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A continuous dependence result for a nonstandard system of phase field equations

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Abstract

The present note deals with a nonstandard systems of differential equations describing a two-species phase segregation. This system naturally arises in the asymptotic analysis carried out recently by the same authors, as the diffusion coefficient in the equation governing the evolution of the order parameter tends to zero. In particular, an existence result has been proved for the limit system in a very general framework. On the contrary, uniqueness was shown by assuming a *constant* mobility coefficient. Here, we generalize this result and prove a continuous dependence property in the case that the mobility coefficient suitably depends on the chemical potential.

1 Introduction

In this paper, we address the system

$$\big(1+2g(\rho)\big)\,\partial_t\mu+\mu\,g'(\rho)\,\partial_t\rho-\operatorname{div}\big(\kappa(\mu)\nabla\mu\big)=0 \tag{1.1}$$

$$\partial_t\rho+f'(\rho)=\mu\,g'(\rho) \tag{1.2}$$

$$(\kappa(\mu)\nabla\mu)\cdot\nu|_\Gamma=0 \tag{1.3}$$

$$\mu(0)=\mu_0 \quad\text{and}\quad \rho(0)=\rho_0 \tag{1.4}$$

of differential equations and boundary and initial conditions in terms of the unknown fields μ and ρ; equations (1.1)–(1.2) are meant to hold in a bounded domain $\Omega\subset\mathbb{R}^3$ with a smooth boundary Γ and in some time interval $(0,T)$, and ν in (1.3) denotes the outward unit normal vector to Γ. The recent paper [2] investigated the existence of solutions to the above system: actually, a solution was found by considering the analogous system in which the ordinary differential equation (1.2) is replaced by the partial differential equation

$$\partial_t\rho-\sigma\Delta\rho+f'(\rho)=\mu\,g'(\rho) \quad\text{with the boundary condition}\quad \partial_\nu\rho|_\Gamma=0, \tag{1.5}$$

and then performing the asymptotic analysis as the diffusive coefficient σ tends to zero.

Such a modified system arises from the model introduced in [9], which describes the phase segregation of two species (atoms and vacancies, say) on a lattice in the presence of diffusion. It turns out to be a modification of the well-known Cahn-Hilliard equations (see, e.g., [7, 8]). The state variables are the *order parameter* ρ (volume density of one of the two species), which must of course attain values in the domain of the nonlinearities g' and f', and the *chemical potential* μ, which is required to be nonnegative for physical reasons. The initial-boundary value problem for the PDE system has been studied in a series of papers with a number of obtained results: here, we confine ourselves to quote the former [4, 5, 6] and latter [2, 3, 1].

In the mentioned papers, the function g is taken as a smooth nonnegative and possibly concave function (like it looks here), while the function f represents a multi-well potential: in this respect, a thermodynamically relevant example for f is the so-called *logarithmic potential*, in which f' is given by the formula

$$f'(\rho) = \ln \frac{1+\rho}{1-\rho} - 2c\rho \quad \text{for } \rho \in (-1,1), \tag{1.6}$$

with $c > 1$ in order that f actually presents a double well. The class of the admissible potentials may be rather wide and include both the standard double-well potential defined by

$$f(\rho) = \frac{1}{4}\,(\rho^2 - 1)^2 \quad \text{for } \rho \in \mathbb{R} \tag{1.7}$$

and potentials whose convex part f_1 is just a proper and lower semicontinuous function, thus possibly non-differentiable in its effective domain. In such a case, the monotone part f_1' of f' is replaced by the (possibly) multivalued subdifferential ∂f_1 and (1.5) has to be read as a differential inclusion. In [2], this wide class of potentials was considered. Moreover, in [2] the mobility coefficient κ in (1.1) and (1.3) was allowed to depend also on ρ.

Therefore, the existence result for system (1.1)–(1.4) proved in [2] is very general. On the other hand, the solution constructed in this way is rather irregular, in principle (due to a lack of regularity for μ). Nevertheless, it has been shown to be unique (and a little smoother than expected) provided that the mobility coefficient κ is a positive constant.

