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On the construction of a class of generalized Kukles systems having at most one limit cycle

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Abstract

Consider the class of planar systems

$$\frac{dx}{dt} = y, \quad \frac{dy}{dt} = -x + \mu \sum_{j=0}^{3} h_j(x,\mu)y^j$$

depending on the real parameter μ. We are concerned with the inverse problem: How to construct the functions h_j such that the system has not more than a given number of limit cycles for μ belonging to some (global) interval. Our approach to treat this problem is based on the construction of suitable Dulac-Cherkas functions $\Psi(x,y,\mu)$ and exploiting the fact that in a simply connected region the number of limit cycles is not greater than the number of ovals contained in the set defined by $\Psi(x,y,\mu) = 0$.

1 Introduction

We consider the following class of planar autonomous differential systems depending on a real parameter μ

$$\frac{dx}{dt} = y, \quad \frac{dy}{dt} = -x + \mu \sum_{j=0}^{3} h_j(x,\mu)y^j, \tag{1}$$

where the functions $h_j : \mathbb{R} \times \mathbb{R} \to \mathbb{R}, j = 0, ..., 3,$ are continuous and continuously differentiable in the first variable, moreover we suppose

$$h_3(x,\mu) \not\equiv 0. \tag{2}$$

System (1) is a generalization of the polynomial system

$$\frac{dx}{dt} = y, \quad \frac{dy}{dt} = -x + \sum_{i+j=2}^{3} a_{ij}x^i y^j$$

which has been studied by I.S. Kukles (see e.g. [7]).

For $\mu = 0$, system (1) presents a linear conservative system having the first integral $x^2 + y^2 = c^2 > 0$, where c is any real number. If the parameter μ crosses zero, then the phenomenon can occur that from some circles $x^2 + y^2 = c_i^2$ limit cycles bifurcate. A famous example is the van der Pol equation

$$\ddot{x} + \mu(x^2 - 1)\dot{x} + x = 0, \tag{3}$$

where a unique limit cycle bifurcates from the circle $x^2 + y^2 = 2$ as μ crosses zero. Concerning this bifurcation problem the question arises: How many limit cycles of system (1) can bifurcate from the continuum of circles surrounding the origin as μ crosses zero.

In this paper we address some inverse problem: How to construct functions $h_j, j = 0, ..., 3$, such that system (1) has not more than a given number $l, l = 0, 1, ...$, of limit cycles for μ belonging to some (global) interval M. If M contains the value 0, then not more than l limit cycles can bifurcate from the continuum of circles with center at the origin as μ crosses 0. Our approach to treat this problem is based on the construction of suitable Dulac-Cherkas functions $\Psi(x, y, \mu)$ and exploiting the fact that in a simply connected region of the phase plane the number of limit cycles of systems (1) is not greater than the number of ovals contained in the set defined by $\Psi(x, y, \mu) = 0$.

The paper is organized as follows. In section 2 we recall some basic properties of a Dulac-Cherkas function. Section 3 contains the description of our general approach. In section 4 we construct systems (1) having no limit cycle, in section 5 we derive systems (1) possessing not more than one limit cycle. In section 6 we present conditions guaranteeing that the systems considered in section 5 have a unique limit cycle.

2 Preliminaries

We consider the planar differential system

$$\frac{dx}{dt} = P(x, y), \; \frac{dy}{dt} = Q(x, y) \tag{1}$$

in some open region $\mathcal{G} \subset \mathbb{R}^2$. First we recall the definition of a Dulac function.

Definition 2.1 *Let $P, Q \in C^1(\mathcal{G}, \mathbb{R})$, let X be the vector field defined by (1). A function $B \in C^1(\mathcal{G}, \mathbb{R})$ is called a Dulac function of system (1) in $\mathcal{G}$ if the expression*

$$div(BX) \equiv \frac{\partial(BP)}{\partial x} + \frac{\partial(BQ)}{\partial y} \equiv (gradB, X) + B\, divX$$

does not change sign in $\mathcal{G}$ and vanishes only on a set $\mathcal{N}$ of measure zero.

The existence of a Dulac function implies the following estimate of the number of limit cycles of system (1) in $\mathcal{G}$ [4].

Proposition 2.2 *Let $\mathcal{G}$ be a p-connected ($p \geq 1$) region in $\mathbb{R}^2$, let $P, Q \in C^1(\mathcal{G}, \mathbb{R})$. If there is a Dulac function B of (1) in $\mathcal{G}$, then (1) has not more than $p - 1$ limit cycles located entirely in $\mathcal{G}$.*

The method of Dulac function has been generalized in different ways. One possibility is to admit that B is not necessarily C^1 at any equilibrium of (1) provided the number of equilibria is finite in $\mathcal{G}$. This generalization has been established by the second author in 1968 (see [9]). Another generalization is due to L. A. Cherkas in 1997 (see [2]). The corresponding generalized Dulac function, which we called Dulac-Cherkas function in our paper [6], is defined as follows.

Definition 2.3 *Let $P, Q \in C^1(\mathcal{G}, \mathbb{R})$. A function $\Psi \in C^1(\mathcal{G}, \mathbb{R})$ is called a Dulac-Cherkas function of system (1) in $\mathcal{G}$ if there exists a real number $k \neq 0$ such that*

$$\Phi := (grad\, \Psi, X) + k\Psi\, div\, X > 0 \quad (< 0) \quad \textit{in} \quad \mathcal{G}. \tag{2}$$

Remark 2.4 *In case $k = 1$, Ψ is a Dulac function.*

Remark 2.5 *Condition (2) can be relaxed by assuming that Φ may vanish in $\mathcal{G}$ on a set of measure zero, and that no simply closed curve (oval) of this set is a limit cycle of (1).*

Remark 2.6 *In case that Φ vanishes identically in $\mathcal{G}$ we get from (2)*

$$\int_0^T divX(x_p(t), y_p(t))dt = 0,$$

where $(x_p(t), y_p(t))$ is a periodic solution of (1) with period T located entirely in $\mathcal{G}$. That means any closed trajectory of (1) located entirely in $\mathcal{G}$ belongs either to a continuum of closed orbits or is a multiple limit cycle.

For the sequel we introduce the subset $\mathcal{W}$ of $\mathcal{G}$ defined by

$$\mathcal{W} := \{(x, y) \in \mathcal{G} : \Psi(x, y) = 0\}. \tag{3}$$

The following theorem can be found in [2].

Theorem 2.7 *Let Ψ be a Dulac-Cherkas function of* (1) *in $\mathcal{G}$. Then any limit cycle Γ of* (1) *located entirely in $\mathcal{G}$ has the following properties:*

(i). *Γ does not intersect $\mathcal{W}$.*

(ii). *Γ is hyperbolic.*

(iii). *The stability of Γ is determined by the sign of the expression $k\Phi\Psi$ on Γ.*

Remark 2.8 *The existence of a Dulac-Cherkas function implies the non-existence of a multiple limit cycle.*

The following result about the upper number of limit cycles has been proved in our paper [6].

