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Derivation of an effective damage model with evolving micro-structure

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Abstract

In this paper rate-independent damage models for elastic materials are considered. The aim is the derivation of an effective damage model by investigating the limit process of damage models with evolving micro-defects. In all presented models the damage is modeled via a unidirectional change of the material tensor. With progressing time this tensor is only allowed to decrease in the sense of quadratic forms. The magnitude of the damage is given by comparing the actual material tensor with two reference configurations, denoting completely undamaged material and maximally damaged material (no complete damage).

The starting point is a microscopic model, where the underlying micro-defects, describing the distribution of either undamaged material or maximally damaged material (but nothing in between), are of a given shape but of different time-dependent sizes. Scaling the micro-structure of this microscopic model by a parameter $\varepsilon > 0$ the limit passage $\varepsilon \to 0$ is preformed via two-scale convergence techniques. Therefore, a regularization approach for piecewise constant functions is introduced to guarantee enough regularity for identifying the limit model. In the limit model the material tensor depends on a damage variable $z : [0, T] \to \mathrm{W}^{1,p}(\Omega)$ taking values between 0 and 1 such that, in contrast to the microscopic model, some kind of intermediate damage for a material point $x \in \Omega$ is possible. Moreover, this damage variable is connected to the material tensor via an explicit formula, namely, a unit cell formula known from classical homogenization results.

1 Introduction

Damage models for elastic materials aim at describing the weakening of the underlying structures when exposed to external loadings. Typical mechanisms are the formation of micro-cracks or defects and the growth of them under further loading. In the framework of continuum damage mechanics these effects are accumulated by an internal variable z of phase field type that represents on a macroscopic level the damage state of a material point x. Typically, the damage variable takes values between zero and one, where $z(t, x) = 1$ means that the material is free from defects and $z(t, x) = 0$ characterizes points, where the damage has reached its maximum state. In many models, see e.g. the models developed in [10], the weakening of the material is encoded in some prescribed dependence of the elasticity tensor on the damage state, for example the ansatz $\mathbb{C}(z) = z\mathbb{C}$, where $\mathbb{C}$ denotes the elasticity tensor of the undamaged material, is frequently used. Inspired by the reference [24] the aim of this paper is to set up and analyze an evolution model for damage processes, where the influence of the damage state on the elasticity tensor is justified by a certain homogenization procedure.

Let us explain this in more detail. As a starting point, we consider a domain $\Omega \subset \mathbb{R}^d$ (the physical body) and assume that at mid points of a given ε-periodic lattice ($\varepsilon > 0$ small) small micro-defects can evolve individually under the presence of time-dependent external forces. Thereby we prescribe the geometry that each micro-defect can take (e.g.

balls). This is done by assuming that each micro-defect coincides with a suitable scaling of a fixed set D. The elastic state of the material is then described by two material tensors, $\mathbb{C}_{\text{strong}}$ and $\mathbb{C}_{\text{weak}}$, where $\mathbb{C}_{\text{weak}}$ characterizes the material properties in the micro-defects, and $\mathbb{C}_{\text{strong}}$ characterizes the properties of the remaining part of Ω. In order to formulate the damage evolution model for $\varepsilon > 0$ we introduce a time-dependent damage function $\chi_\varepsilon : [0,T]\times\Omega \to \{0,1\}$, which is equal to zero on the micro-defects and equal to one on the remaining part of Ω. The elasticity tensor is then given by

$$\mathbb{C}(\chi_\varepsilon(t))(x) := \chi_\varepsilon(t,x)\mathbb{C}_{\text{strong}} + (1-\chi_\varepsilon(t,x))\mathbb{C}_{\text{weak}}.$$

The set of all admissible damage functions is denoted by $\mathbb{X}^D_{\varepsilon\Lambda}(\Omega)$, where $\varepsilon\Lambda$ is periodic lattice giving the "centers" of all micro-inclusions. We emphasize that for fixed $\varepsilon > 0$ this means that the "centers" of all micro-defects are fixed but every micro-inclusion is allowed to evolve independently of the others up to a certain maximal size. See Figure 1 for an illustration of a damaged region yielding an admissible damage function χ_ε. In this way the set of χ_ε being 0 can be interpreted as some kind of micro-structure of the damage which is related to the presumed set D.

The evolution of these microscopic models is given by the energetic formulation for rate-independent problems developed in [16, 17]. This energetic formulation is based on an energy functional $\mathcal{E}_\varepsilon : [0,T]\times\mathrm{H}^1_{\Gamma_{\text{Dir}}}(\Omega)^d\times\mathbb{X}^D_{\varepsilon\Lambda}(\Omega) \to \mathbb{R}_\infty$ depending on the displacement field u and the damage function χ_ε, and a dissipation distance $\mathcal{D}_\varepsilon : \mathbb{X}^D_{\varepsilon\Lambda}(\Omega)\times\mathbb{X}^D_{\varepsilon\Lambda}(\Omega) \to [0,\infty]$ depending only on the damage function. We introduce the energy functional via

$$\mathcal{E}_\varepsilon(t,u_\varepsilon,\chi_\varepsilon) = \tfrac{1}{2}\langle\mathbb{C}(\chi_\varepsilon)\mathbf{e}(u_\varepsilon),\mathbf{e}(u_\varepsilon)\rangle_{\mathrm{L}^2(\Omega)^{d\times d}} + \|R_{\frac{\varepsilon}{2}}(Q_\varepsilon(\chi_\varepsilon))\|^p_{\mathrm{L}^p(\Omega)^d} - \langle\ell(t),u_\varepsilon\rangle, \tag{1.1}$$

where ℓ is a given time-dependent loading and $\|R_{\frac{\varepsilon}{2}}(Q_\varepsilon(\chi_\varepsilon))\|^p_{\mathrm{L}^p(\Omega)^d}$ is a regularization term. This term is introduced in order to obtain better convergence properties when looking for an effective limit damage model. The regularization term is motivated by the theory for broken Sobolev functions, see e.g. [3], and can be interpreted as a discrete gradient.

The dissipated energy is proportional to the growth of the weak material which is modeled by the following dissipation distance $\mathcal{D}_\varepsilon : \mathbb{X}^D_{\varepsilon\Lambda}(\Omega)\times\mathbb{X}^D_{\varepsilon\Lambda}(\Omega) \to [0,\infty]$ given by

$$\mathcal{D}_\varepsilon(\chi_1,\chi_2) = \begin{cases} \displaystyle\int_\Omega \gamma(\chi_1(x)-\chi_2(x))\mathrm{d}x & \text{if } \chi_1 \geq \chi_2 \\ \infty & \text{otherwise} \end{cases}.$$

The quantity $\gamma > 0$ is a material dependent constant and plays the role of an averaged fracture toughness. Observe that the dissipation distance ensures the uni-directionality of the damage, meaning that the damaged region of Ω is only allowed to grow with respect to increasing time.

With this, the evolutionary problem is given by the stability condition (S^ε) and the energy balance (E^ε), which read as:

$$(\mathrm{S}^\varepsilon)\;\; \mathcal{E}_\varepsilon(t,u_\varepsilon(t),\chi_\varepsilon(t)) \leq \mathcal{E}_\varepsilon(t,\widetilde{u},\widetilde{\chi}) + \mathcal{D}_\varepsilon(\chi_\varepsilon(t),\widetilde{\chi}) \quad \text{for all } (\widetilde{u},\widetilde{z}) \in \mathrm{H}^1_{\Gamma_{\text{Dir}}}(\Omega)^d\times\mathbb{X}^D_{\varepsilon\Lambda}(\Omega),$$

$$(\mathrm{E}^\varepsilon)\;\; \mathcal{E}_\varepsilon(t,u_\varepsilon(t),\chi_\varepsilon(t)) + \mathrm{Diss}_{\mathcal{D}_\varepsilon}(\chi_\varepsilon;[0,t]) = \mathcal{E}_\varepsilon(0,u_\varepsilon(0),\chi_\varepsilon(0)) + \int_0^t \partial_t\mathcal{E}_\varepsilon(s,u_\varepsilon(s),\chi_\varepsilon(s))\mathrm{d}s,$$

with $\mathrm{Diss}_{\mathcal{D}_\varepsilon}(\chi;[0,t]) := \sup\sum_{j=1}^N \mathcal{D}_\varepsilon(\chi(s_{j-1}),\chi(s_j))$, where $N\in\mathbb{N}$ and the supremum is taken over all finite partitions of $[0,t]$.

The aim of this paper is to study the limit behavior of the evolution model (S^ε) and (E^ε) as ε tends to zero and to identify the resulting effective model. This is done using evolutionary Γ-convergence methods for rate-independent systems, [15], in combination with two-scale convergence arguments. As the main result (see Theorem 7.7) we obtain a limit damage model with a damage variable $z_0(t)\in\mathrm{W}^{1,p}(\Omega)$, $0\le z_0(t)\le 1$, of phase-field type, where the dependence of the elasticity tensor on the damage variable in form of a suitable cell formula is justified by the limiting procedure. To be more specific: Thanks to a suitable choice of the discrete regularization term $R_{\frac{\varepsilon}{2}}(Q_\varepsilon(\cdot))$ in the energy, see (1.1), for every $t\in[0,1]$ a (sub)sequence of the damage functions $(\chi_\varepsilon(t))_{\varepsilon>0}\subset\mathrm{L}^\infty(\Omega)$ converges to a Sobolev function $z_0(t)\in\mathrm{W}^{1,p}(\Omega;[0,1])$ being the damage variable of a two-scale limit model. Here, every macroscopic point $x\in\Omega$ is associated with a unit cell $x{+}Y$ containing the micro-inclusion of the microscopic models and the value $z_0(t,x)$ is related to the size of the micro-inclusion inside the unit cell $x{+}Y$ at time $t\in[0,T]$. This is modeled by the following two-scale tensor:

$$\begin{aligned}\mathbb{C}_0(z_0(t))(x,y) &= \mathbb{1}_{U(z_0(t,x))}(y)\mathbb{C}_{\mathrm{strong}} + (1-\mathbb{1}_{U(z_0(t,x))}(y))\mathbb{C}_{\mathrm{weak}},\\ U(\theta) = Y\backslash\kappa(\theta)D \quad &\text{and} \quad \kappa^d(\theta) = (\mathrm{vol}(D))^{-1}(1{-}\theta),\end{aligned}\tag{1.2}$$

where $\mathbb{1}_U(\cdot)$ denotes the characteristic function of the set $U\subset Y$ and $\theta\in[0,1]$. As one can see, in the limit we end up with a damage model where the micro-structure of the microscopic damage models is preserved since every unit cell $x{+}Y$ contains a micro-inclusion shaped like the micro-inclusions chosen in the microscopic models for $\varepsilon>0$ (see (1.2)), with a size that is determined through the value $z_0(x,t)$.

The two-scale model can be equivalently described by a one-scale model with an effective tensor $\mathbb{C}_{\mathrm{eff}}(\theta)$ given by the following unit cell problem:

$$\langle\mathbb{C}_{\mathrm{eff}}(\theta)\xi,\xi\rangle_{d\times d} = \min\int_Y\langle\mathbb{C}(\mathbb{1}_{U(\theta)}(y))(\xi+\mathbf{e}_y(v)(y)),\xi+\mathbf{e}_y(v)(y)\rangle_{d\times d}\mathrm{d}y,\tag{1.3}$$

where the minimum is chosen for all functions v belonging to $\mathrm{H}^1_{\mathrm{per}}(Y)^d$ and satisfying $\int_Y v(y)\mathrm{d}y=0$. For a given set D and fixed $\theta\in[0,1]$ this tensor coincides with the effective tensor that is gained by homogenization of a periodic mixture $\mathbb{C}_\varepsilon$ of $\mathbb{C}_{\mathrm{strong}}$ and $\mathbb{C}_{\mathrm{weak}}$ with respect to a periodic geometry with micro-defects of the type $\varepsilon\kappa(\theta)D$.

Let us give a comparison of the developed model and techniques with further homogenization approaches in the literature in the context of damage processes.

In [21, 22] a transformation method is introduced allocating classical homogenization techniques for non-periodic problems. There, coupled reaction-diffusion systems are treated which take place on a domain with non-periodic micro-structure, but which has to be isomorphic, possibly depending on time, to some periodic reference micro-structure. Then this transformation is applied and the homogenization is done in the reference configuration. The reason why this method is not applicable in our case is that there the transformation has to be given (no evolution law for the transformation) whereas in our case the evolution of the micro-structure is part of the model. That means the evolution of the micro-structure is explicitly modeled by one of the unknown variables, namely, the damage function.

In [23] periodic homogenization techniques are used to derive stationary effective models based on a fixed periodic microscopic model, where in every periodicity cell the displace-

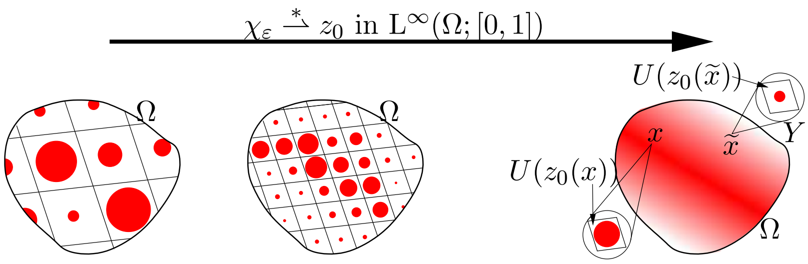


Figure 1: Schematic representation of the limit passage of the micro one-scale model to the two-scale limit model, where the micro inclusions are assumed to be balls

ment is allowed to have jumps on the finite union of given $(d-1)$-dimensional sets. These $(d-1)$-dimensional sets are interpreted as fissures in the material. Then a homogenization parameter dependent energy, consisting of a volume and a surface term, is considered. In dependence of the ratio of the surface term to the homogenization parameter different effective models are obtained.

In the papers [9, 8, 11] the weakening of the material is modeled as a pure mixture of damaged and undamaged material. There, an existence result is proved for a model, where for every $t \in [0,T]$ and any point $x \in \Omega$ the material tensor $A(t,x)$ is an element of the so called G-closure of the two constant tensors $\mathbb{C}_{\text{strong}}$ and $\mathbb{C}_{\text{weak}}$. The result in [11] states the existence of a solution of a damage model based on three variables: the displacement $u(t,x)$, the damage variable $\Theta(t,x)$ and the material tensor $A(t,x)$. In contrast to our model there the material tensor $A(t,x)$ is not uniquely described by the damage variable but is also given by (1.3) except that the set $U(\Theta(t,x))$ could be any set with volume fraction $\Theta(t,x) \in [0,1]$ (no presumed geometry). Compared to our model there the shape of the "micro-inclusion" (not necessarily an inclusion) is allowed to change for different points of Ω, whereas in our model it is a priori given by the choice of the set D. In contrast to our model, the one presented in [8, 11] does not involve a gradient regularization for the damage variable.

The current paper is structured as follows: In Section 2 a discrete gradient for piecewise constant functions on lattices is introduced relying on the theory for broken Sobolev spaces, see for instance [3]. The aim is to construct the discrete gradient in such a way that from sequences of piecewise constant functions on finer and finer lattices, for which the discrete gradient is bounded in $\mathrm{L}^p(\Omega)$, one can extract a subsequence that converges strongly in $\mathrm{L}^p(\Omega)$ to a limit function in $\mathrm{W}^{1,p}(\Omega)$ and where the corresponding discrete gradients converge weakly to the gradient of the limit function. For that purpose, the original definition of a discrete gradient from [3] had to be modified see also the example at the beginning of Section 2. In the subsequent sections this discrete gradient is applied to special piecewise constant functions $Q_\varepsilon\chi_\varepsilon$, that roughly spoken encode the ratio between the damaged and the undamaged region in each cell εY of the lattice. The above described compactness property guarantees enough regularity to identify an effective limit model with a damage variable that belongs to $\mathrm{W}^{1,p}(\Omega)$ and which is the limit of the piecewise constant functions $Q_\varepsilon\chi_\varepsilon$. Moreover, due to the strong convergence of the functions $Q_\varepsilon\chi_\varepsilon$, the information on the shape of the damage set D is preserved in the limit model.

As already mentioned we are going to apply the Γ-convergence theory introduced in [15] for evolutionary problems modeled by the energetic formulation. The theory relies on the construction of certain mutual recovery sequences (see [15] for details). In Section 3 we provide the tools for the construction of the mutual recovery sequence for our damage model. In particular, some kind of mutual recovery sequence for sequences of piecewise constant functions converging to some Sobolev function is constructed, that respect the irreversibility constraint posed on the damage evolution. For that purpose, we extend to the discrete case the ideas from [18], where such sequences were constructed in $W^{1,p}(\Omega)$ in the context of rate-independent damage models.

Section 4 starts with a short summary of the theory for rate-independent problems modeled by the energetic formulation developed in [16, 17]. Moreover, this section contains all damage models, which are considered and discussed in the following.

In Subsection 4.2, for $\varepsilon > 0$ the microscopic damage model based on damage functions χ_ε is introduced. Furthermore, it is defined, in which way the damage function, which is a characteristic function, is identified with such piecewise constant functions considered in the first two sections. This enables us to exploit the theory on discrete gradients stated in the first two sections in the following. Finally, for fixed $\varepsilon > 0$ the existence of at least one solution of the rate-independent damage model is proved with the help of the abstract theory for rate-independent processes summarized in Subsection 4.1.

The introduction of the two limit models is done in Subsection 4.3 and 4.4. The first one is a two-scale model, where the shape of the limit functionals is motivated by the convergence result of Section 6. The second one is a one-scale model which is proven to be equivalent to the first one in the following sense: From any solution of one of these models a solution of the other one can be constructed. Note that in both cases the existence of a solution is proved via the convergence result stated in Section 7, where for a sequence of solutions of the ε-dependent models (S^ε) and (E^ε) the limit $\varepsilon \to 0$ is investigated.

Section 5 is devoted to the theory of two-scale convergence developed by G. Nguetseng in [20] and states the notations, the definitions and the results needed in the following. Here, in this paper we use the so called unfolding technique introduced in [4].

Section 6 is related to the following problem: Assuming suitable boundedness assumptions of a sequence of admissible damage functions $(\chi_\varepsilon)_{\varepsilon>0}$ of the microscopic systems (S^ε) and (E^ε), we identify the limit in the weak$*$ topology and in the strong two-scale topology. As already mentioned, this identification of the strong two-scale limit motivates the definition of the two-scale limit model in Subsection 4.3, by formally replacing all ε-dependent functions in (S^ε) and (E^ε) by the two-scale limits of the associated sequences.

Finally, in Section 7 we prove the main result (Theorem 7.7) of our paper, namely the convergence of the ε-dependent damage model (S^ε) and (E^ε) to the two-scale model introduced in Subsection 4.3. As already mentioned this is done in the setting of evolutionary Γ-convergence as it is developed in [15].

2 Discrete gradients of piecewise constant functions

This section is about the definition and the properties of a discrete gradient for piecewise constant functions. Note, that the following is completely independent of the damage model mentioned in the introduction and investigated in the next sections. That means that this calculus first of all stands on its own concerning the notation and, probably more important, it is not restricted to damage models in its application.

The aim of this section is the definition of a discrete gradient for piecewise constant functions on a lattice in that way that only an overall constant function has gradient zero. Furthermore an in some sense bounded sequence of those piecewise constant functions, where the spacing of the lattice tends to zero, should lead to a limit belonging to a Sobolev-space $\mathrm{W}^{1,p}$. Roughly spoken we want to introduce a penalty term, extracting those sequences of BV-functions that converge strongly in L^p to a Sobolev-function, so that the discrete gradient of these sequences converge weakly in L^p to the gradient of this Sobolev-function.

Before introducing the discrete gradient, we have to start with some definitions. Let d be the space dimension and $\{e_1, e_2, \ldots, e_d\}$ an orthonormal basis of $\mathbb{R}^d$. Furthermore, let

$$\Lambda = \left\{ \lambda \in \mathbb{R}^d \,:\, \lambda = \sum_{i=1}^{d} k_i e_i,\; k_i \in \mathbb{Z} \right\}$$

be a periodic latice and $Y = [0,1)^d$ the associated unit cell. Due to this definition there is only one vertex contained in $\varepsilon(\lambda+Y)$ so that every of those cells is uniquely determined by $\varepsilon > 0$ and the associated vertex $\varepsilon\lambda$. Moreover, according to the definition of the periodic latice Λ we have $\varepsilon\Lambda \subset \frac{\varepsilon}{2}\Lambda$ and due to the choice of the associated unit cell Y for every $\lambda \in \Lambda$ there exist exactly 2^d elements $\lambda_1, \lambda_2, \ldots, \lambda_{2^d} \in \frac{1}{2}\Lambda$ so that

$$\varepsilon(\lambda+Y) = \bigcup_{j=1}^{2^d} \frac{\varepsilon}{2}(\lambda_j+Y). \tag{2.1}$$

Note, that this property (which would not be valid with $Y = [-\frac{1}{2}, \frac{1}{2})^d$ for instance) is crucial for the definition of our discrete gradient.

Finally, for an open set $\Omega \subset \mathbb{R}^d$ the set of piecewise constant functions considered in this paper is given by

$$\mathrm{K}_{\varepsilon\Lambda}(\Omega) := \{ v \in \mathrm{L}^1(\Omega) \,|\, \exists\, \widetilde{v} \in \mathrm{K}_{\varepsilon\Lambda}(\mathbb{R}^d) : \widetilde{v}|_\Omega = v \},$$

where

$$\mathrm{K}_{\varepsilon\Lambda}(\mathbb{R}^d) := \{ \widetilde{v} \in \mathrm{L}^1(\mathbb{R}^d) \,|\, \forall\, \lambda \in \Lambda : \widetilde{v}|_{\varepsilon(\lambda+Y)} = \mathit{const} \}.$$

As already mentioned in Section 1 in the following the open set Ω describes an elastic body undergoing a damage process. Thereto, the body Ω is decomposed in small cells $\varepsilon(\lambda+Y)$ containing the micro-structure of the damage. That is why we introduce the subsets

$$\Lambda_\varepsilon^- := \{ \lambda \in \Lambda : \varepsilon(\lambda+\overline{Y}) \subset \Omega \} \qquad \text{and} \qquad \Lambda_\varepsilon^+ := \{ \lambda \in \Lambda : \varepsilon(\lambda+Y) \cap \Omega \neq \emptyset \}$$

of Λ to define the sets Ω_ε^- and Ω_ε^+ via

$$\Omega_\varepsilon^\pm := \bigcup_{\lambda \in \Lambda_\varepsilon^\pm} \varepsilon(\lambda+Y). \tag{2.2}$$

Observe that $\overline{\Omega_\varepsilon^-}$ is a compact subset of Ω. The set Ω_ε^+ is introduced in order to avoid problems with cells having a non empty intersection with Ω but which are not completely contained in it, i.e. all cells containing a part of the boundary $\partial\Omega$. For the same reason we introduce the extension operator $V_\varepsilon : \mathrm{K}_{\varepsilon\Lambda}(\Omega) \to \mathrm{K}_{\varepsilon\Lambda}(\Omega_\varepsilon^+)$ extending a piecewise constant function $v \in \mathrm{K}_{\varepsilon\Lambda}(\Omega)$ for every $\lambda \in \Lambda_\varepsilon^+ \backslash \Lambda_\varepsilon^-$ on $\varepsilon(\lambda{+}Y)\backslash\Omega$ constantly by the (constant) value of v on $\varepsilon(\lambda{+}Y) \cap \Omega$. From now on we will assume that

$$\Omega \text{ is an open and bounded subset of } \mathbb{R}^d \text{ which satisfies } \mathrm{vol}(\partial\Omega) = 0. \tag{2.3}$$

This guarantees that $\mathrm{vol}(\Omega_\varepsilon^+\backslash\Omega) + \mathrm{vol}(\Omega\backslash\Omega_\varepsilon^-) \to 0$ for $\varepsilon \to 0$ which will be used later. In particular this is crucial when introducing the two-scale convergence with the help of the so called periodic unfolding operator (see [19] Section 2).

With all this, $\mathrm{K}_{\varepsilon\Lambda}(\Omega) \subset \mathrm{BV}(\Omega)$, and we introduce the discrete gradient in the following way:

$$R_{\frac{\varepsilon}{2}} : \mathrm{K}_{\varepsilon\Lambda}(\Omega)^m \to \mathrm{K}_{\frac{\varepsilon}{2}\Lambda}(\Omega_\varepsilon^+)^{m\times d}; \quad v \mapsto \sum_{i=1}^d \widetilde{R}_{\frac{\varepsilon}{2}}^{(i)}(V_\varepsilon v), \tag{2.4}$$

where $\widetilde{R}_{\frac{\varepsilon}{2}}^{(i)} : \mathrm{K}_{\varepsilon\Lambda}(\Omega_\varepsilon^+)^m \to \mathrm{K}_{\frac{\varepsilon}{2}\Lambda}(\Omega_\varepsilon^+)^{m\times d}$ is defined via

$$\widetilde{R}_{\frac{\varepsilon}{2}}^{(i)}(\widetilde{v})(x) := \begin{cases} \frac{1}{\varepsilon}\left\{\widetilde{v}(x{+}\frac{\varepsilon}{2}e_i) - \widetilde{v}(x{-}\frac{\varepsilon}{2}e_i)\right\} \otimes e_i & \text{if } x{+}\frac{\varepsilon}{2}e_i \in \Omega_\varepsilon^+ \text{ and } x{-}\frac{\varepsilon}{2}e_i \in \Omega_\varepsilon^+, \\ 0 & \text{otherwise.} \end{cases} \tag{2.5}$$

This construction of the discrete Gradient is inspired by the so called lifting operator introduced by A. Buffa and C. Ortner in [3] defined via

$$\begin{gathered} \mathrm{R}_\varepsilon^{\mathrm{BO}} : \mathrm{W}_{\varepsilon\Lambda}^{1,p}(\Omega)^m \to S_{\varepsilon\Lambda}^\eta(\Omega)^{m\times d} \\ \int_\Omega \mathrm{R}_\varepsilon^{\mathrm{BO}}(w)(x) : \phi(x)\mathrm{d}x = -\int_{\Gamma_{\mathrm{int}}^\varepsilon} [\![w(s)]\!] : \{\{\phi(s)\}\}\mathrm{d}s \qquad \forall\, \phi \in S_{\varepsilon\Lambda}^\eta(\Omega)^{m\times d}, \end{gathered} \tag{2.6}$$

with $[\![w(s)]\!] = w^+(s) \otimes n^+ + w^-(s) \otimes n^-$ and $\{\{\phi(s)\}\} = \frac{1}{2}(\phi^+(s) + \phi^-(s))$, where $w^\pm$ and $\phi^\pm$ are the traces of w and ϕ with respect to the outward normals $n^\pm$ for $s \in \Gamma_{\mathrm{int}}^\varepsilon := \Omega \cap \bigcup_{\lambda\in\Lambda} \varepsilon(\lambda{+}\partial Y)$. Here, $\mathrm{W}_{\varepsilon\Lambda}^{1,p}(\Omega) := \{w \in \mathrm{L}^1(\Omega) : w|_{\varepsilon(\lambda+Y)\cap\Omega} \in \mathrm{W}^{1,p}(\varepsilon(\lambda{+}Y)\cap\Omega) \quad \forall\, \lambda \in \Lambda\}$ is the so called broken Sobolev space and $S_{\varepsilon\Lambda}^\eta(\Omega)$ denotes the set of all piecewise polynomial functions (in the same sense as in the piecewise constant case) with a degree $\eta \in \mathbb{N}$. Observing $\mathrm{K}_{\varepsilon\Lambda}(\mathbb{R}^d)^m \subset \mathrm{W}_{\varepsilon\Lambda}^{1,p}(\mathbb{R}^d)^m$ one very important difference between our definition (2.4) and the definition (1.5) from [3] is, that their definition leads to the following discrete gradient for piecewise constant functions:

$$\begin{gathered} \mathrm{R}_\varepsilon^{\mathrm{BO}} : \mathrm{K}_{\varepsilon\Lambda}(\mathbb{R}^d)^m \to \mathrm{K}_{\varepsilon\Lambda}(\mathbb{R}^d)^{m\times d} \\ \mathrm{R}_\varepsilon^{\mathrm{BO}}(v)(x) := \sum_{i=1}^d \tfrac{1}{2\varepsilon}\left\{v(x{+}\varepsilon e_i) - v(x{-}\varepsilon e_i)\right\} \otimes e_i \end{gathered} \tag{2.7}$$

Here, we replaced Ω by $\mathbb{R}^d$ such that we do not have to care about what is happening in cells $\varepsilon(\lambda{+}Y)$ intersecting the boundary $\partial\Omega$. With this definition the value of the discrete gradient $(\mathrm{R}_\varepsilon^{\mathrm{BO}}(v)(x))_{k,l}$, $k \in \{1,\dots,m\}$ $l \in \{1,\dots,d\}$, is defined by the values of the function v in the "next" $(v(x{+}\varepsilon e_i))$ and in the "previous" $(v(x{-}\varepsilon e_i))$ cell, but is independent of the value of the "actual" cell $(v(x))$. This leads to the following problems:

1. Considering a periodic piecewise constant function satisfying $v(x{+}\varepsilon e_i) = v(x{-}\varepsilon e_i)$ and $v(x) \neq v(x{+}\varepsilon e_i)$ for every $i \in \{1,\dots,d\}$ we obtain $\mathrm{R}_\varepsilon^{\mathrm{BO}}(v) \equiv 0$ for $v \not\equiv \mathit{const}$.

2. For $d = 1$ the sequence $(v_\varepsilon)_{(\varepsilon>0)} \subset \mathrm{K}_{\varepsilon^p\Lambda}(\mathbb{R})$ of piecewise constant functions ($k \in \mathbb{Z}$) with

$$v_\varepsilon(x) = \begin{cases} 2 & \text{if } x \in \varepsilon^p[2k, 2k+1) \\ -2 & \text{if } x \in -\varepsilon^p[(2|k|+1), 2|k|) \\ 0 & \text{if } x \in \varepsilon^p[(2|k|+1), 2|k|) \end{cases} \tag{2.8}$$

converges weakly in $\mathrm{L}^p_{\mathrm{loc}}(\mathbb{R})$ due to its periodicity to the Heaviside function $H(x) = 1$ for $x \geq 0$ and $H(x) = 0$ otherwise. But H does not belong to $\mathrm{W}^{1,p}_{\mathrm{loc}}(\mathbb{R})$. According to the definition of the lifting operator we have $|\mathrm{R}^{\mathrm{BO}}_\varepsilon(v_\varepsilon)(x)| = \frac{1}{\varepsilon}$ for $x \in [0, \varepsilon^p)$ and $\mathrm{R}^{\mathrm{BO}}_\varepsilon(v_\varepsilon) \equiv 0$ otherwise. This gives $\|\mathrm{R}^{\mathrm{BO}}_\varepsilon(v_\varepsilon)\|_{\mathrm{L}^p(\mathbb{R})} = 1$ which shows that this lifting operator is not the right penalty term in the sense mentioned in the beginning of this section. There is another comment on that in Remark 2.2.

As opposed to this the discrete gradient defined in (2.4) evaluated for v_ε from (2.8) gives us $|R_{\frac{\varepsilon}{2}}(v_\varepsilon)(x)| = \frac{4}{\varepsilon}$ for $x < \frac{\varepsilon^p}{2}$ and $|R_{\frac{\varepsilon}{2}}(v_\varepsilon)(x)| = \frac{2}{\varepsilon}$ otherwise, which leads to $\|R_{\frac{\varepsilon}{2}}(v_\varepsilon)\|^p_{\mathrm{L}^p(\Omega)} \geq \mathrm{vol}(\Omega)(\frac{2}{\varepsilon})^p$ for any bounded subset Ω of $\mathbb{R}$. This shows that this term along $(v_\varepsilon)_{\varepsilon>0}$ is unbounded which correlates with the fact that this sequence does not have a limit belonging to $\mathrm{W}^{1,p}_{\mathrm{loc}}(\mathbb{R})$. That is why in our special case the L^p-norm of the discrete gradient defined in (2.4) is suitable as a penalty term filtering out sequences of piecewise constant functions converging to elements of $\mathrm{W}^{1,p}(\Omega)^m$ as it is stated in the following theorem.

Theorem 2.1 (Compactness result)**.** *For $p \in (1, \infty)$ and every sequence $(v_\varepsilon)_{\varepsilon>0}$ of functions belonging to $\mathrm{K}_{\varepsilon\Lambda}(\Omega)^m$ and satisfying*

$$\sup_{\varepsilon>0} \left(\|v_\varepsilon\|_{\mathrm{L}^p(\Omega)^m} + \|R_{\frac{\varepsilon}{2}}(v_\varepsilon)\|_{\mathrm{L}^p(\Omega^+_\varepsilon)^{m\times d}}\right) \leq C < \infty \tag{2.9}$$

there exist a function $v_0 \in \mathrm{W}^{1,p}(\Omega)^m$ and a sub-sequence $(v_{\varepsilon'})_{\varepsilon'>0}$ of $(v_\varepsilon)_{\varepsilon>0}$ with

$$v_{\varepsilon'} \to v_0 \textit{ in } \mathrm{L}^q(\Omega)^m \quad \textit{and} \quad R_{\frac{\varepsilon}{2}}(v_{\varepsilon'}) \rightharpoonup \nabla v_0 \textit{ in } \mathrm{L}^p(\Omega)^{m\times d},$$

where $1 \leq q < p^*$, and p^* denotes the Sobolev conjugate of p.

Remark 2.2. Our Theorem 2.1 is a modification of Theorem 5.2 from [3]. There, in our condition (2.9) the regularization term $\|R_{\frac{\varepsilon}{2}}(v_\varepsilon)\|_{\mathrm{L}^p(\Omega^+_\varepsilon)^{m\times d}}$ is replaced by the penalty term $\int_{\Gamma^\varepsilon_{\mathrm{int}}} \varepsilon^{1-p}|[\![v_\varepsilon(s)]\!]|^p \mathrm{d}s$, such that the authors of [3] end up with the same convergence result with respect to their discrete gradient $\mathrm{R}^{\mathrm{BO}}_\varepsilon$. But due to this procedure a regularized (ε-dependent) model based on functionals depending on BV-functions has to contain two things to gain a limit model described by functionals depending solely on Sobolev-functions. First, the penalty term $\int_{\Gamma^\varepsilon_{\mathrm{int}}} \varepsilon^{1-p}|[\![v_\varepsilon(s)]\!]|^p \mathrm{d}s$ forcing the sequence $(v_\varepsilon)_{\varepsilon>0}$ of BV-functions to converge to a Sobolev-function, and second, the lifted function $\mathrm{R}^{\mathrm{BO}}_\varepsilon(v_\varepsilon)$ to gain a gradient in the limit. Thereby a further issue arises, namely, the identification and interpretation of the penalty term after passing to the limit. Clearly, due to our replacement this problem is solved. Since the proof of our Theorem 2.1 is based on that of Theorem 5.2 from [3] we need the estimate of Lemma 2.3 below to adapt the proof from [3].

Lemma 2.3. *There exists a constant $C > 0$ such that for every $p \in [1, \infty)$, for every $\varepsilon > 0$ and for all $v \in \mathrm{K}_{\varepsilon\Lambda}(\Omega)^m$ it holds*

$$|Dv|(\Omega) \leq C\left(\int_{\Gamma^\varepsilon_{\mathrm{int}}} \varepsilon^{1-p}|[\![v(s)]\!]|^p \mathrm{d}s\right)^{\frac{1}{p}} \leq C\|R_{\frac{\varepsilon}{2}}(v)\|_{\mathrm{L}^p(\Omega^+_\varepsilon)^{m\times d}},$$

where Dv is the measure representing the distributional derivative of v and $|Dv|(\Omega)$ its total variation. Moreover, $\Gamma^\varepsilon_{\text{int}} := \Omega \cap \bigcup_{\lambda\in\Lambda} \varepsilon(\lambda+\partial Y)$.

Proof. The proof of the first inequality is a straight forward generalization of Theorem 3.26 from [13] to the case of $p \neq 2$ and can be found in [3] (Lemma 2) as a brief sketch, for example.