The aim of the present paper is to generalize the uniqueness proof performed in [2] to the case of a mobility coefficient depending on the chemical potential, exactly as in (1.1) and (1.3). Moreover, the continuous dependence of the solution on the initial data is shown in terms of suitable norms. Of course, in order to accomplish our program, a natural uniform parabolicity condition is required for κ.

The paper is organized as follows. In the next section, we list our assumptions and rewrite problem (1.1)–(1.4) in a precise form. In Section 3, we state and prove our uniqueness and continuous dependence result.

2 Assumptions and notations

We first introduce precise assumptions on the data for the mathematical problem under investigation. We assume Ω to be a bounded connected open set in $\mathbb{R}^3$ with smooth boundary Γ and let $T \in (0,+\infty)$ stand for a final time. We set for brevity

$$V := H^1(\Omega), \quad H := L^2(\Omega), \quad \text{and} \quad Q := \Omega \times (0,T). \tag{2.1}$$

The symbol $\langle\,\cdot\,,\,\cdot\,\rangle$ denotes the duality product between V^*, the dual space of V, and V itself. For the nonlinearities we assume that there exist real constants κ_*, κ^*, ρ_*, ρ^*, ξ_*, and ξ^* such that the combined conditions listed below hold.

$$\kappa : [0, +\infty) \to \mathbb{R} \quad \text{is continuous} \tag{2.2}$$
$$0 < \kappa_* \leq \kappa(m) \leq \kappa^* \quad \text{for every } m \geq 0 \tag{2.3}$$
$$f = f_1 + f_2 \,, \quad f_1 : \mathbb{R} \to [0, +\infty], \quad f_2 : \mathbb{R} \to \mathbb{R} \tag{2.4}$$
$$f_1 \text{ is convex, proper, l.s.c. and } f_2 \text{ is a } C^2 \text{ function} \tag{2.5}$$
$$\beta := \partial f_1 \quad \text{and} \quad \pi := f_2' \tag{2.6}$$
$$g \in C^2(\mathbb{R}), \quad g(r) \geq 0 \text{ and } g''(r) \leq 0 \text{ for } r \in \mathbb{R} \tag{2.7}$$
$$\pi,\ g, \text{ and } g' \text{ are Lipschitz continuous} \tag{2.8}$$
$$\rho_*,\ \rho^* \in D(\beta), \quad \xi_* \in \beta(\rho_*), \quad \text{and} \quad \xi^* \in \beta(\rho^*) \tag{2.9}$$
$$\xi_* + \pi(\rho_*) \leq 0 \leq \xi^* + \pi(\rho^*) \quad \text{and} \quad g'(\rho_*) \geq 0 \geq g'(\rho^*). \tag{2.10}$$

Notice that important potentials like (1.6) and (1.7) fit the above requirements with suitable choices of g and of the constants. For the initial data, we require that

$$\mu_0 \in V \cap L^\infty(\Omega) \quad \text{and} \quad \mu_0 \geq 0 \quad \text{a.e. in } \Omega \tag{2.11}$$
$$\rho_0 \in V \quad \text{and} \quad \rho_* \leq \rho_0 \leq \rho^* \quad \text{a.e. in } \Omega. \tag{2.12}$$

Now, we recall the part that follows from the asymptotic analysis performed in [2] and is of interest for the present paper.

Theorem 2.1. *Assume that* (2.2)–(2.12) *hold. Then there exists at least one triplet* (μ, ρ, ξ) *that satisfies*

$$\mu \in L^\infty(0,T;H) \cap L^2(0,T;V) \cap L^\infty(Q) \quad \text{and} \quad \mu \geq 0 \quad \text{a.e. in } Q \tag{2.13}$$
$$\rho \in L^\infty(0,T;V), \quad \partial_t \rho \in L^\infty(Q), \quad \text{and} \quad \rho_* \leq \rho \leq \rho^* \quad \text{a.e. in } Q \tag{2.14}$$
$$\xi \in L^\infty(Q), \quad \xi \in \beta(\rho) \quad \text{and} \quad \xi_* \leq \xi \leq \xi^* \quad \text{a.e. in } Q \tag{2.15}$$
$$u := \big(1 + 2g(\rho)\big)\mu \ \in W^{1,p}(0,T;V^*) \cap L^2(0,T;W^{1,q}(\Omega)) \tag{2.16}$$