Theorem 2.9 *Let $\mathcal{G}$ be a p-connected region, let Ψ be a Dulac-Cherkas function of* (1) *in $\mathcal{G}$ such that $\mathcal{W}$ has s ovals in $\mathcal{G}$. Then system* (1) *has at most $p - 1 + s$ limit cycles in $\mathcal{G}$, and all limit cycles are hyperbolic.*

Remark 2.10 *In [6] it has been also shown that the differentiability conditions of Ψ in Theorem 2.9 can be weakened in the same manner as in case of a Dulac function.*

3 General approach

For the sequel we suppose $\mathcal{G} \subset \mathbb{R}^2$ to be a simply connected region containing the origin. If we assume that Ψ is a Dulac-Cherkas function of system (1) in $\mathcal{G}$, then Theorem 2.9 implies that the number of ovals of the set $\mathcal{W}$ in $\mathcal{G}$ gives an upper bound for the number of limit cycles of system (1) in $\mathcal{G}$. Since in case of system (1) the set $\mathcal{W}$ depends on the parameter μ, we use in the sequel the notation $\mathcal{W}_\mu$.
For the following we suppose that the Dulac-Cherkas function Ψ is a polynomial in y of degree n

$$\Psi(x, y, \mu) = \sum_{j=0}^{n} \Psi_j(x, \mu) y^j \tag{1}$$

with

$$\Psi_n(x, \mu) \not\equiv 0. \tag{2}$$

Then, the corresponding function Φ defined in (2) is in case of system (1) a polynomial in y of degree m

$$\Phi(x, y, \mu) = \sum_{i=0}^{m} \Phi_i(x, \mu) y^i, \tag{3}$$

where between n and m there holds the relation

$$m = n + 2. \tag{4}$$

From (2), (1) and (1) we get

$$\begin{aligned}
&\Phi(x, y, \mu) \equiv \Big(\Psi_0'(x, \mu) + \Psi_1'(x, \mu) y + ... + \Psi_n'(x, \mu) y^n\Big) y \\
&+ \Big(\Psi_1(x, \mu) + 2\Psi_2(x, \mu) y + ... + n\Psi_n(x, \mu) y^{n-1}\Big) \\
&\times \Big(- x + \mu\Big[h_0(x, \mu) + h_1(x, \mu) y + h_2(x, \mu) y^2 + h_3(x, \mu) y^3\Big]\Big) \\
&+ k\Big(\Psi_0(x, \mu) + \Psi_1(x, \mu) y + ... + \Psi_n(x, \mu) y^n\Big) \\
&\times \mu\Big(h_1(x, \mu) + 2h_2(x, \mu) y + 3h_3(x, \mu) y^2\Big) > 0 \ (< 0),
\end{aligned} \tag{5}$$

where the prime denotes differentiation with respect to x.
The key goal of this paper is to present a method for deriving conditions such that one of the inequalities in (5) is fulfilled in $\mathcal{G}$ for μ belonging to some interval M, that is, $\Psi(x, y, \mu)$ is a Dulac-Cherkas function of system (1) in $\mathcal{G}$ for $\mu \in M$. We treat the cases $n = 1$ in section 4 and $n = 2$ in section 5. Since in case $n = 1$ the set $\mathcal{W}_\mu$ contains no oval, we can conclude by Theorem 2.9 that section 4 is concerned with determining systems (1) having no limit cycle. In case $n = 2$ the set $\mathcal{W}_\mu$ contains at most one oval, thus we can conclude that the systems (1) considered in section 5 have at most one limit cycle. In section 6 we derive additional conditions such that the corresponding systems from section 5 have a unique limit cycle.

4 Construction of systems (1) with no limit cycle

In this section we study the case $n = 1$, that is, the functions Ψ and Φ have the representations

$$\Psi(x, y, \mu) = \Psi_0(x, \mu) + \Psi_1(x, \mu)y \tag{1}$$

with

$$\Psi_1(x, \mu) \not\equiv 0, \tag{2}$$

$$\Phi(x, y, \mu) = \sum_{i=0}^{3} \Phi_i(x, \mu)y^i. \tag{3}$$

The case $n = 1$ implies that the set

$$\mathcal{W}_\mu := \{(x, y) \in \mathcal{G} : \Psi_1(x, y, \mu) + \Psi_2(x, y, \mu)y = 0\} \tag{4}$$

has no oval. Thus, under the condition that $\Psi(x, \mu)$ is a Dulac-Cherkas function in the simply connected region $\mathcal{G}$, Theorem 2.9 implies that system (1) has no limit cycle in $\mathcal{G}$.

From (5) and (3) we obtain

$$\begin{aligned}
\Phi_3(x, \mu) &\equiv (1 + 3k)\mu h_3(x, \mu)\Psi_1(x, \mu), \\
\Phi_2(x, \mu) &\equiv \Psi_1'(x, \mu) + (1 + 2k)\mu h_2(x, \mu)\Psi_1(x, \mu) \\
&\quad + 3k\mu h_3(x, \mu)\Psi_0(x, \mu), \\
\Phi_1(x, \mu) &\equiv \Psi_0'(x, \mu) + (1 + k)\mu h_1(x, \mu)\Psi_1(x, \mu) \\
&\quad + 2k\mu h_2(x, \mu)\Psi_0(x, \mu).
\end{aligned} \tag{5}$$

Concerning the function Φ_0 we get

$$\Phi_0(x, \mu) \equiv -\Psi_1(x, \mu)x + \mu\Big(k\Psi_0(x, \mu)h_1(x, \mu) + \Psi_1(x, \mu)h_0(x, \mu)\Big). \tag{6}$$

We note that this relation is valid for any n.

To derive conditions on the coefficient functions h_j such that one of the inequalities in (5) is fulfilled we study in the following subsections the cases $\Phi(x, y, \mu) \equiv \Phi_0(x, \mu)$ and $\Phi(x, y, \mu) \equiv \Phi_0(x, \mu) + \Phi_2(x, \mu)y^2$.

4.1 Nonexistence of limit cycles if Φ does not depend on y

In this subsection we consider the case that the function Φ does not depend on y. Hence, taking into account (6), the inequalities in (5) read

$$\begin{aligned}
&\Phi(x, y, \mu) \equiv \Phi_0(x, \mu) \\
&\equiv -\Psi_1(x, \mu)x + \mu\Big(k\Psi_0(x, \mu)h_1(x, \mu) + \Psi_1(x, \mu)h_0(x, \mu)\Big) > 0 \ (< 0).
\end{aligned} \tag{7}$$

Since the inequalities should hold also for small μ we have to assume

$$-\Psi_1(x,\mu)x > 0 \ (< 0) \quad \text{for} \quad x \neq 0.$$

Hence, for the following we set

$$\Psi_0(x,\mu) := q \neq 0, \quad \Psi_1(x,\mu) := \mu x, \tag{8}$$

such that we have

$$\Psi(x,\mu) = q + \mu x.$$

Using these relations we get that the inequalities in (7) read as

$$\Phi(x,y,\mu) \equiv \Phi_0(x,\mu) \equiv -\mu(x^2 - kqh_1(x,\mu) - \mu x h_0(x,\mu)) > 0 \ (< 0), \tag{9}$$

and that by (5) the relations $\Phi_i \equiv 0, 1 \leq i \leq 3,$ take the form

$$\begin{aligned} &\Phi_3(x,\mu) \equiv (1+3k)\mu^2 h_3(x,\mu)x \equiv 0, \\ &\Phi_2(x,\mu) \equiv \mu + (1+2k)\mu^2 h_2(x,\mu)x + 3k\mu q h_3(x,\mu) \equiv 0, \\ &\Phi_1(x,\mu) \equiv (1+k)\mu^2 h_1(x,\mu)x + 2k\mu q h_2(x,\mu) \equiv 0. \end{aligned} \tag{10}$$