The second inequality results from the special structure of the discrete gradient. For a better understanding the calculations are split up so that the left hand side of every numbered equations is the same (starting point) and the only changes are on the right hand side. First of all (2.10) is valid since every face of the cell $\varepsilon(\lambda+Y)$ is taken twice when summing up on the right hand side:

$$\int_{\Gamma^\varepsilon_{\text{int}}} \varepsilon^{1-p} |[\![v(s)]\!]|^p \mathrm{d}s = \tfrac{1}{2} \sum_{\lambda\in\Lambda} \int_{\varepsilon(\lambda+\partial Y)} \varepsilon^{1-p} |[\![v(s)]\!]|^p \mathbb{1}_\Omega(s) \mathrm{d}s \tag{2.10}$$

Since the integrand contains the characteristic function $\mathbb{1}_\Omega$, the function $v \in \mathrm{K}_{\varepsilon\Lambda}(\Omega)^m$ can be replaced by any extension $\widetilde{v} \in \mathrm{L}^1(\Omega^+_\varepsilon)$ satisfying $\widetilde{v}|_\Omega = v$. We choose $\widetilde{v} := (V_\varepsilon(v)) \in \mathrm{K}_{\varepsilon\Lambda}(\Omega^+_\varepsilon)^m$ and exploit that due to decomposition (2.1) for every cell $\varepsilon(\lambda+Y) \subset \Omega^+_\varepsilon$ we have $[\![\widetilde{v}(s)]\!] = 0$ for $s \in \frac{\varepsilon}{2}(\lambda_j+\partial Y)\backslash\varepsilon(\lambda+\partial Y)$, since $\widetilde{v} \in \mathrm{K}_{\varepsilon\Lambda}(\Omega^+_\varepsilon)^m$ is constant on $\varepsilon(\lambda+Y)$. That is why the following equality is valid, since there only zeros are added:

$$\int_{\Gamma^\varepsilon_{\text{int}}} \varepsilon^{1-p} |[\![v(s)]\!]|^p \mathrm{d}s = \tfrac{1}{2} \sum_{\lambda\in\Lambda} \sum_{j=1}^{2^d} \varepsilon^{1-p} \int_{\frac{\varepsilon}{2}(\lambda_j+\partial Y)} |[\![\widetilde{v}(s)]\!]|^p \mathbb{1}_\Omega(s) \mathrm{d}s. \tag{2.11}$$

Now we first of all increase the domain of integration in (2.11) by replacing $\mathbb{1}_\Omega$ by $\mathbb{1}_{\overline{\Omega^+_\varepsilon}}$ and then we calculate the integral by splitting $\frac{\varepsilon}{2}(\lambda_j+\partial Y)$ into its $2d$ faces of $\frac{\varepsilon}{2}(\lambda_j+Y)$, afterwards. For $s \in \partial\Omega^+_\varepsilon$ the jump term $[\![\widetilde{v}(s)]\!]$ is not well-defined since $\mathrm{supp}(\widetilde{v}) \subset \overline{\Omega^+_\varepsilon}$. That is why we set $[\![\widetilde{v}(s)]\!] := 0$ for $s \in \partial\Omega^+_\varepsilon$. Since the integrand is constant on every face, the integral gives the constant multiplied with $\left(\frac{\varepsilon}{2}\right)^{d-1}$, which is just the volume of one face. Moreover, the jump term of $\widetilde{v}$ is replaced by its definition, where $v^+ = \widetilde{v}(\frac{\varepsilon}{2}\lambda_j)$, $v^- = \widetilde{v}(\frac{\varepsilon}{2}(\lambda_j+e_i))$ and $n^+ = -n^- = e_i$ is used for one face of $\frac{\varepsilon}{2}(\lambda_j+Y)$ and $v^+ = \widetilde{v}(\frac{\varepsilon}{2}\lambda_j)$, $v^- = \widetilde{v}(\frac{\varepsilon}{2}(\lambda_j-e_i))$ and $n^+ = -n^- = -e_i$ for the opposite one. This results in:

$$\begin{aligned}\int_{\Gamma^\varepsilon_{\text{int}}} \varepsilon^{1-p} |[\![v(s)]\!]|^p \mathrm{d}s \le \tfrac{1}{2} \sum_{\lambda\in\Lambda^+_\varepsilon} \sum_{j=1}^{2^d} \varepsilon^{1-p} \sum_{i=1}^{d} &\left(\tfrac{\varepsilon}{2}\right)^{d-1} \left|\left[\widetilde{v}(\tfrac{\varepsilon}{2}\lambda_j) - \widetilde{v}(\tfrac{\varepsilon}{2}(\lambda_j+e_i))\right]\otimes e_i\right|^p \delta^{(\lambda)}_{i,j} \\ &+ \left(\tfrac{\varepsilon}{2}\right)^{d-1} \left|\left[\widetilde{v}(\tfrac{\varepsilon}{2}(\lambda_j-e_i)) - \widetilde{v}(\tfrac{\varepsilon}{2}\lambda_j)\right]\otimes e_i\right|^p \widetilde{\delta}^{(\lambda)}_{i,j},\end{aligned} \tag{2.12}$$

where

$$\delta^{(\lambda)}_{i,j} := \begin{cases} 0 & \text{if } \frac{\varepsilon}{2}(\lambda_j+e_i) \notin \Omega^+_\varepsilon \\ 1 & \text{otherwise} \end{cases} \qquad \widetilde{\delta}^{(\lambda)}_{i,j} := \begin{cases} 0 & \text{if } \frac{\varepsilon}{2}(\lambda_j-e_i) \notin \Omega^+_\varepsilon \\ 1 & \text{otherwise} \end{cases}.$$

As already mentioned a lot of zeros are added in (2.11) and this results in the following: Observe that for the λ_j as in (2.1) we have $\frac{\varepsilon}{2}\lambda_j \in \varepsilon(\lambda+Y)$. Moreover, either we have $\frac{\varepsilon}{2}(\lambda_j+e_i) \in \varepsilon(\lambda+Y)$ or $\frac{\varepsilon}{2}(\lambda_j-e_i) \in \varepsilon(\lambda+Y)$, which gives us either $\widetilde{v}(\frac{\varepsilon}{2}(\lambda_j+e_i)) = \widetilde{v}(\frac{\varepsilon}{2}\lambda_j)$ or $\widetilde{v}(\frac{\varepsilon}{2}(\lambda_j-e_i)) = \widetilde{v}(\frac{\varepsilon}{2}\lambda_j)$ for fixed $i \in \{1,\ldots,d\}$ and $j \in \{1,\ldots,2^d\}$. With this, always one of the terms of the right hand side of (2.12) is zero and the other can be replaced in the following way:

$$\int_{\Gamma^\varepsilon_{\text{int}}} \varepsilon^{1-p}|[\![v(s)]\!]|^p \mathrm{d}s \leq \sum_{\substack{j=1,\\ \lambda\in\Lambda^+_\varepsilon}}^{2^d} \frac{\varepsilon^d}{2^d} \sum_{i=1}^d \varepsilon^{-p}\Big|\Big[\widetilde{v}\big(\tfrac{\varepsilon}{2}(\lambda_j-e_i)\big) - \widetilde{v}\big(\tfrac{\varepsilon}{2}(\lambda_j+e_i)\big)\Big]\otimes e_i\Big|^p \delta^{(\lambda)}_{i,j}\widetilde{\delta}^{(\lambda)}_{i,j}. \quad (2.13)$$

To make the next step clear let $|\cdot|_F$ denote the p-Frobenius-matrix norm, let $\{\widetilde{e}_1,\dots,\widetilde{e}_m\}$ be the orthonormal basis of $\mathbb{R}^m$ and to shorten notation let $f_\varepsilon(\lambda_j,e_i) = \frac{1}{\varepsilon}[\widetilde{v}(\frac{\varepsilon}{2}(\lambda_j-e_i)) - \widetilde{v}(\frac{\varepsilon}{2}(\lambda_j+e_i))]$. Using the definition of the norm $|\cdot|_F$ in the first and in the last step we have the following identity by first subtracting and then adding zeros:

$$\begin{aligned}
\sum_{i=1}^d \Big|f_\varepsilon(\lambda_j,e_i)\otimes e_i\Big|^p_F &= \sum_{i=1}^d\sum_{k=1}^m\sum_{l=1}^d \Big|\widetilde{e}_k^T[f_\varepsilon(\lambda_j,e_i)\otimes e_i]e_l\Big|^p = \sum_{i=1}^d\sum_{k=1}^m\sum_{l=1}^d \Big|\widetilde{e}_k^T(f_\varepsilon(\lambda_j,e_i)\delta_{il})\Big|^p \\
&= \sum_{k=1}^m\sum_{l=1}^d \Big|\widetilde{e}_k^T(f_\varepsilon(\lambda_j,e_l))\Big|^p = \sum_{k=1}^m\sum_{l=1}^d \Big|\sum_{i=1}^d \widetilde{e}_k^T(f_\varepsilon(\lambda_j,e_i)\delta_{il})\Big|^p \\
&= \sum_{k=1}^m\sum_{l=1}^d \Big|\widetilde{e}_k^T[\sum_{i=1}^d(f_\varepsilon(\lambda_j,e_i)\otimes e_i]e_l\Big|^p = \Big|\sum_{i=1}^d f_\varepsilon(\lambda_j,e_i)\otimes e_i\Big|^p_F,
\end{aligned}$$

This identity turns the right hand side of (2.13) into

$$\int_{\Gamma^\varepsilon_{\text{int}}} \varepsilon^{1-p}|[\![v(s)]\!]|^p \mathrm{d}s \leq \sum_{\substack{j=1,\\ \lambda\in\Lambda^+_\varepsilon}}^{2^d} \frac{\varepsilon^d}{2^d}\Big|\sum_{i=1}^d \frac{1}{\varepsilon}\Big[\widetilde{v}\big(\tfrac{\varepsilon}{2}(\lambda_j-e_i)\big) - \widetilde{v}\big(\tfrac{\varepsilon}{2}(\lambda_j+e_i)\big)\Big]\otimes e_i\Big|^p \delta^{(\lambda)}_{i,j}\widetilde{\delta}^{(\lambda)}_{i,j}.$$

Replacing $\frac{\varepsilon^d}{2^d}$ by the integral over $\frac{\varepsilon}{2}(\lambda_j+Y)$we finally end up with

$$\begin{aligned}
\int_{\Gamma^\varepsilon_{\text{int}}} \varepsilon^{1-p}|[\![v(s)]\!]|^p \mathrm{d}s &\leq \sum_{\substack{j=1,\\ \lambda\in\Lambda^+_\varepsilon}}^{2^d} \int_{\frac{\varepsilon}{2}(\lambda_j+Y)}\Big|\sum_{i=1}^d \frac{1}{\varepsilon}\Big[\widetilde{v}\big(\tfrac{\varepsilon}{2}(\lambda_j-e_i)\big) - \widetilde{v}\big(\tfrac{\varepsilon}{2}(\lambda_j+e_i)\big)\Big]\otimes e_i\Big|^p \delta^{(\lambda)}_{i,j}\widetilde{\delta}^{(\lambda)}_{i,j}\mathrm{d}x \\
&= \sum_{\substack{j=1,\\ \lambda\in\Lambda^+_\varepsilon}}^{2^d} \int_{\frac{\varepsilon}{2}(\lambda_j+Y)}\Big|\sum_{i=1}^d \delta^{(\lambda)}_{i,j}\widetilde{\delta}^{(\lambda)}_{i,j}\frac{1}{\varepsilon}\Big[\widetilde{v}(x-\tfrac{\varepsilon}{2}e_i) - \widetilde{v}(x+\tfrac{\varepsilon}{2}e_i)\Big]\otimes e_i\Big|^p \mathrm{d}x \\
&= \Big\|\sum_{i=1}^d \widetilde{R}^{(i)}_{\frac{\varepsilon}{2}}(\widetilde{v})\Big\|^p_{\mathrm{L}^p(\Omega^+_\varepsilon)^{m\times d}},
\end{aligned}$$

where we used $\widetilde{v}(x\pm\frac{\varepsilon}{2}e_i) \equiv \widetilde{v}(\frac{\varepsilon}{2}\lambda_j\pm\frac{\varepsilon}{2}e_i)$ for $x\in\frac{\varepsilon}{2}(\lambda_j+Y)\subset\Omega^+_\varepsilon$, which is valid for all functions belonging to $\mathrm{K}_{\varepsilon\Lambda}(\Omega^+_\varepsilon)^m$ due to their special structure. Replacing $\widetilde{v}$ by $V_\varepsilon v$ concludes the proof. ☐

Since the proof of Theorem 5.2 in [3] relies on the identity (2.6), in the next Lemma we state a similar identity for our discrete gradient $R_{\frac{\varepsilon}{2}} : \mathrm{K}_{\varepsilon\Lambda}(\Omega)^m \to \mathrm{K}_{\frac{\varepsilon}{2}\Lambda}(\Omega^+_\varepsilon)^{m\times d}$.

Lemma 2.4. *For $\varepsilon>0$ and for all $v\in\mathrm{K}_{\varepsilon\Lambda}(\Omega)^m$ and every $\varphi\in\mathrm{K}_{\varepsilon\Lambda}(\Omega^-_\varepsilon)^{m\times d}$ it holds*

$$\int_\Omega R_{\frac{\varepsilon}{2}}(v)(x) : \varphi^{\text{ex}}(x)\mathrm{d}x = -\int_{\Gamma^\varepsilon_{\text{int}}} [\![v(s)]\!] : \{\{\varphi^{\text{ex}}(s)\}\}\mathrm{d}s, \quad (2.14)$$

where $\varphi^{\text{ex}}\in\mathrm{L}^1(\mathbb{R}^d)$ is the extension with 0 to $\mathbb{R}^d$ of the function $\varphi\in\mathrm{K}_{\varepsilon\Lambda}(\Omega^-_\varepsilon)^{m\times d}$.

Proof. We start with rearranging the right hand side of (2.14). Since we are only testing with functions $\varphi \in \mathrm{K}_{\varepsilon\Lambda}(\Omega_\varepsilon^-)^{m\times d}$, analogously to the proof of Lemma 2.3 the function $v \in \mathrm{K}_{\varepsilon\Lambda}(\Omega)^m$ can be replaced by the extension $\widetilde{v} := (V_\varepsilon(v)) \in \mathrm{K}_{\varepsilon\Lambda}(\Omega_\varepsilon^+)^m$.

Let $\lambda \in \Lambda$ and $s \in \varepsilon(\lambda{+}\partial Y)$. Then $\{\{\varphi^{\mathrm{ex}}(s)\}\} \neq 0$ implies $s \in \Gamma_{\mathrm{int}}^\varepsilon$, which is why the domain of integration can be increased to $\cup_{\lambda\in\Lambda}\varepsilon(\lambda{+}\partial Y)$. Therefore, $\widetilde{v} \in \mathrm{K}_{\varepsilon\Lambda}(\Omega_\varepsilon^+)^m$ needs to be replaced by its extension $\widetilde{v}^{\mathrm{ex}} \in \mathrm{K}_{\varepsilon\Lambda}(\mathbb{R}^d)^m$ extending it with 0 to $\mathbb{R}^d$. Note, that according to $\{\{\varphi^{\mathrm{ex}}(s)\}\} \equiv 0$ for $s \in \partial\Omega_\varepsilon^+$ the additional jump $[\![\widetilde{v}^{\mathrm{ex}}(s)]\!] \neq 0$ does not play any role in the following calculations. On the right hand side of (2.15) below, every face of a cell $\varepsilon(\lambda{+}Y)$ is taken twice when summing up which is why this is an equality:

$$\int_{\Gamma_{\mathrm{int}}^\varepsilon} [\![v(s)]\!] : \{\{\varphi^{\mathrm{ex}}(s)\}\}\mathrm{d}s = \tfrac{1}{2}\sum_{\lambda\in\Lambda}\int_{\varepsilon(\lambda+\partial Y)} [\![\widetilde{v}^{\mathrm{ex}}(s)]\!] : \{\{\varphi^{\mathrm{ex}}(s)\}\}\mathrm{d}s. \tag{2.15}$$

Analogously to the proof of Lemma 2.3 we calculate the integral which gives the factor ε^{d-1}. Furthermore, the jump term of $\widetilde{v}^{\mathrm{ex}}$ and the mean value term of φ^{ex} are replaced by $\big(\widetilde{v}^{\mathrm{ex}}(\varepsilon\lambda) - \widetilde{v}^{\mathrm{ex}}(\varepsilon(\lambda{+}e_i))\big) \otimes e_i$ and $\frac{1}{2}\big(\varphi^{\mathrm{ex}}(\varepsilon\lambda) + \varphi^{\mathrm{ex}}(\varepsilon(\lambda{+}e_i))\big)$ for one face of $\varepsilon(\lambda{+}Y)$ and by $\big(\widetilde{v}^{\mathrm{ex}}(\varepsilon\lambda) - \widetilde{v}^{\mathrm{ex}}(\varepsilon(\lambda{-}e_i))\big) \otimes (-e_i)$ and $\frac{1}{2}\big(\varphi^{\mathrm{ex}}(\varepsilon\lambda) + \varphi^{\mathrm{ex}}(\varepsilon(\lambda{-}e_i))\big)$ for the opposite one:

$$\int_{\Gamma_{\mathrm{int}}^\varepsilon} [\![v(s)]\!] : \{\{\varphi^{\mathrm{ex}}(s)\}\}\mathrm{d}s$$

$$= \tfrac{1}{2}\sum_{\lambda\in\Lambda}\varepsilon^{d-1}\sum_{i=1}^{d}\Big[\big(\widetilde{v}^{\mathrm{ex}}(\varepsilon\lambda) - \widetilde{v}^{\mathrm{ex}}(\varepsilon(\lambda{+}e_i))\big)\otimes e_i : \tfrac{1}{2}\big(\varphi^{\mathrm{ex}}(\varepsilon\lambda) + \varphi^{\mathrm{ex}}(\varepsilon(\lambda{+}e_i))\big) \tag{2.16a}$$

$$+ \ \big(\widetilde{v}^{\mathrm{ex}}(\varepsilon(\lambda{-}e_i)) - \widetilde{v}^{\mathrm{ex}}(\varepsilon\lambda)\big)\otimes e_i : \tfrac{1}{2}\big(\varphi^{\mathrm{ex}}(\varepsilon(\lambda{-}e_i)) + \varphi^{\mathrm{ex}}(\varepsilon\lambda)\big)\Big]. \tag{2.16b}$$

Now, the sums are interchanged and the translation $\lambda^* = \lambda{-}e_i$ is applied to line (2.16b) for every $i = 1, \ldots, d$, such that we end up with

$$\int_{\Gamma_{\mathrm{int}}^\varepsilon} [\![v(s)]\!] : \{\{\varphi^{\mathrm{ex}}(s)\}\}\mathrm{d}s$$

$$= \tfrac{\varepsilon^{d-1}}{2}\sum_{i=1}^{d}\sum_{\lambda\in\Lambda}\big(\widetilde{v}^{\mathrm{ex}}(\varepsilon\lambda) - \widetilde{v}^{\mathrm{ex}}(\varepsilon(\lambda{+}e_i))\big)\otimes e_i : \big(\varphi^{\mathrm{ex}}(\varepsilon\lambda) + \varphi^{\mathrm{ex}}(\varepsilon(\lambda{+}e_i))\big). \tag{2.17}$$

For rearranging the left hand side of (2.14) we introduce $Y_{e_i} = \{y \in Y : y{-}\frac{1}{2}e_i \in Y\}$ ($Y = [0,1)^d \Rightarrow Y_{e_1} = [\frac{1}{2},1){\times}[0,1)^{d-1}$) and $f_\varepsilon^{(i)}(x) := \frac{1}{\varepsilon}\big(\widetilde{v}(x{+}\frac{\varepsilon}{2}e_i) - \widetilde{v}(x{-}\frac{\varepsilon}{2}e_i)\big) \otimes e_i$ to shorten notation. Since $\mathrm{supp}(\varphi) \subset \overline{\Omega_\varepsilon^-}$, again v can be replaced by $\widetilde{v} := V_\varepsilon v$ on the left hand side of (2.14), which leads to

$$\int_\Omega R_{\frac{\varepsilon}{2}}(v)(x) : \varphi^{\mathrm{ex}}(x)\mathrm{d}x = \sum_{\lambda\in\Lambda_\varepsilon^-}\int_{\varepsilon(\lambda+Y)}\sum_{i=1}^{d} f_\varepsilon^{(i)}(x) : \varphi(\varepsilon\lambda)\mathrm{d}x, \tag{2.18}$$

where we already used $\varphi(x) \equiv \varphi(\varepsilon\lambda)$ for $x \in \varepsilon(\lambda{+}Y)$ and $\lambda \in \Lambda_\varepsilon^-$. Observing that

$$f_\varepsilon^{(i)}(x) = \begin{cases} \frac{1}{\varepsilon}\big(\widetilde{v}(\varepsilon(\lambda{+}e_i)) - \widetilde{v}(\varepsilon\lambda)\big) \otimes e_i & \text{if } x \in \varepsilon(\lambda{+}Y_{e_i}), \\ \frac{1}{\varepsilon}\big(\widetilde{v}(\varepsilon\lambda) - \widetilde{v}(\varepsilon(\lambda{-}e_i))\big) \otimes e_i & \text{if } x \in \varepsilon(\lambda{+}Y\backslash Y_{e_i}) \end{cases}$$

we are able to reformulate the right hand side of (2.18) by interchanging integration and summation in the following way:

$$\begin{aligned}
&\int_\Omega R_{\frac{\varepsilon}{2}}(v)(x) : \varphi^{\mathrm{ex}}(x)\mathrm{d}x \\
&\quad= \sum_{\lambda\in\Lambda_\varepsilon^-}\sum_{i=1}^d \Big(\int_{\varepsilon(\lambda+Y_{e_i})} f_\varepsilon^{(i)}(x) : \varphi(\varepsilon\lambda)\mathrm{d}x + \int_{\varepsilon(\lambda+Y\setminus Y_{e_i})} f_\varepsilon^{(i)}(x) : \varphi(\varepsilon\lambda)\mathrm{d}x\Big) \\
&\quad= \sum_{\lambda\in\Lambda_\varepsilon^-}\sum_{i=1}^d \tfrac{1}{2}\varepsilon^d \tfrac{1}{\varepsilon}\big(\widetilde{v}(\varepsilon(\lambda+e_i)) - \widetilde{v}(\varepsilon\lambda)\big)\otimes e_i : \varphi(\varepsilon\lambda) && \text{(2.19a)}\\
&\qquad\qquad + \tfrac{1}{2}\varepsilon^d\tfrac{1}{\varepsilon}\big(\widetilde{v}(\varepsilon\lambda) - \widetilde{v}(\varepsilon(\lambda-e_i))\big)\otimes e_i : \varphi(\varepsilon\lambda). && \text{(2.19b)}
\end{aligned}$$

Here, we already used, that $f_\varepsilon^{(i)}$ is constant on the domain of integration. Since $\varphi^{\mathrm{ex}}(\varepsilon\lambda) = 0$ for all $\lambda \in \Lambda\setminus\Lambda_\varepsilon^-$, the first sum in (2.19) can be replaced by the sum of $\lambda \in \Lambda$. Afterwards, again the sums are interchanged and the translation $\lambda^* = \lambda - e_i$ is applied to line (2.19b) for every $i = 1, \dots, d$, such that we end up with

$$\begin{aligned}
&\int_\Omega R_{\frac{\varepsilon}{2}}(v)(x) : \varphi^{\mathrm{ex}}(x)\mathrm{d}x \\
&\quad= \tfrac{\varepsilon^{d-1}}{2}\sum_{i=1}^d\sum_{\lambda\in\Lambda}\big(\widetilde{v}^{\mathrm{ex}}(\varepsilon(\lambda+e_i)) - \widetilde{v}^{\mathrm{ex}}(\varepsilon\lambda)\big)\otimes e_i : \big(\varphi^{\mathrm{ex}}(\varepsilon\lambda) + \varphi^{\mathrm{ex}}(\varepsilon(\lambda+e_i))\big). \qquad (2.20)
\end{aligned}$$

Comparing (2.20) and (2.17) we find that (2.14) is valid. □

Now we are in the position to prove Theorem 2.1.

Proof. Here, we mainly follow the steps of the proof of Theorem 5.2 of [3] and explain the main differences. We use the uniqueness of the limit of a sequence of BV-function in the same way as it is done in the Sobolev case, i.e. $f_\varepsilon \rightharpoonup f$ in $\mathrm{L}^p(\Omega)$, with $f \in \mathrm{W}^{1,p}(\Omega)$ and $\nabla f_\varepsilon \rightharpoonup g$ in $\mathrm{L}^p(\Omega)^d$ implies $g = \nabla f$.

As already mentioned in [3] the distributional derivative Du of a broken Sobolev function $u \in \mathrm{W}^{1,p}_{\varepsilon\Lambda}(\Omega)^m$ is given by

$$\langle Du, \psi\rangle = \int_\Omega \nabla u : \psi \mathrm{d}x - \int_{\Gamma^\varepsilon_{\mathrm{int}}} [\![u]\!] : \psi \mathrm{d}s \qquad \forall\, \psi \in \mathrm{C}_c^\infty(\Omega)^{m\times d}. \tag{2.21}$$

This can be seen by using integration by parts on each cell $\varepsilon(\lambda+Y)$.

Now, let $(v_\varepsilon)_{\varepsilon>0} \subset \mathrm{K}_{\varepsilon\Lambda}(\Omega)^m$ satisfy condition (2.9) of Theorem 2.1. Since L^p is reflexive ($p \in (1,\infty)$), there exists a subsequence and limit elements $v_0 \in \mathrm{L}^p(\Omega)^m$, $V_0 \in \mathrm{L}^p(\Omega)^{m\times d}$ such that $v_{\varepsilon'} \rightharpoonup v_0$ in $\mathrm{L}^p(\Omega)^m$ and $R_{\frac{\varepsilon'}{2}} v_{\varepsilon'} \rightharpoonup V_0$ in $\mathrm{L}^p(\Omega)^{m\times d}$. The goal is to show that $v_0 \in \mathrm{W}^{1,p}(\Omega)^m$ with $Dv_0 = V_0$. Using (2.21) for $v_\varepsilon \in \mathrm{K}_{\varepsilon\Lambda}(\Omega)^m$ we find with $\psi \in \mathrm{C}_c^\infty(\Omega)^{m\times d}$ arbitrary but fixed

$$\langle Dv_\varepsilon, \psi\rangle = -\int_{\Gamma^\varepsilon_{\mathrm{int}}} [\![v_\varepsilon]\!] : \psi \mathrm{d}s. \tag{2.22}$$

Choosing $\varepsilon_0 > 0$ so small such that $\mathrm{supp}(\psi) \subset \overline{\Omega^-_{\varepsilon_0}}$ we are able to find a sequence $(\varphi_\varepsilon)_{(0<\varepsilon<\varepsilon_0)}$ with $\varphi_\varepsilon \in \mathrm{K}_{\varepsilon\Lambda}(\Omega_\varepsilon^-)^{m\times d}$ such that $\|\psi - \varphi_\varepsilon^{\mathrm{ex}}\|_{\mathrm{L}^\infty(\Omega)^{m\times d}} \to 0$ for $\varepsilon \to 0$. By adding and subtracting $\varphi_\varepsilon^{\mathrm{ex}}$ we find with (2.22)

$$
\begin{aligned}
\langle Dv_\varepsilon, \psi\rangle &= -\int_{\Gamma_{\mathrm{int}}^\varepsilon} [\![v_\varepsilon]\!]:\{\{\psi-\varphi_\varepsilon^{\mathrm{ex}}\}\}\mathrm{d}s - \int_{\Gamma_{\mathrm{int}}^\varepsilon} [\![v_\varepsilon]\!]:\{\{\varphi_\varepsilon^{\mathrm{ex}}\}\}\mathrm{d}s \\
&\overset{(2.14)}{=} -\int_{\Gamma_{\mathrm{int}}^\varepsilon} [\![v_\varepsilon]\!]:\{\{\psi-\varphi_\varepsilon^{\mathrm{ex}}\}\}\mathrm{d}s + \int_\Omega R_{\frac{\varepsilon}{2}}(v_\varepsilon):\varphi_\varepsilon^{\mathrm{ex}}\mathrm{d}x \\
&= -\int_{\Gamma_{\mathrm{int}}^\varepsilon} [\![v_\varepsilon]\!]:\{\{\psi-\varphi_\varepsilon^{\mathrm{ex}}\}\}\mathrm{d}s + \int_\Omega R_{\frac{\varepsilon}{2}}(v_\varepsilon):(\varphi_\varepsilon^{\mathrm{ex}}-\psi)\mathrm{d}x + \int_\Omega R_{\frac{\varepsilon}{2}}(v_\varepsilon):\psi\mathrm{d}x \qquad (2.23)
\end{aligned}
$$

As we will see below, the first two terms of (2.23) are bounded by $C\|\psi-\varphi_\varepsilon^{\mathrm{ex}}\|_{\mathrm{L}^\infty(\Omega)^{m\times d}}$ and hence tend to 0 as $\varepsilon \to 0$. Therefore, since $R_{\frac{\varepsilon'}{2}}v_{\varepsilon'} \rightharpoonup V_0$ in $\mathrm{L}^p(\Omega)^{m\times d}$, we end up with

$$
\lim_{\varepsilon'\to 0}\langle Dv_{\varepsilon'}, \psi\rangle = \int_\Omega V_0 : \psi \mathrm{d}s \qquad \forall\, \psi \in \mathrm{C}_c^\infty(\Omega)^{m\times d}. \qquad (2.24)
$$

To show the boundedness of the first two terms of (2.23) we use Hölder's inequality to conclude with Lemma 2.3

$$
\begin{aligned}
\Big| -\int_{\Gamma_{\mathrm{int}}^\varepsilon} [\![v_\varepsilon]\!] : \{\{\psi-\varphi_\varepsilon^{\mathrm{ex}}\}\}\mathrm{d}s\Big| &\le \|[\![v_\varepsilon]\!]\|_{\mathrm{L}^p(\Gamma_{\mathrm{int}}^\varepsilon)^{m\times d}}\|\{\{\psi-\varphi_\varepsilon^{\mathrm{ex}}\}\}\|_{\mathrm{L}^{p'}(\Gamma_{\mathrm{int}}^\varepsilon)^{m\times d}} \\
&\le \varepsilon^{\frac{p-1}{p}}\|R_{\frac{\varepsilon}{2}}(v_\varepsilon)\|_{\mathrm{L}^p(\Omega_\varepsilon^+)^{m\times d}}\|\psi-\varphi_\varepsilon^{\mathrm{ex}}\|_{\mathrm{L}^\infty(\Omega)^{m\times d}}\mathrm{area}(\Gamma_{\mathrm{int}}^\varepsilon)^{\frac{1}{p'}} \\
&\le \|R_{\frac{\varepsilon}{2}}(v_\varepsilon)\|_{\mathrm{L}^p(\Omega_\varepsilon^+)^{m\times d}}\|\psi-\varphi_\varepsilon^{\mathrm{ex}}\|_{\mathrm{L}^\infty(\Omega)^{m\times d}}(d\mathrm{vol}(\Omega_\varepsilon^+))^{\frac{1}{p'}}
\end{aligned}
$$

and

$$
\begin{aligned}
\Big|\int_\Omega R_{\frac{\varepsilon}{2}}(v_\varepsilon) : (\varphi_\varepsilon^{\mathrm{ex}} - \psi)\mathrm{d}x\Big| &\le \|R_{\frac{\varepsilon}{2}}(v_\varepsilon)\|_{\mathrm{L}^p(\Omega_\varepsilon^+)^{m\times d}}\|\varphi_\varepsilon^{\mathrm{ex}}-\psi\|_{\mathrm{L}^{p'}(\Omega)^{m\times d}} \\
&\le \|R_{\frac{\varepsilon}{2}}(v_\varepsilon)\|_{\mathrm{L}^p(\Omega_\varepsilon^+)^{m\times d}}\|\varphi_\varepsilon^{\mathrm{ex}}-\psi\|_{\mathrm{L}^\infty(\Omega)^{m\times d}}\mathrm{vol}(\Omega)^{\frac{1}{p'}}.
\end{aligned}
$$

Here, we already used $\mathrm{area}(\Gamma_{\mathrm{int}}^\varepsilon) \le d\,\mathrm{vol}(\Omega)\varepsilon^{-1}$, which is valid since $\mathrm{area}(\Gamma_{\mathrm{int}}^\varepsilon)$ is bounded by the product of the number of cells contained in Ω_ε^+, which is $\mathrm{vol}(\Omega_\varepsilon^+)\varepsilon^{-d}$, and the volume of the part of $\Gamma_{\mathrm{int}}^\varepsilon$ contained in one cell, which is $d\varepsilon^{d-1}$. With this, the assumed uniform bound of the term $\|R_{\frac{\varepsilon}{2}}(v_\varepsilon)\|_{\mathrm{L}^p(\Omega_\varepsilon^+)^{m\times d}}$ yields the result.

On the other hand using the definition of the distributional derivative of $v_{\varepsilon'} \in \mathrm{K}_{\varepsilon\Lambda}(\Omega)^m$ and $v_{\varepsilon'} \rightharpoonup v_0$ in $\mathrm{L}^p(\Omega)^m$, we have

$$
\lim_{\varepsilon'\to 0}\langle Dv_{\varepsilon'}, \psi\rangle = \lim_{\varepsilon'\to 0} -\int_\Omega v_{\varepsilon'}\cdot \mathrm{div}\psi\mathrm{d}x = -\int_\Omega v_0\cdot\mathrm{div}\psi\mathrm{d}x \qquad \forall\,\psi\in\mathrm{C}_c^\infty(\Omega)^{m\times d}. \quad (2.25)
$$

Now, combining (2.24) and (2.25) we obtain

$$
\int_\Omega V_0 : \psi\mathrm{d}x = -\int_\Omega v_0\cdot\mathrm{div}\psi\mathrm{d}x \qquad \forall\,\psi\in\mathrm{C}_c^\infty(\Omega)^{m\times d},
$$

which gives us $v_0 \in \mathrm{W}^{1,p}(\Omega)^m$ and $Dv_0 = V_0$.

Finally, we use the fact that $v_{\varepsilon'} \overset{*}{\rightharpoonup} v_0$ in $\mathrm{BV}(\Omega)^m$ implies $v_{\varepsilon'} \to v_0$ in $\mathrm{L}^1(\Omega)^m$ in order to conclude $v_{\varepsilon'} \to v_0$ in $\mathrm{L}^q(\Omega)^m$ for every $q \in [1, p^*)$. Thereby we use the following interpolation inequality obtained by Hölder's inequality for every $\theta \in (0,1)$:

$$
\|v_\varepsilon - v_0\|_{\mathrm{L}^q(\Omega)^m} \le \|v_\varepsilon-v_0\|_{\mathrm{L}^{p^*}(\Omega)^m}^{1-\theta}\|v_\varepsilon-v_0\|_{\mathrm{L}^1(\Omega)^m}^\theta,
$$

and the term $\|v_\varepsilon - v_0\|_{\mathrm{L}^{p^*}(\Omega)^m}$ is bounded due to the following Sobolev-Poincare inequality proved in Theorem 4.1 of [3] and Lemma 2.3:

$$\|v_\varepsilon\|_{\mathrm{L}^{p^*}(\Omega)^m} \le C_S \left(\|v_\varepsilon\|_{\mathrm{L}^1(\Omega)^m} + \left(\int_{\Gamma^\varepsilon_{\mathrm{int}}} \varepsilon^{1-p} |[\![v_\varepsilon(s)]\!]|^p \mathrm{d}s \right)^{\frac{1}{p}} \right).$$

This finishes the proof. □

Definition 2.5 (Projector to piecewise constant functions)**.** Let $\varepsilon > 0$ and $p \in [1, \infty)$. The projector $P_\varepsilon : \mathrm{L}^p(\mathbb{R}^d) \to \mathrm{K}_{\varepsilon\Lambda}(\mathbb{R}^d)$ to piecewise constant functions is defined via

$$P_\varepsilon w(x) := \fint_{\mathcal{N}_\varepsilon(x) + \varepsilon Y} w(\xi) \mathrm{d}\xi,$$

where $\fint_A g(a)\mathrm{d}a := \frac{1}{\mathrm{vol}(A)} \int_A g(a)\mathrm{d}a$ is the average of the function g over A and $\mathcal{N}_\varepsilon : \mathbb{R}^d \to \varepsilon\Lambda$ maps every point $x \in \varepsilon(\lambda + Y) \subset \mathbb{R}^d$ to the lattice point $\varepsilon\lambda \in \varepsilon\Lambda$.

Remark 2.6. Note, that the mapping $\mathcal{N}_\varepsilon : \mathbb{R}^d \to \varepsilon\Lambda$ is well-defined for arbitrary choices of Y, as long as $\bigcup_{\lambda\in\Lambda}(\lambda + Y) = \mathbb{R}^d$ and $(\lambda_1 + Y) \cap (\lambda_2 + Y) = \emptyset$ for all $\lambda_1 \neq \lambda_2$ are fulfilled. In this way $\mathcal{N}_\varepsilon : \mathbb{R}^d \to \varepsilon\Lambda$ does not depend on the choice of Y, so we do not need to worry about it in the following sections.

Moreover note, that $V_\varepsilon((P_\varepsilon w^{\mathrm{ex}})|_\Omega) \equiv (P_\varepsilon w^{\mathrm{ex}})|_{\Omega^+_\varepsilon}$ for $w \in \mathrm{L}^p(\Omega)$.

Theorem 2.7 (Approximation result)**.** *For every function $v_0 \in \mathrm{W}^{1,p}(\Omega)^m$ there exists a sequence $(v_\varepsilon)_{\varepsilon>0} \subset \mathrm{K}_{\varepsilon\Lambda}(\Omega)^m$ so that*

$$\lim_{\varepsilon\to 0} \left(\|v_0 - v_\varepsilon\|_{\mathrm{L}^p(\Omega)^m} + \|\nabla v_0 - R_{\frac{\varepsilon}{2}}(v_\varepsilon)\|_{\mathrm{L}^p(\Omega)^{m\times d}} \right) = 0. \tag{2.26}$$

Proof. Choose $\varepsilon_0 > 0$ and $\delta > 0$ such that for all $\varepsilon \in (0, \varepsilon_0)$ we have $\Omega^+_\varepsilon \subset B_\delta(\Omega)$. Here, $B_\delta(\Omega)$ denotes a δ-neighborhood of Ω. Let $v_0 \in \mathrm{C}^\infty(\Omega)^m \cap \mathrm{W}^{1,p}(\Omega)^m$ and $\widetilde{v}_0 \in \mathrm{W}^{1,p}_0(B_\delta(\Omega))^m$ with $\widetilde{v}_0|_\Omega = v_0$ which exists according to Theorem A 6.12 in [2]. For $\varepsilon \in (0, \varepsilon_0)$ we define $v_\varepsilon := (P_\varepsilon \widetilde{v}^{\mathrm{ex}}_0)|_\Omega$ and prove that the sequence $(v_\varepsilon)_{\varepsilon\in(0,\varepsilon_0)}$ satisfies (2.26).