for some $p,\ q > 1$ *and solves the problem*

$$\langle \partial_t u(t), v \rangle + \int_\Omega \kappa(\mu(t)) \nabla \mu(t) \cdot \nabla v = \int_\Omega \mu(t)\, g'(\rho(t))\, \partial_t \rho(t)\, v$$
$$\text{for all } v \in V \text{ and a.a. } t \in (0,T) \tag{2.17}$$
$$\partial_t \rho + \xi + \pi(\rho) = \mu\, g'(\rho) \quad \text{a.e. in } Q \tag{2.18}$$
$$u(0) = \big(1 + 2g(\rho_0)\big)\mu_0 \quad \text{and} \quad \rho(0) = \rho_0 \quad \text{a.e. in } \Omega. \tag{2.19}$$

Remark 2.2. We notice that (2.17) actually is a weak form of equation (1.1) (with the boundary condition (1.3) since the test function v is free on the boundary). Indeed, whenever μ is smoother with respect to time, one can compute $\partial_t u$ by the Leibniz rule and see that the differential equation hidden in the variational equation (2.17) coincides with (1.1). We also observe that [2] precisely yields $p = 4/3$ and $q = 3/2$ in (2.16). On the other hand, the regularity $u \in L^2(0,T;V)$ follows immediately from $u = \big(1 + 2g(\rho)\big)\mu$ thanks to (2.13)–(2.14), whence one can take $q = 2$.

In [2], an additional result was proved that deals with continuous dependence on the initial datum ρ_0. Here, we adapt the statement to our purposes. Indeed, we also consider possibly different initial data for the chemical potential (even though they do not enter the final estimate, directly).

Proposition 2.3. *Assume that* (2.2)–(2.12) *hold. Let* $(\mu_{0,i}, \rho_{0,i})$*,* $i = 1, 2$*, be two sets of initial data satisfying* (2.11)–(2.12)*, and let* (μ_i, ρ_i, ξ_i)*,* $i = 1, 2$*, be two solutions to the corresponding problem* (2.17)–(2.19) *that satisfy the regularity assumptions* (2.13)–(2.16)*. Then the following estimate holds true:*

$$|(\rho_1 - \rho_2)(t)| + \int_0^t \Big(|\partial_t(\rho_1 - \rho_2)| + |\xi_1 - \xi_2| \Big)(s)\, ds$$
$$\leq C \left(|\rho_{0,1} - \rho_{0,2}| + \int_0^t \big(|\mu_1 - \mu_2| + (1 + \mu_1)|\rho_1 - \rho_2| \big)(s)\, ds \right) \tag{2.20}$$

for every $t \in [0, T]$ *and a.e. in* Ω*, where* C *depends only on the constants and the functions mentioned in our assumptions* (2.2)–(2.10) *on the structure of the system.*

Proof. In order to give just an idea how to obtain (2.20), let us point out that the procedure consists in testing the difference of two equations (2.18) by $\operatorname{sign}(\xi_1 - \xi_2)$. Indeed, setting $w_i = \mu_i g'(\rho_i) - \pi(\rho_i)$, $i = 1, 2$, and multiplying the identity

$$\partial_t(\rho_1 - \rho_2) + (\xi_1 - \xi_2) = w_1 - w_2 \tag{2.21}$$

by $\operatorname{sign}(\xi_1 - \xi_2)$, it is not difficult to infer that

$$\partial_t|\rho_1 - \rho_2| + |\xi_1 - \xi_2| \leq |w_1 - w_2| \quad \text{a.e. in } Q. \tag{2.22}$$