To satisfy the relations (9) and (10) we derive conditions on k and the functions h_j.
Taking into account (2) and (2), we get from (10) that the relation $\Phi_3(x,\mu) \equiv 0$ is equivalent for $\mu x \neq 0$ to

$$k = -\frac{1}{3}. \tag{11}$$

Using (11) we obtain from (10) that the relation $\Phi_1(x,\mu) \equiv 0$ is for $\mu \neq 0$ equivalent to

$$\mu x h_1(x,\mu) - q h_2(x,\mu) \equiv 0$$

from which we get

$$h_2(x,\mu) := \frac{\mu x h_1(x,\mu)}{q}. \tag{12}$$

Taking into account (11) and (12) we obtain from (10) that the relation $\Phi_2(x,\mu) \equiv 0$ is fulfilled if we define h_3 by

$$h_3(x,\mu) := \frac{3q + \mu^2 x^2 h_1(x,\mu)}{3q^2}. \tag{13}$$

Finally, we note that the inequalities in (9) read

$$\Phi(x,y,\mu) \equiv \Phi_0(x,\mu) \equiv -\mu\big(x^2 + \frac{q}{3}h_1(x,\mu) - \mu x h_0(x,\mu)\big) > 0 \ (< 0). \tag{14}$$

Taking into account Remark 2.5 and that system (1) has no limit cycle for $\mu = 0$, we have the result:

Theorem 4.1 *Let q be any given real number different from zero, let $h_0, h_1 : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ be continuous functions, let h_2 and h_3 be defined by (12) and (13), respectively. If there exists an interval M such that for $\mu \in M$ the expression*

$$-x^2 - \frac{q}{3}h_1(x,\mu) + \mu x h_0(x,\mu)$$

has the same sign for all $x \in \mathbb{R}$ and does not vanish identically for any x-interval, then system (1) has no limit cycle for $\mu \in M$.

Proof. Under the assumptions of Theorem 4.1, the function $\Psi(x,\mu)$ defined by (1) and (8) is for $\mu \in M$ a Dulac-Cherkas function in the phase plane and the corresponding set $\mathcal{W}_\mu$ contains no oval. Applying Theorem 2.9 the proof is complete.

□

As an example we consider the case

$$q = -3, \quad h_1(x,\mu) \equiv x^2. \tag{15}$$

By (14) we have

$$\Phi(x,y,\mu) \equiv \mu^2 x h_0(x,\mu),$$

and we can conclude that the function $\Psi(x,\mu) \equiv q + \mu xy$ is a Dulac-Cherkas function for system (1) with

$$h_1(x,\mu) \equiv x^2, \quad h_2(x,\mu) \equiv -\frac{\mu x^3}{3}, \quad h_3(x,\mu) \equiv \frac{-9+\mu^2x^4}{27}$$

in the phase plane for $\mu \neq 0$ under the condition that for given $\mu \neq 0$ $xh_0(x,\mu)$ does not change sign for $x \in \mathbb{R}$ and does not vanish identically for any x-interval. Since the set $\mathcal{W}_\mu$ contains no oval, we have the result:

Corollary 4.2 *The autonomous system*

$$\begin{aligned}\frac{dx}{dt} &= y,\\ \frac{dy}{dt} &= -x + \mu\Big(h_0(x,\mu) + x^2y - \frac{\mu}{3}x^3y^2 + \frac{-9+\mu^2x^4}{27}y^3\Big)\end{aligned}$$

has no limit cycle for any μ provided that for any $\mu \neq 0$ the function $xh_0(x,\mu)$ does not change sign for $x \in \mathbb{R}$ and does vanish identically for any x-interval.

The way we used to derive conditions for system (1) to have no limit cycle can be characterized as an algebraic method: we prescribe Ψ_0 and Ψ_1 and determine conditions for the coefficient functions h_j, $0 \leq j \leq 3$, by solving the identities for $\Phi_3(x,\mu)$, $\Phi_2(x,\mu)$, $\Phi_1(x,\mu)$ in (10) and the inequality $\Phi_0(x,\mu) > 0\ (< 0)$ in (9).

Now we describe another approach based on a combination of the approach used above and the method used in our paper [3]. As in the preceding approach we first determine the number

k in order to satisfy the identity $\Phi_3(x,\mu) \equiv 0$. Then we solve the identities $\Phi_2(x,\mu) \equiv 0$ and $\Phi_1(x,\mu) \equiv 0$ as a system of non-homogeneous linear differential equations for Ψ_0 and Ψ_1. In general it is not possible to get an explicit solution of this system. Under the assumption that we are able to obtain a solution of that system as a function of the coefficient functions h_j, we can plug in this solution into the inequality (7). By this way we derive conditions on the coefficient functions h_j implying that Ψ is a Dulac-Cherkas function. We call this approach an algebraic-differential approach.

As an example we consider system (1) under the condition

$$h_2(x,\mu) \equiv 0. \tag{16}$$

From the first identity in (10) we get $k = -1/3$, the identities for Φ_2 and Φ_1 read

$$\begin{aligned} \Phi_2(x,\mu) &\equiv \Psi_1'(x,\mu) - \mu h_3(x,\mu)\Psi_0(x,\mu) \equiv 0, \\ \Phi_1(x,\mu) &\equiv \Psi_0'(x,\mu) + \frac{2}{3}\mu h_1(x,\mu)\Psi_1(x,\mu) \equiv 0. \end{aligned} \tag{17}$$

We consider (17) as a system of linear homogeneous differential equations to determine Ψ_0 and Ψ_1. If we look for a solution of system (17) satisfying

$$\Psi_1(x,\mu) \equiv \kappa\Psi_0(x,\mu), \tag{18}$$

where κ is some constant which can depend on the parameter μ, we obtain the condition

$$h_3(x,\mu) \equiv -\frac{2}{3}\kappa^2 h_1(x,\mu). \tag{19}$$

Therefore, we get from the last differential equation in (17) the special solution

$$\Psi_0(x,\mu) \equiv \exp\Big(-\frac{2}{3}\mu\kappa\int^x h_1(\xi,\mu)d\xi\Big) \tag{20}$$

which is always positive. Taking into account (18) we obtain from (7)

$$\Phi_0(x,\mu) \equiv \Psi_0(x,\mu)\Big(-\kappa x - \frac{1}{3}\mu h_1(x,\mu) + \mu\kappa h_0(x,\mu)\Big). \tag{21}$$

Setting

$$\kappa = -\frac{1}{3}\mu,$$

and

$$h_1(x,\mu) \equiv x \tag{22}$$

we obtain from (20), (21), (1) and (18)

$$\Psi_0(x,\mu) = \exp\Big(\frac{\mu^2}{9}x^2\Big), \quad \Phi_0(x,\mu) = -\frac{\mu^2}{3}\Psi_0(x,\mu)h_0(x,\mu),$$

$$\Psi(x,y,\mu) = \Psi_0(x,\mu)\Big(1 - \frac{\mu}{3}y\Big).$$

Thus, the function

$$\Psi(x,\mu) := \exp\Big(\frac{\mu^2 x^2}{9}\Big)\Big(1-\frac{\mu}{3}y\Big)$$

is a Dulac-Cherkas function for the system

$$\begin{aligned}\frac{dx}{dt} &= y,\\ \frac{dy}{dt} &= -x+\mu\Big(h_0(x,\mu)+xy-\frac{2}{27}\mu^2xy^3\Big)\end{aligned} \tag{23}$$

under the hypothesis

(H_0). $h_0 : \mathbb{R}\times\mathbb{R}\to\mathbb{R}$ is continuous. For any $\mu\in\mathbb{R}$

(i). $h_0(x,\mu)$ does not change sign for $x\in\mathbb{R}$.