1. Proving $v_\varepsilon \to v_0$ in $\mathrm{L}^p(\Omega)^m$ we start by decomposing Ω into Ω^-_ε and $\Omega\backslash\Omega^-_\varepsilon$, which allows us to exploit $(P_\varepsilon \widetilde{v}^{\mathrm{ex}}_0)|_{\Omega^-_\varepsilon} \equiv (P_\varepsilon v^{\mathrm{ex}}_0)|_{\Omega^-_\varepsilon}$, since $\widetilde{v}_0|_{\Omega^-_\varepsilon} \equiv v_0|_{\Omega^-_\varepsilon}$ by definition. Afterwards we increase the domain of integration and apply the triangle inequality. Then again the domain of integration is increased and at last $\|P_\varepsilon w\|_{\mathrm{L}^p(\Omega^\pm_\varepsilon)} \le \|w\|_{\mathrm{L}^p(\Omega^\pm_\varepsilon)}$ is used for $w \in \mathrm{L}^p(\mathbb{R}^d)$:

$$\begin{aligned}
\|v_0 - P_\varepsilon \widetilde{v}^{\mathrm{ex}}_0\|^p_{\mathrm{L}^p(\Omega)^m} &= \|v_0 - P_\varepsilon v^{\mathrm{ex}}_0\|^p_{\mathrm{L}^p(\Omega^-_\varepsilon)^m} + \|v_0 - P_\varepsilon \widetilde{v}^{\mathrm{ex}}_0\|^p_{\mathrm{L}^p(\Omega\backslash\Omega^-_\varepsilon)^m} \\
&\le \|v_0 - P_\varepsilon v^{\mathrm{ex}}_0\|^p_{\mathrm{L}^p(\Omega)^m} + \|v_0\|^p_{\mathrm{L}^p(\Omega\backslash\Omega^-_\varepsilon)^m} + \|P_\varepsilon \widetilde{v}^{\mathrm{ex}}_0\|^p_{\mathrm{L}^p(\Omega^+_\varepsilon\backslash\Omega^-_\varepsilon)^m} \\
&\le \|v_0 - P_\varepsilon v^{\mathrm{ex}}_0\|^p_{\mathrm{L}^p(\Omega)^m} + \|v_0\|^p_{\mathrm{L}^p(\Omega\backslash\Omega^-_\varepsilon)^m} + \|\widetilde{v}_0\|^p_{\mathrm{L}^p(\Omega^+_\varepsilon\backslash\Omega^-_\varepsilon)^m}.
\end{aligned}$$

Since $P_\varepsilon w^{\mathrm{ex}} \to w$ in $\mathrm{L}^p(\Omega)$ for every $w \in \mathrm{L}^p(\Omega)$ and since $0 \le \mathrm{vol}(\Omega\backslash\Omega^-_\varepsilon) \le \mathrm{vol}(\Omega^+_\varepsilon\backslash\Omega^-_\varepsilon) \to 0$ according to (2.3) this inequality proves $v_\varepsilon \to v_0$ in $\mathrm{L}^p(\Omega)^m$.

2. For $R_{\frac{\varepsilon}{2}}(v_\varepsilon) \to \nabla v_0$ in $\mathrm{L}^p(\Omega)^{m\times d}$ we prove $\lim_{\varepsilon\to 0} \|(\nabla v_0)e_i - (R_{\frac{\varepsilon}{2}}(v_\varepsilon))e_i\|_{\mathrm{L}^p(\Omega)^m} = 0$ for every $i \in \{1, \ldots, d\}$. Thereto, let $i \in \{1, \ldots, d\}$ be fixed. In the following calculations we start by adding and subtracting $(P_\varepsilon(\nabla\widetilde{v}_0)^{\mathrm{ex}})e_i$ to apply the triangle inequality.

$$\begin{aligned}
&\|(\nabla v_0)e_i - (R_{\frac{\varepsilon}{2}}(v_\varepsilon))e_i\|_{\mathrm{L}^p(\Omega)^m} \\
&\quad \le \|(\nabla v_0)e_i - (P_\varepsilon(\nabla\widetilde{v}_0)^{\mathrm{ex}})e_i\|_{\mathrm{L}^p(\Omega)^m} + \|(P_\varepsilon(\nabla\widetilde{v}_0)^{\mathrm{ex}})e_i - (R_{\frac{\varepsilon}{2}}(v_\varepsilon))e_i\|_{\mathrm{L}^p(\Omega)^m}
\end{aligned}$$

Then analogously to step 1 the first term tends to zero when $\varepsilon \to 0$. Moreover, $(R_{\frac{\varepsilon}{2}}(v_\varepsilon))e_i = (\widetilde{R}^{(i)}_{\frac{\varepsilon}{2}}(V_\varepsilon v_\varepsilon))e_i$ see (2.5) and the identity $V_\varepsilon v_\varepsilon = (P_\varepsilon \widetilde{v}_0^{\mathrm{ex}})|_{\Omega_\varepsilon^+}$ can be used to transform the second term in the following way.

$$
\begin{aligned}
&\|(P_\varepsilon(\nabla\widetilde{v}_0)^{\mathrm{ex}})e_i - (R_{\frac{\varepsilon}{2}}(v_\varepsilon))e_i\|_{\mathrm{L}^p(\Omega)^m} \\
&\quad = \|(P_\varepsilon(\nabla\widetilde{v}_0)^{\mathrm{ex}})e_i - (\widetilde{R}^{(i)}_{\frac{\varepsilon}{2}}((P_\varepsilon\widetilde{v}_0^{\mathrm{ex}})|_{\Omega_\varepsilon^+}))e_i\|_{\mathrm{L}^p(\Omega)^m} \\
&\quad \leq \|(P_\varepsilon(\nabla\widetilde{v}_0)^{\mathrm{ex}})e_i - \tfrac{1}{\varepsilon}\big(P_\varepsilon\widetilde{v}_0^{\mathrm{ex}}(\cdot + \tfrac{\varepsilon}{2}e_i) - P_\varepsilon\widetilde{v}_0^{\mathrm{ex}}(\cdot - \tfrac{\varepsilon}{2}e_i)\big)\|_{\mathrm{L}^p(A_\varepsilon)^m} &(2.27)\\
&\qquad + \|(P_\varepsilon(\nabla\widetilde{v}_0)^{\mathrm{ex}})e_i\|_{\mathrm{L}^p(B_\varepsilon)^m}, &(2.28)
\end{aligned}
$$

where, $A_\varepsilon := \{x \in \Omega_\varepsilon^+ \,|\, (x+\frac{\varepsilon}{2}e_i) \in \Omega_\varepsilon^+ \text{ and } (x-\frac{\varepsilon}{2}e_i) \in \Omega_\varepsilon^+\}$ and $B_\varepsilon := \Omega_\varepsilon^+ \backslash A_\varepsilon$ for fixed $i \in \{1, \dots, d\}$. Since $B_\varepsilon \subset \Omega_\varepsilon^+ \backslash \Omega_\varepsilon^-$, the term in line (2.28) is bounded. Moreover,

$$
\|(P_\varepsilon(\nabla\widetilde{v}_0)^{\mathrm{ex}})e_i\|_{\mathrm{L}^p(\Omega_\varepsilon^+\backslash\Omega_\varepsilon^-)^m} \leq \|(\nabla\widetilde{v}_0)e_i\|_{\mathrm{L}^p(\Omega_\varepsilon^+\backslash\Omega_\varepsilon^-)^m} \overset{\varepsilon\to 0}{\longrightarrow} 0,
$$

where again $\|P_\varepsilon w\|_{\mathrm{L}^p(\Omega_\varepsilon^\pm)} \leq \|w\|_{\mathrm{L}^p(\Omega_\varepsilon^\pm)}$ for $w \in \mathrm{L}^p(\mathbb{R}^d)$ and $\mathrm{vol}(\Omega_\varepsilon^+\backslash\Omega_\varepsilon^-) \to 0$ for $\varepsilon \to 0$ is used. The term in line (2.27) can be estimated by increasing the domain of integration, exploiting $\|P_\varepsilon w\|_{\mathrm{L}^p(\Omega_\varepsilon^\pm)} \leq \|w\|_{\mathrm{L}^p(\Omega_\varepsilon^\pm)}$ for $w \in \mathrm{L}^p(\mathbb{R}^d)$ and replacing $\frac{1}{\varepsilon}[\widetilde{v}_0(x+\frac{\varepsilon}{2}e_i) - \widetilde{v}_0(x-\frac{\varepsilon}{2}e_i)]$ by $\frac{1}{2}\int_{-1}^1 \nabla\widetilde{v}_0(x+\frac{\varepsilon}{2}e_i t)e_i \mathrm{d}t$ in the following way

$$
\begin{aligned}
&\|(P_\varepsilon(\nabla\widetilde{v}_0)^{\mathrm{ex}})e_i - \tfrac{1}{\varepsilon}\big(P_\varepsilon\widetilde{v}_0^{\mathrm{ex}}(\cdot + \tfrac{\varepsilon}{2}e_i) - P_\varepsilon\widetilde{v}_0^{\mathrm{ex}}(\cdot - \tfrac{\varepsilon}{2}e_i)\big)\|_{\mathrm{L}^p(A_\varepsilon)^m} \\
&\qquad \leq \|(P_\varepsilon(\nabla\widetilde{v}_0)^{\mathrm{ex}})e_i - \tfrac{1}{\varepsilon}\big(P_\varepsilon\widetilde{v}_0^{\mathrm{ex}}(\cdot + \tfrac{\varepsilon}{2}e_i) - P_\varepsilon\widetilde{v}_0^{\mathrm{ex}}(\cdot - \tfrac{\varepsilon}{2}e_i)\big)\|_{\mathrm{L}^p(\Omega_\varepsilon^+)^m} \\
&\qquad \leq \|(\nabla\widetilde{v}_0)e_i - \tfrac{1}{\varepsilon}\big(\widetilde{v}_0(\cdot + \tfrac{\varepsilon}{2}e_i) - \widetilde{v}_0(\cdot - \tfrac{\varepsilon}{2}e_i)\big)\|_{\mathrm{L}^p(\Omega_\varepsilon^+)^m} \\
&\qquad = \|(\nabla\widetilde{v}_0)e_i - \tfrac{1}{2}\textstyle\int_{-1}^1 (\nabla\widetilde{v}_0(\cdot + \tfrac{\varepsilon}{2}e_i t))e_i \mathrm{d}t\|_{\mathrm{L}^p(\Omega_\varepsilon^+)^m},
\end{aligned}
$$

which is valid for $\varepsilon \in (0, \varepsilon_0)$ small enough such that from $x \in \Omega_\varepsilon^+$ it follows $x+\frac{\varepsilon}{2}e_i \in B_\delta(\Omega)$ and $x-\frac{\varepsilon}{2}e_i \in B_\delta(\Omega)$.

With this estimate it is easy to prove for $v_0 \in \mathrm{C}^\infty(\Omega)^m \cap \mathrm{W}^{1,p}(\Omega)^m$ that the term in line (2.27) converges to zero, too. Then, by density, the claim of Theorem 2.7 holds for arbitrary $v_0 \in \mathrm{W}^{1,p}(\Omega)^m$, too. □

3 Mutual recovery sequence of the damage variable

This section is in preparation of proving the convergence of the microscopic damage models introduced in Subsection 4.2 to the effective damage model introduced in Subsection 4.3. More precisely, it serves as a basis for the construction of the mutual recovery sequence for our microscopic damage models. This in turn enables us to prove stability of the limit with respect to the effective damage model when starting with a sequence of stable states with respect to the microscopic models. Here, everything is done in the context of piecewise constant functions as introduced in Section 2. But, since in this section neither the concrete formula of the discrete gradient $R_{\frac{\varepsilon}{2}}$ nor the micro-structure of the microscopic damage models play any role, here the unit cell Y in principle could be any cube of volume 1. We choose again $Y = [0, 1)^d$.

In the following theorem for two given functions $v_0, \widetilde{v}_0$ satisfying $\widetilde{v}_0 \leq v_0$ and one given sequence $(v_\varepsilon)_{\varepsilon>0}$ the existence of a further sequence $(\widetilde{v}_\varepsilon)_{\varepsilon>0}$ is stated, such that among other things $\widetilde{v}_\varepsilon \leq v_\varepsilon$ is fulfilled. For the given solution $(u_\varepsilon(t), \chi_\varepsilon(t))$ of the microscopic model this will be the crucial relation when constructing the mutual recovery sequence $(\widetilde{\chi}_\varepsilon)_{\varepsilon>0}$ satisfying $\mathcal{D}_\varepsilon(\chi_\varepsilon(t), \widetilde{\chi}_\varepsilon) < \infty$ in Section 7.

Theorem 3.1 (Mutual recovery sequence)**.** *Let $v_0 \in \mathrm{W}^{1,p}(\Omega;[0,1])^m$ and let $(v_\varepsilon)_{\varepsilon>0}$ be a sequence of $\mathrm{K}_{\varepsilon\Lambda}(\Omega;[0,1])^m$ satisfying $v_\varepsilon \to v_0$ in $\mathrm{L}^p(\Omega)^m$ and $R_{\frac{\varepsilon}{2}} v_\varepsilon \rightharpoonup \nabla v_0$ in $\mathrm{L}^p(\Omega)^{m\times d}$. Moreover, let $\widetilde{v}_0 \in \mathrm{W}^{1,p}(\Omega;[0,1])^m$ be arbitrary with $\widetilde{v}_0 \leq v_0$ (component-wise).*

Then there exists a sequence $(\widetilde{v}_\varepsilon)_{\varepsilon>0} \subset \mathrm{K}_{\varepsilon\Lambda}(\Omega;[0,1])^m$ with $\widetilde{v}_\varepsilon \leq v_\varepsilon$ (component-wise), $\widetilde{v}_\varepsilon \to \widetilde{v}_0$ in $\mathrm{L}^p(\Omega)^m$, $R_{\frac{\varepsilon}{2}} \widetilde{v}_\varepsilon \rightharpoonup \nabla \widetilde{v}_0$ in $\mathrm{L}^p(\Omega)^{m\times d}$ and

$$\limsup_{\varepsilon\to 0} \left(\|R_{\frac{\varepsilon}{2}} \widetilde{v}_\varepsilon\|^p_{\mathrm{L}^p(\Omega)^{m\times d}} - \|R_{\frac{\varepsilon}{2}} v_\varepsilon\|^p_{\mathrm{L}^p(\Omega)^{m\times d}}\right) \leq \|\nabla \widetilde{v}_0\|^p_{\mathrm{L}^p(\Omega)^{m\times d}} - \|\nabla v_0\|^p_{\mathrm{L}^p(\Omega)^{m\times d}}. \quad (3.1)$$

Proof. 1. The construction of a mutual recovery sequence is based on that done in [18]. There, the authors constructed a mutual recovery sequence of scalar Sobolev functions. Here, the main steps of the proof stay the same but due to the discrete setting on the ε-level and the vectorial case some new technicalities come into play.

Let $v_0, \widetilde{v}_0 \in \mathrm{W}^p(\Omega;[0,1])^m$ and $(v_\varepsilon)_{\varepsilon>0} \subset \mathrm{K}_{\varepsilon\Lambda}(\Omega;[0,1])^m$ be given as in Theorem 3.1. Following the proof in [18] we introduce the function $\widetilde{v}_\varepsilon \in \mathrm{K}_{\varepsilon\Lambda}(\Omega;[0,1])^m$ decomposed for every component $\widetilde{v}^{(j)}_\varepsilon$, $j \in \{1,2,\ldots,m\}$, in the following way:

$$\widetilde{v}^{(j)}_\varepsilon(x) = \begin{cases} \max\{0, P_\varepsilon \widetilde{v}^{(j)}_0(x) - \delta^{(j)}_\varepsilon\} & \text{if } x \in A^{(j)}_\varepsilon = \Omega \backslash B^{(j)}_\varepsilon \\ v^{(j)}_\varepsilon(x) & \text{if } x \in B^{(j)}_\varepsilon \end{cases}, \quad (3.2)$$

where $B^{(j)}_\varepsilon = \{x \in \Omega : v^{(j)}_\varepsilon(x) < \max\{0, P_\varepsilon \widetilde{v}^{(j)}_0(x) - \delta^{(j)}_\varepsilon\}\}$. The positive constant δ^j_ε will be chosen later in such a way that $\delta^j_\varepsilon \to 0$ for $\varepsilon \to 0$. Definition (3.2) immediately gives us $0 \leq \widetilde{v}_\varepsilon \leq v_\varepsilon$.

2. Now, we prove that $\widetilde{v}_\varepsilon \to \widetilde{v}_0$ in $\mathrm{L}^p(\Omega)^m$. Since $\widetilde{v}_\varepsilon \to \widetilde{v}_0$ in $\mathrm{L}^p(\Omega)^m$ is equivalent to $\widetilde{v}^{(j)}_\varepsilon \to \widetilde{v}^{(j)}_0$ in $\mathrm{L}^p(\Omega)$ for every $j \in \{1,2,\ldots,m\}$ we will restrict ourselves to the case $m = 1$. Hence, let $A_\varepsilon = A^{(j)}_\varepsilon$, $B_\varepsilon = B^{(j)}_\varepsilon$ and $\delta_\varepsilon = \delta^{(j)}_\varepsilon$ to shorten notation. According to $|v_\varepsilon(x) - \widetilde{v}_0(x)| \leq 1$, especially on B_ε, we find

$$\|\widetilde{v}_\varepsilon - \widetilde{v}_0\|_{\mathrm{L}^p(\Omega)} \leq \|\max\{0, P_\varepsilon \widetilde{v}_0 - \delta_\varepsilon\} - \widetilde{v}_0\|_{\mathrm{L}^p(A_\varepsilon)} + (\mathrm{vol}(B_\varepsilon))^{\frac{1}{p}}. \quad (3.3)$$

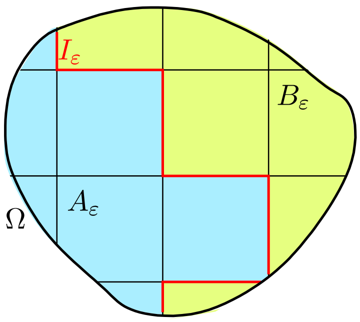
By increasing the domain of integration to Ω, adding zero and applying the triangle inequality, the first term of (3.3) is bounded by $\|\max\{0, P_\varepsilon \widetilde{v}_0 - \delta_\varepsilon\} - P_\varepsilon \widetilde{v}_0\|_{\mathrm{L}^p(\Omega)} + \|P_\varepsilon \widetilde{v}_0 - \widetilde{v}_0\|_{\mathrm{L}^p(\Omega)}$. Together with $P_\varepsilon \widetilde{v}_0 \to \widetilde{v}_0$ in $\mathrm{L}^p(\Omega)$ we have that the right hand side of (3.3) converges to zero if the sequence $(\delta_\varepsilon)_{\varepsilon>0}$ can be chosen such that $\delta_\varepsilon \to 0$ and $\mathrm{vol}(B_\varepsilon) \to 0$.

3. Choice of $\delta_\varepsilon > 0$ such that $\delta_\varepsilon \to 0$ and $\mathrm{vol}(B_\varepsilon) \to 0$: As before let $m = 1$. The crucial point of this construction is, that due to $\widetilde{v}_0 \leq v_0$ the set B_ε satisfies

$$B_\varepsilon \subset \{x \in \Omega : v_\varepsilon(x) < \max\{0, P_\varepsilon v_0(x) - \delta_\varepsilon\}\} \subset \{x \in \Omega : \delta_\varepsilon < |P_\varepsilon v_0(x) - v_\varepsilon(x)|\} =: \widehat{B}_\varepsilon,$$

such that Markov's inequality (**M**) can be exploited in the following way:

$$\mathrm{vol}(B_\varepsilon) \leq \mathrm{vol}(\widehat{B}_\varepsilon) \overset{(\mathbf{M})}{\leq} \tfrac{1}{\delta^p_\varepsilon} \int_\Omega |P_\varepsilon v_0(x) - v_\varepsilon(x)|^p \mathrm{d}x.$$



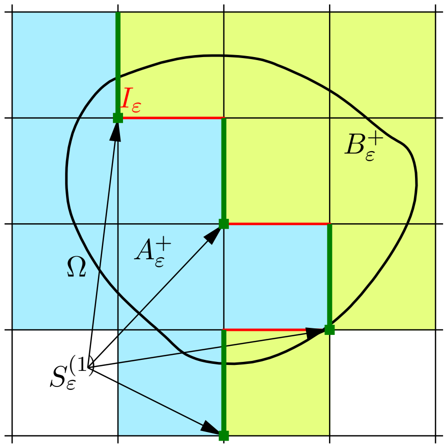


Figure 2: Decomposition of Ω into the subsets A_ε and B_ε.

Choosing $\delta_\varepsilon^p = \|P_\varepsilon v_0 - v_\varepsilon\|_{\mathrm{L}^p(\Omega)} \leq \|P_\varepsilon v_0 - v_0\|_{\mathrm{L}^p(\Omega)} + \|v_0 - v_\varepsilon\|_{\mathrm{L}^p(\Omega)}$, for instance, then $v_\varepsilon \to v_0$ in $\mathrm{L}^p(\Omega)$ yields $\delta_\varepsilon \to 0$ and $\mathrm{vol}(B_\varepsilon) \to 0$ as $\varepsilon \to 0$. As already mentioned in [18], $\delta_\varepsilon > 0$ is necessary to apply Markov's inequality. But in the case of $\delta_\varepsilon = 0$ the assumed convergence $v_\varepsilon \to v_0$ in $\mathrm{L}^p(\Omega)$ implies $P_\varepsilon v_0 - v_\varepsilon \to 0$ in $\mathrm{L}^p(\Omega)$ such that $\lim_{\varepsilon\to 0} \mathrm{vol}(\widehat{B}_\varepsilon) = 0$ results immediately.

4. To show: $\limsup_{\varepsilon\to 0} \big(\|R_{\frac{\varepsilon}{2}} \widetilde{v}_\varepsilon\|^p_{\mathrm{L}^p(\Omega)^d} - \|R_{\frac{\varepsilon}{2}} v_\varepsilon\|^p_{\mathrm{L}^p(\Omega)^d}\big) \leq \|\nabla \widetilde{v}_0\|^p_{\mathrm{L}^p(\Omega)^d} - \|\nabla v_0\|^p_{\mathrm{L}^p(\Omega)^d}$:

Roughly spoken, the fact $\mathrm{vol}(B_\varepsilon) \to 0$ as $\varepsilon \to 0$ means that in the case of a sequence of Sobolev functions ($v_\varepsilon \in \mathrm{W}^{1,p}(\Omega)$) it is sufficient to prove (3.1) for A_ε instead of Ω on the left hand side. But since we are interested in the case of piecewise constant functions we have to pay some special attention to the region around the interface $I_\varepsilon = \partial B_\varepsilon \cap \partial A_\varepsilon$.

Note, that due to the definition of A_ε and B_ε there are disjoint subsets $\Lambda_{A_\varepsilon}, \Lambda_{B_\varepsilon} \subset \Lambda_\varepsilon^+$ such that $A_\varepsilon = \big(\bigcup_{\lambda\in\Lambda_{A_\varepsilon}} \varepsilon(\lambda+Y)\big) \cap \Omega$ and $B_\varepsilon = \big(\bigcup_{\lambda\in\Lambda_{B_\varepsilon}} \varepsilon(\lambda+Y)\big) \cap \Omega$. With this let

$$A_\varepsilon^+ := \bigcup_{\lambda\in\Lambda_{A_\varepsilon}} \varepsilon(\lambda+Y) \qquad \text{and} \qquad B_\varepsilon^+ := \bigcup_{\lambda\in\Lambda_{B_\varepsilon}} \varepsilon(\lambda+Y).$$

Moreover, let $F_{e_i}(\varepsilon\lambda)$ be the face of $\varepsilon(\lambda+Y)$ orthogonal to e_i and contained in $\varepsilon(\lambda+Y)$. Then, the interface I_ε can uniquely be represented as $I_\varepsilon = \big(\bigcup_{i=1}^d \bigcup_{\lambda\in S_\varepsilon^{(i)}} \overline{F}_{e_i}(\varepsilon\lambda)\big) \cap \Omega$, where $S_\varepsilon^{(i)} \subset \Lambda$ is a suitable finite subset and $\bigcup_{\lambda\in S_\varepsilon^{(i)}} \overline{F}_{e_i}(\varepsilon\lambda)$ are all faces of the interface I_ε that are orthogonal to e_i. Observe that $|S_\varepsilon^{(i)}| \leq |\Lambda_{B_\varepsilon}|$ since the number of faces in $S_\varepsilon^{(i)}$ is bounded by the number of all shifted cells $\varepsilon(\lambda+Y)$ contained in B_ε^+. Taking the union of all cells

$$L_\varepsilon := \bigcup_{i=1}^d \bigcup_{\lambda\in S_\varepsilon^{(i)}} \varepsilon(\lambda - \tfrac{1}{2}e_i + Y)$$

containing the face $F_{e_i}(\varepsilon\lambda)$ in the middle (see Figure 3) we have $I_\varepsilon \subset \overline{L}_\varepsilon$,

$$\mathrm{vol}(L_\varepsilon) \leq \sum_{i=1}^d \sum_{\lambda\in S_\varepsilon^{(i)}} \varepsilon^d = \sum_{i=1}^d |S_\varepsilon^{(i)}| \varepsilon^d \leq \sum_{i=1}^d |\Lambda_{B_\varepsilon}| \varepsilon^d = d\,\mathrm{vol}(B_\varepsilon^+) \tag{3.4}$$

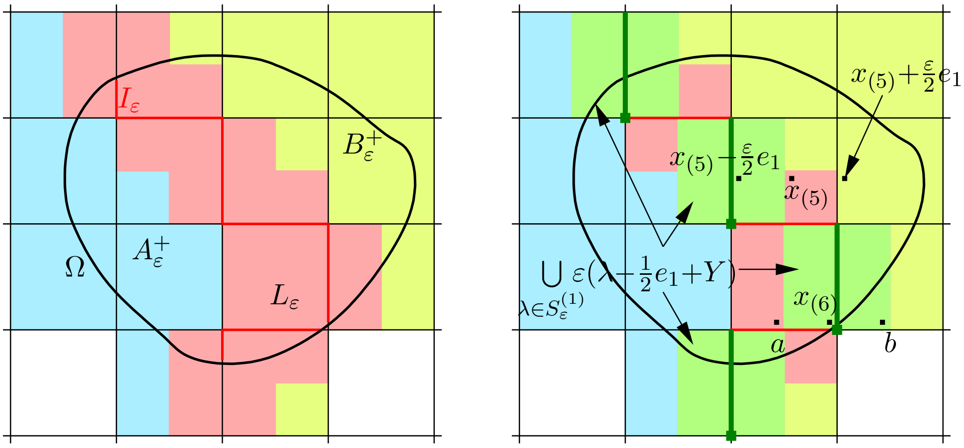


Figure 3: The subscript (5) or (6) of x in the last picture denotes that this is a point x as it is considered in step 5 or 6, respectively.

and

$$R_{\frac{\varepsilon}{2}}\widetilde{v}_\varepsilon = \begin{cases} R_{\frac{\varepsilon}{2}}(\max\{0, P_\varepsilon\widetilde{v}_0 - \delta_\varepsilon\}) & \text{in } A_\varepsilon \backslash L_\varepsilon \\ R_{\frac{\varepsilon}{2}}\widetilde{v}_\varepsilon & \text{in } L_\varepsilon \cap \Omega \, . \\ R_{\frac{\varepsilon}{2}} v_\varepsilon & \text{in } B_\varepsilon \backslash L_\varepsilon \end{cases} \tag{3.5}$$

Keeping (3.1) in mind, we want to estimate $|R_{\frac{\varepsilon}{2}}\widetilde{v}_\varepsilon|^p$ from above by terms depending only on v_ε and $\widetilde{v}_0$. Due to (3.5) we only have to care about the case $x \in L_\varepsilon$. Therefore, we consider every component $(R_{\frac{\varepsilon}{2}}(\widetilde{v}_\varepsilon)(x))e_i$ separately.

5. The case $x \in \big(L_\varepsilon \backslash \bigcup_{\lambda \in S_\varepsilon^{(i)}} \varepsilon(\lambda - \frac{1}{2}e_i + Y)\big) \cap \Omega$ for $i \in \{1, \dots, d\}$ fixed:

In this case either $x+\frac{\varepsilon}{2}e_i \in A_\varepsilon^+$ and $x-\frac{\varepsilon}{2}e_i \in A_\varepsilon^+$ or $x+\frac{\varepsilon}{2}e_i \in B_\varepsilon^+$ and $x-\frac{\varepsilon}{2}e_i \in B_\varepsilon^+$ which immediately results in

$$|(R_{\frac{\varepsilon}{2}}(\widetilde{v}_\varepsilon)(x))e_i| \leq \max\big\{|(R_{\frac{\varepsilon}{2}}(\max\{0, P_\varepsilon\widetilde{v}_0(x) - \delta_\varepsilon\}))e_i|, |(R_{\frac{\varepsilon}{2}}(v_\varepsilon)(x))e_i|\big\}. \tag{3.6}$$

6. The case $x \in \big(\bigcup_{\lambda \in S_\varepsilon^{(i)}} \varepsilon(\lambda - \frac{1}{2}e_i + Y)\big) \cap \Omega$ for $i \in \{1, \dots, d\}$ fixed:

In this case either $x+\frac{\varepsilon}{2}e_i \in A_\varepsilon^+$ and $x-\frac{\varepsilon}{2}e_i \in B_\varepsilon^+$ or $x+\frac{\varepsilon}{2}e_i \in B_\varepsilon^+$ and $x-\frac{\varepsilon}{2}e_i \in A_\varepsilon^+$ according to the definition of $S_\varepsilon^{(i)}$. Without loss of generality let $a = x+\frac{\varepsilon}{2}e_i \in A_\varepsilon^+$ and $b = x-\frac{\varepsilon}{2}e_i \in B_\varepsilon^+$. Then due to definition of A_ε and B_ε we have

$$1 \geq v_\varepsilon(a) \geq \widetilde{v}_\varepsilon(a) = \max\{0, P_\varepsilon\widetilde{v}_0(a) - \delta_\varepsilon\} \geq 0, \tag{3.7}$$

$$1 \geq \max\{0, P_\varepsilon\widetilde{v}_0(b) - \delta_\varepsilon\} > \widetilde{v}_\varepsilon(b) = v_\varepsilon(b) \geq 0. \tag{3.8}$$

Since $a \in A_\varepsilon^+ \backslash A_\varepsilon$ or $b \in B_\varepsilon^+ \backslash B_\varepsilon$ are possible, in (3.7), (3.8) and in the following table every function has to be understood as its extension by the continuation operator $V_\varepsilon : \mathrm{K}_{\varepsilon\Lambda}(\Omega) \to \mathrm{K}_{\varepsilon\Lambda}(\Omega_\varepsilon^+)$ extending a piecewise constant function $v \in \mathrm{K}_{\varepsilon\Lambda}(\Omega)$ for every $\lambda \in \Lambda_\varepsilon^+ \backslash \Lambda_\varepsilon^-$ on $\varepsilon(\lambda+Y)\backslash\Omega$ constantly by the (constant) value of v on $\varepsilon(\lambda+Y)\cap\Omega$. With this the following estimates are valid.

	if $\widetilde{v}_\varepsilon(a) \geq \widetilde{v}_\varepsilon(b)$	if $\widetilde{v}_\varepsilon(a) < \widetilde{v}_\varepsilon(b)$
$\lvert\widetilde{v}_\varepsilon(a)-\widetilde{v}_\varepsilon(b)\rvert$	$= \widetilde{v}_\varepsilon(a)-\widetilde{v}_\varepsilon(b)$ $\overset{(3.7)}{\leq} v_\varepsilon(a)-\widetilde{v}_\varepsilon(b)$ $\overset{(3.8)}{=} v_\varepsilon(a)-v_\varepsilon(b)$	$= \widetilde{v}_\varepsilon(b)-\widetilde{v}_\varepsilon(a)$ $\overset{(3.8)}{<} \max\{0, P_\varepsilon\widetilde{v}_0(b)-\delta_\varepsilon\} - \widetilde{v}_\varepsilon(a)$ $\overset{(3.7)}{=} \max\{0, P_\varepsilon\widetilde{v}_0(b)-\delta_\varepsilon\} - \max\{0, P_\varepsilon\widetilde{v}_0(a)-\delta_\varepsilon\}$

Hence, we also find

$$|(R_{\frac{\varepsilon}{2}}(\widetilde{v}_\varepsilon)(x))e_i| \leq \max\{|(R_{\frac{\varepsilon}{2}}(\max\{0, P_\varepsilon\widetilde{v}_0(x)-\delta_\varepsilon\}))e_i|, |(R_{\frac{\varepsilon}{2}}(v_\varepsilon)(x))e_i|\}, \tag{3.9}$$

for all $x \in \big(\bigcup_{\lambda\in S_\varepsilon^{(i)}} \varepsilon(\lambda-\frac{1}{2}e_i+Y)\big) \cap \Omega$. Combining (3.6) and (3.9) this inequality holds for every $x \in L_\varepsilon \cap \Omega$, which finally results in

$$|R_{\frac{\varepsilon}{2}}\widetilde{v}_\varepsilon|^p \leq \begin{cases} |R_{\frac{\varepsilon}{2}}P_\varepsilon\widetilde{v}_0|^p & \text{in } A_\varepsilon\backslash L_\varepsilon \\ |R_{\frac{\varepsilon}{2}}P_\varepsilon\widetilde{v}_0|^p + |R_{\frac{\varepsilon}{2}}v_\varepsilon|^p & \text{in } L_\varepsilon \cap \Omega\,, \\ |R_{\frac{\varepsilon}{2}}v_\varepsilon|^p & \text{in } B_\varepsilon\backslash L_\varepsilon \end{cases} \tag{3.10}$$

by recalling (3.5), since $|\max\{C_1, C_2\}|^p \leq |C_1|^p + |C_2|^p$ and

$$|R_{\frac{\varepsilon}{2}}\max\{0, P_\varepsilon\widetilde{v}_0(x)-\delta_\varepsilon\}| \leq |R_{\frac{\varepsilon}{2}}(P_\varepsilon\widetilde{v}_0(x)-\delta_\varepsilon)| = |R_{\frac{\varepsilon}{2}}(P_\varepsilon\widetilde{v}_0)(x)|.$$

Now, exploiting (3.10) we conclude in the case $m = 1$ with

$$\begin{aligned}
&\limsup_{\varepsilon\to 0}\big(\|R_{\frac{\varepsilon}{2}}\widetilde{v}_\varepsilon\|^p_{\mathrm{L}^p(\Omega)^d} - \|R_{\frac{\varepsilon}{2}}v_\varepsilon\|^p_{\mathrm{L}^p(\Omega)^d}\big)\\
&\quad\leq \limsup_{\varepsilon\to 0}\bigg(\int_{A_\varepsilon\backslash L_\varepsilon} |R_{\frac{\varepsilon}{2}}P_\varepsilon\widetilde{v}_0(x)|^p - |R_{\frac{\varepsilon}{2}}v_\varepsilon(x)|^p\mathrm{d}x\\
&\qquad\qquad + \int_{B_\varepsilon\backslash L_\varepsilon} |R_{\frac{\varepsilon}{2}}v_\varepsilon(x)|^p - |R_{\frac{\varepsilon}{2}}v_\varepsilon(x)|^p\mathrm{d}x\\
&\qquad\qquad + \int_{L_\varepsilon\cap\Omega} |R_{\frac{\varepsilon}{2}}P_\varepsilon\widetilde{v}_0(x)|^p + |R_{\frac{\varepsilon}{2}}v_\varepsilon(x)|^p - |R_{\frac{\varepsilon}{2}}v_\varepsilon(x)|^p\mathrm{d}x\bigg)\\
&\quad= \limsup_{\varepsilon\to 0}\bigg(\int_{A_\varepsilon\cup(L_\varepsilon\cap\Omega)} |R_{\frac{\varepsilon}{2}}P_\varepsilon\widetilde{v}_0(x)|^p\mathrm{d}x - \int_{A_\varepsilon\backslash L_\varepsilon} |R_{\frac{\varepsilon}{2}}v_\varepsilon(x)|^p\mathrm{d}x\bigg)\\
&\quad\leq \lim_{\varepsilon\to 0}\int_\Omega |R_{\frac{\varepsilon}{2}}P_\varepsilon\widetilde{v}_0(x)|^p\mathrm{d}x - \liminf_{\varepsilon\to 0}\int_\Omega |\mathbb{1}_{A_\varepsilon\backslash L_\varepsilon}(x)R_{\frac{\varepsilon}{2}}v_\varepsilon(x)|^p\mathrm{d}x\\
&\quad= \|\nabla\widetilde{v}_0\|^p_{\mathrm{L}^p(\Omega)^d} - \|\nabla v_0\|^p_{\mathrm{L}^p(\Omega)^d},
\end{aligned}$$

where in the second last line the first term converges to $\|\nabla\widetilde{v}_0\|^p_{\mathrm{L}^p(\Omega)^d}$ which we have seen in the proof of Theorem 2.7. Moreover, weak lower semi-continuity together with $\mathbb{1}_{A_\varepsilon\backslash L_\varepsilon}R_{\frac{\varepsilon}{2}}v_\varepsilon \rightharpoonup \nabla v_0$ in $\mathrm{L}^p(\Omega)^d$ is exploited for the second one. Note, that according to estimate (3.4) we have $\mathbb{1}_{A_\varepsilon\backslash L_\varepsilon} \to \mathbb{1}_\Omega$ in $\mathrm{L}^q(\Omega)$ for every $q \in [1,\infty)$, since $\lim_{\varepsilon\to 0}\mathrm{vol}(B_\varepsilon) = 0$ implies $\lim_{\varepsilon\to 0}\mathrm{vol}(B_\varepsilon^+) = 0$.