Thanks to the Lipschitz continuity properties in (2.8), and integrating (2.22) only with respect to time, we obtain that for $t \in (0, T)$ it holds

$$|\rho_1 - \rho_2|(t) + \int_0^t |\xi_1 - \xi_2|(s)\, ds$$
$$\leq c \left(|\rho_{0,1} - \rho_{0,2}| + \int_0^t \big(|\mu_1 - \mu_2| + (1 + \mu_1)|\rho_1 - \rho_2| \big)(s)\, ds \right)$$

a.e. in Ω. Moreover, note that (2.21) implies

$$\int_0^t \partial_t|\rho_1 - \rho_2|(s)\, ds \leq \int_0^t \big(|w_1 - w_2| + |\xi_1 - \xi_2| \big)(s)\, ds,$$

whence (2.20) easily follows. $\square$

In [2], the uniqueness of the solution given by Theorem 2.1 (as well as the regularity $\partial_t \mu \in L^2(Q)$) was proved under an additional assumption, namely:

Theorem 2.4. *Assume* (2.2)–(2.12) *and that* κ *is a positive constant. Then the solution* (μ, ρ, ξ) *given by Theorem 2.1 is unique.*

The aim of this paper is to improve this result by showing that uniqueness and continuous dependence hold in the more general framework of Theorem 2.1, as stated in the forthcoming Theorem 3.1.

3 Uniqueness and continuous dependence

In this section, we prove the uniqueness and continuous dependence result for the solution to problem (2.17)–(2.19) stated below.

Theorem 3.1. *Assume that the conditions* (2.2)–(2.12) *are satisfied. Then the solution* (μ, ρ, ξ) *given by Theorem 2.1 is unique. Moreover, let* $(\mu_{0,i}, \rho_{0,i})$*,* $i = 1, 2$*, be two sets of initial data satisfying* (2.11)–(2.12)*, and let* (μ_i, ρ_i, ξ_i)*,* $i = 1, 2$*, be the corresponding solutions, which fulfill* (2.17)–(2.19) *with* $\mu_0 = \mu_{0,i}$ *and* $\rho_0 = \rho_{0,i}$*,* $i = 1, 2$*. Then there exists a constant* C*, depending on the data through the structural assumptions, such that*

$$\begin{aligned}\|\mu_1 - \mu_2\|_{L^2(0,T;H)} + \|\rho_1 - \rho_2\|_{L^\infty(0,T;H)} + \|\xi_1 - \xi_2\|_{L^1(Q)} \\ \leq C \left\{ \|\mu_{0,1} - \mu_{0,2}\|_H + \|\rho_{0,1} - \rho_{0,2}\|_H \right\}. \end{aligned} \tag{3.1}$$

Proof. We just prove continuous dependence, with uniqueness as a byproduct. Throughout the proof, we account for the well-known Hölder inequality and for the elementary Young inequality

$$ab \leq \varepsilon a^2 + \frac{1}{4\varepsilon} b^2 \quad \text{for every } a, b \geq 0 \text{ and } \varepsilon > 0.$$

Moreover, in order to simplify the notation, we use the same symbol small-case c for different constants, which may only depend on Ω, the final time T, the nonlinearities κ, f, g, and the solutions under consideration. Thus, the meaning of c may change from line to line and even within the same chain of inequalities. In contrast, we choose capital letters to denote precise constants we want to refer to. Finally, we set

$$Q_t := \Omega \times (0, t) \quad \text{for } t \in (0, T]. \tag{3.2}$$

Our argument relies on a suitable adaptation of the technique developed in [2] with the help of the function $K : [0, +\infty) \to \mathbb{R}$ defined by

$$K(m) := \int_0^m \kappa(m')\, dm' \quad \text{for } m \geq 0. \tag{3.3}$$

We have indeed $K' = \kappa$, whence $\nabla K(\mu) = \kappa(\mu)\nabla\mu$, so that (2.17) becomes, with $k := K(\mu)$,

$$\begin{aligned}\langle \partial_t u(t), v\rangle + \int_\Omega \nabla k(t) \cdot \nabla v = \int_\Omega \mu(t)\, g'(\rho(t))\, \partial_t \rho(t)\, v \\ \text{for all } v \in V \text{ and a.a. } t \in (0, T). \end{aligned} \tag{3.4}$$

