly.

That means, in general relativity and in quantum mechanics we consider different things using the notion "time". It seams to be an illusion to unify things in quantum gravity which are different already in classical quantum mechanics.

- In our theory the gravitational field doesn't play such a fundamental role as in general relativity. It describes no longer the space-time itself and it's topology, because this space-time is independent of the gravitational field and his topology is given and trivial. The gravitational field describes the behaviour of meters, clocks and other physical objects in this absolute space-time. It seams natural to define such a fundamental thing as causality using this absolute space-time and not the gravitational field. Defining absolute space and absolute time is an easy way to define a causality.

- If we have an absolute time we can use the apparatus of the classical quantum theory with states, observables and probabilities for the quantization of gravity. The path integral is a very powerful instrument for quantization, but makes it really sense without states and probabilities? May be renormalization of quantum gravity fails because there are used relativistic invariant schemes for a problem with another symmetry?

- For Einstein general relativity was more fundamental then quantum theory. Now most of physics think that the gravitational field must be quantized. The gravitational field looks like some space-time version of a gauge field which has to be unified with these theories. Bells theorem and it's experimental proof also have shown the power of the fundamental principles of quantum mechanics in relativistic area. I think it's time to say that quantum theory is more fundamental then general relativity. Reviving the absolute time of classical quantum mechanics is a step in this direction.

Using this interpretation we get a new but in some kind "very old" picture of the world. May be you don't like this picture, but there is no contradiction with experimental facts. The most of the predictions of the theory are the same as in general relativity. More, there are additional predictions about the behaviour of quantum superpositional states, which are not possible in general relativity, but coincide with the predictions of the Schroedinger theory.

- There is an absolute (affine) space $\mathbb{R}^3$.
- There is an absolute time t.
- The space is filled by some ether. The ether will be described by the symmetric tensor $g_{ij}(x,t)$ of signature $(1,3)$.
- The motion of the ether will be described by the Einstein equations and the harmonic coordinate equation.
- The factor $d\tau = \sqrt{g_{ij}dx^i dx^j}$ defines the velocity of clocks which is influenced by the ether.
- The velocity of light is some kind of sonic speed of the ether.

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