(ii). There is no interval I_x such that $h_0(x,\mu)$ vanishes identically for $x\in I_x$.

As the set $\mathcal{W}_\mu$ contains no oval, we have the result:

Theorem 4.3 *Under the assumption (H_0), the autonomous system (23) has no limit cycle for any μ.*

In the next subsection we consider the case that the function Φ is an even function of y.

4.2 Nonexistence of limit cycles if Φ_3 and Φ_1 vanish identically

In what follows we assume the identities $\Phi_3\equiv 0$ and $\Phi_1\equiv 0$ to be satisfied such that we have

$$\Phi(x,y,\mu)=\Phi_0(x,\mu)+\Phi_2(x,\mu)y^2. \tag{24}$$

As in the subsection before, we suppose that $\Psi_0(x,\mu)$ and $\Psi_1(x,\mu)$ are defined by (8) such that we have

$$\Psi(x,y,\mu)\equiv q+\mu xy.$$

Solving the identities $\Phi_3\equiv 0$ and $\Phi_1\equiv 0$ in (10) we get the relations (11) and (12). Using these relations we obtain from (10) and (9)

$$\Phi_2(x,\mu)\equiv \mu\Big(1-qh_3(x,\mu)+\frac{\mu^2}{3q}x^2h_1(x,\mu)\Big), \tag{25}$$

$$\Phi_0(x,\mu)\equiv \mu\Big(-x^2-\frac{q}{3}h_1(x,\mu)+\mu xh_0(x,\mu)\Big). \tag{26}$$

By (24) we can conclude that the relation

$$\Phi_2(x,\mu)\Phi_0(x,\mu)\geq 0, \tag{27}$$

is a sufficient condition for Φ to have the same sign. Using (25) and (26) this inequality reads

$$\begin{aligned} \mu^2\Big(-x^2-\frac{q}{3}h_1(x,\mu)+\mu x h_0(x,\mu)\Big) \\ \times\Big(1-qh_3(x,\mu)+\frac{\mu^2}{3q}x^2h_1(x,\mu)\Big)\geq 0. \end{aligned} \tag{28}$$

Analogously to Theorem 4.1 we can prove the following theorem.

Theorem 4.4 *Let q be any given real number different from zero, let $h_0, h_1, h_3 : \mathbb{R}\times\mathbb{R}\to\mathbb{R}$ be continuous functions, let the function h_2 be defined by (12). Suppose the existence of an interval M such that for $\mu\in M$*

- (i). *There is no interval I_x such that Φ_0 and Φ_2 vanish identically for $x\in I_x$.*
- (ii). *The inequality (28) is valid for all $x\in\mathbb{R}$.*

Then system (1) has no limit cycle for $\mu\in M$.

In the special case (15), that is, $q=-3$ and $h_1(x,\mu)\equiv x^2$, we have

$$-x^2-\frac{q}{3}h_1(x,\mu)+\mu x h_0(x,\mu)\equiv \mu x h_0(x,\mu)$$

and

$$1-qh_3(x,\mu)+\frac{\mu^2}{3q}x^2h_1(x,\mu)\equiv 1+3h_3(x,\mu)-\frac{\mu^2}{9}x^4.$$

For the formulation of the following result we introduce the assumption

(H_1). $h_0, h_3 : \mathbb{R}\times\mathbb{R}\to\mathbb{R}$ are continuous functions. There is an interval M such that for $\mu\in M$

- (i). There is no interval I_x such that $h_0(x,\mu)$ and $1+3h_3(x,\mu)-\frac{\mu^2}{9}x^4$ vanish identically for $x\in I_x$.
- (ii). The inequality
$$\mu x h_0(x,\mu)\Big(1+3h_3(x,\mu)-\frac{\mu^2}{9}x^4\Big)\geq 0$$
is valid for $x\in\mathbb{R}$.

Corollary 4.5 *Under the hypothesis (H_1), the autonomous system*

$$\begin{aligned} \frac{dx}{dt} &= y, \\ \frac{dy}{dt} &= -x+\mu\Big(h_0(x,\mu)+x^2y-\frac{\mu x^3}{3}y^2+h_3(x,\mu)y^3\Big) \end{aligned} \tag{29}$$

has no limit cycle.

As in subsection 4.1, we now apply the algebraic-differential approach to derive conditions on Ψ and the functions h_j such that system (1) has no limit cycle. As we noted above, the identity $\Phi_3(x,\mu) \equiv 0$ is equivalent to $k = -1/3$. Concerning the function $\Psi = \Psi_0(x,\mu) + \Psi_1(x,\mu)y$ we assume $\Psi_1(x,\mu) = \kappa\Psi_0(x,\mu)$, where κ is some constant which can depend on μ. We determine Ψ_0 by means of the identity $\Phi_1(x,\mu) \equiv 0$ in (5) which reads

$$\Psi_0' = \frac{2\mu}{3}\Big(h_2(x,\mu) - \kappa h_1(x,\mu)\Big)\Psi_0. \tag{30}$$

A special solution of this differential equation takes the form

$$\Psi_0(x,\mu) = \exp\Big(\frac{2\mu}{3}\int^x (h_2(\xi,\mu) - \kappa h_1(\xi,\mu))d\xi\Big).$$

Using this solution and setting $\kappa = \mu$ we get from (6), (5) and (30)

$$\Phi_0(x,\mu) \equiv \mu\Big(-x - \frac{1}{3}h_1(x,\mu) + \mu h_0(x,\mu)\Big)\Psi_0(x,\mu), \tag{31}$$

$$\Phi_2(x,\mu) \equiv \mu\Big(\mu h_2(x,\mu) - \frac{2}{3}\mu^2 h_1(x,\mu) - h_3(x,\mu)\Big)\Psi_0(x,\mu). \tag{32}$$

For the following we assume

(H_2). $h_0, h_1, h_2, h_3 : \mathbb{R}\times\mathbb{R} \to \mathbb{R}$ are continuous functions. There is an interval M such that for $\mu \in M$

- (i). There is no interval I_x such that the functions $-x - \frac{1}{3}h_1(x,\mu) + \mu h_0(x,\mu)$ and $\mu h_2(x,\mu) - \frac{2}{3}\mu^2 h_1(x,\mu) - h_3(x,\mu)$ vanish identically for $x \in I_x$.
- (ii). The inequality

$$\Big(-x - \frac{1}{3}h_1(x,\mu) + \mu h_0(x,\mu)\Big)\Big(\mu h_2(x,\mu) - \frac{2}{3}\mu^2 h_1(x,\mu) - h_3(x,\mu)\Big) \geq 0$$

  is valid for $x \in \mathbb{R}$.