We remark at once that (2.3) yields that for every $m_1, m_2 \geq 0$ it holds

$$\kappa_* |m_1 - m_2| \leq |K(m_1) - K(m_2)| \leq \kappa^* |m_1 - m_2| \tag{3.5}$$

$$(m_1 - m_2)\big(K(m_1) - K(m_2)\big) \geq \kappa_* (m_1 - m_2)^2. \tag{3.6}$$

In our proof, we use the equation obtained by integrating (3.4) with respect to time rather than (3.4) itself; namely, for every $v \in V$ and $t \in [0, T]$ we have

$$\int_\Omega u(t)\, v - \int_\Omega u(0)\, v + \int_\Omega \nabla \widetilde{k}(t) \cdot \nabla v = \int_\Omega \Big(\textstyle\int_0^t \mu(s)\, g'(\rho(s))\, \partial_t \rho(s)\, ds \Big)\, v, \tag{3.7}$$

where

$$\widetilde{k}(t) := \int_0^t k(s)\,ds = \int_0^t K(\mu(s))\,ds, \tag{3.8}$$

and where $u(0) = (1+2g(\rho_0))\mu_0$ according to the first Cauchy condition (2.19). It is worth observing that the pointwise values of u in the integrals over Ω are well defined. Indeed, the boundedness of u (derived from the boundedness of μ and $g(\rho)$) and the regularity of $\partial_t u$ (cf. (2.13)–(2.14) and (2.16)) ensure that u is weakly continuous, e.g., as an H-valued function. Now, we let $(\mu_{0,i},\rho_{0,i})$, $i=1,2$, be the initial data and pick two solutions (μ_i,ρ_i,ξ_i), $i=1,2$. Then, let us define the corresponding functions u_i, k_i, and $\widetilde{k}_i$ (according to (2.16), (3.4), and (3.8)), as well as the new ones γ_i, as follows:

$$\begin{aligned}
&\gamma_i := 1+2g(\rho_i)\,, \quad u_i := \gamma_i\,\mu_i\,, \quad k_i := K(\mu_i)\,,\\
&\widetilde{k}_i(t) := \int_0^t k_i(s)\,ds \quad \text{for } t\in[0,T], \quad i=1,2,
\end{aligned}$$

in order to simplify the notation. For the same reason, we set

$$\begin{aligned}
&\mu_0 := \mu_{0,1}-\mu_{0,2}\,, \quad \rho_0 := \rho_{0,1}-\rho_{0,2}\\
&\mu := \mu_1-\mu_2\,, \quad \rho := \rho_1-\rho_2\,, \quad \xi := \xi_1-\xi_2\\
&\gamma := \gamma_1-\gamma_2\,, \quad u := u_1-u_2\,, \quad k := k_1-k_2\,, \quad \text{and} \quad \widetilde{k} := \widetilde{k}_1-\widetilde{k}_2\,.
\end{aligned}$$

At this point, we write (3.7) at the time t for both solutions and choose $v = k(t) = \partial_t\widetilde{k}(t)$ in the difference. We obtain

$$\begin{aligned}
&\int_\Omega u(t)k(t) + \int_\Omega \nabla\widetilde{k}(t)\cdot\nabla\partial_t\widetilde{k}(t)\\
&= \int_\Omega \big((1+2g(\rho_{0,1}))\mu_0 + 2\mu_{0,2}(g(\rho_{0,1})-g(\rho_{0,2}))\big)k(t)\\
&\quad + \int_\Omega \Big(\textstyle\int_0^t \big(\mu_1(s)g'(\rho_1(s))\partial_t\rho_1(s) - \mu_2(s)g'(\rho_2(s))\partial_t\rho_2(s)\big)\,ds\Big)k(t).
\end{aligned} \tag{3.9}$$