7. The general case $m > 1$: Up to now, in the case $m > 1$ we have ($j \in \{1, 2, \ldots, m\}$)

$$\limsup_{\varepsilon\to 0}\big(\|R_{\frac{\varepsilon}{2}}\widetilde{v}^{(j)}_\varepsilon\|^p_{\mathrm{L}^p(\Omega)^d} - \|R_{\frac{\varepsilon}{2}}v^{(j)}_\varepsilon\|^p_{\mathrm{L}^p(\Omega)^d}\big) \leq \|\nabla\widetilde{v}^{(j)}_0\|^p_{\mathrm{L}^p(\Omega)^d} - \|\nabla v^{(j)}_0\|^p_{\mathrm{L}^p(\Omega)^d}$$

for every component $\widetilde{v}^{(j)}_\varepsilon, v^{(j)}_\varepsilon, \widetilde{v}^{(j)}_0, v^{(j)}_0$ of the functions $\widetilde{v}_\varepsilon, v_\varepsilon \in \mathrm{K}_{\varepsilon\Lambda}(\Omega; [0,1])^m$ and $\widetilde{v}_0, v_0 \in \mathrm{W}^{1,p}(\Omega; [0,1])^m$. Summing up this inequality for $j = 1, 2, \ldots, m$ we finally get

$$\limsup_{\varepsilon\to 0}\big(\|R_{\frac{\varepsilon}{2}}\widetilde{v}_\varepsilon\|^p_{\mathrm{L}^p(\Omega)^{m\times d}} - \|R_{\frac{\varepsilon}{2}}v_\varepsilon\|^p_{\mathrm{L}^p(\Omega)^{m\times d}}\big) \leq \|\nabla\widetilde{v}_0\|^p_{\mathrm{L}^p(\Omega)^{m\times d}} - \|\nabla v_0\|^p_{\mathrm{L}^p(\Omega)^{m\times d}}.$$

8. $R_{\frac{\varepsilon}{2}}\widetilde{v}_\varepsilon \rightharpoonup \nabla\widetilde{v}_0$ in $\mathrm{L}^p(\Omega)^{m\times d}$: Due to step 7 we can apply Theorem 2.1 and according to step 2 the limit-function can be identified as $\widetilde{v}_0$ so that $R_{\frac{\varepsilon}{2}}\widetilde{v}_\varepsilon \rightharpoonup \nabla\widetilde{v}_0$ in $\mathrm{L}^p(\Omega)^{m\times d}$ follows. ☐

Remark 3.2. Note, that this proof also works when replacing $\widetilde{A}_\varepsilon^{(j)} := A_\varepsilon^{(j)} \cap \Omega_\varepsilon^-$ and $\widetilde{B}_\varepsilon^{(j)} := B_\varepsilon^{(j)} \cup (\Omega\backslash\Omega_\varepsilon^-)$ in

$$\widetilde{v}_\varepsilon^{(j)}(x) = \begin{cases} \max\{0, P_\varepsilon\widetilde{v}_0^{(j)}(x) - \delta_\varepsilon^{(j)}\} & \text{if } x \in \widetilde{A}_\varepsilon^{(j)} \\ v_\varepsilon^{(j)}(x) & \text{if } x \in \widetilde{B}_\varepsilon^{(j)} \end{cases}$$

(step 1), since the crucial condition is $\mathrm{vol}(\widetilde{B}_\varepsilon^{(j)}) \to 0$ for $\varepsilon \to 0$, which according to (2.3) is valid in this case, too. This will become important in Section 7 when proving stability of the limit model.

4 Models

As already mentioned in Section 1 our damage model is rate-independent and set up in the energetic framework introduced in [16, 17]. We start by shortly stating all needed notations and results concerning the energetic formulation.

4.1 Short summary of rate-independent theory

The state space is a product $\mathcal{Q} = \mathcal{U}\times\mathcal{Z}$ of two weakly closed subsets $\mathcal{U}$ and $\mathcal{Z}$ of reflexive Banach spaces. The energetic formulation is based on the stored energy functional $\mathcal{E} : [0,T]\times\mathcal{Q} \to \mathbb{R}_\infty$ and the dissipation distance $\mathcal{D} : \mathcal{Z}\times\mathcal{Z} \to [0,\infty]$ commonly depending only on the second variable. We are looking for the so called energetic solution.

Definition 4.1 (Energetic solution)**.** A function $q=(u,z) : [0,T] \to \mathcal{Q}$ is called an energetic solution for the rate-independent system $(\mathcal{Q},\mathcal{E},\mathcal{D})$, if $\partial_t\mathcal{E}(\cdot,q(\cdot)) \in \mathrm{L}^1((0,T))$ and if $\mathcal{E}(t,q(t)) < \infty$, the stability condition (S) and the energy balance (E) are satisfied for all $t \in [0,T]$.

(S) $\quad \mathcal{E}(t,q(t)) \leq \mathcal{E}(t,\widetilde{q}) + \mathcal{D}(z(t),\widetilde{z}) \quad$ for all $\widetilde{q} = (\widetilde{u},\widetilde{z}) \in \mathcal{Q}$,

(E) $\quad \mathcal{E}(t,q(t)) + \mathrm{Diss}_\mathcal{D}(z;[s,t]) = \mathcal{E}(s,q(s)) + \int_s^t \partial_t\mathcal{E}(\xi,q(\xi))\mathrm{d}\xi$,

with $\mathrm{Diss}_\mathcal{D}(z;[s,t]) := \sup\sum_{j=1}^N \mathcal{D}(z(\xi_{j-1}),z(\xi_j))$, where $N \in \mathbb{N}$ and the supremum is taken over all finite partitions of $[s,t]$.

Definition 4.2 (Set of stable state, stable sequence)**.** The set of stable states at time $t \in [0,T]$ is defined by:

$$\mathcal{S}(t) := \{q \in \mathcal{Q} \,|\, \mathcal{E}(t,q) < \infty \text{ and } \forall\,\widetilde{q} \in \mathcal{Q} : \mathcal{E}(t,q) \leq \mathcal{E}(t,\widetilde{q}) + \mathcal{D}(z,\widetilde{z})\}.$$

A sequence $(t_\delta,q_\delta)_{\delta>0} \subset [0,T]\times\mathcal{Q}$ is called stable sequence if (i) and (ii) hold:

(i) $\sup_{\delta>0}\{\mathcal{E}(t_\delta,q_\delta)\} < \infty$

(ii) $q_\delta \in \mathcal{S}(t_\delta)$ for every $\delta > 0$.

Remark 4.3. Note that here a stable sequence is a tuple of time-steps $t_\delta \in [0,T]$ and functions $q_\delta \in \mathcal{Q}$, since the proof of existence (Theorem 4.4) of a solution of the energetic formulation given in Definition 4.1 is based on a time-discretization (scheme) in the following way. A partition π_N of $[0,T]$ whose fineness tends to zero for $N \to \infty$ is chosen, such that the associated stable sequence $(t_N, q_N)_{N\in\mathbb{N}}$ allows the construction of a sequences $(\bar{q}_N)_{N\in\mathbb{N}}$ of piecewise constant/affine functions $\bar{q}_N : [0,T] \to \mathcal{Q}$ converging in some sense to a function $q : [0,T] \to \mathcal{Q}$ satisfying the energetic formulation for all $t \in [0,T]$. But in the following we already start with a solution $q_\varepsilon : [0,T] \to \mathcal{Q}$ of a energetic formulation of a microscopic model and we are interested in the limit model appearing when letting the micro-structure getting finer and finer ($\varepsilon \to 0$). That is why there is no need of going back to a time discrete description and Definition 4.2 will be adapted in Section 7.

The following four assumptions on $\mathcal{E} : [0,T]\times\mathcal{Q} \to \mathbb{R}_\infty$ and $\mathcal{D} : \mathcal{Z}\times\mathcal{Z} \to [0,\infty]$ guarantee the existence of energetic solutions as it is stated in Theorem 4.4 below ([14] Theorem 3.4).

Compactness of the energy sublevels:

$$\forall\, t \in [0,T]\, \forall\, E \in \mathbb{R} : L_E(t) = \{q \in \mathcal{Q} : \mathcal{E}(t,q) \le E\} \text{ is weakly seq. compact.} \tag{4.1}$$

Uniform control of the power: $\exists\, c_0 \in \mathbb{R}\, \exists\, c_1 > 0\, \forall\, (t_q, q) \in [0,T]\times\mathcal{Q}$ with $\mathcal{E}(t_q, q) < \infty$:

$$\mathcal{E}(\cdot, q) \in \mathrm{C}^1([0,T]) \text{ and } |\partial_t \mathcal{E}(t,q)| \le c_1(c_0 + \mathcal{E}(t,q)) \text{ for all } t \in [0,T]. \tag{4.2}$$

Quasi-distance:

$$\begin{aligned} &\forall\, z_1, z_2, z_3 \in \mathcal{Z} : \mathcal{D}(z_1,z_2) = 0 \quad \Leftrightarrow \quad z_1{=}z_2 \\ &\text{and} \qquad\qquad \mathcal{D}(z_1,z_3) \le \mathcal{D}(z_1,z_2) + \mathcal{D}(z_2,z_3) \end{aligned} \tag{4.3}$$

Semi-continuity:

$$\mathcal{D} : \mathcal{Z}\times\mathcal{Z} \to [0,\infty] \text{ is weakly seq. lower semi-continuous.} \tag{4.4}$$

Theorem 4.4. *[Theorem 3.4 of [14]] Let $\mathcal{U}$ and $\mathcal{Z}$ be weakly closed subsets of reflexive Banach spaces and set $\mathcal{Q} := \mathcal{U}\times\mathcal{Z}$. For all $t \in [0,T]$ the energy functional $\mathcal{E} : [0,T]\times\mathcal{Q} \to \mathbb{R}_\infty$ is assumed to satisfy* (4.1) *and* (4.2). *Moreover, let the following compatibility conditions hold: For every stable sequence $(t_k, u_k, z_k)_{k\in\mathbb{N}}$ with $t_k \to t$, $(u_k, z_k) \rightharpoonup (u,z)$ in $\mathcal{Q}$ we have*

$$\partial_t \mathcal{E}(t_k, u_k, z_k) \to \partial_t \mathcal{E}(t,u,z), \tag{4.5}$$

$$q \in \mathcal{S}(t). \tag{4.6}$$

The dissipation distance $\mathcal{D} : \mathcal{Z}\times\mathcal{Z} \to [0,\infty]$ is assumed to fulfill (4.3) *and* (4.4).

Then for each $(u_0, z_0) \in \mathcal{S}(0)$ there exists an energetic solution $(u,z) : [0,T] \to \mathcal{Q}$ for $(\mathcal{Q}, \mathcal{E}, \mathcal{D})$ satisfying $(u(0), z(0)) = (u_0, z_0)$.

4.2 Microscopic damage model

As already mentioned in Section 1, here damage evolution of a body described by a bounded open Lipschitz domain $\Omega \subset \mathbb{R}^d$ is modeled by the evolution of a damaged region contained in the body. In particular this means, that the body consists of two phases, a weak or damaged and a strong or undamaged one. These phases are modeled by a characteristic function $\mathbb{1}_U$ being 1 in the undamaged region U of Ω and 0 in the damaged part $\Omega\backslash U$ such that the elasticity tensor is given by

$$\mathbb{C}(\mathbb{1}_U) = \mathbb{1}_U\mathbb{C}_{\text{strong}} + (1-\mathbb{1}_U)\mathbb{C}_{\text{weak}},$$

with $\mathbb{C}_{\text{strong}}, \mathbb{C}_{\text{weak}} \in \text{Lin}_{\text{sym}}(\mathbb{R}^{d\times d}_{\text{sym}}, \mathbb{R}^{d\times d}_{\text{sym}})$ such that for some $\alpha > 0$

$$0 < \alpha|\eta|^2_{d\times d} \leq \langle\mathbb{C}_{\text{weak}}\eta, \eta\rangle_{d\times d} \leq \langle\mathbb{C}_{\text{strong}}\eta, \eta\rangle_{d\times d} \quad \text{for all } \eta \in \mathbb{R}^{d\times d}_{\text{sym}}. \tag{4.7}$$

We assume that the damaged region is given by a finite union of scalings and translations of a prescribed damage set $D \subset Y$, assumed to be open and starshaped with respect to the center of the unit cell Y. This leads to one very important notational difference to Section 2, namely, here $Y = [-\frac{1}{2}, \frac{1}{2})^d$ is considered (instead of $Y = [0, 1)^d$) as the associated unit cell, which will expose advantageously in the following. The benefit is that in this case the in Y included damage set D and scalings of it are both star-shaped with respect to $0 \in \mathbb{R}^d$. Otherwise (i.e. $Y = [0, 1)^d$) the damage set would be star-shaped with respect to $(\frac{1}{2}, \dots, \frac{1}{2})^T \in \mathbb{R}^d$ and scalings of this set have to be shifted by $-(\frac{1}{2}, \dots, \frac{1}{2})^T \in \mathbb{R}^d$, scaled and shifted back by $(\frac{1}{2}, \dots, \frac{1}{2})^T \in \mathbb{R}^d$ which would mean a lot of notation for saying not much.

Moreover, we assume $\text{vol}(\partial D) = 0$ and that these damage sets are periodically distributed in body Ω in that way, that the centers of all those damage sets are elements of the periodic latice $\varepsilon\Lambda$ but such that their sizes evolve independently from one cell to the other. Now, the characteristic functions describing the damaged region are elements of the set of admissible damage functions given by

$$\mathbb{X}^D_{\varepsilon\Lambda}(\Omega) := \{\chi \in \text{L}^\infty(\Omega; \{0, 1\}) \,|\, \exists\, \widehat{\chi} \in \mathbb{X}^D_{\varepsilon\Lambda}(\Omega^+_\varepsilon) : \chi = \widehat{\chi}|_\Omega\}, \tag{4.8}$$

where

$$\mathbb{X}^D_{\varepsilon\Lambda}(\Omega^+_\varepsilon) := \{\widehat{\chi} \in \text{L}^\infty(\Omega^+_\varepsilon; \{0, 1\}) \,|\, \forall\, \lambda \in \Lambda^+_\varepsilon \,\exists\, r_{\varepsilon\lambda} \in [0, \tfrac{\varepsilon}{2}] : \widehat{\chi}|_{\varepsilon(\lambda+Y)} = 1-\mathbb{1}_{\varepsilon\lambda+r_{\varepsilon\lambda}D}|_{\varepsilon(\lambda+Y)}\}.$$

Here and in the following $\mathbb{1}_{\mathcal{O}}$ always denotes the characteristic function of the set $\mathcal{O} \subset \mathbb{R}^d$. Since for an element χ of $\mathbb{X}^D_{\varepsilon\Lambda}(\Omega)$ there is no uniqueness in the choice of $\widehat{\chi} \in \mathbb{X}^D_{\varepsilon\Lambda}(\Omega^+_\varepsilon)$ in (4.8) we introduce $M_\varepsilon : \mathbb{X}^D_{\varepsilon\Lambda}(\Omega) \to \mathbb{X}^D_{\varepsilon\Lambda}(\Omega^+_\varepsilon)$ via

$$M_\varepsilon(\chi) := \max\{\widehat{\chi} \in \mathbb{X}^D_{\varepsilon\Lambda}(\Omega^+_\varepsilon) \,|\, \widehat{\chi}|_\Omega = \chi\} \tag{4.9}$$

to fix this problem.

Remark 4.5. Note, that for fixed $\varepsilon > 0$ the set of admissible damage functions is finite dimensional. Hence, the strong and weak topology are the same. For every function $\chi \in \mathbb{X}^D_{\varepsilon\Lambda}(\Omega)$ there exists an associated vector $\vec{r} \in \mathbb{R}^{|\Lambda^+_\varepsilon|}$, whose components are the values $r_{\varepsilon\lambda}$ of definition (4.8), describing the function $M_\varepsilon(\chi)$ uniquely. In this way, convergence of a sequence $(\chi_\delta)_{\delta>0} \subset \mathbb{X}^D_{\varepsilon\Lambda}(\Omega)$ to a function $\chi_0 \in \mathbb{X}^D_{\varepsilon\Lambda}(\Omega)$ is equivalent to the convergence of the sequence of the associated vectors $(\vec{r}_\delta)_{\delta>0} \subset \mathbb{R}^{|\Lambda^+_\varepsilon|}$ to the vector $\vec{r}_0 \in \mathbb{R}^{|\Lambda^+_\varepsilon|}$, uniquely characterizing $M_\varepsilon(\chi_0) \in \mathbb{X}^D_{\varepsilon\Lambda}(\Omega)$. In other words $\chi_\delta \to \chi_0$ in $\mathbb{X}^D_{\varepsilon\Lambda}(\Omega)$ means pointwise convergence, which implies L^p-convergence for $p \in [1, \infty)$ since $0 \leq \chi_\delta \leq 1$ by definition.

Before introducing the energetic formulation of the microscopic damage model we need to introduce a one to one identification of functions in $\mathbb{X}^D_{\varepsilon\Lambda}(\Omega^+_\varepsilon)$ and $\text{K}_{\varepsilon\Lambda}(\Omega^+_\varepsilon)$, since in the end we want to use the regularization theory introduced in Section 2 to gain better convergence results for the damage function χ belonging to $\mathrm{L}^\infty(\Omega)$ helping us identifying an effective damage model.

Let $c_d := 1-\mathrm{vol}(D)$. For fixed $\varepsilon > 0$ an admissible damage function $\widehat{\chi} \in \mathbb{X}^D_{\varepsilon\Lambda}(\Omega^+_\varepsilon)$ is identified with the piecewise constant function $\widehat{z} \in \mathrm{K}_{\varepsilon\Lambda}(\Omega^+_\varepsilon; [c_D, 1])$ appearing by taking the average of $\widehat{\chi}$ in every cell $\varepsilon(\lambda + Y)$. This identification is done by the operator $\widehat{Q}_\varepsilon$: $\mathbb{X}^D_{\varepsilon\Lambda}(\Omega^+_\varepsilon) \to \mathrm{K}_{\varepsilon\Lambda}(\Omega^+_\varepsilon; [c_D, 1])$ given by:

$$\widehat{Q}_\varepsilon(\widehat{\chi}) := (P_\varepsilon(\widehat{\chi}^{\mathrm{ex}}))|_{\Omega^+_\varepsilon}, \tag{4.10}$$

where $P_\varepsilon : \mathrm{L}^p(\mathbb{R}^d) \to \mathrm{K}_{\varepsilon\Lambda}(\mathbb{R}^d)$ denotes the projector to piecewise constant functions introduced in Definition 2.5. Now, the reverse direction, namely the identification of a piecewise constant function $\widehat{z} \in \mathrm{K}_{\varepsilon\Lambda}(\Omega^+_\varepsilon; [c_D, 1])$ with an admissible damage function $\widehat{\chi} \in \mathbb{X}^D_{\varepsilon\Lambda}(\Omega^+_\varepsilon)$ is given by the inverse Operator $\widehat{Q}^{-1}_\varepsilon : \mathrm{K}_{\varepsilon\Lambda}(\Omega^+_\varepsilon; [c_D, 1]) \to \mathbb{X}^D_{\varepsilon\Lambda}(\Omega^+_\varepsilon)$ defined for all $x \in \Omega^+_\varepsilon$ via

$$\widehat{Q}^{-1}_\varepsilon(\widehat{z})(x) := \mathbb{1}_{\mathcal{N}_\varepsilon(x)+\varepsilon U(\widehat{z}(x))}(x), \tag{4.11}$$

where $U(\alpha) := Y\backslash\kappa(\alpha)D$, $\kappa^d(\alpha) := (\mathrm{vol}(D))^{-1}(1-\alpha)$ and $\mathcal{N}_\varepsilon : \mathbb{R}^d \to \varepsilon\Lambda$ is also introduced in Definition 2.5.

Remark 4.6. Considering a damage function $\widehat{\chi} \in \mathbb{X}^D_{\varepsilon\Lambda}(\Omega^+_\varepsilon)$ according to $\widehat{Q}_\varepsilon\widehat{\chi} = \widehat{z}$ the value $\widehat{z}(x)$ is related to the volume of the undamaged region in the cell $x \in \varepsilon(\lambda+Y)$, and the function $\kappa : [c_D, 1] \to [0, 1]$ has to be chosen in such a way that $\widehat{z}(x) = \mathrm{vol}(Y\backslash\kappa(\widehat{z}(x))D)$. But with this, $\widehat{z}$ is bounded by c_D from below and by 1 from above, which is due to the assumption that during the damage evolution the set $\varepsilon(\lambda+D)$ is the biggest possibly appearing damage set in any cell $\varepsilon(\lambda+Y)$. In consequence, this restricts $\widehat{Q}^{-1}_\varepsilon$ to piecewise constant functions bounded by c_D from below, since otherwise $\widehat{z}(x) < c_D$ correlates to a damage function $\widehat{\chi} = \widehat{Q}^{-1}_\varepsilon\widehat{z}$ containing a damage set "bigger" then $\varepsilon(\lambda+D)$ in the cell $x \in \varepsilon(\lambda+Y)$.

Remark 4.7. Alternatively, one could interpret $\widehat{z}(x)$ as the surface area of the damaged set and κ then should be modified in a suitable way. This does not affect the subsequent analysis. Here, it is only important, that κ is a strictly monotone homeomorphism.

Definition 4.8 (Identification of damage functions and piecewise constant functions)**.** For fixed $\varepsilon > 0$ an admissible damage function $\chi \in \mathbb{X}^D_{\varepsilon\Lambda}(\Omega)$ is associated with the piecewise constant function $Q_\varepsilon(\chi) \in \mathrm{K}_{\varepsilon\Lambda}(\Omega; [c_D, 1])$. Here, $Q_\varepsilon : \mathbb{X}^D_{\varepsilon\Lambda}(\Omega) \to \mathrm{K}_{\varepsilon\Lambda}(\Omega; [c_D, 1])$ is an injective mapping defined via

$$Q_\varepsilon(\chi) := (\widehat{Q}_\varepsilon(M_\varepsilon\chi))|_\Omega,$$

where $\widehat{Q}_\varepsilon$ is defined in (4.10) and M_ε is introduced in (4.9).

Conversely, a piecewise constant function $z \in \mathrm{K}_{\varepsilon\Lambda}(\Omega; [c_D, 1])$ is associated with the admissible damage function $N_\varepsilon(z) \in \mathbb{X}^D_{\varepsilon\Lambda}(\Omega)$. Here, $N_\varepsilon : \mathrm{K}_{\varepsilon\Lambda}(\Omega; [c_D, 1]) \to \mathbb{X}^D_{\varepsilon\Lambda}(\Omega)$ is a surjective mapping given by

$$N_\varepsilon(z) := (\widehat{Q}^{-1}_\varepsilon(V_\varepsilon z))|_\Omega,$$

where $\widehat{Q}^{-1}_\varepsilon$ is defined in (4.11) and $V_\varepsilon : \mathrm{K}_{\varepsilon\Lambda}(\Omega) \to \mathrm{K}_{\varepsilon\Lambda}(\Omega^+_\varepsilon)$ denotes the continuation operator extending the piecewise constant function $z \in \mathrm{K}_{\varepsilon\Lambda}(\Omega)$ for every $\lambda \in \Lambda^+_\varepsilon\backslash\Lambda^-_\varepsilon$ on $\varepsilon(\lambda+Y)\backslash\Omega$ constantly by the (constant) value of z on $\varepsilon(\lambda+Y) \cap \Omega$.

According to the in general ambiguous choice of $\widehat{\chi} \in \mathbb{X}^D_{\varepsilon\Lambda}(\Omega^+_\varepsilon)$ in Definition (4.8) the mappings Q_ε and N_ε lose the bijectivity of $\widehat{Q}_\varepsilon$ and $\widehat{Q}_\varepsilon^{-1}$. Nevertheless the following relations hold.

Proposition 4.9. *Let* $Q_\varepsilon : \mathbb{X}^D_{\varepsilon\Lambda}(\Omega) \to \mathrm{K}_{\varepsilon\Lambda}(\Omega; [c_D, 1])$ *and* $N_\varepsilon : \mathrm{K}_{\varepsilon\Lambda}(\Omega; [c_D, 1]) \to \mathbb{X}^D_{\varepsilon\Lambda}(\Omega)$ *be defined as in Definition 4.8. Then:*

$$
\begin{aligned}
N_\varepsilon \circ Q_\varepsilon &: \mathbb{X}^D_{\varepsilon\Lambda}(\Omega) \to \mathbb{X}^D_{\varepsilon\Lambda}(\Omega) \text{ is the identity.}\\
Q_\varepsilon \circ N_\varepsilon &: \mathrm{K}_{\varepsilon\Lambda}(\Omega; [c_D, 1]) \to \mathrm{K}_{\varepsilon\Lambda}(\Omega; [c_D, 1]) \text{ is a projection.}\\
(Q_\varepsilon \circ N_\varepsilon(z))|_{\Omega^-_\varepsilon} &= z|_{\Omega^-_\varepsilon} \text{ for all } z \in \mathrm{K}_{\varepsilon\Lambda}(\Omega; [c_D, 1]).
\end{aligned}
$$

Proof. Using the facts that $V_\varepsilon(\widehat{z}|_\Omega) = \widehat{z}$ for all $\widehat{z} \in \mathrm{K}_{\varepsilon\Lambda}(\Omega^+_\varepsilon)$ and that $(M_\varepsilon\chi)|_\Omega = \chi$ for all $\chi \in \mathbb{X}^D_{\varepsilon\Lambda}(\Omega)$ we find for $\chi \in \mathbb{X}^D_{\varepsilon\Lambda}(\Omega)$ and $z \in \mathrm{K}_{\varepsilon\Lambda}(\Omega; [c_D, 1])$

$$
\begin{aligned}
N_\varepsilon[Q_\varepsilon(\chi)] &= N_\varepsilon[(\widehat{Q}_\varepsilon M_\varepsilon\chi)|_\Omega] = (\widehat{Q}_\varepsilon^{-1} V_\varepsilon[(\widehat{Q}_\varepsilon M_\varepsilon\chi)|_\Omega])|_\Omega\\
&= (\widehat{Q}_\varepsilon^{-1}(\widehat{Q}_\varepsilon M_\varepsilon\chi))|_\Omega = (M_\varepsilon\chi)|_\Omega = \chi
\end{aligned}
$$

and

$$
\begin{aligned}
Q_\varepsilon(N_\varepsilon[Q_\varepsilon(N_\varepsilon z)]) &= Q_\varepsilon(\{\widehat{Q}_\varepsilon^{-1} V_\varepsilon[Q_\varepsilon(N_\varepsilon z)]\}|_\Omega) = Q_\varepsilon(\{\widehat{Q}_\varepsilon^{-1} V_\varepsilon([\widehat{Q}_\varepsilon M_\varepsilon(N_\varepsilon z)]|_\Omega)\}|_\Omega)\\
&= Q_\varepsilon(\{\widehat{Q}_\varepsilon^{-1}[\widehat{Q}_\varepsilon M_\varepsilon(N_\varepsilon z)]\}|_\Omega) = Q_\varepsilon(\{M_\varepsilon(N_\varepsilon z)\}|_\Omega) = Q_\varepsilon(N_\varepsilon z).
\end{aligned}
$$

Let $\bar{Q}_\varepsilon$ and $\bar{Q}_\varepsilon^{-1}$ be defined as in (4.10) and (4.11) except that Ω^+_ε is replaced by Ω^-_ε. Then combining $(Q_\varepsilon(\chi))|_{\Omega^-_\varepsilon} = (\widehat{Q}_\varepsilon M_\varepsilon\chi)|_{\Omega^-_\varepsilon} = \bar{Q}_\varepsilon(\chi|_{\Omega^-_\varepsilon})$ for $\chi \in \mathbb{X}^D_{\varepsilon\Lambda}(\Omega)$ and $(N_\varepsilon(z))|_{\Omega^-_\varepsilon} = (\widehat{Q}_\varepsilon^{-1} V_\varepsilon z)|_{\Omega^-_\varepsilon} = \bar{Q}_\varepsilon^{-1}(z|_{\Omega^-_\varepsilon})$ for $z \in \mathrm{K}_{\varepsilon\Lambda}(\Omega; [c_D, 1])$ results in

$$
(Q_\varepsilon(N_\varepsilon(z)))|_{\Omega^-_\varepsilon} = \bar{Q}_\varepsilon(\bar{Q}_\varepsilon^{-1}(z|_{\Omega^-_\varepsilon})) = z|_{\Omega^-_\varepsilon}.
$$

□

Finally, let $\Gamma_{\mathrm{Dir}} \subset \partial\Omega$ be a part of the boundary of Ω with a positive $(d{-}1)$-dimensional measure, and $\mathrm{H}^1_{\Gamma_{\mathrm{Dir}}}(\Omega)^d := \{u \in \mathrm{H}^1(\Omega)^d \,|\, u_{\Gamma_{\mathrm{Dir}}} = 0\}$. Set

$$
\mathcal{Q}_\varepsilon(\Omega) := \mathrm{H}^1_{\Gamma_{\mathrm{Dir}}}(\Omega)^d \times \mathbb{X}^D_{\varepsilon\Lambda}(\Omega)
$$

to shorten notation.

Now, letting $\langle\cdot,\cdot\rangle : (\mathrm{H}^1_{\Gamma_{\mathrm{Dir}}}(\Omega)^d)^* \times \mathrm{H}^1_{\Gamma_{\mathrm{Dir}}}(\Omega)^d \to \mathbb{R}$ be the dual pairing and denoting by $\mathbf{e}(u) = \mathbf{e}_x(u) = \frac{1}{2}(\nabla u + (\nabla u)^T)$ the linearized strain tensor for $u \in \mathrm{H}^1_{\Gamma_{\mathrm{Dir}}}(\Omega)^d$ the energy functional $\mathcal{E}_\varepsilon : [0,T] \times \mathcal{Q}_\varepsilon(\Omega) \to \mathbb{R}$ is defined via

$$
\mathcal{E}_\varepsilon(t, u, \chi) = \tfrac{1}{2}\langle \mathbb{C}(\chi)\mathbf{e}(u), \mathbf{e}(u)\rangle_{\mathrm{L}^2(\Omega)^{d\times d}} + \|R_{\frac{\varepsilon}{2}}(Q_\varepsilon(\chi))\|^p_{\mathrm{L}^p(\Omega)^d} - \langle \ell(t), u\rangle, \tag{4.12}
$$

where $\ell \in \mathrm{C}^1([0,T]; (\mathrm{H}^1_{\Gamma_{\mathrm{Dir}}}(\Omega)^d)^*)$. Here, $\|R_{\frac{\varepsilon}{2}}(Q_\varepsilon(\chi))\|^p_{\mathrm{L}^p(\Omega)^d}$ is a regularization term yielding better convergence properties when looking for an effective limit damage model. Note, that in this section, where we introduced the mapping Q_ε, we deal with the unit cell $Y = [-\frac{1}{2}, \frac{1}{2})^d$, whereas in Section 2 we choose $Y = [0,1)^d$ when defining $R_{\frac{\varepsilon}{2}}$. But as mentioned before, this is only for technical reasons such that there is no problem with applying

$R_{\frac{\varepsilon}{2}}$ to $Q_\varepsilon(\chi)$. Moreover, this term is neither necessary nor problematic for proving existence of solutions of the microscopic model, i.e. for fixed $\varepsilon > 0$.

The dissipation distance $\mathcal{D}_\varepsilon : \mathbb{X}^D_{\varepsilon\Lambda}(\Omega)\times\mathbb{X}^D_{\varepsilon\Lambda}(\Omega) \to [0,\infty]$ is given by

$$\mathcal{D}_\varepsilon(\chi_1,\chi_2) = \begin{cases} \int_\Omega \chi_1(x) - \chi_2(x)\mathrm{d}x & \text{if } \chi_1 \geq \chi_2, \\ \infty & \text{otherwise,} \end{cases} \tag{4.13}$$

such that the dissipated energy from state χ_1 to χ_2 is proportional to the change of the volume of the damaged region in the body Ω. As already mentioned in Remark 4.7 one could alternatively introduce a dissipation distance that is rate independent with respect to the change of the surface of damage set, for example.

The rate-independent damage evolution is modeled by the ε-dependent energetic formulation (S^ε) and (E^ε), where $\varepsilon > 0$ scales the size of the damage structure.

(S^ε) $\quad \mathcal{E}_\varepsilon(t,u_\varepsilon(t),\chi_\varepsilon(t)) \leq \mathcal{E}_\varepsilon(t,\widetilde{u},\widetilde{\chi}) + \mathcal{D}_\varepsilon(\chi_\varepsilon(t),\widetilde{\chi}) \quad \text{for all } (\widetilde{u},\widetilde{\chi}) \in \mathcal{Q}_\varepsilon(\Omega),$

(E^ε) $\quad \mathcal{E}_\varepsilon(t,u_\varepsilon(t),\chi_\varepsilon(t)) + \mathrm{Diss}_{\mathcal{D}_\varepsilon}(\chi_\varepsilon;[0,t]) = \mathcal{E}_\varepsilon(0,u_\varepsilon(0),\chi_\varepsilon(0)) - \int_0^t \langle \dot{\ell}(s), u_\varepsilon(s))\rangle \mathrm{d}s,$

with $\mathrm{Diss}_{\mathcal{D}_\varepsilon}(\chi_\varepsilon;[s,t]) := \sup \sum_{j=1}^N \mathcal{D}_\varepsilon(\chi_\varepsilon(t_{j-1}),\chi_\varepsilon(t_j))$, where $N \in \mathbb{N}$ and the supremum is taken over all finite partitions of $[s,t]$. Following Definition 4.2 we denote by $\mathcal{S}_\varepsilon(t)$ the subset of all $(u_\varepsilon,\chi_\varepsilon) \in \mathcal{Q}_\varepsilon(\Omega)$ satisfying

$$\mathcal{E}_\varepsilon(t,u_\varepsilon,\chi_\varepsilon)<\infty \quad \text{and} \quad \mathcal{E}_\varepsilon(t,u_\varepsilon,\chi_\varepsilon)\leq\mathcal{E}_\varepsilon(t,(\widetilde{u},\widetilde{\chi})+\mathcal{D}_\varepsilon(\chi_\varepsilon,\widetilde{\chi}) \quad \forall\,(\widetilde{u},\widetilde{\chi})\in\mathcal{Q}_\varepsilon(\Omega).$$

Introducing the subspace $\mathrm{BV}([0,T];\mathrm{L}^1(\Omega))$ of $\mathrm{L}^1([0,T];\mathrm{L}^1(\Omega))$ as all function f belonging to $\mathrm{L}^1([0,T];\mathrm{L}^1(\Omega))$ having a bounded variation

$$\operatorname{ess\,var}_{[0,T]}(f) := \inf\{\operatorname{var}_{[0,T]}(g) \,|\, g = f \ \mathcal{L}^1\text{-a.e. in } [0,T]\},$$

and

$$\operatorname{var}_{[0,T]}(g) := \sup \sum_{j=1}^N \|g(t_{j-1}) - g(t_j)\|_{\mathrm{L}^1(\Omega)},$$

where $N \in \mathbb{N}$ and the supremum is taken over all finite partitions of $[0,T]$, the abstract Theorem 4.4 guarantees the existence of a solution:

Proposition 4.10. *Let $\varepsilon > 0$, $\mathcal{Q}_\varepsilon(\Omega) = \mathrm{H}^1_{\Gamma_{\mathrm{Dir}}}(\Omega)^d\times\mathbb{X}^D_{\varepsilon\Lambda}(\Omega)$ and let condition* (4.7) *be fulfilled. Moreover, let $\mathcal{E}_\varepsilon : [0,T]\times\mathcal{Q}_\varepsilon(\Omega) \to \mathbb{R}$ and $\mathcal{D}_\varepsilon : \mathbb{X}^D_{\varepsilon\Lambda}(\Omega)\times\mathbb{X}^D_{\varepsilon\Lambda}(\Omega) \to [0,\infty]$ be defined via* (4.12) *and* (4.13)*, respectively. Then for a given $(u^0_\varepsilon,\chi^0_\varepsilon) \in \mathcal{S}_\varepsilon(0)$, there exists a solution $(u_\varepsilon,\chi_\varepsilon) : [0,T] \to \mathcal{Q}_\varepsilon(\Omega)$ satisfying $(u_\varepsilon(0),\chi_\varepsilon(0)) = (u^0_\varepsilon,\chi^0_\varepsilon)$ and*

$$u_\varepsilon \in \mathrm{L}^\infty([0,T],\mathrm{H}^1_{\Gamma_{\mathrm{Dir}}}(\Omega)^d) \qquad \text{and} \qquad \chi_\varepsilon \in \mathrm{L}^\infty([0,T],\mathbb{X}^D_{\varepsilon\Lambda}(\Omega)) \cap \mathrm{BV}([0,T],\mathrm{L}^1(\Omega)).$$

Proof. We have to check the conditions (4.1)-(4.6).