Under this assumption, $\Psi(x,\mu) \equiv \Psi_0(x,\mu)(1+\mu y)$ is for $\mu \in M$ a Dulac-Cherkas function for system (1) in the phase plane, and we have the result

Theorem 4.6 *Under the hypothesis (H_2), system (1) has for $\mu \in M$ no limit cycle.*

5 Construction of systems (1) having at most one limit cycle

In this section we consider the case $n = 2$, that is, we use the representations

$$\Psi(x,y,\mu) = \Psi_0(x,\mu) + \Psi_1(x,\mu)y + \Psi_2(x,\mu)y^2, \tag{1}$$

$$\Phi(x,y,\mu) = \sum_{i=0}^{4} \Phi_i(x,\mu)y^i. \tag{2}$$

The case $n = 2$ implies that the set

$$\mathcal{W}_\mu := \{(x,y) \in \mathcal{G} : \Psi(x,y,\mu) = 0\} \tag{3}$$

consists of at most one oval. In the following subsections we consider the case that $\mathcal{W}_\mu$ consists of an oval.

We obtain from (5) and (2) the relations

$$\begin{aligned}
&\Phi_4(x,\mu) \equiv \ (2+3k)\mu h_3(x,\mu)\Psi_2(x,\mu),\\
&\Phi_3(x,\mu) \equiv \ \Psi_2'(x,\mu)\\
&\ + (2k+2)\mu h_2(x,\mu)\Psi_2(x,\mu) + (1+3k)\mu h_3(x,\mu)\Psi_1(x,\mu),\\
&\Phi_2(x,\mu) \equiv \ \Psi_1'(x,\mu) + (1+2k)\mu h_2(x,\mu)\Psi_1(x,\mu)\\
&\ + (2+k)\mu h_1(x,\mu)\Psi_2(x,\mu) + 3k\mu h_3(x,\mu)\Psi_0(x,\mu),\\
&\Phi_1(x,\mu) \equiv \ \Psi_0'(x,\mu) + 2k\mu h_2(x,\mu)\Psi_0(x,\mu)\\
&\ + (k+1)\mu h_1(x,\mu)\Psi_1(x,\mu) + 2\mu h_0(x,\mu)\Psi_2(x,\mu) - 2x\Psi_2(x,\mu).
\end{aligned} \tag{4}$$

Concerning the function Φ_0 we have the same expression as in (6).

To derive conditions on the functions h_j such that one of the inequalities in (5) is fulfilled, we study in the following subsections the cases

$$\begin{aligned}
&\Phi(x,y,\mu) \equiv \Phi_0(x,\mu),\\
&\Phi(x,y,\mu) \equiv \Phi_0(x,\mu) + \Phi_2(x,\mu)y^2,\\
&\Phi(x,y,\mu) \equiv \Phi_0(x,\mu) + \Phi_2(x,\mu)y^2 + \Phi_4(x,\mu)y^4.
\end{aligned}$$

In all cases we apply the algebraic approach, that is, we prescribe the function $\Psi(x,y,\mu)$.

5.1 Existence of at most one limit cycle if Φ does not depend on y

In that case we have

$$\Phi(x,y,\mu) \equiv \Phi_0(x,\mu),$$

$$\Phi_1(x,\mu) \equiv \Phi_2(x,\mu) \equiv \Phi_3(x,\mu) \equiv \Phi_4(x,\mu) \equiv 0.$$

Concerning Ψ we assume

$$\Psi(x,y,\mu) \equiv px^2 - c + \mu xy + py^2, \tag{5}$$

that is

$$\Psi_0(x,\mu) \equiv px^2 - c, \quad \Psi_1(x,\mu) \equiv \mu x, \quad \Psi_2(x,\mu) \equiv p. \tag{6}$$

The set $\mathcal{W}_\mu$ is defined by

$$\mathcal{W}_\mu := \{(x,y) \in \mathbb{R}^2 : px^2 + \mu xy + py^2 = c\}. \tag{7}$$

Thus, under the conditions

$$p > 0, \quad 4p^2 - \mu^2 > 0, \quad c > 0 \tag{8}$$

the set $\mathcal{W}_\mu$ consists exactly of one oval which is an ellipse.

By (2), (2) and (4) we get from the identity $\Phi_4(x,y,\mu) \equiv 0$

$$k = -\frac{2}{3}. \tag{9}$$

Taking into account (9) and (6), we obtain by (4) from the identity $\Phi_3(x,\mu) \equiv 0$

$$h_2(x,\mu) := \frac{3}{2p}\mu x h_3(x,\mu). \tag{10}$$

Using this relation, the identity $\Phi_2(x,\mu) \equiv 0$ is satisfied if we define h_1 by

$$h_1(x,\mu) := \frac{3}{8p^2}\Big(4ph_3(x,\mu)(px^2-c) + h_3(x,\mu)\mu^2x^2 - 2p\Big). \tag{11}$$

Then, by (4) the identity $\Phi_1(x,\mu) \equiv 0$ is valid if we define h_0 by

$$\begin{aligned} h_0(x,\mu) := \frac{\mu}{16p^3}\Big(&12ph_3(x,\mu)x(px^2-c) \\ &- \mu^2 h_3(x,\mu)x^3 + 2px\Big). \end{aligned} \tag{12}$$

Taking into account (12) and (11), we get from (6)

$$\Phi_0(x,\mu) \equiv \frac{\mu}{16p^3}\tilde{\Phi}_0(x,\mu),$$

where

$$\tilde{\Phi}_0(x,\mu) \equiv -x^4 h_3(x,\mu)(4p^2-\mu^2)^2 - x^2 2p(1-4ch_3(x,\mu))(4p^2-\mu^2)$$

$$-8p^2c(1+2ch_3(x,\mu)).$$

A detailed analysis of this expression shows that under the conditions (8) and $h_3(x,\mu) > \frac{1}{16c}$ for $(x,\mu) \in \mathbb{R} \times (-2p, 2p)$, the function $\tilde{\Phi}_0(x,\mu)$ is always negative for $(x,\mu) \in \mathbb{R} \times (-2p, 2p)$. Thus, it holds

Lemma 5.1 *Suppose the following conditions are satisfied:*

(A_1). *Let c and p be given positive numbers, let μ be a number of the interval $(-2p, 2p)$.*

(A_2). *Let $h_3 : \mathbb{R} \times (-2p, 2p) \to \mathbb{R}$ be a continuous function satisfying*

$$h_3(x,\mu) > \frac{1}{16c} \quad \textit{for} \quad (x,\mu) \in \mathbb{R} \times (-2p, 2p). \tag{13}$$

Then the function $\Phi_0(x,\mu)$ is negative (positive) definite for $(x,\mu) \in \mathbb{R} \times (0, 2p)$ $\Big((x,\mu) \in \mathbb{R} \times (-2p, 0)\Big)$.

If what follows we additionally suppose

(A_3). For $j = 0, 1, 2$, the functions $h_j : \mathbb{R} \times (-2p, 2p) \to \mathbb{R}$ are defined by (12), (11) and (10), respectively.