We estimate each term of (3.9) separately. By accounting for (2.7)–(2.8), (2.13)–(2.14), as well as for (3.5)–(3.6), we have

$$\begin{aligned}
&uk = (\gamma_1\mu_1-\gamma_2\mu_2)k = \gamma_1\mu k + \gamma\mu_2 k \ge \kappa_*|\mu|^2 - c|\rho|\,|\mu| \quad \text{a.e. in } Q, \quad \text{whence}\\
&\int_\Omega u(t)k(t) \ge \kappa_*\int_\Omega|\mu(t)|^2 - c\int_\Omega|\rho(t)|\,|\mu(t)| \ge \frac{3\kappa_*}{4}\int_\Omega|\mu(t)|^2 - c\int_\Omega|\rho(t)|^2.
\end{aligned}$$

Next, we clearly see that

$$\int_\Omega \nabla\widetilde{k}(t)\cdot\nabla\partial_t\widetilde{k}(t) = \frac12\,\frac{d}{dt}\int_\Omega|\nabla\widetilde{k}(t)|^2.$$

With the help of (2.8), (2.11)–(2.12) and (3.5), we can control the first term on the right-hand side of (3.9):

$$\begin{aligned}
&\Big|\int_\Omega \big((1+2g(\rho_{0,1}))\mu_0 + 2\mu_{0,2}(g(\rho_{0,1})-g(\rho_{0,2}))\big)k(t)\Big|\\
&\le c\int_\Omega \big(|\mu_0|+|\rho_0|\big)|\mu(t)| \le c\big(\|\mu_0\|_H^2+\|\rho_0\|_H^2\big) + \frac{\kappa_*}{4}\int_\Omega|\mu(t)|^2.
\end{aligned}$$

In order to estimate the second term, we observe that

$$
\begin{aligned}
&|\mu_1 g'(\rho_1)\partial_t\rho_1 - \mu_2 g'(\rho_2)\partial_t\rho_2| \\
&\leq |\mu|\,|g'(\rho_1)|\,|\partial_t\rho_1| + |\mu_2|\,|g'(\rho_1) - g'(\rho_2)|\,|\partial_t\rho_1| + |\mu_2|\,|g'(\rho_2)|\,|\partial_t\rho| \\
&\leq c\bigl(|\mu| + |\rho| + |\partial_t\rho|\bigr) \quad \text{a.e. in } Q\,,
\end{aligned}
$$

thanks to our regularity assumptions on the solutions and on the structure (cf. (2.13)–(2.14) and (2.8)). By owing to Proposition 2.3, we deduce that

$$
\begin{aligned}
&\Bigl|\int_0^t \bigl(\mu_1(s)g'(\rho_1(s))\partial_t\rho_1(s) - \mu_2(s)g'(\rho_2(s))\partial_t\rho_2(s)\bigr)\,ds\Bigr| \\
&\leq c\int_0^t \bigl(|\mu(s)| + |\rho(s)| + |\partial_t\rho(s)|\bigr)\,ds \leq c\Bigl(|\rho_0| + \textstyle\int_0^t \bigl(|\mu(s)| + |\rho(s)|\bigr)\,ds\Bigr) \quad \text{a.e. in } \Omega.
\end{aligned}
$$

Therefore, we have

$$
\begin{aligned}
&\int_\Omega \Bigl(\textstyle\int_0^t \bigl(\mu_1(s)g'(\rho_1(s))\partial_t\rho_1(s) - \mu_2(s)g'(\rho_2(s))\partial_t\rho_2(s)\bigr)\,ds\Bigr)k(t) \\
&\leq c\int_\Omega \Bigl(|\rho_0| + \textstyle\int_0^t (|\mu(s)| + |\rho(s)|)\,ds\Bigr)\,|\mu(t)| \\
&\leq \frac{\kappa_*}{4}\int_\Omega |\mu(t)|^2 + c\int_\Omega \Bigl\{|\rho_0|^2 + \Bigl(\textstyle\int_0^t |\mu(s)|\,ds\Bigr)^2 + \Bigl(\textstyle\int_0^t |\rho(s)|\,ds\Bigr)^2\Bigr\} \\
&\leq \frac{\kappa_*}{4}\int_\Omega |\mu(t)|^2 + c\int_\Omega \Bigl\{|\rho_0|^2 + \textstyle\int_0^t |\mu(s)|^2\,ds + \int_0^t |\rho(s)|^2\,ds\Bigr\} \\
&= \frac{\kappa_*}{4}\int_\Omega |\mu(t)|^2 + c\|\rho_0\|_H^2 + c\int_{Q_t} |\mu|^2 + c\int_{Q_t} |\rho|^2.
\end{aligned}
$$