(4.1): Starting with Korn's inequality and exploiting condition (4.7) afterwards we find the following estimate for every $t \in [0,T]$ and $u \in \mathrm{H}^1_{\Gamma_{\mathrm{Dir}}}(\Omega)^d$:

$$\begin{aligned}\tfrac{1}{2}C_{\mathrm{Korn}}^{-1}\alpha\|u\|^2_{\mathrm{H}^1_{\Gamma_{\mathrm{Dir}}}(\Omega)^d} &\leq \tfrac{1}{2}\alpha\|\mathbf{e}(u)\|^2_{\mathrm{L}^2(\Omega)^{d\times d}}\\ &\leq \tfrac{1}{2}\langle\mathbb{C}_{\mathrm{weak}}\mathbf{e}(u),\mathbf{e}(u)\rangle_{\mathrm{L}^2(\Omega)^{d\times d}}\\ &\leq \mathcal{E}_\varepsilon(t,u,\chi)+\langle\ell(t),u\rangle\\ &\leq \mathcal{E}_\varepsilon(t,u,\chi)+C_\ell\|u\|_{\mathrm{H}^1_{\Gamma_{\mathrm{Dir}}}(\Omega)^d},\end{aligned} \tag{4.14}$$

where $C_\ell := \sup_{t\in[0,T]}\|\ell(t)\|_{(\mathrm{H}^1_{\Gamma_{\mathrm{Dir}}}(\Omega)^d)^*} < \infty$ since $\ell \in \mathrm{C}^1([0,T];(\mathrm{H}^1_{\Gamma_{\mathrm{Dir}}}(\Omega)^d)^*)$. For a sequence $(u_\delta,\chi_\delta)_{\delta>0}$ belonging to the sublevel set $L_E(t)$ this estimate gives us a uniform upper bound of the sequence $(\|u_\delta\|_{\mathrm{H}^1_{\Gamma_{\mathrm{Dir}}}(\Omega)^d})_{\delta>0}$. Due to reflexivity of $\mathrm{H}^1_{\Gamma_{\mathrm{Dir}}}(\Omega)^d$ there exists $u_0 \in \mathrm{H}^1_{\Gamma_{\mathrm{Dir}}}(\Omega)^d$ and a subsequence of $(u_\delta)_{\delta>0}$ converging weakly to u_0 in $\mathrm{H}^1_{\Gamma_{\mathrm{Dir}}}(\Omega)^d$.

Moreover, for $(\chi_\delta)_{\delta>0} \subset \mathbb{X}^D_{\varepsilon\Lambda}(\Omega)$ let $(\vec{r}_\delta)_{\delta>0} \subset \mathbb{R}^{|\Lambda^+_\varepsilon|}$ be the sequence of associated vectors as introduced in Remark 4.5. Then every component of $\vec{r}_\delta$ is bounded by $\frac{\varepsilon}{2}$ by definition. Due to reflexivity of $\mathbb{R}^{|\Lambda^+_\varepsilon|}$ there exists a vector $\vec{r}_0 \in \mathbb{R}^{|\Lambda^+_\varepsilon|}$, which again is associated to a function $\chi_0 \in \mathbb{X}^D_{\varepsilon\Lambda}(\Omega)$, such that a subsequence of $(\chi_\delta)_{\delta>0}$ converges strongly to χ_0 in $\mathrm{L}^p(\Omega)$.

Now, by possibly choosing a further subsequence we have $\mathbf{e}(u_{\delta'}) \rightharpoonup \mathbf{e}(u_0)$ in $\mathrm{L}^2(\Omega)^{d\times d}$ and $\chi_{\delta'} \to \chi_0$ in $\mathrm{L}^p(\Omega)$ for all $p \in [1,\infty)$. Choosing $f(x,\chi,e) := \langle\mathbb{C}(\chi)e,e\rangle_{d\times d}$, all assumptions of Theorem 3.23 of [6] are fulfilled and it states:

$$\liminf_{\delta'\to 0}\int_\Omega\langle\mathbb{C}(\chi_{\delta'})\mathbf{e}(u_{\delta'}),\mathbf{e}(u_{\delta'})\rangle_{d\times d}\mathrm{d}x \geq \int_\Omega\langle\mathbb{C}(\chi_0)\mathbf{e}(u_0),\mathbf{e}(u_0)\rangle_{d\times d}\mathrm{d}x. \tag{4.15}$$

For fixed $\varepsilon > 0$ the operators $R_{\frac{\varepsilon}{2}} : \mathrm{K}_{\varepsilon\Lambda}(\Omega) \to \mathrm{K}_{\frac{\varepsilon}{2}\Lambda}(\Omega)^d$ and $Q_\varepsilon : \mathbb{X}^D_{\varepsilon\Lambda}(\Omega) \to \mathrm{K}_{\varepsilon\Lambda}(\Omega)$ are continuous with respect to the strong L^p-topology for every $p \in [1,\infty)$. Hence, for $\delta \to 0$

$$\|R_{\frac{\varepsilon}{2}}(Q_\varepsilon(\chi_{\delta'}))\|^p_{\mathrm{L}^p(\Omega)^d} \to \|R_{\frac{\varepsilon}{2}}(Q_\varepsilon(\chi_0))\|^p_{\mathrm{L}^p(\Omega)^d} \tag{4.16}$$

Trivially, $\langle\ell(t),u_{\delta'}\rangle \to \langle\ell(t),u_0\rangle$ is valid, since $u_{\delta'} \rightharpoonup u_0$ in $\mathrm{H}^1_{\Gamma_{\mathrm{Dir}}}(\Omega)^d$ which together with (4.15) and (4.16) results in

$$E \geq \liminf_{\delta'\to 0}\mathcal{E}_\varepsilon(t,u_{\delta'},\chi_{\delta'}) \geq \mathcal{E}_\varepsilon(t,u_0,\chi_0),$$

such that the compactness of the energy sublevel sets is proven.

(4.2): Since $\ell \in \mathrm{C}^1([0,T];(\mathrm{H}^1_{\Gamma_{\mathrm{Dir}}}(\Omega)^d)^*)$ we have $|\partial_t\mathcal{E}_\varepsilon(t,u,\chi)| = |\langle\dot{\ell}(t),u\rangle| \leq C_{\dot{\ell}}\|u\|_{\mathrm{H}^1_{\Gamma_{\mathrm{Dir}}}(\Omega)^d}$, where $C_{\dot{\ell}} := \sup_{t\in[0,T]}\|\dot{\ell}(t)\|_{(\mathrm{H}^1_{\Gamma_{\mathrm{Dir}}}(\Omega)^d)^*} < \infty$. Dividing estimate (4.14) by $\|u\|_{\mathrm{H}^1_{\Gamma_{\mathrm{Dir}}}(\Omega)^d}$ gives $\|u\|_{\mathrm{H}^1_{\Gamma_{\mathrm{Dir}}}(\Omega)^d} \leq \widetilde{c}_1(c_0+\mathcal{E}_\varepsilon(t,u,\chi))$ for some constants $c_0,\widetilde{c}_1 > 0$ such that the uniform control of the power is shown.

(4.3): The triangle inequality is trivially fulfilled.

(4.4): Note, that all topologies of $\mathbb{X}^D_{\varepsilon\Lambda}(\Omega)$ are equivalent due to its finite dimensionality. Moreover, convergence in $\mathbb{X}^D_{\varepsilon\Lambda}(\Omega)$ implies L^1-convergence which is why the dissipation distance $\mathcal{D}_\varepsilon : \mathbb{X}^D_{\varepsilon\Lambda}(\Omega)\times\mathbb{X}^D_{\varepsilon\Lambda}(\Omega) \to [0,\infty]$ is lower continuous with respect to the topology of $\mathbb{X}^D_{\varepsilon\Lambda}(\Omega)$ and the lower semi-continuity is proven.

(4.5): Since $\partial_t \mathcal{E}_\varepsilon(t,u,\chi) = -\langle \dot{\ell}(t), u\rangle$ this condition is trivially satisfied.

(4.6): Letting $(u_\delta,\chi_\delta)_{\delta>0}$ be a stable sequence with $u_\delta \rightharpoonup u_0$ in $\mathrm{H}^1_{\Gamma_{\mathrm{Dir}}}(\Omega)^d$ and $\chi_\delta \to \chi_0$ in $\mathbb{X}^D_{\varepsilon\Lambda}(\Omega)$ we have to check $(u_0,\chi_0)\in \mathcal{S}_\varepsilon(t)$. For an arbitrary $\widetilde{\chi}_0 \in \mathbb{X}^D_{\varepsilon\Lambda}(\Omega)$ with $\widetilde{\chi}_0 \le \chi_0$ let $\widetilde{\chi}_\delta := \min\{\widetilde{\chi}_0,\chi_\delta\}$, then $\widetilde{\chi}_\delta \to \widetilde{\chi}_0$ in $\mathbb{X}^D_{\varepsilon\Lambda}(\Omega)$ and $\infty > \mathcal{D}_\varepsilon(\widetilde{\chi}_\delta,\chi_\delta) \to \mathcal{D}_\varepsilon(\widetilde{\chi}_0,\chi_0)$ for $\delta\to 0$. Applying again Theorem 3.23 of [6] to gain the following first estimate we finally have

$$\mathcal{E}_\varepsilon(t,u_0,\chi_0) \le \liminf_{\delta\to 0} \mathcal{E}_\varepsilon(t,u_\delta,\chi_\delta) \le \lim_{\delta\to 0}\left(\mathcal{E}_\varepsilon(t,\widetilde{u},\widetilde{\chi}_\delta) + \mathcal{D}_\varepsilon(\chi_\delta,\widetilde{\chi}_\delta)\right) = \mathcal{E}_\varepsilon(t,\widetilde{u},\widetilde{\chi}_0) + \mathcal{D}_\varepsilon(\chi_0,\widetilde{\chi}_0)$$

for arbitrary $(\widetilde{u},\widetilde{\chi}_0) \in \mathcal{Q}_\varepsilon(\Omega)$. Here, the second inequality is due to the stability of the sequence $(u_\delta,\chi_\delta)_{\delta>0}$. This concludes the proof.

Finally, letting $(u_\varepsilon,\chi_\varepsilon):[0,T]\to\mathcal{Q}_\varepsilon(\Omega)$ be a solution of (S^ε) and (E^ε) its time regularity needs to be proven. Since (E^ε) is fulfilled for all $t\in[0,T]$ and its right hand side is finite we have $\mathcal{E}_\varepsilon(t,u_\varepsilon(t),\chi_\varepsilon) < \infty$ and $\mathrm{Diss}_{\mathcal{D}_\varepsilon}(\chi_\varepsilon;[0,T]) < \infty$. The first estimate immediately yields $u_\varepsilon \in \mathrm{L}^\infty([0,T],\mathrm{H}^1_{\Gamma_{\mathrm{Dir}}}(\Omega)^d)$ analogously to inequality (4.14). Moreover, the second estimate means that $\chi_\varepsilon : [0,T]\to \mathbb{X}^D_{\varepsilon\Lambda}(\Omega)$ is a monotone decreasing function such that

$$\mathrm{var}_{[0,T]}(\chi_\varepsilon) = \mathrm{Diss}_{\mathcal{D}_\varepsilon}(\chi_\varepsilon;[0,T]) < \infty,$$

by definition. Trivially, $\chi_\varepsilon \in \mathrm{L}^\infty([0,T],\mathbb{X}^D_{\varepsilon\Lambda}(\Omega))$.

□

4.3 Two-scale limit damage model

In this section we introduce a two-scale damage model $(\mathbf{S}^0)$ and $(\mathbf{E}^0)$ which will turn out to be the limit model of (S^ε) and (E^ε) for $\varepsilon\to 0$. For a damage variable $z_0 \in \mathrm{W}^{1,p}(\Omega;[c_D,1])$ here the damage of the body in the point $x\in\Omega$ is described by the two-scale elasticity tensor

$$\mathbb{C}_0(z_0)(x,y) := \mathbb{C}(\mathbb{1}_{U(z_0(x))}(y))$$

meaning that in every point $x\in\Omega$ there is a unit cell Y containing a scaled damage set D. The scaling of the included damage set is related to the value $z_0(x)$. Hence, firstly the micro-structure of the microscopic model survives in the effective model and secondly the percentage of the damage of the body in point x is given by the size of the scaling of the damage set, ergo $z_0(x)$.

Since the energetic formulation (S^ε) and (E^ε) is solely based on functionals, this approach is well adapted to the theory of Γ-convergence when looking for an effective limit model. Recalling $c_D := 1-\mathrm{vol}(D)$ and letting $\mathcal{Y} := \mathbb{R}^d/_\Lambda$ denote the periodicity cell, the limit function space $\mathbf{Q}$ has the following structure:

$$\mathbf{Q} := \mathrm{H}^1_{\Gamma_{\mathrm{Dir}}}(\Omega)^d \times \mathrm{L}^2(\Omega;\mathrm{H}^1_{\mathrm{av}}(\mathcal{Y}))^d \times \mathrm{W}^{1,p}(\Omega;[c_D,1]),$$

since we are going to apply the theory of Γ-convergence to $\mathcal{E}_\varepsilon(t,u_\varepsilon,\chi_\varepsilon)$ with respect to the two-scale topology for the displacement component u_ε (see Proposition 5.5) and with respect to the topology implied by Theorem 2.1 for the damage component $Q_\varepsilon(\chi_\varepsilon)$.

For $(u_0,U_1)\in \mathrm{H}^1_{\Gamma_{\mathrm{Dir}}}(\Omega)^d \times \mathrm{L}^2(\Omega;\mathrm{H}^1_{\mathrm{av}}(\mathcal{Y}))^d$ we define $\widetilde{\mathbf{e}}(u_0,U_1) := \mathbf{e}_x(u_0) + \mathbf{e}_y(U_1)$ and for $\theta\in[c_D,1]$ we set $U(\theta) := Y\backslash\kappa(\theta)D$ and $\kappa^d(\theta) := (\mathrm{vol}(D))^{-1}(1-\theta)$. Then the two-scale energy functional $\mathbf{E}(t,u_0,U_1,z_0):[0,T]\times\mathbf{Q}\to\mathbb{R}$ is defined via

$$\mathbf{E}(t,u_0,U_1,z_0)=\tfrac{1}{2}\langle\mathbb{C}_0(z_0)\widetilde{\mathbf{e}}(u_0,U_1),\widetilde{\mathbf{e}}(u_0,U_1)\rangle_{\mathrm{L}^2(\Omega\times\mathcal{Y})^{d\times d}}+\|\nabla z_0\|^p_{\mathrm{L}^p(\Omega)^d}-\langle\ell(t),u_0\rangle,$$

where for $W\in\mathrm{L}^2(\Omega\times\mathcal{Y})^{d\times d}$

$$\langle\mathbb{C}_0(\zeta)W,W\rangle_{\mathrm{L}^2(\Omega\times\mathcal{Y})^{d\times d}}=\int_{\Omega\times Y}\langle\mathbb{C}(\mathbb{1}_{U(\zeta(x))}(y))W(x,y),W(x,y)\rangle_{d\times d}\mathrm{d}y\mathrm{d}x,$$

and for $\zeta\in\mathrm{L}^\infty(\Omega;[c_D,1])$

$$\mathbb{C}_0(\zeta)(x,y)=\mathbb{C}(\mathbb{1}_{U(\zeta(x))}(y))=\mathbb{1}_{U(\zeta(x))}(y)\mathbb{C}_{\mathrm{strong}}+(1-\mathbb{1}_{U(\zeta(x))}(y))\mathbb{C}_{\mathrm{weak}}.$$

Furthermore, the dissipation distance $\mathbf{D}:\mathrm{W}^{1,p}(\Omega)\times\mathrm{W}^{1,p}(\Omega)\to[0,\infty]$ is given by

$$\mathbf{D}(z_1,z_2)=\begin{cases}\int_\Omega z_1(x)-z_2(x)\mathrm{d}x & \text{if } z_1\geq z_2\\ \infty & \text{otherwise}\end{cases}.\tag{4.17}$$

The rate-independent damage evolution is modeled by the energetic formulation ($\mathbf{S}^0$) and ($\mathbf{E}^0$):

($\mathbf{S}^0$) $\mathbf{E}(t,u_0(t),U_1(t),z_0(t))\leq\mathbf{E}(t,\widetilde{u},\widetilde{U},\widetilde{z})+\mathbf{D}(z_0(t),\widetilde{z})$ for all $(\widetilde{u},\widetilde{U},\widetilde{z})\in\mathbf{Q}$,

($\mathbf{E}^0$) $\mathbf{E}(t,u_0(t),U_1(t),z_0(t))+\mathrm{Diss}_{\mathbf{D}}(z_0;[0,t])=\mathbf{E}(0,u_0(0),U_1(0),z_0(0))-\int_0^t\langle\dot{\ell}(s),u_0(s))\rangle\mathrm{d}s$,

with $\mathrm{Diss}_{\mathbf{D}}(z_0;[0,t]):=\sup\sum_{j=1}^N\mathbf{D}(z_0(t_{j-1}),z_0(t_j))$, where $N\in\mathbb{N}$ and the supremum is taken over all finite partitions of $[0,t]$. Following Definition 4.2 we denote by $\mathbf{S}_0(t)$ the subset of all $(u_0,U_1,z_0)\in\mathbf{Q}$ satisfying

$$\mathbf{E}(t,u_0,U_1,z_0)<\infty\quad\text{and}\quad\mathbf{E}(t,u_0,U_1,z_0)\leq\mathbf{E}(t,(\widetilde{u},\widetilde{U},\widetilde{z})+\mathbf{D}(z_0,\widetilde{z})\quad\forall\,(\widetilde{u},\widetilde{U},\widetilde{z})\in\mathbf{Q}.$$

Remark 4.11. Existence of a solution of the two-scale damage model is proven indirectly via the convergence result in Section 7, where for $\varepsilon\to0$ the convergence of solutions $(u_\varepsilon,\chi_\varepsilon):[0,T]\to\mathcal{Q}_\varepsilon$ of the microscopic damage models (S^ε) and (E^ε) to a function $(u_0,U_1,z_0):[0,T]\to\mathbf{Q}$ satisfying ($\mathbf{S}^0$) and ($\mathbf{E}^0$) is shown.

4.4 One-scale model

In this subsection we introduce a one-scale model which is equivalent to the two-scale model introduced in Subsection 4.3 in the following sense: From any solution of one of those systems a solution of the other model can be constructed.

For a given function $(u_0,U_1,z_0)\in\mathbf{S}_0(t)$ by choosing $(\widetilde{u},\widetilde{z})=(u_0,z_0)$ in the stability condition ($\mathbf{S}^0$) we find that U_1 is the unique solution of the following minimization problem:

$$\min\{\mathbf{E}(t,u_0,U,z_0)\,|\,U\in\mathrm{L}^2(\Omega;\mathrm{H}^1_{\mathrm{av}}(\mathcal{Y}))^d\}.\tag{4.18}$$

This motivates the introduction of an effective tensor given by the following unit cell problem. For $\xi\in\mathbb{R}^{d\times d}_{\mathrm{sym}}$ and $\theta\in[c_D,1]$ let

$$\begin{aligned}C_{\mathrm{eff}}(\theta,\xi)&:=\min_{v\in\mathrm{H}^1_{\mathrm{av}}(\mathcal{Y})^d}I(\theta,\xi,v)\\ I(\theta,\xi,v)&:=\int_Y\langle\mathbb{C}(\mathbb{1}_{U(\theta)}(y))(\xi+\mathbf{e}_y(v)(y)),\xi+\mathbf{e}_y(v)(y)\rangle_{d\times d}\mathrm{d}y\\ U(\theta)&:=Y\backslash\kappa(\theta)D\quad\text{and}\quad\kappa^d(\theta):=(\mathrm{vol}(D))^{-1}(1-\theta)\end{aligned}$$

Proposition 4.12. *For a given $\theta \in [c_D, 1]$ there exists $\mathbb{C}_{\text{eff}}(\theta) \in \text{Lin}_{\text{sym}}(\mathbb{R}^{d\times d}_{\text{sym}}, \mathbb{R}^{d\times d}_{\text{sym}})$ such that*

$$\forall \xi \in \mathbb{R}^{d\times d}_{\text{sym}}: \qquad C_{\text{eff}}(\theta, \xi) = \langle \mathbb{C}_{\text{eff}}(\theta)\xi, \xi\rangle_{d\times d}.$$

Proof. For given $\xi \in \mathbb{R}^{d\times d}_{\text{sym}}$ and $\theta \in [c_D, 1]$ according to (4.7) the functional $I(\theta, \xi, \cdot) : \mathrm{H}^1_{\text{av}}(\mathcal{Y})^d \to \mathbb{R}$ is strictly convex and continuous. Hence, there exists a unique minimizer $v^* \in \mathrm{H}^1_{\text{av}}(\mathcal{Y})^d$ fulfilling the Euler-Lagrange equation

$$D_{\nabla v}(I(\theta, \xi, v))[\widetilde{v}] = 0 \qquad \forall \, \widetilde{v} \in \mathrm{H}^1_{\text{av}}(\mathcal{Y})^d. \tag{4.19}$$

Letting $B_{\theta\xi} := D_{\nabla v}(I(\theta, \xi, \cdot))[\cdot] : \mathrm{H}^1_{\text{av}}(\mathcal{Y})^d \times \mathrm{H}^1_{\text{av}}(\mathcal{Y})^d \to \mathbb{R}$, due to the Lemma of Lax-Milgram there exists a linear operator $\mathcal{L}^\xi_\theta : (\mathrm{H}^1_{\text{av}}(\mathcal{Y})^d)^* \to \mathrm{H}^1_{\text{av}}(\mathcal{Y})$ such that $\mathcal{L}^\xi_\theta(0) = v^*$. With this we are able to define a linear mapping $\mathcal{L}_\theta := \mathcal{L}^{(\cdot)}_\theta(0) : \mathbb{R}^{d\times d}_{\text{sym}} \to \mathrm{H}^1_{\text{av}}(\mathcal{Y})^d$ satisfying $\mathcal{L}_\theta(\xi) = v^*$.

For $e_{ij} \in \mathbb{R}^{d\times d}$, where $(e_{ij})_{kl} := \delta_{ij,kl}$ and $\delta_{ij,kl}$ for $i, j, k, l \in \{1, \dots, d\}$ is the Kronecker delta, we define

$$\mathbb{C}_{\text{eff}_{ijkl}}(\theta) := \int_{\mathcal{Y}} \langle \mathbb{C}(\mathbb{1}_{U(\theta)}(y))(e_{ij} + \mathbf{e}_y(\mathcal{L}_\theta e_{ij})(y)), e_{kl} + \mathbf{e}_y(\mathcal{L}_\theta e_{kl})(y)\rangle_{d\times d} \mathrm{d}y.$$

First of all we have $\mathbb{C}_{\text{eff}}(\theta) \in \text{Lin}_{\text{sym}}(\mathbb{R}^{d\times d}_{\text{sym}}, \mathbb{R}^{d\times d}_{\text{sym}})$ and we find

$$C_{\text{eff}}(\theta, \xi) = \langle \mathbb{C}_{\text{eff}}(\theta)\xi, \xi\rangle_{d\times d}$$

by writing $\xi \in \mathbb{R}^{d\times d}_{\text{sym}}$ as $\xi = \sum_{i,j=1}^d \xi_{ij} e_{ij}$. ☐

For

$$\mathcal{Q}_0(\Omega) := \mathrm{H}^1_{\Gamma_{\text{Dir}}}(\Omega)^d \times \mathrm{W}^{1,p}(\Omega; [c_D, 1])$$

the one-scale model is based on the one-scale energy functional $\mathcal{E}_0 : [0, T] \times \mathcal{Q}_0(\Omega) \to \mathbb{R}_\infty$ defined in the following way:

$$\mathcal{E}_0(t, u_0, z_0) := \tfrac{1}{2}\langle \mathbb{C}_{\text{eff}}(z_0)\mathbf{e}(u_0), \mathbf{e}(u_0)\rangle_{\mathrm{L}^2(\Omega)^{d\times d}} + \|\nabla z_0\|^p_{\mathrm{L}^p(\Omega)^d} - \langle \ell(t), u_0\rangle.$$

Moreover, the one-scale dissipation distance $\mathcal{D}_0 : \mathrm{W}^{1,p}(\Omega) \times \mathrm{W}^{1,p}(\Omega) \to [0, \infty]$ reads as follows

$$\mathcal{D}_0(z_1, z_2) = \begin{cases} \displaystyle\int_\Omega z_1(x) - z_2(x) \mathrm{d}x & \text{if } z_1 \geq z_2, \\ \infty & \text{otherwise.} \end{cases} \tag{4.20}$$

Trivially $\mathcal{D}_0(z_1, z_2) = \mathbf{D}_0(z_1, z_2)$ for all $z_1, z_2 \in \mathrm{W}^{1,p}(\Omega; [c_D, 1])$ by definition. With this, the energetic formulation (S^0) and (E^0) of the rate-independent system $(\mathcal{Q}_0(\Omega), \mathcal{E}_0, \mathcal{D}_0)$ reads as follows:

(S^0) $\quad \mathcal{E}_0(t, u_0(t), z_0(t)) \leq \mathcal{E}_0(t, \widetilde{u}, \widetilde{z}) + \mathcal{D}_0(z_0(t), \widetilde{z}) \quad$ for all $(\widetilde{u}, \widetilde{z}) \in \mathcal{Q}_0(\Omega)$,

(E^0) $\quad \mathcal{E}_0(t, u_0(t), z_0(t)) + \text{Diss}_{\mathcal{D}_0}(z_0; [0, t]) = \mathcal{E}_0(0, u_0(0), z_0(0)) - \int_0^t \langle \dot{\ell}(s), u_0(s))\rangle \mathrm{d}s$,

with $\text{Diss}_{\mathcal{D}_0}(z_0; [s, t]) := \sup \sum_{j=1}^N \mathcal{D}_0(z_0(t_{j-1}), z_0(t_j))$, where $N \in \mathbb{N}$ and the supremum is taken over all finite partitions of $[s, t]$. Furthermore, we define the set of stable states $\mathcal{S}_0(t)$ for $t \in [0, T]$ via

$$\mathcal{S}_0(t) := \{(u_0, z_0) \in \mathcal{Q}_0(\Omega) \,|\, \mathcal{E}_0(t, u_0, z_0) \leq \mathcal{E}_0(t, \widetilde{u}, \widetilde{z}) + \mathcal{D}_0(z_0, \widetilde{z}) \quad \forall \, (\widetilde{u}, \widetilde{z}) \in \mathcal{Q}_0(\Omega)\}$$

Proposition 4.13. *The following two statements hold:*

(i) $(u_0, U_1, z_0) \in \mathbf{S}_0(t) \quad \Leftrightarrow \quad U_1 = \mathcal{L}_{z_0}(\mathbf{e}_x(u_0))$ *and* $(u_0, z_0) \in \mathcal{S}_0(t)$

(ii) U_1 *unique solution of* (4.18) $\Leftrightarrow \quad U_1 = \mathcal{L}_{z_0}(\mathbf{e}_x(u_0)) \quad \Leftrightarrow \quad \mathbf{E}_0(u_0, U_1, z_0) = \mathcal{E}_0(u_0, z_0)$

Proof. 1. We start with proving the first equivalence of part (ii). For given $(u_0, z_0) \in \mathcal{Q}_0(\Omega)$ let $U_1 \in \mathrm{L}^2(\Omega; \mathrm{H}^1_{\mathrm{av}}(\mathcal{Y}))^d$ be the unique solution of the Euler-Lagrange equation

$$D_{\nabla_y U}(I_1(z_0, u_0, U))[\widetilde{U}] = 0 \qquad \forall\, \widetilde{U} \in \mathrm{L}^2(\Omega, \mathrm{H}^1_{\mathrm{av}}(\mathcal{Y}))^d, \tag{4.21}$$

where

$$I_1(z_0, u_0, U) := \int_{\Omega \times Y} \langle \mathbb{C}_0(z_0)(x,y)(\widetilde{\mathbf{e}}(u_0, U)(x,y)), \widetilde{\mathbf{e}}(u_0, U)(x,y) \rangle_{d\times d} \mathrm{d}y \mathrm{d}x.$$

With $\mathcal{L}_\theta : \mathbb{R}^{d\times d}_{\mathrm{sym}} \to \mathrm{H}^1_{\mathrm{av}}(\mathcal{Y})^d$ as in the proof of Proposition 4.12 we now show that the function $U_1^*(x,y) := \mathcal{L}_{z_0(x)}(\mathbf{e}_x(u_0)(x))(y)$ is also a solution of the Euler-Lagrange equation (4.21). This then results in $U_1 = U_1^*$ according to the uniqueness of the minimizer.

For fixed $x \in \Omega$ and $\theta = z_0(x)$ the function $U_1^*(x,y)$ solves (4.19) by definition. Since this is valid for almost every $x \in \Omega$ this result stays valid by multiplying (4.19) with a scalar function $f_i \in \mathrm{L}^2(\Omega)$, $i \in \{1, \dots, d\}$, and integrating everything over Ω afterwards:

$$\int_\Omega D_{\nabla v}(I(z_0(x), u_0(x), U_1^*(x,\cdot)))[f_i(x)\widetilde{v}]\mathrm{d}x = 0 \qquad \forall\, \widetilde{v} \in \mathrm{H}^1_{\mathrm{av}}(\mathcal{Y})^d.$$

Choosing $\widetilde{v} = v_i e_i$ for an arbitrary function $v_i \in \mathrm{H}^1_{\mathrm{av}}(\mathcal{Y})$, repeating this procedure for every $i \in \{1, \dots, d\}$ and summing up everything afterwards, we finally end up with

$$D_{\nabla_y U}(I_1(z_0, u_0, U_1^*))[\widehat{U}] = 0 \qquad \forall\, \widehat{U} = (f_1 v_1, \dots, f_d v_d)^T,$$

where $f_i \in \mathrm{L}^2(\Omega)$ for every $i \in \{1, \dots, d\}$. Since $\{(f_1 v_1, \dots, f_d v_d)^T \,|\, f_i \in \mathrm{L}^2(\Omega), v_i \in \mathrm{H}^1_{\mathrm{av}}(\mathcal{Y})\}$ is dense in $\mathrm{L}^2(\Omega; \mathrm{H}^1_{\mathrm{av}}(\mathcal{Y}))^d$ we finally obtain that U_1^* solves (4.21) for a given function $(u_0, z_0) \in \mathcal{Q}_0(\Omega)$.

2. Following the trivial transformations below for given $(u_0, z_0) \in \mathcal{Q}_0(\Omega)$ we find

$$U_1 = \mathcal{L}_{z_0}(\mathbf{e}_x(u_0)) \quad \Leftrightarrow \quad \mathbf{E}_0(t, u_0, U_1, z_0) = \mathcal{E}_0(t, u_0, z_0).$$

Indeed:

$$\begin{aligned}
\mathcal{E}_0(t, u_0, z_0) &= \tfrac{1}{2}\langle \mathbb{C}_{\mathrm{eff}}(z_0)\mathbf{e}_x(u_0), \mathbf{e}_x(u_0)\rangle_{\mathrm{L}^2(\Omega)^{d\times d}} + \|\nabla z_0\|^p_{\mathrm{L}^p(\Omega)^d} - \langle \ell(t), u_0 \rangle && (4.22a)\\
&= \tfrac{1}{2}\int_\Omega I(z_0(x), \mathbf{e}_x(u_0)(x), \mathcal{L}_{z_0(x)}(\mathbf{e}_x(u_0)(x)))\mathrm{d}x + \|\nabla z_0\|^p_{\mathrm{L}^p(\Omega)^d} - \langle \ell(t), u_0 \rangle \\
&= \tfrac{1}{2} I_1(z_0, u_0, \mathcal{L}_{z_0}(\mathbf{e}_x(u_0))) + \|\nabla z_0\|^p_{\mathrm{L}^p(\Omega)^d} - \langle \ell(t), u_0 \rangle && (4.22b)\\
&= \mathbf{E}_0(t, u_0, U_1, z_0) && (4.22c)
\end{aligned}$$

In the case of "⇒" line (4.22c) is equal to line (4.22a) by assumption and $U_1 = \mathcal{L}_{z_0}(\mathbf{e}_x(u_0))$ follows by comparing line (4.22b) and (4.22c). On the other hand in the case of "⇐" in line (4.22c) $U_1 = \mathcal{L}_{z_0}(\mathbf{e}_x(u_0))$ was exploited.

Note, that in the case of $U_1 = \mathcal{L}_{z_0}(\mathbf{e}_x(u_0))$ the function U_1 is the unique minimizer of (4.18) such that there is no function $\widehat{U}_1 \neq U_1$ satisfying $\mathbf{E}_0(t, u_0, \widehat{U}_1, z_0) = \mathcal{E}_0(t, u_0, z_0)$

3. It remains to prove part (i). "⇒": As we already mentioned in the beginning of this section $(u_0, U_1, z_0) \in \mathcal{S}_0(t)$ implies that $U_1 \in \mathrm{L}^2(\Omega; \mathrm{H}^1_{\mathrm{av}}(\mathcal{Y}))^d$ is the minimizer of (4.18). Hence, we have $U_1 = \mathcal{L}_{z_0}(\mathbf{e}_x(u_0))$ according to part (ii) and

$$\mathbf{E}_0(t, u_0, U_1, z_0) \leq \min_{\widetilde{U}} \mathbf{E}_0(t, \widetilde{u}, \widetilde{U}, \widetilde{z}) + \mathbf{D}_0(z_0, \widetilde{z}) \qquad \forall\, (\widetilde{u}, \widetilde{z}) \in \mathcal{Q}_0(\Omega)$$

by taking the minimum over all $\widetilde{U} \in \mathrm{L}^2(\Omega; \mathrm{H}^1_{\mathrm{av}}(\mathcal{Y}))^d$ on the right hand side of the stability condition ($\mathbf{S}^0$). But with this we are able to exploit part (ii) on the left hand side as well as on the right hand side such that we end up with

$$\mathcal{E}_0(t, u_0, z_0) \leq \mathcal{E}_0(t, \widetilde{u}, \widetilde{z}) + \mathcal{D}_0(z_0, \widetilde{z}) \qquad \forall\, (\widetilde{u}, \widetilde{z}) \in \mathcal{Q}_0(\Omega).$$

"⇐": Due to part (ii) we have

$$\mathbf{E}_0(t, u_0, U_1, z_0) \overset{(ii)}{=} \mathcal{E}_0(t, u_0, z_0) \leq \mathcal{E}_0(t, \widetilde{u}, \widetilde{z}) + \mathcal{D}_0(z_0, \widetilde{z}) \qquad \forall\, (\widetilde{u}, \widetilde{z}) \in \mathcal{Q}_0(\Omega).$$

Moreover, also according to part (ii) $\mathcal{E}_0(t, \widetilde{u}, \widetilde{z}) \leq \mathbf{E}_0(t, \widetilde{u}, \widetilde{U}, \widetilde{z})$ for all $\widetilde{U} \in \mathrm{L}^2(\Omega; \mathrm{H}^1_{\mathrm{av}}(\mathcal{Y}))^d$ since there is equality for the unique minimizer of (4.18). This finally gives us

$$\mathbf{E}_0(t, u_0, U_1, z_0) \leq \mathbf{E}_0(t, \widetilde{u}, \widetilde{U}, \widetilde{z}) + \mathbf{D}_0(z_0, \widetilde{z}) \qquad \forall\, (\widetilde{u}, \widetilde{z}) \in \mathbf{Q}$$

and Proposition 4.13 is proven. □

Corollary 4.14. *The following two statements are equivalent:*

(i) $(u_0, U_1, z_0) : [0, T] \to \mathbf{Q}$, $u_0 \in \mathrm{L}^\infty([0, T], \mathrm{H}^1_{\Gamma_{\mathrm{Dir}}}(\Omega)^d)$, $U_1 \in \mathrm{L}^\infty([0, T], \mathrm{L}^2(\Omega; \mathrm{H}^1_{\mathrm{av}}(\mathcal{Y}))^d)$ *and* $z_0 \in \mathrm{L}^\infty([0, T], \mathrm{W}^{1,p}(\Omega; [c_D, 1])) \cap \mathrm{BV}([0, T], \mathrm{L}^1(\Omega))$,*is a solution of* ($\mathbf{S}^0$) *and* ($\mathbf{E}^0$) *satisfying* $(u_0(0), U_1(0), z_0(0)) = (u_0^0, U_1^0, z_0^0) \in \mathcal{S}_0(0)$.

(ii) $(u_0, z_0) \in \mathrm{L}^\infty([0, T], \mathrm{H}^1_{\Gamma_{\mathrm{Dir}}}(\Omega)^d) \times \left[\mathrm{L}^\infty([0, T], \mathrm{W}^{1,p}(\Omega; [c_D, 1])) \cap \mathrm{BV}([0, T], \mathrm{L}^1(\Omega))\right]$ *is a solution of* (S^0) *and* (E^0) *satisfying* $(u_0(0), z_0(0)) = (u_0^0, z_0^0) \in \mathcal{S}_0(0)$ *and* $U_1 = \mathcal{L}_{z_0}(\mathbf{e}_x(u_0))$.

Proof. This is an easy consequence of Proposition 4.13 □

5 Two-scale convergence

This section introduces everything needed in the following sections concerning the theory of folding/unfolding and two-scale convergence and does not claim completeness. For further details we recommend to [1, 5, 4]. Here, we refer to the notation introduced in Section 2, extend it if necessary and state the main results concerning the two-scale convergence needed in the following.

As already introduced in Section 2 let Λ be a periodic latice and Y the so called associated unit cell. But contrary to Section 2, Λ is allowed to be defined via an arbitrary basis $\{b_1, b_2, \ldots, b_d\}$ of $\mathbb{R}^d$, with no need of orthonormality. In particular, the unit cell Y is the d-parallelotope whose axis are the basis vectors. Only

$$\mathrm{vol}(Y) = 1$$

needs to be satisfied to make the following statements valid without any normalization coefficients. Moreover, one could think of any $\widetilde{Y} := Y - \widetilde{y}$ for a fixed $\widetilde{y} \in Y$ as the unit cell.