Then we can conclude that under the assumptions $(A_1) - (A_3)$ the function Ψ defined in (5) is for $\mu \in (-2p, 2p) \setminus \{0\}$ a Dulac-Cherkas function for system (1) in the phase plane, and the set $\mathcal{W}_\mu$ consists of exactly one oval. Thus, according to Theorem 2.7 and Theorem 2.9, and the fact that system (1) has for $\mu = 0$ a continuum of circles centered at the origin as orbits, we have the following result:

Theorem 5.2 *Under the assumptions $(A_1) - (A_3)$, system (1) has for $\mu \in (-2p, 2p)$ at most one limit cycle in the phase plane. If system (1) has a limit cycle Γ_μ, then it is hyperbolic and contains the ellipse $\mathcal{W}_\mu$ in its interior.*

5.2 Existence of at most one limit cycle if Φ_4, Φ_3 and Φ_1 vanish identically

In the case under consideration we have

$$\Phi(x,y,\mu) \equiv \Phi_0(x,\mu) + \Phi_2(x,\mu)y^2, \tag{14}$$

$$\Phi_1(x,\mu) \equiv \Phi_3(x,\mu) \equiv \Phi_4(x,\mu) \equiv 0.$$

Concerning the function Ψ we assume

$$\Psi(x, y, \mu) \equiv px^2 + py^2 - c, \tag{15}$$

that is,

$$\Psi_0(x, \mu) \equiv px^2 - c, \quad \Psi_1(x, \mu) \equiv 0, \quad \Psi_2(x, \mu) \equiv p, \tag{16}$$

where p and c are positive numbers.

As in the subsection before we get from the identity $\Phi_4(x, \mu) \equiv 0$ by (4) the relation $k = -2/3$. According to (16) we obtain from $\Phi_3(x, \mu) \equiv 0$ and $\Phi_1(x, \mu) \equiv 0$

$$h_2(x, \mu) \equiv 0, \tag{17}$$

$$h_0(x, \mu) \equiv 0, \tag{18}$$

respectively.

Taking into account the relations obtained before we get from (4) and (6)

$$\Phi_2(x, \mu) \equiv \mu\Big(\frac{4}{3}h_1(x, \mu)p - 2h_3(x, \mu)(px^2 - c)\Big), \tag{19}$$

$$\Phi_0(x, \mu) \equiv \mu\Big(-\frac{2}{3}(px^2 - c)h_1(x, \mu)\Big). \tag{20}$$

For the following we assume

(H_3). There are intervals M_i, $i = 1, 2, ...$, such that for $\mu \in M_i$ the following conditions are satisfied:

(i). The function $\Phi_0(x, \mu)$ does not change sign for $x \in \mathbb{R}$.

(ii). $\Phi_2(x, \mu)$ has the same sign as $\Phi_0(x, \mu)$ for $x \in \mathbb{R}$.

(iii). There is no interval I_x such that $\Phi_0(x, \mu)$ and $\Phi_2(x, \mu)$ vanish identically for $x \in I_x$.

Under this assumption, the function Ψ defined in (15) is for $\mu \in M_i$, $i = 1, 2, ...$, a Dulac-Cherkas function of (1) in the phase plane, and we have the result:

Theorem 5.3 *Let the functions $h_1, h_3 : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ be continuous, let h_0 and h_2 be defined by (18) and (17). Under the assumption (H_3), system (1) has for $\mu \in M_i$, $i = 1, 2, ...$, at most one limit cycle. If such a limit cycle exists, then it is hyperbolic and contains the oval $\mathcal{W}_\mu$ in its interior.*

In the special case

$$h_1(x,\mu) := px^2 - c, \quad h_3(x,\mu) := px^2 - c + \frac{2}{3}p \tag{21}$$

it holds

$$\Phi_2(x,\mu) \equiv -2\mu(px^2 - c)^2, \quad \Phi_0(x,\mu) \equiv -\frac{2}{3}\mu(px^2 - c)^2. \tag{22}$$

Therefore, we have the result:

Corollary 5.4 *The autonomous system*

$$\begin{aligned}\frac{dx}{dt} &= y,\\ \frac{dy}{dt} &= -x + \mu\Big[(px^2 - c)y + \Big(px^2 - c + \frac{2}{3}p\Big)y^3\Big]\end{aligned} \tag{23}$$

has for any positive numbers p and c and $\mu \in \mathbb{R}$ at most one limit cycle.

5.3 Existence of at most one limit cycle if Φ_3 and Φ_1 vanish identically

In this case we have

$$\Phi(x,y,\mu) \equiv \Phi_0(x,\mu) + \Phi_2(x,\mu)y^2 + \Phi_4(x,\mu)y^4, \tag{24}$$

$$\Phi_1(x,\mu) \equiv \Phi_3(x,\mu) \equiv 0.$$

As Ψ we choose the function

$$\Psi(x,y,\mu) := x^2 + y^2 - 1, \tag{25}$$

that is

$$\Psi_0(x,\mu) \equiv x^2 - 1, \quad \Psi_1(x,\mu) \equiv 0, \quad \Psi_2(x,\mu) \equiv 1, \tag{26}$$

and the set $\mathcal{W}_\mu$ consists of the unit circle.
By (26) and (4) we have

$$\begin{aligned}\Phi_3(x,\mu) &\equiv (2k+2)\mu h_2(x,\mu),\\ \Phi_1(x,\mu) &\equiv 2\mu\Big(kh_2(x,\mu)(x^2-1) + h_0(x,\mu)\Big).\end{aligned} \tag{27}$$

To fulfill the identity $\Phi_3(x,\mu) \equiv 0$ we choose

$$k = -1. \tag{28}$$

The identity $\Phi_1(x,\mu) \equiv 0$ holds if we set

$$h_0(x,\mu) := h_2(x,\mu)(x^2 - 1). \tag{29}$$

From (4), (26), (28) and (29) we obtain

$$\Phi_4(x,\mu) \equiv -\mu h_3(x,\mu), \quad \Phi_2(x,\mu) \equiv \mu h_1(x,\mu) - 3\mu h_3(x,\mu)(x^2-1), \\ \Phi_0(x,\mu) \equiv -\mu h_1(x,\mu)(x^2-1). \tag{30}$$

Our goal is to derive conditions on the functions h_j such that Φ does not change sign. For this purpose we introduce the assumption

(H_4). There are intervals $M_i, i = 1, 2, ...,$ such that for $\mu \in M_i$ one of the following conditions is satisfied:

(i). The functions $\Phi_0(x,\mu)$ and $\Phi_4(x,\mu)$ do not change sign for $x \in \mathbb{R}$, the functions $\Phi_0(x,\mu)$, $\Phi_2(x,\mu)$ and $\Phi_4(x,\mu)$ have the same sign for $x \in \mathbb{R}$, there is no interval I_x such that $\Phi_0(x,\mu)$, $\Phi_2(x,\mu)$ and $\Phi_4(x,\mu)$ simultaneously vanish identically for $x \in I_x$.

(ii). The functions $\Phi_0(x,\mu)$ and $\Phi_4(x,\mu)$ do not change sign for $x \in \mathbb{R}$, the inequality

$$\Phi_2^2(x,\mu) - 4\Phi_0(x,\mu)\Phi_4(x,\mu) \le 0$$

holds for $x \in \mathbb{R}$, and there is no interval I_x such that $\Phi_0(x,\mu)$, $\Phi_2(x,\mu)$ and $\Phi_4(x,\mu)$ vanish identically for $x \in I_x$.