By combining the above equalities and inequalities with (3.9), we infer that

$$
\begin{aligned}
&\frac{\kappa_*}{4}\int_\Omega |\mu(t)|^2 + \frac{1}{2}\,\frac{d}{dt}\int_\Omega |\nabla\widetilde{k}(t)|^2 \\
\leq\;\; & c\bigl(\|\mu_0\|_H^2 + \|\rho_0\|_H^2\bigr) + c\int_\Omega |\rho(t)|^2 + c\int_{Q_t} |\mu|^2 + c\int_{Q_t} |\rho|^2\,,
\end{aligned}
$$

and an integration with respect to time yields

$$
\begin{aligned}
&\frac{\kappa_*}{4}\int_{Q_t} |\mu|^2 + \frac{1}{2}\int_\Omega |\nabla\widetilde{k}(t)|^2 \\
&\leq c\bigl(\|\mu_0\|_H^2 + \|\rho_0\|_H^2\bigr) + c\int_{Q_t} |\rho|^2 + \int_0^t \Bigl(\int_{Q_s} |\mu|^2 + \int_{Q_s} |\rho|^2\Bigr)\,ds \\
&\leq c\bigl(\|\mu_0\|_H^2 + \|\rho_0\|_H^2\bigr) + c\int_0^t \Bigl(\int_{Q_s} |\mu|^2\Bigr)\,ds + c\int_{Q_t} |\rho|^2. \qquad (3.10)
\end{aligned}
$$

Now, let us consider (2.20). Squaring and integrating over Ω, then applying Hölder's inequality on the right-hand side, we easily obtain that

$$
\int_\Omega |\rho(t)|^2 \leq D\|\rho_0\|_H^2 + D\int_{Q_t} |\mu|^2 + D\int_{Q_t} |\rho|^2 \qquad (3.11)
$$

for some positive constant D. Moreover, by integrating (2.20) over Ω and then squaring, we arrive at

$$\left(\int_{Q_t} |\xi|\right)^2 \leq D\|\rho_0\|_H^2 + D\int_{Q_t} |\mu|^2 + D\int_{Q_t} |\rho|^2 \,, \tag{3.12}$$

where D is the same constant as before, without loss of generality. Hence, we multiply (3.10) by $12D/\kappa_*$ and add it to (3.11) and (3.12). This computation leads to

$$\begin{aligned} &D\|\mu\|_{L^2(Q_t)}^2 + \|\rho(t)\|_H^2 + \|\xi\|_{L^1(Q_t)}^2 \\ &\leq c\big(\|\mu_0\|_H^2 + \|\rho_0\|_H^2\big) + c\int_0^t \|\mu\|_{L^2(Q_s)}^2 \, ds + c\int_0^t \|\rho(s)\|_H^2 \, ds. \end{aligned} \tag{3.13}$$

At this point, it suffices to apply the Gronwall lemma to deduce a variation of (3.1) with the squared norms. Therefore, (3.1) is completely proved. $\square$

Remark 3.2. Clearly, just a few of the assumptions (2.2)–(2.12) are used in the above proof. The whole set of hypotheses has been listed in the statement of Theorem 3.1 in order to ensure both the existence of a solution satisfying (2.13)–(2.16) and the validity of estimate (2.20), according to Theorem 2.1 and Proposition 2.3.

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