Before defining the two-scale convergence with the help of the so called periodic unfolding operator we start by introducing the mappings $[\cdot]_\Lambda$ and $\{\cdot\}_Y$ on $\mathbb{R}^d$ so that

$$[\cdot]_\Lambda : \mathbb{R}^d \to \Lambda, \quad \{\cdot\}_Y : \mathbb{R}^d \to Y, \quad \text{and} \quad x = [x]_\Lambda + \{x\}_Y \quad \text{for all } x \in \mathbb{R}^d.$$

Let $\lambda \in \Lambda$ and let $x \in \mathbb{R}^d$ be in the cell $\lambda + Y$, then $[x]_\Lambda = \lambda$ and $\{x\}_Y$ is determinable as $\{x\}_Y = x - [x]_\Lambda$. For $\varepsilon > 0$ and $x \in \mathbb{R}^d$ we have the following decomposition:

$$x = \mathcal{N}_\varepsilon(x) + \varepsilon \mathcal{V}_\varepsilon(x), \quad \text{with } \mathcal{N}_\varepsilon(x) = \varepsilon \left[\frac{x}{\varepsilon}\right]_\Lambda \text{ and } \mathcal{V}_\varepsilon(x) = \left\{\frac{x}{\varepsilon}\right\}_Y,$$

where $\mathcal{N}_\varepsilon(x)$ denotes the macroscopic center of the cell $\mathcal{N}_\varepsilon(x) + \varepsilon Y$ that contains x and $\mathcal{V}_\varepsilon(x)$ is the microscopic part of x in Y. At last we want to distinguish the unit cell Y from the periodicity cell $\mathcal{Y} := \mathbb{R}^d/_\Lambda$. Following Ref. [25], we introduce the mappings $\mathcal{V}_\varepsilon$ and $\mathcal{S}_\varepsilon$ as follows:

$$\mathcal{D}_\varepsilon : \begin{cases} \mathbb{R}^d & \to \quad \mathbb{R}^d \times \mathcal{Y}, \\ x & \mapsto \quad (\mathcal{N}_\varepsilon(x), \mathcal{V}_\varepsilon(x)), \end{cases} \qquad \mathcal{S}_\varepsilon : \begin{cases} \mathbb{R}^d \times \mathcal{Y} & \to \quad \mathbb{R}^d, \\ (x, y) & \mapsto \quad \mathcal{N}_\varepsilon(x) + \varepsilon y, \end{cases}$$

where in the last sum $y \in \mathcal{Y}$ is identified with $y \in Y \subset \mathbb{R}^d$.

Two-scale convergence is linked to a suitable two-scale embedding of $\mathrm{L}^p(\Omega)$ in the two-scale space $\mathrm{L}^p(\mathbb{R}^d \times \mathcal{Y})$. Such an embedding is called periodic unfolding operator. Here and in the following

$$\Omega \subset \mathbb{R}^d \text{ is assumed to be open and bounded and satisfies } \mathrm{vol}(\partial\Omega) = 0. \tag{5.1}$$

The following definition of a periodic unfolding operator was given in Ref. [4].

Definition 5.1. (Ref. [4]) Let $\Omega \subset \mathbb{R}^d$ be open, $\varepsilon > 0$ and $p \in [1, \infty]$. Then the natural candidate of a periodic unfolding operator $\mathcal{T}_\varepsilon$ is defined via:

$$\mathcal{T}_\varepsilon : \mathrm{L}^p(\Omega) \to \mathrm{L}^p(\mathbb{R}^d \times \mathcal{Y});\ v \mapsto v^{\mathrm{ex}} \circ \mathcal{S}_\varepsilon,$$

where $v^{\mathrm{ex}} \in \mathrm{L}^p(\mathbb{R}^d)$ is the extension of the function v by 0 to all of $\mathbb{R}^d$.

With this definition the following product rule is valid: Let $p, q, r \in [1, \infty]$ such that

$$\tfrac{1}{p} + \tfrac{1}{q} = \tfrac{1}{r}, v_1 \in \mathrm{L}^p(\Omega), v_2 \in \mathrm{L}^q(\Omega) \Longrightarrow \mathcal{T}_\varepsilon(v_1 v_2) = (\mathcal{T}_\varepsilon v_1)(\mathcal{T}_\varepsilon v_2) \in \mathrm{L}^r(\mathbb{R}^d \times \mathcal{Y}).$$

Note that $\overline{[\Omega \times Y]_\varepsilon} := \overline{\mathcal{S}_\varepsilon^{-1}(\Omega)} = \overline{\{(x, y) | \mathcal{S}_\varepsilon(x, y) \in \Omega\}}$ is the support of $\mathcal{T}_\varepsilon v$, and this is not contained in $\Omega \times Y$, in general.

Following the lines in Ref. [19] we now will use this periodic unfolding operator to introduce the kind of two-scale convergence, which is used here; the strong and weak two-scale convergence, respectively. But before that, we define the folding operator $\mathcal{F}_\varepsilon$. For details see [19].

Definition 5.2. (Ref. [19]) Let $\Omega \subset \mathbb{R}^d$ be open, $\varepsilon > 0$ and $p \in [1, \infty)$. Then the folding operator $\mathcal{F}_\varepsilon$ is defined via:

$$\mathcal{F}_\varepsilon : \mathrm{L}^p(\mathbb{R}^d \times \mathcal{Y}) \to \mathrm{L}^p(\Omega);\ V \mapsto \big(P_\varepsilon(\mathbb{1}_{[\Omega \times Y]_\varepsilon} V) \circ \mathcal{D}_\varepsilon\big)|_\Omega.$$

Definition 5.3. (Ref. [19]) Let $p \in (1,\infty)$ and let $(v_\varepsilon)_{\varepsilon>0}$ be a sequence in $\mathrm{L}^p(\Omega)$. Then

(a) v_ε converges strongly two-scale to $V \in \mathrm{L}^p(\Omega\times\mathcal{Y})$ in $\mathrm{L}^p(\Omega\times\mathcal{Y})$, $v_\varepsilon \overset{s}{\to} V$ in $\mathrm{L}^p(\Omega\times\mathcal{Y})$, if $\mathcal{T}_\varepsilon v_\varepsilon \to V^{\mathrm{ex}}$ in $\mathrm{L}^p(\mathbb{R}^d\times\mathcal{Y})$.

(b) v_ε converges weakly two-scale to $V \in \mathrm{L}^p(\Omega\times\mathcal{Y})$ in $\mathrm{L}^p(\Omega\times\mathcal{Y})$, $v_\varepsilon \overset{w}{\rightharpoonup} V$ in $\mathrm{L}^p(\Omega\times\mathcal{Y})$, if $\mathcal{T}_\varepsilon v_\varepsilon \rightharpoonup V^{\mathrm{ex}}$ in $\mathrm{L}^p(\mathbb{R}^d\times\mathcal{Y})$.

Referring to definition (2.2) we have that for all $\varepsilon > 0$ the support of the function $\mathcal{T}_\varepsilon v_\varepsilon$ is contained in $\overline{[\Omega\times Y]_\varepsilon} \subset \overline{\Omega}_\varepsilon^+ \times Y$ which results in the fact that the support of a possible accumulation point U of the sequence $(\mathcal{T}_\varepsilon v_\varepsilon)_{\varepsilon>0}$ has to be in $\overline{\Omega}\times Y$, since $\mathrm{vol}(\Omega_\varepsilon^+\backslash\Omega) \to 0$. Due to $\mathrm{vol}(\partial\Omega) = 0$ we also have $\mathrm{L}^p(\Omega\times\mathcal{Y}) = \mathrm{L}^p(\overline{\Omega}\times\mathcal{Y})$ and so every accumulation point of $(\mathcal{T}_\varepsilon v_\varepsilon)_{\varepsilon>0}$ can be uniquely identified with an element of $\mathrm{L}^p(\Omega\times\mathcal{Y})$. But notice that it is important to determine the convergence in $\mathrm{L}^p(\mathbb{R}^d\times\mathcal{Y})$ and not in $\mathrm{L}^p(\Omega\times\mathcal{Y})$. We refer to Ref. [19], where it is shown in Example 2.3 that convergence in $\mathrm{L}^p(\Omega\times\mathcal{Y})$ is not sufficient.

Notice, according to the definition of the two-scale convergence in $\mathrm{L}^p(\Omega\times\mathcal{Y})$ via the convergence of the unfolded sequence in $\mathrm{L}^p(\mathbb{R}^d\times\mathcal{Y})$ all convergence properties known for L^p-convergence are transmitted. For a summary of those properties we refer to Proposition 2.4 in [19]. For the convenience of the reader we state here only those properties used in the following.

Proposition 5.4. (Ref. [19]) *Let* $p \in (1,\infty)$, $p' = \frac{p}{p-1}$ *and* $\varepsilon > 0$. *Then*

(a) *If* $v_\varepsilon \overset{w}{\rightharpoonup} V_0$ *in* $\mathrm{L}^p(\Omega\times\mathcal{Y})$ *and* $w_\varepsilon \overset{s}{\to} W_0$ *in* $\mathrm{L}^{p'}(\Omega\times\mathcal{Y})$ *then* $\langle v_\varepsilon, w_\varepsilon\rangle_{\mathrm{L}^2(\Omega)} \to \langle V_0, W_0\rangle_{\mathrm{L}^2(\Omega\times\mathcal{Y})}$.

(b) *If* $v_\varepsilon \to v_0$ *in* $\mathrm{L}^p(\Omega)$ *then* $v_\varepsilon \overset{s}{\to} Ev_0$ *in* $\mathrm{L}^p(\Omega\times\mathcal{Y})$, *where* $E : \mathrm{L}^p(\Omega) \to \mathrm{L}^p(\Omega\times\mathcal{Y})$ *is defined via* $Ev(x,y) := v(x)$.

(c) *If* $v_\varepsilon \overset{s}{\to} V_0$ *in* $\mathrm{L}^p(\Omega\times\mathcal{Y})$ *and if* $(m_\varepsilon)_{\varepsilon>0}$ *is a bounded sequence of* $\mathrm{L}^\infty(\Omega)$ *so that* $\mathcal{T}_\varepsilon m_\varepsilon(x,y) \to M(x,y)$ *for almost every* $(x,y) \in \Omega\times\mathcal{Y}$. *Then* $m_\varepsilon v_\varepsilon \overset{s}{\to} M_0 V_0$ *in* $\mathrm{L}^p(\Omega\times\mathcal{Y})$.

In Section 7 we are going to prove Γ-convergence results with respect to the weak two-scale topology for the functional based evolution models introduced in Subsection 4.2. There the following integral identity for $v \in \mathrm{L}^1(\Omega)$ will be central.

$$\int_\Omega v(x)\mathrm{d}x = \int_{[\Omega\times Y]_\varepsilon} \mathcal{T}_\varepsilon v(x,y)\mathrm{d}y\mathrm{d}x \tag{5.2}$$

Moreover, this identity immediately gives us the norm-preservation of the periodic unfolding operator $\mathcal{T}_\varepsilon$ and it is proved by decomposing $\mathbb{R}^d$ into cells $\varepsilon(\lambda+Y)$ for $\lambda \in \Lambda$:

$$\begin{aligned}
\int_{[\Omega\times Y]_\varepsilon} \mathcal{T}_\varepsilon v(x,y)\mathrm{d}y\mathrm{d}x &= \int_{\mathbb{R}^d\times Y} \mathbb{1}_{[\Omega\times Y]_\varepsilon}(x,y)\mathcal{T}_\varepsilon v(x,y)\mathrm{d}y\mathrm{d}x \\
&= \int_{\mathbb{R}^d\times Y} \mathcal{T}_\varepsilon \mathbb{1}_\Omega(x,y)\mathcal{T}_\varepsilon v(x,y)\mathrm{d}y\mathrm{d}x \\
&= \sum_{\lambda\in\Lambda} \int_{\varepsilon(\lambda+Y)} \int_Y \mathbb{1}_\Omega^{\mathrm{ex}}(\mathcal{N}_\varepsilon(x){+}\varepsilon y) v^{\mathrm{ex}}(\mathcal{N}_\varepsilon(x){+}\varepsilon y)\mathrm{d}y\mathrm{d}x \\
&= \sum_{\lambda\in\Lambda} \varepsilon^d \int_Y \mathbb{1}_\Omega^{\mathrm{ex}}(\varepsilon(\lambda{+}y)) v^{\mathrm{ex}}(\varepsilon(\lambda{+}y))\mathrm{d}y \\
&= \sum_{\lambda\in\Lambda} \int_{\varepsilon(\lambda+Y)} \mathbb{1}_\Omega^{\mathrm{ex}}(x) v^{\mathrm{ex}}(x)\mathrm{d}x \\
&= \int_\Omega v(x)\mathrm{d}x.
\end{aligned}$$

Since our model introduced in Subsection 4.2 contains the deformation gradient we now will consider bounded sequences of $\mathrm{W}^{1,p}(\Omega)$ and state the main two-scale convergence results for these. In particular we will need the function space

$$\mathrm{W}^{1,p}_{\mathrm{av}}(\mathcal{Y}) = \left\{ v \in \mathrm{W}^{1,p}(\mathcal{Y}) \,\middle|\, \int_{\mathcal{Y}} v(y)\mathrm{d}y = 0 \right\}.$$

To describe the weak two-scale convergence of gradients we introduce the function space $\mathrm{L}^p(\Omega; \mathrm{W}^{1,p}_{\mathrm{av}}(\mathcal{Y}))$, which is the space of functions $V \in \mathrm{L}^p(\Omega{\times}\mathcal{Y}) = \mathrm{L}^p(\Omega; \mathrm{L}^p(\mathcal{Y}))$, with $\int_{\mathcal{Y}} V(x,y)\mathrm{d}y = 0$ for almost every $x \in \Omega$ and $\nabla_y V \in \mathrm{L}^p(\Omega{\times}\mathcal{Y})^d$ in the sense of distributions. We equip this space with the norm $\|V\|_{\mathrm{L}^p(\Omega;\mathrm{W}^{1,p}(\mathcal{Y}))} := \|\nabla_y V\|_{\mathrm{L}^p(\Omega\times\mathcal{Y})^d}$.

With this we have the following compactness result used for the convergence of the displacement component of the microscopic models in the following.

Proposition 5.5. *Let $(v_\varepsilon)_{\varepsilon>0}$ be a bounded sequence in $\mathrm{W}^{1,p}(\Omega)$. Then there exists a subsequence $(v_{\varepsilon'})_{\varepsilon'>0}$ of $(v_\varepsilon)_{\varepsilon>0}$ and functions $v_0 \in \mathrm{W}^{1,p}(\Omega)$ and $V_1 \in \mathrm{L}^p(\Omega; \mathrm{W}^{1,p}_{\mathrm{av}}(\mathcal{Y}))$ so that:*

$$\begin{aligned} v_{\varepsilon'} &\rightharpoonup v_0 && \text{in } \mathrm{W}^{1,p}(\Omega),\\ v_{\varepsilon'} &\overset{s}{\to} Ev_0 && \text{in } \mathrm{L}^p(\Omega{\times}\mathcal{Y}),\\ \nabla v_{\varepsilon'} &\overset{w}{\rightharpoonup} \nabla_x Ev_0 + \nabla_y V_1 && \text{in } \mathrm{L}^p(\Omega{\times}\mathcal{Y})^d, \end{aligned}$$

where $E : \mathrm{L}^p(\Omega) \to \mathrm{L}^p(\Omega{\times}\mathcal{Y})$ is defined via $Ev(x,y) := v(x)$.

Proof. Theorem 3.1.4 in Ref. [21] yields the result. □

For the construction of the displacement component of the joint recovery sequence the following density result is important.

Proposition 5.6. *Let $(w_0, W_1) \in \mathrm{W}^{1,p}_0(\Omega){\times}\mathrm{L}^p(\Omega; \mathrm{W}^{1,p}_{\mathrm{av}}(\mathcal{Y}))$ be given. Moreover, for every $\varepsilon > 0$ let $w_\varepsilon \in \mathrm{W}^{1,p}_0(\Omega)$ be the solution of the following elliptic problem:*

$$\int_\Omega ((w_\varepsilon - \mathcal{F}_\varepsilon(Ew_0)^{\mathrm{ex}})w + \langle \nabla w_\varepsilon - \mathcal{F}_\varepsilon(\nabla_x Ew_0 + \nabla_y W_1)^{\mathrm{ex}}, \nabla v\rangle_d)\mathrm{d}x = 0 \quad \forall\, v \in \mathrm{W}^{1,p'}_0(\Omega).$$

Then

$$\begin{aligned} w_\varepsilon &\to w_0 && \text{in } \mathrm{W}^{1,p}_0(\Omega),\\ w_\varepsilon &\overset{s}{\to} Ew_0 && \text{in } \mathrm{L}^p(\Omega{\times}\mathcal{Y}),\\ \nabla w_\varepsilon &\overset{s}{\to} \nabla_x Ew_0 + \nabla_y W_1 && \text{in } \mathrm{L}^p(\Omega{\times}\mathcal{Y})^d, \end{aligned}$$

Proof. Proposition 2.11 in Ref. [12] yields the result. □

6 Two-scale limit identification

This section is about the identification of a two-scale limit function when considering a special kind of sequences of characteristic functions, namely $(\chi_\varepsilon)_{\varepsilon>0} \subset \mathbb{X}^D_{\varepsilon\Lambda}(\Omega)$. In this section $Y = [-\frac{1}{2}, \frac{1}{2})^d$ (instead of $Y = [0,1)^d$) is considered as the associated unit cell, which already exposed to be beneficial in Subsection 4.2.

Recalling the definition of the identification operator $Q_\varepsilon : \mathbb{X}^D_{\varepsilon\Lambda}(\Omega) \to \mathrm{K}_{\varepsilon\Lambda}(\Omega; [c_D, 1])$ (see Definition 4.8) the following theorem claims a special structure of the two-scale limit of a sequence of admissible damage functions, when assuming a certain regularity of the associated sequence of piecewise constant functions.

Theorem 6.1 (Two-scale limit identification)**.** *For every sequence $(\chi_\varepsilon)_{\varepsilon>0}$ of functions satisfying $\chi_\varepsilon \in \mathbb{X}^D_{\varepsilon\Lambda}(\Omega)$ and $Q_\varepsilon\chi_\varepsilon \to z_0$ almost everywhere in Ω for some $z_0 \in \mathrm{L}^\infty(\Omega)$ we have*

$$\chi_\varepsilon \overset{*}{\rightharpoonup} z_0 \text{ in } \mathrm{L}^\infty(\Omega) \quad \text{and} \quad \mathcal{T}_\varepsilon\chi_\varepsilon(x,y) \to \mathbb{1}_{U(z_0^{\mathrm{ex}}(x))}(y) \quad \text{for almost every } (x,y) \in \mathbb{R}^d{\times}Y.$$

Proof. We start by proving $\chi_\varepsilon \overset{*}{\rightharpoonup} z_0$ in $\mathrm{L}^\infty(\Omega)$. Thereto, let $(\chi_\varepsilon)_{\varepsilon>0} \subset \mathbb{X}^D_{\varepsilon\Lambda}(\Omega)$ with $Q_\varepsilon\chi_\varepsilon \to z_0$ almost everywhere in Ω. Moreover, let $\varepsilon_0 > 0$ be fixed and $\varphi \in \mathrm{K}_{\varepsilon_0\Lambda}(\Omega^-_{\varepsilon_0})$. Looking at the product $\int_\Omega \varphi^{\mathrm{ex}}\chi_{\varepsilon_k}\mathrm{d}x$ due to $\mathrm{supp}(\varphi) \subset \overline{\Omega^-_{\varepsilon_0}}$ the function $\chi_\varepsilon \in \mathbb{X}^D_{\varepsilon\Lambda}(\Omega)$ can be replaced by any extension $\widehat{\chi}_\varepsilon \in \mathbb{X}^D_{\varepsilon\Lambda}(\Omega^+_\varepsilon)$ satisfying $\widehat{\chi}_\varepsilon|_{\Omega^-_{\varepsilon_0}} \equiv \chi_{\varepsilon_k}|_{\Omega^-_{\varepsilon_0}}$. Here, we choose $\widehat{\chi}_\varepsilon := M_\varepsilon\chi_\varepsilon$ which is defined in (4.9). Choosing now $\varepsilon_k := \frac{\varepsilon_0}{2^k}$ and exploiting the piecewise constancy of $\varphi \in \mathrm{K}_{\varepsilon_0\Lambda}(\Omega^-_{\varepsilon_0})$, namely, $\varphi(x) \equiv \varphi(\mathcal{N}_\varepsilon(x))$ for $x \in \Omega^-_{\varepsilon_0}$, we obtain

$$\begin{aligned}\int_\Omega \varphi^{\mathrm{ex}}(x)\chi_{\varepsilon_k}(x)\mathrm{d}x &= \sum_{\lambda\in\Lambda^-_{\varepsilon_k}\cap\Omega^-_{\varepsilon_0}} \int_{\varepsilon_k(\lambda+Y)} \varphi(\mathcal{N}_{\varepsilon_k}(x))M_{\varepsilon_k}\chi_{\varepsilon_k}(x)\mathrm{d}x \\ &= \sum_{\lambda\in\Lambda^-_{\varepsilon_k}\cap\Omega^-_{\varepsilon_0}} \varphi(\varepsilon_k\lambda)\varepsilon_k^d P_\varepsilon\big((M_{\varepsilon_k}\chi_{\varepsilon_k})^{\mathrm{ex}}\big)(\varepsilon_k\lambda). \qquad (6.1)\end{aligned}$$

Observing $Q_\varepsilon\chi_\varepsilon = \big(P_\varepsilon(M_\varepsilon\chi_\varepsilon)^{\mathrm{ex}}\big)|_\Omega$ by definition (see Definition 4.8 and (4.10)), equation (6.1) is equal to

$$\sum_{\lambda\in\Lambda^-_{\varepsilon_k}\cap\Omega^-_{\varepsilon_0}} \varepsilon_k^d\varphi(\varepsilon_k\lambda)Q_{\varepsilon_k}\chi_{\varepsilon_k}(\varepsilon_k\lambda) = \sum_{\lambda\in\Lambda^-_{\varepsilon_k}\cap\Omega^-_{\varepsilon_0}} \int_{\varepsilon_k(\lambda+Y)} \varphi(x)Q_{\varepsilon_k}\chi_{\varepsilon_k}(x)\mathrm{d}x, \qquad (6.2)$$

where on the right hand side the constancy of φ and $Q_{\varepsilon_k}\chi_{\varepsilon_k}$ on $\varepsilon_k(\lambda{+}Y)$ was exploited. Combining (6.1) and (6.2) we have

$$\int_\Omega \varphi^{\mathrm{ex}}(x)\chi_{\varepsilon_k}(x)\mathrm{d}x = \int_\Omega \varphi^{\mathrm{ex}}(x)Q_{\varepsilon_k}\chi_{\varepsilon_k}(x)\mathrm{d}x \overset{\varepsilon_k\to0}{\longrightarrow} \int_\Omega \varphi^{\mathrm{ex}}(x)z_0(x)\mathrm{d}x$$

for every $\varphi \in \mathrm{K}_{\varepsilon_0\Lambda}(\Omega^-_{\varepsilon_0})$ according to the assumed convergence $z_{\varepsilon_k} \to z_0$ almost everywhere in Ω. Since this kind of test-functions ($\mathrm{K}_{\varepsilon_0\Lambda}(\Omega^-_{\varepsilon_0})$) are dense in $\mathrm{L}^1(\Omega)$ and since the sequence $(Q_\varepsilon\chi_\varepsilon)_{\varepsilon>0}$ is bounded in $\mathrm{L}^\infty(\Omega)$ we conclude $\chi_\varepsilon \overset{*}{\rightharpoonup} z_0$ in $\mathrm{L}^\infty(\Omega)$.

Denoting by $\widehat{\mathcal{T}}_\varepsilon : \mathrm{L}^p(\Omega^+_\varepsilon) \to \mathrm{L}^p(\mathbb{R}^d{\times}\mathcal{Y})$ the periodic unfolding operator analogously defined to that one given in Definition 5.1 for

$$\widehat{\chi}_\varepsilon := M_\varepsilon\chi_\varepsilon \in \mathbb{X}^D_{\varepsilon\Lambda}(\Omega^+_\varepsilon) \qquad (6.3)$$

defined in (4.9) we find $\mathcal{T}_\varepsilon\chi_\varepsilon - \widehat{\mathcal{T}}_\varepsilon\widehat{\chi}_\varepsilon \to 0$ almost everywhere in $\mathbb{R}^d{\times}Y$. This is due to the fact that $\mathcal{T}_\varepsilon\chi_\varepsilon$ and $\widehat{\mathcal{T}}_\varepsilon\widehat{\chi}_\varepsilon$ coincide on $(\mathbb{R}^d{\times}Y)\backslash((\Omega^+_\varepsilon\backslash\Omega^-_\varepsilon){\times}Y)$. Hence, for every

$(x,y) \notin \partial\Omega\times Y$ there exists $\varepsilon_0 > 0$ such that $\mathcal{T}_\varepsilon\chi_\varepsilon(x,y) = \widehat{\mathcal{T}}_\varepsilon\widehat{\chi}_\varepsilon(x,y)$ for all $\varepsilon \in (0,\varepsilon_0)$ which proves the convergence almost everywhere according to assumption (2.3). Instead of proving $\mathcal{T}_\varepsilon\chi_\varepsilon(x,y) \to \mathbb{1}_{U(z_0^{\mathrm{ex}}(x))}(y)$ for almost every $(x,y) \in \mathbb{R}^d\times Y$ we will now prove that $\widehat{\mathcal{T}}_\varepsilon\widehat{\chi}_\varepsilon(x,y) \to \mathbb{1}_{U(z_0^{\mathrm{ex}}(x))}(y)$ for almost every $(x,y) \in \mathbb{R}^d\times Y$, which avoids any problems with cells $\varepsilon(\lambda+Y)$ intersecting the boundary $\partial\Omega$.

Thereto, we start by rearranging the two-scale function $\widehat{\mathcal{T}}_\varepsilon\widehat{\chi}_\varepsilon$ to find a simpler description to work with. Let

$$\widehat{z}_\varepsilon := \widehat{Q}_\varepsilon(\widehat{\chi}_\varepsilon) \in \mathrm{K}_{\varepsilon\Lambda}(\Omega_\varepsilon^+). \tag{6.4}$$

Then in (6.5) we just apply the definitions of the operators. In the second line (6.6) we already exploited for every $x \in \mathbb{R}^d$ firstly the special structure of the functions $\mathcal{N}_\varepsilon$ and $\widehat{z}_\varepsilon^{\mathrm{ex}}$, namely, that $\mathcal{N}_\varepsilon(\mathcal{N}_\varepsilon(x)+\varepsilon y) = \mathcal{N}_\varepsilon(x)$ and $\widehat{z}_\varepsilon^{\mathrm{ex}}(\mathcal{N}_\varepsilon(x)+\varepsilon y) = \widehat{z}_\varepsilon^{\mathrm{ex}}(x)$ holds and secondly $(\widehat{Q}_\varepsilon^{-1}\widehat{z})^{\mathrm{ex}}(x) = \mathbb{1}_{\mathcal{N}_\varepsilon(x)+U(\widehat{z}^{\mathrm{ex}}(x))}(x)$ (see (4.11)). In the last line (6.7) we first translate the function by $-\mathcal{N}_\varepsilon(x)$ and scale it by $\frac{1}{\varepsilon}$ afterwards.

$$\widehat{\mathcal{T}}_\varepsilon\widehat{\chi}_\varepsilon(x,y) = \widehat{\mathcal{T}}_\varepsilon(\widehat{Q}_\varepsilon^{-1}\widehat{z}_\varepsilon)(x,y) = (\widehat{Q}_\varepsilon^{-1}\widehat{z}_\varepsilon)^{\mathrm{ex}}(\mathcal{N}_\varepsilon(x)+\varepsilon y) \tag{6.5}$$
$$= \mathbb{1}_{\mathcal{N}_\varepsilon(x)+\varepsilon U(\widehat{z}_\varepsilon^{\mathrm{ex}}(x))}(\mathcal{N}_\varepsilon(x)+\varepsilon y) \tag{6.6}$$
$$= \mathbb{1}_{\varepsilon U(\widehat{z}_\varepsilon^{\mathrm{ex}}(x))}(\varepsilon y) = \mathbb{1}_{U(\widehat{z}_\varepsilon^{\mathrm{ex}}(x))}(y). \tag{6.7}$$

1. First of all for almost every $x \in \mathbb{R}^d$ we want to show $\mathbb{1}_{U(\widehat{z}_\varepsilon^{\mathrm{ex}}(x))} \to \mathbb{1}_{U(z_0^{\mathrm{ex}}(x))}$ almost everywhere in Y.

The case $x \in \mathbb{R}^d\backslash\overline{\Omega}$:

Since $\mathrm{vol}(\partial\Omega) = 0$ for fixed $x \in \mathbb{R}^d\backslash\overline{\Omega}$ there is $\varepsilon_0 > 0$ such that $\widehat{z}_\varepsilon^{\mathrm{ex}}(x) \equiv 0$ for all $\varepsilon \in (0,\varepsilon_0)$. This immediately results in $\mathbb{1}_{U(\widehat{z}_\varepsilon^{\mathrm{ex}}(x))} \to \mathbb{1}_{U(0)} = \mathbb{1}_{U(z_0^{\mathrm{ex}}(x))}$ almost everywhere in Y.

The case $x \in \Omega$:

By assumption we have $\widehat{z}_\varepsilon|_\Omega = \widehat{Q}_\varepsilon\widehat{\chi}_\varepsilon|_\Omega = Q_\varepsilon\chi_\varepsilon \to z_0$ almost everywhere in Ω (see Definition 4.8 and (6.3), (6.4)). Hence, there is a set $N \subset \Omega$ with $\mathrm{vol}(N) = 0$ such that

$$\forall\,\delta > 0 \quad \forall\,x \in \Omega\backslash N \quad \exists\,\varepsilon^* > 0 \quad \forall\,\varepsilon \in (0,\varepsilon^*): \quad |z_0(x) - \widehat{z}_\varepsilon(x)| \leq \delta. \tag{6.8}$$

Let now $x \in \Omega\backslash N$ be fixed. Hence, (6.8) for all $\varepsilon \in (0,\varepsilon^*)$ results in

$$\kappa(\min\{z_0(x)+\delta, 1\})D \subset \kappa(\widehat{z}_\varepsilon(x))D \subset \kappa(\max\{z_0(x)-\delta, c_D\})D, \tag{6.9}$$

since $\kappa : [c_D, 1] \to [0,1]$ is strictly monotonously decreasing and $\beta D \subset \beta' D$ for $0 \leq \beta < \beta'$ (the damage set D is star-shaped with respect to $0 \in \mathbb{R}^d$).

We consider the following two cases:

First let $y \in \kappa(z_0(x))D$, which in fact implies $z_0(x) < 1$ since $\kappa(1) = 0$. By assumption, D is an open and star-shaped set (with respect to the center of the unit cell) such that there exists a positive constant $\gamma > 0$ so that $(1+\gamma)y \in \kappa(z_0(x))D$ $(\Leftrightarrow y \in \frac{1}{1+\gamma}\kappa(z_0(x))D)$. Since $\kappa : [c_D, 1] \to [0,1]$ is continuous and strictly monotonously decreasing, there exists $\delta > 0$ such that $\frac{1}{1+\gamma}\kappa(z_0(x))D = \kappa(z_0(x)+\delta)D$ which finally gives us $y \in \kappa(\widehat{z}_\varepsilon(x))D$ for all $\varepsilon \in (0,\varepsilon^*)$ by exploiting inclusion (6.9). Note, that the equality $\emptyset \neq \frac{1}{1+\gamma}\kappa(z_0(x))D = \kappa(z_0(x)+\delta)D$ implies $z_0(x)+\delta < 1$.

Secondly let $y \notin \kappa(z_0(x))\overline{D}$. Analogously to the first case we find that there exists $\varepsilon^* > 0$ such that $y \notin \kappa(\widehat{z}_\varepsilon(x))\overline{D}$ for all $\varepsilon \in (0, \varepsilon^*)$.

By using condition (6.8) these two cases show $\mathbb{1}_{U(\widehat{z}_\varepsilon(x))}(y) \to \mathbb{1}_{U(z_0(x))}(y)$ for fixed $x \in \Omega\backslash N$ and for all $y \in Y\backslash\kappa(z_0(x))\partial D$.

2. Up to now we showed $\widehat{\mathcal{T}}_\varepsilon\widehat{\chi}_\varepsilon(x,y) \to \mathbb{1}_{U(z_0^{\mathrm{ex}}(x))}(y)$ for all points (x,y) belonging to $(\mathbb{R}^d \times Y)\backslash(\partial\Omega\times Y \cup N\times Y \cup M)$, where $M := \bigcup_{x\in\Omega\backslash N}\{x\}\times\kappa(z_0(x))\partial D$. That means

$$(\mu_d \otimes \mu_d)(M) = 0$$

would finish the proof. But this is an easy consequence of

$$(\mu_d \otimes \mu_d)(M) = \int_\Omega \mu_d(M_x)\mathrm{d}x = \int_\Omega \mu_d(\kappa(z_0(x))\partial D)\mathrm{d}x = 0, \tag{6.10}$$

where $M_x := \{y \in Y : (x,y) \in M\}$ (Satz 1.5 in [7]). The only thing that needs to be checked to apply (6.10) is the Lebesgue measurability of M.

3. In the following the measurability of M is shown by considering measurable parameter dependent sets containing M. Considering the intersection of all these sets by letting the parameter going to zero the measurability of M follows.

Let $M_\gamma^\pm := \bigcup_{x\in\Omega\backslash N}\{x\}\times\kappa(z_0(x)\mp\gamma)D$. Then $M \subset \bigcap_{\gamma>0} M_\gamma^+\backslash M_\gamma^-$ by definition. The opposite inclusion is shown by the following contradiction argument.

Assume that there exists $(x,y) \in \bigcap_{\gamma>0} M_\gamma^+\backslash M_\gamma^-$ so that $(x,y) \notin M$. That means $(x,y) \in M_\gamma^+\backslash M_\gamma^-$ for all $\gamma > 0$. Assume $y = (y_1, 0, \ldots, 0)^T \in \mathbb{R}^d$, $y_1 > 0$, which is always possible by rotating the system. Then $(x,y) \in M_\gamma^+\backslash M_\gamma^-$ is equivalent to

$$y \in \kappa(z_0(x)-\gamma)D\backslash\kappa(z_0(x)+\gamma)D \quad \Leftrightarrow \quad y_1 \in [\kappa(z_0(x)+\gamma)s, \kappa(z_0(x)-\gamma)s), \tag{6.11}$$

where $s = \sup\{\alpha \in \mathbb{R} : (\alpha, 0 \ldots, 0)^T \in D\}$.

Due to the continuity of $\kappa : [c_D, 1] \to [0,1]$ we now find $y_1 = \kappa(z_0(x))s$ which is equivalent to $y \in \partial D$. But this contradicts our assumption so that $M = \bigcap_{\gamma>0} M_\gamma^+\backslash M_\gamma^-$ results. Now, the measurability of M follows by showing the measurability of the sets $M_\gamma^\pm$, since a countable intersection of measurable sets is measurable.

To show the measurability of $M_\gamma^\pm$ we start by choosing a sequence $(z_\delta)_{(\delta>0)}$ of simple functions ($z_\delta(x) = \sum_{k=1}^{N_\delta} \mathbb{1}_{A_k^\delta}(x) z_k^\delta$ with $z_k^\delta = \mathit{const}$, A_k^δ measurable and $\dot{\bigcup}_{k=1}^{N_\delta} A_k = \Omega$) with $z_\delta(x) \nearrow z_0(x)$ for all $x \in \Omega\backslash N$. Let $M_\gamma^\pm(\delta) := \bigcup_{x\in\Omega\backslash N}\{x\}\times\kappa(z_\delta(x)\mp\gamma)D$. Then $M_\gamma^\pm(\delta) \subset M_\gamma^\pm$ by definition and $M_\gamma^\pm \subset \bigcup_{\delta>0} M_\gamma^\pm(\delta)$ can be shown by a contradiction argument in a similar way to that from above, hence $M_\gamma^\pm = \bigcup_{\delta>0} M_\gamma^\pm(\delta)$.

Since

$$M_\gamma^\pm(\delta) = \dot{\bigcup}_{k=1}^{N_\delta}\Big(\bigcup_{x\in A_k^\delta}\{x\}\times\kappa(z_k^\delta \mp \gamma)D\Big) = \dot{\bigcup}_{k=1}^{N_\delta}\Big(A_k^\delta\times\kappa(z_k^\delta \mp \gamma)D\Big),$$

$M_\gamma^\pm(\delta)$ is the disjoint union of countable many measurable sets and ergo measurable. This implies the measurability of $M_\gamma^\pm$ and the proof is finished. □

7 Convergence results

Recalling

$$\mathcal{Q}_\varepsilon(\Omega) := \mathrm{H}^1_{\Gamma_{\mathrm{Dir}}}(\Omega)^d \times \mathbb{X}^D_{\varepsilon\Lambda}(\Omega)$$

and

$$\mathbf{Q} := \mathrm{H}^1_{\Gamma_{\mathrm{Dir}}}(\Omega)^d \times \mathrm{L}^2(\Omega; \mathrm{H}^1_{\mathrm{av}}(\mathcal{Y}))^d \times \mathrm{W}^{1,p}(\Omega; [c_D, 1])$$

this section contains the proof that subsequences of solutions $(u_\varepsilon, \chi_\varepsilon) : [0,T] \to \mathcal{Q}_\varepsilon(\Omega)$ of the microscopic damage models (S^ε) and (E^ε) converge to a solution $(u_0, U_1, z_0) : [0,T] \to \mathbf{Q}$ of the effective two-scale damage model ($\mathbf{S}^0$) and ($\mathbf{E}^0$). As already mentioned in [15] proving stability of the limit function $(u_0, U_1, z_0) : [0,T] \to \mathbf{Q}$ is the crucial point. That is why we start by recalling the definition of a stable sequence, but now in the context of the model introduced in Subsection 4.2. Afterwards, we introduce the mutual recovery sequence, which will turn out to solve our problem of proving stability of the limit.