Under this assumption, the function Ψ defined in (25) is for $\mu \in M_i$, $i = 1, 2, ...,$ a Dulac-Cherkas function for (1) in the phase plane, and we have the result:

Theorem 5.5 *Let $h_1, h_2, h_3 : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ be continuous functions, let the function h_0 be defined by (29). Suppose assumption (H_4) to be valid. Then system (1) has for $\mu \in M_i$, $i = 1, 2, ...,$ at most one limit cycle. If such a limit cycle exists, then it is hyperbolic and contains the unit circle in its interior.*

For the special case

$$h_1(x,\mu) := x^2 - 1 \tag{31}$$

and

$$h_3(x,\mu) := \frac{x^2}{3} \tag{32}$$

we get from (30) that condition (i) from the assumption (H_4) is satisfied.

In the special case (31) the condition (ii) in assumption (H_4) are fulfilled provided the inequality

$$\frac{1}{9} \le h_3(x,\mu) \le 1 \tag{33}$$

holds.

6 Conditions for the existence of a unique limit cycle

In section 5 we derived conditions on the functions h_j such that the corresponding system (1) has at most one limit cycle. In this section we will show that if we improve the smoothness of the functions h_j with respect to μ, then we are able to derive sufficient conditions for the existence of a unique limit cycle. Our approach is based on a known perturbation (bifurcation) theorem. To be able to formulate the corresponding result we introduce the following condition:

(A). The functions $h_j : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$, $0 \leq j \leq 3$, can be represented in the form

$$h_j(x, \mu) = h_j^0(x) + \tilde{h}_j(x, \mu)\mu,$$

where $h_j^0(x) := h_j(x, 0)$, and the functions $\tilde{h}_j : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ are continuous.

Under this assumption, system (1) can be written in the following form

$$\frac{dx}{dt} = y, \quad \frac{dy}{dt} = -x + \mu q(x, y) + \mu^2 h(x, y, \mu), \tag{34}$$

where

$$q(x, y) := \sum_{j=0}^{3} h_j^0(x) y^j, \quad h(x, y, \mu) := \sum_{j=0}^{3} \tilde{h}_j(x, \mu) y^j.$$

The application of a well-known theorem (see [1], Theorem 75) implies the result:

Theorem 6.1 *Suppose the assumption (A) to be valid. If the equation (r, φ are polar coordinates)*

$$\int_0^{2\pi} q(r\cos\varphi, r\sin\varphi)\sin\varphi \, d\varphi = 0 \tag{35}$$

has a positive root $r = r_*$ satisfying

$$\int_0^{2\pi} \frac{\partial q(r_*\cos\varphi, r_*\sin\varphi)}{\partial y} \, d\varphi \neq 0, \tag{36}$$

then system (34) has for sufficiently small μ a unique limit cycle near the circle centered at the origin with radius r_* which is hyperbolic.

In the following subsections we apply this result to the autonomous systems studied in the subsections of section 5.

6.1 Existence of a unique limit cycle in the class of systems (1) considered in subsection 5.1

In section 5.1 we studied system (1) by means of the function

$$\Psi(x, y, \mu) \equiv px^2 - c + \mu xy + py^2,$$

where the functions h_0, h_1, h_2 are defined by the function h_3 (see (10), (11), (12)). For the sequel we suppose

(A_4). The function h_3 can be written in the form

$$h_3(x, \mu) = h_3^0(x) + \mu\tilde{h}_3(x, \mu),$$

where $\tilde{h}_3 : \mathbb{R} \times (-2p, 2p) \to \mathbb{R}$ is continuous.

Thus, assumption (A) is fulfilled. Taking into account (10), (11), (12), it holds

$$h_0^0(x) \equiv h_2^0(x) \equiv 0, h_1^0(x) \equiv \frac{3}{4p}\big[h_3^0(x)2(px^2 - c) - 1\big]y + h_3^0(x)y^3.$$

Hence, we have

$$q(x, y) := \frac{3}{4p}\big[h_3^0(x)2(px^2 - c) - 1\big]y + h_3^0(x)y^3. \tag{37}$$

Now we consider equation (35) and inequality (36), where the function q is defined in (37). We get

$$\begin{aligned}
&\int_0^{2\pi} q(r\cos\varphi, r\sin\varphi)\sin\varphi\, d\varphi \\
&= \int_0^{2\pi} \Big(\frac{3}{4p}\Big[2h_3^0(r\cos\varphi)(pr^2\cos^2\varphi - c) - 1\Big]r\sin\varphi + h_3^0(r\cos\varphi)r^3\sin^3\varphi\Big)\sin\varphi\, d\varphi \\
&= -\frac{3r}{4p}\int_0^{2\pi}\sin^2\varphi\, d\varphi - \frac{3cr}{2p}\int_0^{2\pi} h_3^0(r\cos\varphi)\sin^2\varphi\, d\varphi \\
&+ \frac{3r^3}{2}\int_0^{2\pi} h_3^0(r\cos\varphi)\sin^2\varphi\cos^2\varphi\, d\varphi + r^3\int_0^{2\pi} h_3^0(r\cos\varphi)\sin^4\varphi\, d\varphi \\
&= -\frac{3r}{2p}\Big(\frac{\pi}{2} + c\int_0^{2\pi} h_3^0(r\cos\varphi)\sin^2\varphi\, d\varphi\Big) \\
&+ \frac{r^3}{2}\Big(3\int_0^{2\pi} h_3^0(r\cos\varphi)\sin^2\varphi\cos^2\varphi\, d\varphi + 2\int_0^{2\pi} h_3^0(r\cos\varphi)\sin^4\varphi\, d\varphi\Big) = 0,
\end{aligned} \tag{38}$$

$$
\begin{aligned}
&\int_0^{2\pi} \frac{\partial q(r_* \cos\varphi, r_* \sin\varphi)}{\partial y}\, d\varphi \\
&= -\frac{3\pi}{2p} - \frac{3c}{2p}\int_0^{2\pi} h_3^0(r_*\cos\varphi)\, d\varphi + \frac{3r_*^2}{2}\int_0^{2\pi} h_3^0(r_*\cos\varphi)\cos^2\varphi d\varphi \\
&+ 3r_*^2\int_0^{2\pi} h_3^0(r_*\cos\varphi)\sin^2\varphi\, d\varphi \neq 0.
\end{aligned} \tag{39}
$$

For the following we assume

(A_5). The equation

$$
r^2\Big(3\int_0^{2\pi} h_3^0(r\cos\varphi)\sin^2\varphi\cos^2\varphi\, d\varphi + 2\int_0^{2\pi} h_3^0(r\cos\varphi)\sin^4\varphi\, d\varphi\Big)
$$

$$
= \frac{3}{p}\Big(\frac{\pi}{2} + c\int_0^{2\pi} h_3^0(r\cos\varphi)\sin^2\varphi\, d\varphi\Big)
$$

has a positive root r_* satisfying

$$
-\frac{3\pi}{2p} - \frac{3c}{2p}\int_0^{2\pi} h_3^0(r_*\cos\varphi)\, d\varphi
$$

$$
+\frac{3r_*^2}{2}\int_0^{2\pi} h_3^0(r_*\cos\varphi)\cos^2\varphi d\varphi + 3r_*^2\int_0^{2\pi} h_3^0(r_*\cos\varphi)\sin^2\varphi\, d\varphi \neq 0.
$$