Definition 7.1 (Stable sequence in the context of the model of Subsection 4.2)**.** A sequence $(u_\varepsilon, \chi_\varepsilon)_{\varepsilon>0} \subset \mathcal{Q}_\varepsilon(\Omega)$ is called stable sequence with respect to the fixed time $t \in [0,T]$ if (i) and (ii) hold:

(i) There exists a function $(u_0, U_1, z_0) \in \mathbf{Q}$ such that:

$$\begin{aligned} u_\varepsilon &\rightharpoonup u_0 && \text{in } \mathrm{H}^1_{\Gamma_{\mathrm{Dir}}}(\Omega)^d, & \chi_\varepsilon &\overset{*}{\rightharpoonup} z_0 && \text{in } \mathrm{L}^\infty(\Omega),\\ u_\varepsilon &\overset{s}{\to} Eu_0 && \text{in } \mathrm{L}^2(\Omega\times\mathcal{Y})^d, & Q_\varepsilon(\chi_\varepsilon) &\to z_0 && \text{in } \mathrm{L}^p(\Omega),\\ \nabla u_\varepsilon &\overset{w}{\rightharpoonup} \nabla_x E u_0 + \nabla_y U_1 && \text{in } \mathrm{L}^2(\Omega\times\mathcal{Y})^{d\times d}. & R_{\frac{\varepsilon}{2}}(Q_\varepsilon(\chi_\varepsilon)) &\rightharpoonup \nabla z_0 && \text{in } \mathrm{L}^p(\Omega)^d, \end{aligned}$$

(ii) $(u_\varepsilon, \chi_\varepsilon) \in \mathcal{S}_\varepsilon(t)$ for every $\varepsilon > 0$.

Remark 7.2. Note, that here the uniform boundedness of the energy functional $\mathcal{E}_\varepsilon(t,\cdot,\cdot) : \mathcal{Q}_\varepsilon(\Omega) \to \mathbb{R}$ with respect to the sequence $(u_\varepsilon, \chi_\varepsilon)_{\varepsilon>0}$ (see Definition 4.2 (i)) is replaced by a convergence statement. But as we already saw in the proof of Proposition 4.10, the energy sublevels are weakly sequentially compact, which enables us to apply Proposition 5.5 for the displacements $(u_\varepsilon)_{\varepsilon>0} \subset \mathrm{H}^1_{\Gamma_{\mathrm{Dir}}}(\Omega)^d$ and Theorem 2.1 for the damage functions $(Q_\varepsilon \chi_\varepsilon)_{\varepsilon>0} \subset \mathrm{K}_{\varepsilon\Lambda}(\Omega)$. In this sense, condition (i) of Definition 4.2 and Definition 7.1 are equivalent by choosing a subsequence.

Definition 7.3 (Mutual recovery condition and mutual recovery sequence)**.** The functionals $\mathcal{E}_\varepsilon$ and $\mathcal{D}_\varepsilon$ fulfill the mutual recovery condition, if for every function $(\widetilde{u}_0, \widetilde{U}_1, \widetilde{z}_0) \in \mathbf{Q}$ and for every stable sequence $(u_\varepsilon, \chi_\varepsilon)_{\varepsilon>0} \subset \mathcal{Q}_\varepsilon(\Omega)$ the following holds:

There exists a sequence $(\widetilde{u}_{\varepsilon'}, \widetilde{\chi}_{\varepsilon'})_{\varepsilon'>0} \subset \mathcal{Q}_{\varepsilon'}(\Omega)$ satisfying

$$\begin{aligned} \widetilde{u}_{\varepsilon'} &\rightharpoonup \widetilde{u}_0 && \text{in } \mathrm{H}^1_{\Gamma_{\mathrm{Dir}}}(\Omega)^d, & \widetilde{\chi}_{\varepsilon'} &\overset{*}{\rightharpoonup} \widetilde{z}_0 && \text{in } \mathrm{L}^\infty(\Omega),\\ \widetilde{u}_{\varepsilon'} &\overset{s}{\to} E\widetilde{u}_0 && \text{in } \mathrm{L}^2(\Omega\times\mathcal{Y})^d, & Q_{\varepsilon'}(\widetilde{\chi}_{\varepsilon'}) &\to \widetilde{z}_0 && \text{in } \mathrm{L}^p(\Omega),\\ \nabla \widetilde{u}_{\varepsilon'} &\overset{w}{\rightharpoonup} \nabla_x E \widetilde{u}_0 + \nabla_y \widetilde{U}_1 && \text{in } \mathrm{L}^2(\Omega\times\mathcal{Y})^{d\times d}, & R_{\frac{\varepsilon}{2}}(Q_{\varepsilon'}(\widetilde{\chi}_{\varepsilon'})) &\rightharpoonup \nabla \widetilde{z}_0 && \text{in } \mathrm{L}^p(\Omega)^d, \end{aligned}$$

such that

$$\limsup_{\varepsilon'\to 0} \left(\mathcal{E}_\varepsilon(t, \widetilde{u}_{\varepsilon'}, \widetilde{\chi}_{\varepsilon'}) - \mathcal{E}_\varepsilon(t, u_{\varepsilon'}, \chi_{\varepsilon'})\right) \le \mathbf{E}(t, \widetilde{u}_0, \widetilde{U}_1, \widetilde{z}_0) - \mathbf{E}(t, u_0, U_1, z_0) \tag{7.1}$$

and

$$\lim_{\varepsilon'\to 0} \mathcal{D}_\varepsilon(\chi_{\varepsilon'}, \widetilde{\chi}_{\varepsilon'}) = \mathbf{D}(z_0, \widetilde{z}_0) \tag{7.2}$$

where $(u_{\varepsilon'}, \chi_{\varepsilon'})_{\varepsilon'>0}$ is a subsequence of $(u_\varepsilon, \chi_\varepsilon)_{\varepsilon>0}$. Such a sequence $(\widetilde{u}_{\varepsilon'}, \widetilde{\chi}_{\varepsilon'})_{\varepsilon'>0} \subset \mathcal{Q}_{\varepsilon'}(\Omega)$ is called mutual recovery sequence.

Theorem 7.4 (Mutual recovery sequence)**.** *Let $(u_\varepsilon, \chi_\varepsilon)_{\varepsilon>0} \subset \mathcal{Q}_\varepsilon(\Omega)$ be a stable sequence in sense of Definition 7.1 with limit $(u_0, U_1, z_0) \in \mathbf{Q}$. Then:*

(a) For every $(\widetilde{u}_0, \widetilde{U}_1, \widetilde{z}_0) \in \mathbf{Q}$ there exists a mutual recovery sequence.

(b) $(u_0, U_1, z_0) \in \mathbf{S}_0(t)$.

Proof. 1. Part (a): For a given function $(\widetilde{u}_0, \widetilde{U}_1, \widetilde{z}_0) \in \mathbf{Q}$ we start by constructing the mutual recovery sequence $(\widetilde{u}_\varepsilon, \widetilde{\chi}_\varepsilon)_{\varepsilon>0} \subset \mathcal{Q}_\varepsilon(\Omega)$.

First, the displacement-component $\widetilde{u}_\varepsilon \in \mathrm{H}^1_{\Gamma_{\mathrm{Dir}}}(\Omega)^d$ is constructed. Adopting the notation of Proposition 5.6 let $w_\varepsilon \in \mathrm{H}^1_0(\Omega)^d$ be the solution of the elliptic problem stated there with $w_0 = 0 \in \mathrm{H}^1_0(\Omega)^d$ and $W_1 = \widetilde{U}_1 \in \mathrm{L}^2(\Omega; \mathrm{H}^1_{\mathrm{av}}(\mathcal{Y}))^d$. Then according to Proposition 5.6 we have $w_\varepsilon \to 0$ in $\mathrm{H}^1_0(\Omega)^d$, $w_\varepsilon \overset{s}{\to} 0$ in $\mathrm{L}^2(\Omega\times\mathcal{Y})^d$ and $\nabla w_\varepsilon \overset{s}{\to} \nabla_y \widetilde{U}_1$ in $\mathrm{L}^2(\Omega\times\mathcal{Y})^{d\times d}$. Now, the displacement-component of the mutual recovery sequence is defined via

$$\widetilde{u}_\varepsilon := \widetilde{u}_0 + w_\varepsilon.$$

Using property (b) of Proposition 5.4 and the convergence results for $(w_\varepsilon)_{\varepsilon>0}$ we find

$$\begin{aligned} \widetilde{u}_\varepsilon &\rightharpoonup \widetilde{u}_0 && \text{in } \mathrm{H}^1_{\Gamma_{\mathrm{Dir}}}(\Omega)^d, \\ \widetilde{u}_\varepsilon &\overset{s}{\to} E\widetilde{u}_0 && \text{in } \mathrm{L}^2(\Omega\times\mathcal{Y})^d, \\ \nabla\widetilde{u}_\varepsilon &\overset{s}{\to} \nabla_x E\widetilde{u}_0 + \nabla_y \widetilde{U}_1 && \text{in } \mathrm{L}^2(\Omega\times\mathcal{Y})^{d\times d}. \end{aligned}$$

2. For the construction of the damage-component $\widetilde{\chi}_\varepsilon \in \mathbb{X}^D_{\varepsilon\Lambda}(\Omega)$ of the mutual recovery sequence assume $\widetilde{z}_0 \le z_0$, since otherwise $\mathbf{D}(z_0, \widetilde{z}_0) = \infty$. The construction of $\widetilde{\chi}_\varepsilon$ is based on the construction of a sequence of piecewise constant functions $(\widetilde{v}_\varepsilon)_{\varepsilon>0} \subset \mathrm{K}_{\varepsilon\Lambda}(\Omega; [0,1])$ in sense of Theorem 3.1. Adopting the notation of Theorem 3.1 we introduce the following functions: $v_0 := z_0 - c_D \in \mathrm{W}^{1,p}(\Omega; [0,1])$, $v_\varepsilon := Q_\varepsilon(\chi_\varepsilon) - c_D \in \mathrm{K}_{\varepsilon\Lambda}(\Omega; [0,1])$ and $\widetilde{v}_0 := \widetilde{z}_0 - c_D \in \mathrm{W}^{1,p}(\Omega; [0,1])$. By first subtracting c_D and then adding c_D after applying Theorem 3.1 guarantees that we are able to construct $\widetilde{\chi}_\varepsilon$ with the help of the Operator $N_\varepsilon : \mathrm{K}_{\varepsilon\Lambda}(\Omega; [c_D, 1]) \to \mathbb{X}^D_{\varepsilon\Lambda}(\Omega)$ as we will see below.

Letting $\widetilde{z}_\varepsilon := \widetilde{v}_\varepsilon + c_D$ first of all ensures $\widetilde{z}_\varepsilon(x) \in [c_D, 1]$ for all $x \in \Omega$ such that we are able to apply the operator $N_\varepsilon : \mathrm{K}_{\varepsilon\Lambda}(\Omega; [c_D, 1]) \to \mathbb{X}^D_{\varepsilon\Lambda}(\Omega)$ to it and define

$$\widetilde{\chi}_\varepsilon := \begin{cases} N_\varepsilon(\widetilde{z}_\varepsilon) & \text{on } \Omega^-_\varepsilon \\ \chi_\varepsilon & \text{on } \Omega\backslash\Omega^-_\varepsilon \end{cases}$$

Then exploiting Proposition 4.9 we find

$$Q_\varepsilon\widetilde{\chi}_\varepsilon = \left\{ \begin{array}{ll} Q_\varepsilon(N_\varepsilon(\widetilde{z}_\varepsilon)) = \widetilde{z}_\varepsilon & \text{on } \Omega^-_\varepsilon \\ Q_\varepsilon\chi_\varepsilon = v_\varepsilon + c_D & \text{on } \Omega\backslash\Omega^-_\varepsilon \end{array} \right\} = \widetilde{v}_\varepsilon + c_D,$$

where we used Remark 3.2 to justify the last equality ($\widetilde{v}_\varepsilon = v_\varepsilon$ on $\Omega\backslash\Omega_\varepsilon^-$). With this equality and Theorem 3.1 we now have:

$$\begin{aligned} Q_\varepsilon\widetilde{\chi}_\varepsilon = &\ \widetilde{v}_\varepsilon + c_D \le v_\varepsilon + c_D &&= Q_\varepsilon\chi_\varepsilon, \\ Q_\varepsilon\widetilde{\chi}_\varepsilon = &\ \widetilde{v}_\varepsilon + c_D \to \widetilde{v}_0 + c_D &&= \widetilde{z}_0 \qquad \text{in } \mathrm{L}^p(\Omega), \\ R_{\frac{\varepsilon}{2}}(Q_\varepsilon(\widetilde{\chi}_\varepsilon)) = R_{\frac{\varepsilon}{2}}(\widetilde{v}_\varepsilon + c_D) = &\ R_{\frac{\varepsilon}{2}}\widetilde{v}_\varepsilon \rightharpoonup \nabla\widetilde{v}_0 &&= \nabla\widetilde{z}_0 \qquad \text{in } \mathrm{L}^p(\Omega)^d, \end{aligned}$$

and

$$\limsup_{\varepsilon\to 0}\left(\|R_{\frac{\varepsilon}{2}}(Q_\varepsilon(\widetilde{\chi}_\varepsilon))\|^p_{\mathrm{L}^p(\Omega)^d} - \|R_{\frac{\varepsilon}{2}}(Q_\varepsilon(\chi_\varepsilon))\|^p_{\mathrm{L}^p(\Omega)^d}\right) \le \|\nabla\widetilde{z}_0\|^p_{\mathrm{L}^p(\Omega)^d} - \|\nabla z_0\|^p_{\mathrm{L}^p(\Omega)^d}. \quad (7.3)$$

3. Proving (7.2) is an easy consequence of Theorem 6.1. Due to Definition 7.1 and step 2 we have $Q_\varepsilon(\chi_\varepsilon) \to z_0$ in $\mathrm{L}^p(\Omega)$ and $Q_\varepsilon(\widetilde{\chi}_\varepsilon) \to \widetilde{z}_0$ in $\mathrm{L}^p(\Omega)$. This implies convergence almost everywhere of a subsequence in both cases ($(\chi_\varepsilon)_{\varepsilon>0}$ and $(\widetilde{\chi}_\varepsilon)_{\varepsilon>0}$) such that we are able to apply Theorem 6.1 saying

$$\begin{aligned} &\chi_{\varepsilon'} \overset{*}{\rightharpoonup} z_0 \quad \text{in } \mathrm{L}^\infty(\Omega) \quad \text{and} \quad \mathcal{T}_\varepsilon\chi_{\varepsilon'}(x,y) \to \mathbb{1}_{U(z_0(x))}(y) \quad \text{for almost every } (x,y)\in\mathbb{R}^d\times Y, \\ &\widetilde{\chi}_{\varepsilon'} \overset{*}{\rightharpoonup} \widetilde{z}_0 \quad \text{in } \mathrm{L}^\infty(\Omega) \quad \text{and} \quad \mathcal{T}_\varepsilon\widetilde{\chi}_{\varepsilon'}(x,y) \to \mathbb{1}_{U(\widetilde{z}_0(x))}(y) \quad \text{for almost every } (x,y)\in\mathbb{R}^d\times Y. \end{aligned}$$

But this weak*-convergences immediately give us $\lim_{\varepsilon'\to 0}\mathcal{D}_\varepsilon(\chi_{\varepsilon'},\widetilde{\chi}_{\varepsilon'}) = \mathbf{D}(z_0,\widetilde{z}_0)$.

4. In the end we have to prove the lim sup-inequality stated in (7.1). According to the assumption and step 1 we have $u_\varepsilon \rightharpoonup u_0$ in $\mathrm{H}^1_{\Gamma_{\mathrm{Dir}}}(\Omega)^d$ and $\widetilde{u}_\varepsilon \rightharpoonup \widetilde{u}_0$ in $\mathrm{H}^1_{\Gamma_{\mathrm{Dir}}}(\Omega)^d$ which giving us

$$\lim_{\varepsilon\to 0}\left(\langle\ell(t),u_\varepsilon\rangle - \langle\ell(t),\widetilde{u}_\varepsilon\rangle\right) = \langle\ell(t),u_0\rangle - \langle\ell(t),\widetilde{u}_0\rangle. \quad (7.4)$$

Next we prove that

$$\begin{aligned} &\limsup_{\varepsilon'\to 0}\left(\langle\mathbb{C}(\widetilde{\chi}_{\varepsilon'})\mathbf{e}(\widetilde{u}_{\varepsilon'}),\mathbf{e}(\widetilde{u}_{\varepsilon'})\rangle_{\mathrm{L}^2(\Omega)^{d\times d}} - \langle\mathbb{C}(\chi_{\varepsilon'})\mathbf{e}(u_{\varepsilon'}),\mathbf{e}(u_{\varepsilon'})\rangle_{\mathrm{L}^2(\Omega)^{d\times d}}\right) \quad (7.5) \\ &\le \langle\mathbb{C}_0(\widetilde{z}_0)\widetilde{\mathbf{e}}(\widetilde{u}_0,\widetilde{U}_1),\widetilde{\mathbf{e}}(\widetilde{u}_0,\widetilde{U}_1)\rangle_{\mathrm{L}^2(\Omega\times\mathcal{Y})^{d\times d}} - \langle\mathbb{C}_0(z_0)\widetilde{\mathbf{e}}(u_0,U_1),\widetilde{\mathbf{e}}(u_0,U_1)\rangle_{\mathrm{L}^2(\Omega\times\mathcal{Y})^{d\times d}}. \end{aligned}$$

Combining this with the convergence results (7.3) and (7.4) implies the lim sup-inequality (7.1). Adopting the notation of Proposition 5.4 let $m_\varepsilon := \mathbb{C}(\widetilde{\chi}_\varepsilon)$, $M_0 := \mathbb{C}_0(\widetilde{z}_0)$ and $v_\varepsilon := \mathbf{e}(\widetilde{u}_\varepsilon)$, $V_0 := \widetilde{\mathbf{e}}(\widetilde{u}_0,\widetilde{U}_1)$. Then Proposition 5.4(c) together with the convergence results for $(\widetilde{\chi}_\varepsilon)_{\varepsilon>0}$ and $(\widetilde{u}_\varepsilon)_{\varepsilon>0}$ give $w_{\varepsilon'} := \mathbb{C}(\widetilde{\chi}_{\varepsilon'})\mathbf{e}(\widetilde{u}_{\varepsilon'}) \overset{s}{\to} \mathbb{C}_0(\widetilde{z}_0)\widetilde{\mathbf{e}}(\widetilde{u}_0,\widetilde{U}_1) =: W_0$ in $\mathrm{L}^2(\Omega\times\mathcal{Y})$. With this, Proposition 5.4(a) gives

$$\lim_{\varepsilon'\to 0}\langle\mathbb{C}(\widetilde{\chi}_{\varepsilon'})\mathbf{e}(\widetilde{u}_{\varepsilon'}),\mathbf{e}(\widetilde{u}_{\varepsilon'})\rangle_{\mathrm{L}^2(\Omega)^{d\times d}} = \langle\mathbb{C}_0(\widetilde{z}_0)\widetilde{\mathbf{e}}(\widetilde{u}_0,\widetilde{U}_1),\widetilde{\mathbf{e}}(\widetilde{u}_0,\widetilde{U}_1)\rangle_{\mathrm{L}^2(\Omega\times\mathcal{Y})^{d\times d}}.$$

To prove (7.5), it is sufficient to show:

$$\liminf_{\varepsilon'\to 0}\langle\mathbb{C}(\chi_{\varepsilon'})\mathbf{e}(u_{\varepsilon'}),\mathbf{e}(u_{\varepsilon'})\rangle_{\mathrm{L}^2(\Omega)^{d\times d}} \ge \langle\mathbb{C}_0(z_0)\widetilde{\mathbf{e}}(u_0,U_1),\widetilde{\mathbf{e}}(u_0,U_1)\rangle_{\mathrm{L}^2(\Omega\times\mathcal{Y})^{d\times d}} \quad (7.6)$$

Thereto, we start with the following integral identity valid according to identity (5.2) and the product rule for the unfolding operator $\mathcal{T}_\varepsilon$:

$$\langle\mathbb{C}(\chi_\varepsilon)\mathbf{e}(u_\varepsilon),\mathbf{e}(u_\varepsilon)\rangle_{\mathrm{L}^2(\Omega)^{d\times d}} = \langle\mathbb{C}(\mathcal{T}_\varepsilon\chi_\varepsilon)\mathcal{T}_\varepsilon\mathbf{e}(u_\varepsilon),\mathcal{T}_\varepsilon\mathbf{e}(u_\varepsilon)\rangle_{\mathrm{L}^2([\Omega\times\mathcal{Y}]_\varepsilon)^{d\times d}} \quad (7.7)$$

Due to $\mathcal{T}_\varepsilon \chi_{\varepsilon'} \to \mathbb{1}_{U(z_0)}$ almost everywhere in $\mathbb{R}^d \times Y$, $\operatorname{supp}(\mathcal{T}_\varepsilon \chi_\varepsilon) \subset \overline{[\Omega \times Y]}_\varepsilon$ and $0 \le \mathcal{T}_\varepsilon \chi_\varepsilon \le 1$ we have $\mathcal{T}_\varepsilon \chi_{\varepsilon'} \to \mathbb{1}_{U(z_0)}$ in $\mathrm{L}^p(\mathbb{R}^d \times \mathcal{Y})$ for all $p \in [1, \infty)$. Moreover, according to the definition of two-scale convergence it holds $\mathcal{T}_\varepsilon \mathbf{e}(u_\varepsilon) \rightharpoonup \widetilde{\mathbf{e}}(u_0, U_1)$ in $\mathrm{L}^2(\mathbb{R}^d \times \mathcal{Y})^{d \times d}$, which enables us to apply Theorem 3.23 of [6] for $f(x, \mathcal{X}, e) := \langle \mathbb{C}(\mathcal{X}) e, e \rangle_{d \times d}$ yielding ($\mathrm{L}^2 := \mathrm{L}^2(\mathbb{R}^d \times \mathcal{Y})^{d \times d}$)

$$\liminf_{\varepsilon' \to 0} \langle \mathbb{C}(\mathcal{T}_\varepsilon \chi_{\varepsilon'}) \mathcal{T}_\varepsilon \mathbf{e}(u_{\varepsilon'}), \mathcal{T}_\varepsilon \mathbf{e}(u_{\varepsilon'}) \rangle_{\mathrm{L}^2} \ge \langle \mathbb{C}_0(z_0^{\mathrm{ex}}) \widetilde{\mathbf{e}}(u_0^{\mathrm{ex}}, U_1^{\mathrm{ex}}), \widetilde{\mathbf{e}}(u_0^{\mathrm{ex}}, U_1^{\mathrm{ex}}) \rangle_{\mathrm{L}^2}.$$

Taking into account that $\operatorname{supp}(\mathbb{1}_{U(z_0^{\mathrm{ex}})}) \subset \Omega \times Y$ this inequality together with (7.7) gives (7.6) and the proof of point (a) is done.

5. Point (b) is a consequence of (a). Due to the stability of $(u_\varepsilon, \chi_\varepsilon)_{\varepsilon > 0} \subset \mathcal{Q}_\varepsilon(\Omega)$ the left hand side of the limsup-inequality is greater or equal to 0, such that this also holds for the right hand side. But the right hand side of the limsup-inequality is nothing else than the stability condition for $(u_0, U_1, z_0) \in \mathbf{Q}$. □

Remark 7.5. Note, that here only in the step 4 the stability of $(u_\varepsilon, \chi_\varepsilon)_{\varepsilon > 0} \subset \mathcal{Q}_\varepsilon(\Omega)$ is used, i.e. the steps 1, 2 and 3 are valid for all sequences $(u_\varepsilon, \chi_\varepsilon)_{\varepsilon > 0} \subset \mathcal{Q}_\varepsilon(\Omega)$ satisfying only point (i) of Definition 7.1.

The following theorem states that $\mathbf{E} : [0, T] \times \mathbf{Q} \to \mathbb{R}$ is the Γ-limit of $\mathcal{E}_\varepsilon : [0, T] \times \mathcal{Q}_\varepsilon(\Omega) \to \mathbb{R}$ with respect to our special topology.

Theorem 7.6 (Γ-convergence of $\mathcal{E}_\varepsilon$)**.** *Let $(u_\varepsilon, \chi_\varepsilon)_{\varepsilon > 0}$ be a sequence in $\mathcal{Q}_\varepsilon(\Omega)$ such that*

$$\begin{aligned} u_\varepsilon &\rightharpoonup u_0 && \text{in } \mathrm{H}^1_{\Gamma_{\mathrm{Dir}}}(\Omega)^d, & \chi_\varepsilon &\overset{*}{\rightharpoonup} z_0 && \text{in } \mathrm{L}^\infty(\Omega), \\ u_\varepsilon &\overset{s}{\to} E u_0 && \text{in } \mathrm{L}^2(\Omega \times \mathcal{Y})^d, & Q_\varepsilon(\chi_\varepsilon) &\to z_0 && \text{in } \mathrm{L}^p(\Omega), \\ \nabla u_\varepsilon &\overset{w}{\rightharpoonup} \nabla_x E u_0 + \nabla_y U_1 && \text{in } \mathrm{L}^2(\Omega \times \mathcal{Y})^{d \times d}, & R_{\frac{\varepsilon}{2}}(Q_\varepsilon(\chi_\varepsilon)) &\rightharpoonup \nabla z_0 && \text{in } \mathrm{L}^p(\Omega)^d. \end{aligned}$$

Then for every $t \in [0, T]$ it holds $\liminf_{\varepsilon \to 0} \mathcal{E}_\varepsilon(t, u_\varepsilon, \chi_\varepsilon) \ge \mathbf{E}(t, u_0, U_1, z_0)$.

Moreover, for every $(\widetilde{u}_0, \widetilde{U}_1, \widetilde{z}_0) \in \mathbf{Q}$ there exists a sequence $(\widetilde{u}_\varepsilon, \widetilde{\chi}_\varepsilon)_{\varepsilon > 0} \subset \mathcal{Q}_\varepsilon(\Omega)$ such that

$$\begin{aligned} \widetilde{u}_\varepsilon &\to \widetilde{u}_0 && \text{in } \mathrm{H}^1_{\Gamma_{\mathrm{Dir}}}(\Omega)^d, & \widetilde{\chi}_\varepsilon &\overset{*}{\rightharpoonup} \widetilde{z}_0 && \text{in } \mathrm{L}^\infty(\Omega), \\ \widetilde{u}_\varepsilon &\overset{s}{\to} E \widetilde{u}_0 && \text{in } \mathrm{L}^2(\Omega \times \mathcal{Y})^d, & Q_\varepsilon(\widetilde{\chi}_\varepsilon) &\to \widetilde{z}_0 && \text{in } \mathrm{L}^p(\Omega), \\ \nabla \widetilde{u}_\varepsilon &\overset{s}{\to} \nabla_x E \widetilde{u}_0 + \nabla_y \widetilde{U}_1 && \text{in } \mathrm{L}^2(\Omega \times \mathcal{Y})^{d \times d}, & R_{\frac{\varepsilon}{2}}(Q_\varepsilon(\widetilde{\chi}_\varepsilon)) &\rightharpoonup \nabla \widetilde{z}_0 && \text{in } \mathrm{L}^p(\Omega)^d \end{aligned}$$

and $\limsup_{\varepsilon \to 0} \mathcal{E}_\varepsilon(t, \widetilde{u}_\varepsilon, \widetilde{\chi}_\varepsilon) \le \mathbf{E}(t, \widetilde{u}_0, \widetilde{U}_1, \widetilde{z}_0)$.

Proof. lim inf-inequality: According to the assumption we already have $\lim_{\varepsilon \to 0} \langle \ell(t), u_\varepsilon \rangle = \langle \ell(t), u_0 \rangle$ and $\liminf_{\varepsilon \to 0} \| R_{\frac{\varepsilon}{2}}(Q_\varepsilon(\chi_\varepsilon)) \|_{\mathrm{L}^p(\Omega)^d} \ge \| \nabla z_0 \|_{\mathrm{L}^p(\Omega)^d}$.

Moreover, Theorem 6.1 together with $0 \le \mathcal{T}_\varepsilon \chi_\varepsilon \le 1$ and $\operatorname{supp}(\mathcal{T}_\varepsilon \chi_\varepsilon) \subset \overline{[\Omega \times Y]}_\varepsilon$ yields $\mathcal{T}_\varepsilon \chi_\varepsilon \to \mathbb{1}_{U(z_0^{\mathrm{ex}})}$ in $\mathrm{L}^p(\mathbb{R}^d \times Y)$ for every $p \in [1, \infty)$. Now, with $f(x, \mathcal{X}, e) := \langle \mathbb{C}(\mathcal{X}) e, e \rangle_{d \times d}$, we are in the position to apply Theorem 3.23 of [6] yielding ($\mathrm{L}^2 := \mathrm{L}^2(\mathbb{R}^d \times \mathcal{Y})^{d \times d}$)

$$\liminf_{\varepsilon \to 0} \langle \mathbb{C}(\mathcal{T}_\varepsilon \chi_\varepsilon) \mathcal{T}_\varepsilon \mathbf{e}(u_\varepsilon), \mathcal{T}_\varepsilon \mathbf{e}(u_\varepsilon) \rangle_{\mathrm{L}^2} \ge \langle \mathbb{C}_0(z_0^{\mathrm{ex}}) \widetilde{\mathbf{e}}(u_0^{\mathrm{ex}}, U_1^{\mathrm{ex}}), \widetilde{\mathbf{e}}(u_0^{\mathrm{ex}}, U_1^{\mathrm{ex}}) \rangle_{\mathrm{L}^2}.$$

Altogether we proved $\liminf_{\varepsilon \to 0} \mathcal{E}_\varepsilon(t, u_\varepsilon, \chi_\varepsilon) \ge \mathbf{E}(t, u_0, U_1, z_0)$ for every $t \in [0, T]$, by taking the integral identity (5.2) and $\operatorname{supp}(\mathbb{1}_{U(z_0^{\mathrm{ex}})}) \subset \Omega \times Y$ into account.

lim sup-inequality: For a given function $(\widetilde{u}_0, \widetilde{U}_1, \widetilde{z}_0) \in \mathbf{Q}$ choosing the displacement component $\widetilde{u}_\varepsilon \in \mathrm{H}^1_{\Gamma_{\mathrm{Dir}}}(\Omega)^d$ as in step 1 of the proof of Theorem 7.4 yields the stated convergence results.

Moreover, keeping Remark 7.5 in mind we are able to construct $\widetilde{\chi}_\varepsilon \in \mathbb{X}^D_{\varepsilon\Lambda}(\Omega)$ as in step 2 of the proof of Theorem 7.4, where $(\chi_\varepsilon)_{\varepsilon>0} \subset \mathbb{X}^D_{\varepsilon\Lambda}(\Omega)$ is chosen such that $\chi_\varepsilon \equiv 1$ and $z_0 \in \mathrm{W}^{1,p}(\Omega)$ such that $z_0 \equiv 1$. Then according to step 2 of the proof of Theorem 7.4 we have $Q_\varepsilon \widetilde{\chi}_\varepsilon \to \widetilde{z}_0$ in $\mathrm{L}^p(\Omega)$, $R_{\frac{\varepsilon}{2}}(Q_\varepsilon(\widetilde{\chi}_\varepsilon)) \rightharpoonup \nabla \widetilde{z}_0$ in $\mathrm{L}^p(\Omega)^d$ and

$$\limsup_{\varepsilon\to 0} \|R_{\frac{\varepsilon}{2}}(Q_\varepsilon(\widetilde{\chi}_\varepsilon))\|^p_{\mathrm{L}^p(\Omega)^d} \le \|\nabla \widetilde{z}_0\|^p_{\mathrm{L}^p(\Omega)^d} \tag{7.8}$$

(see (7.3)). Finally, according to Theorem 6.1 we have $\mathcal{T}_\varepsilon \widetilde{\chi}_\varepsilon \to \mathbb{1}_{U(\widetilde{z}_0^{\mathrm{ex}})}$ almost everywhere in $\mathbb{R}^d \times Y$ at least for a subsequence (not relabeled).

By adopting the notation of Proposition 5.4, the combination of the results for the displacements and the damage functions gives $w_\varepsilon := \mathbb{C}(\widetilde{\chi}_\varepsilon)\mathbf{e}(\widetilde{u}_\varepsilon) \xrightarrow{s} \mathbb{C}_0(\widetilde{z}_0)\widetilde{\mathbf{e}}(\widetilde{u}_0, \widetilde{U}_1) =: W_0$ in $\mathrm{L}^2(\Omega\times\mathcal{Y})$ by applying Proposition 5.4(c) for $m_\varepsilon := \mathbb{C}(\widetilde{\chi}_\varepsilon)$, $M_0 := \mathbb{C}_0(\widetilde{z}_0)$ and $v_\varepsilon := \mathbf{e}(\widetilde{u}_\varepsilon)$, $V_0 := \widetilde{\mathbf{e}}(\widetilde{u}_0, \widetilde{U}_1)$. Finally, this gives

$$\lim_{\varepsilon\to 0} \langle \mathbb{C}(\widetilde{\chi}_\varepsilon)\mathbf{e}(\widetilde{u}_\varepsilon), \mathbf{e}(\widetilde{u}_\varepsilon)\rangle_{\mathrm{L}^2(\Omega)^{d\times d}} = \langle \mathbb{C}_0(\widetilde{z}_0)\widetilde{\mathbf{e}}(\widetilde{u}_0, \widetilde{U}_1), \widetilde{\mathbf{e}}(\widetilde{u}_0, \widetilde{U}_1)\rangle_{\mathrm{L}^2(\Omega\times\mathcal{Y})^{d\times d}}, \tag{7.9}$$

by exploiting Proposition 5.4(a). Combining (7.8), (7.9) and $\lim_{\varepsilon\to 0}\langle \ell(t), \widetilde{u}_\varepsilon\rangle = \langle \ell(t), \widetilde{u}_0\rangle$ finishes the proof. □

Now we are in the position to state the final result of this section, saying that the solutions of the microscopic model (S^ε) and (E^ε) introduced in Subsection 4.2 converges to a solution of the effective two-scale model ($\mathbf{S}^0$) and ($\mathbf{E}^0$) introduced in Subsection 4.3.

Theorem 7.7 (Convergence result). *Let $\ell \in \mathrm{C}^1([0,T]; (\mathrm{H}^1_{\Gamma_{\mathrm{Dir}}}(\Omega)^d)^*)$. Furthermore, let $(u_\varepsilon, \chi_\varepsilon) : [0,T] \to \mathcal{Q}_\varepsilon(\Omega)$ be a solution of (S^ε) and (E^ε) with $(u_\varepsilon(0), \chi_\varepsilon(0)) = (u^0_\varepsilon, \chi^0_\varepsilon)$ and assume that there exists a triple $(u^0_0, U^0_1, z^0_0) \in \mathbf{Q}$ such that the initial values satisfy the following:*

$$\begin{aligned} u^0_\varepsilon &\to u^0_0 && \text{in } \mathrm{H}^1_{\Gamma_{\mathrm{Dir}}}(\Omega)^d, & \chi^0_\varepsilon &\overset{*}{\rightharpoonup} z^0_0 && \text{in } \mathrm{L}^\infty(\Omega),\\ u^0_\varepsilon &\xrightarrow{s} E u^0_0 && \text{in } \mathrm{L}^2(\Omega\times\mathcal{Y})^d, & Q_\varepsilon(\chi^0_\varepsilon) &\to z^0_0 && \text{in } \mathrm{L}^p(\Omega),\\ \nabla u^0_\varepsilon &\xrightarrow{s} \nabla_x E u^0_0 + \nabla_y U^0_1 && \text{in } \mathrm{L}^2(\Omega\times\mathcal{Y})^{d\times d}, & R_{\frac{\varepsilon}{2}}(Q_\varepsilon(\chi^0_\varepsilon)) &\to \nabla z^0_0 && \text{in } \mathrm{L}^p(\Omega)^d. \end{aligned}$$

Then there exists $(u_0, U_1, z_0) : [0,T] \to \mathbf{Q}$ and a subsequence of $(\varepsilon)_{\varepsilon>0}$ (not relabeled) satisfying for all $t \in [0,T]$

$$\begin{aligned} u_\varepsilon(t) &\to u_0(t) && \text{in } \mathrm{H}^1_{\Gamma_{\mathrm{Dir}}}(\Omega)^d, & \chi_\varepsilon(t) &\overset{*}{\rightharpoonup} z_0(t) && \text{in } \mathrm{L}^\infty(\Omega),\\ u_\varepsilon(t) &\xrightarrow{s} E u_0(t) && \text{in } \mathrm{L}^2(\Omega\times\mathcal{Y})^d, & Q_\varepsilon(\chi_\varepsilon(t)) &\to z_0(t) && \text{in } \mathrm{L}^p(\Omega),\\ \nabla u_\varepsilon(t) &\xrightarrow{s} \nabla_x E u_0(t) + \nabla_y U_1(t) && \text{in } \mathrm{L}^2(\Omega\times\mathcal{Y})^{d\times d}, & R_{\frac{\varepsilon}{2}}(Q_\varepsilon(\chi_\varepsilon(t))) &\to \nabla z_0(t) && \text{in } \mathrm{L}^p(\Omega)^d \end{aligned}$$

and solving ($\mathbf{S}^0$) and ($\mathbf{E}^0$) with $(u_0(0), U_1(0), z_0(0)) = (u^0_0, U^0_1, z^0_0)$. Moreover, for all $t \in [0,T]$ it holds

$$\begin{aligned} &\lim_{\varepsilon\to 0} \mathcal{E}_\varepsilon(t, u_\varepsilon(t), \chi_\varepsilon(t)) = \mathbf{E}(t, u_0(t), U_1(t), z_0(t)),\\ &\lim_{\varepsilon\to 0} \mathrm{Diss}_{\mathcal{D}_\varepsilon}(\chi_\varepsilon; [0,t]) = \mathrm{Diss}_{\mathbf{D}}(z_0; [0,t]). \end{aligned}$$

Proof. 1. Let $(u_\varepsilon, \chi_\varepsilon) : [0, T] \to \mathcal{Q}_\varepsilon(\Omega)$ be a solution of (S^ε) and (E^ε) with $(u_\varepsilon(0), \chi_\varepsilon(0)) = (u_\varepsilon^0, \chi_\varepsilon^0)$. We start by proving a-priori estimates. According to (4.14) we obtain inequality (7.10) below which is further estimated by exploiting the non-negativity of $\mathrm{Diss}_{\mathcal{D}_\varepsilon}(\chi_\varepsilon; [0, t])$ in the energy balance (E^ε).

$$
\begin{aligned}
\widehat{C}\|u_\varepsilon(t)\|^2_{\mathrm{H}^1_{\Gamma_{\mathrm{Dir}}}(\Omega)^d} &\leq \mathcal{E}_\varepsilon(t, u_\varepsilon(t), \chi_\varepsilon(t)) + C_\ell \|u_\varepsilon(t)\|_{\mathrm{H}^1_{\Gamma_{\mathrm{Dir}}}(\Omega)^d} \\
&\overset{(\mathrm{E}^\varepsilon)}{\leq} \mathcal{E}_\varepsilon(0, u_\varepsilon^0, \chi_\varepsilon^0) - \int_0^t \langle \dot{\ell}(s), u_\varepsilon(s)\rangle \mathrm{d}s + C_\ell \|u_\varepsilon(t)\|_{\mathrm{H}^1_{\Gamma_{\mathrm{Dir}}}(\Omega)^d}. \qquad (7.10)
\end{aligned}
$$

According to the assumption on $(u_\varepsilon^0, \chi_\varepsilon^0)_{\varepsilon>0}$ there exists a constant $\widetilde{C} > 0$ such that $\mathcal{E}_\varepsilon(0, u_\varepsilon^0, \chi_\varepsilon^0) \leq \widetilde{C}$ for all $\varepsilon > 0$. Taking the supremum $\sup_{t\in[0,T]}$ on both sides we end up with

$$
\sup_{t\in[0,T]} \widehat{C}\|u_\varepsilon(t)\|^2_{\mathrm{H}^1_{\Gamma_{\mathrm{Dir}}}(\Omega)^d} \leq \widetilde{C} + (TC_{\dot{\ell}} + C_\ell) \sup_{t\in[0,T]} \|u_\varepsilon(t)\|_{\mathrm{H}^1_{\Gamma_{\mathrm{Dir}}}(\Omega)^d}, \qquad (7.11)
$$

where $C_{\dot{\ell}} := \sup_{t\in[0,T]} \|\dot{\ell}(t)\|_{(\mathrm{H}^1_{\Gamma_{\mathrm{Dir}}}(\Omega)^d)^*}$. This immediately gives us that the right hand side of the energy balance (E^ε) is uniformly bounded which results in a uniform bound of the total dissipation $\mathrm{Diss}_{\mathcal{D}_\varepsilon}(\chi_\varepsilon; [0, t])$ on the left hand side. But in this case the total dissipation simplifies to

$$
\mathrm{Diss}_{\mathcal{D}_\varepsilon}(\chi_\varepsilon; [0, t]) = \int_\Omega \chi_\varepsilon(0, x) - \chi_\varepsilon(t, x)\mathrm{d}x, \qquad (7.12)
$$

which is bounded by the value $\mathrm{vol}(\Omega)$ since $0 \leq \chi_\varepsilon \leq 1$ and since $\chi_\varepsilon : [0, T] \to \mathbb{X}^D_{\varepsilon\Lambda}(\Omega)$ is monotonously decreasing.

Estimating $\|R_{\frac{\varepsilon}{2}}(Q_\varepsilon(\chi_\varepsilon(t)))\|^p_{\mathrm{L}^p(\Omega_\varepsilon^+)^d}$ in the same way as in (7.10) we have

$$
\sup_{t\in[0,T]} \|R_{\frac{\varepsilon}{2}}(Q_\varepsilon(\chi_\varepsilon(t)))\|^p_{\mathrm{L}^p(\Omega_\varepsilon^+)^d} \leq C,
$$

where $C > 0$ only depends on $T > 0$ and $\ell \in \mathrm{C}^1([0, T]; (\mathrm{H}^1_{\Gamma_{\mathrm{Dir}}}(\Omega)^d)^*)$ and satisfies $\sup_{t\in[0,T]} \|u_\varepsilon(t)\|_{\mathrm{H}^1_{\Gamma_{\mathrm{Dir}}}(\Omega)^d} \leq C$ according to (7.11). Moreover, $\|Q_\varepsilon(\chi_\varepsilon(t))\|^p_{\mathrm{L}^p(\Omega)} \leq \mathrm{vol}(\Omega)$ for every $\varepsilon > 0$ and all $t \in [0, T]$ since $0 \leq Q_\varepsilon(\chi_\varepsilon(t)) \leq 1$ by definition. Taking all together this results in the following uniform bound of the solution $(u_\varepsilon, \chi_\varepsilon)$: There exists a constant $0 < C < \infty$ such that for all $\varepsilon > 0$ it holds:

$$
\sup_{t\in[0,T]} \left(\|u_\varepsilon(t)\|_{\mathrm{H}^1_{\Gamma_{\mathrm{Dir}}}(\Omega)^d} + \|Q_\varepsilon(\chi_\varepsilon(t))\|^p_{\mathrm{L}^p(\Omega)} + \|R_{\frac{\varepsilon}{2}}(Q_\varepsilon(\chi_\varepsilon(t)))\|^p_{\mathrm{L}^p(\Omega_\varepsilon^+)^d}\right) \leq C. \qquad (7.13)
$$

2. Now we prove convergence of the damage functions of the same subsequence for every $t \in [0, T]$. For this, we define $\delta_\varepsilon : [0, T] \to \mathbb{R}$ via

$$
\delta_\varepsilon(t) := \int_\Omega \chi_\varepsilon(t, x)\mathrm{d}x. \qquad (7.14)
$$

Then δ_ε is monotonously decreasing with respect to t and uniformly bounded by $\mathrm{vol}(\Omega)$, since for all $\varepsilon > 0$ we have $0 \leq \chi_\varepsilon \leq 1$ everywhere on $[0, T]\times\Omega$. This allows us to apply the Helly selection principle, saying that there exists a monotonously decreasing function $\delta_0 \in \mathrm{BV}([0, T]; \mathbb{R})$ and a subsequence $(\varepsilon')_{\varepsilon'>0}$ of $(\varepsilon)_{\varepsilon>0}$ such that for all $t \in [0, T]$

$$
\delta_{\varepsilon'}(t) \to \delta_0(t). \qquad (7.15)
$$

Let $J_0 \subset [0,T]$ be the jump set of δ_0, which is at most countable since $\delta_0 \in \mathrm{BV}([0,T];\mathbb{R})$ is monotone. Then let $K \subset [0,T]\backslash J_0$ be a dense and countable subset and choose $(t_n)_{n\in\mathbb{N}}$ such that $(t_n)_{n\in\mathbb{N}} = K \cup J_0$. According to (7.13) we are able to apply Theorem 2.1 and Theorem 6.1 afterwards such that there exists a sequence of functions $(z_0^{(t_n)})_{n\in\mathbb{N}} \subset \mathrm{W}^{1,p}(\Omega)$ for which we are able to choose a subsequence $(\varepsilon'')_{\varepsilon''>0}$ of $(\varepsilon')_{\varepsilon'>0}$ via a diagonalization argument satisfying for all $n \in \mathbb{N}$ and for $\varepsilon'' \to 0$:

$$\chi_{\varepsilon''}(t_n) \overset{*}{\rightharpoonup} z_0^{(t_n)} \quad \text{in } \mathrm{L}^\infty(\Omega), \tag{7.16a}$$

$$Q_{\varepsilon''}(\chi_{\varepsilon''}(t_n)) \to z_0^{(t_n)} \quad \text{almost everywhere in } \Omega, \tag{7.16b}$$

$$R_{\frac{\varepsilon''}{2}}(Q_{\varepsilon''}(\chi_{\varepsilon''}(t_n))) \rightharpoonup \nabla z_0^{(t_n)} \quad \text{in } \mathrm{L}^p(\Omega)^d. \tag{7.16c}$$

Note, that according to (7.16a) for all $n \in \mathbb{N}$ we have $\int_\Omega \chi_{\varepsilon''}(t_n,x)\mathrm{d}x \overset{\varepsilon''\to 0}{\longrightarrow} \int_\Omega z_0^{(t_n)}(x)\mathrm{d}x$ which results in $\delta_0(t_n) = \int_\Omega z_0^{(t_n)}(x)\mathrm{d}x$ by keeping (7.15) in mind. Hence, for $s,t \in K$ we find

$$\|z_0^{(s)} - z_0^{(t)}\|_{\mathrm{L}^1(\Omega)} = \mathrm{sign}(s-t)\int_\Omega z_0^{(s)}(x) - z_0^{(t)}(x)\mathrm{d}x = \mathrm{sign}(s-t)(\delta_0(s)-\delta_0(t))$$

converging to 0 for $s \to t$. This means that the function $\zeta_0 : K \to \mathrm{W}^{1,p}(\Omega)$ defined by $\zeta_0(t_n) := z_0^{(t_n)}$ for all $t_n \in K$ is continuous with respect to $\|\cdot\|_{\mathrm{L}^1(\Omega)}$. With this let $z_0 : [0,T] \to \mathrm{L}^1(\Omega)$ be defined via

(a) $z_0(t_n) = z_0^{(t_n)}$ for all $n \in \mathbb{N}$

(b) $z_0|_{[0,T]\backslash J_0}$ is the continuous extension of ζ_0 with respect to $\|\cdot\|_{\mathrm{L}^1(\Omega)}$

Now we prove $\chi_{\varepsilon''}(t) \overset{*}{\rightharpoonup} z_0(t)$ in $\mathrm{L}^\infty(\Omega)$ for all $t \in [0,T]$. Firstly, since $0 \le Q_{\varepsilon''}(\chi_{\varepsilon''}(t_n)) \le 1$ and (7.16b) we have $0 \le \zeta_0(t_n) \le 1$ almost everywhere on Ω. Exploiting the continuity of $z_0|_{[0,T]\backslash J_0}$ for all $t \in [0,T]$ this results in $0 \le z_0(t) \le 1$ almost everywhere on Ω (contradiction argument) which implies

$$\|z_0(t)\|_{\mathrm{L}^\infty(\Omega)} \le 1 \qquad \text{for all } t \in [0,T]. \tag{7.17}$$

Moreover, for $t \in [0,T]\backslash(t_n)_{n\in\mathbb{N}}$ and $\varphi \in \mathrm{C}_\mathrm{c}^\infty(\Omega)$ by choosing $t_m \in K$ such that $t < t_m$ we have

$$\begin{aligned}
&\lim_{\varepsilon''\to 0} |\langle\varphi, \chi_{\varepsilon''}(t) - z_0(t)\rangle| \\
&\quad \le \lim_{\varepsilon''\to 0}\Big(\|\varphi\|_{\mathrm{L}^\infty(\Omega)}\|\chi_{\varepsilon''}(t) - \chi_{\varepsilon''}(t_m)\|_{\mathrm{L}^1(\Omega)} + \langle\varphi, \chi_{\varepsilon''}(t_m) - z_0(t_m)\rangle\Big) \\
&\qquad + \|\varphi\|_{\mathrm{L}^\infty(\Omega)}\|z_0(t_m) - z_0(t)\|_{\mathrm{L}^1(\Omega)} \\
&\quad = \lim_{\varepsilon''\to 0}\Big(\|\varphi\|_{\mathrm{L}^\infty(\Omega)}(\delta_{\varepsilon''}(t) - \delta_{\varepsilon''}(t_m))\Big) + \|\varphi\|_{\mathrm{L}^\infty(\Omega)}\|z_0(t_m) - z_0(t)\|_{\mathrm{L}^1(\Omega)} \\
&\quad = \|\varphi\|_{\mathrm{L}^\infty(\Omega)}(\delta_0(t) - \delta_0(t_m)) + \|\varphi\|_{\mathrm{L}^\infty(\Omega)}\|z_0(t_m) - z_0(t)\|_{\mathrm{L}^1(\Omega)},
\end{aligned} \tag{7.18}$$

where we already exploited (7.14), (7.16a) and (7.15). Since δ_0 and z_0 are continuous on $[0,T]\backslash J_0$ we can choose $t_m \in K$ with $t < t_m$ such that (7.18) gets arbitrarily small. Combining (7.17) and (7.18) we finally have for all $t \in [0,T]$

$$\chi_{\varepsilon''}(t) \overset{*}{\rightharpoonup} z_0(t) \text{ in } \mathrm{L}^\infty(\Omega).$$

On the other hand, according to (7.13) we are able to apply Theorem 2.1 and Theorem 6.1 afterwards again, such that for arbitrary but fixed $t \in [0,T] \backslash (t_n)_{n\in\mathbb{N}}$ there exist a function $\xi^{(t)} \in \mathrm{W}^{1,p}(\Omega)$ and a subsequence $(\varepsilon''')_{\varepsilon'''>0}$ of $(\varepsilon'')_{\varepsilon''>0}$ satisfying

$$\chi_{\varepsilon'''}(t) \overset{*}{\rightharpoonup} \xi^{(t)} \quad \text{in } \mathrm{L}^\infty(\Omega), \tag{7.19a}$$

$$Q_{\varepsilon'''}(\chi_{\varepsilon'''}(t)) \to \xi^{(t)} \quad \text{almost everywhere in } \Omega, \tag{7.19b}$$

$$R_{\frac{\varepsilon'''}{2}}(Q_{\varepsilon'''}(\chi_{\varepsilon'''}(t))) \rightharpoonup \nabla\xi^{(t)} \quad \text{in } \mathrm{L}^p(\Omega)^d. \tag{7.19c}$$

Since $t \in [0,T] \backslash (t_n)_{n\in\mathbb{N}}$ was chosen arbitrarily and we already proved $\chi_{\varepsilon''}(t) \overset{*}{\rightharpoonup} z_0(t)$ in $\mathrm{L}^\infty(\Omega)$ for all $t \in [0,T]$, this first of all gives $z_0(t) \in \mathrm{W}^{1,p}(\Omega)$ for all $t \in [0,T]$. Secondly, with $\xi^{(t)} = z_0(t)$ the convergence result (7.19) is valid for all converging subsequences of $(\varepsilon'')_{\varepsilon''>0}$ such that we conclude that (7.19) holds for the whole sequence $(\varepsilon'')_{\varepsilon''>0}$.

Recapitulating all results proven in this step we now have that there exists a monotone function $z_0 \in \mathrm{BV}([0,T];\mathrm{L}^1(\Omega)) \cap \mathrm{L}^\infty([0,T];\mathrm{W}^{1,p}(\Omega;[0,1]))$ and a subsequence of $(\varepsilon)_{\varepsilon>0}$ (not relabeled) such that the following is valid for all $t \in [0,T]$ and for $\varepsilon \to 0$:

$$\chi_\varepsilon(t) \overset{*}{\rightharpoonup} z_0(t) \quad \text{in } \mathrm{L}^\infty(\Omega), \tag{7.20a}$$

$$\mathcal{T}_\varepsilon(\chi_\varepsilon(t))(x,y) \to \mathbb{1}_{U(z_0^{\mathrm{ex}}(t,x))}(y) \quad \text{for almost every } (x,y) \in \mathbb{R}^d \times Y, \tag{7.20b}$$

$$Q_\varepsilon(\chi_\varepsilon(t)) \to z_0(t) \quad \text{almost everywhere in } \Omega, \tag{7.20c}$$

$$R_{\frac{\varepsilon}{2}}(Q_\varepsilon(\chi_\varepsilon(t))) \rightharpoonup \nabla z_0(t) \quad \text{in } \mathrm{L}^p(\Omega)^d, \tag{7.20d}$$

where we added the second convergence result (7.20b) of Theorem 6.1.

3. Now we prove convergence of the displacements of the same subsequence constructed in step 2 for every $t \in [0,T]$. Therefore, let the functions $u_0 : [0,T] \to \mathrm{H}^1_{\Gamma_{\mathrm{Dir}}}(\Omega)^d$ and $U_1 : [0,T] \to \mathrm{L}^2(\Omega;\mathrm{H}^1_{\mathrm{av}}(\mathcal{Y}))^d$ be uniquely defined by

$$u_0(t) \in \arg\min_{u\in\mathrm{H}^1_{\Gamma_{\mathrm{Dir}}}(\Omega)^d} \mathcal{E}_0(t,u,z_0(t)), \tag{7.21}$$

$$U_1(t) := \mathcal{L}_{z_0(t)}(\mathbf{e}_x(u_0(t))) \qquad \text{(see Proposition 4.13)}$$

where $z_0 : [0,T] \to \mathrm{W}^{1,p}(\Omega)$ is the function defined in step 2.

On the other hand for fixed $t \in [0,T]$ we have $(u_\varepsilon(t), \chi_\varepsilon(t)) \in \mathcal{S}_\varepsilon(t)$ by assumption. According to (7.13) and Proposition 5.5 there exist $(u_0^{(t)}, U_1^{(t)}) \in \mathrm{H}^1_{\Gamma_{\mathrm{Dir}}}(\Omega)^d \times \mathrm{L}^2(\Omega;\mathrm{H}^1_{\mathrm{av}}(\mathcal{Y}))^d$ and a subsequence $(\varepsilon')_{\varepsilon'>0}$ of the sequence $(\varepsilon)_{\varepsilon>0}$ constructed in (7.20) such that

$$u_{\varepsilon'}(t) \rightharpoonup u_0^{(t)} \quad \text{in } \mathrm{H}^1_{\Gamma_{\mathrm{Dir}}}(\Omega)^d, \tag{7.22a}$$

$$u_{\varepsilon'}(t) \overset{s}{\to} Eu_0^{(t)} \quad \text{in } \mathrm{L}^2(\Omega\times\mathcal{Y})^d, \tag{7.22b}$$

$$\nabla u_{\varepsilon'}(t) \overset{w}{\rightharpoonup} \nabla_x Eu_0^{(t)} + \nabla_y U_1^{(t)} \quad \text{in } \mathrm{L}^2(\Omega\times\mathcal{Y})^{d\times d}. \tag{7.22c}$$

With this, all assumptions of Theorem 7.4 are fulfilled and $(u_0^{(t)}, U_1^{(t)}, z_0(t)) \in \mathbf{S}_0(t)$ due to point (b). Following Proposition 4.13(i) this is equivalent to

$$U_1^{(t)} = \mathcal{L}_{z_0(t)}(\mathbf{e}_x(u_0^{(t)})) \quad \text{and} \quad (u_0^{(t)}, z_0(t)) \in \mathcal{S}_0(t). \tag{7.23}$$

By choosing $\widetilde{z} = z_0(t)$ in the stability condition (S^0) we find

$$u_0^{(t)} \in \arg\min_{u\in\mathrm{H}^1_{\Gamma_{\mathrm{Dir}}}(\Omega)^d} \mathcal{E}_0(t,u,z_0(t)). \tag{7.24}$$

Comparing (7.21) and (7.24) we obtain $u_0^{(t)} = u_0(t)$ and $U_1^{(t)} = U_1(t)$ according to (7.23). Since this is valid for all converging subsequences in (7.22) we conclude

$$\begin{aligned} u_\varepsilon(t) &\rightharpoonup u_0(t) && \text{in } \mathrm{H}^1_{\Gamma_{\mathrm{Dir}}}(\Omega)^d, && (7.25a)\\ u_\varepsilon(t) &\stackrel{s}{\to} E u_0(t) && \text{in } \mathrm{L}^2(\Omega\times\mathcal{Y})^d, && (7.25b)\\ \nabla u_\varepsilon(t) &\stackrel{w}{\rightharpoonup} \nabla_x E u_0(t) + \nabla_y U_1(t) && \text{in } \mathrm{L}^2(\Omega\times\mathcal{Y})^{d\times d}, && (7.25c) \end{aligned}$$

where $(\varepsilon)_{\varepsilon>0}$ is the sequence constructed in (7.20).

Note, that in this step we already proved $(u_0(t), U_1(t), z_0(t)) \in \mathbf{S}_0(t)$ for all $t \in [0,T]$. Moreover, due to (7.13) we have $u_0 \in \mathrm{L}^\infty([0,T]; \mathrm{H}^1_{\Gamma_{\mathrm{Dir}}}(\Omega)^d)$ and $U_1 \in \mathrm{L}^\infty([0,T]; \mathrm{L}^2(\Omega; \mathrm{H}^1_{\mathrm{av}}(\mathcal{Y}))^d)$.

4. For proving that $(u_0, U_1, z_0) : [0,T] \to \mathbf{Q}$ satisfies the energy balance $(\mathbf{E}^0)$ we pass in (E^ε) to the limit $\varepsilon \to 0$. We start with the right hand side. Due to the uniform bound (7.13) we have $|\langle \dot\ell(s), u_\varepsilon(s)\rangle| \le C_{\dot\ell} C$ for every $\varepsilon > 0$ and all $s \in [0,t]$ such that we obtain

$$\lim_{\varepsilon\to0} \int_0^t \langle \dot\ell(s), u_\varepsilon(s)\rangle \mathrm{d}s = \int_0^t \lim_{\varepsilon\to0} \langle \dot\ell(s), u_\varepsilon(s)\rangle \mathrm{d}s = \int_0^t \langle \dot\ell(s), u_0(s)\rangle \mathrm{d}s \qquad (7.26)$$

by applying Lebesgue's Theorem of dominated convergence and making use of $u_\varepsilon(s) \rightharpoonup u_0(s)$ in $\mathrm{H}^1_{\Gamma_{\mathrm{Dir}}}(\Omega)^d$ for all $s \in [0,t]$.

Adopting the notation of Proposition 5.4 let $m_\varepsilon := \mathbb{C}(\chi_\varepsilon^0)$, $M_0 := \mathbb{C}_0(z_0^0)$ and $v_\varepsilon := \mathbf{e}(u_\varepsilon^0)$, $V_0 := \widetilde{\mathbf{e}}(u_0^0, U_1^0)$. Then Proposition 5.4(c) together with the assumptions for the initial values $(\chi_\varepsilon^0)_{\varepsilon>0}$ and $(u_\varepsilon^0)_{\varepsilon>0}$ give $w_\varepsilon := \mathbb{C}(\chi_\varepsilon^0)\mathbf{e}(u_\varepsilon^0) \stackrel{s}{\to} \mathbb{C}_0(z_0^0)\widetilde{\mathbf{e}}(u_0^0, U_1^0) =: W_0$ in $\mathrm{L}^2(\Omega\times\mathcal{Y})$. With this, Proposition 5.4(a) gives

$$\lim_{\varepsilon\to0} \langle \mathbb{C}(\chi_\varepsilon^0)\mathbf{e}(u_\varepsilon^0), \mathbf{e}(u_\varepsilon^0)\rangle_{\mathrm{L}^2(\Omega)^{d\times d}} = \langle \mathbb{C}_0(z_0^0)\widetilde{\mathbf{e}}(u_0^0, U_1^0), \widetilde{\mathbf{e}}(u_0^0, U_1^0)\rangle_{\mathrm{L}^2(\Omega\times\mathcal{Y})^{d\times d}},$$

which finally results in $\lim_{\varepsilon\to0} \mathcal{E}_\varepsilon(t, u_\varepsilon^0, \chi_\varepsilon^0) = \mathbf{E}(t, u_0^0, U_1^0, z_0^0)$.

5. Left hand side of (E^ε): According to the convergence results of step 2 and 3 all assumptions of Theorem 7.6 are fulfilled, such that for all $t \in [0,T]$ we have

$$\liminf_{\varepsilon\to0} \mathcal{E}_\varepsilon(t, u_\varepsilon(t), \chi_\varepsilon(t)) \ge \mathbf{E}(t, u_0(t), U_1(t), z_0(t)). \qquad (7.27)$$

Exploiting the structure (7.12) of $\mathrm{Diss}_{\mathcal{D}_\varepsilon}(\chi_\varepsilon; [0,t])$ and the convergence result of step 2 we have

$$\mathrm{Diss}_{\mathcal{D}_\varepsilon}(\chi_\varepsilon; [0,t]) = \int_\Omega \chi_\varepsilon(0) - \chi_\varepsilon(t) \mathrm{d}x \stackrel{\varepsilon\to0}{\longrightarrow} \int_\Omega z_0(0) - z_0(t) \mathrm{d}x = \mathrm{Diss}_{\mathbf{D}}(z_0; [0,t]), \qquad (7.28)$$

where we exploited the monotonicity of $z_0 : [0,T] \to \mathrm{W}^{1,p}(\Omega)$ for the last equality.

Adding (7.27) and (7.28) and combing this with the convergence results of step 4 for all $t \in [0,T]$ we end up with

$$\mathbf{E}(t, u_0(t), U_1(t), z_0(t)) + \mathrm{Diss}_{\mathbf{D}}(z_0; [0,t]) \le \mathbf{E}(t, u_0(0), U_1(0), z_0(0)) - \int_0^t \langle \dot\ell(s), u_0(s)\rangle \mathrm{d}s \qquad (7.29)$$

by letting ε tend to 0 in (E^ε). Due to the stability $(u_0(t), U_1(t), z_0(t)) \in \mathcal{S}_0(t)$ proved in step 3 we immediately obtain the opposite inequality to (7.29) according to Proposition

2.4 of [15], such that finally $(u_0, U_1, z_0) : [0,T] \to \mathbf{Q}$ satisfies for all $t \in [0,T]$ the energy balance

$$\mathbf{E}(t, u_0(t), U_1(t), z_0(t)) + \mathrm{Diss}_{\mathbf{D}}(z_0; [0,t]) = \mathbf{E}(t, u_0(0), U_1(0), z_0(0)) - \int_0^t \langle \dot{\ell}(s), u_0(s)\rangle \mathrm{d}s.$$

This implies that (7.27) has to be an equality. Moreover, by subtracting $\mathrm{Diss}_{\mathcal{D}_\varepsilon}(\chi_\varepsilon; [0,t])$ from (E^ε) we see that the limit of the right hand side of (E^ε) exists according to (7.28) and step 4. That means, that the limit of the left hand side also exists such that we finally have for all $t \in [0,T]$

$$\lim_{\varepsilon\to 0} \mathcal{E}_\varepsilon(t, u_\varepsilon(t), \chi_\varepsilon(t)) = \mathbf{E}(t, u_0(t), U_1(t), z_0(t)). \tag{7.30}$$

6. So far we proved that $(u_0(t), U_1(t), z_0(t)) \in \mathbf{Q}$ is a solution of ($\mathbf{S}^0$) and ($\mathbf{E}^0$) and it remains to prove the strong convergence properties. To prove $R_{\frac{\varepsilon}{2}}(Q_\varepsilon(\chi_\varepsilon(t))) \to \nabla z_0(t)$ in $\mathrm{L}^p(\Omega)^d$ let $a_\varepsilon(t) := \langle \mathbb{C}(\chi_\varepsilon(t))\mathbf{e}(u_\varepsilon(t)), \mathbf{e}(u_\varepsilon(t))\rangle_{\mathrm{L}^2(\Omega)^{d\times d}}$ and $b_\varepsilon(t) := \|R_{\frac{\varepsilon}{2}}(Q_\varepsilon(\chi_\varepsilon(t)))\|^p_{\mathrm{L}^p(\Omega)^d}$ and start by recalling

$$\liminf_{\varepsilon\to 0} a_\varepsilon(t) \geq \langle \mathbb{C}_0(z_0(t))\widetilde{\mathbf{e}}(u_0(t), U_1(t)), \widetilde{\mathbf{e}}(u_0(t), U_1(t))\rangle_{\mathrm{L}^2(\Omega\times\mathcal{Y})^{d\times d}} =: a(t) \tag{7.31}$$

analogously to (7.6) and

$$\liminf_{\varepsilon\to 0} b_\varepsilon(t) \geq \|\nabla z_0(t)\|^p_{\mathrm{L}^p(\Omega)^d} =: b(t), \tag{7.32}$$

since $R_{\frac{\varepsilon}{2}}(Q_\varepsilon(\chi_\varepsilon(t))) \rightharpoonup \nabla z_0(t)$ in $\mathrm{L}^p(\Omega)^d$ was proved in step 2. This together with (7.30) yields

$$a(t)+b(t) \leq \liminf_{\varepsilon\to 0} a_\varepsilon(t) + \liminf_{\varepsilon\to 0} b_\varepsilon(t) \leq \lim_{\varepsilon\to 0}(a_\varepsilon(t)+b_\varepsilon(t)) \overset{(7.30)}{=} a(t)+b(t),$$

stating that (7.31) and (7.32) actually are equalities. But with this, via a contradiction argument we even have $\lim_{\varepsilon\to 0} a_\varepsilon(t) = a(t)$ and $\lim_{\varepsilon\to 0} b_\varepsilon(t) = b(t)$. Since weak convergence combined with norm convergence gives strong convergence (see [2] Exercise 6.6), we finally proved that in (7.19c) we have actually have strong convergence, namely, $R_{\frac{\varepsilon}{2}}(Q_\varepsilon(\chi_\varepsilon(t))) \to \nabla z_0(t)$ in $\mathrm{L}^p(\Omega)^d$.

7. To shorten notation let $(u^t_\varepsilon, \chi^t_\varepsilon, u^t_0, U^t_1, z^t_0) := (u_\varepsilon(t), \chi_\varepsilon(t), u_0(t), U_1(t), z_0(t))$. To prove $\nabla u^t_\varepsilon \to \nabla u^t_0$ in $\mathrm{L}^2(\Omega)^{d\times d}$ and $\nabla u^t_\varepsilon \overset{s}{\to} \nabla_x E u^t_0 + \nabla_y U^t_1$ in $\mathrm{L}^2(\Omega\times\mathcal{Y})^{d\times d}$, choose $\widetilde{u}^t_\varepsilon := u^t_0 + v^t_\varepsilon$, where $v^t_\varepsilon \in \mathrm{H}^1_0(\Omega)^d$ is the solution of the elliptic problem stated in Proposition 5.6 with $v^t_0 = 0 \in \mathrm{H}^1_0(\Omega)^d$ and $V^t_1 = U^t_1 \in \mathrm{L}^2(\Omega; \mathrm{H}^1_{\mathrm{av}}(\mathcal{Y}))^d$. Observe that

$$\widetilde{u}^t_\varepsilon \to u^t_0 \quad \text{in } \mathrm{H}^1_{\Gamma_{\mathrm{Dir}}}(\Omega)^d, \tag{7.33a}$$
$$\widetilde{u}^t_\varepsilon \overset{s}{\to} E u^t_0 \quad \text{in } \mathrm{L}^2(\Omega\times\mathcal{Y})^d, \tag{7.33b}$$
$$\nabla\widetilde{u}^t_\varepsilon \overset{s}{\to} \nabla_x E u^t_0 + \nabla_y U^t_1 \quad \text{in } \mathrm{L}^2(\Omega\times\mathcal{Y})^{d\times d}, \tag{7.33c}$$

due to Proposition 5.6. Then analogously to step 3 of the proof of Theorem 7.4 we have

$$\lim_{\varepsilon\to 0}\langle \mathbb{C}(\chi^t_\varepsilon)\mathbf{e}(\widetilde{u}^t_\varepsilon), \mathbf{e}(\widetilde{u}^t_\varepsilon)\rangle_{\mathrm{L}^2(\Omega)^{d\times d}} = \langle \mathbb{C}_0(z^t_0)\widetilde{\mathbf{e}}(u^t_0, U^t_1), \widetilde{\mathbf{e}}(u^t_0, U^t_1)\rangle_{\mathrm{L}^2(\Omega\times\mathcal{Y})^{d\times d}}.$$

Note, that here $(\widetilde{u}_0, \widetilde{U}_1, \widetilde{z}_0)$ is replaced by (u_0^t, U_1^t, z_0^t). Moreover, according to (7.33a) we have $\lim_{\varepsilon\to 0}\langle \ell(t), \widetilde{u}_\varepsilon^t\rangle = \langle \ell(t), u_0^t\rangle$ and $\lim_{\varepsilon\to 0}\|R_{\frac{\varepsilon}{2}}(Q_\varepsilon(\chi_\varepsilon^t))\|_{\mathrm{L}^p(\Omega)^d} = \|\nabla z_0^t\|_{\mathrm{L}^p(\Omega)^d}$ was proved in step 6. All this together results in

$$\lim_{\varepsilon\to 0}\mathcal{E}_\varepsilon(t, \widetilde{u}_\varepsilon^t, \chi_\varepsilon^t) = \mathbf{E}(t, u_0^t, U_1^t, z_0^t). \tag{7.34}$$

According to assumption (4.7) we obtain ($\mathrm{L}^2 := \mathrm{L}^2(\Omega)^{d\times d}$)

$$\begin{aligned}
\alpha\|\mathbf{e}(u_\varepsilon^t - \widetilde{u}_\varepsilon^t)\|_{\mathrm{L}^2}^2 & \\
&\le \langle \mathbb{C}(\chi_\varepsilon^t)\mathbf{e}(u_\varepsilon^t - \widetilde{u}_\varepsilon^t), \mathbf{e}(u_\varepsilon^t - \widetilde{u}_\varepsilon^t)\rangle_{\mathrm{L}^2} \\
&= \langle \mathbb{C}(\chi_\varepsilon^t)\mathbf{e}(u_\varepsilon^t), \mathbf{e}(u_\varepsilon^t)\rangle_{\mathrm{L}^2} - \langle \mathbb{C}(\chi_\varepsilon^t)\mathbf{e}(\widetilde{u}_\varepsilon^t), \mathbf{e}(\widetilde{u}_\varepsilon^t)\rangle_{\mathrm{L}^2} + 2\langle \mathbb{C}(\chi_\varepsilon^t)\mathbf{e}(\widetilde{u}_\varepsilon^t), \mathbf{e}(\widetilde{u}_\varepsilon^t - u_\varepsilon^t)\rangle_{\mathrm{L}^2} \\
&= 2\mathcal{E}_\varepsilon(t, u_\varepsilon^t, \chi_\varepsilon^t) - 2\mathcal{E}_\varepsilon(t, \widetilde{u}_\varepsilon^t, \chi_\varepsilon^t) + 2\langle \ell(t), u_\varepsilon^t - \widetilde{u}_\varepsilon^t\rangle + 2\langle \mathbb{C}(\chi_\varepsilon^t)\mathbf{e}(\widetilde{u}_\varepsilon^t), \mathbf{e}(\widetilde{u}_\varepsilon^t - u_\varepsilon^t)\rangle_{\mathrm{L}^2}.
\end{aligned} \tag{7.35}$$

According to condition (7.20b) and (7.33c) and Proposition 5.4(c) the term $\mathbb{C}(\chi_\varepsilon^t)\mathbf{e}(\widetilde{u}_\varepsilon^t)$ converges strongly in $\mathrm{L}^2(\Omega\times\mathcal{Y})^{d\times d}$ in the two-scale sense. Hence, due to Proposition 5.4(a) the last term of (7.35) converges to zero, since $\mathbf{e}(\widetilde{u}_\varepsilon^t - u_\varepsilon^t) \xrightarrow{w} 0$ in $\mathrm{L}^2(\Omega\times\mathcal{Y})^{d\times d}$ according to step 3 and again (7.33c). Trivially the second last term of (7.35) converges to zero, too. Recalling (7.30) and (7.34) also the first two terms of (7.35) sum up to zero in the limit such that we end up with

$$\|\nabla(u_\varepsilon^t - \widetilde{u}_\varepsilon^t)\|_{\mathrm{L}^2(\Omega)^{d\times d}}^2 \le C_{\mathrm{Korn}}\|\mathbf{e}(u_\varepsilon^t - \widetilde{u}_\varepsilon^t)\|_{\mathrm{L}^2(\Omega)^{d\times d}}^2 \xrightarrow{\varepsilon\to 0} 0, \tag{7.36}$$

where we already exploited the Korn inequality. Now we conclude the proof by the following two estimates, where we start by adding zero to apply the triangle inequality afterwards.

$$\begin{aligned}
&\|\nabla u_\varepsilon^t - \nabla u_0^t\|_{\mathrm{L}^2(\Omega)^{d\times d}} \\
&\quad \le \|\nabla(u_\varepsilon^t - \widetilde{u}_\varepsilon^t)\|_{\mathrm{L}^2(\Omega)^{d\times d}} + \|\nabla\widetilde{u}_\varepsilon^t - \nabla u_0^t\|_{\mathrm{L}^2(\Omega)^{d\times d}}, && (7.37) \\
&\|\mathcal{T}_\varepsilon(\nabla u_\varepsilon^t) - (\nabla_x E u_0^t + \nabla_y U_1^t)^{\mathrm{ex}}\|_{\mathrm{L}^2(\mathbb{R}^d\times\mathcal{Y})^{d\times d}} \\
&\quad \le \|\mathcal{T}_\varepsilon(\nabla(u_\varepsilon^t - \widetilde{u}_\varepsilon^t))\|_{\mathrm{L}^2(\mathbb{R}^d\times\mathcal{Y})^{d\times d}} + \|\mathcal{T}_\varepsilon(\nabla\widetilde{u}_\varepsilon^t) - (\nabla_x E u_0^t + \nabla_y U_1^t)^{\mathrm{ex}}\|_{\mathrm{L}^2(\mathbb{R}^d\times\mathcal{Y})^{d\times d}}. && (7.38)
\end{aligned}$$

The first terms of (7.37) and (7.38) converge to zero according to (7.36), where we already exploited the norm preservation of the unfolding operator $\mathcal{T}_\varepsilon : \mathrm{L}^2(\Omega)^{d\times d} \to \mathrm{L}^2(\mathbb{R}^d\times\mathcal{Y})^{d\times d}$. Furthermore, according to (7.33a) and (7.33c) the last terms of (7.37) and (7.38) converges to zero, which concludes the proof. □

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