Taking into account Theorem 5.2 and Theorem 6.1 we have the global result

Theorem 6.2 *Suppose the assumptions $(A_1) - (A_5)$ to be valid. Then for sufficiently small $|\mu| \neq 0$ system (34) has a unique limit cycle Γ_μ which is hyperbolic and tends to the circle centered at the origin with radius r_* as μ tends to zero.*

Remark 6.3 *The stability of the limit cycle Γ_μ can be determined by means of Theorem 2.7: if the expression $k\Psi\Phi\ |_{\Gamma_\mu}$ is negative (positive), then Γ_μ is orbitally stable (unstable). From (9) we get $k < 0$, from (5) and from the fact that $\mathcal{W}_\mu$ is located in the region bounded by Γ_μ we obtain $\Psi > 0$ at Γ_μ. Finally we get from Lemma 5.1 $\Phi\ |_{\Gamma_\mu} < 0 (> 0)$ for $\mu > 0$ $(\mu < 0)$. Therefore, Γ_μ is orbitally stable (unstable) for $\mu < 0$ $(\mu > 0)$.*

Remark 6.4 *By Theorem 2.9, the limit cycle Γ_μ contains the ellipse $\mathcal{W}_\mu$ defined in (7) in its interior. If we ask for the behavior of Γ_μ as μ tends to $\pm 2p$ we can conclude from (7) that the diameter of the ellipse $\mathcal{W}_\mu$ tends to ∞ as μ tends to $\pm 2p$, therefore the amplitude of the limit cycle Γ_μ tends also to ∞ as μ tends to $\pm 2p$.*

6.2 Existence of a unique limit cycle in the class of systems (1) considered in subsection 5.2

In subsection 5.2 we studied the system

$$\begin{aligned} \frac{dx}{dt} &= y, \\ \frac{dy}{dt} &= -x + \mu\Big((px^2 - c)y + (px^2 - c + \frac{2}{3}p)y^3\Big), \end{aligned} \tag{40}$$

where c and p are positive numbers.

The function $q(x, y)$ belonging to that system reads

$$q(x, y) := (px^2 - c)y + (px^2 - c + \frac{2}{3}p)y^3. \tag{41}$$

Using this relation, equation (35) reads

$$\int_0^{2\pi} q(r\cos\varphi, r\sin\varphi)\sin\varphi d\varphi = r\pi\Big(\frac{p}{8}r^4 + \frac{3}{4}(p - c)r^2 - c\Big) = 0. \tag{42}$$

This equation has the unique positive solution $r_* = \sqrt{\frac{3(c-p)+4\sqrt{D}}{p}}$, where $D = \frac{9(p-c)^2+8pc}{16}$. It is easy to check that this root satisfies the inequality

$$\begin{aligned} &\int_0^{2\pi} \frac{\partial q(r_*\cos\varphi, r_*\sin\varphi)}{\partial y} d\varphi = \\ &\int_0^{2\pi} \Big(pr_*^2\cos^2\varphi - c + (3pr_*^2\cos^2\varphi + 2p - 3c)r_*^2\sin^2\varphi\Big)d\varphi \neq 0. \end{aligned} \tag{43}$$

Thus, we have the result

Theorem 6.5 *Let c and p be any positive numbers. For sufficiently small $|\mu| \neq 0$, system (40) has a unique limit cycle Γ_μ in the phase plane which is hyperbolic and tends to the circle centered at the origin with radius r_* as μ tends to zero.*

6.3 Existence of a unique limit cycle in a class of systems (1)

In subsection 5.3 we considered the system

$$\begin{aligned} \frac{dx}{dt} &= y, \\ \frac{dy}{dt} &= -x + \mu\Big(h_2(x,\mu)(x^2 - 1) + (x^2 - 1)y + h_2(x,\mu)y^2 + \frac{x^2}{3}y^3\Big) \end{aligned} \tag{44}$$

depending on the continuous function h_2 by means of the function $\Psi(x,y) \equiv x^2 + y^2 - 1$. In that case we have

$$\Phi_4(x,y) \equiv -\mu\frac{x^2}{3}, \Phi_2(x,y) \equiv \Phi_0(x,y) \equiv -\mu(x^2-1)^2,$$

that is condition $(H_4), (i)$ is fulfilled for $\mu < 0$ and $\mu > 0$, and we have by Theorem 4.4 the result

Theorem 6.6 *Suppose $h_2 : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ is continuous. Then system (44) has for any $\mu \neq 0$ and for any positive c and p at most one limit cycle.*

In what follows we derive conditions on the function h_2 guaranteeing the existence of a limit cycle in system (44) by means of Theorem 6.1. To this end we suppose

(A_6). The function h_2 can be written in the form

$$h_2(x,\mu) = h_2^0(x) + \mu\tilde{h}_2(x,\mu) \quad \text{for} \quad (x,\mu) \in \mathbb{R}^2,$$

where $\tilde{h}_2$ is continuous.

Furthermore, we introduce the function

$$q(x,y) := (x^2-1)h_2(\varphi,\mu) + (x^2-1)y + h_2(\varphi,\mu)y^2 + \frac{x^2}{3}y^3, \tag{45}$$

and suppose

(A_7). There exists a unique root $r_* > 1$ of the equation

$$\begin{aligned} &\int_0^{2\pi} q(r\cos\varphi, r\sin\varphi)\sin\varphi\, d\varphi = \\ &(r^2-1)\int_0^{2\pi} h_2^0(r\cos\varphi)\sin\varphi\, d\varphi + \pi r\Big(\frac{r^4}{24} + \frac{r^2}{4} - 1\Big) = 0 \end{aligned} \tag{46}$$

satisfying

$$\begin{aligned} &\int_0^{2\pi} \frac{\partial q(r_*\cos\varphi, r_*\sin\varphi)}{\partial y} d\varphi = \\ &\int_0^{2\pi} ((r_*^2\cos^2\varphi - 1) + 2h_2^0(r_*\cos\varphi)r_*\sin\varphi + r_*^4\cos^2\varphi\sin^2\varphi\, d\varphi \neq 0. \end{aligned} \tag{47}$$

By Theorem 6.1 and Theorem 6.6 we have the result

Theorem 6.7 *Suppose the assumptions (A_6) and (A_7) to be valid. Then for sufficiently small $|\mu| \neq 0$ system (44) has a unique limit cycle Γ_μ which is hyperbolic and tends to the circle centered at the origin with radius r_* as μ tends to zero.*

In case that h_2^0 is an even function, equation (46) reads

$$\frac{r^4}{24} + \frac{r^2}{4} - 1 = 0,$$

which has a unique root $r = r_*$ satisfying $1 < r_*^2 < 3$. Inequality (47) takes the form

$$\frac{\pi}{2}(8 - r_*^2) > 0.$$

Thus, we have

Corollary 6.8 *Suppose the assumption (H_4) is valid, and h_2^0 is an even function. Then for sufficiently small $|\mu| \neq 0$ system (44) has a unique limit cycle Γ_μ which is hyperbolic and tends to the circle centered at the origin with radius r_* as μ tends to zero.*

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